Abstract

We present an algorithm for constructing a depth-first search tree in planar digraphs; the algorithm can be implemented in the complexity class $\text{AC}^1(\text{UL} \cap \text{co-UL})$, which is contained in $\text{AC}^2$. Prior to this (for more than a quarter-century), the fastest uniform deterministic parallel algorithm for this problem was $O(\log^3 n)$ (corresponding to the complexity class $\text{AC}^{10} \subseteq \text{NC}^{11}$).

We also consider the problem of computing depth-first search trees in other classes of graphs, and obtain additional new upper bounds.

1 Introduction

Depth-first search trees (DFS trees) constitute one of the most useful items in the algorithm designer’s toolkit, and for this reason they are a standard part of the undergraduate algorithmic curriculum around the world. When attention shifted to parallel algorithms in the 1980’s, the question arose of whether NC algorithms for DFS trees exist. An early negative result was that the problem of constructing the lexicographically least DFS tree in a given digraph is complete for $\text{P}$ [20]. But soon thereafter significant advances were made in developing parallel algorithms for DFS trees, culminating in the $\text{RNC}^7$ algorithm of Aggarwal, Anderson, and Kao [1]. This remains the fastest parallel algorithm for the problem of constructing DFS trees in general graphs, in the probabilistic setting, or in the setting of nonuniform circuit complexity. It remains unknown if this problem lies in (deterministic) $\text{NC}$ (and we do not solve that problem here).

More is known for various restricted classes of graphs. For directed acyclic graphs (DAGs), the lexicographically-least DFS tree from a given vertex can be computed in $\text{AC}^1$ [10]. (See also [11, 7, 13, 19, 16, 15].) For undirected planar graphs, an $\text{AC}^1$ algorithm for DFS trees was presented by Hagerup [14]. For more general planar directed graphs Kao and Klein presented an $\text{AC}^{10}$ algorithm. Kao subsequently presented an $\text{AC}^5$ algorithm for DFS in strongly connected planar digraphs. In stating the complexity results for this prior work
in terms of complexity classes (such as $\mathcal{AC}^1, \mathcal{AC}^{10}$, etc.), we are ignoring an aspect that
was of particular interest to the authors of this earlier work: minimizing the number of
processors. This is because our focus is on classifying the complexity of constructing DFS
trees in terms of complexity classes. Thus, if we reduce the complexity of a problem from
$\mathcal{AC}^{10}$ to $\mathcal{AC}^2$, then we view this as a significant advance, even if the $\mathcal{AC}^2$ algorithm uses many
more processors (so long as the number of processors remains bounded by a polynomial).
Indeed, our algorithms rely on the logspace algorithm for undirected reachability [21], which
does not directly translate into a processor-efficient algorithm. We suspect that our approach
can be modified to yield a more processor-efficient $\mathcal{AC}^3$ algorithm, but we leave that for
others to investigate.

1.1 Our Contributions

First, we observe that, given a DAG $G$, computation of a DFS tree in $G$ logspace reduces to
the problem of reachability in $G$. Thus, for general DAGs, computation of a DFS tree lies in
$\mathcal{NL}$, and for planar DAGs, the problem lies in $\mathcal{UL} \cap \text{co-UL}$ [8, 23]. For classes of graphs where
the reachability problem lies in $\mathcal{L}$, so does the computation of DFS trees. One such class
of graphs is planar DAGs with a single source (see [2], where this class of graphs is called
SMPDs, for Single-source, Multiple-sink, Planar DAGs).

For undirected planar graphs, it was shown in [4] that the approach of Hagerup’s $\mathcal{AC}^1$
DFS algorithm [14] can be adapted in order to show that construction of a DFS tree in a
planar undirected graph logspace-reduces to computing the distance between two nodes in
a planar digraph. Since this latter problem lies in $\mathcal{UL} \cap \text{co-UL}$ [24], so does the problem of
DFS for planar undirected graphs.

Our main contribution in the current paper is to show that a more sophisticated application
of the ideas in [14] leads to an $\mathcal{AC}^1(\mathcal{UL} \cap \text{co-UL})$ algorithm for construction of DFS trees in
planar directed graphs. (That is, we show DFS trees can be constructed by unbounded fan-in
log-depth circuits that have oracle gates for a set in $\mathcal{UL} \cap \text{co-UL}$.) Since $\mathcal{UL} \subseteq \mathcal{NL} \subseteq \mathcal{SAC}^1 \subseteq
\mathcal{AC}^1$, the $\mathcal{AC}^1(\mathcal{UL} \cap \text{co-UL})$ algorithm can be implemented in $\mathcal{AC}^2$. Thus this is a significant
improvement over the best previous parallel algorithm for this problem: the $\mathcal{AC}^{10}$ algorithm
of [18], which has stood for 28 years.

2 Preliminaries

We assume that the reader is familiar with depth-first search trees (DFS trees).

We further assume that the reader is familiar with the standard complexity classes $\mathcal{L}, \mathcal{NL}$
and $\mathcal{P}$ (see e.g. the text [6]). We will also make frequent reference to the logspace-uniform
circuit complexity classes $\mathcal{NC}^k$ and $\mathcal{AC}^k$. $\mathcal{NC}^k$ is the class of problems for which there is a
logspace-uniform family of circuits $\{C_n\}$ consisting of AND, OR, and NOT gates, where
the AND and OR gates have fan-in two and each circuit $C_n$ has depth $O(\log^k n)$. (The
logspace-uniformity condition implies that each $C_n$ has only $n^{O(1)}$ gates.) $\mathcal{AC}^k$ is defined
similarly, although the AND and OR gates are allowed unbounded fan-in. An equivalent
characterization of $\mathcal{AC}^k$ is in terms of concurrent-read concurrent-write PRAMs with running
time $O(\log^k n)$, using $n^{O(1)}$ processors. For more background on these circuit complexity
classes, see, e.g., the text [26].

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1 An earlier version of this work claimed a stronger upper bound, but there was an error in one of the
lemmas in that version [3].
A nondeterministic Turing machine is said to be unambiguous if, on every input $x$, there is at most one accepting computation path. If we consider logspace-bounded nondeterministic Turing machines, then unambiguous machines yield the class UL. A set $A$ is in co-UL if and only if its complement lies in UL.

The construction of DFS trees is most naturally viewed as a function that takes a graph $G$ and a vertex $v$ as input, and produces as output an encoding of a DFS tree in $G$ rooted at $v$. But the complexity classes mentioned above are all defined as sets of languages, instead of as sets of functions. Since our goal is to place DFS tree construction into the appropriate complexity classes, it is necessary to discuss how the complexity of functions fits into the framework of complexity classes.

When $C$ is one of $\{L, \overline{P}\}$, it is fairly obvious what is meant by “$f$ is computable in $C$”; the classes of logspace-computable functions and polynomial-time-computable functions should be familiar to the reader. However, the reader might be less clear as to what is meant by “$f$ is computable in NL”. As it turns out, essentially all of the reasonable possibilities are equivalent. Let us denote by FN$L$ the class of functions that are computable in NL; it is shown in [17] each of the three following conditions is equivalent to “$f \in$ FN$L$”.

1. $f$ is computed by a logspace machine with an oracle from NL.
2. $f$ is computed by a logspace-uniform NC$^1$ circuit family with oracle gates for a language in NL.
3. $f(x)$ has length bounded by a polynomial in $|x|$, and the set $\{(x, i, b) : \text{the } i^{th} \text{ bit of } f(x) \text{ is } b\}$ is in NL.

Rather than use the unfamiliar notation “FN$L$”, we will abuse notation slightly and refer to certain functions as being “computable in NL”.

The proof of the equivalence above relies on the fact that NL is closed under complement.

Thus it is far less clear what it should mean to say that a function is “computable in UL” since it remains an open question if UL is closed under complement (although it is widely conjectured that UL = NL) ([22, 5]). However the proof from [17] carries over immediately to the class UL $\cap$ co-UL. That is, the following conditions are equivalent:

1. $f$ is computed by a logspace machine with an oracle from UL $\cap$ co-UL.
2. $f$ is computed by a logspace-uniform NC$^1$ circuit family with oracle gates for a language in UL $\cap$ co-UL.
3. $f(x)$ has length bounded by a polynomial in $|x|$, and the set $\{(x, i, b) : \text{the } i^{th} \text{ bit of } f(x) \text{ is } b\}$ is in UL $\cap$ co-UL.

Thus, if any of those conditions hold, we will say that “$f$ is computable in UL $\cap$ co-UL”.

The important fact that the composition of two logspace-computable functions is also logspace-computable (see, e.g., [6]) carries over with an identical proof to the functions computable in LC for any oracle C. Thus the class of functions computable in UL $\cap$ co-UL is also closed under composition. We make implicit use of this fact frequently when presenting our algorithms. For example, we may say that a colored labeling of a graph $G$ is computable in UL $\cap$ co-UL, and that, given such a colored labeling, a decomposition of the graph into layers is also computable in logspace, and furthermore, that – given such a decomposition of $G$ into layers – an additional coloring of the smaller graphs is computable in UL $\cap$ co-UL, etc.

The reader need not worry that a logspace-bounded machine does not have adequate space to store these intermediate representations; the fact that the final result is also computable in UL $\cap$ co-UL follows from closure under composition. In effect, the bits of these intermediate representations are re-computed each time we need to refer to them.

Finally, we will consider AC$^k$ circuits augmented with oracle gates for an oracle in UL $\cap$ co-UL, which we denote by AC$^k$(UL $\cap$ co-UL).
3 DFS in DAGs logspace reduces to Reachability

In this section, we observe that constructing the lexicographically-least DFS tree in a DAG $G$ can be done in logspace given an oracle for reachability in $G$. But first, let us define what we mean by the lexicographically-first DFS tree in $G$:

Definition 1. Let $G$ be a DAG, with the neighbours of the vertices given in some order in the input. (For example, with adjacency lists, we can consider the ordering in which the neighbors are presented in the list). Then the lexicographic first DFS traversal of $G$ is the traversal done by the following procedure:

```
Input: (G, v)
Output: Sequence of edges in DFS tree
visited[|v|] ← 1
for every out neighbour w of v, in the given order do
  if visited[w] = 0 then
    print(v, w)
    DFS(G, w)
end
```

Algorithm 1 Static DFS routine

That is, the lexicographically-first DFS tree is merely a DFS tree, but with the (very natural) condition that the children of every vertex are explored in the order given in the input.

When we apply this procedure as part of our algorithm for DFS in planar graphs, we will need to apply it to directed acyclic multigraphs (i.e., graphs with parallel edges between vertices) where there is a logspace-computable function $f(v, e)$ that computes the ordering of the neighbors of vertex $v$, assuming that $v$ is entered using edge $e$. (That is, if the DFS tree visits vertex $v$ from vertex $x$, and there are several parallel edges from $x$ to $v$, then the ordering of the neighbors of $v$ may be different, depending on which edge is followed from $x$ to $v$.)

As is observed in [10], the unique path from $s$ to another vertex $v$ in the lexicographically-least DFS tree in $G$ rooted at $s$ is the lexicographically-least path in $G$ from $s$ to $t$.

Now consider the following simple algorithm for constructing the lexicographically-least path in a DAG $G$ from $s$ to $v$:

```
Input: (G, s, v, f)
Output: Lex least path from s to v under f
current ← s; e ← null;
while (current ≠ v) do
  child ← first child of current (in the order given by f(current, e))
  while (REACH(child, v) ≠ TRUE) do
    child ← next child of current (in the order given by f(current, e))
  end
  e ← (current, child); current = child;
end
```

Algorithm 2 DAG DFS routine
The correctness of this algorithm is essentially shown by the proof of Theorem 11 of [10]. The algorithm for computing the lexicographically-least DFS tree rooted at $s$ can thus be presented as the composition of two functions $g$ and $h$, where $g(G, s) = (G, s, L)$, where $L$ is a list of the lexicographically-least paths from $s$ to each vertex $v$. Note that the set of edges in the DFS tree in $G$ rooted at $s$ is exactly the set of edges that occur in the list $L$ in $g(G, s) = (G, s, L)$. Then $h(G, s, L)$ is just the result of removing from $G$ each edge that does not appear in $L$. The function $h$ is computable in logspace, whereas $g$ is computable in logspace with an oracle for reachability in $G$.

Since reachability in DAGs is a canonical complete problem for $\text{NL}$, we obtain the following corollary:

\begin{corollary}
Construction of lexicographically-first DFS trees for DAGs lies in $\text{NL}$.
\end{corollary}

Similarly, since reachability in planar directed (not-necessarily acyclic) graphs lies in $\text{UL} \cap \text{co-UL}$ [8, 23], we obtain:

\begin{corollary}
Construction of lexicographically-first DFS trees for planar DAGs lies in $\text{UL} \cap \text{co-UL}$.
\end{corollary}

A planar DAG $G$ is said to be an SMPD if it contains at most one vertex of indegree zero. Reachability in SMPDs is known to lie in $L$ [2].

\begin{corollary}
Construction of lexicographically-first DFS trees for SMPDs lies in $L$.
\end{corollary}

## 4 Layering the graph

The main algorithmic insight that led us to the current algorithm was a generalization of the layering algorithm that Hagerup developed for undirected graphs [14]. We show that this approach can be modified to yield a useful decomposition of directed graphs, where the layers of the graph have a restricted structure that can be exploited. More specifically, the strongly-connected components of each layer are what we call meshes, which enable us easily to construct paths (which will end up being paths in the DFS trees we construct) whose removal partitions the graph into significantly smaller strongly connected components.

The high-level structure of the algorithm is thus:

1. Construct a planar embedding of $G$.
2. Partition the graph planar $G$ into layers (each of which is surrounded by a directed cycle).
3. Identify one such cycle $C$ that has properties that will allow us to partition the graph into smaller weakly connected components.
4. Depending on which properties $C$ satisfies, create a path $p$ from the exterior face either to a vertex on $C$ or to one of the meshes that reside in the layer just inside $C$. Removal of $p$ partitions $G$ into weakly connected components, where each strongly-connected component therein is smaller than $G$ by a constant factor.
5. Let the vertices on this path $p$ be $v_1, v_2, \ldots, v_k$. The DFS tree will start with the path $p$, and append DFS trees for subgraphs $G_1, G_2, \ldots, G_k$ to this path, where $G_i$ consists of all of the vertices that are reachable from $v_i$ that are not reachable from $v_j$ for any $j > i$. (This is obviously a tree, and it will follow that it is a DFS tree.) Further, decompose each $G_i$ into a DAG of strongly-connected components. Build a DFS of that DAG, and then work on building DFS trees of the remaining (smaller) strongly-connected components.
6. Each of the steps above can be accomplished in $\text{UL} \cap \text{co-UL}$, which means that there is an $\text{AC}^0$ circuit with oracle gates from $\text{UL} \cap \text{co-UL}$ that takes $G$ as input and produces the list of much smaller graphs $G_1, \ldots, G_k$, as well as the path $p$ that forms the spine.
of the DFS tree. We now recursively apply this procedure (in parallel) to each of these
smaller graphs. The construction is complete after $O(\log n)$ phases, yielding the desired
AC$^1(\text{UL} \cap \text{co-UL})$ circuit family.

In the exposition below, we first layer the graph in terms of clockwise cycles (which we
will henceforth call red cycles), and obtain a decomposition of the original graph into smaller
pieces. We then apply a nested layering in terms of counterclockwise cycles (which we will
henceforth call blue cycles); ultimately we decompose the graph into units that are structured
as a DAG, which we can then process using the tools from the earlier sections of the paper.
The more detailed presentation follows.

4.1 Degree Reduction and Expansion

\textbf{Definition 5.} (of $\text{Exp}^\triangle(G)$ and $\text{Exp}^\star(G)$) Let $G$ be a planar digraph. The “expanded”
digraph $\text{Exp}^\triangle(G)$ (respectively, $\text{Exp}^\star(G)$) is formed by replacing each vertex $v$ of total degree
d$(v) > 3$ by a clockwise (respectively, counterclockwise) cycle $C_v$ on $d(v)$ vertices such that
the endpoint of the the $i$-th edge incident on $v$ is now incident on the the $i$-th vertex of the
cycle.

$\text{Exp}^\triangle(G)$ and $\text{Exp}^\star(G)$ each have maximum degree bounded by 3; i.e., they are subcubic.
Next we define the clockwise (and counterclockwise) dual for such a graph and also a notion
distance.

Recall that for an undirected plane graph $H$, the dual (multigraph) $H^*$ is formed by
placing, for every edge $e \in E(H)$, a dual edge $e^*$ between the face(s) on either side of $e$ (see
Section 4.6 from [12] for more details). Faces $f$ of $H$ and the vertices $f^*$ of $H^*$ correspond
to each other as do vertices $v$ of $H$ and faces $v^*$ of $H^*$.

\textbf{Definition 6.} (of Duals $G^\triangle$ and $G^\star$) Let $G$ be a plane digraph, then the clockwise dual
$G^\triangle$ (respectively, counterclockwise dual $G^\star$) is a weighted bidirected version of the undirected
dual of the underlying undirected graph of $G$ in a way so that the orientation formed by
rotating the corresponding directed edge of $G$ in a clockwise (respectively, counterclockwise)
way gets a weight of 1 and the other orientation gets weight 0. We inherit the definition of
dual vertices and faces from the underlying undirected dual.

\textbf{Definition 7.} For a plane subcubic digraph $G$, let $f_0$ be the external face. Define the type
$\text{type}^\triangle(f)$ (respectively, $\text{type}^\star(f)$) of a face to be the singleton set consisting of the distance
at which $f$ lies from $f_0$ in $G^\triangle$: $\{d^\triangle(f_0, f)\}$ (respectively, $\{d^\star(f_0, f)\}$). Generalise this to
edges $e$ by defining $\text{type}^\triangle(e)$ (respectively $\text{type}^\star(e)$) as the set consisting of the union of the
type$^\triangle$ (respectively, type$^\star$) of the two faces adjacent to $e$, and to vertices $v$ by defining as
the type$^\triangle(v)$ (respectively type$^\star(v)$) union of the type$^\triangle$ (respectively, type$^\star$) of the faces
incident on the vertex $v$.

The following is a direct consequence of subcubicity and the triangle inequality:

\textbf{Lemma 8.} In every subcubic graph $G$, the cardinality $|\text{type}^\triangle(x)|$, $|\text{type}^\star(x)|$ where $x$
is a face, edge or a vertex is at least one and at most 2 and in the latter case consists of
consecutive non-negative integers.

Further, if $v \in V(G)$ is such that $|\text{type}^\star(v)| = 2$, then there exist unique $u, w \in V(G)$,
such that $(u, v), (v, w) \in E(G)$ and $|\text{type}^\star(u, v)| = |\text{type}^\star(v, w)| = 2$.

We first need a simple lemma:
Lemma 9. Suppose \((f_1, f_2)\) is a dual edge with weight 1 (and \((f_2, f_1)\) is of weight 0) then, 
\(d^\triangleright(f_0, f_1) \leq d^\triangleright(f_0, f_2) \leq d^\triangleright(f_0, f_1) + 1.\)

Proof. From the triangle inequality \(d^\triangleright(f_0, f_1) \leq d^\triangleright(f_0, f_2) + d^\triangleright(f_2, f_1) = d^\triangleright(f_0, f_2).\) Similarly, \(d^\triangleright(f_0, f_2) \leq d^\triangleright(f_0, f_1) + d^\triangleright(f_1, f_2) \leq d^\triangleright(f_0, f_1) + 1.\)

Proof. (of Lemma 8) Since each vertex \(v \in V(G)\) of a subcubic graph is incident on at most 3 faces the only case in which \(|\text{type}^\triangleright(v)| > 2\) corresponds to three distinct faces \(f_1, f_2, f_3\) being incident on a vertex. But here the undirected dual edges form a triangle such that in the directed dual the edges with weight 1 are oriented either as a cycle or acyclically. In the former case by three applications of the first half of Lemma 9 we get that \(d^\triangleright(f_0, f_1) \leq d^\triangleright(f_0, f_2) \leq d^\triangleright(f_0, f_3) \leq d^\triangleright(f_0, f_1),\) hence all 3 distances are the same.

Therefore \(|\text{type}^\triangleright(v)| = 1.\)

In the latter case, suppose the edges of weight 1 are \((f_1, f_2), (f_2, f_3), (f_1, f_3),\) then by Lemma 9 we get: \(d^\triangleright(f_0, f_1) \leq d^\triangleright(f_0, f_2), d^\triangleright(f_0, f_3) \leq d^\triangleright(f_0, f_1) + 1.\) Thus, both \(d^\triangleright(f_0, f_2), d^\triangleright(f_0, f_3)\) are sandwiched between two consecutive values \(d^\triangleright(f_0, f_1), d^\triangleright(f_0, f_1) + 1.\) Hence \(d^\triangleright(f_0, f_1), d^\triangleright(f_0, f_2), d^\triangleright(f_0, f_3)\) must take at most two distinct values, and thus \(|\text{type}^\triangleright(v)| \leq 2.\) Moreover either \(\text{type}^\triangleright(f_1) \neq \text{type}^\triangleright(f_2) = \text{type}^\triangleright(f_3)\) or \(\text{type}^\triangleright(f_1) = \text{type}^\triangleright(f_2) \neq \text{type}^\triangleright(f_3).\) Let \(e_1, e_2, e_3\) be such that, \(e_1^\triangleright = (f_2, f_3), e_2^\triangleright = (f_1, f_3), e_3^\triangleright = (f_1, f_2).\) Then the two cases correspond to \(|\text{type}^\triangleright(e_1)| = |\text{type}^\triangleright(e_2)| = 2, |\text{type}^\triangleright(e_3)| = 1\) and to \(|\text{type}^\triangleright(e_1)| = 1, |\text{type}^\triangleright(e_2)| = |\text{type}^\triangleright(e_3)| = 2\) respectively. Noticing that \(e_1, e_3\) are both incoming or both outgoing edges of \(v\) completes the proof for the clockwise case. The proof for the counterclockwise case is formally identical.

Definition 10. For a plane subcubic graph \(G\) as above, we refer to vertices and edges with a type of cardinality two in \(G^\triangleright\) (respectively, in \(G^\triangleleft\)) as red (respectively, blue) while the ones with a cardinality of one as white. The resulting colored graphs are called \(\text{red}(G)\) and \(\text{blue}(G)\) respectively.

We will see later how to apply both the duals in \(G\) to get red and blue layerings of a given input graph.

Also note that a red (respectively blue) edge must have red (respectively blue) end points, as they are adjacent to the same faces as the edge between is.

We enumerate some properties of \(\text{red}(G), \text{blue}(G)\) (\(G\) is subcubic):

Lemma 11. 1. Red vertices and edges in \(\text{red}(G)\) form disjoint clockwise cycles.
2. No clockwise cycle in \(\text{red}(G)\) consists of only white edges (and hence white vertices).

Similar properties hold for \(\text{blue}(G)\).

Proof. 1. Firstly, note that a red edge must have red end point vertices, as they are adjacent to the same faces that the edge between them is adjacent to. It is immediate from Lemma 8 that if \(v\) is a red vertex, it has exactly one red incoming edge and one red outgoing edge, proving this part.

2. Suppose \(C\) is a clockwise cycle. Consider the shortest path in \(G^\triangleright\) from the external face to a face enclosed by \(C\). From the Jordan curve theorem (Theorem 4.1.1 [12]), it must cross the cycle \(C\). The edge dual to the crossing must be red.

The definitions above, which apply only to subcubic plane graphs, can now be extended to a general plane graph \(G\), by considering the subcubic graphs \(\text{Exp}^\triangleright(G)\) (and \(\text{Exp}^\triangleleft(G)\)). But first, we must make a simple observation about \(\text{red}(\text{Exp}^\triangleright(G))\) (respectively about \(\text{blue}(\text{Exp}^\triangleright(G))\)).
Lemma 12. Let $v \in V(G)$ be a vertex of degree more than 3. Let $C_v$ be the corresponding expanded cycle in $\text{Exp}^\langle\rangle(G)$. Suppose at least one edge of $C_v$ is white in $\text{red}(\text{Exp}^\langle\rangle(G))$ then there is a unique red cycle $C$ that shares edges with $C_v$.

Proof. First we note that $C_v$ does not contain anything inside it since it is an expanded cycle. By lemma 11 we know that $C_v$ has at least one red edge. Suppose it shares one or more edges with a red cycle $R_1$. Since both cycles are clockwise and $C_v$ has nothing inside, the cycle $R_1$ must enclose $C_v$. Now suppose there is another red cycle $R_2$ that shares one or more edges with $C_v$. Then $R_2$ must also enclose $C_v$. But two cycles cannot enclose a cycle whilst sharing edges with it without touching each other, which contradicts the above lemma that all red cycles in a subcubic graph are vertex disjoint.

The last two lemmas allow us to consistently contract the red cycles in $\text{red}(\text{Exp}^\langle\rangle(G))$:

Definition 13. The colored graph $\text{Col}^\langle\rangle(G)$ (respectively, $\text{Col}^\langle\rangle(G)$) is obtained by labeling a degree more than 3 vertex $v \in V(G)$ as red iff the cycle $C_v$ in $\text{red}(\text{Exp}^\langle\rangle(G))$ has at least one red edge and at least one white edge. Else the color of $v$ is white. All the low degree vertices and edges of $G$ inherit their colors from $\text{red}(\text{Exp}^\langle\rangle(G))$. The coloring of $\text{Col}^\langle\rangle(G)$ is similar.

Figure 1 An example of contracting expanded cycles. The figure on right shows the graph after contracting the expanded cycles $C_1, C_2, C_3$ according to definition 13.

We can now characterize the colorings in the graph $\text{Col}^\langle\rangle(G)$:

Lemma 14. The following hold:

1. A red cycle in $\text{Col}^\langle\rangle(G)$ is vertex disjoint from every red cycle contained in its interior.
2. Every 2-connected component of the red subgraph of $\text{Col}^\langle\rangle(G)$ is a simple clockwise cycle.

Proof. For $v \in V(G)$, let $C_v \subseteq \text{Exp}^\langle\rangle(G)$ be the expanded cycle. If it has a red vertex it is immediately enclosed by a unique red cycle $R$ in $\text{Exp}^\langle\rangle(G)$ by Lemma 12. Assuming $C_v$ is not all red, it consists of alternating red subpaths and white subpaths. On contracting $C_v$ we get a collection of clockwise red cycles outside sharing a common cut-vertex $v$. Notice that the new collection of red cycles consists of edges that $R$ did not share with $C_v$. Also notice
4.2 Layering the colored graphs

Definition 15. Let \( x \in V(\text{Col}^\bigodot(G)) \cup E(\text{Col}^\bigodot(G)) \). Let \( \ell^\bigodot(x) \) be one more than the minimum integer that occurs in \( \text{type}^\bigodot(x') \), for each \( x' \in V(\text{Exp}^\bigodot(G)) \cup E(\text{Exp}^\bigodot(G)) \) that is contracted to \( x \). Further let \( \mathcal{L}^k(\text{Col}^\bigodot(G)) = \{ x \in V(\text{Col}^\bigodot(G)) \cup E(\text{Col}^\bigodot(G)) : \ell^\bigodot(x) = k \} \). Similarly define, \( \ell^\bigodot(x), \mathcal{L}^k(\text{Col}^\bigodot(G)) \).

We call \( \mathcal{L}^k(\text{Col}^\bigodot(G)) \) the \( k \)th layer of the graph.

It is easy to see the following from Lemma 14:

Proposition 16. For every \( x \in V(\text{Col}^\bigodot(G)) \cup E(\text{Col}^\bigodot(G)) \) the quantity \( \ell^\bigodot(x) \) is one more than the number of red cycles that strictly enclose \( x \) in \( \text{Col}^\bigodot(G) \). All the vertices and edges of a red cycle of \( \text{Col}^\bigodot(G) \) lie in the same layer \( \mathcal{L}^{k+1}(\text{Col}^\bigodot(G)) \) for the enclosure depth \( k \) of the cycle.

We had already noted above that the red subgraph of \( G \) had simple clockwise cycles as its biconnected components. We note a few more lemmas about the structure of a layer of \( G \):

Lemma 17. We have:
1. A red cycle in a layer \( \mathcal{L}^{k+1}(\text{Col}^\bigodot(G)) \) does not contain any vertex/edge of the same layer inside it.
2. Any clockwise cycle in a layer consists of only red vertices and edges.
Dually, a blue cycle in a layer does not contain any vertex or edge of the same layer inside it.

Remark 18. Notice that the conclusion in the second part of the lemma fails to hold if we allow cycles spanning more than one layer.

Proof. The first part is a direct consequence of proposition 16. For the second part we mimic the proof of the second part of Lemma 11. Consider a clockwise cycle \( C \subseteq \mathcal{L}^{k+1}(\text{Col}^\bigodot(G)) \) that contains a white edge \( e \). Every face adjacent to \( C \) from the outside must have \( \text{type}^\bigodot = k \) because \( C \) is contained in layer \( k + 1 \). Then the \( \text{type}^\bigodot \) of the faces on either side of \( e \) is the same and therefore must be \( k \). Let \( f \) be a face enclosed by \( C \) that has \( \text{type}^\bigodot = k \). Thus it must be adjacent to a face of \( \text{type}^\bigodot = k - 1 \). But this contradicts that every face inside and adjacent to \( C \) must have \( \text{type}^\bigodot \) at least \( k \).

The lemmas above show that the strongly connected components of the red subgraph of a layer consist of red cycles touching each other without nesting, in a tree like structure. This prompts the following definition:

Definition 19. For a red cycle \( R \subseteq \mathcal{L}^k(\text{Col}^\bigodot(G)) \) we denote by \( G_R \), the graph induced by vertices of \( \mathcal{L}^{k+1}(\text{Col}^\bigodot(G)) \) enclosed by \( R \).

Now we combine Definitions 13 and 15:
Definition 20. Each vertex or edge $x \in V(G) \cup E(G)$ gets a red layer number $k + 1$ if it belongs to $L^{k+1}(\text{Col}^R(G))$ and a blue layer number $l + 1$, if it belongs to $L^{l+1}(\text{Col}^B(G_R))$ where $R \subseteq L^k(\text{Col}^R(G))$ is the red cycle immediately enclosing $x$.

Moreover this defines the colored graph $\text{Col}(G)$ by giving $x$ the color red if it is red in $\text{Col}^R(G)$ and/or blue in $\text{Col}^B(G_R)$ (notice it could be both red and blue) and lastly white if it is white in both the graphs. In this case, we say that $x$ belongs to sublayer $L^{k+1,l+1}(\text{Col}(G))$.

By proposition 16, we can also say that a sublayer $L^{k+1,l+1}(\text{Col}(G))$ thus consists of edges/vertices that are strictly enclosed inside $k$ red cycles and inside $l$ blue cycles that are contained inside the first enclosing red cycle.

We’ll see some observations and lemmas regarding the structure of a sublayer now.

Since every edge/vertex in $L^{k+1,l+1}(\text{Col}(G))$ has the same red AND blue layer number, it is clear that there can be no nesting of colored cycles. Also we have:

Lemma 21. Every clockwise cycle in a sublayer $L^{k+1,l+1}(\text{Col}(G))$ consists of all red edges and vertices and any every counterclockwise cycle in the sublayer consists of all blue vertices and edges. (Some edges/vertices of the cycle can be both red as well as blue)

Proof. This is a direct consequence of Lemma 17 applied to the sublayer $L^{k+1,l+1}(\text{Col}(G))$, which is a (counterclockwise) layer in graph $G_R$ for some red cycle $R$.

Thus we can refer to clockwise cycles and counterclockwise cycles as red and blue cycles respectively.

Definition 22. For a red or blue colored cycle $C$ of layer $L^{k,l}(\text{Col}(G))$, we denote by $G_C$ the graph induced by vertices of $L^{k,l}(\text{Col}(G))$ enclosed by $C$, where $\{k',l'\} = \{k + 1, l\}$ or $\{k, l + 1\}$ according to whether $C$ is a red or a blue cycle respectively.

Note that:

Proposition 23. Two cycles of the same color in $L^{k+1,l+1}(G)$ cannot share an edge

This is since neither is enclosed by the other as they belong to the same layer, and as they also have the same orientation. Cycles of different colors can share edges but we note:

Lemma 24. Two cycles of a sublayer $L^{k+1,l+1}(\text{Col}(G))$ can only share one contiguous segment of edges.

Proof. Let a red cycle $R$ and a blue cycle $B$ in a sublayer share two vertices $u, v$ but let the paths $R(u, v), B(u, v)$ in the two cycles be disjoint. Notice that the graph $(R \setminus R(u, v)) \cup B(u, v)$ is also a clockwise cycle that encloses the edges of $R(u, v)$ contradicting the first part of Lemma 17.

We consider the strongly connected components of a sublayer and note the following lemmas regarding them:

Lemma 25. The trivial strongly connected components of a sublayer (those that consist of a single vertex) are white vertices. The non-trivial strongly connected components of a sublayer have the following properties:

1. Every vertex/edge in them is blue or red (possibly both).
2. Every face, except possibly the outer face, is a directed cycle.
3. Every face other than the outer face has at least one edge adjacent to the outer face.

Proof. 1. In a non-trivial strongly connected graph every vertex and edge lies on a cycle and therefore by Lemma 21 must be colored red or blue (or both).
Figure 2 Figure (a) is a graph $G$. Figure (b) is the graph in (a) after labelling red edges using clockwise dual. We omit the cycle expansion and contraction procedure here.

Figure (c) shows $G$ after applying blue labellings to each red layer we obtained in the previous figure. The vertices and edges colored purple are those that are red as well as blue. Figure (d) represents the sublayer $(1,1)$. The dashed edges and empty vertices are not part of the layer. Figure (e) figure represents the sublayer $(2,1)$. Figure (f) represents the sublayer $(3,1)$. 
2. Suppose there is a face \( f \) the boundary of which is not a directed cycle. Look at a
directed dual (say clockwise) of the strongly connected component (just the component
independently). This dual must be a DAG since the primal is strongly connected. The
vertex \( f^* \) in the dual corresponding to face \( f \) of the strongly connected component has
in degree at least one and out degree at least one since it has boundary edges of both
orientations, hence the edges adjacent to \( f^* \) do not form a directed cut of the dual.
Let \( \sigma^* \) denote the dual vertex corresponding to the outer face of the SCC. In order to
prove the claim, it is sufficient to show the existence of a directed cut \( C^* \) that separates
\( f^* \) and \( \sigma^* \), since it would imply by cut cycle duality that there is a directed cycle \( C \) in
the primal SCC that encloses the face \( f \) w.r.t the outer face and since the boundary of \( f \)
is not a directed cycle, \( C \) must strictly enclose at least one edge of the boundary of \( f \)
contradicting Lemma 17. To see the cut, consider a topological sort ordering of the dual
(it is a DAG). Let the number of a dual vertex \( v^* \) in the ordering be denoted by \( n(v^*) \).
W.l.o.g, let \( n(f^*) < n(\sigma^*) \). Consider the partition of the dual vertices:
\[
A = \{ v^* | n(v^*) \leq n(f^*) \}, \quad B = \{ v^* | n(v^*) > n(f^*) \}
\]
By definition of topological sort, all edges across this partition must be directed from \( A \)
to \( B \), hence it is a directed cut, and therefore it must also contain a subset which is a
minimal directed cut. But clearly the minimal cut is not the set of edges adjacent to \( f^* \)
since it has both out and in degree at least one, hence proving the claim. Hence every
face in the SCC of a sublayer must be a directed (hence colored) cycle (by Lemma 21).

3. Let \( H \) be an SCC of the sublayer. We observed from the proof above that no vertex in
the dual of \( H \), except possibly the vertex corresponding to the outer face of \( H \), can have
both in degree and out degree more than one. (i.e. every dual vertex, except the outer
face is a source or a sink). Therefore if any dual vertex \( f^* \) has a directed path to \( \sigma^* \) or
vice versa, then the path must be an edge and we are done. Suppose there is no directed
path from \( f^* \) to \( \sigma^* \) and w.l.o.g. let \( f^* \) be a source. Consider the trivial directed cut \( C_1 \):
\[
A = \{ f^* \}, \quad B = V(H) \setminus A
\]
This is a cut since there are no edges from \( B \) to \( A \), and this cut clearly corresponds to
the directed cycle which is the boundary of face \( f \) in \( H \).
Now consider the cut \( C_2 \):
\[
A' = \{ v^* | v^* \text{ is reachable from } f^* \}, \quad B' = V(H) \setminus A'
\]
Clearly this is a \( f^* - \sigma^* \) cut with no edge from a vertex in \( A' \) to a vertex in \( B' \) and \( \sigma^* \in B' \).
But this \( f^* - \sigma^* \) cut is different from \( C_1 \) since \( f^* \) is a source vertex and hence \( A' \) has at
least one more vertex than just \( f^* \). Hence this corresponds to a directed cycle in \( H \) that
strictly encloses at least some edge of \( f \), and we again get a contradiction of Lemma 17.

The strongly connected components of a sublayer hence consist of intersecting red and blue
facial cycles, with every face having at least one boundary edge adjacent to the outer face of
the component.

Definition 26. We call the strongly connected components of a sublayer \( \mathcal{L}(k,l) \) meshes.

5 Mesh Properties

Definition 27. Given a subgraph \( H \) of \( G \) embedded in the plane, we define the closure of
\( H \), denoted by \( \bar{H} \), to be the induced graph on the vertices of \( H \) together with the vertices of
\( G \) that lie in the interior of faces of \( H \) (except for the outer face of \( H \)).
For convenience, we call a face of a graph that is not the outer face an \textit{internal face}.

From Lemmas 21 and 25, we have a bijection: every face of a mesh, except possibly its outer face, is a directed cycle, and every directed cycle in a mesh is the boundary of a face of the mesh.

\textbf{Definition 28.} Let $0 < \alpha < 1$. An \textit{\(\alpha\)-separator of a digraph} \(H\) that is a subgraph of a digraph \(G\) is a set of vertices of \(H\) whose removal from \(H\) separates \(H\) into subgraphs, where no strongly connected component has size greater than \(\alpha|G|\).

A \textit{path separator} is a sequence of vertices \((v_1, \ldots, v_n)\) that is a separator and also is a directed path.

\textbf{Definition 29.} Let \(G\) be a graph and let \(M\) be a mesh in a sublayer \(G\).

For an internal face \(f\) of \(M\), we define \(wt(f)\) to be \(|V(\tilde{f})|\).

Let \(wt(H)\) where \(H\) is a subgraph of \(M\) be defined as \(|V(\tilde{H})|\).

\textbf{Definition 30.} For a mesh \(M\), we call a vertex that is adjacent to the outer face of \(M\) an external vertex, and a vertex that is not adjacent to the outer face an internal vertex.

Also, we call vertices of degree more than two junction vertices.

If \(p = (v_1, v_2, \ldots, v_k)\) is a directed path such that \(v_2, \ldots, v_{k-1}\) are all vertices of degree two, but \(v_1, v_k\) have degree more than two, then we call \(p\) a \textit{segment}. We call \(v_k\) the out junction neighbour of \(v_1\) and \(v_1\) the in junction neighbour of \(v_k\).

We call a \textit{segment} with all edges adjacent to the outer face an \textit{external segment}, and a \textit{segment} with no edge adjacent to the outer face an \textit{internal segment}.

If the end points of an internal segment are both internal vertices also, we call the segment an \textit{i-i-segment}.

The rest of this section is devoted to a proof of the following lemma, which asserts that we can construct a path separator in a mesh, assuming that no internal face of the mesh is too large.

\textbf{Lemma 31.} Suppose \(wt(f) < wt(G)/12\) holds for every internal face \(f\) of a mesh \(M\) that is a subgraph of \(G\). Then from any external vertex \(r\) of \(M\), we can find (in UL \(\cap\) co-UL) an \(\frac{11}{12}\) \textit{path separator} of \(M\), starting at \(r\).

The vertices of \(M\) with degree two (in-degree 1 and out-degree 1 because \(M\) is strongly connected) are not important since they can be seen as just “subdivision” vertices. Now we will look at the structure of a mesh around an internal junction vertex, and the way the rest of the mesh is attached to that structure. Also, we state here that we will abuse the notion of 3-connected components by ignoring the non-junction vertices for convenience.

\textbf{Lemma 32.} If \(v\) is an internal junction vertex of a mesh and \(e_1, \ldots, e_k\) are the edges adjacent to \(v\) in the cyclic order of embedding, then the edges alternate in directions i.e. if \(e_1\) is outgoing from \(v\), then \(e_2\) is incoming to \(v\) and \(e_3\) is outgoing and so on. Consequently, \(v\) has even degree (at least 4).

\textbf{Proof.} Let \(e_i, e_{i+1}\) be two edges adjacent to \(v\), that are also adjacent in the cyclic order of the drawing. Since they are adjacent in the drawing, they must enclose between them, a region, and hence a face, which is not the outer face. But the boundary of every non-outer face in a mesh is a directed cycle, hence \(v, e_1, e_{i+1}\) lie on a directed cycle, with both edges adjacent to \(v\). Hence one of them must be an out edge from \(v\), and the other incident towards \(v\).
Definition 33. Let \( v \) be an internal junction vertex of degree 2\( d \) in a mesh \( M \), and let its junction neighbours be \( \{u_1, w_1, u_2, w_2, \ldots, u_d, w_d\} \) in clockwise order starting from edge \( \langle u_1, v \rangle \) (the \( w_i \)'s are out neighbours, and \( u_i \)'s the in neighbours, since junction neighbours alternate).

Every adjacent pair of edges incident to \( v \) borders a face that is not the outer face. Let \( f_{u,v,w} \) denote the face bordered by \( v \) and the junction neighbours \( u \) and \( w \) of \( v \) which are adjacent in cyclic order around \( v \). The boundary of \( f_{u,v,w} \) can be written as three disjoint parts (except for endpoints), segment \( \langle u, v \rangle \) + segment \( \langle v, w \rangle \) + petal \( w, u \), where the third part denotes a simple path from \( w \) to \( u \) along the face boundary. We will use the notation petal \( w, u \) to denote the corresponding boundary for any face \( f_{u,v,w} \) adjacent to \( v \).

We define \( \text{flower}(v) \) as \( \bigcup \{ \text{vertices on boundary of faces adjacent to } v \} \).

We note the following property of petals:

Proposition 34. For all adjacent junction neighbour pairs \( u_i, u_j \) of internal vertex \( v \), petal \( w_i, u_j \) are disjoint, except possibly the end points.

Proof. Petals of two faces must be internally disjoint because the corresponding faces share the vertex \( v \) and two faces cannot have a non-contiguous intersection, by Lemma 24.

For an internal junction vertex \( v \), the union of the petals around \( \text{flower}(v) \) thus form an undirected cycle around \( v \), with at least four alternations in directions. Now we define bridges of the cycle, which are components of \( M - \text{flower}(v) \), along with the points of attachment. We use the definition of bridges from [25]:

Definition 35. For a subgraph \( H \) of \( M \), a vertex of attachment of \( H \) is a vertex of \( H \) that is incident with some edge of \( M \) not belonging to \( H \).

Let \( J \) be an undirected cycle of \( M \). We define a bridge of \( J \) in \( M \) as a subgraph \( B \) of \( M \) with the following properties:
1. each vertex of attachment of \( B \) is a vertex of \( J \).
2. \( B \) is not a subgraph of \( J \).
3. no proper subgraph of \( B \) has both the above properties.

We denote by 2-bridge, bridges with exactly two vertices of attachment to the specified cycle, and by 3-bridge, bridges with three or more vertices of attachment.

Lemma 36. 1. The vertices of attachment of a 2-bridge of \( \text{flower}(v) \) must both lie on one petal of \( \text{flower}(v) \).
2. The vertices of attachment of a 3-bridge of \( \text{flower}(P) \) can lie on one or, at most two adjacent petals. Moreover, in the latter case the junction neighbour of \( v \) common to both petals must be a vertex of attachment of the 3-bridge.
3. For an internal vertex \( v \), and an external vertex \( r \) of \( M \), let \( p = \langle r, \ldots, u_1, v \rangle \) be a simple path from \( r \) to \( v \), where \( u_1 \) is an in junction neighbour of \( v \). Let the other junction neighbours of \( v \) be named as in Lemma 33 in cyclic order from \( u_1 \). For \( j \in \{i, i+1\} \), consider an extended path of \( p \), \( p_{w_i, u_j} = \langle r, \ldots, u_1, v, w_i \rangle + \text{petal} \( w_i, u_j \) + \langle u_j, \ldots, v \rangle \), excluding the last edge incident to \( v \) in the sequence. That is, \( p_{w_i, u_j} \) goes from \( r \) to \( v \), then to an out junction neighbour \( w_i \), and then wraps around \( f_{u_j, v, w_i} \) by taking petal \( w_i, u_j \) and then the segment back towards \( v \) from \( u_j \).
Let $M - p_{w_i,u_i}$ denote the induced graph on $V(M) \setminus V(p_{w_i,u_i})$. Then $V(M) \setminus V(p_{w_i,u_i})$ can be partitioned into four disconnected parts, say $V_{\text{left}}$ and $V_{\text{right}}$, $V_f$, $V_{\text{rem}}$ such that:

$$V_{\text{left}} = \{ \text{petal}_{w_i,u_i} \cup \text{petal}_{w_i,u_2} \cup \text{petal}_{w_i,u_2} \cup \cdots \cup \text{petal}_{w_i,u_i-1,u_i-1} \}$$

$$\cup \{ \text{petal}_{w_i,u_i} \text{if } j = i + 1 \}$$

$$\cup \{ \text{all vertices in the bridges attached to these petals} \}$$

$$V_{\text{right}} = \{ \text{petal}_{w_i,u_i+1,u_{i+1}} \cup \text{petal}_{w_i,u_i+1,u_{i+1}} \cup \text{petal}_{w_i,u_i} \}$$

$$\cup \{ \text{petal}_{w_i,u_{i+1}} \text{if } j = i \}$$

$$\cup \{ \text{all vertices in the bridges attached to these petals} \}$$

$$V_f = f_{u_j,v,w_i} \setminus V(p_{w_i,u_j})$$

$$V_{\text{rem}} = \bigcup \{ \text{vertices of all bridges that have vertices of attachment only in petal}_{w_i,u_i} \}.$$  

such that there is no undirected path between any vertex of one of these four sets to any vertex in another. The path $p_{w_i,u_i}$ is therefore a path separator that gives these components.

Proof. 1. Let $x, y$ be the two vertices of attachment of the 2-bridge $B$ on $\text{flower}(v)$. Since bridges are connected graphs without the edges of the corresponding cycle, there must be an undirected path, $p$ in the bridge connecting $x,y$, without using any edge of $\text{flower}(v)$. If $x$ and $y$ were not on the same petal, then this path along with the boundary of $\text{flower}(v)$ must clearly enclose a junction neighbour of $v$, say $w$. Thus $w$ is not adjacent to the outer face. Now since $w$ is an internal junction vertex, and two of its adjacent faces are also adjacent to $v$, look at another face $f$ adjacent to $w$ and not adjacent to $v$. (Internal junction vertices have at least four adjacent faces.) The boundary of this face cannot touch $B$ since that would make it a part of $B$ and consequently $w$ a vertex of attachment of $B$ to $\text{flower}(v)$. Therefore the boundary of $f$ is enclosed within the paths $p$ and the part of $\text{flower}(v)$ that is also enclosed by $p$. Therefore $f$ has no external edge, contradicting Lemma 25.

2. Let $x_1, x_2, \ldots, x_k$ be the vertices of attachment of the bridge $B$ on $\text{flower}(v)$, in the cyclic order of boundary of $\text{flower}(v)$. Clearly if the vertices of attachment lie on more than two petals of $v$, then at least one petal will be completely enclosed by $B$, which is not possible since every petal must have at least one external edge. Let say they lie on two adjacent petals, and the junction neighbour common to both of them is $w$. By the same argument as above, $w$ must have an edge other than those of adjacent petals of $v$, that connect it to $B$. Therefore $w$ must be a vertex of attachment of $B$ to $\text{flower}(v)$.

3. First we note that $\text{petal}_{u_j,v}$ will have an external vertex in it since the boundary of every face has at least one external vertex (Lemma 25), and segments $(u_j,v)$ and $(v,w_i)$ are internal. Let $z$ be an external vertex on $\text{petal}_{u_j,v}$. The path $p$ starts at external vertex $r$, comes to $u_1,v,w_i$, and reaches external vertex $z$ on its way back to $v$. It will clearly divide $M$ into at least two parts by Jordan Curve
We introduce another notation for an extension of a bridge:

\begin{definition}
For a bridge $B$ of \text{flower}(v), we define $B^\circ$ as $B$ along with segments of \text{flower}(v) that lie between consecutive vertices of attachment of $B$. We call this the closed bridge of $B$.
\end{definition}

Now we will give definitions/lemmas regarding the “internal structure” of meshes, that will be useful to define the “center” of a mesh.

\begin{definition}
For a mesh $M$, we call its internal-skeleton, denoted by $I(M)$, the induced subgraph on the vertices of internal segments of $M$.
\end{definition}

\begin{lemma}
1. For a mesh $M$, the graph $I(M)$ is a forest.
2. If $H$ is a 3-connected induced subgraph of $M$ (ignoring subdivision vertices), then $I(H)$ is a tree.
\end{lemma}

\begin{proof}
1. Suppose there were an undirected cycle in $M$ of all internal segments, then this cycle must enclose a face whose boundaries are also all internal segments. This contradicts Lemma 25 as it states that every face must have at least one external edge, and hence segment. Hence there can be no cycle (directed or undirected) consisting of all internal segments, an consequently, no cycle (directed or undirected) of all internal vertices.

2. Let $H$ be a 3-connected induced subgraph of $M$. By definition, $I(H)$ is obtained from $M$ by removing all external edges and external non-junction vertices. Suppose $I(H)$ is not a tree, and hence consists of two or more disconnected trees. Let $T_1$ and $T_2$ be any two trees in $I(H)$. Let $x$ be a vertex in $T_1$ and $y$ be a vertex in $T_2$. Since $H$ is 3-connected, there must be at least three disjoint paths (undirected) between $x$ and $y$. Clearly in a planar graph, if there are three disjoint paths between two vertices, one of the paths must be strictly enclosed in the closed region formed by the other two. Therefore there must a path between $x$ and $y$ that is strictly enclosed inside the boundary of $H$, and hence does not contain any edge or vertex adjacent to the outer face of $H$. Hence $x$ and $y$ cannot become disconnected after removing external edges and external non-junction vertices leading to a contradiction that $I(H)$ is disconnected. Therefore $I(H)$ must be a tree.
\end{proof}

We state a well-known proposition about a vertex separator in a tree $T$ with weighted nodes.

\begin{proposition}
Suppose $T$ is a tree with each node having a weight assigned to it. Let $\text{wt}(T')$ denote sum of weights of each node in a subgraph $T'$ of $T$. Then there exists a node $v$, or a pair of adjacent nodes $v_1, v_2$, such that after removing it (or them in case of a pair), no connected component in the remaining forest has weight more than $\frac{1}{2}\text{wt}(T)$.
\end{proposition}

\begin{proof}
Folklore.
\end{proof}

We will next give a procedure to define a “center” of a mesh.
Lemma 41. For a mesh $M$, let $T_M$ denote the tree obtained by the 1,2-clique sum decomposition of $M$. The nodes of $T_M$ are of two types, clique nodes (cut vertices or separating pairs), and piece nodes, which are either 3-connected parts or cycles. Every piece node is adjacent to a clique node and vice-versa. (See [9, Section 3.1] for background about this decomposition.)

Consider the $\frac{1}{2}$ separator node of $T_M$ as described in Proposition 40. If it is a separating pair, a cut vertex, or a face cycle, we call that subgraph the center of $M$.

If it is a 3-connected node $P$, look at its internal skeleton $I(P)$. We construct a new graph $I'(P)$ which is isomorphic to $I(P)$ but has edges directed differently. Let $u,v$ be two adjacent internal junction vertices of $M$. To give direction to a segment $(u,v)$ in $I'(P)$, we consider the unique bridge $B$ of flower(v) that contains $v$ as a point of attachment; we denote the closed bridge of $B$ by $B^c(x)$, $B^c(y)$ is defined analogously. We orient $(u,v)$ in the direction of the howler of $B^c(x)$ and $B^c(y)$ (breaking ties arbitrarily), where the weights of $B^c(x)$, $B^c(y)$ are $|B^c(x)|$ and $|B^c(y)|$, respectively.

The center of $M$ is defined to be flower(v) in this case, where $v$ is the sink node of $I'(P)$.

We show why $I'(P)$ cannot have more than one sink.

Lemma 42. The tree $I'(P)$ defined above will have exactly one sink vertex.

Proof. Suppose $I'(P)$ has two junction vertices $x$ and $y$ that are sinks. They cannot be adjacent, so consider the unique undirected path in $I'(P)$ between $x$ and $y$. There must be a source $z$ on the path. Let neighbours of $z$ be $x', y'$, lying on the path from $x$ to $z$ and from $z$ to $y$, respectively.

Let $B^c(x')$ and $B^c(y')$ denote the bridges of flower(v) with points of attachments $x'$ and $y'$ respectively. Then by the orientations of the edges we have: $|B^c(x')| \geq |B^c(z)|$ which gives $|B^c(x')| > |B^c(y')|$ since $B^c(y')$ is clearly a proper subgraph of $B^c(z)$ and $|B^c(y')| \geq |B^c(z)|$ which gives $|B^c(y')| > |B^c(x')|$ which is clearly a contradiction. ▲

Lemma 43. If the center of $M$ is flower(v), and $w$ is an out neighbor of $v$, then $wt(B^c(w)) \leq \frac{1}{4} (wt(M) - wt(V_{rem}(u,w)))$, where $u$ is either of the two in neighbors of $v$ that are adjacent to $w$ around flower(v).

Proof. Since the center is flower(v), we have that $wt(B^c(w)) \leq wt(B^c(v))$. But $V_{rem}(u,w)$ has empty intersection with each of $B^c(v)$ and $B^c(w)$. Thus $wt(B^c(w)) + wt(B^c(v)) \leq wt(M) - wt(V_{rem}(u,w))$. The lemma follows. ▲

Lemma 44. 1. If the center of $M$ is not of the form flower(v) where $v$ is an internal node of a 3-connected component, then removing it from $M$ disconnects $M$ into weakly connected components, each with weight less than $\frac{1}{2} wt(M)$.

2. If the center of $M$ is flower(v) for an internal node $v$ of a 3-connected component $P$, then on removing flower(v) from $M$, no weakly connected component has weight more than $\frac{1}{2} wt(M)$. ▲

Proof. 1. This follows directly from the vertex separator lemma for trees with weighted vertices.

2. This follows from the $v$ being the sink node of $I'(P)$. ▲

Lemma 45. For every possible path $p_{u,v}$ around $v$ as defined in Lemma 36, $V_{rem}$ consists of a disjoint union of weakly-connected components, each of which has weight $\leq \frac{1}{2} (wt(M))$. ▲
Figure 3 An example of a mesh

Figure 4 The internal skeleton of the mesh. One of its components is a single node.

Figure 5 The tree decomposition of the mesh using 1,2-clique sums. The nodes encircled red are clique separator nodes.

Figure 6 An example of a path separator. The vertex $v$ is a central node, and the green path is a separator.

Proof. A (weakly connected) component of $V_{rem}$ is a bridge, attached to $petal_{w_i, u_i}$ or to $petal_{w_i, u_{i+2}}$ via its vertices of attachment. In the clique sum decomposition, these vertices of attachment will always contain a 1 or 2 separating clique, since if a bridge is attached to a petal via three or more nodes, the first and the last vertices of attachment form a separating pair that separates the bridge from flower$(v)$. Hence it is a branch remaining in $T_M$ after removing the 3-connected piece node that is central in $T_M$. Since every branch after removal of the central piece of $T_M$ has weight $\leq \frac{1}{2}(wt(M))$, every (weakly) connected component of $V_{rem}$ has weight $\leq \frac{1}{2}(wt(M))$. □

For a path $p_{w_i, u_j}$ (where $j \in \{i, i+1\}$) we sometimes use the notation $V_{rem}(w_i, u_j)$ to specify the petal where the bridges of $V_{rem}$ are attached.

5.1 Mesh Separator Algorithm

Now we give the algorithm to find an $\alpha$ separator in a mesh $M(G)$, assuming the hypothesis of Lemma 31.

1. Find the decomposition tree, $T_M$ of $M$ with 2-cliques and 1-cliques as the separating sets.
2. Find the center of the mesh $M$. It will either be a cut vertex, a separating pair, a cycle, or flower$(v)$ for some internal vertex $v$. 
3. If it is a cut vertex, we just find a path from the root $r$ to it. If it is a separating pair $(u, v)$, both the vertices must lie on a same face, which is a directed cycle. In both this case, and also the case in which the center is a cycle, find a path from the root to any vertex of the face that touches it the first time, and then extend the path by encircling the cycle.

4. If it is flower $v$ for some internal vertex $v$, find a path $p = \langle r, \ldots, u_1,v \rangle$ to $v$. Let the junction neighbours of $v$ in clockwise order starting from $(u_1, v)$, be $u_1, u_2, u_3, \ldots, u_d$, with the $u_i$’s being out junction neighbours and the $u_i$’s being in junction neighbours.

Starting clockwise from segment $\langle u_i,v \rangle$, find the first index $i$ and $j \in \{i,i+1\}$ s.t. after removing the extended path $p_{u_i,u_j}$ (defined in Lemma 36) the remaining strongly connected components are smaller than $\frac{1}{12} wt(G)$.

The algorithm above can clearly be implemented in logspace with an oracle for planar reachability, and thus it can be implemented in $\text{UL} \cap \text{co-UL}$.

It remains to show that the “first $i$” mentioned in the final step actually exists.

**Lemma 46.** If the center of $M$ is flower $v$ for some internal vertex $v$, then there will always exist an adjacent face $f_{u_i,v,w}$, s.t. the path $p_{w_i,u_j}$ is a $\frac{1}{12}$-separator.

**Proof.** There are following two cases

1. For some $i$ and $j \in \{i,i+1\}$, $p_{w_i,u_j}$, $wt(V_{\text{rem}}(w_i,u_j)) \geq \frac{1}{2} wt(M)$.

Then by Lemma 45, $p_{w_i,u_j}$ separates $V_{\text{rem}}(w_i,u_j)$ from the rest of the graph, and also every weakly connected component in $V_{\text{rem}}(w_i,u_j)$ has weight $\leq \frac{1}{2} wt(M)$. Hence every weakly connected component in $M$ after removing $p_{w_i,u_j}$ has weight $\leq \frac{1}{2} wt(M)$.

2. For every $p_{w_i,u_j}$, $wt(V_{\text{rem}}(w_i,u_j)) \leq \frac{1}{2} wt(M)$.

We know that for any index $i$ and $j \in \{i,i+1\}$, if $f = f_{u_i,v,w}$, then $wt(V_f) \leq wt(G)/12$ by the hypothesis of Lemma 31. Starting clockwise from $p_{u_i,w_1}$ at first $V_{\text{left}}$ is small, and on shifting from $p_{w_i,u_j}$ to $p_{w_i,u_{j+1}}$ or from $p_{w_i,u_{j+1}}$ to $p_{w_i,u_{j+1}}$, the increase in $V_{\text{left}}$ is bounded above by $wt(V_f) + wt(V_{\text{rem}}(w_i,u_j)) + wt(B^{w_1}_{\text{left}}(u_i))$.

Recall that

a. $wt(V_f) \leq wt(G)/12$ (by the hypothesis of Lemma 31).

b. $wt(V_{\text{rem}}(w_i,u_j)) \leq \frac{1}{2} wt(M)$ (by hypothesis for this case).

c. $wt(B^{w_1}_{\text{left}}(u_i)) \leq \frac{1}{4} (wt(M) - wt(V_{\text{rem}}(w_i,u_j)))$ (by Lemma 43).

Thus the increase to $V_{\text{left}}$ in each iteration is $\leq \frac{1}{12} wt(G) + \frac{1}{2} wt(V_{\text{rem}}(w_i,u_j)) + \frac{1}{4} (wt(M)) - \frac{1}{2} wt(V_f)$, which is equal to $\frac{1}{12} wt(G) + \frac{1}{2} wt(V_{\text{rem}}(w_i,u_j)) + \frac{1}{4} (wt(M)) \leq \frac{1}{12} wt(G) + \frac{1}{4} wt(M)$. Hence we can stop the first time $wt(V_{\text{left}})$ becomes greater than $wt(G)/12$. At this point, we have $wt(V_{\text{left}}) \leq \frac{1}{12} wt(G) + \frac{1}{2} wt(M) \leq \frac{11}{12} wt(G)$.

Thus we have an upper bound of $\frac{11}{12} wt(G)$ on all the disconnected components. Hence $p_{x_i,u_i}$ is a $\frac{11}{12}$ path separator.

## 6 Path separator in a planar digraph

Having seen how to construct a path separator in a mesh, we now show how to use that to construct an $\frac{11}{12}$ separator in any planar digraph.

1. Given a graph $G$, first embed the graph so that the root $r$ lies on the outer face. Through the root, draw a virtual directed cycle $C_0$ that encloses the entire graph, and orient it, say clockwise. Find the layering described in Section 4 and output it on a transducer. Cycle $C_0$ will be colored red and will be in the sublayer $(0,0)$. 

2. In the laminar family of red/blue cycles, find the cycle $C$ s.t. $wt(C)$ is more than $|G|/12$, but no colored cycle $C'$ in the interior of $C$ has the same property. Such a cycle will clearly exist (it could be the virtual cycle $C_0$). Let the sublayer of $C$ be $(k, l)$.

3. Find a path $p$ from the root $r$ to any vertex $r_C$ of the cycle $C$ such that no other vertex of $C$ is in the path. As seen above in Lemma 25, the graph in the interior of $C$ and belonging to the immediately next sublayer ($(k + 1, l)$ if $C$ is clockwise and $(k, l + 1)$ if $C$ is counter-clockwise) is a DAG of meshes. There are two cases possible:

   a. The graph $\tilde{C}$ has no strongly connected components of weight larger than $|G|/12$. In this case we simply extend the path $p$ from $r_C$ by encircling the cycle $C$ till the last vertex and stop.

   b. The graph $\tilde{C}$ has a strongly connected component of weight more than $|G|/12$. In this case, we extend $p$ from $r_C$ by encircling $C$ till the last vertex $u$ on $C$ that can reach any such component $M_C$. Then extend the path from $u$ to any vertex of $M_C$ and apply the mesh separator lemma (Lemma 31) to obtain the desired separator. (Observe that $M_C$ satisfies the hypothesis of Lemma 31.)

   ▶ Lemma 47. The path $p$ obtained by the above procedure is an $11/12$ separator.

   Proof. We look at the two cases:

   1. In this case it is clear that the interior and exterior of cycle $C$ are disconnected by $p$. The exterior of $C$ has size $\leq 11/24 |G|$ (by definition of $C$), and in its interior every strongly-connected component has weight at most $|G|/12$. Thus this satisfies the definition of an $11/12$ separator.

   2. We took the last edge in $C$ from $r_C$ that can reach the mesh $M_C$, and extended the path to $M_C$. Thus after removing $p$, one weakly-connected component consists of the exterior of $G$, along with (possibly) some vertices in the interior of $C$ that cannot reach any “large” mesh in the interior. Since $M_C$ has weight greater than $11/24 |G|$, no strongly-connected component embedded outside of $M_C$ can have weight more than $11/24 |G|$. Also, after removing path $p$, Lemma 31 guarantees that no other strongly-connected component will have weight more than $11/24 |G|$. Thus this satisfies the definition of an $11/12$ separator.

   Hence overall we can guarantee an $11/12$ path separator in $G$. □

7 Building a DFS tree using path separators

We give a recursive divide and conquer algorithm for DFS:

1. Given a planar drawing of $G$ and a root vertex on the outer face $r$, find an $11/12$ path separator $p = (r, v_1, v_2, \ldots, v_k)$. Path $p$ is included in the DFS tree.

2. Let $R(v)$ denote the set of vertices of $G$ reachable from $v$. Now for every vertex $v_i$ in $p$ compute in parallel:

   $R'(v_i) = R(v) \setminus (\bigcup_{j=i+1}^{k} R(v_j))$

   Our DFS will correspond to first traveling along $p$ to $v_k$, doing DFS on $R(v_k)$, and then while backtracking on $p$, do DFS on $R'(v_i)$ for $i$ from $k - 1$ downto 1. Given $G$, the encodings of $p$ and $R'(v_i)$ can all be computed in $AC^0(UL \cap co-UL)$.

3. For any $v_i$, $R'(v_i)$ can be written as a DAG of SCCs (strongly connected components), where each SCC is smaller than $11/24 |G|$. In $AC^0(UL \cap co-UL)$ we can compute this DAG and we can compute an encoding of the tuple $(i, M, v)$ where $M$ is a SCC in $R'(i)$ and $v$ is a vertex in $M$. Recursively, in parallel, we compute a DFS tree of $M$ for each tuple $(i, M, v)$, using $v$ is the root. Now we need to show how to sew together (some of) these trees, to form a DFS tree for $G$ with root $r$. 

Given a triple \((i, M, v)\), let \(x_0, x_1, \ldots, x_r\) be the order in which the vertices of \(M\) appear in a DFS traversal where the root \(x_0 = v\). Our DFS will correspond to first following the edges from \(x_0\) that lead to other SCCs in \(R(v_i)\). (No vertex reachable in this way can reach any \(x_j\), or else that vertex would also be in \(M\).) And then we will move on to \(x_1\) and repeat the process, etc. Thus let \(R''_{i,M,v}(x_j) = (R'(x_j) \setminus M) \setminus (\bigcup_{k<j} R'(x_k))\).

Our DFS tree is composed by computing a DFS tree \(T\) of the DAG of meshes (considering each mesh to be a vertex) using the algorithm of Section 3. A logspace machine can do a DFS traversal of \(T\), starting with the node containing \(v_i\) as the root, and using (as auxiliary information) the DFS tree that was computed for \((i, M, v_i)\). If this traversal contains an edge \((M, M')\) (where \(M\) and \(M'\) are SCCs in \(R'(v_i)\)), then there is exactly one \(j\) such that there is an edge from \(x_j\) in the DFS tree for \((i, M, v_i)\) to a vertex (call it \(v_{M'}\)) in \(M' \cap R''_{i,M,v}(x_j)\). [Namely, \(x_j\) is the first vertex in this tree that has an edge to \(M'\).] The edge from \(x_j\) to \(v_{M'}\) will be in our DFS tree, as will the DFS tree that was computed for \((i, M', v_{M'})\). We then continue the traversal of \(T\), and process each node of the DAG in the same way. All of this can be accomplished in \(\text{AC}^0(\text{UL} \cap \text{co-UL})\).

The final DFS tree for \(R_i\) consists of all of the edges that appear in the trees for tuples \((i, M, v)\) that were utilized in the traversal of \(T\). The tree for \(G\) consists of \(p\) together with the trees for each \(R_i\).

References

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