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Capacity Lower Bounds via Productization

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Abstract

The purpose of this note is to state and prove a lower bound on the capacity of a real stable polynomial $p(x)$ which is based only on its value and gradient at $x = 1$. This can be seen as a quantitative generalization of the fact that the capacity of p is maximized when the normalized gradient of p at 1 is equal to 1. This result implies a sharp improvement to a similar inequality proved by Linial-Samorodnitsky-Wigderson in 2000 [6]. Such inequalities have played an important role in the recent work on operator scaling and its generalizations and applications [3]. Our bound is also similar to one used very recently by Karlin-Klein-Oveis Gharan to give an improved approximation factor for metric TSP [5]. While our bound does not immediately improve upon theirs, we believe our techniques will help to achieve such an improvement when applied more directly to their situation.

1 Introduction

The purpose of this note is to state and prove a lower bound on the capacity of a real stable polynomial $p(x)$ which is based only on its value and gradient at $x = \mathbf{1}$. This can be seen as a quantitative generalization of the fact that the capacity of p is maximized when the normalized gradient of p at $\mathbf{1}$ is equal to $\mathbf{1}$ (see Proposition 4.4). We state this result now.

Theorem 1.1. *Let $p(x)$ be a homogeneous real stable polynomial of degree n in n variables. If $p(\mathbf{1}) = 1$ and $\|1 - \nabla p(\mathbf{1})\|_1 < 2$, then*

$$\inf_{x_1,\ldots,x_n>0} \frac{p(x)}{x_1 \cdots x_n} \geq \left(1 - \frac{\|1 - \nabla p(\mathbf{1})\|_1}{2}\right)^n.$$

This result implies a sharp improvement (see Corollary 5.6) of a similar inequality proved in [6] in the case that $\|1-\nabla p(\mathbf{1})\|_2 < \frac{1}{\sqrt{n}}$. Such inequalities have also played an important role in the recent work on operator scaling and its generalizations and applications [3]. This result also gives a bound similar to one used very recently in [5] to give an improved approximation factor for metric TSP (see Corollary 2.1). Our bound does not immediately imply an improvement over theirs, but we believe our techniques will help to make such an improvement when applied more directly to their situation. We discuss this further in Section 2.

We now give a high-level overview of the proof of the main inequality, which is split into two parts. First, we prove the result for a more restricted class of polynomials, those of the form

$$f(x) = \prod_{i=1}^{n} (Ax)_i$$

for a given matrix A with non-negative entries and row sums equal to 1 (see Theorem 5.5). Proving the capacity bound in this case relies heavily on combinatorics related to bounding the permanent of A. Second, we then show that the bound can be generalized to real stable polynomials by demonstrating a certain "productization" result for real stable polynomials (see Theorem 6.2). In particular we show that for any real stable p and any x in the positive orthant, we can find a matrix A with non-negative entries such that

$$p(x) = \prod_{i=1}^{n} (Ax)_i \qquad \text{and} \qquad \nabla p(\mathbf{1}) = \nabla \left[\prod_{i=1}^{n} (Ax)_i\right]\Bigg|_{x=\mathbf{1}}.$$

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2 Application to Metric TSP

In a recent paper [5], Karlin, Klein and Oveis Gharan give an improved approximation factor for metric TSP. Their proof relies on bounds of a similar spirit to that of Theorem 1.1. In this section, we discuss how our bound relates to their bounds.

First we need to set up a bit of their notation. Let μ be a probability distribution on $\{0,1\}^m$, and let the corresponding probability generating function be given by

$$p_\mu(z) := \sum_{S \subseteq [m]} \mathbb{P}(1_S) z^S.$$

Such a distribution μ is called *strongly Rayleigh (SR)* when the polynomial p is real stable. Let X be a random variable distributed according to μ. We then want to investigate random variables $A_1, \ldots, A_n$ which are defined via sets $S_1 \sqcup \cdots \sqcup S_n = [m]$ by

$$A_i := \sum_{s \in S_i} X_s.$$

That is, A_i is the random variable given by summing the entries of X corresponding to S_i. Our main result then implies the following bound.

Corollary 2.1. *Let μ be a strongly Rayleigh distribution on $\{0,1\}^m$, and let $A_1, \ldots, A_n$ be random variables corresponding to sets $S_1, \ldots, S_n$ as described above. Define $\beta_i := \mathbb{E}[A_i]$, and let d be the size of the largest set that μ assigns a non-zero probability. If $\|\beta - \mathbf{1}\|_1 < 1 - \epsilon$, then*

$$\mathbb{P}[\forall i : A_i = 1] > e^{-n} \epsilon^d.$$

Proof. To prove this, we translate the above discussion into the language of polynomials. Given μ and $\mathcal{S} = (S_1, \ldots, S_n)$, define

$$p_{\mu,\mathcal{S}}(x_1, \ldots, x_n) := p_\mu(z)\big|_{z_i = x_j \text{ for } i \in S_j}.$$

So $p_{\mu,\mathcal{S}}$ is a polynomial in n variables of degree at most m, and the coefficient of x^κ is the probability that $(A_1, \ldots, A_n) = \kappa$. In particular, we want to bound the coefficient of $x^{\mathbf{1}}$ in $p_{\mu,\mathcal{S}}$. Further, we also define $P_{\mu,\mathcal{S}}(x_1, \ldots, x_n, x_{n+1})$ as the homogenization of $p_{\mu,\mathcal{S}}$, and $d \leq m$ is its degree. Since setting variables equal and homogenization are operations which preserve real stability when the coefficients are non-negative, we have that $P_{\mu,\mathcal{S}}$ is real stable when μ is SR. We finally define

$$Q_{\mu,\mathcal{S}}(x_1, \ldots, x_d) := P_{\mu,\mathcal{S}}\left(x_1, \ldots, x_n, \frac{x_{n+1} + \cdots + x_d}{d-n}\right)$$

which is then also a real stable homogeneous polynomial, of degree d in d variables.

We now apply our bound to the polynomial $Q_{\mu,\mathcal{S}}$. Since $\beta := \nabla p_{\mu,\mathcal{S}}(\mathbf{1})$, we have

$$\alpha := \nabla Q_{\mu,\mathcal{S}}(\mathbf{1}) = \left(\beta_1, \ldots, \beta_n, \frac{d - \|\beta\|_1}{d-n}\right).$$

Note further that $Q_{\mu,\mathcal{S}}(\mathbf{1}) = 1$. Now, $\|\beta - \mathbf{1}\|_1 < 1 - \epsilon$ then implies

$$\begin{aligned}
\|\alpha - \mathbf{1}\|_1 &= \|\beta - \mathbf{1}\|_1 + (d-n)\left|\frac{d - \|\beta\|_1}{d-n} - 1\right| \\
&= \|\beta - \mathbf{1}\|_1 + \left|\sum_{i=1}^n 1 - \beta_i\right| \\
&< 2(1-\epsilon).
\end{aligned}$$

Applying our theorem then gives

$$\inf_{x_1,\ldots,x_d>0} \frac{Q_{\mu,\mathcal{S}}}{x_1\cdots x_d} \geq \left(1 - \frac{\|1-\alpha\|_1}{2}\right)^d > \epsilon^d.$$

Additionally, it is easy to see that the $x^{\mathbf{1}}$ coefficients of $Q_{\mu,\mathcal{S}}$ and $p_{\mu,\mathcal{S}}$ are related via

$$\langle x^{\mathbf{1}}\rangle p_{\mu,\mathcal{S}} = \frac{(d-n)^{d-n}}{(d-n)!} \cdot \langle x^{\mathbf{1}}\rangle Q_{\mu,\mathcal{S}}.$$

We finally apply Gurvits' original coefficient bound (see Theorem 1.4 of [5]) to get

$$\langle x^{\mathbf{1}}\rangle Q_{\mu,\mathcal{S}} \geq \frac{d!}{d^d} \cdot \inf_{x_1,\ldots,x_d>0} \frac{Q_{\mu,\mathcal{S}}}{x_1\cdots x_d} > \frac{d!}{d^d}\cdot \epsilon^d.$$

Combining everything then gives

$$\mathbb{P}[\forall i : A_i = 1] = \langle x^{\mathbf{1}}\rangle p_{\mu,\mathcal{S}} > \frac{(d-n)^{d-n}}{(d-n)!} \cdot \frac{d!}{d^d} \cdot \epsilon^d \geq e^{-n}\epsilon^d,$$

as desired. □

Our bound compares favorably to that of [5] when d is of order at most 2^n. However, the main problem with our bound is its dependence on d, which could be as large as m. In [5], the authors were able to achieve a bound that was independent of m (and d), and this was important to their applications. While we have not quite achieved this, we believe our techniques should yield an improvement to their doubly exponential bound. In particular since the bound of Theorem 1.1 is tight, we believe applying our techniques to non-homogeneous polynomials directly should yield a singly exponential bound. (Note that it is already suggested in [5] that singly exponential dependence should be possible.) We leave this to future work.

3 Notation

We let $\mathbb{R}, \mathbb{R}_+, \mathbb{Z}, \mathbb{Z}_+, \mathbb{N}, \mathbb{C}$ denote the reals, non-negative reals, integers, non-negative integers, positive integers, and complex numbers respectively. We further let $\mathbb{K}^d[x] = \mathbb{K}^d[x_1,\ldots,x_n]$ denote the set of homogeneous polynomials of degree d in n variables with coefficients in $\mathbb{K}$. For a polynomial p in n variables, the *support* of p, denoted $\mathrm{supp}(p)$, is the set of all $\mu \in \mathbb{Z}_+^n$ such that x^μ has non-zero coefficient in p. Further, the *Newton polytope* of p, denoted $\mathrm{Newt}(p)$, is the convex hull of $\mathrm{supp}(p)$. We also denote $\|\alpha\|_1 := \sum_{i=1}^n |\alpha_i|$ for $\alpha \in \mathbb{R}_+^n$ as usual. We now define all of the various classes of matrices and polynomials we will consider.

Definition 3.1. Given an $\alpha \in \mathbb{R}_+^n$ with $\|\alpha\|_1 = n$, we define $\mathrm{Mat}_n(\alpha)$ to be the set of $n \times n$ matrices A with non-negative entries such that the row sums of A are all 1 and the column sums of A are given by α.

Definition 3.2. Given $p \in \mathbb{R}_+^d[x_1,\ldots,x_n]$, we say that p is *real stable* if $p(x) = p(x_1,\ldots,x_n) \neq 0$ whenever $x_1,\ldots,x_n$ are all in the complex upper half-plane.

Definition 3.3. Given $p \in \mathbb{R}_+^d[x_1,\ldots,x_n]$, we say that p is *strongly log-concave* if $\nabla_{v_1}\cdots\nabla_{v_k} p$ is either identically zero or log-concave in the positive orthant for all $k \geq 0$ and all choices of $v_1,\ldots,v_k \in \mathbb{R}_+^n$. (These polynomials also go by the names *completely log-concave* and *Lorentzian*; see [1] and [2].)

Definition 3.4. For $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}_+^n$, we define the following classes of polynomials, ordered by inclusion:

1. $\mathrm{Prod}_n(\alpha)$ is the set of all polynomials of the form $p(x) = \prod_{i=1}^n (Ax)_i$, where $A \in \mathrm{Mat}_n(\alpha)$. Note that $p(\mathbf{1}) = 1$ and $\nabla p(\mathbf{1}) = \alpha$ for all such polynomials.

2. $\mathrm{HStab}_n(\alpha)$ is the set of all real stable polynomials in $\mathbb{R}_+^n[x_1,\ldots,x_n]$ for which $p(\mathbf{1}) = 1$ and $\nabla p(\mathbf{1}) = \alpha$.

3. $\mathrm{SLC}_n(\alpha)$ is the set of all strongly log-concave polynomials in $\mathbb{R}_+^n[x_1, \ldots, x_n]$ for which $p(\mathbf{1}) = 1$ and $\nabla p(\mathbf{1}) = \alpha$.
4. $\mathrm{LC}_n(\alpha)$ is the set of all polynomials in $\mathbb{R}_+^n[x_1, \ldots, x_n]$ which are log-concave in the open positive orthant and for which $p(\mathbf{1}) = 1$ and $\nabla p(\mathbf{1}) = \alpha$.

We also give a special name to such matrices and polynomials whenever $\alpha = \mathbf{1}$.

Definition 3.5. We refer to matrices in $\mathrm{Mat}_n(\mathbf{1})$ as *doubly stochastic*. Similarly, if $p \in \mathbb{R}_+^n[x_1, \ldots, x_n]$ such that $p(\mathbf{1}) = 1$ and $\nabla p(\mathbf{1}) = \mathbf{1}$, then we say p is *doubly stochastic*.

Finally, we define the key quantity we study in this note.

Definition 3.6. Given a polynomial $p \in \mathbb{R}_+^n[x_1, \ldots, x_n]$, we define the *capacity* of p as

$$\mathrm{Cap}_{\mathbf{1}}(p) := \inf_{x>0} \frac{p(x)}{x^{\mathbf{1}}} = \inf_{x_1,\ldots,x_n>0} \frac{p(x)}{x_1 \cdots x_n}.$$

4 Basic Results

We state here a few standard basic results concerning polynomials, matrices, and capacity.

Proposition 4.1. *Polynomials of the form $\prod_{i=1}^n (Ax)_i$ for a given matrix A with non-negative coefficients are homogeneous real stable, and homogeneous real stable polynomials are strongly log-concave.*

Proposition 4.2 (see e.g. [2])**.** *For any strongly log-concave $p \in \mathbb{R}_+^d[x_1, \ldots, x_n]$ and any $\mu \in \mathbb{Z}_+^n$, we have that $\mu \in \mathrm{supp}(p)$ if and only if $\mu \in \mathrm{Newt}(p)$.*

Proposition 4.3 (Symmetric exchange; see e.g. [2], Section 3.3)**.** *Let $p \in \mathbb{R}_+^d[x_1, \ldots, x_n]$ be strongly log-concave, and let $\mu, \nu \in \mathrm{supp}(p)$ such that $\mu_i > \nu_i$ for some $i \in [n]$. Then there exists $j \in [n]$ such that $\mu_j < \nu_j$ and $(\mu - \delta_i + \delta_j), (\nu + \delta_i - \delta_j) \in \mathrm{supp}(p)$.*

Proposition 4.4 ([4])**.** *If p is doubly stochastic then $\mathrm{Cap}_{\mathbf{1}}(p) = 1$. In particular, if A is doubly stochastic then $\mathrm{Cap}_{\mathbf{1}}(\sum_{i=1}^n (Ax)_i) = 1$. More generally, if $p(\mathbf{1}) = 1$ and $\nabla p(\mathbf{1}) = \alpha$, then $\mathrm{Cap}_{\alpha}(p) = 1$.*

Proposition 4.5. *Given a polynomial $p \in \mathbb{R}_+^n[x_1, \ldots, x_n]$, we have that $\mathrm{Cap}_{\mathbf{1}}(p) > 0$ if and only if $\mathbf{1} \in \mathrm{Newt}(p)$.*

Corollary 4.6. *Given an $n \times n$ matrix A with non-negative entries, the following are equivalent:*

1. $\mathrm{per}(A) = 0$.
2. $\mathrm{Cap}_{\mathbf{1}}(\prod_{i=1}^n (Ax)_i) = 0$.
3. *Up to permutation, the bottom-left $i \times j$ block of A is 0 for some $i + j > n$.*

Proof. (1) $\iff$ (2). Let us denote $p(x) := \prod_{i=1}^n (Ax)_i$. Recall that

$$\mathrm{per}(A) = \partial_{x_1} \cdots \partial_{x_n} p.$$

From this expression, it is clear to see that $\mathrm{per}(A) > 0$ iff $\mathbf{1} \in \mathrm{supp}(p)$. Further, since p is a real stable polynomial, we know that $\mathbf{1} \in \mathrm{supp}(p)$ iff $\mathbf{1} \in \mathrm{Newt}(p)$ by Proposition 4.2. Finally, $\mathrm{Cap}_{\mathbf{1}}(p) > 0$ iff $\mathbf{1} \in \mathrm{Newt}(p)$ by Proposition 4.5.

(1) $\iff$ (3). Follows from Hall's marriage theorem. □

Lemma 4.7. *For any $c \in \mathbb{R}_+^n$, we have*

$$\mathrm{Cap}_{\mathbf{1}}((c \cdot x)^n) = n^n \prod_{i=1}^n c_i.$$

5 Capacity Bound for Product Polynomials

In this section we prove the main result (Theorem 1.1) for polynomials in $\mathrm{Prod}_n(\alpha)$. To simplify notation, we define the following for $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}^n_+$:

$$L_n(\alpha) \equiv L_n^{\mathrm{Prod}}(\alpha) := \min_{p \in \mathrm{Prod}_n(\alpha)} \mathrm{Cap}_{\mathbf{1}}(p) \qquad \text{and} \qquad L_n^{\mathrm{SLC}}(\alpha) := \min_{p \in \mathrm{SLC}_n(\alpha)} \mathrm{Cap}_{\mathbf{1}}(p).$$

Before proving the result, we need Proposition 5.4, which has some resemblance to Corollary 4.6. In particular note that both results give equivalent conditions for capacity bounds and Hall-like properties. First though, we need a few lemmas.

Definition 5.1. Given a polynomial $p \in \mathbb{R}^n_+[x_1, \ldots, x_n]$, we say that p is a *Hall polynomial* if for all $S \subseteq [n]$ we have $\deg_S(p) \geq |S|$ where $\deg_S(p)$ is the total degree of p involving variables with index in S.

Lemma 5.2. *For any $p \in \mathbb{R}^n_+[x_1, \ldots, x_n]$, if $\mathrm{Cap}_{\mathbf{1}}(p) > 0$ then p is a Hall polynomial. If p is strongly log-concave, then these conditions are equivalent.*

Proof. We prove the contrapositive of the first statement. Let $S \subseteq [n]$ be such that $\deg_S(p) < |S|$. So for every degree vector v which shows up in p we have

$$\sum_{i \in S} v_i < |S|,$$

and therefore the same inequality holds for every $v \in \mathrm{Newt}(p)$. In particular, $\mathbf{1} \notin \mathrm{Newt}(p)$ and so $\mathrm{Cap}_{\mathbf{1}}(p) = 0$ by Proposition 4.5.

Now suppose that p is strongly log-concave and that p is a Hall polynomial. We set out to show that $q := \partial_{x_n} p|_{x_n=0}$ is a Hall polynomial. Fix $S \subseteq [n-1]$, and let $\mu \in \mathrm{supp}(p)$ be such that $\deg_S(x^\mu) \geq |S|$. We have three cases.

Case 1: $\deg_n(x^\mu) \geq 1$. Let $\nu \in \mathrm{supp}(p)$ be such that $\deg_{[n-1]}(x^\nu) \geq n-1$, so that $\deg_n(x^\nu) \leq 1$. By applying symmetric exchange (Proposition 4.3) from μ to ν, there exists $\mu' \in \mathrm{supp}(p)$ such that $\deg_S(x^{\mu'}) \geq |S|$ and $\deg_n(x^{\mu'}) = 1$. This implies $\deg_S(q) \geq |S|$.

Case 2: $\deg_n(x^\mu) = 0$ and $\deg_S(x^\mu) > |S|$. Let $\nu \in \mathrm{supp}(p)$ be such that $\deg_n(x^\nu) \geq 1$. By applying symmetric exchange from ν to μ, there exists $\mu' \in \mathrm{supp}(p)$ such that $\deg_S(x^{\mu'}) \geq |S|$ and $\deg_n(x^{\mu'}) = 1$. This implies $\deg_S(q) \geq |S|$.

Case 3: $\deg_n(x^\mu) = 0$ and $\deg_S(x^\mu) = |S|$. Let $\nu \in \mathrm{supp}(p)$ be such that $\deg_{S \cup \{n\}}(x^\nu) \geq |S|+1$. Letting $T := [n-1] \setminus S$, we have that $\deg_T(x^\mu) = |T| + 1$ and $\deg_T(\nu) \leq |T|$. Apply symmetric exchange from μ to ν, choosing indices from T to remove from μ, until we have $\mu' \in \mathrm{supp}(p)$ such that $\deg_j(x^{\mu'}) \leq \deg_j(x^\nu)$ for all $j \in T$. This implies $\deg_T(x^{\mu'}) \leq \deg_T(x^\nu) \leq |T| = n - 1 - |S|$ and $\deg_S(x^{\mu'}) \geq |S|$. Therefore either $\deg_n(x^{\mu'}) \geq 1$ or $\deg_S(x^{\mu'}) > |S|$, and so one of the previous two cases can be applied to $x^{\mu'}$.

In any case we have $\deg_S(q) \geq |S|$, and therefore q is a Hall polynomial. Since q is also strongly log-concave, we inductively have $\mathrm{Cap}_{\mathbf{1}}(q) > 0$. By Euler's identity, this implies

$$\mathrm{Cap}_{\mathbf{1}}(p) \geq \frac{1}{n} \sum_{i=1}^{n} \mathrm{Cap}_{\mathbf{1}}(x_i \cdot \partial_{x_i} p|_{x_i=0}) > 0.$$

$\square$

Lemma 5.3. *Let $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ be such that $p(\mathbf{1}) = 1$, and let $\nabla p(\mathbf{1}) = \alpha$. For all $S \subseteq [n]$, we have*

$$\sum_{i \in S} \alpha_i \leq \deg_S(p).$$

Proof. By plugging in $x_i = 1$ for all $i \notin S$, we may assume that $S = [n]$. (Note that we are not assuming p is homogeneous.) Letting P be the homogenization of p, we have that $\nabla P(\mathbf{1}) = (\alpha_1, \ldots, \alpha_n, \beta)$ and $\deg(P) = \deg(p)$. Therefore

$$\beta + \sum_{i=1}^{n} \alpha_i = \deg(P) \implies \sum_{i=1}^{n} \alpha_i \leq \deg(p).$$

This completes the proof. □

Proposition 5.4. *Given $\alpha \in \mathbb{R}_+^n$ such that $\sum_i \alpha_i = n$, the following are equivalent:*

1. $\|\mathbf{1} - \alpha\|_1 < 2$.
2. $L_n^{\mathrm{SLC}}(\alpha) > 0$.
3. $L_n(\alpha) \equiv L_n^{\mathrm{Prod}}(\alpha) > 0$.
4. $\sum_{i \in F} \alpha_i > |F| - 1$ *for all* $F \subseteq [n]$.
5. *Every* $p \in \mathrm{SLC}_n(\alpha)$ *is a Hall polynomial.*

Proof. Let $\delta := 1 - \alpha$, so that $\sum_i \delta_i = 0$. For any $F \subseteq [n]$, let $F = F_+ \sqcup F_-$ such that $\delta_i \geq 0$ for $i \in F_+$ and $\delta_i < 0$ for $i \in F_-$.

(1) $\implies$ (4). For any F, we have

$$\sum_{i \in F} \alpha_i = |F| - \sum_{i \in F} \delta_i = |F| + \sum_{i \in F_-} \delta_i - \sum_{i \in F_+} \delta_i.$$

Since $\sum_i \delta_i = 0$ and $\sum_i |\delta_i| < 2$, we have that $\sum_{i \in F_+} \delta_i < 1$. This implies

$$\sum_{i \in F} \alpha_i = |F| + \sum_{i \in F_-} \delta_i - \sum_{i \in F_+} \delta_i > |F| - 1.$$

(4) $\implies$ (1). Letting $F = [n]$ with F_+, F_- defined as above, we have

$$|F_+| - \sum_{i \in F_+} \delta_i = \sum_{i \in F_+} \alpha_i > |F_+| - 1 \implies \sum_{\delta_i \geq 0} \delta_i < 1.$$

Since $\sum_i \delta_i = 0$, this implies

$$\|1 - \alpha\|_1 = \sum_i |\delta_i| = \sum_{\delta_i \geq 0} \delta_i - \sum_{\delta_i < 0} \delta_i < 2.$$

(2) $\implies$ (3). Trivial.

(3) $\implies$ (4). So as to get a contradiction, suppose there is some $k \in [n]$ such that $\alpha_1 + \cdots + \alpha_k \leq k - 1$. We now construct a matrix $A \in \mathrm{Mat}_n(\alpha)$ such that $\mathrm{Cap}_{\mathbf{1}}(\prod_i (Ax)_i) = 0$. Let A_1 be a $(k-1) \times k$ matrix with column sums $\alpha_1, \ldots, \alpha_k$ and row sums all equal to $\beta := \frac{\alpha_1 + \cdots + \alpha_k}{k-1}$. Since $\beta \leq 1$, we can define

$$A := \begin{bmatrix} A_1 & * \\ 0 & * \end{bmatrix} \in \mathrm{Mat}_n(\alpha),$$

where the bottom-left $(n - k + 1) \times k$ block of A is 0. Since $(n - k + 1) + k > 0$, Corollary 4.6 implies $\mathrm{Cap}_{\mathbf{1}}(\prod_i (Ax)_i) = 0$. Therefore $L_n(\alpha) = 0$.

(4) $\implies$ (5). Follows from Lemma 5.3.

(5) $\implies$ (2). Follows from Lemma 5.2.

□

We now state the main result of this section: a lower bound on the capacity of polynomials in $\mathrm{Prod}_n(\alpha)$.

Theorem 5.5. *Fix $n \in \mathbb{N}$, $\alpha \in \mathbb{R}_+^n$, and $p \in \mathrm{Prod}_n(\alpha)$. If $\|1-\alpha\|_1 < 2$, then*

$$\mathrm{Cap}_{\mathbf{1}}(p) \geq \left(1 - \frac{\|1-\alpha\|_1}{2}\right)^n.$$

Proof. Define $\delta := 1 - \alpha$. We just need to prove

$$L_n(\alpha) \geq \left(1 - \frac{\|\delta\|_1}{2}\right)^n.$$

Define

$$S := \{(\gamma, D) \in \mathbb{R}_+ \times \mathrm{Mat}_n(\mathbf{1}) \ : \ A - \gamma D \geq 0 \ \text{ entrywise}\},$$

and further define $(\gamma_0, D_0) \in S$ to be such that γ_0 is maximized. (This maximum exists by compactness of $\mathrm{Mat}_n(\mathbf{1})$, the Birkhoff polytope.) Now consider the matrix

$$M = \frac{A - \gamma_0 D_0}{1 - \gamma_0},$$

which is an element of $\mathrm{Mat}_n(\tilde{\alpha})$ for $\tilde{\alpha} = \frac{\alpha - \gamma_0}{1-\gamma_0}$. We now show that $\mathrm{per}(M) = 0$. If not, then there is some permutation matrix P and some $\epsilon > 0$ such that

$$A - (\gamma_0 + \epsilon) \cdot \frac{\gamma_0 D_0 + \epsilon P}{\gamma_0 + \epsilon}$$

is entrywise non-negative. Since $\frac{\gamma_0 D_0 + \epsilon P}{\gamma_0 + \epsilon} \in \mathrm{Mat}_n(\mathbf{1})$, this contradicts the maximality of γ_0. So in fact $\mathrm{per}(M) = 0$, and therefore $L_n(\tilde{\alpha}) = 0$ by Corollary 4.6. By Lemma 5.4, this implies

$$\left\| \frac{\delta}{1-\gamma_0} \right\|_1 = \|1 - \tilde{\alpha}\|_1 \geq 2 \implies \gamma_0 \geq 1 - \frac{\|\delta\|_1}{2}.$$

Since $D_0 \in \mathrm{Mat}_n(\mathbf{1})$, we have that $\mathrm{Cap}_{\mathbf{1}}(\prod_i (D_0 x)_i) = 1$ by Proposition 4.4. The fact that $A \geq \gamma_0 D_0$ entrywise then implies

$$\mathrm{Cap}_{\mathbf{1}}\left[\prod_{i=1}^n (Ax)_i\right] \geq \mathrm{Cap}_{\mathbf{1}}\left[\prod_{i=1}^n (\gamma_0 D_0 x)_i\right] \geq \gamma_0^n \geq \left(1 - \frac{\|\delta\|_1}{2}\right)^n.$$

$\square$

We finally state the following corollary of Lemma 5.4 and Theorem 5.5, which gives a similar result for the 2-norm instead of the 1-norm.

Corollary 5.6. *Fix $n \in \mathbb{N}$, $\alpha \in \mathbb{R}_+^n$, and $p \in \mathrm{Prod}_n(\alpha)$. If $\|1-\alpha\|_2 < \frac{2}{\sqrt{n}}$, then*

$$\mathrm{Cap}_{\mathbf{1}}(p) \geq \left(1 - \frac{\sqrt{n} \cdot \|1-\alpha\|_2}{2}\right)^n.$$

Proof. Follows from $\|x\|_1 \leq \sqrt{n}\|x\|_2$ for $x \in \mathbb{R}^n$. $\square$

The corollary is a sharp improvement of the following similar inequality proved in [6] in the case that $\|1-\alpha\|_2 < \frac{1}{\sqrt{n}}$:

$$L_n(\alpha) \geq \left(1 - \sqrt{n} \cdot \|1-\alpha\|_2\right)^n.$$

This last inequality also plays a key role in the recent work on the operator scaling and its generalizations and applications, see [3].

6 Productization of Real Stable Polynomials

In this section we prove the productization result for polynomials in $\mathrm{HStab}_n(\alpha)$. This result immediately implies the main result (Theorem 1.1) for polynomials in $\mathrm{HStab}_n(\alpha)$ as a corollary (see Corollary 6.3).

To actually prove the productization result, we need a way to associate matrices in $\mathrm{Mat}_n(\alpha)$ to polynomials in $\mathrm{HStab}_n(\alpha)$. For the case of $\alpha = \mathbf{1}$, this statement was conjectured by Gurvits in the slightly different form given below. The conjecture was motivated by the case of determinantal polynomials, where the desired element of $\mathrm{Mat}_n(\mathbf{1})$ can be constructed from the matrices in the determinant. We now state this result, the proof of which was told to us by Petter Brändén in personal correspondence.

Theorem 6.1 (Brändén)**.** *Fix $p \in \mathrm{HStab}_n(\mathbf{1})$, and let $\lambda(x)$ denote the roots of $f(t) = p(\mathbf{1}t - x)$ for any $x \in \mathbb{R}^n$. Then for any $x \in \mathbb{R}^n$, there exists some $D \in \mathrm{Mat}_n(\mathbf{1})$ such that $Dx = \lambda(x)$.*

We now utilize this result to prove the productization result for polynomials in $\mathrm{HStab}_n(\alpha)$.

Theorem 6.2. *Fix $n \in \mathbb{N}$, $x, \alpha \in \mathbb{R}^n_+$, and $p \in \mathrm{HStab}_n(\alpha)$. There exists $f \in \mathrm{Prod}_n(\alpha)$ such that $p(x) = f(x)$.*

Proof. We first prove the result in the case that α is rational. Denote $\alpha = \left(\frac{k_1}{N}, \ldots, \frac{k_n}{N}\right)$ for some $k_1, \ldots, k_n \in \mathbb{Z}_+$ and $N \in \mathbb{N}$. Considering variables $y_{1,1}, \ldots, y_{1,k_1}, y_{2,1}, \ldots, y_{n,k_n}$, we define

$$q(y) := p\left(\frac{y_{1,1} + \cdots + y_{1,k_1}}{k_1}, \ldots, \frac{y_{n,1} + \cdots + y_{n,k_n}}{k_n}\right)^N,$$

so that $q \in \mathbb{R}^{nN}_+[y_{1,1}, \ldots, y_{n,k_n}]$. Since $q(1) = 1$ and

$$\partial_{y_{i,j}} q(1) = \left[\frac{N}{k_i}(\partial_{x_i} p) p^{N-1}\right](1) = 1,$$

we in fact have $q \in \mathrm{HStab}_{nN}(\mathbf{1})$. Letting $y \in \mathbb{R}^{nN}_+$ be such that $y_{i,j} = x_i$ for all i, j, note that the roots of $q(\mathbf{1}t - y)$ will consist of N copies of the n roots of $p(\mathbf{1}t - x)$. So by Theorem 6.1, there exists $D \in \mathrm{Mat}_{nN}(\mathbf{1})$ such that

$$Dy = (\lambda_1(x), \ldots, \lambda_1(x), \lambda_2(x), \ldots, \lambda_2(x), \ldots, \lambda_n(x), \ldots, \lambda_n(x))$$

where the roots are all repeated N times. Let D' be the $n \times n$ matrix formed by summing the elements of each $N \times k_i$ block of D and dividing by N. We then have

$$\prod_{i=1}^n (D'x)_i = \prod_{i=1}^n \lambda_i(x) = (-1)^n p(1 \cdot 0 - x) = p(x).$$

Since the row sums of D' are all 1 and the column sums are given by $\frac{k_i}{N}$, we have that $D' \in \mathrm{Mat}_n(\alpha)$ which proves the result for p.

We now handle the case of irrational α. First if $\alpha_k = 0$ for some k, then p does not depend on x_k and the result follows by induction. So we may assume that $\alpha_k > 0$ for all $k \in [n]$. By [7], the set of homogeneous real stable polynomials of degree n in n variables is the closure of its interior with respect to the Euclidean topology on coefficients. Define the map $M(q) := \nabla q(\mathbf{1})$ on the space of $q \in \mathbb{R}^n[x_1, \ldots, x_n]$ for which $q(\mathbf{1}) = 1$, and note that this map is linear and surjects onto the affine subspace of $\mathbb{R}^n$ consisting of vectors whose entries sum to n. Choosing a small neighborhood U about p, surjectivity and linearity imply $M(U)$ contains a small open ball about α in the range of M. We can therefore choose a sequence $p_j \in \mathrm{HStab}_n(\alpha^j)$ such that $\alpha^j \to \alpha$, $p_j \to p$, and α^j is rational for all j. The previous arguments then imply there exists $A_j \in \mathrm{Mat}_n(\alpha^j)$ such that $p_j(x) = \prod_{i=1}^n (A_j x)_i$ for all j. By compactness of the set of all $n \times n$ matrices with non-negative entries and row sums all equal to 1, we can assume that A_j is a convergent subsequence with limit A. Therefore $A \in \mathrm{Mat}_n(\alpha)$ and $p(x) = \prod_{i=1}^n (Ax)_i$, and this completes the proof.

□

The perturbation argument at the end of the above proof can also be replaced by a different argument which uses the fact that $r \mapsto \nabla p(r \cdot x)|_{x=\mathbf{1}}$ maps the strict positive orthant to the relative interior of the Newton polytope of p. With this, we can choose $p_j \in \mathrm{HStab}_n(\alpha^j)$ for α^j rational by choosing particular values of r^j which limit to $\mathbf{1}$. As a note, both arguments work for both log-concave and strongly log-concave polynomials (topological properties of the set of strongly log-concave polynomials follow from results of [2]).

We now prove the capacity bound for real stable polynomials. Recall the definition of $L_n(\alpha)$ given in Section 5.

Corollary 6.3. *For $p \in \mathrm{HStab}_n(\alpha)$, we have*

$$\mathrm{Cap}_{\mathbf{1}}(p) \geq L_n(\alpha) \geq \left(1 - \frac{\|1-\alpha\|_1}{2}\right)^n.$$

Proof. The second inequality is given by Theorem 5.5, so we just need to prove the first inequality. For any $x \in \mathbb{R}^n_+$, let $f \in \mathrm{Prod}_n(\alpha)$ be such that $p(x) = f(x)$ according to Theorem 6.2. With this, we have

$$\mathrm{Cap}_{\mathbf{1}}(p) = \inf_{x>0} \frac{p(x)}{x^{\mathbf{1}}} \geq \inf_{x>0} \min_{f \in \mathrm{Prod}_n(\alpha)} \frac{f(x)}{x^{\mathbf{1}}} = L_n(\alpha).$$

□

Note that to get this lower bound on capacity, we actually only needed a lower bound for the productization. That is, we only used the fact that for any $p \in \mathrm{HStab}_n(\alpha)$ and any $x \in \mathbb{R}^n_+$, there is some $f \in \mathrm{Prod}_n(\alpha)$ such that $p(x) \geq f(x)$. Of course, having equality in the productization is a nice fact on its own.

7 An Algorithm for Computing $L_n(\alpha)$

In this section, we give an algorithm for computing the minimum capacity value $L_n(\alpha)$ for any fixed $\alpha > 0$. Recall

$$L_n(\alpha) = \min_{p \in \mathrm{Prod}_n(\alpha)} \mathrm{Cap}_{\mathbf{1}}(p) = \min_{p \in \mathrm{HStab}_n(\alpha)} \mathrm{Cap}_{\mathbf{1}}(p),$$

where the second equality follows from the results of the previous section. To compute this minimum, first note that

$$f(M) := \log \mathrm{Cap}_{\mathbf{1}} \left(\prod_{i=1}^n (Mx)_i \right) = \inf_{x>0} \left(\sum_{i=1}^n \log(Mx)_i - \alpha_i \log x_i \right)$$

is concave as a function of $M \in \mathrm{Mat}_n(\alpha)$. This follows from the fact that $\sum_{i=1}^n \log(Mx)_i - \alpha_i \log x_i$ is a concave function in M for all x, and concavity is preserved under taking inf. Therefore to compute

$$L_n(\alpha) = \min_{p \in \mathrm{Prod}_n(\alpha)} \mathrm{Cap}_{\mathbf{1}}(p) = \min_{M \in \mathrm{Mat}_n(\alpha)} e^{f(M)},$$

we just need to minimize $f(M)$ over the extreme points of $\mathrm{Mat}_n(\alpha)$. The support (non-zero entries) of the extreme points of $\mathrm{Mat}_n(\alpha)$ then correspond to bipartite forests on $2n$ vertices. (To see this, note that if the support of M contains a cycle, then one can perturb the corresponding entries by $\pm\epsilon$ with alternating sign to show that M is not extreme.) Now let M be an extreme point of $\mathrm{Mat}_n(\alpha)$ with support F which is a bipartite forest on $2n$ vertices. Then there is some row or column of M which contains exactly one non-zero element, corresponding to an edge connected to a leaf of F. The appropriate row or column sum then forces a specific value for this entry of M. Remove that edge from F, and remove the corresponding row or column from M. Since F is still a forest after this, we can recursively apply the above argument. This implies the entries of M are actually determined by F. So for every bipartite forest F on $2n$ vertices, there is at most one M with support F. The above argument also describes the algorithm for constructing the matrix M from F. (If at any point a row or column sum is violated, it means there is no such M with support F.) These observations then yield an algorithm for computing $L_n(\alpha)$, given as follows.

1. Iterate over all bipartite forests F on $2n$ vertices.
2. Construct the matrix $M \in \mathrm{Mat}_n(\alpha)$ associated to F, or skip this F if no such M exists.
3. Compute $f(M)$, keeping track of the minimum value.

This algorithm has running time on the order the number of spanning forests of the complete bipartite graph on $2n$ vertices, which is at least $n^{2(n-1)}$.

8 Bounds for Other Classes of Polynomials

We have not yet been able to prove the same capacity bound for log-concave or strongly log-concave polynomials. In this section, we discuss a number of results and observations which suggest that such a bound should be possible. The main thing we are missing is a productization result for strongly log-concave polynomials. For real stable polynomials, we were able to explicitly construct the matrices which gave rise to the productization. The first lemma here shows that a productization result already follows from a bound by the min and max product polynomials.

Lemma 8.1. *Fix $n \in \mathbb{N}$, $x, \alpha \in \mathbb{R}^n_+$, and $p \in \mathbb{R}^n_+[x_1, \ldots, x_n]$ such that $p(\mathbf{1}) = 1$ and $\nabla p(\mathbf{1}) = \alpha$. Suppose further that*

$$\max_{f \in \mathrm{Prod}_n(\alpha)} f(x) \geq p(x) \geq \min_{f \in \mathrm{Prod}_n(\alpha)} f(x).$$

Then there exists $f \in \mathrm{Prod}_n(\alpha)$ such that $p(x) = f(x)$.

Proof. Define the map $P : \mathrm{Mat}_n(\alpha) \to \mathbb{R}_+$ via

$$P(A) := \prod_{i=1}^{n} (Ax)_i.$$

Since $\mathrm{Mat}_n(\alpha)$ is a closed convex polytope, its image under P is a closed interval. The result follows. □

Further, we actually only need the productization lower bound to obtain capacity lower bounds. That is, to prove a capacity bound for (strongly) log-concave polynomials, we just need to prove for any x and α that

$$\min_{p \in \mathrm{LC}_n(\alpha)} p(x) \geq \min_{f \in \mathrm{Prod}_n(\alpha)} f(x).$$

That said, we now state various relations between these upper and lower bounds for the various classes of polynomials.

Proposition 8.2. *Fix $n \in \mathbb{N}$ and $y \in \mathbb{R}^n_+$. Given $p \in \mathbb{R}^n_+[x_1, \ldots, x_m]$, define $\gamma = \nabla \frac{p(y \cdot x)}{p(y)}\Big|_{x=\mathbf{1}}$. For any $x \in \mathbb{R}^m_+$, we have*

1. $\dfrac{p(x)}{p(y)} \leq \dfrac{1}{n} \displaystyle\sum_{i=1}^{m} \gamma_i \left(\dfrac{x_i}{y_i}\right)^n$ *for all $p \in \mathbb{R}^n_+[x_1, \ldots, x_m]$.*
2. $\dfrac{p(x)}{p(y)} \leq \left(\dfrac{1}{n} \displaystyle\sum_{i=1}^{m} \gamma_i \cdot \dfrac{x_i}{y_i}\right)^n$ *for all $p \in \mathbb{R}^n_+[x_1, \ldots, x_m]$ which is log-concave in $\mathbb{R}^m_+$.*
3. $\dfrac{p(x)}{p(y)} \geq \left(\dfrac{x_1}{y_1}\right)^{\gamma_1} \cdots \left(\dfrac{x_m}{y_m}\right)^{\gamma_m}$ *for all $p \in \mathbb{R}^n_+[x_1, \ldots, x_m]$.*

Proof. Since $\frac{p(x)}{p(y)} = \frac{p(y\cdot x)}{p(y)}$ and variable scaling preserves the various classes of polynomials, we only need to prove the bounds for $y = \mathbf{1}$ and $p(\mathbf{1}) = 1$.

(1). By AM-GM, we have $x^\mu \leq \sum_{i=1}^m \frac{\mu_i}{n} x_i^n$ for any $\mu \in \mathbb{Z}_+^n$ and any $x \in \mathbb{R}_+^n$. Therefore

$$p(x) = \sum_\mu p_\mu x^\mu \leq \sum_\mu p_\mu \sum_{i=1}^m \frac{\mu_i x_i^n}{n} = \sum_{i=1}^m \frac{x_i^n}{n} \sum_\mu \mu_i p_\mu = \sum_{i=1}^m \frac{x_i^n}{n} \cdot \gamma_i.$$

(2). By log-concavity, $p^{\frac{1}{n}}$ is concave in the positive orthant. Further, we have

$$p^{\frac{1}{n}}(\mathbf{1}) = \frac{\sum_{i=1}^m \gamma_i \cdot 1}{n} \qquad \text{and} \qquad \nabla|_{x=\mathbf{1}}(p^{\frac{1}{n}}) = \left(\frac{\gamma_1}{n}, \ldots, \frac{\gamma_m}{n}\right) = \nabla|_{x=\mathbf{1}} \left(\frac{\sum_{i=1}^n \gamma_i x_i}{n}\right).$$

Since $\frac{1}{n}\sum_{i=1}^n \gamma_i x_i$ is linear and $p^{\frac{1}{n}}$ is concave, this immediately implies $p(x) \leq \left(\frac{1}{n}\sum_{i=1}^n \gamma_i x_i\right)^n$.

(3). Proposition 4.4 implies $\frac{p(x)}{x^\gamma} \geq 1$, which gives the bound. ☐

An immediate corollary of (2) in the above proposition is that the maximizing polynomial for log-concave polynomials is a product polynomial, stated formally as follows.

Corollary 8.3. *For $n \in \mathbb{N}$ and $x, \alpha \in \mathbb{R}_+^n$, we have*

$$\max_{p \in \mathrm{Prod}_n(\alpha)} p(x) = \left(\frac{\sum_{i=1}^n \alpha_i x_i}{n}\right)^n = \max_{p \in \mathrm{LC}_n(\alpha)} p(x).$$

Note also that (1) in the above proposition implies there is no such relationship between product polynomials and polynomials $p \in \mathbb{R}_+^n[x_1, \ldots, x_n]$ in general.

Of course what we really care about here is the lower bound, which in general is more difficult. In particular, it seems unlikely that the minimums will have explicit formulas like the maximums did in Corollary 8.3. One thing we can say in the direction of a lower bound follows from (3) in the above proposition, stated as follows.

Corollary 8.4. *For $n \in \mathbb{N}$, $x \in \mathbb{R}_+^n$, and non-negative integer vector $\alpha \in \mathbb{Z}_+^n$, we have*

$$\min_{p \in \mathrm{Prod}_n(\alpha)} p(x) = \prod_{i=1}^n x_i^{\alpha_i} = \min_{p \in \mathbb{R}_+^n[x_1,\ldots,x_n]} p(x).$$

Proof. Follows from the fact that $\prod_{i=1}^n x_i^{\alpha_i}$ is a product polynomial when $\alpha \in \mathbb{Z}_+^n$. ☐

9 The General Minimization Problem

In general, the minimization problem for general polynomials $p \in \mathbb{R}_+^n[x_1, \ldots, x_m]$ for which $p(\mathbf{1}) = 1$ and $\nabla p(\mathbf{1}) = \alpha$ can be written as the linear program

$$\min_{\substack{p_\mu \geq 0 \\ \sum_\mu \mu \cdot p_\mu = \alpha}} \quad \sum_\mu p_\mu t^\mu$$

where $t > 0$ is fixed. (Note that homogeneity makes the $p(\mathbf{1}) = 1$ condition equivalent to $\sum_i \alpha_i = n$.) One thing we can do is characterize the support of the minimizers of the above linear program.

Proposition 9.1. *For $t, \alpha \in \mathbb{R}_+^m$, a support set S is the support of a polynomial $p \in \mathbb{R}_+^n[x_1, \ldots, x_m]$ which minimizes the above linear program if and only if the following hold.*

1. *α is in the convex hull of S.*

2. *There exists $\beta \in \mathbb{R}^m$ such that*

$$(t_1 x_1 + \cdots + t_m x_m)^n - (\beta_1 x_1 + \cdots + \beta_m x_m)(x_1 + \cdots + x_m)^{n-1}$$

is supported outside of S and has non-negative coefficients.

Proof. ($\implies$). Property (1) is immediate. For property (2), consider the standard dual linear program, along with an equivalent formulation with slack variables:

$$\min_{\substack{y \in \mathbb{R}^m \\ \sum_{i=1}^m \mu_i y_i \leq t^\mu}} \sum_{i=1}^m \alpha_i y_i \qquad \iff \qquad \min_{\substack{y \in \mathbb{R}^m \\ \sum_{i=1}^m \mu_i y_i + s_\mu = t^\mu \\ s_\mu \geq 0}} \sum_{i=1}^m \alpha_i y_i.$$

Now suppose $p^\star$ is an optimum for the primal program with support S, and let $y^\star, s^\star$ be an optimum for the dual program with slack variables. Strong duality then implies

$$\alpha \cdot y^\star = \sum_\mu p_\mu^\star t^\mu = \sum_\mu p_\mu^\star \left[\mu \cdot y^\star + s_\mu^\star\right] = \alpha \cdot y^\star + \sum_\mu p_\mu^\star s_\mu^\star \implies \sum_\mu p_\mu^\star s_\mu^\star = 0.$$

Therefore $s_\mu^\star = 0$ for $\mu \in S$, which implies $\sum_{i=1}^m \mu_i y_i^\star = t^\mu$ for $\mu \in S$. With this, define

$$\begin{aligned} q(x) := \sum_\mu \binom{n}{\mu} x^\mu \left[t^\mu - \sum_{i=1}^m \mu_i y_i^\star\right] &= (t_1 x_1 + \cdots t_m x_m)^n - n \sum_{i=1}^m y_i^\star x_i \sum_\mu \binom{n-1}{\mu - \delta_i} x^{\mu - \delta_i} \\ &= (t_1 x_1 + \cdots t_m x_m)^n - n(y_1^\star x_1 + \cdots y_m^\star x_m) \cdot (x_1 + \cdots x_m)^{n-1}. \end{aligned}$$

Letting $\beta := n y^\star$ completes this direction of the proof.

($\impliedby$). Now let p be a polynomial with support S for which $\nabla p(\mathbf{1}) = \alpha$, and let

$$q(x) = (t_1 x_1 + \cdots + t_m x_m)^n - (\beta_1 x_1 + \cdots + \beta_m x_m)(x_1 + \cdots + x_m)^{n-1}.$$

Using the Bombieri (Fischer-Fock, etc.) inner product, we have

$$0 = \langle p, q \rangle = p(t) - \frac{\beta}{n} \cdot \nabla p(\mathbf{1}) = p(t) - \frac{\alpha \cdot \beta}{n} \implies p(t) = \frac{\alpha \cdot \beta}{n}.$$

Further, for any polynomial p with non-negative coefficients and $\nabla p(\mathbf{1}) = \alpha$ we have

$$0 \leq \langle p, q \rangle = p(t) - \frac{\alpha \cdot \beta}{n} \implies p(t) \geq \frac{\alpha \cdot \beta}{n}.$$

Therefore p minimizes the primal linear program, and this completes this direction of the proof. $\square$

For rational gradient, a completely general lower bound in terms of product polynomials is unlikely to hold. To see this, note the following support condition implied by the above results.

Lemma 9.2. *Fix $n \in \mathbb{N}$ and $\alpha \in \mathbb{R}_+^n$, and let $p \in \mathbb{R}_+^n[x_1, \ldots, x_n]$ be such that $p(\mathbf{1}) = 1$ and $\nabla p(\mathbf{1}) = \alpha$. If $\mathrm{Cap}_{\mathbf{1}}(p) \geq L_n(\alpha)$, then either $\mathbf{1} \in \mathrm{Newt}(p)$ or $\|\mathbf{1} - \alpha\|_1 \geq 2$.*

Proof. If $\mathrm{Cap}_{\mathbf{1}}(p) > 0$, then $\mathbf{1} \in \mathrm{Newt}(p)$ by Proposition 4.5. Otherwise, $0 = \mathrm{Cap}_{\mathbf{1}}(p) \geq L_n(\alpha)$ which implies $\|\mathbf{1} - \alpha\|_1 \geq 2$ by Lemma 5.4. $\square$

That is, a general lower bound by product polynomials is contradicted by the existence of a polynomial with non-negative coefficients for which $\nabla p(\mathbf{1})$ is close to $\mathbf{1}$ but $\mathbf{1} \notin \mathrm{Newt}(p)$. Such a polynomial likely exists. On the other hand, the well-known matroidal support conditions of strongly log-concave polynomials imply that this polynomial cannot be in $\mathrm{SLC}_n(\alpha)$, and so a lower bound by product polynomials is still possible in the strongly log-concave case.

10 Capacity Upper Bounds

Although we are mainly interested in lower bounds on capacity in this paper, upper bounds can also be achieved using similar methods. In this section we use the results of Section 8 to prove upper bounds on the capacity of various classes of polynomials. We present these observations for the interested reader, but say nothing further about them. The first is a tight bound for $p \in \mathrm{LC}_n(\alpha)$ which is also tight for $\mathrm{Prod}_n(\alpha)$.

Proposition 10.1. *For $n \in \mathbb{N}$, $\alpha \in \mathbb{R}_+^n$, and $p \in \mathrm{LC}_n(\alpha)$, we have*

$$\mathrm{Cap}_{\mathbf{1}}(p) \leq \prod_{i=1}^n \alpha_i.$$

This bound is tight over all $p \in \mathrm{Prod}_n(\alpha)$, and hence tightness also holds for $p \in \mathrm{LC}_n(\alpha)$, $p \in \mathrm{SLC}_n(\alpha)$, and $p \in \mathrm{HStab}_n(\alpha)$.

Proof. By (2) of Proposition 8.2 and Lemma 4.7, we have

$$\mathrm{Cap}_{\mathbf{1}}(p) = \mathrm{Cap}_{\mathbf{1}}\left(\left(\frac{1}{n}\sum_{i=1}^n \alpha_i x_i\right)^n\right) = \prod_{i=1}^n \alpha_i.$$

Tightness follows from the fact that $\left(\frac{1}{n}\sum_{i=1}^n \alpha_i x_i\right)^n \in \mathrm{Prod}_n(\alpha)$. ☐

Interestingly this bound does not hold for polynomials in general, for which we obtain a different tight bound.

Proposition 10.2. *For $n \in \mathbb{N}$, $\alpha \in \mathbb{R}_+^n$, and $p \in \mathbb{R}_+^n[x_1, \ldots, x_n]$ such that $p(\mathbf{1}) = 1$ and $\nabla p(\mathbf{1}) = \alpha$, we have*

$$\mathrm{Cap}_{\mathbf{1}}(p) \leq \left(\prod_{i=1}^n \alpha_i\right)^{\frac{1}{n}},$$

and this bound is tight.

Proof. By (1) of Proposition 8.2, we have

$$\mathrm{Cap}_{\mathbf{1}}(p) \leq \mathrm{Cap}_{\mathbf{1}}\left(\frac{1}{n}\sum_{i=1}^n \alpha_i x_i^n\right).$$

We then compute

$$0 = \nabla \log\left(\frac{\frac{1}{n}\sum_{i=1}^n \alpha_i e^{nx_i}}{e^{\mathbf{1}\cdot x}}\right) = \nabla\left[\log\left(\frac{1}{n}\sum_{i=1}^n \alpha_i e^{nx_i}\right) - \mathbf{1}\cdot x\right] \iff \frac{\alpha_i e^{nx_i}}{\frac{1}{n}\sum_{i=1}^n \alpha_i e^{nx_i}} = 1 \quad \forall i.$$

This is equivalent to saying that $\alpha_i e^{nx_i} = \alpha_j e^{nx_j}$ for all i, j, which occurs when $x_i = \frac{-\log \alpha_i}{n}$ for all i. Plugging this in to the objective function gives

$$\log \mathrm{Cap}_{\mathbf{1}}(p) \leq \log\left(\frac{\frac{1}{n}\sum_{i=1}^n \alpha_i \cdot \alpha_i^{-1}}{(\alpha_1 \cdots \alpha_n)^{-\frac{1}{n}}}\right) = \log\left((\alpha_1 \cdots \alpha_n)^{\frac{1}{n}}\right).$$

Exponentiating gives the bound. Tightness is immediate. ☐

Again, since this paper is predominantly about lower bounds on polynomial capacity, we say nothing further about these upper bounds.

References

[1] Nima Anari, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials, entropy, and a deterministic approximation algorithm for counting bases of matroids. *arXiv preprint arXiv:1807.00929*, 2018.

[2] Petter Brändén and June Huh. Lorentzian polynomials. *arXiv preprint arXiv:1902.03719*, 2019.

[3] Ankit Garg, Leonid Gurvits, Rafael Oliveira, and Avi Wigderson. Algorithmic and optimization aspects of brascamp-lieb inequalities, via operator scaling. *Geometric and Functional Analysis*, 28(1):100–145, 2018.

[4] Leonid Gurvits. Van der waerden/schrijver-valiant like conjectures and stable (aka hyperbolic) homogeneous polynomials: one theorem for all. *the electronic journal of combinatorics*, 15(1):66, 2008.

[5] Anna R Karlin, Nathan Klein, and Shayan Oveis Gharan. A (slightly) improved approximation algorithm for metric tsp. *arXiv preprint arXiv:2007.01409*, 2020.

[6] Nathan Linial, Alex Samorodnitsky, and Avi Wigderson. A deterministic strongly polynomial algorithm for matrix scaling and approximate permanents. *Combinatorica*, 20(4):545–568, 2000.

[7] Wim Nuij. A note on hyperbolic polynomials. *Mathematica Scandinavica*, 23(1):69–72, 1969.

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