Electronic Colloquium on Computational Complexity, Report No. 110 (2020)

Capacity Lower Bounds via Productization

Leonid Gurvits and Jonathan Leake

July 16, 2020

1 Introduction

The purpose of this note is to state and prove a lower bound on the capacity of a real stable polynomial $p(x)$ which is based only on its value and gradient at $x = \mathbf{1}$. This can be seen as a quantitative generalization of the fact that the capacity of p is maximized when the normalized gradient of p at $\mathbf{1}$ is equal to $\mathbf{1}$ (see Proposition 3.3). We state this result now.

Theorem 1.1. *Let $p(x)$ be a homogeneous real stable polynomial of degree n in n variables. If $p(\mathbf{1}) = 1$ and $\|1 - \nabla p(\mathbf{1})\|_1 < 2$, then*

$$\inf_{x_1,\ldots,x_n>0} \frac{p(x)}{x_1 \cdots x_n} \geq \left(1 - \frac{\|1 - \nabla p(\mathbf{1})\|_1}{2}\right)^n.$$

This result implies a sharp improvement (see Corollary 4.3) to a similar inequality proved in [5] in the case that $\|1 - \nabla p(\mathbf{1})\|_2 < \frac{1}{\sqrt{n}}$. Such inequalities have also played an important role in the recent work on operator scaling and its generalizations and applications [3].

We now give a high-level overview of the proof of the main inequality, which is split into two parts. First, we prove the result for a more restricted class of polynomials, those of the form

$$f(x) = \prod_{i=1}^n (Ax)_i$$

for a given matrix A with non-negative entries and row sums equal to 1 (see Theorem 4.2). Proving the capacity bound in this case relies heavily on combinatorics related to bounding the permanent of A. Second, we then show that the bound can be generalized to real stable polynomials by demonstrating a certain "productization" result for real stable polynomials (see Theorem 5.2). In particular we show that for any real stable p and any x in the positive orthant, we can find a matrix A with non-negative entries such that

$$p(x) = \prod_{i=1}^n (Ax)_i \qquad \text{and} \qquad \nabla p(\mathbf{1}) = \nabla \left[\prod_{i=1}^n (Ax)_i\right]\Bigg|_{x=\mathbf{1}}.$$

In what follows, we present the proof of these results and leave further discussion and explication to later work.

2 Notation

We let $\mathbb{R}, \mathbb{R}_+, \mathbb{Z}, \mathbb{Z}_+, \mathbb{N}, \mathbb{C}$ denote the reals, non-negative reals, integers, non-negative integers, positive integers, and complex numbers respectively. We further let $\mathbb{K}^d[x] = \mathbb{K}^d[x_1, \ldots, x_n]$ denote the set of homogeneous polynomials of degree d in n variables with coefficients in $\mathbb{K}$. For a polynomial p in n variables, the *support* of p, denoted $\operatorname{supp}(p)$, is the set of all $\mu \in \mathbb{Z}_+^n$ such that x^μ has non-zero coefficient in p. Further, the *Newton polytope* of p, denoted $\operatorname{Newt}(p)$, is the convex hull of $\operatorname{supp}(p)$. We also denote $\|\alpha\|_1 := \sum_{i=1}^n |\alpha_i|$ for $\alpha \in \mathbb{R}_+^n$ as usual. We now define all of the various classes of matrices and polynomials we will consider.

ISSN 1433-8092

Definition 2.1. Given an $\alpha \in \mathbb{R}_+^n$ with $\|\alpha\|_1 = n$, we define $\mathrm{Mat}_n(\alpha)$ to be the set of $n \times n$ matrices A with non-negative entries such that the row sums of A are all 1 and the column sums of A are given by α.

Definition 2.2. Given $p \in \mathbb{R}_+^d[x_1, \ldots, x_n]$, we say that p is *real stable* if $p(x) = p(x_1, \ldots, x_n) \neq 0$ whenever $x_1, \ldots, x_n$ are all in the complex upper half-plane.

Definition 2.3. Given $p \in \mathbb{R}_+^d[x_1, \ldots, x_n]$, we say that p is *strongly log-concave* if $\nabla_{v_1} \cdots \nabla_{v_k} p$ is either identically zero or log-concave in the positive orthant for all $k \geq 0$ and all choices of $v_1, \ldots, v_k \in \mathbb{R}_+^n$. (These polynomials also go by the names *completely log-concave* and *Lorentzian*; see [1] and [2].)

Definition 2.4. For $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}_+^n$, we define the following classes of polynomials:

1. $\mathrm{Prod}_n(\alpha)$ is the set of all polynomials of the form $p(x) = \prod_{i=1}^n (Ax)_i$, where $A \in \mathrm{Mat}_n(\alpha)$. Note that $p(\mathbf{1}) = 1$ and $\nabla p(\mathbf{1}) = \alpha$ for all such polynomials.
2. $\mathrm{HStab}_n(\alpha)$ is the set of all real stable polynomials in $\mathbb{R}_+^n[x_1, \ldots, x_n]$ for which $p(\mathbf{1}) = 1$ and $\nabla p(\mathbf{1}) = \alpha$.
3. $\mathrm{SLC}_n(\alpha)$ is the set of all strongly log-concave polynomials in $\mathbb{R}_+^n[x_1, \ldots, x_n]$ for which $p(\mathbf{1}) = 1$ and $\nabla p(\mathbf{1}) = \alpha$.

We also give a special name to such matrices and polynomials whenever $\alpha = \mathbf{1}$.

Definition 2.5. We refer to matrices in $\mathrm{Mat}_n(\mathbf{1})$ as *doubly stochastic*. Similarly, if $p \in \mathbb{R}_+^n[x_1, \ldots, x_n]$ such that $p(\mathbf{1}) = 1$ and $\nabla p(\mathbf{1}) = \mathbf{1}$, then we say p is *doubly stochastic*.

Finally, we define the key quantity we study in this note.

Definition 2.6. Given a polynomial $p \in \mathbb{R}_+^n[x_1, \ldots, x_n]$, we define the *capacity* of p as

$$\mathrm{Cap}_{\mathbf{1}}(p) := \inf_{x>0} \frac{p(x)}{x^{\mathbf{1}}} = \inf_{x_1,\ldots,x_n>0} \frac{p(x)}{x_1 \cdots x_n}.$$

3 Basic Results

We state here a few standard basic results concerning polynomials, matrices, and capacity.

Proposition 3.1. *Polynomials of the form $\prod_{i=1}^n (Ax)_i$ for a given matrix A with non-negative coefficients are homogeneous real stable, and homogeneous real stable polynomials are strongly log-concave.*

Proposition 3.2 ([2])**.** *For any strongly log-concave $p \in \mathbb{R}_+^d[x_1, \ldots, x_n]$ and any $\mu \in \mathbb{Z}_+^n$, we have that $\mu \in \mathrm{supp}(p)$ if and only if $\mu \in \mathrm{Newt}(p)$.*

Proposition 3.3 ([4])**.** *If p is doubly stochastic then $\mathrm{Cap}_{\mathbf{1}}(p) = 1$. In particular, if A is doubly stochastic then $\mathrm{Cap}_{\mathbf{1}}(\sum_{i=1}^n (Ax)_i) = 1$.*

Proposition 3.4. *Given a polynomial $p \in \mathbb{R}_+^n[x_1, \ldots, x_n]$, we have that $\mathrm{Cap}_{\mathbf{1}}(p) > 0$ if and only if $\mathbf{1} \in \mathrm{Newt}(p)$.*

Corollary 3.5. *Given an $n \times n$ matrix A with non-negative entries, the following are equivalent:*

1. $\mathrm{per}(A) = 0$.
2. $\mathrm{Cap}_{\mathbf{1}}(\prod_{i=1}^n (Ax)_i) = 0$.
3. *Up to permutation, the bottom-left $i \times j$ block of A is 0 for some $i + j > n$.*

Proof. (1) $\iff$ (2). Let us denote $p(x) := \prod_{i=1}^n (Ax)_i$. Recall that

$$\mathrm{per}(A) = \partial_{x_1} \cdots \partial_{x_n} p.$$

From this expression, it is clear to see that $\mathrm{per}(A) > 0$ iff $\mathbf{1} \in \mathrm{supp}(p)$. Further, since p is a real stable polynomial, we know that $\mathbf{1} \in \mathrm{supp}(p)$ iff $\mathbf{1} \in \mathrm{Newt}(p)$ by Proposition 3.2. Finally, $\mathrm{Cap}_{\mathbf{1}}(p) > 0$ iff $\mathbf{1} \in \mathrm{Newt}(p)$ by Proposition 3.4.

(1) $\iff$ (3). Follows from Hall's marriage theorem. □

4 Capacity Bound for Product Polynomials

In this section we prove the main result (Theorem 1.1) for polynomials in $\mathrm{Prod}_n(\alpha)$. To simplify notation, we define the following for $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}_+^n$:

$$L_n(\alpha) := \min_{p \in \mathrm{Prod}_n(\alpha)} \mathrm{Cap}_{\mathbf{1}}(p).$$

Before proving the result, we need a lemma which has some vague resemblance to Corollary 3.5. In particular note that in both results, (2) is a condition on the capacity and (3) is a Hall-like condition.

Lemma 4.1. *Given $\alpha \in \mathbb{R}_+^n$ such that $\sum_i \alpha_i = n$, the following are equivalent:*

1. $\|1 - \alpha\|_1 < 2$.
2. $L_n(\alpha) > 0$.
3. $\sum_{i \in F} \alpha_i > |F| - 1$ *for all* $F \subseteq [n]$.

Proof. Let $\delta := 1 - \alpha$, so that $\sum_i \delta_i = 0$. For any $F \subseteq [n]$, let $F = F_+ \sqcup F_-$ such that $\delta_i \geq 0$ for $i \in F_+$ and $\delta_i < 0$ for $i \in F_-$.

(1) $\implies$ (3). For any F, we have

$$\sum_{i \in F} \alpha_i = |F| - \sum_{i \in F} \delta_i = |F| + \sum_{i \in F_-} \delta_i - \sum_{i \in F_+} \delta_i.$$

Since $\sum_i \delta_i = 0$ and $\sum_i |\delta_i| < 2$, we have that $\sum_{i \in F_+} \delta_i < 1$. This implies

$$\sum_{i \in F} \alpha_i = |F| + \sum_{i \in F_-} \delta_i - \sum_{i \in F_+} \delta_i > |F| - 1.$$

(3) $\implies$ (1). Letting $F = [n]$ with F_+, F_- defined as above, we have

$$|F_+| - \sum_{i \in F_+} \delta_i = \sum_{i \in F_+} \alpha_i > |F_+| - 1 \implies \sum_{\delta_i \geq 0} \delta_i < 1.$$

Since $\sum_i \delta_i = 0$, this implies

$$\|1 - \alpha\|_1 = \sum_i |\delta_i| = \sum_{\delta_i \geq 0} \delta_i - \sum_{\delta_i < 0} \delta_i < 2.$$

(2) $\implies$ (3). So as to get a contradiction, suppose there is some $k \in [n]$ such that $\alpha_1 + \cdots + \alpha_k \leq k - 1$. We now construct a matrix $A \in \mathrm{Mat}_n(\alpha)$ such that $\mathrm{Cap}_{\mathbf{1}}(\prod_i (Ax)_i) = 0$. Let A_1 be a $(k-1) \times k$ matrix with column sums $\alpha_1, \ldots, \alpha_k$ and row sums all equal to $\beta := \frac{\alpha_1 + \cdots + \alpha_k}{k-1}$. Since $\beta \leq 1$, we can define

$$A := \begin{bmatrix} A_1 & * \\ 0 & * \end{bmatrix} \in \mathrm{Mat}_n(\alpha),$$

where the bottom-left $(n-k+1) \times k$ block of A is 0. Since $(n-k+1) + k > 0$, Corollary 3.5 implies $\mathrm{Cap}_{\mathbf{1}}(\prod_i (Ax)_i) = 0$. Therefore $L_n(\alpha) = 0$.

(3) $\implies$ (2). So as to get a contradiction, suppose $L_n(\alpha) = 0$ By Corollary 3.5, this implies there is some $A \in \mathrm{Mat}_n(\alpha)$ for which the bottom-left $i \times j$ block of A is 0 for some $i + j > n$ (up to permutation of the entries of α). Now, the total sum of all entries in the first j columns of A is equal to $\alpha_1 + \cdots + \alpha_j$ (by the column sums and the block of zeros), and also is at most $n - i \leq j - 1$ (by the row sums and the block of zeros). That is, $\alpha_1 + \cdots + \alpha_j \leq j - 1$, which contradicts (3). ∎

We now state the main result of this section: a lower bound on the capacity of polynomials in $\mathrm{Prod}_n(\alpha)$.

Theorem 4.2. *Fix $n \in \mathbb{N}$, $\alpha \in \mathbb{R}_+^n$, and $p \in \mathrm{Prod}_n(\alpha)$. If $\|1-\alpha\|_1 < 2$, then*

$$\mathrm{Cap}_{\mathbf{1}}(p) \geq \left(1 - \frac{\|1-\alpha\|_1}{2}\right)^n.$$

Proof. Define $\delta := 1 - \alpha$. We just need to prove

$$L_n(\alpha) \geq \left(1 - \frac{\|1-\alpha\|_1}{2}\right)^n.$$

Define

$$S := \{(\gamma, D) \in \mathbb{R}_+ \times \mathrm{Mat}_n(\mathbf{1}) \ : \ A - \gamma D \geq 0 \ \text{ entrywise}\},$$

and further define $(\gamma_0, D_0) \in S$ to be such that γ_0 is maximized. (This maximum exists by compactness of $\mathrm{Mat}_n(\mathbf{1})$, the Birkhoff polytope.) Now consider the matrix

$$M = \frac{A - \gamma_0 D_0}{1 - \gamma_0},$$

which is an element of $\mathrm{Mat}_n(\tilde{\alpha})$ for $\tilde{\alpha} = \frac{\alpha - \gamma_0}{1-\gamma_0}$. We now show that $\mathrm{per}(M) = 0$. If not, then there is some permutation matrix P and some $\epsilon > 0$ such that

$$A - (\gamma_0 + \epsilon) \cdot \frac{\gamma_0 D_0 + \epsilon P}{\gamma_0 + \epsilon}$$

is entrywise non-negative. Since $\frac{\gamma_0 D_0 + \epsilon P}{\gamma_0 + \epsilon} \in \mathrm{Mat}_n(\mathbf{1})$, this contradicts the maximality of γ_0. So in fact $\mathrm{per}(M) = 0$, and therefore $L_n(\tilde{\alpha}) = 0$ by Corollary 3.5. By Lemma 4.1, this implies

$$\left\| \frac{\delta}{1-\gamma_0} \right\|_1 = \|1 - \tilde{\alpha}\|_1 \geq 2 \implies \gamma_0 \geq 1 - \frac{\|\delta\|_1}{2}.$$

Since $D_0 \in \mathrm{Mat}_n(\mathbf{1})$, we have that $\mathrm{Cap}_{\mathbf{1}}(\prod_i (D_0 x)_i) = 1$ by Proposition 3.3. The fact that $A \geq \gamma_0 D_0$ entrywise then implies

$$\mathrm{Cap}_{\mathbf{1}}\left[\prod_{i=1}^n (Ax)_i\right] \geq \mathrm{Cap}_{\mathbf{1}}\left[\prod_{i=1}^n (\gamma_0 D_0 x)_i\right] \geq \gamma_0^n \geq \left(1 - \frac{\|\delta\|_1}{2}\right)^n.$$

□

We finally state the following corollary of Lemma 4.1 and Theorem 4.2, which gives a similar result for the 2-norm instead of the 1-norm.

Corollary 4.3. *Fix $n \in \mathbb{N}$, $\alpha \in \mathbb{R}_+^n$, and $p \in \mathrm{Prod}_n(\alpha)$. If $\|1-\alpha\|_2 < \frac{2}{\sqrt{n}}$, then*

$$\mathrm{Cap}_{\mathbf{1}}(p) \geq \left(1 - \frac{\sqrt{n} \cdot \|1-\alpha\|_2}{2}\right)^n.$$

Proof. Follows from $\|x\|_1 \leq \sqrt{n}\|x\|_2$ for $x \in \mathbb{R}^n$. □

The corollary is a sharp improvement of the following similar inequality proved in [5] in the case that $\|1-\alpha\|_2 < \frac{1}{\sqrt{n}}$:

$$L_n(\alpha) \geq \left(1 - \sqrt{n} \cdot \|1-\alpha\|_2\right)^n.$$

This last inequality also plays a key role in the recent work on the operator scaling and its generalizations and applications, see [3].

5 Productization of Real Stable Polynomials

In this section we prove the productization result for polynomials in $\mathrm{HStab}_n(\alpha)$. This result immediately implies the main result (Theorem 1.1) for polynomials in $\mathrm{HStab}_n(\alpha)$ as a corollary (see Corollary 5.3).

To actually prove the productization result, we need a way to associate matrices in $\mathrm{Mat}_n(\alpha)$ to polynomials in $\mathrm{HStab}_n(\alpha)$. For the case of $\alpha = \mathbf{1}$, this statement was conjectured by Gurvits in the slightly different form given below. The conjecture was motivated by the case of determinantal polynomials, where the desired element of $\mathrm{Mat}_n(\mathbf{1})$ can be constructed from the matrices in the determinant. We now state this result, the proof of which was told to us by Petter Brändén in personal correspondence.

Theorem 5.1 (Brändén)**.** *Fix $p \in \mathrm{HStab}_n(\mathbf{1})$, and let $\lambda(x)$ denote the roots of $f(t) = p(\mathbf{1}t - x)$ for any $x \in \mathbb{R}^n$. Then for any $x \in \mathbb{R}^n$, there exists some $D \in \mathrm{Mat}_n(\mathbf{1})$ such that $Dx = \lambda(x)$.*

We now utilize this result to prove the productization result for polynomials in $\mathrm{HStab}_n(\alpha)$.

Theorem 5.2. *Fix $n \in \mathbb{N}$, $x, \alpha \in \mathbb{R}^n_+$, and $p \in \mathrm{HStab}_n(\alpha)$. There exists $f \in \mathrm{Prod}_n(\alpha)$ such that $p(x) = f(x)$.*

Proof. We first prove the result in the case that α is rational. Denote $\alpha = \left(\frac{k_1}{N}, \ldots, \frac{k_n}{N}\right)$ for some $k_1, \ldots, k_n \in \mathbb{Z}_+$ and $N \in \mathbb{N}$. Considering variables $y_{1,1}, \ldots, y_{1,k_1}, y_{2,1}, \ldots, y_{n,k_n}$, we define

$$q(y) := p\left(\frac{y_{1,1} + \cdots + y_{1,k_1}}{k_1}, \ldots, \frac{y_{n,1} + \cdots + y_{n,k_n}}{k_n}\right)^N,$$

so that $q \in \mathbb{R}^{nN}_+[y_{1,1}, \ldots, y_{n,k_n}]$. Since $q(1) = 1$ and

$$\partial_{y_{i,j}} q(1) = \left[\frac{N}{k_i}(\partial_{x_i} p)p^{N-1}\right](1) = 1,$$

we in fact have $q \in \mathrm{HStab}_{nN}(\mathbf{1})$. Letting $y \in \mathbb{R}^{nN}_+$ be such that $y_{i,j} = x_i$ for all i, j, note that the roots of $q(\mathbf{1}t - y)$ will consist of N copies of the n roots of $p(\mathbf{1}t - x)$. So by Theorem 5.1, there exists $D \in \mathrm{Mat}_{nN}(\mathbf{1})$ such that

$$Dy = (\lambda_1(x), \ldots, \lambda_1(x), \lambda_2(x), \ldots, \lambda_2(x), \ldots, \lambda_n(x), \ldots, \lambda_n(x))$$

where the roots are all repeated N times. Let D' be the $n \times n$ matrix formed by summing the elements of each $N \times k_i$ block of D and dividing by N. We then have

$$\prod_{i=1}^n (D'x)_i = \prod_{i=1}^n \lambda_i(x) = (-1)^n p(1 \cdot 0 - x) = p(x).$$

Since the row sums of D' are all 1 and the column sums are given by $\frac{k_i}{N}$, we have that $D' \in \mathrm{Mat}_n(\alpha)$ which proves the result for p.

We now handle the case of irrational α. First if $\alpha_k = 0$ for some k, then p does not depend on x_k and the result follows by induction. So we may assume that $\alpha_k > 0$ for all $k \in [n]$. By [6], the set of homogeneous real stable polynomials of degree n in n variables is the closure of its interior with respect to the Euclidean topology on coefficients. Define the map $M(q) := \nabla q(\mathbf{1})$ on the space of $q \in \mathbb{R}^n[x_1, \ldots, x_n]$ for which $q(\mathbf{1}) = 1$, and note that this map is linear and surjects onto the subspace of $\mathbb{R}^n$ consisting of vectors whose entries sum to n. So for small enough $\epsilon > 0$, we have that $M^{-1}(B_\epsilon(\alpha))$ is an open subset of the set of homogeneous real stable polynomials q such that $q(\mathbf{1}) = 1$. Choosing any small neighborhood $U \subseteq M^{-1}(B_\epsilon(\alpha))$ about p, surjectivity implies that $M(U)$ is full-dimensional in the range of M. We can therefore choose a sequence $p_j \in \mathrm{HStab}_n(\alpha^j)$ such that $\alpha^j \to \alpha$, $p_j \to p$, and α^j is rational for all j. With this, the previous arguments imply the following for all j:

$$\min_{q \in \mathrm{Prod}_n(\alpha^j)} q(x) \leq p_j(x) \leq \max_{q \in \mathrm{Prod}_n(\alpha^j)} q(x).$$

Now let $Y_j \in \mathrm{Mat}_n(\alpha^j)$ and $Z_j \in \mathrm{Mat}_n(\alpha^j)$ be such that

$$\prod_{i=1}^n (Y_j x)_i = \min_{q \in \mathrm{Prod}_n(\alpha^j)} q(x) \qquad \text{and} \qquad \prod_{i=1}^n (Z_j x)_i = \max_{q \in \mathrm{Prod}_n(\alpha^j)} q(x).$$

By compactness of the set of all $n \times n$ matrices with non-negative entries and row sums all equal to 1, we can assume that Y_j and Z_j are convergent subsequences with respective limits Y and Z. Since the maps

$$A \mapsto (\text{column sums of } A) \qquad \text{and} \qquad A \mapsto \prod_{i=1}^n (Ax)_i$$

are continuous, we have that $Y, Z \in \mathrm{Mat}_n(\alpha)$ and

$$\prod_{i=1}^n (Yx)_i = \lim_{j \to \infty} \min_{q \in \mathrm{Prod}_n(\alpha^j)} q(x) \leq \lim_{j \to \infty} p_j(x) \leq \lim_{j \to \infty} \max_{q \in \mathrm{Prod}_n(\alpha^j)} q(x) = \prod_{i=1}^n (Zx)_i.$$

Since $p(x) = \lim_{j \to \infty} p_j(x)$, Lemma 6.1 implies the result. $\square$

From this follows the capacity bound for real stable polynomials. Recall the definition of $L_n(\alpha)$ given in Section 4.

Corollary 5.3. *For $p \in \mathrm{HStab}_n(\alpha)$, we have*

$$\mathrm{Cap}_{\mathbf{1}}(p) \geq L_n(\alpha) \geq \left(1 - \frac{\|1 - \alpha\|_1}{2}\right)^n.$$

Proof. The second inequality is given by Theorem 4.2, so we just need to prove the first inequality. For any $x \in \mathbb{R}_+^n$, let $f \in \mathrm{Prod}_n(\alpha)$ be such that $p(x) = f(x)$ according to Theorem 5.2. With this, we have

$$\mathrm{Cap}_{\mathbf{1}}(p) = \inf_{x > 0} \frac{p(x)}{x^{\mathbf{1}}} \geq \inf_{x > 0} \min_{f \in \mathrm{Prod}_n(\alpha)} \frac{f(x)}{x^{\mathbf{1}}} = L_n(\alpha).$$

$\square$

6 Productization of Strongly Log-Concave Polynomials

We do not currently have a productization theorem for strongly log-concave polynomials. However, there are a few results which seem promising in the direction of obtaining such a result. That said, the arguments of this section deviate from those of Section 5 in the fact that we do not attempt to "construct" the matrix which gives rise to the productization. Instead we try to show that evaluations of polynomials in $\mathrm{SLC}_n(\alpha)$ are bounded by those of polynomials in $\mathrm{Prod}_n(\alpha)$ and use convexity. In particular, the first lemma will give a sense of this idea.

Lemma 6.1. *Fix $n \in \mathbb{N}$, $x, \alpha \in \mathbb{R}_+^n$, and $p \in \mathbb{R}_+^n[x_1, \ldots, x_n]$ such that $p(\mathbf{1}) = 1$ and $\nabla p(\mathbf{1}) = \alpha$. Suppose further that there exist $f, g \in \mathrm{Prod}_n(\alpha)$ such that $f(x) \leq p(x) \leq g(x)$, then there exists $h \in \mathrm{Prod}_n(\alpha)$ such that $p(x) = h(x)$.*

Proof. Define the map $P : \mathrm{Mat}_n(\alpha) \to \mathbb{R}_+$ via

$$P(A) := \prod_{i=1}^n (Ax)_i.$$

Since $\mathrm{Mat}_n(\alpha)$ is a closed convex polytope, its image under P is a closed interval. The result follows. $\square$

We now go about trying to obtain such upper and lower bounds for any $p \in \mathrm{SLC}_n(\alpha)$. First, we prove the upper bound.

Proposition 6.2. *Fix $n \in \mathbb{N}$ and $x, \alpha \in \mathbb{R}^n_+$. We have*

$$\max_{p \in \mathrm{Prod}_n(\alpha)} p(x) = \left(\frac{\sum_{i=1}^n \alpha_i x_i}{n}\right)^n = \max_{p \in \mathrm{SLC}_n(\alpha)} p(x).$$

Proof. The first equality follows from the AM-GM inequality. For any $A \in \mathrm{Mat}_n(\alpha)$, we have

$$\prod_{i=1}^n (Ax)_i \leq \left(\frac{\sum_{i=1}^n (Ax)_i}{n}\right)^n = \left(\frac{\sum_{i=1}^n \alpha_i x_i}{n}\right)^n.$$

For the second equality, by homogeneity we can rewrite the optimization problem as

$$\max_{\Delta_\alpha} \max_{p \in \mathrm{SLC}_n(\alpha)} p(x) = 1 \iff \max_{p \in \mathrm{SLC}_n(\alpha)} \max_{\Delta_\alpha} p(x) = 1,$$

where $\Delta_\alpha := \{x \geq 0 \ : \ \sum_i \alpha_i x_i = n\}$. That is, for any $p \in \mathrm{SLC}_n(\alpha)$ we want to show that

$$\max_{\Delta_\alpha} p(x) = 1.$$

The fact that $\nabla p(\mathbf{1}) = \alpha$ means that the gradient of p at $x = \mathbf{1}$ projected onto Δ_α is 0. By log-concavity of p in $\mathbb{R}^n_+$, this implies p is maximized on Δ_α at $x = \mathbf{1}$. The fact that $p(\mathbf{1}) = 1$ then completes the proof. ☐

This only leaves the lower bound. Unfortunately, in this case the problem is much more difficult. In particular, it seems unlikely that the minimums will have explicit formulas like the maximums did in Proposition 6.2. We leave further exploration of the potential lower bounds to future work.

References

[1] Nima Anari, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials, entropy, and a deterministic approximation algorithm for counting bases of matroids. *arXiv preprint arXiv:1807.00929*, 2018.

[2] Petter Brändén and June Huh. Lorentzian polynomials. *arXiv preprint arXiv:1902.03719*, 2019.

[3] Ankit Garg, Leonid Gurvits, Rafael Oliveira, and Avi Wigderson. Algorithmic and optimization aspects of brascamp-lieb inequalities, via operator scaling. *Geometric and Functional Analysis*, 28(1):100–145, 2018.

[4] Leonid Gurvits. Van der waerden/schrijver-valiant like conjectures and stable (aka hyperbolic) homogeneous polynomials: one theorem for all. *the electronic journal of combinatorics*, 15(1):66, 2008.

[5] Nathan Linial, Alex Samorodnitsky, and Avi Wigderson. A deterministic strongly polynomial algorithm for matrix scaling and approximate permanents. *Combinatorica*, 20(4):545–568, 2000.

[6] Wim Nuij. A note on hyperbolic polynomials. *Mathematica Scandinavica*, 23(1):69–72, 1969.

ECCC ISSN 1433-8092

https://eccc.weizmann.ac.il