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One-Tape Turing Machine and Branching Program Lower Bounds for MCSP

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Abstract

For a size parameter $s\colon \mathbb{N} \to \mathbb{N}$, the Minimum Circuit Size Problem (denoted by MCSP$[s(n)]$) is the problem of deciding whether the minimum circuit size of a given function $f\colon \{0,1\}^n \to \{0,1\}$ (represented by a string of length $N := 2^n$) is at most a threshold $s(n)$. A recent line of work exhibited "hardness magnification" phenomena for MCSP: A very weak lower bound for MCSP implies a breakthrough result in complexity theory. For example, McKay, Murray, and Williams (STOC 2019) implicitly showed that, for some constant $\mu_1 > 0$, if MCSP$[2^{\mu_1 \cdot n}]$ cannot be computed by a one-tape Turing machine (with an additional one-way read-only input tape) running in time $N^{1.01}$, then P $\neq$ NP.

In this paper, we present the following new lower bounds against one-tape Turing machines and branching programs:

1. A randomized two-sided error one-tape Turing machine (with an additional one-way read-only input tape) cannot compute MCSP$[2^{\mu_2 \cdot n}]$ in time $N^{1.99}$, for some constant $\mu_2 > \mu_1$.
2. A non-deterministic (or parity) branching program of size $o(N^{1.5}/\log N)$ cannot compute MKTP, which is a time-bounded Kolmogorov complexity analogue of MCSP. This is shown by directly applying the Nechiporuk method to MKTP, which previously appeared to be difficult.

These results are the first non-trivial lower bounds for MCSP and MKTP against one-tape Turing machines and non-deterministic branching programs, and essentially match the best-known lower bounds for any explicit functions against these computational models.

The first result is based on recent constructions of pseudorandom generators for read-once oblivious branching programs (ROBPs) and combinatorial rectangles (Forbes and Kelley, FOCS 2018; Viola 2019). En route, we obtain several related results:

1. There exists a (local) hitting set generator with seed length $\widetilde{O}(\sqrt{N})$ secure against read-once polynomial-size non-deterministic branching programs on N-bit inputs.
2. Any read-once co-non-deterministic branching program computing MCSP must have size at least $2^{\widetilde{\Omega}(N)}$.

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1 Introduction

The Minimum Circuit Size Problem (MCSP) asks whether a given Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ can be computed by some Boolean circuit of size at most a given threshold s. Here the function f is represented by the truth table of f, i.e., the string of length $N := 2^n$ that is obtained by concatenating all the outputs of f. For a size parameter $s\colon \mathbb{N} \to \mathbb{N}$, its parameterized version is denoted by MCSP[s]: That is, MCSP[s] asks if the minimum circuit size of a function $f\colon \{0,1\}^n \to \{0,1\}$ is at most $s(n)$.

MCSP is one of the most fundamental problems in complexity theory, because of its connection to various research areas, such as circuit complexity [38, 27, 24, 33, 23, 2], learning theory [9], and cryptography [38, 18, 20]. It is easy to see that MCSP $\in \mathsf{NP}$ because, given a circuit C of size s as an NP certificate, one can check whether C computes the given function f in time $N^{O(1)}$. On the other hand, its NP-completeness is a long-standing open question, which dates back to the introduction of the theory of NP-completeness (cf. [4]), and it has an application to the equivalence between the worst-case and average-case complexity of NP (cf. [20]).

Recently, a line of work exhibited surprising connections between very weak lower bounds of MCSP and important open questions of complexity theory, informally termed as "hardness magnification" phenomena. Oliveira and Santhanam [37] (later with Pich [36]) showed that, if an approximation version of MCSP cannot be computed by a circuit of size $N^{1.01}$, then $\mathsf{NP} \not\subseteq \mathsf{P/poly}$ (in particular, $\mathsf{P} \neq \mathsf{NP}$ follows). Similarly, McKay, Murray, and Williams [32] showed that, if MCSP[$s(n)$] cannot be computed by a 1-pass streaming algorithm of poly $(s(n))$ space and poly $(s(n))$ update time, then $\mathsf{P} \neq \mathsf{NP}$. Therefore, in order to obtain a breakthrough result, it is sufficient to obtain a very weak lower bound for MCSP.

Are hardness magnification phenomena plausible approaches for resolving the P versus NP question? We do not know the answer yet. However, it should be noted that, as argued in [3, 37], hardness magnification phenomena appear to bypass the *natural proof* barrier of Razborov and Rudich [38], which is one of the major barriers of complexity theory for resolving the P versus NP question. Most of lower bound proof techniques of complexity theory are "natural" in the following sense: Given a lower bound proof for a circuit class $\mathfrak{C}$, one can interpret it as an efficient average-case algorithm for solving $\mathfrak{C}$-MCSP (i.e., one can efficiently decide whether a given Boolean function f can be computed by a small $\mathfrak{C}$-circuit when the input f is chosen uniformly at random; cf. Hirahara and Santhanam [22]). Razborov and Rudich [38] showed that such a "natural proof" technique is unlikely to resolve $\mathsf{NP} \not\subseteq \mathsf{P/poly}$; thus we need to develop fundamentally new proof techniques. There seems to be no simple argument that naturalizes proof techniques of hardness magnification phenomena; hence, investigating hardness magnification phenomena could lead us to a new non-natural proof technique.

1.1 Our results

1.1.1 Lower bounds against one-tape Turing machines

Motivated by hardness magnification phenomena, we study the time required to compute MCSP by using a one-tape Turing machine. We first observe that the hardness magnification phenomena of [32] imply that a barely superlinear time lower bound for a one-tape Turing machine is sufficient for resolving the P versus NP question.

▶ **Theorem 1** (A corollary of McKay, Murray, and Williams [32]; see Appendix A)**.** *There exists a small constant* $\mu > 0$ *such that if* $\mathrm{MCSP}[2^{\mu \cdot n}] \notin \mathsf{DTIME}_1\left[N^{1.01}\right]$*, then* $\mathsf{P} \neq \mathsf{NP}$*.*

Here, we denote by $\mathsf{DTIME}_1[t(N)]$ the class of languages that can be computed by a Turing machine equipped with a one-way read-only input tape and a two-way read/write work tape running in time $O(t(N))$ on inputs of length N. We note that it is rather counter-intuitive that there is a *universal* constant $\mu > 0$; it is instructive to state Theorem 1 in the following logically equivalent way: If $\mathrm{MCSP}[2^{\mu \cdot n}] \notin \mathsf{DTIME}_1\left[N^{1.01}\right]$ for *all constants* $\mu > 0$, then $\mathsf{P} \neq \mathsf{NP}$.[1]

One of our main results is a nearly quadratic lower bound on the time complexity of a *randomized* one-tape Turing machine (with one additional read-only one-way input tape) computing MCSP.

▶ **Theorem 2.** *There exists some constant* $0 < \mu < 1$ *such that* $\mathrm{MCSP}[2^{\mu \cdot n}] \notin \mathsf{BPTIME}_1\left[N^{1.99}\right]$*.*

Here, $\mathsf{BPTIME}_1[t(N)]$ denotes the class of languages that can be computed by a *two-sided-error randomized* Turing machine equipped with a one-way read-only input tape and a two-way read/write work tape running in time $t(N)$ on inputs of length N; we say that a two-sided-error randomized algorithm *computes* a problem if it outputs a correct answer with high probability (say, with probability at least $2/3$) over the internal randomness of the algorithm.

Previously, no non-trivial lower bound on the time complexity required for computing MCSP by a Turing machine was known. Moreover, Theorem 2 essentially matches the best-known lower bound for this computational model; namely, the lower bound due to Kalyanasundaram and Schnitger [28], who showed that Element Distinctness is not in $\mathsf{BPTIME}_1\left[o\left(N^2/\log N\right)\right]$.

Our lower bound against $\mathsf{BPTIME}_1\left[N^{1.99}\right]$ is much stronger than the required lower bound (i.e, $\mathsf{DTIME}_1\left[N^{1.01}\right]$) of the hardness magnification phenomenon of Theorem 1. However, Theorem 2 falls short of the hypothesis of the hardness magnification phenomenon of Theorem 1 because of the choice of the size parameter. In the hardness magnification phenomenon, we need to choose the size parameter to be $2^{\mu \cdot n}$ for some small constant $\mu > 0$, whereas, in our lower bound, we will choose μ to be some constant close to 1. That is, what is missing for proving $\mathsf{P} \neq \mathsf{NP}$ is to decrease the size parameter from $2^{(1-o(1)) \cdot n}$ to $2^{o(n)}$ in Theorem 2, or to increase the size parameter from $2^{o(n)}$ to $2^{(1-o(1)) \cdot n}$ in Theorem 1.

Next, we investigate the question of whether hardness magnification phenomena on $\mathrm{MCSP}[s(n)]$ such as Theorem 1 can be proved when the size parameter $s(n)$ is large, as posed by Chen, Jin, and Williams [11]. As observed in [10], most existing proof techniques on hardness magnification phenomena are shown by constructing an oracle algorithm which makes short queries to some oracle. For example, behind the hardness magnification

[1] Observe that $\exists \mu, (P(\mu) \Rightarrow Q)$ is logically equivalent to $\exists \mu, (\neg P(\mu) \vee Q)$, which is equivalent to $\neg(\forall \mu, P(\mu)) \vee Q$.

phenomena of Theorem 1 is a nearly-linear-time oracle algorithm that solves MCSP[$2^{o(n)}$] by making queries of length $2^{o(n)}$ to some PH oracle (see Corollary 17 for a formal statement). Chen, Hirahara, Oliveira, Pich, Rajgopal, and Santhanam [10] showed that most lower bound proof techniques can be generalized to such an oracle algorithm, thereby explaining the difficulty of combining hardness magnification phenomena with lower bound proof techniques. Following [10], we observe that our lower bound (Theorem 3) can be generalized to a lower bound against an oracle algorithm which makes short queries.

▶ **Theorem 3.** *Let $O \subseteq \{0,1\}^*$ be any oracle. Then, for every constant $1/2 < \mu < 1$,* MCSP[$2^{\mu \cdot n}$] *on truth tables of size $N := 2^n$ is not in* $\mathsf{BPTIME}_1^O\left[N^{1+\mu'}\right]$ *for some constant $\mu' > 0$, where all of the strings queried to O are of length $N^{o(1)}$.*

Theorem 3 can be seen as a partial answer to the question posed by [11]: It is impossible to extend the hardness magnification phenomena of Theorem 1 to MCSP[$2^{\mu n}$] for $\mu > 1/2$ by using similar techniques used in [32]. Recall that the proof techniques behind [32] are to construct a nearly-linear-time oracle algorithm that solves MCSP[$2^{\mu n}$] by making short queries to some oracle; the existence of such an oracle algorithm is ruled out by Theorem 3 when $\mu > 1/2$. Therefore, in order to obtain a hardness magnification phenomenon for MCSP[$2^{0.51n}$], one needs to develop a completely different proof technique that does not rely on constructing an oracle algorithm that makes short queries.

1.1.2 Lower bounds against branching programs

Another main result of this work is a lower bound against non-deterministic branching programs. We make use of *Nečiporuk's method*, which is a standard proof technique for proving a lower bound against branching programs. However, it appeared previously that Nečiporuk's method is not directly applicable to the problems such as MCSP [22]. In this paper, we develop a new proof technique for applying Nečiporuk's method to a variant of MCSP, called MKTP. MKTP is the problem of deciding whether $\mathrm{KT}(x) \leq s$ given (x, s) as input. Here $\mathrm{KT}(x)$ is defined as the minimum, over all programs M and integers t, of $|M| + t$ such that, for every i, M outputs the ith bit of x in time t given an index i as input [1]. We prove lower bounds against general branching programs and non-deterministic branching programs by using Nečiporuk's method.

▶ **Theorem 4.** *The size of a branching program computing* MKTP *is at least $\Omega(N^2/\log^2 N)$. The size of a non-deterministic branching program or a parity branching program computing* MKTP *is at least $\Omega(N^{1.5}/\log N)$.*

We mention that the same lower bound can be obtained for MKtP, which is an exponential-time analogue of MKTP.

Theorem 4 gives the first non-trivial lower bounds against non-deterministic and parity branching programs for MKTP and MKtP and, in addition, these are the best lower bounds which can be obtained by using Nečiporuk's method (cf. [7]). Previously, by using a pseudorandom generator for branching programs constructed by [25], it was shown in [36, 12] that (deterministic) branching programs requires $N^{2-o(1)}$ size to compute MCSP, MKTP and MKtP.[2] However, it is not known whether there is a pseudorandom generator

[2] It is worthy of note that Theorem 4 mildly improves the lower bounds of [36, 12] to $\Omega(N^2/\log^2 N)$ by directly applying Nečiporuk's method, which matches the state-of-the-art lower bound for any explicit function up to a constant factor.

for non-deterministic or parity branching programs. As a consequence, no non-trivial lower bound for MKTP (nor its exponential-time version MKtP) against these models was known before. Surprisingly, Theorem 4 is proved without using a pseudorandom generator nor a weaker object called a hitting set generator. We emphasize that it is surprising that a lower bound for MKtP can be obtained without using a hitting set generator; indeed, the complexity of MKtP is closely related to a hitting set generator, and in many settings (especially when the computational model is capable of computing XOR), a lower bound for MKtP and the existence of a hitting set generator are equivalent [20, 21].

A *hitting set generator* (HSG) $H\colon \{0,1\}^{\lambda(N)} \to \{0,1\}^N$ for a circuit class $\mathfrak{C}$ is a function such that, for any circuit C from $\mathfrak{C}$ that accepts at least $(1/2)\cdot 2^N$ strings of length N, there exists some seed $z \in \{0,1\}^{\lambda(N)}$ such that C accepts $H(z)$. A hitting set generator enables us to derandomize one-sided-error randomized algorithms by trying all $2^{\lambda(N)}$ seeds.

Along the way, we obtain several new results regarding a lower bound for MCSP and a hitting set generator. We present a hitting set generator secure against read-once non-deterministic branching programs, based on a pseudorandom generator constructed by Forbes and Kelley [14].

▶ **Theorem 5.** *There exists an explicit construction of a (local) hitting set generator* $H\colon \{0,1\}^{\widetilde{O}\left(\sqrt{N\cdot\log s}\right)} \to \{0,1\}^N$ *for read-once non-deterministic branching programs of size* s.

Previously, Andreev, Baskakov, Clementi, and Rolim [6] constructed a hitting set generator with non-trivial seed length for read-k-times non-deterministic branching programs, but their seed length is as large as $N - o(N)$. Theorem 5 improves the seed length to $\widetilde{O}(\sqrt{N\cdot\log s})$. As an immediate corollary, we obtain a lower bound for MCSP against read-once non-deterministic branching programs.

▶ **Corollary 6.** *Any read-once co-non-deterministic branching program that computes* MCSP *must have size at least* $2^{\widetilde{\Omega}(N)}$.

1.2 Our techniques

1.2.1 Local HSGs for MCSP lower bounds

For a circuit class $\mathfrak{C}$, a general approach for obtaining a $\mathfrak{C}$-lower bound for MCSP is by constructing a "local" hitting set generator (or a pseudorandom generator (PRG), which is a stronger notion) secure against $\mathfrak{C}$. Here, we say that a function $G\colon \{0,1\}^s \to \{0,1\}^N$ is *local* if, for every z, the ith bit of $G(z)$ is "easy to compute" from the index i; more precisely, for every seed z, there exists some circuit C of size at most s such that C outputs the ith bit of $G(z)$ on input $i \in [N]$. Note here that $G(z)$ is a YES instance of MCSP$[s]$, whereas a string w chosen uniformly at random is a NO instance of MCSP$[s]$ with high probability. This means that any $\mathfrak{C}$-algorithm that computes MCSP$[s]$ distinguishes the pseudorandom distribution $G(z)$ from the uniform distribution w, and hence the existence of $\mathfrak{C}$-algorithm for MCSP$[s]$ implies that there exists no local hitting set generator secure against $\mathfrak{C}$. This approach has been used in several previous works, e.g., [38, 1, 22, 12]. In fact, it is worthy of note that, in some sense, this is *the only approach* — at least for a general polynomial-size circuit class $\mathfrak{C} = \mathsf{P/poly}$, because Hirahara [20] showed that a lower bound for an approximation version of MCSP is equivalent to the existence of a local HSG.

At the core of our results is the recent breakthrough result of Forbes and Kelley [14], who constructed the first pseudorandom generator with $\mathsf{polylog}(n)$ seed length that fools

unknown-order read-once oblivious branching programs. Viola [42] used their construction to obtain a pseudorandom generator that fools *deterministic* Turing machines (DTMs). Herein, we generalize his result to the case of *randomized* Turing machine (RTMs), and the case of *two-sided-error* randomized Turing machine ($\mathsf{BPTIME}_1[t(N)]$).[3] At a high level, our crucial idea is that Viola's proof does not exploit the uniformity of Turing machines, and hence a good coin flip sequence of a randomized oracle algorithm and all of its [small enough] oracle queries and corresponding answers can be fixed as non-uniformity (Lemma 21). In addition, by a careful examination of the Forbes-Kelley PRG, we show that their PRG is local; this gives rise to a local PRG that fools $\mathsf{BPTIME}_1[t(N)]$, which will complete a proof of our main result (Theorem 3).

We note that the proof above implicitly shows an exponential-size lower bound for MCSP against read-once oblivious branching programs, which was previously not known. Corollary 6 generalizes this lower bound to the case of co-non-deterministic read-once (not necessarily oblivious) branching program. In order to prove this, we make use of PRGs that fool combinatorial rectangles (e.g., [14, 29]). We present a general transformation from a PRG for combinatorial rectangles into a HSG for non-deterministic read-once branching program, by using the proof technique of Borodin, Razborov, and Smolensky [8]; see Theorem 5.

1.2.2 Nečiporuk's method for MKTP lower bounds

In order to apply Nečiporuk's method to MKTP, we need to give a lower bound on the number of distinct subfunctions that can be obtained by fixing all but $O(\log n)$ bits.

The idea of counting distinct subfunctions of MKTP is to show that a random restriction which leaves $O(\log n)$ variables free induces different subfunctions with high probability. Specifically, partition the input variables $[n]$ into $m := n/O(\log n)$ blocks, pick $m-1$ strings $\rho := \rho_2 \cdots \rho_m \in (\{0,1\}^{O(\log n)})^{m-1}$ randomly, and consider the restricted function $f\restriction_\rho(\rho_1) := \mathrm{MKTP}(\rho_1\rho_2\cdots\rho_m, \theta)$ for some appropriate threshold function θ. Then, the string $\rho_i\rho_2\cdots\rho_m$ is compressible when $i \in \{2, \cdots, m\}$ whereas the string $\rho_1\rho_2\cdots\rho_m$ is not compressible when ρ_1 is chosen randomly. This holds as, in the former case, there exists a $2 \leq k \leq m$ such that $\rho_i = \rho_k$ and this yields a description for the string $\rho_i\rho_2\cdots\rho_m$ that is shorter than most of its descriptions in the latter case. Let now θ be the KT complexity of $\rho_i\rho_2\cdots\rho_m$ in the case where $i \in \{2, \cdots, m\}$. Therefore, $f\restriction_\rho(\rho_i) = 1$ for any $i \in \{2, \cdots, m\}$ and $f\restriction_\rho(\rho_1) = 0$ with high probability over a random ρ_1. This implies that, with high probability over the random restrictions ρ and ρ', it is the case that $f\restriction_\rho \not\equiv f\restriction_{\rho'}$. This is so as, for every $i \in \{2, \ldots, m\}$, the probability over the random restrictions ρ and ρ' that the string ρ_i is such that $f\restriction_{\rho'}(\rho_i) = f\restriction_\rho(\rho_i)$ is small, by the fact that $f\restriction_\rho(\rho_i) = 1$ and the fact that $f\restriction_{\rho'}(\rho_i) = 0$ with high probability over a random ρ_i [and therefore with high probability over a random ρ as well].

Unfortunately, the probability that $f\restriction_\rho \equiv f\restriction_{\rho'}$ holds may not be exponentially small. As a consequence, a lower bound on the number of distinct subfunctions that can be directly obtained from this fact may not be exponential. In contrast, we need to prove an exponential lower bound on the number of distinct subfunctions in order to obtain the state-of-the-art lower bound via Nečiporuk's method.

In order to make the argument work, we exploit symmetry of information for (resource-unbounded) Kolmogorov complexity and Kolmogorov-randomness. Instead of picking ρ and

[3] We emphasize that the notion of PRGs secure against these three computational models is different. See Definition 10, Definition 12, and Lemma 14.

ρ' randomly, we keep a set P which contains restrictions ρ that induce distinct subfunctions. Starting from $P := \emptyset$, we add one Kolmogorov-random restriction ρ to P so that the property of P is preserved. By using symmetry of information for Kolmogorov complexity, we can argue that one can add a restriction to P until P becomes as large as $2^{\Omega(n)}$, which proves that the number of distinct subfunctions of MKTP is exponentially large. Details can be found in Section 6.

1.3 Related work

Chen, Jin, and Williams [11] generalized hardness magnification phenomena to arbitrary sparse languages in NP. Note that MCSP$[2^{\mu n}]$ is a *sparse* language in the sense that the number of YES instances of MCSP$[2^{\mu n}]$ is at most $2^{\widetilde{O}(2^{\mu n})}$, which is much smaller than the number 2^{2^n} of all the instances of length 2^n. Hirahara [21] proved that a super-linear-size lower bound on co-non-deterministic branching programs for computing an approximation and space-bounded variant of MKtP implies the existence of a hitting set generator secure against read-once branching programs (and, in particular, $\mathsf{RL} = \mathsf{L}$).

Regarding unconditional lower bounds for MCSP, Razborov and Rudich [38] showed that there exists no AC^0-natural property useful against $\mathsf{AC}^0[\oplus]$, which in particular implies that MCSP $\notin \mathsf{AC}^0$; otherwise, the complement of MCSP would yield an AC^0-natural property useful against $\mathsf{P/poly} \supseteq \mathsf{AC}^0[\oplus]$. Hirahara and Santhanam [22] proved that MCSP essentially requires quadratic-size de Morgan formulas. Cheraghchi, Kabanets, Lu, and Myrisiotis [12] proved that MCSP essentially requires cubic-size de Morgan formulas as well as quadratic-size (general, unconstrained) branching programs. Golovnev, Ilango, Impagliazzo, Kabanets, Kolokolova, and Tal [16] proved that, for any prime p, MCSP requires constant-depth circuits, that are augmented with MOD_p gates, of weakly-exponential size.

The state-of-the-art time lower bound against DTMs on inputs of size n is $\Omega\big(n^2\big)$, proved by Maass [30], for the Polydromes function (which is a generalization of Palindromes). Regarding the case when the considered DTMs have a two-way read-only input tape, Maass and Schorr [31] proved that there is some problem in $\Sigma_2\mathsf{TIME}[n]$ that requires $\Omega\big(n^{3/2}/\log^6 n\big)$ time to compute on such machines. As mentioned earlier, in Section 1.1, the state-of-the-art time lower bound against RTMs is due to Kalyanasundaram and Schnitger [28], who showed that Element Distinctness is not in $\mathsf{BPTIME}_1\big[o\big(N^2/\log N\big)\big]$.

Viola [42] gave a PRG that fools RTMs that run in time $n^{1+\Omega(1)}$; this also yields a $n^{1+\Omega(1)}$ time lower bound against such machines. To do this, Viola extended prior work [31, 41] on simulating any RTM by a sum of ROBPs [see Lemma 19] and then employed the PRG by Haramaty, Lee, and Viola [17] that fools ROBPs;[4] it is a straightforward observation [42], then, that the Forbes-Kelley PRG [14] [which appeared afterwards and was inspired by the PRG by Haramaty, Lee, and Viola] yields a PRG of nearly quadratic stretch that fools RTMs and, therefore, a nearly quadratic lower bound against the same model as well. Moreover, Viola [42] showed that there exists some problem in $\Sigma_3\mathsf{TIME}[n]$ that requires $n^{1+\Omega(1)}$ time to compute on any RTM that has the extra feature of a two-way read-only input tape; one of the ingredients of this result, is again the PRG by Haramaty, Lee, and Viola [17].

For the case of one-tape TMs with no extra tapes, Hennie [19] proved in 1965 that the Palindromes function requires $\Omega\big(n^2\big)$ time to compute. Van Melkebeek and Raz [41] observed fixed-polynomial time lower bounds for SAT against non-deterministic TMs with

[4] It should be noted that before Haramaty, Lee, and Viola [17] and Viola [42], the problem of designing PRGs of polynomial stretch that fool RTMs was wide open despite intense research efforts.

a d-dimensional read/write two-way work tape and a random access read-only input tape; these lower bounds depend on d.

1.4 Organization

In Section 2, we give the necessary background. The main ideas of Theorem 3 and Theorem 5 are described in Section 3 and Section 4, respectively. In Section 5, we show that the pseudorandom generators we make use of are local, which will complete the proofs of Theorem 3 and Theorem 5. We prove Theorem 1 in Appendix A.

2 Preliminaries

2.1 Circuit complexity

Let $f : \{0,1\}^n \to \{0,1\}$. We define the *circuit complexity of* f, denoted by $\mathrm{CC}(f)$, to be equal to the size (i.e., the number of gates) of the smallest bounded fan-in unbounded fan-out Boolean circuit, over the $\{\mathrm{AND}, \mathrm{OR}, \mathrm{NOT}\} = \{\wedge, \vee, \neg\}$ basis, that, on input x, outputs $f(x)$. For a string $y \in \{0,1\}^{2^n}$, we denote by $\mathrm{CC}(y)$ the circuit complexity of the function $f_y : \{0,1\}^n \to \{0,1\}$ encoded by y; i.e., $f_y(x) = y_x$, for any $x \in \{0,1\}^n$.

A standard counting argument shows that a random function attains nearly maximum circuit complexity with high probability.

▶ **Proposition 7** ([39]). *For any function $s\colon \mathbb{N} \to \mathbb{N}$ with $s(n) = o(2^n/n)$, it holds that*

$$\mathop{\mathbf{Pr}}_{x \sim \{0,1\}^{2^n}}[\mathrm{CC}(x) \leq s(n)] = o(1),$$

for all large $n \in \mathbb{N}$.

▶ **Definition 8** (Minimum Circuit Size Problem [27]). *We define* MCSP *as*

$$\mathrm{MCSP} := \left\{(x, \theta) \in \{0,1\}^{2^n} \times \{0,1\}^n \mid \mathrm{CC}(x) \leq \theta\right\}_{n \in \mathbb{N}},$$

and its parameterized version *as*

$$\mathrm{MCSP}[s(n)] := \left\{x \in \{0,1\}^{2^n} \mid \mathrm{CC}(x) \leq s(n)\right\}_{n \in \mathbb{N}},$$

for a size parameter $s\colon \mathbb{N} \to \mathbb{N}$.

2.2 Turing machines

Throughout this paper, we consider a Turing machine that has one work tape and a one-way input tape. In this context, "one-way" means that the tape-head may move only from left to right.

A *deterministic Turing machine (DTM)* is a Turing machine with two tapes: A two-way read/write work tape and a one-way read-only input tape. Let $x \in \{0,1\}^*$ and M be a DTM; we write $M(x)$ to denote the output of M when its input tape is initialized with x and its work tape is empty. Let $t\colon \mathbb{N} \to \mathbb{N}$ be time-constructible. The class of languages $L \subseteq \{0,1\}^*$ decided by some $O(1)$-state time-t DTM is denoted by $\mathsf{DTIME}_1[t]$.

We also consider a randomized variant of DTMs. A *randomized Turing machine (RTM)* is a Turing machine with three tapes: A two-way read/write work tape, a one-way read-only input tape, and a one-way read-only random tape. Let $x, r \in \{0,1\}^*$ and M be a RTM;

we write $M(x, r)$ to denote the output of M when its input tape contains x, its work tape is empty, and its random tape contains r. Let $t\colon \mathbb{N} \to \mathbb{N}$ be time-constructible. For a language $L \subseteq \{0,1\}^*$ and a RTM M, we say that M *decides* L *with two-sided error* if $\Pr_r[M(x,r) = 1] \geq \frac{2}{3}$ for every input $x \in L$ and $\Pr_r[M(x,r) = 0] \geq \frac{2}{3}$ for every input $x \notin L$. The class of languages $L \subseteq \{0,1\}^*$ decided by some $O(1)$-state time-t RTM with two-sided error is denoted by $\mathsf{BPTIME}_1[t]$.

A *randomized oracle Turing machine (oracle RTM)* is a Turing machine with four tapes: A two-way read/write work tape, a one-way read-only input tape, a one-way read-only random tape, and an oracle tape. This model is identical to the randomized Turing machine model apart from the oracle tape, which is a standard oracle tape. The class of languages $L \subseteq \{0,1\}^*$ decided by some $O(1)$-state time-t oracle RTM, with access to some oracle $O \subseteq \{0,1\}^*$, with two-sided error is denoted by $\mathsf{BPTIME}_1^O[t]$.

2.3 Streaming algorithms

A *space-$s(n)$ streaming algorithm* with update time $u(n)$ on an input $x \in \{0,1\}^n$ has a working storage of $s(n)$ bits. At any point the algorithm can either choose to perform one operation on $O(1)$ bits in storage or it can choose to read the next bit from the input. The total time between two next-bit reads is at most $u(n)$ and the final outcome is reported in $O(u(n))$ time.

▶ **Lemma 9.** *Any one-pass streaming algorithm with $t(N)$ update time, on inputs of length N, can be simulated by a one-tape Turing machine with a one-way read-only input tape running in time $O(N \cdot \mathrm{poly}(t(N)))$.*

Proof. Recall that a streaming algorithm reads one bit of its input from left to right, and each consecutive read operation occurs within $t(N)$ time steps. Thus, it takes $N \cdot \mathrm{poly}(t(N))$ time-steps in total to finish the computation on inputs of length N in the standard multi-tape Turing machine model, as the size of the input is N and $\mathrm{poly}(t(N))$ time-steps suffice for some multi-tape Turing machine to perform an update [13]. For any time constructible function $T : \mathbb{N} \to \mathbb{N}$, a one-tape Turing machine can simulate a $T(n)$-time multi-tape Turing machine within $O(T(n)^2)$ steps. Thus, a streaming algorithm can be simulated in time $N \cdot (\mathrm{poly}(t(N)))^2 = N \cdot \mathrm{poly}(t(N))$ by a one-tape Turing machine. ◀

2.4 Branching programs

A *branching program (BP)* is a layered acyclic directed graph with three special vertices: a start vertex s (the *source*) and two finish vertices, namely an accepting vertex h_1 and a rejecting vertex h_0 (the *sinks*). Each layer has at most w vertices (w here stands for *width*); s is the sole vertex of the first layer and h_1 and h_0 are the only vertices of the last layer.

On input $x \in \{0,1\}^n$, the computation starts at s and follows a directed path from s to some h_b, with $b \in \{0,1\}$. On this occasion, the output of the computation is b. In each step, the computation queries some input x_i associated with layer i, for $i \in [n]$, and then visits the next layer, depending on the value of the variable just queried, namely 0 or 1, through an edge with label "$x_i = 0$" or "$x_i = 1$," respectively.

A branching program P *decides a language* $L \subseteq \{0,1\}^*$ in the natural way, i.e., $x \in L$ if and only if, on input x, the computation path that P follows starts at s and finishes at h_1. If the variable queried within each layer is the same, then the branching program is called *oblivious*. If the branching program queries each variable at most once, then the branching program is called a *read-once branching program (ROBP)*. If the branching program is

oblivious and always queries the variables in some known order, where it is known beforehand which variable x_i is queried at layer i, along any s-h_0 or any s-h_1 path (these are called *source-to-sink* paths), then the branching program is called *known-order*, else it is called *unknown-order*.

If there are multiple edges emanating from some vertex, and these edges have the same label, then the branching program is called *non-deterministic*. Non-deterministic branching programs may also have *free* (i.e., unlabelled) edges, as well. A *non-deterministic branching program computes a function* $f\colon \{0,1\}^n \to \{0,1\}$ if, for every $x \in \{0,1\}^n$ such that $f(x) = 1$, there is some s-h_1 path and for every $x \in \{0,1\}^n$ such that $f(x) = 0$, all source-to-sink paths are s-h_0 paths. A *co-non-deterministic branching program computes a function* $f\colon \{0,1\}^n \to \{0,1\}$ if, for every $x \in \{0,1\}^n$ such that $f(x) = 1$, all source-to-sink paths are s-h_1 paths and for every $x \in \{0,1\}^n$ such that $f(x) = 0$, there exists some s-h_0 path. The *size* of a branching program is the number of its labelled edges.

2.5 Pseudorandom generators and hitting set generators

We recall the standard notions of pseudorandom generators and hitting set generators.

▶ **Definition 10.** *Let* $s\colon \mathbb{N} \to \mathbb{N}$ *be a function,* $\mathfrak{C}$ *be a circuit class, and* $0 < \varepsilon < 1$. *A* pseudorandom generator (PRG) that ε-fools $\mathfrak{C}$ *is a function* $G\colon \{0,1\}^{s(n)} \to \{0,1\}^n$ *such that*

$$\left| \mathop{\mathbf{Exp}}_{x\sim\{0,1\}^n} [f(x)] - \mathop{\mathbf{Exp}}_{y\sim\{0,1\}^{s(n)}} [f(G(y))] \right| \leq \varepsilon,$$

for any circuit $C \in \mathfrak{C}$. *The value* $s(n)$ *is referred to as the* seed length *of* G.

▶ **Definition 11.** *Let* $s\colon \mathbb{N} \to \mathbb{N}$ *be a function,* $\mathfrak{C}$ *be a circuit class, and* $0 < \varepsilon < 1$. *A* hitting set generator (HSG) ε-secure against $\mathfrak{C}$ *is a function* $G\colon \{0,1\}^{s(n)} \to \{0,1\}^n$ *such that*

$$\mathop{\mathbf{Pr}}_{x\sim\{0,1\}^n}[C(x) = 1] \geq \varepsilon \implies C(H(y)) = 1 \text{ for some } y \in \{0,1\}^{s(n)},$$

for any circuit $C \in \mathfrak{C}$. *By default, we choose* $\varepsilon := 1/2$.

For our purpose, it is useful to extend the notion of PRG to a pseudorandom generator that fools *randomized* algorithms.

▶ **Definition 12.** *For a function* $s\colon \mathbb{N} \to \mathbb{N}$ *and a parameter* $0 < \varepsilon < 1$, *a function* $G\colon \{0,1\}^{s(n)} \to \{0,1\}^n$ *is said to be a pseudorandom generator that* ε-fools q-state time-t RTMs *if*

$$\left| \mathop{\mathbf{Exp}}_{\substack{x\sim\{0,1\}^n,\\ r\sim\{0,1\}^t}} [M(x,r)] - \mathop{\mathbf{Exp}}_{\substack{y\sim\{0,1\}^{s(n)},\\ r\sim\{0,1\}^t}} [M(G(y),r)] \right| \leq \varepsilon,$$

for any q*-state time-*t *RTM* M.

2.6 MCSP lower bounds from local HSGs

For a function $G\colon \{0,1\}^s \to \{0,1\}^n$, we say that G is *local* [12] if $\mathrm{CC}(G(z)) \leq s$ for every string $z \in \{0,1\}^s$. We make use of the following standard fact.

▶ **Lemma 13.** *Let $s\colon \mathbb{N} \to \mathbb{N}$ be a function such that $s(n) = o(2^n/n)$, and $N := 2^n$. Suppose that there exists a local hitting set generator $H\colon \{0,1\}^{s(n)} \to \{0,1\}^N$ for a circuit class $\mathfrak{C}$. Then,* $\mathrm{MCSP}[s(n)] \notin \mathsf{co}\mathfrak{C}$.

Proof. We prove the contrapositive. Let $C \in \mathsf{co}\mathfrak{C}$ be a circuit that computes $\mathrm{MCSP}[s(n)]$. Since $\mathrm{CC}(H(z)) \leq s(n)$, we have $H(z) \in \mathrm{MCSP}[s(n)]$; thus $C(H(z)) = 1$, for every $z \in \{0,1\}^{s(n)}$. For a random $w \sim \{0,1\}^N$, it follows from Proposition 7 that $w \notin \mathrm{MCSP}[s(n)]$ with probability $1 - o(1)$; hence $C(w) = 0$ for most w. Therefore, $\neg C \in \mathfrak{C}$ accepts at least a half of $\{0,1\}^N$ but rejects every string in the range of H, which contradicts the security of the hitting set generator H. ◀

We observe that a local pseudorandom generator for time-t RTMs also "fools" $\mathsf{BPTIME}_1[t(N)]$ in the following sense.

▶ **Lemma 14.** *Let $s, t\colon \mathbb{N} \to \mathbb{N}$ be functions, such that $s(n) = o(2^n/n)$, and $N := 2^n$. Suppose that there exists a family of local pseudorandom generators $G = \{G_n : \{0,1\}^{s(n)} \to \{0,1\}^N\}_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, G_n $(1/6)$-fools time-$t(N)$ RTMs. Then,* $\mathrm{MCSP}[s(n)]$ *is not in* $\mathsf{BPTIME}_1[t(N)]$.

Proof. We prove the contrapositive. Let M be a time-t RTM that decides $\mathrm{MCSP}[s(n)]$. Fix any $n \in \mathbb{N}$. For any seed $z \in \{0,1\}^{s(n)}$, we have $G_n(z) \in \mathrm{MCSP}[s(n)]$ since G_n is local. Thus, $\Pr_r[M(G_n(z), r) = 1] \geq 2/3$. On the other hand, pick a string $w \in \{0,1\}^N$ chosen uniformly at random. By the counting argument of Proposition 7, we get $\Pr_w[w \notin \mathrm{MCSP}[s(n)]] \geq 1 - o(1)$. Thus, we have $\Pr_{w,r}[M(w, r) = 1] \leq o(1) + 1/3 < 1/2$. Therefore,

$$\Pr_{z,r}[M(G_n(z), r) = 1] - \Pr_{w,r}[M(w, r) = 1] > \frac{2}{3} - \frac{1}{2} = \frac{1}{6},$$

which means that G_n does not fool RTMs. ◀

3 MCSP lower bounds against one-tape oracle RTMs

In this section, we present a proof of our main result.

▶ **Theorem 15** (Theorem 3, restated). *Let $O \subseteq \{0,1\}^*$ be any language. Then, for every constant $1/2 < \mu < 1$,* $\mathrm{MCSP}[2^{\mu\cdot n}]$ *on truth tables of size $N := 2^n$ is not in* $\mathsf{BPTIME}_1^O\left[N^{2\cdot(\mu' - o(1))}\right]$, *for all $1/2 < \mu' < \mu$, where all of the strings queried to O are of length $N^{o(1)}$.*

3.1 Connections to hardness magnification

As discussed in Section 1.1.1, Theorem 15 implies that establishing hardness magnification phenomena for MCSP, when the circuit size threshold parameter is $2^{0.51n}$, would require the development of new techniques; see Remark 18. To explain why this is true, we shall first require the following result by McKay, Murray, and Williams [32] that gives an oracle streaming algorithm for MCSP.

▶ **Lemma 16** ([32, Theorem 1.2]). *Let $s : \mathbb{N} \to \mathbb{N}$ be a size function, with $s(n) \geq n$ for all n, and $N := 2^n$. Then, there is a one-pass streaming algorithm for* $\mathrm{MCSP}[s(n)]$ *on N-bit inputs running in $N \cdot \widetilde{O}(s(n))$ time with $\widetilde{O}\left(s(n)^2\right)$ update time and $\widetilde{O}(s(n))$ space, using an oracle for* $\Sigma_3\mathrm{SAT}$ *with queries of length $\widetilde{O}(s(n))$.*

A corollary of Lemma 16 and Lemma 9 is the following.

▶ **Corollary 17** (Consequences of hardness magnification from currently known techniques)**.** *Let $s : \mathbb{N} \to \mathbb{N}$ be a size function. Then,* $\mathrm{MCSP}[s(n)]$ *on truth tables of length $N := 2^n$ is in* $\mathsf{DTIME}_1^O[N \cdot \mathrm{poly}(s(n))]$*, for some $O \in \Sigma_3^{\mathsf{P}}$, where all of the strings queried to O are of length at most* $\mathrm{poly}(s(n))$*.*

The following remark summarizes the main idea of this subsection.

▶ Remark 18. By Corollary 17, we see that if $s(n) = 2^{\mu \cdot n}$, for $\mu = o(1)$, then $\mathrm{MCSP}[s(n)]$ is in $\mathsf{DTIME}_1^O\left[N^{1+o(1)}\right]$, where all of the strings queried to O are of length $N^{o(1)}$. In light of this observation, Theorem 15 is important for the following reason. As $\mathsf{DTIME}_1^O\left[N^{1+o(1)}\right]$ is a subset of $\mathsf{BPTIME}_1^O\left[N^{2\cdot(\mu'-o(1))}\right]$ for all $1/2 < \mu' < 1$ and all languages $O \subseteq \{0,1\}^*$, Theorem 15 shows that establishing hardness magnification phenomena for $\mathrm{MCSP}[s(n)]$ like that of Theorem 1, when $s(n) = 2^{\mu \cdot n}$ for any constant $1/2 < \mu < 1$, would require the development of techniques that do not rely on designing oracle algorithms that make short oracle queries.

3.1.1 Comparison with the locality barrier

Chen, Hirahara, Oliveira, Pich, Rajgopal, and Santhanam [10] introduced the "locality barrier" to explain why it will be difficult to acquire a major complexity breakthrough through the lens of hardness magnification. Their reasoning goes as follows:

> Existing magnification theorems unconditionally show that problems, against which some circuit lower bound implies a complexity-theoretic breakthrough, admit highly efficient small fan-in oracle circuits, while lower bound techniques against weak circuit models quite often easily extend to circuits containing such oracles.

Our Remark 18, therefore, is close in spirit to the results of Chen et al. [10]: We make use of a lower bound (Theorem 15) to motivate the development of new techniques for proving hardness magnification phenomena while Chen et al. make use of hardness magnification phenomena to motivate the development of new techniques for acquiring lower bounds; a notable difference is that we consider one-tape Turing machines while they consider Boolean circuits.

3.2 Proof of Theorem 15

In order to prove Theorem 15, our goal is to construct a local pseudorandom generator that fools oracle RTMs and then apply Lemma 14. Viola [42] constructed a pseudorandom generator that fools the one-tape Turing machine model (DTM).[5] We will show that, in fact, the same construction fools oracle RTMs as well. In order to do so, we recall the idea of Viola [42]. The idea is that, in order to fool DTMs, it is sufficient to use a PRG that ε-fools ROBPs for an exponentially small ε. This is because time-t DTMs can be written as the sum of an exponential number of ROBPs.

[5] We note that our definition of PRG is different from that of [42] in that a random tape is not regarded as an input tape.

▶ **Lemma 19** (Viola [42]). *Let $n \in \mathbb{N}$ and M be a q-state time-t DTM. Then, there is a family $\{P_\alpha\}_{\alpha \in A}$ of n-input ROBPs of width $\exp\big(O\big(\sqrt{t} \cdot \log(tq)\big)\big)$ such that, for any $x \in \{0,1\}^n$,*

$$M(x) = \sum_{\alpha \in A} P_\alpha(x),$$

where $|A| \leq (tq)^{O(\sqrt{t})}$.

By a simple calculation (cf. Section 5), any pseudorandom generator that $\varepsilon/|A|$-fools ROBPs also ε-fools DTMs. Viola [42] then used the pseudorandom generator of Forbes and Kelley [14] that fools ROBPs. By a careful examination, we will show that the Forbes-Kelley pseudorandom generator is local; see Corollary 39 in Section 5.

▶ **Theorem 20** (Forbes-Kelley PRG is local). *There exists a local pseudorandom generator with seed length $\widetilde{O}\big((\sqrt{t} + \log(1/\varepsilon)) \cdot \log q\big)$ that ε-fools q-state time-t n-input DTMs for any $t \geq n$.*

Our main idea for obtaining an oracle randomized Turing machine lower bound is that Viola's reduction can be applied to *non-uniform computational models*, i.e., q-state Turing machines where q can become large as the input length becomes large. More specifically, it is possible to incorporate all possible oracle queries [along with their answers] and any good coin flip sequence r into the internal states of DTMs.

▶ **Lemma 21.** *For an input length $n \in \mathbb{N}$, for any q-state time-t oracle RTM M, that only queries strings of length at most ℓ to its oracle O, and a coin flip sequence $r \in \{0,1\}^t$, there exists some $\big(q \cdot 2^\ell \cdot t\big)$-state time-$t$ DTM M' such that $M'(x) = M^O(x, r)$ for every input $x \in \{0,1\}^n$.*

Proof. Let Q_M denote the set of the states of M. We define the set of the states of M' as

$$Q_{M'} := \Big\{(q, s, b, i) \in Q_M \times \{0,1\}^\ell \times \{0,1\} \times [t] \mid O(s) = b\Big\}.$$

The transition from the state $(q, s, b, i) \in Q_{M'}$ can be defined in a natural way, by using the i-th bit of r, namely r_i, the state q, and the fact that $O(s) = b$. ◀

▶ **Corollary 22.** *There exists a local pseudorandom generator with seed length $\sigma(t, q, \varepsilon) = \widetilde{O}\big((\sqrt{t} + \log(1/\varepsilon)) \cdot \log(q \cdot 2^\ell \cdot t)\big)$ that ε-fools q-state time-t n-input oracle RTMs that may only query strings of length at most ℓ to their oracle, for any $t \geq n$.*

Proof. We hard-code the oracle queries and their answers in the internal states and, moreover, we use an averaging argument to fix one good coin flip sequence r. Let M be any q-state time-t oracle RTM that may query to its oracle O strings of length at most ℓ. Let G be a PRG from Theorem 20. We have that

$$\begin{aligned}
&\left| \mathop{\mathbf{Exp}}_{r \sim \{0,1\}^t} \left[\mathop{\mathbf{Exp}}_{x \sim \{0,1\}^n} \big[M^O(x, r)\big] \right] - \mathop{\mathbf{Exp}}_{r \sim \{0,1\}^t} \left[\mathop{\mathbf{Exp}}_{y \sim \{0,1\}^{\sigma(t,q,\varepsilon)}} \big[M^O(G(y), r)\big] \right] \right| \\
&\quad = \left| \mathop{\mathbf{Exp}}_{r} \left[\mathop{\mathbf{Exp}}_{x} \big[M^O(x, r)\big] - \mathop{\mathbf{Exp}}_{y} \big[M^O(G(y), r)\big] \right] \right| \\
&\quad \leq \mathop{\mathbf{Exp}}_{r} \left[\left| \mathop{\mathbf{Exp}}_{x} \big[M^O(x, r)\big] - \mathop{\mathbf{Exp}}_{y} \big[M^O(G(y), r)\big] \right| \right] \\
&\quad \leq \left| \mathop{\mathbf{Exp}}_{x} \big[M^O(x, r^*)\big] - \mathop{\mathbf{Exp}}_{y} \big[M^O(G(y), r^*)\big] \right|,
\end{aligned}$$

for some $r^* \in \{0,1\}^t$, by an averaging argument. By applying Lemma 21, for M^O, O, and r^*, we obtain an equivalent $\big(q \cdot 2^\ell \cdot t\big)$-state time-$t$ DTM M'. The result now follows from Theorem 20. Specifically,

$$\left|\mathop{\mathbf{Exp}}_{x}\big[M^O(x,r^*)\big] - \mathop{\mathbf{Exp}}_{y}\big[M^O(G(y),r^*)\big]\right| = \left|\mathop{\mathbf{Exp}}_{x}[M'(x)] - \mathop{\mathbf{Exp}}_{y}[M'(G(y))]\right| \leq \varepsilon. \qquad \blacktriangleleft$$

Proof of Theorem 15. Take the local pseudorandom generator G of Corollary 22 with parameter $\varepsilon := 1/6$. Let $1/2 < \mu' < \mu < 1$ be arbitrary constants. Let $t, s, \ell \colon \mathbb{N} \to \mathbb{N}$ be functions such that $t(N) = N^{2\cdot\left(\mu' - o(1)\right)}$, $s(n) = 2^{\mu \cdot n}$, and $\ell(n) = 2^{o(n)}$. Then, the seed length of G is at most

$$\widetilde{O}\left(\sqrt{t(N)} \cdot (\log q + \ell(n))\right) \leq \widetilde{O}(N^{\mu' - o(1) + o(1)}) \leq s(n),$$

where $N = 2^n$. Since $s(n) = o(2^n/n)$, by Lemma 14, we obtain that $\mathrm{MCSP}[s(n)] \notin \mathsf{BPTIME}_1^O[t(N)]$, where all of the strings queried to O are of length $N^{o(1)}$. $\blacktriangleleft$

4 HSGs against non-deterministic ROBPs

In this section, we present a construction of hitting set generator secure against non-deterministic ROBPs.

▶ **Theorem 23** (Theorem 5, restated). *There exists a local hitting set generator $H \colon \{0,1\}^{\widetilde{O}\left(\sqrt{n \cdot \log s}\right)} \to \{0,1\}^n$ for n-input size-s read-once non-deterministic branching programs.*

As a corollary, we obtain an exponential-size lower bound for co-non-deterministic read-once branching programs that compute MCSP.

▶ **Corollary 24** (Corollary 6, restated). *Any read-once co-non-deterministic branching program that computes* MCSP *must have size at least $2^{\widetilde{\Omega}(N)}$.*

Proof. Immediate from Lemma 13 and Theorem 23. $\blacktriangleleft$

Herein, we present a general connection from a pseudorandom generator for combinatorial rectangles to a hitting set generator for non-deterministic ROBPs. Below, for $x \in \{0,1\}^n$ and $S \subseteq [n]$, we denote by $x|_S$ the $|S|$-bit string obtained by concatenating x_i for each $i \in S$.

▶ **Definition 25** (Combinatorial rectangles). *Let $n \in \mathbb{N}$. A* combinatorial rectangle of k products and width m *is a function $\pi : \{0,1\}^n \to \{0,1\}$ of the form*

$$\pi(x) = \bigwedge_{j=1}^{k} f_j\Big(x|_{S_j}\Big),$$

for every $x \in \{0,1\}^n$, where $k \in \mathbb{N}$, $f_j : \{0,1\}^m \to \{0,1\}$ and $|S_j| \leq m$, for all $1 \leq j \leq k$, and the sets $\{S_j\}_{j=1}^k$ are disjoint subsets of $[n]$.

▶ **Theorem 26** (Local PRG for combinatorial rectangles). *There exists a local pseudorandom generator $G \colon \{0,1\}^r \to \{0,1\}^n$ that ε-fools the class of combinatorial rectangles of k products and width m, where the seed length r is $\widetilde{O}(m + \log(k/\varepsilon)) \cdot \mathsf{polylog}(n)$.*

We defer the proof of Theorem 26 to Section 5 (cf. Theorem 40). For the proof of Theorem 23, we shall invoke the following lemma first, by Borodin, Razborov, and Smolensky [8].

▶ **Lemma 27** (Borodin, Razborov, and Smolensky [8]). *Let $n, k, s \in \mathbb{N}$, $m := n/k \geq 1$, and $t := O\big((2s)^{2k}\big)$. Let $f\colon \{0,1\}^n \to \{0,1\}$ be a function that can be computed by a read-once non-deterministic branching program of size s. Then, there exist combinatorial rectangles $\pi_1, \ldots, \pi_t$ of k products and width m such that, for all $x \in \{0,1\}^n$,*

$$f(x) = \bigvee_{i=1}^{t} \pi_i(x)\,.$$

Proof of Theorem 23. Let $G\colon \{0,1\}^r \to \{0,1\}^n$ be the local PRG of Theorem 26 that fools k-product combinatorial rectangles of width m. We prove that G is a hitting set generator secure against read-once non-deterministic branching programs of size s. To this end, let f be any read-once non-deterministic branching programs of size s such that $f(G(z)) = 0$ for every seed $z \in \{0,1\}^r$. We claim that $\mathbf{Pr}_w[f(w) = 1] < \frac{1}{2}$. By Lemma 27, f can be written as an OR of t combinatorial rectangles $\pi_1, \ldots, \pi_t$. By the assumption, for every seed z, we have $\bigvee_i \pi_i(G(z)) = f(G(z)) = 0$, and thus $\pi_i(G(z)) = 0$. By the fact that the pseudorandom generator G ε-fools combinatorial rectangles, it holds that, for all $1 \leq i \leq t$,

$$\left| \mathop{\mathbf{Pr}}_{x \sim \{0,1\}^n}[\pi_i(x) = 1] - \mathop{\mathbf{Pr}}_{z \sim \{0,1\}^r}[\pi_i(G(z)) = 1] \right| \leq \varepsilon$$

or

$$\mathop{\mathbf{Pr}}_{x \sim \{0,1\}^n}[\pi_i(x) = 1] \leq \varepsilon.$$

Hence,

$$\mathop{\mathbf{Pr}}_{x \sim \{0,1\}^n}[f(x) = 1] = \mathop{\mathbf{Pr}}_{x \sim \{0,1\}^n}\left[\bigvee_{i=1}^{t} \pi_i(x) = 1\right] \leq \sum_{i=1}^{t} \mathop{\mathbf{Pr}}_{x \sim \{0,1\}^n}[\pi_i(x) = 1] \leq t \cdot \varepsilon.$$

Choosing $\varepsilon := 1/(4t)$, this probability is bounded by $1/4$. It remains to choose the parameter k so that the seed length r is minimized. We have

$$r = \mathsf{polylog}(n) \cdot \widetilde{O}(m + \log(k/\varepsilon)) = \widetilde{O}(n/k + k \cdot \log s) = \widetilde{O}\left(\sqrt{n \cdot \log s}\right),$$

by setting $k := \sqrt{n/\log s}$. ◀

5 The Forbes-Kelley PRG is local

In this section, we show that the Forbes-Kelley PRG is local (up to some overhead in the seed length), which will complete the proofs of Section 3 and Section 4.

In the following proofs, for each pseudorandom generator $G\colon \{0,1\}^s \to \{0,1\}^n$ with seed length s, we will show that $\mathrm{CC}(G(z)) \leq \widetilde{O}(s) \cdot \mathsf{polylog}(n)$ for every seed $z \in \{0,1\}^s$. This naturally gives rise to a local pseudorandom generator of seed length $\widetilde{O}(s) \cdot \mathsf{polylog}(n)$, by simply ignoring a part of the seed.

We first recapitulate biased and almost k-wise independent distributions; these are primitives used in Forbes-Kelley PRG.

5.1 Biased and almost k-wise independent distributions

▶ **Definition 28.** *Let $n \in \mathbb{N}$ and $0 < \delta < 1$. A random variable $X = (x_1, \ldots, x_n)$ over $\{0,1\}^n$ is said to be* δ-biased with respect to the distribution D *if, for any $S \subseteq [n]$, it is the*

case that

$$\left| \Pr_{X \sim D}\left[\prod_{i \in S}(-1)^{x_i} = 1\right] - \Pr_{X \sim \{0,1\}^n}\left[\prod_{i \in S}(-1)^{x_i} = -1\right] \right| \leq \delta.$$

On this occasion, we also say that D is a δ-biased distribution over $\{0,1\}^n$.

Below, we define almost k-wise independent distributions.

▶ **Definition 29.** *Let $n \in \mathbb{N}$, $0 < \gamma < 1$, and $k > 0$. A random variable $X = (x_1, \ldots, x_n)$ over $\{0,1\}^n$ is said to be* γ-almost k-wise independent with respect to the distribution D *if, for any k indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, it is the case that*

$$\sum_{\alpha \in \{0,1\}^k} \left| \Pr_{X \sim D}[x_{i_1} \cdots x_{i_k} = \alpha] - \Pr_{X \sim \{0,1\}^n}[x_{i_1} \cdots x_{i_k} = \alpha] \right| \leq \gamma.$$

On this occasion, we also say that D is a γ-almost k-wise independent distribution over $\{0,1\}^n$.

▶ **Definition 30.** *Let $n, k \in \mathbb{N}$, with $k \leq n$. A random variable $X = (x_1, \ldots, x_n)$ over $\{0,1\}^n$ is said to be* k-wise independent with respect to the distribution D *if X is 0-almost k-wise independent with respect to the distribution D. On this occasion, we also say that D is a* k-wise independent distribution.

It is known that a k-wise independent distribution over $\{0,1\}^n$ may be sampled by using $O(k \log n)$ random bits [40]. The following lemma upper-bounds the circuit complexity of some k-wise independent distribution.

▶ **Theorem 31** ([12, 15, 40])**.** *There exists a local k-wise independent generator $G\colon \{0,1\}^s \to \{0,1\}^n$ with seed length $s = k \cdot \widetilde{O}(\log n)$.*

We next show that there exists a local ε-biased generator.

▶ **Lemma 32** (The complexity of multiplication; cf. [15])**.** *For an integer $m > 0$, let the elements in $\mathbb{F}_{2^m}$ be represented by m-bit strings. Then, there exists a circuit of size $\widetilde{O}(m)$ that, on input $x, y \in \mathbb{F}_{2^m}$, outputs the m-bit representation of the product $x \cdot y$.*

▶ **Theorem 33.** *There exists a local ε-biased generator $G\colon \{0,1\}^s \to \{0,1\}^n$ with seed length $s = \widetilde{O}(\log(n/\varepsilon)) \cdot \log n$.*

Proof. We use the standard construction of an ε-biased generator G_0 of [5]. Let $m := \log(n/\varepsilon)$, and take a field of size 2^m. For random seeds $a, b \in \mathbb{F}_{2^m}$ of length $2m$, the ith bit of $G_0(a,b)$ is defined as $\langle a^i, b \rangle$, i.e., the inner product of the binary representations of a^i and b. It was shown in [5] that, for a, b chosen uniformly at random from $\mathbb{F}_{2^m}$, the distribution of $G_0(a,b)$ is ε-biased.

We claim that, for every $a, b \in \mathbb{F}_{2^m}$, there exists a circuit of size $O(\log(n/\varepsilon) \cdot \log n)$ that takes $i \in [n]$ as an input of length $\log n$ and computes $\langle a^i, b \rangle$. Indeed, we hardwire a^{2^j} for all $j \leq \log n$ into a circuit; given an input $i \in [n]$, one can compute a^i by multiplying a^{2^j} for all j such that the j-th bit of the binary representation of i is 1. This can be done with a circuit of size $\widetilde{O}(m) \cdot \log n$ by using Lemma 32. It remains to compute the inner product of a^i and b, which can be done with a linear-size circuit. Overall, we obtain a circuit of size $\widetilde{O}(m) \cdot \log n$.

The local ε-biased generator G is defined as a version of G_0 such that a part c of the seed is ignored: i.e., $G(a,b,c) := G_0(a,b)$, where $a, b \in \mathbb{F}_{2^m}$ and $|c| = \widetilde{O}(m) \cdot \log n$. ◀

Finally, we present a local δ-almost k-wise independent generator.

▶ **Lemma 34** ([34]). *Let $0 < \delta < 1$ and D be an δ-biased distribution over n-bit strings. Then, for any $k \in \mathbb{N}$, D is a $\left(2^{k/2} \cdot \delta\right)$-almost k-wise independent distribution over n-bit strings.*

▶ **Theorem 35.** *There exists a local δ-almost k-wise independent generator $G\colon \{0,1\}^s \to \{0,1\}^n$ with seed length $s = \widetilde{O}(k + \log(n/\delta)) \cdot \log n$.*

Proof. By setting $\varepsilon := 2^{-k/2} \cdot \delta$, the local ε-biased pseudorandom generator of Theorem 33 is a local δ-almost k-wise independent generator by using Lemma 34. ◀

5.2 The Forbes-Kelley PRG

▶ **Definition 36** (Forbes-Kelley PRG [14]). *Let n, δ, γ, r, k be some parameters chosen later. Let $D_1, \ldots, D_r$ be r independent δ-biased distributions and let $T_1, \ldots, T_r$ be r independent γ-almost k-wise independent distributions over $\{0,1\}^n$. We define G_r^{FK} inductively as follows. Let G_1^{FK} be some $320k$-wise independent distribution and let*

$$G_{i+1}^{\mathrm{FK}} := D_i + T_i \wedge G_i^{\mathrm{FK}},$$

for all $1 \leq i \leq r-1$, where $\wedge$ denotes bitwise AND and $+$ denotes bitwise XOR.

▶ **Theorem 37** (Correctness of Forbes-Kelley PRG [14]). *For parameters $n, w \in \mathbb{N}$ and $\varepsilon > 0$, choose the parameters as follows: $r := \lceil \lg n \rceil$, $k := \lceil 3 \lg(nw/\varepsilon) \rceil$, $\gamma := (nw/\varepsilon)^{-9}$, and $\delta := (nw\mathcal{L}/\varepsilon)^{-3}$, where $\mathcal{L} := \binom{n}{k} w^{1/2}$. Then, $G_r^{\mathrm{FK}}\colon \{0,1\}^s \to \{0,1\}^n$ is a pseudorandom generator that ε-fools unknown-order ROBPs of width w, where the seed length s is $O\left(\log(nw/\varepsilon) \cdot \log^2 n\right)$.*

▶ **Theorem 38.** *There exists a local pseudorandom generator of seed length $\widetilde{O}\left(\log(nw/\varepsilon) \cdot \log^3 n\right)$ that ε-fools unknown-order n-input ROBPs of width w.*

Proof. We instantiate the Forbes-Kelley pseudorandom generator G_r^{FK} by using the parameters of Theorem 37 and using the construction of local generators of Theorem 33, Theorem 35, and Theorem 31 for D_i, T_i, and G_1^{FK}, respectively.

For a distribution $\mathcal{D}$, we denote by $\mathrm{CC}(\mathcal{D})$ the maximum of $\mathrm{CC}(D)$ over all strings D in the range of $\mathcal{D}$. We claim that $\mathrm{CC}\left(G_r^{\mathrm{FK}}\right)$ is at most $\widetilde{O}\left(\log(nw/\varepsilon) \cdot \log^3 n\right)$. By Theorem 33 and Theorem 35, we have $\mathrm{CC}(D_i) \leq \widetilde{O}(\log(n/\delta)) \cdot \log n$ and $\mathrm{CC}(T_i) \leq \widetilde{O}(k + \log(n/\gamma)) \cdot \log n$. Therefore, since $G_{i+1}^{\mathrm{FK}} = D_i + T_i \wedge G_i^{\mathrm{FK}}$, we obtain

$$\mathrm{CC}\left(G_{i+1}^{\mathrm{FK}}\right) \leq \mathrm{CC}(D_i) + \mathrm{CC}(T_i) + \mathrm{CC}\left(G_i^{\mathrm{FK}}\right) + O(1)$$

for any $i \in [r-1]$. We also have $\mathrm{CC}\left(G_1^{\mathrm{FK}}\right) \leq k \cdot \widetilde{O}(\log n)$ from Theorem 31. Therefore,

$$\mathrm{CC}\left(G_r^{\mathrm{FK}}\right) \leq r \cdot \widetilde{O}(k + \log(n/\gamma\delta)) \cdot \log n + k \cdot \widetilde{O}(\log n) = \widetilde{O}(\log(nw/\varepsilon) \log^3(n)).$$ ◀

▶ **Corollary 39** (A restatement of Theorem 20). *There exists a local pseudorandom generator with seed length $\widetilde{O}\left((\sqrt{t} + \log(1/\varepsilon)) \cdot \log q\right)$ that ε-fools q-state time-t n-input DTMs, for any $t \geq n$.*

Proof. By Lemma 19, any q-state time-t DTM M can be written as the sum of ROBPs $\{P_\alpha\}_{\alpha \in A}$ so that $M(x) = \sum_{\alpha \in A} P_\alpha(x)$, where $|A| \leq (tq)^{O(\sqrt{t})}$. We set $\varepsilon' := \varepsilon/|A|$ and use the local pseudorandom generator G_r^{FK} of Theorem 38 that ε'-fools ROBPs of width w, where $w = (tq)^{O(\sqrt{t})}$. Then, G_r^{FK} is a PRG that ε-fools DTMs because

$$\left|\mathop{\mathbf{Exp}}_{z,w}\left[M(G_r^{\mathrm{FK}}(z)) - M(w)\right]\right| \leq \sum_{\alpha \in A} \left|\mathop{\mathbf{Exp}}_{z,w}\left[P_\alpha(G_r^{\mathrm{FK}}(z)) - P_\alpha(w)\right]\right| \leq \varepsilon.$$ ◀

5.3 Local PRG for combinatorial rectangles

▶ **Theorem 40** (Local PRG for combinatorial rectangles). *There exists a local pseudorandom generator $G\colon \{0,1\}^s \to \{0,1\}^n$ that ε-fools the class of combinatorial rectangles of k products and width m, where the seed length s is $\widetilde{O}\big((m+\log(k/\varepsilon))\cdot \log^3 n\big)$.*

We mention that Lee [29] showed that in the case of product tests (which contain combinatorial rectangles as a special case), the analysis of the Forbes-Kelley PRG can be improved, and obtained a PRG with nearly optimal seed length. This optimization would improve our results at most a $\mathsf{polylog}(n)$ factor; for the sake of simplicity, we make use of the Forbes-Kelley PRG.

Proof of Theorem 40. Suppose that a function f is computed by a combinatorial rectangle of k products and width m. Namely, there exist some functions f_i and disjoint subsets $S_1, \cdots, S_k \subseteq [n]$, where $|S_i| \leq m$, such that $f(x) = \bigwedge_{i\in[k]} f_i(x|_{S_i})$, for every x. Since any function f_i on m inputs can be written as a width-2^m ROBP, one can observe that f can be computed by a ROBP of width 2^m+1. Using the local PRG of Theorem 38 for width $w := 2^m+1$, we obtain a local PRG for combinatorial rectangles. ◀

6 MKTP lower bounds against branching programs

In this section, we develop a proof technique for applying Nečiporuk's method to MKTP and prove Theorem 4. The KT-complexity is formally defined as follows.

▶ **Definition 41.** *Let U be an efficient universal Turing machine. For a string $x \in \{0,1\}^*$, the* KT-complexity *of x is defined as follows.*

$$\mathrm{KT}(x) := \min\{|d| + t \mid U^d(i) \text{ outputs } x_i \text{ in time } t \text{ for every } i \in [|x|+1]\}.$$

Here we define x_i as the ith bit of x if $i \leq |x|$ and $\perp$ otherwise.

For a threshold $\theta\colon \mathbb{N} \to \mathbb{N}$, we denote by $\mathrm{MKTP}[\theta]$ the problem of deciding whether $\mathrm{KT}(x) \leq \theta(|x|)$ given a string $x \in \{0,1\}^*$ as input.

For a function $f\colon \{0,1\}^n \to \{0,1\}$, we partition the input variables $[n]$ into disjoint blocks $V_1, \cdots, V_m$, where $|V_i| = v$ for each $i \in [m]$ and $n = vm$. ($v = O(\log n)$ will be chosen later.) The idea of the Nečiporuk's method is to lower-bound the number of subfunctions. For each $i \in [m]$, we define $c_i(f)$ to be the number of distinct functions $f\restriction_\rho$ such that $\rho\colon [n] \to \{0,1,*\}$ is a restriction with $\rho^{-1}(*) = V_i$. [Recall that, for any $x \in \{0,1\}^n$, $f\restriction_\rho(x) := f(y)$ where $y \in \{0,1\}^n$ and, for all $1 \leq i \leq n$, $y_i := \rho(i)$ if $\rho(i) \in \{0,1\}$, else $y_i := x_i$.]

The Nečiporuk method can be then summarized as follows.

▶ **Theorem 42** (Nečiporuk [35]; cf. [26, Theorem 15.1]). *The size of a branching program computing f is at least $\Omega\left(\sum_{i=1}^m \log c_i(f)/\log\log c_i(f)\right)$. The size of a non-deterministic branching program or a parity branching program computing f is at least $\Omega\left(\sum_{i=1}^m \sqrt{\log c_i(f)}\right)$.*

Our main technical result of this section is the following.

▶ **Theorem 43.** *Let $f\colon \{0,1\}^n \to \{0,1\}$ be $\mathrm{MKTP}[\theta]$ on n-bit inputs for $\theta := n - 3c\log n - 4$, where $c > 0$ is a universal constant. Then, for every $i \in [m]$, it holds that $c_i(f) = 2^{\Omega(n)}$.*

The lower bounds for branching programs (Theorem 4) immediately follow from Theorem 43 and Theorem 42.

In our proof of Theorem 43, we only need the following two properties of KT-complexity.

1. The resource-unbounded Kolmogorov complexity[6] provides a lower bound on the KT-complexity. That is, $\mathrm{K}(x) \leq \mathrm{KT}(x)$ for any $x \in \{0,1\}^*$.
2. For any strings $\rho_1, \cdots, \rho_m \in \{0,1\}^v$ such that there exist distinct indices $i \neq j \in [m]$ such that $\rho_i = \rho_j$, we have $\mathrm{KT}(\rho_1 \cdots \rho_m) \leq (m-1) \cdot v + O(\log n)$. This is because each bit of the string $\rho_1 \cdots \rho_m$ can be described by the strings $\{\rho_1, \cdots, \rho_m\} \setminus \{\rho_j\}$ and the index $j \in [m]$ in time $O(\log n)$.[7]

For simplicity, we focus on the case when $i = 1$; the other case can be proved similarly. The idea of the proof is the following. Imagine that we pick $\rho \in \{*\}^{V_1} \times \{0,1\}^{V_2 \cup \cdots \cup V_m}$ uniformly at random. (Here we identify a restriction with a string in $\{0,1,*\}^{[n]}$.) We denote by $\rho_2 \in \{0,1\}^{V_2}, \cdots, \rho_m \in \{0,1\}^{V_m}$ the random bits such that $\rho = *^{V_1}\rho_2 \cdots \rho_m$. We will sometimes identify $\rho_2 \cdots \rho_m$ with ρ.

Consider the function $f\restriction_\rho \colon \{0,1\}^{V_1} \to \{0,1\}$ obtained by restricting f by ρ. Then, we expect that $f\restriction_\rho(\rho_i) = 1$ for any $i \in \{2, \cdots, m\}$ since $\mathrm{KT}(\rho_i \rho_2 \cdots \rho_m)$ is small, whereas $f\restriction_\rho(U) = 0$ for a random $U \sim \{0,1\}^{V_1}$ with high probability. Thus, the function $f\restriction_\rho$ is likely to be distinct for a randomly chosen ρ.

In order to make the argument formal, we proceed as follows. Pick ρ randomly. Then we add it to a set P while keeping the promise that the map $\rho \in P \mapsto f\restriction_\rho$ is injective. We will show that one can keep adding ρ until the size of P becomes exponentially large.

We will make use of symmetry of information of (resource-unbounded) Kolmogorov complexity.

▶ **Lemma 44.** *There exists a constant $c > 0$ such that, for any strings $x, y \in \{0,1\}^*$,*

$$\mathrm{K}(xy) \geq \mathrm{K}(x) + \mathrm{K}(y \mid x) - c \log \mathrm{K}(xy)\,.$$

We focus on restrictions ρ such that ρ is Kolmogorov-random. To this end, define

$$R := \{\rho \in \{0,1\}^{V_2 \cup \cdots \cup V_m} \mid \mathrm{K}(\rho) \geq |\rho| - 1\}$$

as the set of Kolmogorov-random restrictions ρ. By the standard counting argument, we have

$$\Pr_\rho[\rho \notin R] \leq \sum_{i=1}^{|\rho|-2} 2^i / 2^{|\rho|} \leq \frac{1}{2}.$$

The following lemma is the key for counting the number of distinct subfunctions.

▶ **Lemma 45.** *Let $\rho' \in R$ be an arbitrary restriction and define $\theta := n - v + c \log n$. If $f\restriction_\rho \equiv f\restriction_{\rho'}$, then $\mathrm{K}(\rho_i \mid \rho') \leq 2c \log n + 1$ for any $i \in \{2, \cdots, m\}$.*

Proof. For each $i \in [m] \setminus \{1\}$,

$$\mathrm{KT}(\rho_i \rho_2 \cdots \rho_m) \leq |\rho_2| + \cdots + |\rho_m| + O(\log n) \leq (m-1) \cdot v + c \log n \leq \theta.$$

This means that $\rho_i \rho_2 \cdots \rho_m$ is a YES instance of $\mathrm{MKTP}[\theta]$. Therefore, we have $1 = f\restriction_\rho(\rho_i) = f\restriction_{\rho'}(\rho_i)$, which implies that $\mathrm{KT}(\rho_i \rho'_2 \cdots \rho'_m) \leq \theta$. By the symmetry of information,

$$\theta \geq \mathrm{KT}(\rho_i \rho'_2 \cdots \rho'_m) \geq \mathrm{K}(\rho_i \rho'_2 \cdots \rho'_m) \geq \mathrm{K}(\rho'_2 \cdots \rho'_m) + \mathrm{K}(\rho_i \mid \rho'_2 \cdots \rho'_m) - c \log n.$$

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[6] Let U be an efficient universal Turing machine. For a string $x \in \{0,1\}^*$, the *resource-unbounded Kolmogorov complexity of x* is defined as $\mathrm{K}(x) := \min\{|d| \mid U^d(i) \text{ outputs } x_i \text{ for every } i \in [|x|+1]\}$.

[7] Here we assume that the universal Turing machine is efficient. If the universal Turing machine is slower and the time is $\mathrm{polylog}(n)$, we obtain a branching program size lower bound of $n^2/\mathrm{polylog}(n)$.

Since $\rho' \in R$, we have $\mathrm{K}(\rho'_2 \cdots \rho'_m) \geq v(m-1) - 1 = n - v - 1$. Therefore,

$$\mathrm{K}(\rho_i \mid \rho'_2 \cdots \rho'_m) \leq \theta + c \log n - (n - v - 1) = 2c \log n + 1. \qquad \blacktriangleleft$$

Now we set $v := 4c \log n + 4$. Then, for any $\rho' \in R$,

$$\begin{aligned}
\Pr_{\rho}[f\restriction_{\rho} \equiv f\restriction_{\rho'}] &\leq \Pr[\forall i \in [m] \setminus \{1\},\ \mathrm{K}(\rho_i \mid \rho') \leq v/2 - 1] \\
&\leq (2^{v/2}/2^v)^{m-1} \\
&= 2^{-n/2+v/2} \\
&\leq 2^{-n/3}.
\end{aligned}$$

In particular, for any $P \subseteq R$, by the union bound, we obtain

$$\Pr_{\rho}[\exists \rho' \in P,\ f\restriction_{\rho} \equiv f\restriction_{\rho'}] \leq |P| \cdot 2^{-n/3}.$$

Therefore,

$$\Pr_{\rho}[\rho \notin R \text{ or } \exists \rho' \in P,\ f\restriction_{\rho} \equiv f\restriction_{\rho'}] \leq 1/2 + |P| \cdot 2^{-n/3},$$

which is strictly less than 1 if $|P| < 2^{n/3-1}$. To summarize, we established the following property.

▶ **Corollary 46.** *For any $P \subseteq R$ such that $|P| < 2^{n/3-1}$, there exists a restriction ρ such that $\rho \in R$ and $f\restriction_{\rho} \not\equiv f\restriction_{\rho'}$ for any $\rho' \in P$.*

In light of this, we can construct a large set P such that the map $\rho \in P \mapsto f\restriction_{\rho}$ is injective as follows: Starting from $P := \emptyset$, add a restriction $\rho \in R$ such that $f\restriction_{\rho} \not\equiv f\restriction_{\rho'}$ for any $\rho' \in P$, whose existence is guaranteed by Corollary 46 if $|P| < 2^{n/3-1}$. In this way, we obtain a set P such that $|P| \geq 2^{n/3-1}$ and each $f\restriction_{\rho}$ is distinct for any $\rho \in P$. We conclude that $c_1(f) \geq |P| \geq 2^{n/3-1}$. This completes the proof of Theorem 43.

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A Hardness magnification for one-tape Turing machines

In this section, we obtain the following hardness magnification result for one-tape Turing machines.

▶ **Theorem 47** (A corollary of McKay, Murray, and Williams [32]; Theorem 1, restated)**.** *There exists a constant $\mu > 0$ such that, if* $\mathrm{MCSP}[2^{\mu n}]$ *is not in* $\mathsf{DTIME}_1\left[N^{1.01}\right]$*, then* $\mathsf{P} \neq \mathsf{NP}$*.*

Proof. McKay, Murray, and Williams [32, Theorem 1.3] showed that if $\mathsf{P} = \mathsf{NP}$, then there exists a polynomial p such that, for any time-constructible function $s(n)$, there exists a one-pass streaming algorithm with update time $p(s(n))$ that computes $\mathrm{MCSP}[s(n)]$. By Lemma 9, we obtain $\mathrm{MCSP}[s(n)] \in \mathsf{DTIME}_1[N \cdot p(s(n))]$, where $N = 2^n$. Depending on p, we choose a small constant $\mu > 0$ and set $s(n) := 2^{\mu n}$ so that $N \cdot p(s(n)) = N^{1+O(\mu)} \leq N^{1.01}$.

To summarize, we have proved that if $\mathsf{P} = \mathsf{NP}$, then for some constant $\mu > 0$, $\mathrm{MCSP}[2^{\mu n}] \in \mathsf{DTIME}_1[N^{1.01}]$. This statement is logically equivalent to the following: There exists a constant $\mu > 0$ such that $\mathsf{P} = \mathsf{NP}$ implies that $\mathrm{MCSP}[2^{\mu n}] \notin \mathsf{DTIME}_1[N^{1.01}]$ (because the statement that $\mathsf{P} = \mathsf{NP}$ is independent of μ). Taking its contrapositive, we obtain the desired result. ◀

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