

# Classification of the streaming approximability of Boolean CSPs

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## Abstract

A Boolean constraint satisfaction problem (CSP),  $\text{Max-CSP}(f)$ , is a maximization problem specified by a constraint  $f : \{-1, 1\}^k \rightarrow \{0, 1\}$ . An instance of the problem consists of  $m$  constraint applications on  $n$  Boolean variables, where each constraint application applies the constraint to  $k$  literals chosen from the  $n$  variables and their negations. The goal is to compute the maximum number of constraints that can be satisfied by a Boolean assignment to the  $n$  variables. In the  $(\gamma, \beta)$ -approximation version of the problem for parameters  $\gamma \geq \beta \in [0, 1]$ , the goal is to distinguish instances where at least  $\gamma$  fraction of the constraints can be satisfied from instances where at most  $\beta$  fraction of the constraints can be satisfied.

In this work we completely characterize the approximability of all Boolean CSPs in the streaming model. Specifically, given  $f$ ,  $\gamma$  and  $\beta$  we show that either (1) the  $(\gamma, \beta)$ -approximation version of  $\text{Max-CSP}(f)$  has a probabilistic streaming algorithm using  $O(\log n)$  space, or (2) for every  $\varepsilon > 0$  the  $(\gamma - \varepsilon, \beta + \varepsilon)$ -approximation version of  $\text{Max-CSP}(f)$  requires  $\Omega(\sqrt{n})$  space for probabilistic streaming algorithms. Previously such a separation was known only for  $k = 2$ . We stress that for  $k = 2$ , there are only finitely many distinct problems to consider.

Our positive results show wider applicability of bias-based algorithms used previously by [GVV17], [CGV20] by giving a systematic way to explore biases. Our negative results combine the Fourier analytic methods of [KKS15], which we extend to a wider class of CSPs, with a rich collection of reductions among communication complexity problems that lie at the heart of the negative results.

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# 1 Introduction

In this paper we give a complete characterization of the approximability of Boolean constraint satisfaction problems (CSPs) described by a single constraint in the streaming setting. We describe the exact class of problems below, and give a brief history of previous work before giving our results.

## 1.1 Boolean CSPs

In this paper we use  $\mathbb{N}$  to denote the set of natural numbers  $\{1, 2, 3, \dots\}$ . For  $n \in \mathbb{N}$  we use  $[n]$  to denote the set  $\{1, 2, \dots, n\}$ . We refer to a variable taking values in  $\{-1, 1\}$  as a Boolean variable. Given a Boolean variable  $X$ , we refer to  $X$  and  $-X$  as the literals associated with  $X$ . For vectors  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$  we use  $\mathbf{a} \odot \mathbf{b}$  to denote their coordinate-wise product. I.e., if  $\mathbf{a} = (a_1, \dots, a_n)$  and  $\mathbf{b} = (b_1, \dots, b_n)$  then  $\mathbf{a} \odot \mathbf{b} = (a_1 b_1, \dots, a_n b_n)$ .

In this paper, a Boolean CSP is a maximization problem,  $\text{Max-CSP}(f)$ , specified by a single constraint function  $f : \{-1, 1\}^k \rightarrow \{0, 1\}$  for some positive integer  $k$ . Given  $n$  Boolean variables  $x_1, \dots, x_n$ , an application of the constraint function  $f$  to these variables, which we term simply a *constraint*, is given by two  $k$ -tuples  $\mathbf{j} = (j_1, \dots, j_k) \in [n]^k$  and  $\mathbf{b} = (b_1, \dots, b_k) \in \{-1, 1\}^k$  where the  $j_i$ 's are distinct, and represents the application of the constraint function  $f$  to the literals  $b_1 x_{j_1}, \dots, b_k x_{j_k}$ . Specifically an assignment  $\sigma = (\sigma_1, \dots, \sigma_n) \in \{-1, 1\}^n$  satisfies a constraint given by  $(\mathbf{j}, \mathbf{b})$  if  $f(b_1 \sigma_{j_1}, \dots, b_k \sigma_{j_k}) = 1$ . For a constraint  $C = (\mathbf{j}, \mathbf{b})$  and assignment  $\sigma$  we use  $\sigma|_{\mathbf{j}}$  as shorthand for  $(\sigma_{j_1}, \dots, \sigma_{j_k})$  and  $C(\sigma)$  as shorthand for  $f(\mathbf{b} \odot \sigma|_{\mathbf{j}}) = f(b_1 \sigma_{j_1}, \dots, b_k \sigma_{j_k})$ . An instance  $\Psi$  of  $\text{Max-CSP}(f)$  consists of  $m$  constraints  $C_1, \dots, C_m$  applied to  $n$  variables  $x_1, \dots, x_n$ . The value of an assignment  $\sigma \in \{-1, 1\}^n$  on an instance  $\Psi = (C_1, \dots, C_m)$ , denoted  $\text{val}_{\Psi}(\sigma)$  is the fraction of constraints satisfied by  $\sigma$ , i.e.,  $\text{val}_{\Psi}(\sigma) = \frac{1}{m} \sum_{i \in [m]} C_i(\sigma)$ . The goal of the *exact* problem is to compute the maximum, over all assignments, of the value of the assignment on the input instance, i.e., to compute, given  $\Psi$ , the quantity  $\text{val}_{\Psi} = \max_{\sigma \in \{-1, 1\}^n} \{\text{val}_{\Psi}(\sigma)\}$ . (We note that the literature on CSPs has several generalizations: one may allow an entire set of constraint functions, not just a single one. One may restrict the constraint applications to be applicable only to variables and not literals. And finally one can of course consider non Boolean CSPs. We do not do any of those in this paper, though extending our techniques to classes of functions seems immediately feasible. See more discussion in [Section 1.7](#).)

In this work we consider the approximation version of  $\text{Max-CSP}(f)$ , which we study in terms of the ‘‘gapped promise problems’’. Specifically given  $0 \leq \beta < \gamma \leq 1$ , the  $(\gamma, \beta)$ -approximation version of  $\text{Max-CSP}(f)$ , abbreviated  $(\gamma, \beta)$ - $\text{Max-CSP}(f)$ , is the task of distinguishing between instances from  $\Gamma = \{\Psi \mid \text{opt}(\Psi) \geq \gamma\}$  and instances from  $B = \{\Psi \mid \text{opt}(\Psi) \leq \beta\}$ . It is well-known that this distinguishability problem is a refinement of the usual study of approximation which usually studies the ratio of  $\gamma/\beta$  for tractable versions of  $(\gamma, \beta)$ - $\text{Max-CSP}(f)$ . See [Proposition 2.10](#) for a formal statement in the context of streaming approximability of  $\text{Max-CSP}(f)$  problems.

## 1.2 Streaming algorithms

We consider streaming algorithms that take as input instances  $\Psi$  of  $\text{Max-CSP}(f)$  on  $n$  variables and  $m$  clauses for  $m, n \in \mathbb{N}$ .  $m$  and  $n$  are given to our algorithms initially and then the constraints  $C_1, \dots, C_m$  arrive one at a time. Our algorithms are allowed to use internal randomness and  $s$  bits of space. The algorithms output a single bit at the end. They are said to solve the  $(\gamma, \beta)$ -approximation problem correctly if they output the correct answer with probability at least  $2/3$

(i.e., they err with probability at most  $1/3$ ).

Our main dividing line is between algorithms that work with space  $O(\text{poly log } n)$ , versus algorithms that require space at least  $n^\varepsilon$  for some  $\varepsilon > 0$ . In informal usage we refer to a streaming problem as “easy” if it can be solved with polylogarithmic space (the former setting) and “hard” if it requires polynomial space (the latter setting).

### 1.3 Past work

To our knowledge, streaming algorithms for Boolean CSPs have not been investigated extensively. On the positive side, it may be surprising that there exists any non-trivial algorithm at all. Here, and later, we describe algorithms solving the  $(1, \rho(f) - \varepsilon)$ -approximation problem for  $\varepsilon > 0$  as “trivial”, where  $\rho(f) = 2^{-k} \sum_{\mathbf{a} \in \{-1, 1\}^k} f(\mathbf{a})$  is the fraction of clauses satisfied by a random assignment. Note that the algorithm that always outputs 1s solves  $(1, \rho(f) - \varepsilon)$ -approximation problem.

It turns out that there do exist some non-trivial approximation algorithms for Boolean CSPs. This was established by the work of Guruswami, Velingker, and Velusamy [GVV17] who, in our notation, gave an algorithm for the  $(\gamma, 2\gamma/5 - \varepsilon)$ -approximation version of Max-2AND, for every  $\gamma \in [0, 1]$ . (Max-2AND is the Max-CSP( $f$ ) problem corresponding to  $f(a, b) = 1$  if  $a = b = 1$  and 0 otherwise.) A central ingredient in their algorithm is the ability of streaming algorithms to approximate the  $\ell_1$  norm of a vector in the turnstile model, which allows them to estimate the “bias” of  $n$  variables (how often they occur positively in constraints, as opposed to negatively). Subsequently, the work of Chou, Golovnev, and Velusamy [CGV20] further established the utility of such algorithms, which we refer to as bias-based algorithms, by giving optimal algorithms for all Boolean CSPs on 2 variables. In particular they give a better (optimal!) analysis of bias-based algorithms for Max-2AND, and show that Max-2SAT also has an optimal algorithm based on bias. (We note that Max-2SAT is again not covered by the results of the current paper since it involves two functions corresponding to clauses of length 1, and clauses of length 2.)

On the negative side, the problem that has been explored the most is Max-CUT, or in our language Max-2XOR, which corresponds to  $f(x, y) = x \oplus y = (1 - xy)/2$ .<sup>1</sup> Kapralov, Khanna, and Sudan [KKS15] showed that Max-2XOR does not have a  $(1, 1/2 + \varepsilon)$ -approximation algorithm using  $o(\sqrt{n})$ -space. This was subsequently improved upon by Kapralov, Khanna, Sudan, and Velingker [KKS17], and Kapralov and Krachun [KK19]. The final paper [KK19] completely resolves Max-CUT and Max-2XOR showing that  $(1, 1/2 + \varepsilon)$ -approximation for these problems requires  $\Omega(n)$  space. Turning to other problems the work by [GVV17] notices that the  $(1, 1/2 + \varepsilon)$ -inapproximability of Max-2XOR immediately yields  $(1, 1/2 + \varepsilon)$ -inapproximability of Max-2AND as well. In [CGV20] more sophisticated reductions are used to improve the inapproximability result for Max-2AND to a  $(\gamma, 4\gamma/9 + \varepsilon)$ -inapproximability for some positive  $\gamma$ , which turns out to be the optimal ratio by their algorithm and analysis. As noted earlier their work gives optimal algorithms for all functions  $f : \{-1, 1\}^2 \rightarrow \{0, 1\}$ .

### 1.4 Our results

Our main theorem is a dichotomy for approximating all Boolean CSPs in the streaming setting.

**Theorem 1.1.** *For every  $k \in \mathbb{N}$ , for every function  $f : \{-1, 1\}^k \rightarrow \{0, 1\}$  and for every  $0 \leq \beta < \gamma \leq 1$ , at least one of the following always holds:*

<sup>1</sup>Strictly speaking this work does not include Max-CUT, which does not allow constraints to be placed on arbitrary literals. Max-2XOR is however very closely related and in particular is harder than Max-CUT.

1.  $(\gamma, \beta)$ -Max-CSP( $f$ ) has a  $O(\log n)$ -space streaming algorithm.
2. For every  $\varepsilon > 0$ ,  $(\gamma - \varepsilon, \beta + \varepsilon)$ -Max-CSP( $f$ ) requires  $\Omega(\sqrt{n})$  space. If  $\gamma = 1$ , then  $(1, \beta + \varepsilon)$ -Max-CSP( $f$ ) requires  $\Omega(\sqrt{n})$  space.

Furthermore, given the truth-table of  $f$  and  $\gamma$  and  $\beta$  as  $\ell$ -bit rationals<sup>2</sup>, it can be decided in polynomial space  $\text{poly}(2^k, \ell)$  which one of the two conditions holds.

In analogy with the terminology used in the study of CSP approximation in polynomial time, we define a problem to be “(streaming)-approximation-resistant” if it is hard to beat random assignment with  $n^{o(1)}$ -space. Recall  $\rho(f)$  denotes the fraction of assignments that satisfy a function  $f$ . We say that Max-CSP( $f$ ) is *streaming-approximation-resistant* if, for every  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $(1, \rho + \varepsilon)$ -Max-CSP( $f$ ) requires  $\Omega(n^\delta)$  space. (We suppress the qualifier “streaming-” for most of the paper.) We get the following dichotomy for streaming-approximation-resistance.

**Corollary 1.2.** *For every  $k \in \mathbb{N}$ , for every function  $f : \{-1, 1\}^k \rightarrow \{0, 1\}$ , if Max-CSP( $f$ ) is streaming-approximation-resistant then for every  $\varepsilon > 0$  the  $(1, \rho + \varepsilon)$ -approximation version of Max-CSP( $f$ ) requires  $\Omega(\sqrt{n})$  space. If Max-CSP( $f$ ) is not streaming-approximation-resistant then there exists  $\varepsilon > 0$  such that  $(1 - \varepsilon, \rho + \varepsilon)$ -Max-CSP( $f$ ) can be solved in logarithmic space. Furthermore given the truth-table of the function  $f$  there is an algorithm running in space  $\text{poly}(2^k)$  that decides if Max-CSP( $f$ ) is streaming-approximation-resistant or not.*

In [Section 2.4](#), we show how to apply our theorem above to get a full characterization of the approximation profile of the Max-2AND problem (i.e., the Max-CSP( $f$ ) problem for  $f(x, y) = 1$  if  $x = y = 1$  and 0 otherwise). This reproduces the result shown in [\[CGV20\]](#) while giving a more refined picture of the approximability. See [Example 2.11](#).

We remark that while our dichotomy theorems are in some sense “explicit” (formalized best by the assertion that they can be decided in PSPACE given the truth table of  $f : \{-1, 1\}^k \rightarrow \{0, 1\}$ ,  $\gamma$ , and  $\beta$ ), they do not necessarily resolve questions about the approximation resistance of an infinite family of functions such as the linear threshold functions. But they can be applied to get some uniform classes of results. We mention one below.

Say that a function  $f$  *supports 1-wise independence* if there exists a distribution  $D$  supported on the satisfying assignments to  $f$ , i.e., on  $f^{-1}(1) \subseteq \{-1, 1\}^k$  such that its marginals are all uniform, i.e., for every  $j \in [k]$ , we have  $\mathbb{E}_{\mathbf{a} \sim D}[a_j] = 0$ . Our main theorem immediately yields the following corollary.

**Corollary 1.3.** *If  $f : \{-1, 1\}^k \rightarrow \{0, 1\}$  supports 1-wise independence then Max-CSP( $f$ ) is streaming-approximation-resistant.*

We also give a (very) partial converse, showing that symmetric functions are approximation resistant if and only if they support 1-wise independence (see [Lemma 2.12](#)). While we do believe that there are other streaming-approximation-resistant problems, we do not know of one (and in particular do not give one in this paper). We discuss this more in the next section.

<sup>2</sup> $\alpha \in \mathbb{R}$  is said to be an  $\ell$ -bit rational if there exist integers  $-2^\ell < p, q < 2^\ell$  such that  $\alpha = p/q$ .

## 1.5 Contrast with dichotomies in the polynomial time setting

The literature on dichotomies of  $\text{Max-CSP}(f)$  problems is vast. One broad family of results here [Sch78, Bul17, Zhu17] considers the exact satisfiability problems (corresponding to distinguishing between instances from  $\{\Psi \mid \text{opt}(\Psi) = 1\}$  and instances from  $\{\Psi \mid \text{opt}(\Psi) < 1\}$ ). Another family of results [Rag08, AM09, KTW14] considers the approximation versions of  $\text{Max-CSP}(f)$  and gets “near dichotomies” along the lines of this paper — i.e., they either show that the  $(\gamma, \beta)$ -approximation is easy (in polynomial time), or for every  $\varepsilon > 0$  the  $(\gamma - \varepsilon, \beta + \varepsilon)$ -approximation version is hard (in some appropriate sense). Our work resembles the latter series of works both in terms of the nature of results obtained, the kinds of characterizations used to describe the “easy” and “hard” classes, and also in the proof approaches (though of course the streaming setting is much easier to analyze, allowing for much simpler proofs overall). We summarize their results giving comparisons to our theorem and then describe a principal contrast.

In a seminal work, Raghavendra [Rag08] gave a characterization of the polynomial time approximability of the  $\text{Max-CSP}(f)$  problems based on the unique games conjecture [Kho02]. Our [Theorem 1.1](#) is analogous to his theorem, though restricted to a single function, with Boolean variables, with ability to complement variables. A characterization of approximation resistant functions is given by Khot, Tulsiani and Worah [KTW14]. Our [Corollary 1.2](#) is analogous to this. Austrin and Mossel [AM09] show that all functions supporting a pairwise independent distribution are approximation-resistant. Our [Corollary 1.3](#) is analogous to this theorem.

While our results run in parallel to the work on polynomial time approximability our characterizations are not immediately comparable. Indeed there are some significant differences which we highlight below. Of course there is the obvious difference that our negative results are unconditional (and not predicated on a complexity theoretic assumption like the unique games conjecture or  $\text{P} \neq \text{NP}$ ). But more significantly our characterization is a bit more “explicit” than those of [Rag08] and [KTW14]. In particular the former only shows decidability of the problem which take  $\varepsilon$  as an input (in addition to  $\gamma, \beta$  and  $f$ ) and distinguishes  $(\gamma, \beta)$ -approximable problems from  $(\gamma - \varepsilon, \beta + \varepsilon)$ -inapproximable problems. The running time of their decision procedure grows with  $1/\varepsilon$ . In contrast our distinguishability separates  $(\gamma, \beta)$ -approximability from “ $\forall \varepsilon > 0, (\gamma - \varepsilon, \beta + \varepsilon)$ -inapproximability” — so our algorithm does not require  $\varepsilon$  as an input - it merely takes  $\gamma, \beta$  and  $f$  as input. Indeed this difference is key to the understanding of approximation resistance. Due to the stronger form of our main theorem ([Theorem 1.1](#)), our characterization of streaming-approximation-resistance is explicit (decidable in  $\text{PSPACE}$ ), whereas a decidable characterization of approximation-resistance in the polynomial time setting seems to be still open.

Our characterizations also seem to differ from the previous versions in terms of the features being exploited to distinguish the two classes. This leads to some strange gaps in our knowledge. For instance, it would be natural to suspect that (conditional) inapproximability in the polynomial time setting should also lead to (unconditional) inapproximability in the streaming setting. But we don’t have a formal theorem proving this.<sup>3</sup> One (unfulfilling) consequence of this gap in knowledge is that we do not yet have an streaming-approximation-resistant problem that is not covered by [Corollary 1.3](#). In the polynomial time setting, Potechin [Pot19] gives a balanced linear threshold function that is approximation-resistant. Balanced linear threshold functions do not support 1-wise independence and so his function would be a good candidate for a streaming-approximation-resistant function that is not covered by [Corollary 1.3](#).

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<sup>3</sup>Of course, if this were false, it would be a breakthrough result giving a polynomial time (even log space) algorithm for the unique games!

## 1.6 Overview of our analysis

At the heart of our characterization is a family of algorithms for  $\text{Max-CSP}(f)$  in the streaming setting. We will describe this family soon, but the main idea of our proof is that if no algorithm in this family solves  $(\gamma, \beta)\text{-Max-CSP}(f)$ , then we can extract a single pair of instances, roughly a  $\gamma$ -satisfiable “yes” instance and an at most  $\beta$ -satisfiable “no” instance, that certify this inability. We then show how this pair of instances can be exploited as gadgets in a negative result. Up to this part our approach resembles that in [Rag08] (though of course all the steps are quite different). The main difference is that we are able to use the structure of the algorithm and the lower bound construction to show that we can afford to consider only instances on  $k$  variables. (This step involves a non-trivial choice of definitions that we elaborate on shortly.) This bound on the number of variables allows us to get a very “decidable” separation between approximable and inapproximable problems. Specifically we show that distinction between approximable setting and the inapproximable one can be expressed by a quantified formula over the reals with a constant number of quantifiers over  $2^k$  variables and equations — a problem that is known to be solvable in PSPACE. We give more details below.

**Bias-based algorithms.** For every  $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$  we define the  $\lambda$ -bias measure of an instance  $\Psi$  of  $\text{Max-CSP}(f)$  as follows. Let  $p_{ij}$  denote the number of occurrences of the literal  $x_i$  as the  $j$ th variable in a constraint, and let  $n_{ij}$  denote the same quantity for the literal  $-x_i$ . Let  $\text{bias}_{i,j} = \frac{1}{m}(p_{ij} - n_{ij})$ . We define the  $\lambda$ -bias of the  $i$ th variable to be a weighted sum of  $\text{bias}_{i,j}$  as follows:  $\text{bias}_\lambda(\Psi)_i = \sum_{j=1}^k \lambda_j \text{bias}_{i,j}$ . Let the bias vector of the instance  $\Psi$  be  $\text{bias}_\lambda(\Psi) = (\text{bias}_\lambda(\Psi)_1, \dots, \text{bias}_\lambda(\Psi)_n)$ . It turns out that the ability to estimate the  $\ell_1$  norm of a vector in the “turnstile model” implies that for any given  $\lambda$  vector, we can estimate the  $\ell_1$  norm of  $\text{bias}_\lambda(\Psi)$  (to within a multiplicative factor of  $(1 \pm \varepsilon)$  for arbitrarily small  $\varepsilon > 0$ ). We refer to an algorithm that aims to solve the  $(\gamma, \beta)\text{-Max-CSP}(f)$  using only an estimate of the  $\ell_1$  norm of  $\text{bias}_\lambda(\Psi)$  (for some  $\lambda$  based on  $f, \gamma, \beta$ ) as a “bias-based algorithm”. A priori it is not clear how to choose a  $\lambda$  vector for a given problem. The crux of our analysis is to identify two (bounded, closed) convex sets  $K_\gamma^Y, K_\beta^N \subseteq \mathbb{R}^k$  such that if the two sets are disjoint then the hyperplane separating them gives us the desired  $\lambda$ .

We now give some insight into the sets  $K_\gamma^Y$  and  $K_\beta^N$ . Roughly these sets capture properties of instances of  $\text{Max-CSP}(f)$  on  $k$  variables, say  $x_1, \dots, x_k$ . The instances we consider are special in that  $x_i$  always appears as the  $i$ th variable in every constraint: the only variability being in whether it appears positively or negatively. The set  $K_\gamma^Y$  consists of the bias vectors  $\text{bias}_\lambda(\Psi)$  of all instances  $\Psi$  that have  $\text{val}_\Psi(1^k) \geq \gamma$ , i.e., the assignment of all 1’s satisfied  $\gamma$  fraction of the constraints of  $\Psi$ . The set  $K_\beta^N$  is similarly supposed to capture the biases  $\text{bias}_\lambda(\Psi)$  of instances  $\Psi$  for which the value is at most  $\beta$ . Determining exactly which assignments achieve this bounded value turns out to be subtle and we defer describing it here. But given our choice, our analysis roughly works as follows: Given an instance  $\Psi$  on  $n$  variables, we create a distribution  $\mathcal{D}(\Psi) \in \Delta(\{-1, 1\}^k)$  and its projection  $\mu$  onto  $\mathbb{R}^k$  such that if  $\Psi$  is a YES instance, then  $\mu$  ends up being in  $K_\gamma^Y$ , while if  $\Psi$  is a NO instance,  $\mu \in K_\beta^N$ . Most crucially, the  $\ell_1$  norm of  $\text{bias}_\lambda(\Psi)$  exactly corresponds to the distance from  $\mu$  to the hyperplane separating  $K_\gamma^Y$  and  $K_\beta^N$ , which allows us to distinguish the YES and NO cases. Details of the definition of sets can be found in [Section 2](#) and the analysis of the algorithm can be found in [Section 4](#).

**Communication complexity of hidden partitions.** Hardness results in streaming are usually obtained by appealing to lower bounds for one-way communication complexity and our work is no different. The rough idea is to create instances  $\Psi$  that are divided into a (large) constant number of sub-instances  $\Psi_1, \dots, \Psi_T$  that are on the same set of variables,  $x_1, \dots, x_n$ . In YES instances, the sub-instances are chosen so that a planted assignment chosen uniformly satisfies  $\gamma$  fraction of the constraints. In NO instances, the sub-instances are chosen “randomly” so that no assignment is very likely to satisfy  $\beta$  fraction of the constraints. The division into sub-instances is used as follows: no two constraints within a sub-instance share variables - so an algorithm with limited memory when facing the stream corresponding to  $\Psi_t$  would not really see any interesting patterns locally, and so would need to remember “details” about  $\Psi_1, \dots, \Psi_{t-1}$ . However, and this is where our sets  $K^Y$  and  $K^N$  come into play, remembering univariate marginals (or how often  $x_i$  appeared positively or negatively) would hopefully be of no use since both the YES and the NO distributions would have exactly the same marginals.

Implementing this reduction to the communication complexity problem is mostly straightforward given previous works. We don’t describe the reduction but only the reduced communication problem. We consider a two-player one-way communication problem, which we call the *Randomized Mask Detection* (RMD) problem where Alice gets a vector  $\mathbf{x}^* \in \{-1, 1\}^n$  chosen uniformly at random, and Bob gets a random  $k$ -uniform hypermatching  $M$  with  $\alpha n$  hyperedges on  $[n]$ , along with a vector  $\mathbf{z} \in \{-1, 1\}^{k\alpha n}$  whose distribution depends on whether we are in the YES case or NO case. (Here,  $\alpha$  is some small but positive constant.) Specifically,  $\mathbf{z}$  specifies the  $x$ -values of the vertices touched by  $M$ , but this information is hidden partially by picking for each edge (independently) a masking vector  $\mathbf{b}$  and letting  $\mathbf{z}$  for this edge be the information for  $\mathbf{x}^*$  masked by (xor’ing with)  $\mathbf{b}$ . (See [Section 5.2](#) for a mathematically precise statement.) The key difference between the YES instance and the NO instance is the distribution of  $\mathbf{b}$ : In the YES case, it is chosen according to some distribution  $\mathcal{D}_Y$  supported on  $\{-1, 1\}^k$  whose marginals are in  $K_\gamma^N$ ; and in the NO case, they come from the distribution  $\mathcal{D}_N$  whose marginals are in  $K_\beta^Y$ . Of course, we apply this reduction only in the setting where the two sets of marginals intersect, so for our purpose we can ignore  $K_\gamma^N$  and  $K_\beta^Y$ , and just consider two arbitrary distributions  $\mathcal{D}_Y$  and  $\mathcal{D}_N$  with matching marginals. The technical meat of our negative result is proving that for an arbitrary pair of distributions  $\mathcal{D}_Y$  and  $\mathcal{D}_N$  with matching marginals, any one-way communication protocol with  $o(\sqrt{n})$  communication has  $o(1)$ -advantage in distinguishing the YES and NO cases. See [Theorem 5.3](#).

The work of Kapralov, Khanna, and Sudan [[KKS15](#)] seeds our quest by showing that  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD is hard on the special case where  $\mathcal{D}_Y$  is uniform on  $\{(1, 1), (-1, -1)\}$  and  $\mathcal{D}_N$  is uniform on  $\{-1, 1\}^2$ . Strictly speaking their formalism is slightly different<sup>4</sup> — and one in which we are not able to express all our problems, but their proof for this case certainly applies to our formalism. The proof of [[KKS15](#)] is Fourier analytic, based on prior work of Gavinsky, Kempe, Kerenidis, Raz, and Wolf [[GKK<sup>+</sup>09](#)]. The first step of our analysis extends this Fourier analytic approach to the case of distributions over  $\{-1, 1\}^k$  for all values of  $k$ , and to all distributions  $\mathcal{D}_Y$  and  $\mathcal{D}_N$  that have *uniform marginals*. This is reported in [Section 6](#).

Somewhat to our surprise we were unable to extend the Fourier analytic proof to the case where  $\mathcal{D}_Y$  and  $\mathcal{D}_N$  have arbitrary but matching marginals. To get the full case, we turn to reductions. Specifically we show that while we cannot directly prove the indistinguishability of general  $\mathcal{D}_Y$  and

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<sup>4</sup>In order to handle the general Max-CSP problem, in RMD we extend the previous framework with a more detailed encoding of the hypermatching  $M$ , and also allow for a general masking vector  $\mathbf{b}$ . Due to these extensions, we cannot immediately conclude hardness of RMD from previous results, and we prove it from scratch.



$\mathcal{D}_N$  with matching marginals, we can use the indistinguishability for uniform marginals as a tool (via reductions) to show indistinguishability of some restricted pairs of distributions  $(\mathcal{D}, \mathcal{D}')$ . The key to the final result is that for any pair of distributions  $\mathcal{D}_Y$  and  $\mathcal{D}_N$  with matching marginals, there is a path from one to the other of finite length (our upper bound is  $\text{poly}(k!)$ ) such that every adjacent pair of distributions on the path is indistinguishable by our aforementioned reductions for restricted pairs. (We remark that while  $\mathcal{D}_Y$  and  $\mathcal{D}_N$  are typically chosen to have interesting properties with respect to their value on various assignments, the intermediate distributions may not have any interesting properties for the underlying optimization problem! But the generality of the framework turns out to be a strength in that we can refer to these problems anyway and use their indistinguishability features.) The path from  $\mathcal{D}_Y$  to  $\mathcal{D}_N$  allows us to use triangle-inequality for indistinguishability to get the final result on indistinguishability of RMD on distributions with matching marginals. Details of this part can be found in [Section 7](#).

## 1.7 Future questions and work

Some of the main questions left open in this work are:

1. Can the methods be extended to handle the case where the constraints come from a family of functions, rather than a single function? We believe this should be straightforward to achieve.
2. Can we further extend the results to the setting where the constraints are not placed on literals, but rather only on variables? Such an extension seems to require new ideas beyond those in this paper.
3. Can we extend the results to the non-Boolean setting, i.e., when the variables take on values from an arbitrary finite set, as opposed to  $\{-1, 1\}$ . We stress that both the positive and negative results in this paper exploit restrictions of the Boolean setting! In this direction, Guruswami and Tao [[GT19](#)] proved that  $(1/p + \epsilon)$ -approximation for the unique games with alphabet size  $p$  requires  $\tilde{\Omega}(\sqrt{n})$  space in the streaming setting.
4. Can the lower bound for the hard problems be improved to linear? Such an improvement was given by Kapralov and Krachun [[KK19](#)] for the Max-2LIN problem (Max-CSP( $f$ ) where  $f(x, y) = x \oplus y$ ) in a technical tour-de-force. Extending this work to other optimization problems seems non-trivially challenging.
5. Finally, our work and all the questions above only consider the setting of single-pass streaming algorithms. Once this is settled, it would make sense to extend the analyses to multi-pass algorithms. While there are several multi-pass streaming algorithms and lower bounds (see, e.g., [[Cha20](#), [McG14](#), [GM08](#)] and references therein), we note that Assadi, Kol, Saxena, and Yu [[AKSY20](#)] recently suggested a multi-round version of the Boolean Hidden Hypermatching problem that allows to extend some previous single-pass results (including a lower bound for approximate Max-2LIN) to the multi-pass setting.

## 1.8 Structure of rest of the paper

In [Section 2](#), we describe our result in detail. In particular we give an explicit criterion to distinguish the easy and hard Max-CSP( $f$ ) problems in the streaming setting. [Section 3](#) contains some of the preliminary background used in the rest of the paper. In [Section 4](#), we describe and analyze our

algorithm that yields our easiness result. In [Section 5](#), we define the central family of communication problems that lie at the heart of our negative results and prove the negative result for streaming problems assuming the communication problems are hard. In [Section 6](#), we establish the desired lower bounds for a subclass of the problems by Fourier analytic methods. In [Section 7](#), we establish reductions between the communication problems that allows us to extend our negative results to the entire set.

## 2 Our Results

We start with some notation needed to state our results. We use  $\mathbb{R}^{\geq 0}$  to denote the set of non-negative real numbers. For a finite set  $\Omega$ , let  $\Delta(\Omega)$  denote the space of all probability distributions over  $\Omega$ , i.e.,

$$\Delta(\Omega) = \{\mathcal{D} : \Omega \rightarrow \mathbb{R}^{\geq 0} \mid \sum_{\omega \in \Omega} \mathcal{D}(\omega) = 1\}.$$

We view  $\Delta(\Omega)$  as being contained in  $\mathbb{R}^{|\Omega|}$ . We use  $X \sim \mathcal{D}$  to denote a random variable drawn from the distribution  $\mathcal{D}$ .

### 2.1 Key definitions

The main objects that allow us to derive our characterization are the space of distributions on constraints that either allow a large number of constraints to be satisfied, or only a few constraints to be satisfied. To see where the distributions come from, note that distributions of constraints over  $n$  variables can naturally be identified with instances of weighted constraint satisfaction problem (where the weight associated with a constraint is simply its probability). In what follows we will consider instances on exactly  $k$  variables  $x_1, \dots, x_k$ . Furthermore all constraints will use  $x_i$  as the  $i$ th variable. Thus a constraint on  $k$  variables is specified by  $\mathbf{b} \in \{-1, 1\}^k$ , specifying the constraint  $f(b_1x_1, \dots, b_kx_k)$ . Thus in what follows we will equate “instances on  $k$  variables” with distributions on  $\{-1, 1\}^k$ .

Given  $0 \leq \beta \leq \gamma \leq 1$  we will consider two sets of instances/distributions. The first set  $S_\gamma^Y = S_\gamma^Y(f)$  will be instances where  $\gamma$  fraction of the constraints are satisfied by the assignment  $1^k$ . The second set  $S_\beta^N = S_\beta^N(f)$  is a bit more subtle: it consists of instances where no “independent identical distribution” on the variables satisfies more than  $\beta$ -fraction of the clauses. To elaborate, recall that the only distributions on a single variable taking values in  $\{-1, 1\}$  are the Bernoulli distributions. Let  $\text{Bern}(p)$  denote the distribution that takes the value 1 with probability  $1 - p$  and  $-1$  with probability  $p$ . Then an instance belongs to  $S_\beta^N$  if for every  $p$ , when  $(x_1, \dots, x_k)$  gets a random assignment chosen according to  $\text{Bern}(p)^k$ , the expected fraction of satisfied clauses is at most  $\beta$ . The following is our formal definition.

**Definition 2.1** (Space of Yes/No Distributions). *For  $\gamma, \beta \in \mathbb{R}$ , we define*

$$S_\gamma^Y = S_\gamma^Y(f) = \{\mathcal{D}_Y \in \Delta(\{-1, 1\}^k) \mid \mathbb{E}_{\mathbf{b} \sim \mathcal{D}_Y} [f(\mathbf{b})] \geq \gamma\}$$

and  $S_\beta^N = S_\beta^N(f) = \{\mathcal{D}_N \in \Delta(\{-1, 1\}^k) \mid \mathbb{E}_{\mathbf{b} \sim \mathcal{D}_N} \mathbb{E}_{\mathbf{a} \sim \text{Bern}(p)^k} [f(\mathbf{b} \odot \mathbf{a})] \leq \beta, \forall p\}.$

For  $\gamma > \beta$  the sets  $S_\gamma^Y$  and  $S_\beta^N$  are clearly disjoint. But their marginals, when projected to single coordinates need not be, and this is the crux of our characterization. In what follows, we define sets  $K_\gamma^Y$  and  $K_\beta^N$  to be the marginals of the distributions in  $S_\gamma^Y$  and  $S_\beta^N$  respectively. For a distribution  $\mathcal{D} \in \Delta(\{-1, 1\}^k)$ , let  $\boldsymbol{\mu}(\mathcal{D})$  denote its marginals, i.e.,  $\boldsymbol{\mu}(\mathcal{D}) = (\mu_1, \dots, \mu_k)$  where  $\mu_i = \mathbb{E}_{\mathbf{b} \sim \mathcal{D}}[b_i]$ .

**Definition 2.2** (Marginals of Yes/No Distributions). *For  $\gamma, \beta \in \mathbb{R}$ , we define*

$$K_\gamma^Y = K_\gamma^Y(f) = \{ \boldsymbol{\mu}(\mathcal{D}_Y) \mid \mathcal{D}_Y \in S_\gamma^Y \}$$

$$\text{and } K_\beta^N = K_\beta^N(f) = \{ \boldsymbol{\mu}(\mathcal{D}_N) \mid \mathcal{D}_N \in S_\beta^N \}.$$

With the two definitions above in hand we are ready to describe our characterizations of easy vs. hard approximation versions of  $\text{Max-CSP}(f)$ .

## 2.2 The characterization

Our main result, stated formally below, roughly says that the  $\text{Max-CSP}(f)$  problem is  $(\gamma, \beta)$ -approximable if and only if the sets  $K_\gamma^Y$  and  $K_\beta^N$  do not intersect.

**Theorem 2.3.** *For every function  $f : \{-1, 1\}^k \rightarrow \{0, 1\}$  and for every  $0 \leq \beta < \gamma \leq 1$ , the following hold:*

1. *If  $K_\gamma^Y(f) \cap K_\beta^N(f) = \emptyset$ , then  $(\gamma, \beta)$ -Max-CSP( $f$ ) admits a probabilistic streaming algorithm that uses  $O(\log n)$  space.*
2. *If  $K_\gamma^Y(f) \cap K_\beta^N(f) \neq \emptyset$ , then for every  $\varepsilon > 0$ , the  $(\gamma - \varepsilon, \beta + \varepsilon)$ -approximation version of Max-CSP( $f$ ) requires  $\Omega(\sqrt{n})$  space<sup>5</sup>. Furthermore, if  $\gamma = 1$ , then  $(1, \beta + \varepsilon)$ -Max-CSP( $f$ ) requires  $\Omega(\sqrt{n})$  space.*

*Proof of Theorem 2.3.* Part (1) of the theorem is restated and proved as [Theorem 4.1](#) in [Section 4](#). Part (2) is proved as [Theorem 5.1](#) in [Section 5.4](#).  $\square$

We now turn to the implications of this theorem. First, to get [Theorem 1.1](#) from [Theorem 2.3](#), we need to show that the question “Is  $K_\gamma^Y \cap K_\beta^N = \emptyset$ ?” can be decided in polynomial space. To this end, we first make the following observation.

**Lemma 2.4.** *For every  $\beta, \gamma \in [0, 1]$  the sets  $S_\gamma^Y, S_\beta^N, K_\gamma^N$  and  $K_\beta^Y$  are bounded, closed and convex. Furthermore,  $K_\gamma^Y \cap K_\beta^N = \emptyset$  can be expressed in the quantified theory of reals with 2 quantifier alternations,  $O(2^k)$  variables, and polynomials of degree at most  $k + 1$ .*

*Proof.* We start by considering the sets  $S_\gamma^Y$  and  $S_\beta^N$ . It is straightforward to see that  $S_\gamma^Y$  is a bounded and convex polytope in  $\mathbb{R}^{2^k}$ .  $S_\beta^N$  is a bit more subtle due to the universal quantification over  $p \in [0, 1]$ . It is now specified by infinitely many linear inequalities in  $\mathbb{R}^{2^k}$  and so is still a bounded and convex set (though not necessarily a polytope).  $K_\gamma^Y$  (resp.  $K_\beta^N$ ) is obtained by a linear projection from  $\mathbb{R}^{2^k}$  to  $\mathbb{R}^k$ . So  $K_\gamma^Y$  is a bounded, closed, and convex polytope in  $\mathbb{R}^k$ , while  $K_\beta^N$  is still a bounded, closed, and convex set.

<sup>5</sup>The constant hidden in the  $\Omega$  notation may depend on  $k$  and  $\varepsilon$ .

To get an intersection detection algorithm we use one more property. Note that for variable  $p$ , the condition  $\mathbb{E}_{\mathbf{a} \sim \mathcal{D}_N} \mathbb{E}_{\mathbf{b} \sim \text{Bern}(p)^k} [f(\mathbf{b} \odot \mathbf{a})] \leq \beta$  is a polynomial inequality in  $p$  of degree at most  $k$ , with coefficients that are linear forms in  $\mathcal{D}_N(\mathbf{b})$ ,  $\mathbf{b} \in \{-1, 1\}^k$ . This allows us to express the condition  $K_\gamma^Y \cap K_\beta^N \neq \emptyset$  using the following system of quantified polynomial inequalities:

$$\begin{aligned} \exists \mathcal{D}_Y, \mathcal{D}_N \in \mathbb{R}^{2^k}, \forall p \in [0, 1] \text{ s.t.} \\ \mathcal{D}_Y, \mathcal{D}_N \text{ are distributions,} \end{aligned} \tag{2.5}$$

$$\forall i \in [k], \mathbb{E}_{\mathbf{b} \sim \mathcal{D}_Y} [b_i] = \mathbb{E}_{\mathbf{b} \sim \mathcal{D}_N} [b_i], \tag{2.6}$$

$$\mathbb{E}_{\mathbf{b} \sim \mathcal{D}_Y} [f(\mathbf{b})] \geq \gamma, \tag{2.7}$$

$$\mathbb{E}_{\mathbf{b} \sim \mathcal{D}_N} \mathbb{E}_{\mathbf{a} \sim \text{Bern}(p)^k} [f(\mathbf{a} \odot \mathbf{b})] \leq \beta. \tag{2.8}$$

Note that Equations (2.5), (2.6) and (2.7) are just linear inequalities in the variables  $\mathcal{D}_Y, \mathcal{D}_N$  and do not depend on  $p$ . As noticed above Equation (2.8) is an inequality in  $p$ , and  $\mathcal{D}_N$ , of degree  $k$  in  $p$ , and 1 in  $\mathcal{D}_N$ . We thus get that the intersection problem can be expressed in the quantified theory of the reals by an expression with two quantifier alternations,  $2^k$  variables and  $O(2^k)$  polynomial inequalities, with polynomials of degree at most  $k + 1$ . (Most of the inequalities are of the form  $\mathcal{D}_Y(\mathbf{b}) \geq 0$  or  $\mathcal{D}_N(\mathbf{b}) \geq 0$ . Only  $O(k)$  inequalities are not of that form; and of these, only one is non-linear.)  $\square$

The quantified theory of the reals is known to be solvable in PSPACE. In particular we may use the following theorem.

**Theorem 2.9** ([BPR06, Theorem 14.11, see also Remark 13.10]). *The truth of a quantified formula with  $w$  quantifier alternations over  $K$  variables and polynomial (potentially strict) inequalities can be decided in space  $K^{O(w)}$  and time  $2^{K^{O(w)}}$ .*

(Specifically, Theorem 14.11 in [BPR06] asserts the time complexity above, and Remark 13.10 yields the space complexity.)

[Theorem 1.1](#) now follows immediately.

*Proof of Theorem 1.1.* [Theorem 2.3](#) asserts that the  $(\gamma, \beta)$ -approximation version of Max-CSP( $f$ ) is easy if and only if  $K_\gamma^Y \cap K_\beta^N = \emptyset$ . [Lemma 2.4](#) asserts that this condition is in turn expressible in the quantified theory of the reals with 2 quantifier alternations. Finally [Theorem 2.9](#) asserts that this can be decided in polynomial space. The theorem follows.  $\square$

We note that the literature on approximation algorithms usually considers a single parameter version of the problem. In our context we would say that an algorithm  $A$  is a  $\alpha$ -approximation algorithm for Max-CSP( $f$ ) if for every instance  $\Psi$ , we have

$$\alpha \cdot \text{val}_\Psi \leq A(\Psi) \leq \text{val}_\Psi.$$

The following proposition converts our main theorem in terms of this standard notion.

**Proposition 2.10.** *Fix  $f : \{-1, 1\}^k$  and let  $K_\gamma^Y$  and  $K_\beta^N$  denote the space of marginals for this function  $f$ . Let*

$$\alpha = \inf_{\beta \in [0, 1]} \left\{ \sup_{\gamma \in (\beta, 1] \text{ s.t. } K_\gamma^Y \cap K_\beta^N = \emptyset} \{\beta/\gamma\} \right\}.$$

Then for every  $\varepsilon > 0$ , there is an  $(\alpha - \varepsilon)$ -approximation algorithm for  $\text{Max-CSP}(f)$  that uses  $O(\log n)$  space. Conversely every  $(\alpha + \varepsilon)$ -approximation algorithm for  $\text{Max-CSP}(f)$  requires  $\Omega(\sqrt{n})$  space.

*Proof.* For the positive result, let  $\tau \triangleq \varepsilon \cdot \rho(f)/2$ , where  $\rho(f) = 2^{-k} \sum_{\mathbf{a} \in \{-1,1\}^k} f(\mathbf{a})$  is the fraction of clauses satisfied by a random assignment. Let

$$A_\tau = \{(i\tau, j\tau) \in [0, 1]^2 \mid i, j \in \mathbb{Z}^{\geq 0}, i > j, K_{i\tau}^Y \cap K_{j\tau}^N = \emptyset\}.$$

By [Theorem 2.3](#), for every  $(\gamma, \delta) \in A_\tau$  there is a  $O(\log n \log(1/\tau))$ -space algorithm for  $(\gamma, \beta)$ - $\text{Max-CSP}(f)$  with error probability  $1/(10\tau^2)$ , which we refer to as the  $(\gamma, \beta)$ -distinguisher below. In the following we consider the case where all  $O(\tau^{-2})$  distinguishers output correct answers, which happens with probability at least  $2/3$ .

Our  $O(\tau^{-2} \log(1/\tau) \log n) = O(\log n)$  space  $(\alpha - \varepsilon)$ -approximation algorithm for  $\text{Max-CSP}(f)$  is the following: On input  $\Psi$ , run in parallel all the  $(\gamma, \beta)$ -distinguishers on  $\Psi$ , for every  $(\gamma, \beta) \in A_\tau$ . Let

$$\beta_0 = \arg \max_{\beta} [\exists \gamma \text{ such that the } (\gamma, \beta)\text{-distinguisher outputs YES on } \Psi].$$

Output  $\beta' = \max\{\rho(f), \beta_0\}$ .

We now prove that this is an  $(\alpha - \varepsilon)$ -approximation algorithm. First note that by the correctness of the distinguisher we have  $\beta' \leq \text{val}_\Psi$ . Let  $\gamma_0$  be the smallest multiple of  $\tau$  satisfying  $\gamma_0 \geq (\beta_0 + \tau)/\alpha$ . By the definition of  $\alpha$ , we have that  $K_{\gamma_0}^Y \cap K_{\beta_0 + \tau}^N = \emptyset$ . So  $(\gamma_0, \beta_0 + \tau) \in A_\tau$  and so the  $(\gamma_0, \beta_0 + \tau)$ -distinguisher must have output NO on  $\Psi$  (by the maximality of  $\beta_0$ ). By the correctness of this distinguisher we conclude  $\text{val}_\Psi \leq \gamma_0 \leq (\beta_0 + \tau)/\alpha + \tau \leq (\beta' + \tau)/\alpha + \tau$ . We now verify that  $(\beta' + \tau)/\alpha + \tau \leq \beta'/(\alpha - \varepsilon)$  and this gives us the desired approximation guarantee. We have

$$(\beta' + \tau)/\alpha + \tau \leq (\beta' + 2\tau)/\alpha \leq (\beta'/\alpha) \cdot (1 + 2\tau/\rho(f)) = (\beta'/\alpha)(1 + \varepsilon) \leq (\beta'/(\alpha(1 - \varepsilon))),$$

where the first inequality uses  $\alpha \leq 1$ , the second uses  $\beta' \geq \rho(f)$ , the equality comes from the definition of  $\tau$  and the final inequality uses  $(1 + \varepsilon)(1 - \varepsilon) \leq 1$ . This concludes the positive result.

The negative result is simpler. Given  $\gamma, \beta$  with  $\beta/\gamma \geq \alpha + \varepsilon$ , we can use an  $(\alpha + \varepsilon)$ -approximation algorithm  $A$  to solve the  $(\gamma, \beta)$ - $\text{Max-CSP}(f)$ , by outputting YES if  $A(\Psi) \geq \beta$  and NO otherwise.  $\square$

### 2.3 Approximation resistance

We now turn to [Corollary 1.2](#). Recall that for a function  $f : \{-1, 1\}^k \rightarrow \{0, 1\}$ , we define  $\rho(f) = 2^{-k} \cdot |\{\mathbf{a} \in \{-1, 1\}^k : f(\mathbf{a}) = 1\}|$  to be the probability that a uniformly random assignment satisfies  $f$ . Recall further that  $f$  is approximation-resistant if for every  $\varepsilon > 0$ , the  $(1, \rho(f) + \varepsilon)$ -approximation version of  $\text{Max-CSP}(f)$  requires polynomial space.

*Proof of Corollary 1.2.* By [Theorem 2.3](#) we have that  $\text{Max-CSP}(f)$  is approximation-resistant if and only if  $K_1^Y \cap K_{\rho(f) + \varepsilon}^N \neq \emptyset$  for every  $\varepsilon > 0$ . In turn, this is equivalent to saying  $\text{Max-CSP}(f)$  is approximation resistant if and only if  $K_1^Y \cap K_{\rho(f)}^N \neq \emptyset$ . If  $K_1^Y \cap K_{\rho(f)}^N = \emptyset$ , then by the property that these sets are closed, we have that there must exist  $\varepsilon > 0$  such that  $K_{1-\varepsilon}^Y \cap K_{\rho(f) + \varepsilon}^N = \emptyset$ . In turn this implies, again by [Theorem 2.3](#), that the  $(1 - \varepsilon, \rho(f) + \varepsilon)$ -approximation version of  $\text{Max-CSP}(f)$  can be solved by a streaming algorithm with  $O(\log n)$  space. Finally, from [Lemma 2.4](#) and [Theorem 2.9](#) the condition “Is  $K_1^Y \cap K_{\rho(f)}^N = \emptyset$ ?” can be checked in polynomial space.  $\square$

To get [Corollary 1.3](#), we perform some basic reasoning about the sets  $K_1^Y$  and  $K_{\rho(f)}^N$ .

*Proof of Corollary 1.3.* We argue that the vector  $0^k$  belongs to both  $K_1^Y$  and  $K_{\rho(f)}^N$ . [Theorem 2.3](#) now implies the assertion.

Let  $\mathcal{D}_Y$  be the distribution proving that  $f$  supports a 1-wise independent distribution, i.e.,  $\mathcal{D}_Y$  is supported on  $f^{-1}(1)$  and satisfies  $\mathbb{E}_{\mathbf{b} \in \mathcal{D}_Y}[b_i] = 0$  for every  $i \in [k]$ . It follows that  $\mathcal{D}_Y \in S_1^Y$  and  $0^k \in K_1^Y$ .

Let  $\mathcal{D}_N$  be the uniform distribution on  $\{-1, 1\}^k$ . Note that for every  $\mathbf{a} \in \{-1, 1\}^k$  we have  $\mathbf{a} \odot \mathbf{b}$  is uniformly distributed over  $\{-1, 1\}^k$  if  $\mathbf{b} \sim \mathcal{D}_N$ . Consequently, for every  $\mathbf{a}$  we get  $\mathbb{E}_{\mathbf{b} \sim \mathcal{D}_N}[f(\mathbf{b} \odot \mathbf{a})] = \rho(f)$ , and so for every  $p \in [0, 1]$ , we have

$$\mathbb{E}_{\mathbf{a} \sim \text{Bern}(p)^k} \mathbb{E}_{\mathbf{b} \sim \mathcal{D}_N} [f(\mathbf{b} \circ \mathbf{a})] = \rho(f).$$

We conclude the  $\mathcal{D}_N \in S_{\rho(f)}^N$  and so  $0^k \in K_{\rho(f)}^N$ .

We conclude from [Theorem 2.3](#) that  $\text{Max-CSP}(f)$  is not  $(1, \rho(f) - \varepsilon)$ -approximable and so is approximation-resistant.  $\square$

## 2.4 Examples

We illustrate the applicability of our theorem with two examples. The first is of the specific function  $\text{Max-2AND}$ , i.e.,  $\text{Max-CSP}(f)$  for  $f(a, b) = a \wedge b$ , i.e.,  $f(a, b) = 1$  if and only if  $a = b = 1$ .

**Example 2.11.** *For the function  $f : \{-1, 1\} \rightarrow \{0, 1\}$  given by  $f(1, 1) = 1$  and  $f(a, b) = 0$  otherwise, we have for  $0 \leq \gamma \leq 1$ ,*

$$K_\gamma^Y = \{(\mu_1, \mu_2) \mid 2\gamma - 1 \leq \mu_1, \mu_2 \leq 1\}.$$

*It is easy to see that for this function,  $K_\gamma^Y$  intersects  $K_\beta^N$  if and only if  $(2\gamma - 1, 2\gamma - 1) \in K_\beta^N$ . From this we obtain that  $K_\gamma^Y \cap K_\beta^N \neq \emptyset$  in the following cases*

$$\left[ \begin{array}{ll} \beta \leq 1 - 2\gamma & \text{and } 1/4 \leq \gamma < 1/3 \\ \beta \leq \frac{\gamma^2}{4\gamma - 1} & \text{and } 1/3 \leq \gamma < 1/2 \\ \beta \leq \frac{(1-\gamma)^2}{3-4\gamma} & \text{and } 1/2 \leq \gamma < 2/3 \\ \beta \leq 2\gamma - 1 & \text{and } 2/3 \leq \gamma. \end{array} \right.$$

*The quantity  $\alpha(\beta) = \sup_{\gamma \mid K_\gamma^Y \cap K_\beta^N \neq \emptyset} \beta/\gamma$  is minimized at  $\beta = 4/15$ . At this point  $\alpha = 4/9$ , which is consistent with the findings in [\[CGV20\]](#) for the  $\text{MAX-2AND}$  problem. Our more refined analysis also shows that  $\alpha(\beta)$  approaches 1 as  $\beta \rightarrow 1$  (suggesting that “almost-satisfiable” instances are better approximated).*

The second example we consider includes an entire family of functions.

**Lemma 2.12.** *For a symmetric function  $f : \{-1, 1\}^k \rightarrow \{0, 1\}$ ,  $\text{Max-CSP}(f)$  is approximation resistant if and only if it supports a 1-wise independent distribution.*

*Proof.* One direction of the implication directly follows from [Corollary 1.2](#). For the other direction, we use Fourier analysis. The necessary definitions are included in [Section 3.4](#). A symmetric function  $f$  is given by a set of “levels”  $L = \{\ell_1, \dots, \ell_t\} \subseteq \{-k, \dots, k\}$  such that  $f(a_1, \dots, a_k) = 1$

if and only if  $\|\mathbf{a}\|_1 = \sum_{i=1}^k a_i \in L$ . If  $L$  contains 0, or if  $L$  contains both positive and negative elements, then  $f$  supports a 1-wise independent distribution.<sup>6</sup> So we conclude  $L$  contains only positive elements or only negative elements. Without loss of generality we consider the case where  $L$  contains only positive elements.

Let  $\rho = \rho(f)$ , first note that both  $K_1^Y$  and  $K_\rho^N$  are symmetric since  $f$  is symmetric. Thus, by the convexity of the sets, it suffices to consider vectors of the form  $\mu^k = (\mu, \mu, \dots, \mu)$  in  $K_1^Y$  and  $K_\rho^N$ . Since  $L$  contains only positive elements, it follows that for  $\mu^k \in K_1^Y$ , we must have  $\mu > 0$ . To prove that  $\text{Max-CSP}(f)$  is not approximation resistant, it suffices to show that for  $\mu > 0$ ,  $\mu^k$  is not contained in  $K_\rho^N$ . Consider a distribution  $\mathcal{D} \in S_\rho^N$  with  $\mu(\mathcal{D}) = \mu^k$ . It can be shown by elementary Fourier analysis that if  $\mathbf{a} \sim \text{Bern}(1/2 + \varepsilon)^k$  and  $\mathbf{b} \sim \mathcal{D}$  then

$$\mathbb{E}_{\mathbf{b} \sim \mathcal{D}} \mathbb{E}_{\mathbf{a} \sim \text{Bern}(p)^k} [f(\mathbf{b} \odot \mathbf{a})] = \rho + \Omega(\mu\tau\varepsilon) - O(\varepsilon^2),$$

where  $\tau$  is the sum of the first level Fourier coefficients of  $f$  (i.e.,  $\tau = \sum_{\|w\|_1=1} \hat{f}(w)$ ), and the  $\Omega(\cdot)$  and  $O(\cdot)$  notations hide constants depending on  $f$  and  $\mathcal{D}$ , but not on  $\varepsilon > 0$ . Due to the symmetry of  $f$ , all the first level Fourier coefficients are equal, and due to the positive of  $L$ , all these coefficients are positive. It follows that for some sufficiently small  $\varepsilon > 0$ , the expected probability of satisfying a constraint is strictly larger than  $\rho$  thus proving  $\mu^k \notin K_\rho^N$ . We conclude  $K_1^Y \cap K_\rho^N = \emptyset$ , and so  $\text{Max-CSP}(f)$  is not approximation-resistant.  $\square$

### 3 Preliminaries

We will follow the convention that  $n$  denotes the number of variables in the CSP as well as the communication game,  $m$  denotes the number of constraints in the CSP, and  $k$  denotes the arity of the CSP. We use  $\mathbb{N}$  to denote the set of natural numbers  $\{1, 2, 3, \dots\}$  and use  $[n]$  to denote the set  $\{1, 2, \dots, n\}$ . By default, the Boolean variable in this paper takes value in  $\{-1, 1\}$ .

For variables of a vector form, we write them in boldface, e.g.,  $\mathbf{x} \in \{-1, 1\}^n$ , and its  $i$ -th entry is written without boldface, e.g.,  $x_i$ . For variable being a vector of vectors, we write it, for example, as  $\mathbf{b} = (\mathbf{b}(1), \mathbf{b}(2), \dots, \mathbf{b}(m))$  where  $\mathbf{b}(i) \in \{-1, 1\}^k$ . The  $j$ -th entry of the  $i$ -th vector of  $\mathbf{b}$  is then written as  $\mathbf{b}(i)_j$ . Let  $\mathbf{x}$  and  $\mathbf{y}$  be two vectors of the same length,  $\mathbf{x} \odot \mathbf{y}$  denotes the entry-wise product of them.

For every  $p \in [0, 1]$ ,  $\text{Bern}(p)$  denotes the Bernoulli distribution that takes value 1 with probability  $p$  and takes value  $-1$  with probability  $1 - p$ .

#### 3.1 Approximate Constraint Satisfaction

Let  $f : \{-1, 1\}^k \rightarrow \{0, 1\}$  be a Boolean constraint function of arity  $k$  and  $x_1, \dots, x_n$  be variables. A constraint  $C$  consists of  $\mathbf{j} = (j_1, \dots, j_k) \in [n]^k$  and  $\mathbf{b} = (b_1, \dots, b_k) \in \{-1, 1\}^k$  where the  $j_i$ 's are distinct. The constraint  $C$  reads as requiring  $f(\mathbf{b} \odot \mathbf{x}|_{\mathbf{j}}) = f(b_1 x_{j_1}, \dots, b_k x_{j_k}) = 1$ . A  $\text{Max-CSP}(f)$  instance  $\Psi$  contains  $m$  constraints  $C_1, \dots, C_m$  where  $C_i = (\mathbf{j}(i), \mathbf{b}(i))$  for each  $i \in [m]$ . For an assignment  $\sigma \in \{-1, 1\}^n$ , the value  $\text{val}_\Psi(\sigma)$  of  $\sigma$  on  $\Psi$  is the fraction of constraints satisfied by  $\sigma$ , i.e.,  $\text{val}_\Psi(\sigma) = \frac{1}{m} \sum_{i \in [m]} f(\mathbf{b}(i) \odot \sigma|_{\mathbf{j}(i)})$ . The optimal value of  $\Psi$  is defined as  $\text{val}_\Psi = \max_{\sigma \in \{-1, 1\}^n} \text{val}_\Psi(\sigma)$ . The approximation version of  $\text{Max-CSP}(f)$  is defined as follows.

<sup>6</sup>Indeed, if  $\ell_1, \ell_2 \in L$ , where  $\ell_1 < 0$  and  $\ell_2 > 0$ , then a distribution  $\mathcal{D}$  that with probability  $p = \ell_2 / (\ell_2 - \ell_1)$  samples a random  $\mathbf{a}$  of Hamming weight  $\|\mathbf{a}\|_1 = \ell_1$  and with probability  $1 - p$  samples a random  $\mathbf{a}$  of weight  $\|\mathbf{a}\|_1 = \ell_2$  is 1-wise independent and is supported on  $f^{-1}(1)$ .

**Definition 3.1** ( $(\gamma, \beta)$ -Max-CSP( $f$ )). Let  $f: \{-1, 1\}^k \rightarrow \{0, 1\}$  be a constraint function and  $0 \leq \beta < \gamma \leq 1$ . For each  $m \in \mathbb{N}$ , let  $\Gamma_m = \{\Psi = (C_1, \dots, C_m) \mid \text{val}_\Psi \geq \gamma\}$  and  $B_m = \{\Psi = (C_1, \dots, C_m) \mid \text{val}_\Psi \leq \beta\}$ .

The task of  $(\gamma, \beta)$ -Max-CSP( $f$ ) is to distinguish between instances from  $\Gamma = \cup_{m \leq \text{poly}(n)} \Gamma_m$  and instances from  $B = \cup_{m \leq \text{poly}(n)} B_m$ .

For  $\alpha \in [0, 1]$ , an algorithm **ALG** is an  $\alpha$ -approximation to the Max-CSP( $f$ ) problem if **ALG** can solve  $(\gamma, \beta)$ -Max-CSP( $f$ ) with success probability at least  $2/3$  for every  $0 \leq \beta < \gamma \leq 1$  such that  $\beta/\gamma \leq \alpha$ .

Let  $\rho(f) = 2^{-k} \cdot |\{\mathbf{a} \in \{-1, 1\}^k \mid f(\mathbf{a}) = 1\}|$  denote the probability that a uniformly random assignment satisfies  $f$ . We say  $f$  is *streaming-approximation-resistant* if for every  $\varepsilon > 0$ , the  $(1, \rho(f) + \varepsilon)$ -Max-CSP( $f$ ) requires  $\Omega(n^\delta)$  space for some constant  $\delta > 0$ .

### 3.2 Total variation distance

The total variation distance between probability distributions plays an important role in our analysis.

**Definition 3.2** (Total variation distance of discrete random variables). Let  $\Omega$  be a finite probability space and  $X, Y$  be random variables with support  $\Omega$ . The total variation distance between  $X$  and  $Y$  is defined as follows.

$$\|X - Y\|_{\text{tvd}} := \frac{1}{2} \sum_{\omega \in \Omega} |\Pr[X = \omega] - \Pr[Y = \omega]|.$$

We will use the triangle and data processing inequalities for the total variation distance.

**Proposition 3.3** (E.g., [KKS15, Claim 6.5]). For random variables  $X, Y$  and  $W$ :

- (Triangle inequality)  $\|X - Y\|_{\text{tvd}} \geq \|X - W\|_{\text{tvd}} - \|Y - W\|_{\text{tvd}}$ .
- (Data processing inequality) If  $W$  is independent of both  $X$  and  $Y$ , and  $f$  is a function, then  $\|f(X, W) - f(Y, W)\|_{\text{tvd}} \leq \|X - Y\|_{\text{tvd}}$ .

### 3.3 Concentration inequality

We will use the following concentration inequality which is essentially an Azuma-Hoeffding style inequality for submartingales. The form we use is from [KK19].

**Lemma 3.4** ([KK19, Lemma 2.5]). Let  $X = \sum_{i \in [N]} X_i$  where  $X_i$  are Bernoulli random variables such that for any  $k \in [N]$ ,  $\mathbb{E}[X_k \mid X_1, \dots, X_{k-1}] \leq p$  for some  $p \in (0, 1)$ . Let  $\mu = Np$ . For any  $\Delta > 0$ ,

$$\Pr[X \geq \mu + \Delta] \leq \exp\left(-\frac{\Delta^2}{2\mu + 2\Delta}\right).$$

### 3.4 Fourier analysis

We will need the following basic notions from Fourier analysis over the Boolean hypercube (see, for instance, [O'D14]). For a Boolean function  $f: \{-1, 1\}^k \rightarrow \mathbb{R}$  its Fourier coefficients are defined by  $\hat{f}(\mathbf{v}) = \mathbb{E}_{\mathbf{a} \in \{-1, 1\}^k} [f(\mathbf{a}) \cdot (-1)^{\mathbf{v}^\top \mathbf{a}}]$ , where  $\mathbf{v} \in \{0, 1\}^k$ . We need the following two important tools.



**Lemma 3.5** (Parseval’s identity). *For every function  $f : \{-1, 1\}^k \rightarrow \mathbb{R}$ ,*

$$\|f\|_2^2 = \frac{1}{2^k} \sum_{\mathbf{a} \in \{-1, 1\}^k} f(\mathbf{a})^2 = \sum_{\mathbf{v} \in \{0, 1\}^k} \widehat{f}(\mathbf{v})^2.$$

Note that for every distribution  $f$  on  $\{-1, 1\}^k$ ,  $\widehat{f}(0^k) = 2^{-k}$ . For the uniform distribution  $U$  on  $\{-1, 1\}^k$ ,  $\widehat{U}(\mathbf{v}) = 0$  for every  $\mathbf{v} \neq 0^k$ . Thus, by [Lemma 3.5](#), for any distribution  $f$  on  $\{-1, 1\}^k$ :

$$\|f - U\|_2^2 = \sum_{\mathbf{v} \in \{0, 1\}^k} \left( \widehat{f}(\mathbf{v}) - \widehat{U}(\mathbf{v}) \right)^2 = \sum_{\mathbf{v} \in \{0, 1\}^k \setminus \{0^k\}} \widehat{f}(\mathbf{v})^2. \quad (3.6)$$

Next, we will use the following consequence of hypercontractivity for Boolean functions as given in [\[GKK<sup>+</sup>09, Lemma 6\]](#) which it turns relies on a lemma from [\[KKL88\]](#).

**Lemma 3.7.** *Let  $f : \{-1, 1\}^n \rightarrow \{-1, 0, 1\}$  and  $A = \{\mathbf{a} \in \{-1, 1\}^n \mid f(\mathbf{a}) \neq 0\}$ . If  $|A| \geq 2^{n-c}$  for some  $c \in \mathbb{N}$ , then for every  $\ell \in \{1, \dots, 4c\}$ , we have*

$$\frac{2^{2n}}{|A|^2} \sum_{\substack{\mathbf{v} \in \{0, 1\}^n \\ \|\mathbf{v}\|_1 = \ell}} \widehat{f}(\mathbf{v})^2 \leq \left( \frac{4\sqrt{2}c}{\ell} \right)^\ell.$$

## 4 Streaming Algorithm

In this section we give our main algorithmic result — a  $O(\log n)$ -space streaming algorithm for the  $(\gamma, \beta)$ -Max-CSP( $f$ ) if  $K_\gamma^Y = K_\gamma^Y(f)$  and  $K_\beta^N = K_\beta^N(f)$  are disjoint. (See [Definition 2.2](#).)

We state our main theorem of this section which simply repeats Part (1) of [Theorem 2.3](#).

**Theorem 4.1.** *For every function  $f : \{-1, 1\}^k \rightarrow \{0, 1\}$  and for every  $0 \leq \beta < \gamma \leq 1$ , if  $K_\gamma^Y(f) \cap K_\beta^N(f) = \emptyset$ , then  $(\gamma, \beta)$ -Max-CSP( $f$ ) admits a probabilistic streaming algorithm that uses  $O(\log n)$  space and succeeds with probability at least  $2/3$ .*

The overview of the algorithm is as follows: We use the separability of  $K_\gamma^Y$  and  $K_\beta^N$  to obtain a hyperplane with normal vector  $\boldsymbol{\lambda}$  that separates the two sets. We then estimate a  $\boldsymbol{\lambda}$ -weighted bias of a given instance  $\Psi$  and accept  $\Psi$  if this bias falls on the  $K_\gamma^Y$  side of the hyperplane. We note that the bias can be approximated arbitrarily well using well-known  $\ell_1$ -norm approximators in the turnstile model. The bulk of the work is in analyzing the correctness of our algorithm.

We will use the following streaming algorithm for approximating the  $\ell_1$  norm of a vector.

**Proposition 4.2** ([\[Ind00\]](#), [\[KNW10, Theorem 2.1\]](#)). *Given a stream  $S$  of  $\text{poly}(n)$  updates  $(i, v) \in [n] \times \{-M, -(M-1), \dots, M-1, M\}$  where  $M = \text{poly}(n)$ , let  $x_i = \sum_{(i, v) \in S} v$  for  $i \in [n]$ . For every  $\varepsilon > 0$ , there exists a streaming algorithm that uses  $O(\log n)$  bits of memory and outputs a  $(1 \pm \varepsilon)$ -approximation to the value  $\|x\|_1 = \sum_i |x_i|$  with probability at least  $2/3$ .*

### 4.1 Algorithm

Let us start with the definition of  $\boldsymbol{\lambda}$ -bias.

**Definition 4.3** (Bias (vector)). For  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$ , and instance  $\Psi = (C_1, \dots, C_m)$  of *Max-CSP(f)* where  $C_i = (\mathbf{j}(i), \mathbf{b}(i))$ , we let the  $\boldsymbol{\lambda}$ -bias vector of  $\Psi$ , denoted  $\text{bias}_{\boldsymbol{\lambda}}(\Psi)$ , be the vector in  $\mathbb{R}^n$  given by

$$\text{bias}_{\boldsymbol{\lambda}}(\Psi)_\ell = \frac{1}{m} \cdot \sum_{i \in [m], t \in [k]: j(i)_t = \ell} \lambda_t b_t(i),$$

for  $\ell \in [n]$ . The  $\boldsymbol{\lambda}$ -bias of  $\Psi$ , denoted  $B_{\boldsymbol{\lambda}}(\Psi)$ , is the  $\ell_1$  norm of  $\text{bias}_{\boldsymbol{\lambda}}(\Psi)$ , i.e.,  $B_{\boldsymbol{\lambda}}(\Psi) = \sum_{\ell=1}^n |\text{bias}_{\boldsymbol{\lambda}}(\Psi)_\ell|$ .

By directly applying the known  $\ell_1$ -sketching algorithm (i.e., [Proposition 4.2](#)), the following lemma shows that  $\boldsymbol{\lambda}$ -bias can be estimated in  $O(\log n)$  space.

**Lemma 4.4.** For every vector  $\boldsymbol{\lambda} \in \mathbb{R}^k$  and  $\varepsilon > 0$ , there exists a  $O(\log n)$  space algorithm  $\mathcal{A}$  that, on input  $\Psi$  outputs a  $(1 \pm \varepsilon)$ -approximation to  $B_{\boldsymbol{\lambda}}(\Psi)$ , i.e., for every  $\Psi$ ,  $(1 - \varepsilon)B_{\boldsymbol{\lambda}}(\Psi) \leq \mathcal{A}(\Psi) \leq (1 + \varepsilon)B_{\boldsymbol{\lambda}}(\Psi)$ , with probability at least  $2/3$ .

*Proof.* Note that since here  $k$  and  $\varepsilon$  are constants with respect to  $n$ , we can without loss of generality assume each entry of  $\boldsymbol{\lambda}$  is an integer and  $\varepsilon$  has constant bit complexity.<sup>7</sup>

Next, for each  $i \in [m]$  and  $t \in [k]$ , let  $(j(i)_t, \lambda_t)$  be an update. Since  $m = \text{poly}(n)$  and  $k$  is a constant, we know that there are only  $\text{poly}(n)$  updates and each update is constant integer. Thus, by [Proposition 4.2](#), there exists a streaming algorithm that uses  $O(\log n)$  bits of memory and outputs a  $(1 \pm \varepsilon)$ -approximation to the value

$$\sum_{i \in [m], t \in [k]: j(i)_t = \ell} \lambda_t b_t(i)$$

with probability at least  $2/3$ . Now, we design a streaming algorithm  $\mathcal{A}$  which gets the  $(1 \pm \varepsilon)$ -approximation  $A$  to the above quantity and calculates the value  $m$ . Finally, let  $\mathcal{A}(\Psi) = A/m$  and by [Proposition 4.2](#) and the definition of  $\boldsymbol{\lambda}$ -bias, we know that  $\mathcal{A}(\Psi)$  is a  $(1 \pm \varepsilon)$ -approximation to  $B_{\boldsymbol{\lambda}}(\Psi)$  using  $O(\log n)$  space with probability at least  $2/3$ .  $\square$

We will use the following form of the hyperplane separation theorem for convex bodies (see, e.g., [\[BV04, Exercise 2.22\]](#)).

**Proposition 4.5.** Let  $K^Y$  and  $K^N$  be two disjoint nonempty closed convex sets in  $\mathbb{R}^k$  at least one of which is compact. Then there exists a nonzero vector  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k)$  and real numbers  $\tau_Y > \tau_N$  such that

$$\forall \mathbf{x} \in K^Y, \langle \boldsymbol{\lambda}, \mathbf{x} \rangle \geq \tau_Y \quad \text{and} \quad \forall \mathbf{x} \in K^N, \langle \boldsymbol{\lambda}, \mathbf{x} \rangle \leq \tau_N.$$

We are now ready to describe our algorithm for  $(\gamma, \beta)$ -Max-CSP(f).

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<sup>7</sup>Concretely, round  $\varepsilon$  to  $2^{-t}$  where  $t$  is the smallest integer such that  $\varepsilon \geq 2^{-t}$ . As for  $\boldsymbol{\lambda}$ , let  $\lambda_{\min} = \min_{j \in [k]} |\lambda_j|$  and round it with the same way as we did for  $\varepsilon$ . Next, for each  $j \in [k]$ , scale and round  $\lambda_j$  to  $\lceil \frac{4\lambda_j}{\lambda_{\min}} \rceil$ . It is not difficult to verify that scaling down the new  $\boldsymbol{\lambda}$ -bias by a factor of  $\lambda_{\min}/4$ , it is a  $(1 \pm \varepsilon/2)$ -approximation to the original  $\boldsymbol{\lambda}$ -bias.

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**Algorithm 1** A streaming algorithm for  $(\gamma, \beta)$ -Max-CSP( $f$ )

---

**Input:**  $\Psi$ —an instance of Max-CSP( $f$ ).

- 1: Let  $\boldsymbol{\lambda} \in \mathbb{R}^k$  and  $\tau_N < \tau_Y$  be as given by [Proposition 4.5](#) separating  $K_\gamma^Y(f)$  and  $K_\beta^N(f)$ .
  - 2: Let  $\varepsilon = \frac{\tau_Y - \tau_N}{2(\tau_Y + \tau_N)}$ .
  - 3: Compute  $\tilde{B}$  to be a  $(1 \pm \varepsilon)$  approximation to  $B_\lambda(\Psi)$ , i.e.,  $(1 - \varepsilon)B_\lambda(\Psi) \leq \tilde{B} \leq (1 + \varepsilon)B_\lambda(\Psi)$  with probability at least  $2/3$ .
  - 4: **if**  $\tilde{B} \leq \tau_N(1 + \varepsilon)$  **then**  
**Output:** NO.
  - 5: **else**  
**Output:** YES.
- 

It is clear that the algorithm above runs in  $O(\log n)$  space (in particular by using [Proposition 4.2](#) for Step 3). We now turn to analyzing the correctness of the algorithm.

## 4.2 Analysis of the correctness of Algorithm 1

**Lemma 4.6.** *Algorithm 1 correctly solves  $(\gamma, \beta)$ -Max-CSP( $f$ ), if  $K_\gamma^Y(f)$  and  $K_\beta^N(f)$  are disjoint. Specifically, for every  $\Psi$ , let  $\tau_Y, \tau_N, \varepsilon, \boldsymbol{\lambda}, \tilde{B}$  be as given in [Algorithm 1](#), we have:*

$$\begin{aligned} \text{val}_\Psi \geq \gamma &\Rightarrow B_\lambda(\Psi) \geq \tau_Y \text{ and } \tilde{B} > \tau_N(1 + \varepsilon), \\ \text{and } \text{val}_\Psi \leq \beta &\Rightarrow B_\lambda(\Psi) \leq \tau_N \text{ and } \tilde{B} \leq \tau_N(1 + \varepsilon), \end{aligned}$$

provided  $(1 - \varepsilon)B_\lambda(\Psi) \leq \tilde{B} \leq (1 + \varepsilon)B_\lambda(\Psi)$ .

In the rest of this section, we will prove [Lemma 4.6](#). The key to our analysis is a distribution  $\mathcal{D}(\Psi^{\mathbf{a}}) \in \Delta(\{-1, 1\}^k)$  that we associate with every instance  $\Psi$  and assignment  $\mathbf{a} \in \{-1, 1\}^n$  to the variables of  $\Psi$ . Recall that in [Definition 2.2](#) we define  $\boldsymbol{\mu}(\mathcal{D}) = (\mu_1, \dots, \mu_k)$  where  $\mu_i = \mathbb{E}_{\mathbf{b} \sim \mathcal{D}}[b_i]$ . If  $\Psi$  is  $\gamma$ -satisfied by assignment  $\mathbf{a}$ , we prove that  $\boldsymbol{\mu}(\mathcal{D}(\Psi^{\mathbf{a}})) \in K_\gamma^Y$ . On the other hand, if  $\Psi$  is not  $\beta$ -satisfiable by any assignment, we prove that for every  $\mathbf{a}$ ,  $\boldsymbol{\mu}(\mathcal{D}(\Psi^{\mathbf{a}})) \in K_\beta^N$ . Finally we also show that the bias  $B_\lambda(\Psi)$  relates to  $\boldsymbol{\lambda}(\mathcal{D}(\Psi^{\mathbf{a}})) \triangleq \langle \boldsymbol{\mu}(\mathcal{D}(\Psi^{\mathbf{a}})), \boldsymbol{\lambda} \rangle$ , where the latter quantity is exactly what needs to be computed (by [Proposition 4.5](#)) to distinguish the membership of  $\boldsymbol{\mu}(\mathcal{D}(\Psi^{\mathbf{a}}))$  in  $K_\gamma^Y$  versus the membership in  $K_\beta^N$ .

We start with recalling some notations. For an instance  $\Psi = (C_1, \dots, C_m)$  on  $n$  variables with  $C_i = (\mathbf{j}(i), \mathbf{b}(i))$ , and an assignment  $\mathbf{a} \in \{-1, 1\}^n$ , let  $\Psi^{\mathbf{a}}$  denote the new instance obtained by flipping the variables according to  $\mathbf{a}$ . Specifically  $\Psi^{\mathbf{a}} = (C_1^{\mathbf{a}}, \dots, C_m^{\mathbf{a}})$  where  $C_i^{\mathbf{a}} = (\mathbf{j}(i), \mathbf{a}|_{\mathbf{j}(i)} \odot \mathbf{b}(i))$ .

Given instance  $\Psi$ , let  $\mathcal{D}(\Psi) \in \Delta(\{-1, 1\}^k)$  be the distribution obtained by sampling a constraint at random from  $\Psi$  and outputting the “negation pattern”. Formally to sample a random vector  $\mathbf{b} \sim \mathcal{D}(\Psi)$ , we sample  $i \in [m]$  uniformly and output  $\mathbf{b}(i)$  where  $C_i = (\mathbf{j}(i), \mathbf{b}(i))$ .

The next lemma relates the  $\boldsymbol{\lambda}$ -bias vector of  $\Psi$  to  $\boldsymbol{\lambda}(\mathcal{D}(\Psi^{\mathbf{a}}))$  and uses this to relate the bias of  $\Psi$  to the maximum over  $\mathbf{a}$  of  $\boldsymbol{\lambda}(\mathcal{D}(\Psi^{\mathbf{a}}))$ .

**Lemma 4.7.** *For every vector  $\mathbf{a} \in \{-1, 1\}^n$ , we have  $\boldsymbol{\lambda}(\mathcal{D}(\Psi^{\mathbf{a}})) = \langle \mathbf{a}, \text{bias}_\lambda(\Psi) \rangle$ . Consequently we have  $B_\lambda(\Psi) = \max_{\mathbf{a} \in \{-1, 1\}^n} \{\boldsymbol{\lambda}(\mathcal{D}(\Psi^{\mathbf{a}}))\}$ .*

*Proof.* We start with the first equality. Fix  $\mathbf{a} \in \{-1, 1\}^n$ . We have

$$\begin{aligned}
\boldsymbol{\lambda}(\mathcal{D}(\Psi^{\mathbf{a}})) &= \langle \boldsymbol{\mu}(\mathcal{D}(\Psi^{\mathbf{a}})), \boldsymbol{\lambda} \rangle \quad (\text{By definition of } \boldsymbol{\lambda}(\cdot)) \\
&= \mathbb{E}_{\mathbf{y} \sim \mathcal{D}(\Psi^{\mathbf{a}})} [\langle \mathbf{y}, \boldsymbol{\lambda} \rangle] \quad (\text{By definition of } \boldsymbol{\mu}(\mathcal{D}) \text{ and linearity of inner product}) \\
&= \mathbb{E}_{i \sim \text{Unif}\{[m]\}} [\langle \mathbf{b}^{\mathbf{a}}(i), \boldsymbol{\lambda} \rangle] \quad (\text{By definition of } \mathcal{D}(\Psi^{\mathbf{a}})) \\
&= \mathbb{E}_{i \sim \text{Unif}\{[m]\}} \left[ \sum_{t \in [k]} b^{\mathbf{a}}(i)_t \cdot \lambda_t \right] \quad (\text{Expanding the inner product}) \\
&= \frac{1}{m} \sum_{i \in [m]} \sum_{\ell \in [n]} \sum_{t \in [k]} \mathbb{1}_{j(i)_t = \ell} \cdot \lambda_t \cdot a_\ell \cdot b(i)_t \quad (\text{Using definition of } \Psi^{\mathbf{a}}) \\
&= \frac{1}{m} \sum_{\ell \in [n]} a_\ell \sum_{t \in [k]} \lambda_t \sum_{i \in [m]} \mathbb{1}_{j(i)_t = \ell} \cdot b(i)_t \quad (\text{Exchanging summations}) \\
&= \sum_{\ell \in [n]} a_\ell \cdot \text{bias}_{\boldsymbol{\lambda}}(\Psi)_\ell \quad (\text{By definition of } \text{bias}_{\boldsymbol{\lambda}}(\cdot)) \\
&= \langle \mathbf{a}, \text{bias}_{\boldsymbol{\lambda}}(\Psi) \rangle,
\end{aligned}$$

yielding the first equality.

The second part is immediate from the observation that for every vector  $\mathbf{v} \in \mathbb{R}^n$ , we have  $\|\mathbf{v}\|_1 = \max_{\mathbf{a} \in \{-1, 1\}^n} \langle \mathbf{a}, \mathbf{v} \rangle$  and so

$$B_{\boldsymbol{\lambda}}(\Psi) = \|\text{bias}_{\boldsymbol{\lambda}}(\Psi)\|_1 = \max_{\mathbf{a} \in \{-1, 1\}^n} \{\langle \mathbf{a}, \text{bias}_{\boldsymbol{\lambda}}(\Psi) \rangle\} = \max_{\mathbf{a} \in \{-1, 1\}^n} \{\boldsymbol{\lambda}(\mathcal{D}(\Psi^{\mathbf{a}}))\}.$$

□

We now turn to connecting  $\text{val}_{\Psi}$  to properties of  $\mathcal{D}(\Psi^{\mathbf{a}})$ .

**Lemma 4.8.** *For every  $\Psi$  and  $\mathbf{a}$ , if  $\text{val}_{\Psi}(\mathbf{a}) \geq \gamma$  then  $\mathcal{D}(\Psi^{\mathbf{a}}) \in S_{\gamma}^Y$ .*

*Proof.* Follows from the fact that

$$\mathbb{E}_{\mathbf{b} \sim \mathcal{D}(\Psi^{\mathbf{a}})} [f(\mathbf{b})] = \frac{1}{m} \sum_{i \in [m]} f(\mathbf{b}(i)) \odot \mathbf{a}|_{j(i)} = \frac{1}{m} \sum_{i \in [m]} C_i(\mathbf{a}) = \text{val}_{\Psi}(\mathbf{a}) \geq \gamma,$$

implying  $\mathcal{D}(\Psi^{\mathbf{a}}) \in S_{\gamma}^Y$ . □

**Lemma 4.9.** *For every  $\Psi$ , if  $\text{val}_{\Psi} \leq \beta$ , then for all  $\mathbf{a}$ , we have  $\mathcal{D}(\Psi^{\mathbf{a}}) \in S_{\beta}^N$ .*

*Proof.* We claim if  $\text{val}_{\Psi} \leq \beta$ , then  $\mathcal{D}(\Psi) \in S_{\beta}^N$ . This suffices to prove the lemma, since for every  $\mathbf{a} \in \{-1, 1\}^n$  we have  $\text{val}_{\Psi^{\mathbf{a}}} = \text{val}_{\Psi}$ . So if  $\text{val}_{\Psi} \leq \beta$  then  $\text{val}_{\Psi^{\mathbf{a}}} \leq \beta$  and so by the claim above applied to  $\Psi^{\mathbf{a}}$ , we have  $\mathcal{D}(\Psi^{\mathbf{a}}) \in S_{\beta}^N$ .

We prove the contrapositive, i.e., we assume  $\mathcal{D}(\Psi) \notin S_{\beta}^N$  and show this implies  $\text{val}_{\Psi} > \beta$ . Then there exists  $p \in [0, 1]$  such that  $\mathbb{E}_{\mathbf{b} \sim \mathcal{D}(\Psi)} \mathbb{E}_{\mathbf{c} \sim \text{Bern}(p)^k} [f(\mathbf{b} \odot \mathbf{c})] > \beta$ . But this implies, as we show

below, that if  $\sigma \sim \text{Bern}(p)^n$ , then  $\mathbb{E}_{\sigma \sim \text{Bern}(p)^n} [\text{val}_\Psi(\sigma)] > \beta$ . We have:

$$\begin{aligned}
\mathbb{E}_{\sigma \sim \text{Bern}(p)^n} [\text{val}_\Psi(\sigma)] &= \mathbb{E}_{\sigma \sim \text{Bern}(p)^n} \mathbb{E}_{i \sim \text{Unif}\{[m]\}} [C_i(\sigma)] \quad (\text{By definition of } \Psi) \\
&= \mathbb{E}_{\sigma \sim \text{Bern}(p)^n} \mathbb{E}_{i \sim \text{Unif}\{[m]\}} [f(\mathbf{b}(i) \odot \sigma|_{\mathbf{j}(i)})] \quad (\text{By definition of } C_i) \\
&= \mathbb{E}_{i \sim \text{Unif}\{[m]\}} \mathbb{E}_{\sigma|_{\mathbf{j}(i)} \sim \text{Bern}(p)^k} [f(\mathbf{b}(i) \odot \sigma|_{\mathbf{j}(i)})] \quad (\text{Exchanging summations}) \\
&= \mathbb{E}_{i \sim \text{Unif}\{[m]\}} \mathbb{E}_{\mathbf{c} \sim \text{Bern}(p)^k} [f(\mathbf{b}(i) \odot \mathbf{c})] \quad (\text{Renaming variables}) \\
&= \mathbb{E}_{\mathbf{b} \sim \mathcal{D}(\Psi)} \mathbb{E}_{\mathbf{c} \sim \text{Bern}(p)^k} [f(\mathbf{b} \odot \mathbf{c})] \quad (\text{By definition of } \mathcal{D}(\Psi)) \\
&> \beta \quad (\text{By the contrapositive assumption})
\end{aligned}$$

Since  $\text{val}_\Psi \triangleq \max_{\sigma} \{\text{val}_\Psi(\sigma)\} \geq \mathbb{E}_{\sigma \sim \text{Bern}(p)^n} [\text{val}_\Psi(\sigma)]$  we get a contradiction to  $\text{val}_\Psi \leq \beta$ . This concludes the proof of the claim and hence the lemma.  $\square$

Before turning to the proof of [Lemma 4.6](#), we first do a quick post-mortem of the proof above. The proof above is the key reason why the definition of  $S_\beta^N$  is chosen as it is: In particular from the fact that there was an i.i.d. distribution, namely  $\text{Bern}(p)^k$ , according to which a random assignment satisfied the ‘‘instance’’ underlying  $\mathcal{D}(\Psi)$  with value more than  $\beta$  allowed us to extend this to a (again i.i.d., but this was not necessary) distribution over assignments to  $\Psi$  that also achieved value of at least  $\beta$ . Note that the mere existence of an assignment of value greater than  $\beta$  on  $\mathcal{D}(\Psi)$  would have been insufficient for this step to go through, explaining our choice of definition of  $S_\beta^N$ .

We are now ready to prove [Lemma 4.6](#).

*Proof of Lemma 4.6.* Let  $\text{val}_\Psi \geq \gamma$ . Then there exists  $\mathbf{a} \in \{-1, 1\}^n$  such that  $\text{val}_\Psi(\mathbf{a}) \geq \gamma$ . By [Lemma 4.8](#), we have that  $\mathcal{D}(\Psi^{\mathbf{a}}) \in S_\gamma^Y$ . By our choice of  $\lambda$ , we have  $\lambda(\mathcal{D}) \geq \tau_Y$  for every  $\mathcal{D} \in S_\gamma^Y$  and so in particular we have  $\lambda(\mathcal{D}(\Psi^{\mathbf{a}})) \geq \tau_Y$ . By [Lemma 4.7](#) we have  $B_\lambda(\Psi) = \max_{\mathbf{c} \in \{-1, 1\}^n} \{\lambda(\mathcal{D}(\Psi^{\mathbf{c}}))\}$ . Putting these together we have

$$B_\lambda(\Psi) = \max_{\mathbf{c} \in \{-1, 1\}^n} \{\lambda(\mathcal{D}(\Psi^{\mathbf{c}}))\} \geq \lambda(\mathcal{D}(\Psi^{\mathbf{a}})) \geq \tau_Y.$$

Finally, since  $\tilde{B} \geq (1 - \varepsilon)B_\lambda(\Psi)$ , we get  $\tilde{B} \geq (1 - \varepsilon)\tau_Y \geq (1 + \varepsilon)\tau_N$ , where the final inequality holds by our choice of  $\varepsilon$ .

The case  $\text{val}_\Psi \leq \beta$  is similar. In this case, by [Lemma 4.9](#) we have  $\mathcal{D}(\Psi^{\mathbf{a}}) \in S_\beta^N$  for every  $\mathbf{a}$ . Now applying [Lemma 4.7](#) we get that for every  $\mathbf{a}$ ,  $\langle \mathbf{a}, \text{bias}_\lambda \rangle = \lambda(\mathcal{D}(\Psi^{\mathbf{a}})) \leq \tau_N$ . We conclude that  $B_\lambda(\Psi) = \max_{\mathbf{a} \in \{-1, 1\}^n} \{\langle \mathbf{a}, \text{bias}_\lambda \rangle\} \leq \tau_N$ . And so since  $\tilde{B} \leq (1 + \varepsilon)B_\lambda(\Psi)$ , we get  $\tilde{B} \leq (1 + \varepsilon)\tau_N$ .  $\square$

We now conclude the section with a formal proof of [Theorem 4.1](#).

*Proof of Theorem 4.1.* The desired algorithm is Algorithm 1. Its space complexity is bounded by the space required for Step 3, which by [Lemma 4.4](#) is  $O(\log n)$ . Assuming Step 3 works correctly, which happens with probability at least  $2/3$ , [Lemma 4.6](#) shows that it correctly solves  $(\gamma, \beta)$ -Max-CSP( $f$ ) whenever  $K_\gamma^Y(f) \cap K_\beta^N(f) = \emptyset$ .  $\square$

## 5 Streaming Space Lower Bound from Communication Games

In this section, we prove the following theorem, which is simply a restatement of the “hard” part of [Theorem 2.3](#).

**Theorem 5.1.** *For every function  $f : \{-1, 1\}^k \rightarrow \{0, 1\}$  and for every  $0 \leq \beta < \gamma \leq 1$ , if  $K_\gamma^Y(f) \cap K_\beta^N(f) \neq \emptyset$ , then for every  $\varepsilon > 0$ ,  $(\gamma - \varepsilon, \beta + \varepsilon)$ -Max-CSP( $f$ ) requires  $\Omega(\sqrt{n})$  space<sup>8</sup>. Furthermore, if  $\gamma = 1$ , then  $(1, \beta + \varepsilon)$ -Max-CSP( $f$ ) requires  $\Omega(\sqrt{n})$  space.*

To prove this theorem, we introduce the *Randomized Mask Detection (RMD)* communication game below. We then state a lower bound for the communication complexity of this game ([Theorem 5.3](#)), and use the lower bound to prove [Theorem 5.1](#). The proof of [Theorem 5.3](#) appears in [Section 7](#).

### 5.1 2-Player Communication Games and The Randomized Mask Detection Problem

In this section, and most of the rest of this paper, we will be considering the complexity of 2-player 1-way communication games. Broadly such games are described by two (parametrized set of) distributions  $\mathcal{Y}$  and  $\mathcal{N}$ . An instance of the game is a pair  $(X, Y)$  either drawn from  $\mathcal{Y}$  or from  $\mathcal{N}$  and  $X$  is given as input to Alice and  $Y$  to Bob. A (one-way communication) protocol  $\Pi = (\Pi_A, \Pi_B)$  is a pair of functions with  $\Pi_A(X) \in \{0, 1\}^c$  denoting Alice’s message to Bob, and  $\Pi_B(\Pi_A(X), Y) \in \{\mathbf{YES}, \mathbf{NO}\}$  denoting the protocol’s output. We denote this output by  $\Pi(X, Y)$ . The complexity of this protocol is the parameter  $c$  specifying the length of  $\Pi_A(X)$  (maximized over all  $X$ ). The advantage of the protocol  $\Pi$  is the quantity

$$\left| \Pr_{(X,Y) \sim \mathcal{Y}}[\Pi(X, Y) = \mathbf{YES}] - \Pr_{(X,Y) \sim \mathcal{N}}[\Pi(X, Y) = \mathbf{YES}] \right|.$$

The Randomized Mask Detection (RMD) communication game is an instance of such a communication game. Let  $n, k \in \mathbb{N}$  and  $\alpha \in (0, 1)$  with  $k \leq n$  and  $\alpha k \leq 1$ . Alice receives a private input  $\mathbf{x}^*$  drawn uniformly at random from  $\{-1, 1\}^n$  while Bob receives private inputs a  $k$ -uniform hypermatching of size  $\alpha n$  and a vector  $\mathbf{z} \in \{-1, 1\}^{\alpha k n}$  of the form  $\mathbf{z} = (\mathbf{z}(1), \dots, \mathbf{z}(\alpha n))$  where  $\mathbf{z}(i) \in \{-1, 1\}^k$  for each  $i \in [\alpha n]$ . Alice’s input  $\mathbf{x}^*$  encodes a random bipartition of the vertex set according to the  $\pm 1$  pattern. Bob’s  $k$ -uniform hypermatching is encoded by a matrix  $M \in \{0, 1\}^{\alpha k n \times n}$  where the  $(k(i-1) + 1)$ -th to the  $(ki)$ -th rows encode the  $i$ -th hyperedge by putting exactly one 1 in each row to the corresponding vertices. During the game, Alice sends a message to Bob and Bob has to discover the hidden structure of the vector  $\mathbf{z}$ . The following formally describes the problem.

**Definition 5.2** (Randomized Mask Detection (RMD) Problem). *For  $k \in \mathbb{N}$ ,  $\alpha \in (0, 1/k]$  and a pair of distributions  $\mathcal{D}_Y, \mathcal{D}_N \in \Delta(\{-1, 1\}^k)$ , the  $(\mathcal{D}_Y, \mathcal{D}_N; \alpha, k)$ -RMD problem is the 2-player communication game given by a family of instances  $(\mathcal{Y}_n, \mathcal{N}_n)_{n \in \mathbb{N}, n \geq 1/\alpha}$  where for a given  $n$ ,  $\mathcal{Y} = \mathcal{Y}_n$  and  $\mathcal{N} = \mathcal{N}_n$  are as follows: Both  $\mathcal{Y}$  and  $\mathcal{N}$  are supported on triples  $(\mathbf{x}^*, M, \mathbf{z})$  where  $\mathbf{x}^* \in \{-1, 1\}^n$ ,  $M \in \{0, 1\}^{k \alpha n \times n}$  and  $\mathbf{z} \in \{-1, 1\}^{k \alpha n}$ , where  $\mathbf{x}^*$  is Alice’s input and the pair  $(M, \mathbf{z})$  are Bob’s inputs. We now specify the distributions of  $\mathbf{x}^*, M$  and  $\mathbf{z}$  in  $\mathcal{Y}$  and  $\mathcal{N}$ :*

- In both  $\mathcal{Y}$  and  $\mathcal{N}$ ,  $\mathbf{x}^*$  is distributed uniformly over  $\{-1, 1\}^n$ .

<sup>8</sup>The constant hidden in the  $\Omega$  notation may depend on  $k$  and  $\varepsilon$ .

- In both  $\mathcal{Y}$  and  $\mathcal{N}$  the matrix  $M \in \{0, 1\}^{\alpha kn \times n}$  is chosen uniformly (and independently of  $\mathbf{x}^*$ ) among matrices with exactly one 1 per row and at most one 1 per column. (Thus  $M$  represents a  $k$ -hypermatching where each block of  $k$  rows describes a hyperedge.)
- The vector  $\mathbf{z}$  is obtained by “masking” (i.e., xor-ing)  $M\mathbf{x}^*$  by a random vector  $\mathbf{b} \in \{-1, 1\}^{\alpha kn}$  whose distribution differs in  $\mathcal{Y}$  and  $\mathcal{N}$ . Specifically let  $\mathbf{b} = (\mathbf{b}(1), \dots, \mathbf{b}(\alpha n))$  be sampled from one of the following distributions (independent of  $\mathbf{x}^*$  and  $M$ ):

- $\mathcal{Y}$ : Each  $\mathbf{b}(i) \in \{-1, 1\}^k$  is sampled independently according to  $\mathcal{D}_Y$ .
- $\mathcal{N}$ : Each  $\mathbf{b}(i) \in \{-1, 1\}^k$  is sampled independently according to  $\mathcal{D}_N$ .

We now set  $\mathbf{z} = (M\mathbf{x}^*) \odot \mathbf{b}$  (recall that  $\odot$  denotes coordinatewise product).

We will typically suppress  $k$  and  $\alpha$  from the notation when they are clear from context and simply refer to the  $(\mathcal{D}_Y \mathcal{D}_N)$ -RMD. We will refer to  $n$  as the length parameter or refer to “instances of length  $n$ ” when the instances are drawn from  $\mathcal{Y}_n$  vs.  $\mathcal{N}_n$ . The goal of a protocol solving RMD is to distinguish between case where the masks are sampled from  $\mathcal{D}_Y$  from the case where the masks are sampled from  $\mathcal{D}_N$  and advantage measures this probability of distinguishing.

We note that our communication game is slightly different from those in previous works: Specifically the problem studied in [GKK<sup>+</sup>09, KKS15] is called the *Boolean Hidden Matching (BHM)* problem from [GKK<sup>+</sup>09] and the works [KKS17, KK19] study a variant called the *Implicit Hidden Partition* problem. While these problems are similar, they are less expressive than our formulation, and specifically do not seem to capture the many different all Max-CSP( $f$ ) problems.

There are two main differences between the previous settings and our setting. The first difference is the way to encode the matching matrix  $M$ . In all the previous works, each edge (or hyperedge) is encoded by a single row in  $M$  where the corresponding columns are assigned to 1, so that  $m = \alpha n$ . However, it turns out that this encoding hides too much information and hence we do not know how to reduce the problem to general Max-CSP. We unfold the encoding by using  $k$  rows to encode a single  $k$ -hyperedge (leading to the setting of  $m = k\alpha n$  in our case). The second difference is that we allow the masking vector  $\mathbf{b}$  to be sampled from a more general distribution. This is also for the purpose of establishing a reduction to general Max-CSP. That being said, it is possible to describe some of the previous results in our language: all the papers consider the complexity of distinguishing the distribution  $\mathcal{D}_Y = \text{Unif}(\{(1, 1), (-1, -1)\})$  from the distribution  $\mathcal{D}_N = \text{Unif}(\{-1, 1\}^2)$ . This problem is shown to have a communication lower bound of  $\Omega(\sqrt{n})$  in [GKK<sup>+</sup>09]. And a variant of this problem (not captured by our formulation above) is shown to have an  $\Omega(n)$  lower bound in [KK19].

Due to the above two differences, it is not clear how to derive communication lower bounds for general  $\mathcal{D}_Y$  and  $\mathcal{D}_N$  by reduction from the previous works. The main technical contribution of this part of the paper is a communication lower bound for RMD for general  $\mathcal{D}_Y$  and  $\mathcal{D}_N$ . We summarize the result in the following theorem.

**Theorem 5.3** (RMD Lower bound for distributions with matching marginals). *For every  $k \in \mathbb{N}$ , there exists  $\alpha_0 > 0$  such that for every  $\alpha \in (0, \alpha_0)$  and  $\delta > 0$  the following holds: For every pair of distributions  $\mathcal{D}_Y, \mathcal{D}_N \in \Delta(\{-1, 1\}^k)$  with  $\boldsymbol{\mu}(\mathcal{D}_Y) = \boldsymbol{\mu}(\mathcal{D}_N)$  there exists  $\tau > 0$  and  $n_0$  such that for every  $n \geq n_0$ , every protocol for  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD achieving advantage  $\delta$  on instances of length  $n$  requires  $\tau\sqrt{n}$  bits of communication.*

We prove [Theorem 5.3](#) in two parts. First, in [Section 6](#), we prove a communication lower bound for the special case where the marginals of  $\mathcal{D}_Y$  and  $\mathcal{D}_N$  are all zero. While this captures many new cases, it fails to capture the more interesting scenarios (involving non-approximation resistant problems). To get lower bounds for the general case, we reduce the 0-marginal case to the general case in [Section 7](#).

In the rest of this section, we use [Theorem 5.3](#) to prove the main theorem ([Theorem 5.1](#)) of this section. We first perform a standard step on bootstrapping the number of hyperedges (which corresponds to the number of clauses in Max-CSP) in [Section 5.2](#). Next, we present the reduction to Max-CSP in [Section 5.3](#). Finally, we wrap up the proof for [Theorem 5.1](#) in [Section 5.4](#).

## 5.2 Streaming (in)distinguishability games and hardness

The hardness of RMD suggests a natural path for hardness of  $\text{Max-CSP}(f)$  problems in the streaming model. Such a reduction would take two distributions  $\mathcal{D}_Y \in S_\gamma^Y$  and  $\mathcal{D}_N \in S_\beta^N$  with matching marginals, construct distributions  $\mathcal{Y}$  and  $\mathcal{N}$  of RMD, and then interpret these distributions (in a natural way) as distributions over instances of  $\text{Max-CSP}(f)$  that are indistinguishable to small space algorithms. While the exact details of this “interpretation” need to be spelled out, every step in this path can be achieved. Unfortunately this does not mean any hardness for  $\text{Max-CSP}(f)$  since the CSPs generated by this reduction would consist of instances that have at most one constraint per variable! Indeed to go from the hardness of RMD to hardness of CSPs we need the hardness of distinguishing a  $T$ -fold concatenation of streams drawn according to  $\mathcal{Y}$  from a  $T$ -fold concatenation of streams drawn according to  $\mathcal{N}$ . (The concatenation now allows us to appeal to the membership  $\mathcal{D}_Y \in S_\gamma^Y$  to conclude that instances  $\Psi$  drawn from  $\mathcal{Y}^T$  have high  $\text{val}_\Psi$  whereas for instances  $\Psi$  drawn from  $\mathcal{N}^T$ , the fact that  $\mathcal{D}_N \in S_\beta^N$  will imply that  $\text{val}_\Psi$  is low for large but constant  $T$ .)

In what follows we define the  $T$ -fold concatenated streaming problem associated with  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD, which we call the  $(\mathcal{D}_Y, \mathcal{D}_N, T)$ -streaming-RMD, formally. We then show that this problem remains indistinguishable which allows us to implement the plan alluded to above. We note this part of our reduction is standard in prior works. In particular we follow the presentation in [[CGV20](#)].

The general framework defines two distributions  $\mathcal{Y}_{\text{stream}}$  and  $\mathcal{N}_{\text{stream}}$  over streams. A streaming algorithm **ALG** processes the streams with space  $s$  and is required to output a verdict in  $\{\text{YES}, \text{NO}\}$ . The advantage of **ALG** is defined as usual to be  $|\Pr_{\sigma \sim \mathcal{Y}_{\text{stream}}}[\text{ALG}(\sigma) = \text{YES}] - \Pr_{\sigma \sim \mathcal{N}_{\text{stream}}}[\text{ALG}(\sigma) = \text{YES}]|$ .

**Definition 5.4** ( $(\mathcal{D}_Y, \mathcal{D}_N, T)$ -streaming-RMD). *For  $k, T \in \mathbb{N}$ ,  $\alpha \in (0, 1/k]$ , distributions  $\mathcal{D}_Y, \mathcal{D}_N$  over  $\{-1, 1\}^k$ , the streaming problem  $(\mathcal{D}_Y, \mathcal{D}_N, T; \alpha, k)$ -streaming-RMD is the task of distinguishing, for every  $n$ ,  $\sigma \sim \mathcal{Y}_{\text{stream}, n}$  from  $\sigma \sim \mathcal{N}_{\text{stream}, n}$  where for a given length parameter  $n$ , the distributions  $\mathcal{Y}_{\text{stream}} = \mathcal{Y}_{\text{stream}, n}$  and  $\mathcal{N}_{\text{stream}} = \mathcal{N}_{\text{stream}, n}$  are defined as follows:*

- *Let  $\mathcal{Y}$  be the distribution over instances of length  $n$ , i.e., triples  $(\mathbf{x}^*, M, \mathbf{z})$ , from the definition of  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD. For  $\mathbf{x} \in \{-1, 1\}^n$ , let  $\mathcal{Y}|_{\mathbf{x}}$  denote the distribution  $\mathcal{Y}$  conditioned on  $\mathbf{x}^* = \mathbf{x}$ . The stream  $\sigma \sim \mathcal{Y}_{\text{stream}}$  is sampled as follows: Sample  $\mathbf{x}^*$  uniformly from  $\{-1, 1\}^n$ . Let  $(M^{(1)}, \mathbf{z}^{(1)}), \dots, (M^{(T)}, \mathbf{z}^{(T)})$  be sampled independently according to  $\mathcal{Y}|_{\mathbf{x}^*}$ . Let  $\sigma^{(t)}$  be the pair  $(M^{(t)}, \mathbf{z}^{(t)})$  presented as a stream of edges with labels in  $\{-1, 1\}^k$ . Specifically for  $t \in [T]$  and  $i \in [\alpha n]$ , let  $\sigma^{(t)}(i) = (e^t(i), \mathbf{z}^{(t)}(i))$  where  $e^t(i)$  is the  $i$ th hyperedge of  $M^t$ , i.e.,  $e^t(i) = (j^t(k(i-1)+1), \dots, j^t(k(i-1)+k))$  and  $j^t(\ell)$  is the unique index  $j$  such that  $M_{j,\ell}^{(t)} = 1$ . Finally we let  $\sigma = \sigma^{(1)} \circ \dots \circ \sigma^{(T)}$  be the concatenation of the  $\sigma^{(t)}$ s.*



- $\sigma \sim \mathcal{N}_{stream}$  is sampled similarly except we now sample  $(M^{(1)}, \mathbf{z}^{(1)}), \dots, (M^{(T)}, \mathbf{z}^{(T)})$  independently according to  $\mathcal{N}|_{\mathbf{x}^*}$  where  $\mathcal{N}|_{\mathbf{x}}$  is the distribution  $\mathcal{N}$  condition on  $\mathbf{x}^* = \mathbf{x}$ .

Again when  $\alpha$  and  $k$  are clear from context we suppress them and simply refer to the  $(\mathcal{D}_Y, \mathcal{D}_N, T)$ -streaming-RMD problem.

**Lemma 5.5.** *Let  $T, k \in \mathbb{N}$ ,  $\mathcal{D}_Y, \mathcal{D}_N$  be two distributions over  $\{-1, 1\}^k$  and let  $\alpha \in (0, 1/k]$ . Suppose that a streaming algorithm **ALG** solves  $(\mathcal{D}_Y, \mathcal{D}_N, T)$ -streaming-RMD on instances of length  $n$  with advantage  $\Delta$  and space  $s$ , then there is a one-way protocol for  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD on instances of length  $n$  using at most  $sT$  bits of communication achieving advantage at least  $\Delta/T$ .*

The proof of [Lemma 5.5](#) is based on a hybrid argument (e.g., [\[KKS15, Lemma 6.3\]](#)). We provide a proof here based on the proof of [\[CGV20, Lemma 4.11\]](#). (We note that previous lemmas of this form only considered the case where  $\mathcal{D}_N$  is the uniform distribution, and the proofs used some special properties of this setting. Generalizing it to arbitrary  $\mathcal{D}_N$  involves a little extra care as we do below.) Later, in [Section 5.3](#), we show a reduction from  $(\mathcal{D}_Y, \mathcal{D}_N, T)$ -streaming-RMD to Max-CSP( $f$ ) thus completing the objective of this section.

*Proof of Lemma 5.5.* Note that since we are interested in distributional advantage, we can fix the randomness in **ALG** so that it becomes a deterministic algorithm. By an averaging argument the randomness can be chosen to ensure the advantage does not decrease. Let  $\Gamma$  denote the evolution of function of **ALG** as it processes a block of  $\alpha n$  edges. I.e., if the algorithm is in state  $s$  and receives a stream  $\sigma$  of length  $\alpha n$  then it ends in state  $\Gamma(s, \sigma)$ . Let  $s_0$  denote its initial state.

We consider the following collection of (jointly distributed) random variables: Let  $\mathbf{x}^* \sim \text{Unif}(\{-1, 1\}^n)$ . Let  $\sigma_Y^{(1)}, \dots, \sigma_Y^{(T)} \sim \mathcal{Y}|_{\mathbf{x}^*}$  be chosen independently conditioned on  $\mathbf{x}^*$ . Similarly, let  $\sigma_N^{(1)}, \dots, \sigma_N^{(T)} \sim \mathcal{N}|_{\mathbf{x}^*}$  be chosen independently conditioned on  $\mathbf{x}^*$ . Let  $S_t^Y$  denote the state of **ALG** after processing  $\sigma_Y^{(1)}, \dots, \sigma_Y^{(t)}$ , i.e.,  $S_0^Y = s_0$  and  $S_t^Y = \Gamma(S_{t-1}^Y, \sigma_Y^{(t)})$ . Similarly let  $S_t^N$  denote the state of **ALG** after processing  $\sigma_N^{(1)}, \dots, \sigma_N^{(t)}$ .

Let  $S_{a:b}^Y$  denote the sequence of states  $(S_a^Y, \dots, S_b^Y)$  and similarly for  $S_{a:b}^N$ . Now let  $\Delta_t = \|S_{0:t}^Y - S_{0:t}^N\|_{tvd}$ . Observe that  $\Delta_0 = 0$  while  $\Delta_T \geq \Delta$ . (The latter is based on the fact that **ALG** distinguishes the two distributions with advantage  $\Delta$ .) Thus  $\Delta \leq \Delta_T - \Delta_0 = \sum_{t=0}^{T-1} (\Delta_{t+1} - \Delta_t)$  and so there exists  $t^* \in \{0, 1, \dots, T-1\}$  such that

$$\Delta_{t^*+1} - \Delta_{t^*} = \|S_{0:t^*+1}^Y - S_{0:t^*+1}^N\|_{tvd} - \|S_{0:t^*}^Y - S_{0:t^*}^N\|_{tvd} \geq \frac{\Delta}{T}.$$

Now consider the random variable  $\tilde{S} = \Gamma(S_{t^*}^Y, \sigma_N^{(t^*+1)})$  (so the previous state is from the **YES** distribution and the input from the **NO** distribution). We claim below that  $\mathbb{E}_{A \sim_d S_{0:t^*}^Y} [\|S_{t^*+1}^Y |_{S_{0:t^*}^Y = A} - \tilde{S} |_{S_{0:t^*}^Y = A}\|_{tvd}] \geq \Delta_{t^*+1} - \Delta_{t^*}$ . Once we have the claim, we show how to get a space  $Ts$  protocol for  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD with advantage  $\Delta_{t^*+1} - \Delta_{t^*}$  concluding the proof of the lemma.

**Claim 5.6.**  $\mathbb{E}_{A \sim_d S_{0:t^*}^Y} [\|S_{t^*+1}^Y |_{S_{0:t^*}^Y = A} - \tilde{S} |_{S_{0:t^*}^Y = A}\|_{tvd}] \geq \Delta_{t^*+1} - \Delta_{t^*}$ .

*Proof.* We use the following equivalent definition of total variation distance. Random variables  $X$  and  $Y$  satisfy  $\|X - Y\|_{tvd} \leq \tau$  if and only if there exists a coupling distribution  $\mathcal{D}_{\text{couple}}$  such that  $(\tilde{X}, \tilde{Y}) \sim \mathcal{D}_{\text{couple}}$  satisfy (1)  $\tilde{X} \sim_d X$ , (2)  $\tilde{Y} \sim_d Y$  and (3)  $\Pr[\tilde{X} \neq \tilde{Y}] \leq \tau$ .

Since  $\|S_{0:t^*}^Y - S_{0:t^*}^N\|_{tvd} \leq \Delta_{t^*}$  we have  $\mathcal{D}_{\text{couple}}$  such that  $(A, B) \sim \mathcal{D}_{\text{couple}}$  satisfy  $A \sim_d S_{0:t^*}^Y$ ,  $B \sim_d S_{0:t^*}^N$ , and  $\Pr[A \neq B] \leq \Delta_{t^*}$ .

Now assume the claim is not true. Then for every  $A$  we have a coupling distribution  $\mathcal{D}_A^*$  such that for  $(X, Y) \sim \mathcal{D}_A^*$  we have  $X \sim_d S_{t^*+1}^Y |_{S_{0:t^*}^Y=A}$ ,  $Y \sim_d \tilde{S} |_{S_{0:t^*}^Y=A}$ , and  $\mathbb{E}_{A \sim_d S_{0:t^*}^Y} [\Pr_{(X,Y) \sim \mathcal{D}_A^*} [X \neq Y]] < \Delta_{t^*+1} - \Delta_{t^*}$ .

We now describe a distribution  $\tilde{\mathcal{D}}_{\text{couple}}$  coupling  $S_{0:t^*+1}^Y$  and  $S_{0:t^*+1}^N$  showing their total variation distance is less than  $\Delta_{t^*+1}$  thus achieving a contradiction. We describe the sampling procedure sampling  $(\tilde{A}, \tilde{B}) \sim \tilde{\mathcal{D}}_{\text{couple}}$ : We first sample  $(A, B) \sim \mathcal{D}_{\text{couple}}$ . If  $A = B$  we sample  $(X, Y) \sim \mathcal{D}_A^*$ . Else we sample  $X \sim_d S_{t^*+1}^Y |_{S_{0:t^*}^Y=A}$  and (independently)  $Y \sim_d S_{t^*+1}^N |_{S_{0:t^*}^N=B}$ . We let  $\tilde{A} = (A, X)$  and  $\tilde{B} = (B, Y)$ . It is easy to verify that  $\tilde{A} \sim_d S_{0:t^*+1}^Y$  and  $\tilde{B} \sim_d S_{0:t^*+1}^N$ . Finally note that the probability that  $\tilde{A} \neq \tilde{B}$  is upper bounded by

$$\begin{aligned} \Pr_{(\tilde{A}, \tilde{B}) \sim \tilde{\mathcal{D}}_{\text{couple}}} [\tilde{A} \neq \tilde{B}] &= \Pr_{((A,X),(B,Y)) \sim \tilde{\mathcal{D}}_{\text{couple}}} [A \neq B] + \Pr_{((A,X),(B,Y)) \sim \tilde{\mathcal{D}}_{\text{couple}}} [(A = B) \text{ and } X \neq Y] \\ &\leq \Pr_{(A,B) \sim \mathcal{D}_{\text{couple}}} [A \neq B] + \Pr_{(A,B) \sim \mathcal{D}_{\text{couple}}, (X,Y) \sim \mathcal{D}_A^*} [X \neq Y] \\ &< \Delta_{t^*} + (\Delta_{t^*+1} - \Delta_{t^*}) \\ &< \Delta_{t^*+1} \end{aligned}$$

which implies  $\|S_{0:t^*+1}^Y - S_{0:t^*+1}^N\|_{\text{tvd}} < \Delta_{t^*+1}$  and hence contradicts to the definition of  $\Delta_{t^*+1}$ .  $\square$

We now show how a protocol can be designed for  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD that achieves advantage at least  $\theta = \mathbb{E}_{A \sim_d S_{0:t^*}^Y} [\|S_{t^*+1}^Y |_{S_{0:t^*}^Y=A} - \tilde{S} |_{S_{0:t^*}^Y=A}\|_{\text{tvd}}] \geq \Delta_{t^*+1} - \Delta_{t^*}$  concluding the proof of the lemma. The protocol uses the distinguisher  $T_A : \{0, 1\}^s \rightarrow \{0, 1\}$  such that  $\mathbb{E}_{A, S_{t^*+1}^Y, \tilde{S}} [T_A(S_{t^*+1}^Y)] - \mathbb{E}[T_A(\tilde{S})] \geq \theta$  which is guaranteed to exist by the definition of total variation distance.

Our protocol works as follows: Let Alice receive input  $\mathbf{x}^*$  and Bob receive inputs  $(M, \mathbf{z})$  sampled from either  $\mathcal{Y}|_{\mathbf{x}^*}$  or  $\mathcal{N}|_{\mathbf{x}^*}$ .

1. Alice samples  $\sigma^{(1)}, \dots, \sigma^{(t^*)} \sim \mathcal{Y}|_{\mathbf{x}^*}$  independently and computes  $A = S_{0:t^*}^Y \in \{0, 1\}^{(t^*+1)s}$  and sends  $A$  to Bob.
2. Bob extracts  $S_{t^*}^Y$  from  $A$ , computes  $\hat{S} = \Gamma(S_{t^*}^Y, \sigma)$ , where  $\sigma$  is the encoding of  $(M, \mathbf{z})$  as a stream, and outputs **YES** if  $T_A(\hat{S}) = 1$  and **NO** otherwise.

Note that if  $(M, \mathbf{z}) \sim \mathcal{Y}|_{\mathbf{x}^*}$  then  $\hat{S} \sim_d S_{t^*+1}^Y |_{S_{0:t^*}^Y=A}$  while if  $(M, \mathbf{z}) \sim \mathcal{N}|_{\mathbf{x}^*}$  then  $\hat{S} \sim \tilde{S} |_{S_{0:t^*}^Y=A}$ . It follows that the advantage of the protocol above exactly equals  $\mathbb{E}_A [T_A(S_{t^*+1}^Y)] - \mathbb{E}_A [T_A(\tilde{S})] \geq \theta \geq \Delta_{t^*+1} - \Delta_{t^*} \geq \Delta/T$ . This concludes the proof of the lemma.  $\square$

By combining [Lemma 5.5](#) with [Theorem 5.3](#), we immediately have the following corollary.

**Lemma 5.7.** *For  $k \in \mathbb{N}$  let  $\alpha_0(k)$  be as given by [Theorem 5.3](#). Let  $T, k \in \mathbb{N}$ ,  $\alpha \in (0, \alpha_0(k))$ , and  $\mathcal{D}_Y, \mathcal{D}_N$  be two distributions over  $\{-1, 1\}^k$  with  $\mu(\mathcal{D}_Y) = \mu(\mathcal{D}_N)$ . Then every streaming algorithm **ALG** solving  $(\mathcal{D}_Y, \mathcal{D}_N, T)$ -streaming-RMD with advantage  $1/8$  for all lengths uses space  $\Omega(\sqrt{n})$ .*

*Proof.* We get the lemma by combining [Lemma 5.5](#) and [Theorem 5.3](#). Let **ALG** be an algorithm using space  $s$  solving  $(\mathcal{D}_Y, \mathcal{D}_N, T)$ -streaming-RMD with advantage  $1/8$ . Then by [Lemma 5.5](#), there exists a one-way protocol for  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD using at most  $sT$  bits of communication with advantage at least  $1/(8T)$ . Applying [Theorem 5.3](#) with  $\delta = 1/(8T) > 0$ , we now get that  $s = \Omega(\sqrt{n})$ .  $\square$

### 5.3 Reduction from $(\mathcal{D}_Y, \mathcal{D}_N, T)$ -streaming-RMD to approximating Max-CSP( $f$ )

We now complete the sequence of reductions from RMD to approximating Max-CSP( $f$ ) by reducing streaming-RMD to Max-CSP( $f$ ). To this end note that an instance  $\sigma$  of streaming-RMD is a sequence  $(\sigma(1), \dots, \sigma(m))$  where each  $\sigma(i) = (\mathbf{j}(i), \mathbf{z}(i))$  with  $\mathbf{j}(i) \in [n]^k$  and  $\mathbf{z}(i) \in \{-1, 1\}^k$  is already syntactically very close to the description of a Max-CSP( $f$ ) instance. The only missing ingredient is any reference to the function  $f$  itself! Indeed the reduction from streaming-RMD to Max-CSP( $f$ ) involves just applying this function  $f$  to the literals indicated by  $\sigma(i)$ .

Given an instance  $\sigma = (\sigma(1), \dots, \sigma(m))$  of streaming-RMD, let  $\Psi(\sigma)$  denote the instance of Max-CSP( $f$ ) on variables  $\mathbf{x} = (x_1, \dots, x_n)$  with the constraints  $C_1, \dots, C_m$  with  $C_i = \sigma(i) = (\mathbf{j}(i), \mathbf{z}(i))$  is the constraint satisfied if  $f(\mathbf{z}(i) \odot \mathbf{x}|_{\mathbf{j}(i)}) = 1$ .

In what follows we show that if  $\mathcal{D}_Y \in S_\gamma^Y$  then for all sufficiently large constant  $T$  and sufficiently large  $n$  if we draw  $\sigma \sim \mathcal{Y}_{\text{stream}, n}$  then with high probability  $\Psi(\sigma)$  has value at least  $\gamma - o(1)$ . Conversely if  $\mathcal{D}_N \in S_\beta^N$  then for all sufficiently large  $n$  if we draw  $\sigma \sim \mathcal{N}_{\text{stream}, n}$  then with high probability  $\Psi(\sigma)$  has value at most  $\beta + o(1)$ . This essentially completes our reduction to Max-CSP( $f$ ).

**Lemma 5.8.** *For every  $k \in \mathbb{N}$ ,  $f : \{-1, 1\}^k \rightarrow \{0, 1\}$ ,  $0 \leq \beta < \gamma \leq 1$ ,  $\varepsilon > 0$ , distributions  $\mathcal{D}_Y, \mathcal{D}_N \in \Delta(\{-1, 1\}^k)$ ,  $\alpha \in (0, 1/(100k))$ , there exists an integer  $T_0$  such that for every  $T \geq T_0$  the following holds:*

1. *If  $\mathcal{D}_Y \in S_\gamma^Y$  then for every sufficiently large  $n$ , the  $(\mathcal{D}_Y, \mathcal{D}_N, T)$ -streaming-RMD **YES** instance  $\sigma \sim \mathcal{Y}_{\text{stream}, n}$  satisfies  $\Pr[\text{val}_{\Psi(\sigma)} < (\gamma - \varepsilon)] \leq \exp(-n)$ .*
2. *If  $\mathcal{D}_N \in S_\beta^N$  then for every sufficiently large  $n$ , the  $(\mathcal{D}_Y, \mathcal{D}_N, T)$ -streaming-RMD **NO** instance  $\sigma \sim \mathcal{N}_{\text{stream}, n}$  satisfies  $\Pr[\text{val}_{\Psi(\sigma)} > (\beta + \varepsilon)] \leq \exp(-n)$ .*

Furthermore, if  $\gamma = 1$  then  $\Pr_{\sigma \sim \mathcal{Y}_{\text{stream}, n}}[\text{val}_{\Psi(\sigma)} = 1] = 1$ .

*Proof.* Roughly our proof uses the fact that the definitions of  $S_\gamma^Y$  is setup so that  $\Psi(\sigma)$  achieves value  $\gamma$  under the “planted” assignment  $\mathbf{x}^*$ . Similarly  $S_\beta^N$  is setup so that for every assignment, with any ratio of 1s to  $-1$ s, the expected value is not more than  $\beta$ .

We recall that the condition  $\mathcal{D}_Y \in S_\gamma^Y$  implies that  $\mathbb{E}_{\mathbf{a} \sim \mathcal{D}_Y}[f(\mathbf{a})] \geq \gamma$ . Now consider a random **YES** instance  $\sigma \sim \mathcal{Y}_{\text{stream}, n}$  of  $(\mathcal{D}_Y, \mathcal{D}_N, T)$ -streaming-RMD and let  $\mathbf{x}^*$  denote the underlying vector corresponding to this draw. We show that for  $\Psi = \Psi(\sigma)$  we have  $\text{val}_{\Psi}(\mathbf{x}^*) \geq \gamma - \varepsilon$  with high probability. We consider the constraints given by  $\sigma(i)$  one at a time. Let  $m = \alpha n T$  denote the total number of constraints of  $\Psi$ . Let  $Z_i = C_i(\mathbf{x}^*) = f(\mathbf{z}(i) \odot \mathbf{x}^*|_{\mathbf{j}(i)})$  denote the indicator of the event that the  $i$ th constraint is satisfied by  $\mathbf{x}^*$ . By construction of  $\mathbf{z}(i)$  (from Definition 5.2 and passed through Definition 5.4), we have  $\mathbf{z}(i) = \mathbf{b}(i) \odot \mathbf{x}^*|_{\mathbf{j}(i)}$  where  $\mathbf{b}(i) \sim \mathcal{D}_Y$  independently of all other choices. We thus have  $Z_i = f(\mathbf{b}(i) \odot \mathbf{x}^*|_{\mathbf{j}(i)} \odot \mathbf{x}^*|_{\mathbf{j}(i)}) = f(\mathbf{b}(i))$ . Thus  $Z_i$  is a random variable, chosen independent of  $Z_1, \dots, Z_{i-1}$ , with expectation  $\mathbb{E}[Z_i | Z_1, \dots, Z_{i-1}] = \mathbb{E}_{\mathbf{b} \sim \mathcal{D}_Y}[f(\mathbf{b})] \geq \gamma$ . By applying a concentration bound (Lemma 3.4 suffices, though even simpler Chernoff bounds would suffice) we get that  $\mathbb{E}_{\sigma \sim \mathcal{Y}_{\text{stream}, n}}[\text{val}_{\Psi(\sigma)} = \frac{1}{m} \sum_{i=1}^m Z_i < (\gamma - \varepsilon)] \leq \exp(-\varepsilon^2 m) = \exp(-\varepsilon^2 \alpha T n)$ . This yields Part (1) of the lemma.

Note also that if  $\gamma = 1$  then  $Z_i = 1$  deterministically for every  $i$  and so we get  $\text{val}_{\Psi} = 1$  with probability 1, yielding the furthermore part of the lemma.

We now turn to the analysis of the **NO** case. Here the condition  $\mathcal{D}_N \in S_\beta^N$  implies that for every  $p \in [0, 1]$  we have  $\mathbb{E}_{\mathbf{b} \sim \mathcal{D}_N} \mathbb{E}_{\mathbf{a} \sim \text{Bern}(p)^k}[f(\mathbf{b} \odot \mathbf{a})] \leq \beta$ . Now consider any fixed assignment

$\nu \in \{-1, 1\}^n$ . In what follows we show that for a random **NO** instance  $\sigma \sim \mathcal{N}_n$  of  $(\mathcal{D}_Y, \mathcal{D}_N, T)$ -streaming-RMD if we let  $\Psi = \Psi(\sigma)$ , then  $\Pr[\text{val}_\Psi(\nu) > (\beta + \varepsilon)] \leq c^{-n}$  for  $c > 2$ . This allows us to take a union bound over the  $2^n$  possible  $\nu$ 's to claim  $\Pr[\text{val}_\Psi > (\beta + \varepsilon)] \leq 2^n \cdot c^{-n}$ .

We thus turn to analyzing  $\text{val}_\Psi(\nu)$ . Recall that  $\sigma$  is chosen by picking  $\mathbf{x}^* \in \{-1, 1\}^n$  uniformly and then picking  $\sigma(i)$ 's based on this choice — but our analysis will work for every choice of  $\mathbf{x}^* \in \{-1, 1\}^n$ . Fix such a choice and let  $\nu^* = \nu \odot \mathbf{x}^*$ . Now for  $i \in [m]$  (where  $m = \alpha n T$ ) let  $Z_i$  denote the indicator of the event that  $\nu$  satisfies  $C_i$ . We have  $Z_i = f(\mathbf{b}(i) \odot \mathbf{x}_{\mathbf{j}(i)}^*) \odot \nu_{\mathbf{j}(i)} = f(\mathbf{b}(i) \odot \nu^*_{|\mathbf{j}(i)})$ . We show below that  $\mathbb{E}[Z_i | Z_1, \dots, Z_{i-1}] \leq \beta + \varepsilon/2$ . (This time the  $Z_i$ 's are dependent, but we will show the conditioning does not hurt.) This allows us to apply [Lemma 3.4](#) to conclude  $\Pr[\text{val}_\Psi(\nu) = \frac{1}{m}(\sum_{i=1}^m Z_i) > (\beta + \varepsilon)] \leq \exp(-\varepsilon^2 \alpha T n) \leq c^{-n}$  for and constant  $c$  of our choice (for correspondingly large  $T$ ).

We thus turn to the final remaining step, i.e., to show  $\mathbb{E}[Z_i | Z_1, \dots, Z_{i-1}] \leq \beta + \varepsilon/2$ . Note that the only affect of the conditioning  $Z_1, \dots, Z_i$  is that this influences the distribution of  $\mathbf{j}(i)$ . Recall from the construction in [Definition 5.4](#) that  $\sigma = \sigma^{(1)} \circ \dots \circ \sigma^{(T)}$  is a concatenation of  $T$  streams that are independent conditioned on  $\mathbf{x}^*$ . Say that  $\sigma(i)$  belongs to the  $t$ -th component i.e.,  $\sigma^{(t)}$ . Then the only variables that affect  $Z_i$  are the  $Z_{i'}$ 's where  $i' < i$  and  $i'$  also is a part of  $\sigma^{(t)}$ . This effect is in turn passed through the conditioning of  $\mathbf{j}(i)$ . Let us fix  $\mathbf{j}(i')$  for every  $i' < i$  with  $i'$  being part of  $\sigma^{(t)}$ . Note there are at most  $\alpha n$  such  $i'$ 's. Now let  $S = [n] \setminus \cup_{i'} \mathbf{j}(i')$  be the remaining vertices. Note that conditioned on the fixed  $\mathbf{j}(i')$ 's,  $\mathbf{j}(i)$  is a uniformly chosen sequence of  $k$  distinct elements of  $S$ . Note that since  $\alpha < 1/(100k)$  we have  $|S| \geq n - k\alpha n \geq .99n$ . Let  $p = p_S$  be the fraction 1's in  $\nu^*|_S$ . We have  $\mathbb{E}[Z_i | S] \leq \mathbb{E}_{\mathbf{j}(i), \mathbf{b}(i)}[f(\mathbf{b}(i) \odot \nu^*_{|\mathbf{j}(i)})] \leq \mathbb{E}_{\mathbf{b}(i) \sim \mathcal{D}_N} \mathbb{E}_{\mathbf{a} \sim \text{Bern}(p)^k}[f(\mathbf{b}(i) \odot \mathbf{a}) + k^2/|S|] \leq \beta + \varepsilon/2$  where the second inequality uses the fact that sampling  $\mathbf{j}(i)_1, \dots, \mathbf{j}(i)_k$  independently from  $S$  leads to  $\nu^*_{\mathbf{j}(i)_1}, \dots, \nu^*_{\mathbf{j}(i)_k}$  that are distributed independently according to  $\text{Bern}(p)$  while  $\mathbf{j}(i)_1, \dots, \mathbf{j}(i)_k$  are distinct with probability at least  $1 - k^2/|S|$ . This concludes the proof of the lemma.  $\square$

## 5.4 Proof of [Theorem 5.1](#)

We are now ready to prove [Theorem 5.1](#).

*Proof of [Theorem 5.1](#).* Let  $\alpha_0 = \alpha_0(k)$  be as given by [Theorem 5.3](#). Given  $\varepsilon > 0$ , let  $\alpha = \alpha_0/2$  and  $T = \lceil 10000/(\varepsilon^2 \alpha) \rceil$ . Suppose there exists a streaming algorithm **ALG** that solves  $(\gamma - \varepsilon, \beta + \varepsilon)$ -Max-CSP( $f$ ). Let  $\mathcal{D}_Y, \mathcal{D}_N$  be distributions such that  $\mathbb{E}_{\mathbf{a} \sim \mathcal{D}_Y}[f(\mathbf{a})] \geq \gamma$ ,  $\mathbb{E}_{\mathbf{a} \sim \mathcal{D}_N} \mathbb{E}_{\mathbf{c} \sim \text{Bern}(p)^k}[f(\mathbf{a} \odot \mathbf{c})] \leq \beta$  for all  $p \in [0, 1]$ , and  $\mu(\mathcal{D}_Y) = \mu(\mathcal{D}_N)$ . Let  $n$  be sufficiently large and let  $\mathcal{Y}_{\text{stream}, n}$  and  $\mathcal{N}_{\text{stream}, n}$  denote the distributions of **YES** and **NO** instances of  $(\mathcal{D}_Y, \mathcal{D}_N, T)$ -streaming-RMD of length  $n$ . Since  $\alpha$  and  $T$  satisfy the conditions of [Lemma 5.8](#), we have for every sufficiently large  $n$

$$\Pr_{\sigma \sim \mathcal{Y}_{\text{stream}, n}} [\text{val}_\Psi(\sigma) < (\gamma - \varepsilon)] = o(1) \quad \text{and} \quad \Pr_{\sigma \sim \mathcal{N}_{\text{stream}, n}} [\text{val}_\Psi(\sigma) > (\beta + \varepsilon)] = o(1).$$

We conclude that **ALG** can distinguish **YES** instances of Max-CSP( $f$ ) from **NO** instances with advantage at least  $1/4 - o(1) \geq 1/8$ . However, since  $\mathcal{D}_Y, \mathcal{D}_N$  and  $\alpha$  satisfy the conditions of [Lemma 5.7](#) (in particular  $\mu(\mathcal{D}_Y) = \mu(\mathcal{D}_N)$  and  $\alpha \in (0, \alpha_0)$ ) such an algorithm requires space at least  $\Omega(\sqrt{n})$ . Thus, we conclude that any streaming algorithm that solves  $(\gamma - \varepsilon, \beta + \varepsilon)$ -Max-CSP( $f$ ) requires  $\Omega(\sqrt{n})$  space.

Finally, note that if  $\gamma = 1$  then in [Lemma 5.8](#), we have  $\text{val}_\Psi = 1$  with probability one. Repeating the above reasoning with this information, shows that  $(1, \beta + \varepsilon)$ -Max-CSP( $f$ ) requires  $\Omega(\sqrt{n})$ -space.  $\square$

## 6 Communication Lower Bound: A Special Case of 1-wise Independence

The goal of this section is to prove a special case of [Theorem 5.3](#) where the matching marginals are 0s, i.e., the distributions are 1-wise independent. Concretely, the main result of this section is summarized as follows.

**Theorem 6.1** (Lower bound for 1-wise distributions). *For every  $k \geq 2$ , there exists an  $\alpha_0 > 0$  such that for every  $\alpha \in (0, 1/\alpha_0)$ ,  $\delta \in (0, 1/2)$  and every  $\mathcal{D}_Y, \mathcal{D}_N \in \Delta(\{-1, 1\}^k)$  with  $\mu(\mathcal{D}_Y) = \mu(\mathcal{D}_N) = 0^k$  there exists  $\tau > 0$  and  $n_0$  such that for every  $n \geq n_0$  we have that every protocol for  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD with parameter  $\alpha$  that achieves advantage  $\delta$  requires at least  $\tau\sqrt{n}$  bits of communication on instances of length  $n$ .*

Our proof of [Theorem 6.1](#) follows the methodology of [[GKK<sup>+</sup>09](#)] with minor modifications as required by the RMD formulation. Their proof uses Fourier analysis to reduce the task of proving a communication lower bound to that of proving some combinatorial identities about randomly chosen matchings. We follow the same approach and this leads us to slightly different conditions about randomly chosen hypermatchings which requires a fresh analysis (though at the end our bounds are qualitatively similar to those in [[GKK<sup>+</sup>09](#)]).

The proof is by contradiction. We show that if the number of bits of communication is  $o(\sqrt{n})$ , then the *posterior distribution* of Bob’s input  $\mathbf{z}$  is close to the uniform distribution in total variation distance, hence contradicts the assumed advantage of the protocol. In [Theorem 6.2](#) we show that this total variation distance is small, when Alice’s message is a “typical” one, in that the number of Alice inputs leading to this message is not too small. We show immediately after stating [Theorem 6.2](#) how to go from the case of typical messages to all messages, and this gives a proof of [Theorem 6.1](#).

For each  $k$ -uniform hypermatching  $M$ , distribution  $\mathcal{D}$  over  $\{-1, 1\}^k$ , and a fixed Alice’s message, the posterior distribution function  $p_{M, \mathcal{D}} : \{-1, 1\}^{\alpha kn} \rightarrow [0, 1]$  is defined as follows. For each  $\mathbf{z} \in \{-1, 1\}^{\alpha kn}$ , let

$$p_{M, \mathcal{D}}(\mathbf{z}) := \Pr_{\substack{\mathbf{x}^* \in A \\ \mathbf{b} \sim \mathcal{D}^{\alpha n}}} [\mathbf{z} = (M\mathbf{x}^*) \odot \mathbf{b} \mid M, \text{ Alice's message}] = \mathbb{E}_{\mathbf{x}^* \in A} \mathbb{E}_{\mathbf{b} \sim \mathcal{D}^{\alpha n}} [\mathbf{1}_{\mathbf{z} = (M\mathbf{x}^*) \odot \mathbf{b}}],$$

where  $A \subset \{-1, 1\}^n$  is the set of Alice’s inputs that correspond to the message. If the number of communication bits is at most  $c$ , then there exists a message such that  $A \subseteq \{-1, 1\}^n$  and  $|A| \geq 2^{n-c}$ .

**Theorem 6.2.** *For every  $k \in \mathbb{N}$  there exists  $\alpha_0 > 0$  such that for every  $\alpha \in (0, \alpha_0)$ ,  $\delta \in (0, 1/2)$ , there exists a  $\tau_0 > 0$  such that the following holds for every sufficiently large  $n$ . Let  $A \subseteq \{-1, 1\}^n$  be a set satisfying  $|A| \geq 2^{n-\tau_0\sqrt{n}}$  and let  $\mathcal{D}$  be a distribution over  $\{-1, 1\}^k$  satisfying  $\mathbb{E}_{\mathbf{a} \sim \mathcal{D}}[a_j] = 0$  for all  $j \in [k]$ . Then*

$$\mathbb{E}_M [\|p_{M, \mathcal{D}} - U\|_{\text{tvd}}^2] \leq \delta^2 \tag{6.3}$$

where  $U$  denotes the uniform distribution over  $\{-1, 1\}^{k\alpha n}$ .

We prove the theorem in the rest of this section. But first we show how [Theorem 6.1](#) follows.

*Proof of [Theorem 6.1](#).* Let  $\delta$  be as in the theorem statement and let  $\delta' = \delta/8$ . Let  $\tau_0$  be the constant given by [Theorem 6.2](#) when invoked with parameter  $\alpha$  and  $\delta'$ . Let  $\tau = \tau_0/2$ ,  $c' = \tau_0\sqrt{n}$  and  $c = c' - \log(1/\delta')$ . Note that for large enough  $n$  we have  $c \geq \tau\sqrt{n}$ .

We will prove this theorem for this choice of  $\tau$ . The proof is by contradiction. Suppose there exists a protocol for  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD on instances of length  $n$  with advantage at least  $\delta$  using at most  $c$  bits of communication. Let  $\mathcal{D}_{unif}$  be the uniform distribution over  $\{-1, 1\}^k$ . By triangle inequality, there is a protocol for either  $(\mathcal{D}_Y, \mathcal{D}_{unif})$ -RMD or  $(\mathcal{D}_N, \mathcal{D}_{unif})$ -RMD with advantage at least  $\frac{\delta}{2}$  using at most  $c$  bits of communication. Without loss of generality, suppose the protocol is for  $(\mathcal{D}_Y, \mathcal{D}_{unif})$ -RMD and hence we have

$$\|p_{M, \mathcal{D}_Y} - p_{M, \mathcal{D}_{unif}}\|_{tvd} \geq \frac{\delta}{2}.$$

Next, by Yao's principle [Yao77] we may assume Alice is deterministic. Namely, the protocol partitions the set  $\{-1, 1\}^n$  of  $\mathbf{x}^*$  into  $2^c$  sets  $A_1, A_2, \dots, A_{2^c}$ . By a simple counting argument, with probability at least  $1 - \delta'$  the message sent by Alice corresponds to a set  $A_i \subset \{-1, 1\}^n$  of size at least  $2^{n-c-\log 1/\delta'} \geq 2^{n-c'}$ . We call such an event

$$\text{GOOD} = \bigcup_{i \in [2^c]: |A_i| \geq 2^{n-c'}} A_i.$$

Now for each  $A_i$  with  $|A_i| \geq 2^{n-c'}$  we invoke [Theorem 6.2](#) with parameters  $\alpha$  and  $\delta'$  to get

$$\|p_{M, \mathcal{D}_Y} - p_{M, \mathcal{D}_{unif}}\|_{tvd} |_{\mathbf{x}^* \in A_i} = \mathbb{E}_M[\|p_{M, \mathcal{D}_Y} - U\|_{tvd} |_{\mathbf{x}^* \in A_i}] \leq \delta'.$$

Now, consider  $\mathbf{x}^* \sim \text{Unif}(\{-1, 1\}^n)$ , we have

$$\begin{aligned} \|p_{M, \mathcal{D}_Y} - U\|_{tvd} &= \Pr[\mathbf{x}^* \in \text{GOOD}] \cdot \|p_{M, \mathcal{D}_Y} - U\|_{tvd} |_{\mathbf{x}^* \in \text{GOOD}} \\ &\quad + \Pr[\mathbf{x}^* \notin \text{GOOD}] \cdot \|p_{M, \mathcal{D}_Y} - U\|_{tvd} |_{\mathbf{x}^* \notin \text{GOOD}} \\ &\leq 1 \cdot \delta' + \delta' \cdot 1 = \frac{\delta}{4} < \frac{\delta}{2}. \end{aligned}$$

But this contradicts our assumption that

$$\|p_{M, \mathcal{D}_Y} - U\|_{tvd} = \|p_{M, \mathcal{D}_Y} - p_{M, \mathcal{D}_{unif}}\|_{tvd} \geq \frac{\delta}{2}.$$

This completes the proof of [Theorem 6.1](#). □

The rest of this section is devoted to the proof of [Theorem 6.2](#). In [Section 6.1](#), we reduce the upper bound for [Equation 6.3](#) to a combinatorial problem. Next, we analyze the combinatorial problem in [Section 6.2](#) and wrap up the proof of [Theorem 6.2](#) in [Section 6.3](#).

## 6.1 Reduction to a combinatorial problem

Let  $A \subseteq \{-1, 1\}^n$  be the set of Alice's inputs that correspond to the message, we naturally define  $f : \{-1, 1\}^n \rightarrow \{0, 1\}$  to be the indicator function of  $A$  such that  $f(\mathbf{x}^*) = 1$  iff  $\mathbf{x}^* \in A$ . In this subsection, we apply Fourier analysis on the left hand side of [Equation 6.3](#) and get an upper bound in terms of some combinatorics related to random matching as well as the Fourier coefficients of  $f$ . The reduction is summarized in the following lemma.

In what follows we will write a vector  $\mathbf{s} \in \{0, 1\}^{\alpha n}$  as a concatenation of  $\alpha n$  vectors, i.e.,  $\mathbf{s} = (\mathbf{s}(1), \dots, \mathbf{s}(\alpha n))$  where  $\mathbf{s}(i) \in \{0, 1\}^k$ . We use  $|\mathbf{s}(i)|$  to denote the Hamming weight of  $\mathbf{s}(i)$ .

**Lemma 6.4.** Let  $A \subseteq \{-1, 1\}^n$  and  $f : \{-1, 1\}^n \rightarrow \{0, 1\}$  be its indicator function. Let  $k \in \mathbb{N}$  and  $\alpha \in (0, 1/100k)$ . Let  $\mathcal{D}$  be a distribution over  $\{-1, 1\}^k$  such that  $\mathbb{E}_{\mathbf{a} \sim \mathcal{D}}[a_j] = 0$  for all  $j \in [k]$ . For each  $\ell \in [n]$ , let us denote by  $\mathbf{v}_\ell \in \{0, 1\}^n$  the vector where the first  $\ell$  entries are 1 and the remaining entries are 0.

$$\mathbb{E}_M [\|p_{M, \mathcal{D}} - U\|_{\text{tvd}}^2] \leq \frac{2^{2n}}{|A|^2} \sum_{\ell \geq 2} g(\ell) \cdot \sum_{\substack{\mathbf{v} \in \{0, 1\}^n \\ |\mathbf{v}| = \ell}} \widehat{f}(\mathbf{v})^2,$$

where

$$g(\ell) = \Pr_M \left[ \exists \mathbf{s} \in \{0, 1\}^{\alpha kn} \setminus \{0^{\alpha kn}\}, |\mathbf{s}(i)| \neq 1 \forall i, M^\top \mathbf{s} = \mathbf{v}_\ell \right].$$

*Proof.* By Cauchy–Schwarz inequality and Equation 3.6,

$$\begin{aligned} \mathbb{E}_M [\|p_{M, \mathcal{D}} - U\|_{\text{tvd}}^2] &\leq 2^{2\alpha kn} \mathbb{E}_M [\|p_{M, \mathcal{D}} - U\|_2^2] \\ &= 2^{2\alpha kn} \mathbb{E}_M \left[ \sum_{\mathbf{s} \in \{0, 1\}^{\alpha kn} \setminus \{0^{\alpha kn}\}} \widehat{p_{M, \mathcal{D}}}(\mathbf{s})^2 \right]. \end{aligned} \quad (6.5)$$

The following claim shows that the Fourier coefficient of the posterior distribution  $p_{M, \mathcal{D}}$  can be nicely upper bounded by a certain Fourier coefficient of the indicator function  $f$ .

**Claim 6.6.**

$$\mathbb{E}_M [\|p_{M, \mathcal{D}} - U\|_{\text{tvd}}^2] \leq \frac{2^{2n}}{|A|^2} \sum_{\mathbf{s} \in \text{GOOD} \setminus \{0^{\alpha kn}\}} \mathbb{E}_M \left[ \widehat{f}(M^\top \mathbf{s})^2 \right].$$

*Proof.* We first rewrite the Fourier coefficients of  $p_{M, \mathcal{D}}$ . For each  $\mathbf{s} \in \{0, 1\}^{\alpha kn} \setminus \{0^{\alpha kn}\}$ , express  $\mathbf{s} \in \{0, 1\}^{\alpha kn}$  as  $\alpha n$  blocks  $\mathbf{s}(1), \dots, \mathbf{s}(\alpha n) \in \{0, 1\}^k$  of length  $k$ . Observe that

$$\widehat{p_{M, \mathcal{D}}}(\mathbf{s}) = \frac{1}{2^{\alpha kn}} \sum_{\mathbf{z} \in \{-1, 1\}^{\alpha kn}} p_{M, \mathcal{D}}(\mathbf{z}) \prod_{\substack{i \in [\alpha n], j \in [k] \\ \mathbf{s}(i)_j = 1}} z(i)_j.$$

Recall that  $p_{M, \mathcal{D}}(\mathbf{z}) = \mathbb{E}_{\mathbf{x}^* \in A} \mathbb{E}_{\mathbf{b} \sim \mathcal{D}^{\alpha n}} [\mathbf{1}_{\mathbf{z} = M\mathbf{x}^* \odot \mathbf{b}}]$ , the equation becomes

$$= \frac{1}{2^{\alpha kn}} \cdot \mathbb{E}_{\mathbf{x}^* \in A} \left[ \prod_{\substack{i \in [\alpha n], j \in [k] \\ \mathbf{s}(i)_j = 1}} (M\mathbf{x}^*)_{i,j} \right] \mathbb{E}_{\mathbf{b} \sim \mathcal{D}^{\alpha n}} \left[ \prod_{\substack{i \in [\alpha n], j \in [k] \\ \mathbf{s}(i)_j = 1}} b(i)_j \right].$$

Observe that because  $\mathbb{E}_{\mathbf{a} \sim \mathcal{D}}[a_j] = 0$  for all  $j \in [k]$ , the right most sum is 0 if there exists  $i$  such that  $|\mathbf{s}(i)| = 1$ . Define  $\text{GOOD} := \{\mathbf{s} \in \{0, 1\}^{\alpha kn} \mid |\mathbf{s}(i)| \neq 1 \forall i\}$ , the equation becomes

$$\leq \frac{1}{2^{\alpha kn}} \cdot \left| \mathbb{E}_{\mathbf{x}^* \in A} \left[ \prod_{\substack{i \in [\alpha n], j \in [k] \\ \mathbf{s}(i)_j = 1}} (M\mathbf{x}^*)_{i,j} \right] \right| \cdot \mathbf{1}_{\mathbf{s} \in \text{GOOD}}.$$

Note that as each row and column of  $M$  has at most 1 non-zero entry, we have

$$= \frac{1}{2^{\alpha kn}} \cdot \left| \mathbb{E}_{\mathbf{x}^* \in A} \left[ \prod_{\substack{i \in [n] \\ (M^\top \mathbf{s})_i = 1}} \mathbf{x}_i^* \right] \right| \cdot \mathbf{1}_{\mathbf{s} \in \text{GOOD}}$$

Now we relate the above quantity to the Fourier coefficients of  $f$ . Recall that  $f$  is the indicator function of the set  $A$  and hence for each  $\mathbf{v} \in \{0, 1\}^n$ , we have

$$\widehat{f}(\mathbf{v}) = \frac{1}{2^n} \sum_{\mathbf{x}^*} f(\mathbf{x}^*) \prod_{i \in [n]: v_i = 1} \mathbf{x}_i^* = \frac{1}{2^n} \sum_{\mathbf{x}^* \in A} \prod_{i \in [n]: v_i = 1} \mathbf{x}_i^*.$$

Thus, the Fourier coefficient of  $p_M$  can be bounded as follows.

$$\widehat{p_{M, \mathcal{D}}}(\mathbf{s}) \leq \frac{1}{2^{\alpha kn}} \cdot \frac{2^n}{|A|} \widehat{f}(M^\top \mathbf{s}) \cdot \mathbf{1}_{\mathbf{s} \in \text{GOOD}}. \quad (6.7)$$

By plugging Equation 6.7 into Equation 6.5, we have the desired bound and complete the proof of Claim 6.6.  $\square$

Next, by Claim 6.6, we have

$$\mathbb{E}_M [\|p_{M, \mathcal{D}} - U\|_{\text{tvd}}^2] \leq \frac{2^{2n}}{|A|^2} \sum_{\mathbf{s} \in \text{GOOD} \setminus \{0^{\alpha kn}\}} \mathbb{E}_M \left[ \widehat{f}(M^\top \mathbf{s})^2 \right].$$

Since for a fixed  $M$ , the map  $M^\top$  is injective, the right hand side of the above inequality has the following nice combinatorial form.

$$= \frac{2^{2n}}{|A|^2} \sum_{\mathbf{v} \in \{0, 1\}^n \setminus \{0^n\}} \Pr_M \left[ \exists \mathbf{s} \in \text{GOOD} \setminus \{0\}, M^\top \mathbf{s} = \mathbf{v} \right] \widehat{f}(\mathbf{v})^2.$$

By symmetry, the above probability term will be the same for  $\mathbf{v}$  and  $\mathbf{v}'$  having the same Hamming weight. For each  $\ell \in [n]$ , denote  $g(\ell) = \Pr_M \left[ \exists \mathbf{s} \in \text{GOOD} \setminus \{0\}, M^\top \mathbf{s} = \mathbf{v} \right]$  where  $|\mathbf{v}| = \ell$ , the equation becomes

$$= \frac{2^{2n}}{|A|^2} \sum_{\ell \geq 1} g(\ell) \cdot \sum_{\substack{\mathbf{v} \in \{0, 1\}^n \\ |\mathbf{v}| = \ell}} \widehat{f}(\mathbf{v})^2.$$

Note that for  $\ell = 1$  and every  $\ell > \alpha kn$ ,  $g(\ell) = 0$  by definition. Thus, this expression simplifies to the following.

$$= \frac{2^{2n}}{|A|^2} \sum_{\ell \geq 2} g(\ell) \cdot \sum_{\substack{\mathbf{v} \in \{0, 1\}^n \\ |\mathbf{v}| = \ell}} \widehat{f}(\mathbf{v})^2.$$

This completes the proof of Lemma 6.4.  $\square$



## 6.2 An upper bound for the combinatorial problem

In this subsection, we upper bound the combinatorial term  $g(\ell)$  in [Lemma 6.4](#). The result is summarized in the following lemma.

**Lemma 6.8.** *For every  $k$  there exists an  $\alpha_0 > 0$  such that for every  $\alpha \in (0, \alpha_0)$  and for every  $n$  and  $\ell \leq n/2$ , we have*

$$g(\ell) = \Pr_M \left[ \exists \mathbf{s} \neq 0, |\mathbf{s}(i)| \neq 1 \forall i, M^\top \mathbf{s} = \mathbf{v}_\ell \right] \leq \left( \frac{\ell}{n} \right)^{\ell/2}.$$

*Proof.* We set  $\alpha_0 = (1/(2e^2k))^k$  so that  $2\alpha_0^{1/k}e^{3/2}k \leq 1$ .

We first reformulate our events. Instead of fixing  $\mathbf{v} = \mathbf{v}_\ell$  and picking the matching  $M$  at random, we note that it is equivalent to fix the matching  $M$  and letting  $\mathbf{v}$  be a uniformly random vector of weight  $\ell$ . We thus let  $M$  be the matching  $e_1, \dots, e_{\alpha n}$  where  $e_i = \{(i-1)k+1, \dots, (i-1)k+k\}$ . Letting  $V$  denote the support of the vector  $\mathbf{v}$ , the event we wish to consider is: “ $V \subseteq [k\alpha n]$  and  $|V \cap e_i| \geq 1$  for every  $i \in [\alpha n]$ .”

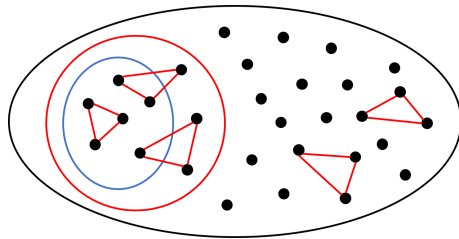


Figure 1: An example of  $n = 30$ ,  $k = 3$ ,  $\alpha = 0.5$ , and  $\ell = 6$ . The red circle denotes the choice of the set  $T$  with  $t = 3$  and the blue circle denotes the choice of the set  $V$ . Note that the figure illustrates some overcounting in proof of the lemma - the set  $V$  actually intersects one of the edges just once and so should not be counted. Our counting will nevertheless include the set since it is contained in at most  $\ell/2 = 3$  edges.

We bound the probability as follows. Let  $T = \{i \in [\alpha n] \mid e_i \cap V \neq \emptyset\}$  denote the set of edges that touch  $V$ , and let  $|T| = t$ . Note that  $\ell/k \leq t \leq \ell/2$  where the latter inequality following from the fact that every intersection is of size at least 2. We pick  $V$  by first picking  $T$  (there are at most  $\binom{\alpha n}{t}$  ways of doing this), and then picking  $V$  as a subset of the vertices incident to the edges of  $T$  (there are  $\binom{kt}{\ell}$  ways of doing this). (See [Figure 1](#).) Summing over  $t$  and dividing out by the total number of choices of  $V$  gives the final bound. We give the calculations below (which use the

inequalities  $(a/b)^b \leq \binom{a}{b} \leq (ea/b)^b$ .

$$\begin{aligned}
\Pr_V[V \subseteq [kan], |V \cap e_i| \geq 1] &\leq \frac{\sum_{t=\ell/k}^{\ell/2} \binom{\alpha n}{t} \binom{k t}{\ell}}{\binom{n}{\ell}} \\
&\leq \sum_{t=\ell/k}^{\ell/2} \left(\frac{e\alpha n}{t}\right)^t \cdot \left(\frac{ekt}{\ell}\right)^\ell \cdot \left(\frac{n}{\ell}\right)^{-\ell} \\
&= \sum_{t=\ell/k}^{\ell/2} e^{t+\ell} \alpha^t k^\ell (t/n)^{\ell-t} \\
&\leq \alpha^{\ell/k} e^{3\ell/2} k^\ell (\ell/n)^{\ell/2} \sum_{t'=0}^{\infty} (\ell/n)^{t'} \\
&\leq 2(\alpha^{1/k} e^{3/2} k)^\ell (\ell/n)^{\ell/2} \\
&\leq (2\alpha^{1/k} e^{3/2} k)^\ell (\ell/n)^{\ell/2} \\
&\leq (\ell/n)^{\ell/2}.
\end{aligned}$$

□

### 6.3 Proof of Theorem 6.2

*Proof of Theorem 6.2.* By Lemma 6.4 and Lemma 6.8, we have

$$\mathbb{E}_M[\|p_{M,\mathcal{D}} - U\|_{tvd}^2] \leq \frac{2^{2n}}{|A|^2} \cdot \sum_{\ell=2}^{\alpha kn} \frac{\ell^{\ell/2}}{n^{\ell/2}} \sum_{\substack{\mathbf{v} \in \{0,1\}^n \\ |\mathbf{v}|=\ell}} \widehat{f}(\mathbf{v})^2.$$

We use Lemma 3.7 to upper bound the Fourier coefficients for small  $\ell$  as follows. Let  $c = \tau_0 \sqrt{n}$  so that  $|A| \geq 2^{n-c}$ . For  $\ell \in [4c]$ , we have

$$\frac{2^{2n}}{|A|^2} \sum_{\substack{\mathbf{v} \in \{0,1\}^n \\ |\mathbf{v}|=\ell}} \widehat{f}(\mathbf{v})^2 \leq \left(\frac{4\sqrt{2}c}{\ell}\right)^\ell.$$

Next, we apply Parseval's inequality (Lemma 3.5) and have  $\sum_{\mathbf{v}} \widehat{f}(\mathbf{v})^2 \leq 1$ . Thus,

$$\mathbb{E}_M[\|p_{M,\mathcal{D}} - U\|_{tvd}^2] \leq \sum_{\ell=2}^{4c} \frac{\ell^{\ell/2}}{n^{\ell/2}} \cdot \left(\frac{4\sqrt{2}c}{\ell}\right)^\ell + \frac{2^{2n}}{|A|^2} \cdot \max_{4c < \ell \leq \alpha kn} \left\{ \frac{\ell^{\ell/2}}{n^{\ell/2}} \right\}$$

The second term on the right hand side is maximized by  $\ell = 4c + 1$  (since  $\ell \leq n$ ) and hence

$$\begin{aligned}
&\leq \sum_{\ell=2}^{4c} \left(\frac{32c^2}{\ell \cdot n}\right)^{\ell/2} + \left(\frac{8c}{n}\right)^{2c} \\
&\leq \sum_{\ell=2}^{4c} (2^{10} \tau_0^2)^\ell + (8\tau_0)^{2c} \\
&\leq \delta^2.
\end{aligned}$$

where the final expression determines our choice of  $\tau_0$ . Specifically we set  $\tau_0 = \delta/2^6$  so that each of the terms is at most  $\delta^2/2$ . This completes the proof of [Theorem 6.2](#).  $\square$

## 7 Communication Lower Bound: General Case

In this section we finally prove [Theorem 5.3](#). In other words we show that for every  $\mathcal{D}_Y, \mathcal{D}_N \in \Delta(\{-1, 1\}^k)$  with matching marginals, any protocol for  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD with positive advantage requires  $\Omega(\sqrt{n})$  bits of communication. We start with an overview.

The first step is to observe that we can prove indistinguishability of *some* distributions with matching non-zero marginals. For example, given that  $\mathcal{D}_1 = \text{Unif}(\{(-1, -1), (1, 1)\})$  is indistinguishable from  $\mathcal{D}_2 = \text{Unif}(\{-1, 1\}^2)$ , it can also be shown that  $\mathcal{D}'_1 = \frac{1}{2}\{(1, 1)\} + \frac{1}{2}\mathcal{D}_1$  is indistinguishable from  $\mathcal{D}'_2 = \frac{1}{2}\{(1, 1)\} + \frac{1}{2}\mathcal{D}_2$  (see [Lemma 7.7](#) for a related statement). Note that  $\mathcal{D}'_1$  and  $\mathcal{D}'_2$  are distributions with non-zero but matching marginals.

The bulk of this section is devoted to proving that for every pair of distributions  $\mathcal{D}_Y$  and  $\mathcal{D}_N$ , we can find a path (a sequence) of intermediate distributions  $\mathcal{D}_Y = \mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_L = \mathcal{D}_N$  such that adjacent pairs in this sequence are indistinguishable by a “basic” argument, where a basic argument is a combination of an indistinguishability result from [Theorem 6.1](#) and a shifting argument formalized in [Lemma 7.7](#). Our proof comes in the following steps:

1. For every marginal vector  $\boldsymbol{\mu}$ , we identify a *canonical* distribution  $\mathcal{D}_{\boldsymbol{\mu}}$  that we use as the endpoint of the path. So it suffices to prove that for all  $\mathcal{D}$ ,  $\mathcal{D}$  is indistinguishable from  $\mathcal{D}_{\boldsymbol{\mu}(\mathcal{D})}$ , i.e., there is a path of finite length from  $\mathcal{D}$  to  $\mathcal{D}_{\boldsymbol{\mu}(\mathcal{D})}$ .
2. We identify a measure  $\Phi(\mathcal{D})$  associated with distributions that helps measure progress on a path. Among distributions with marginal  $\boldsymbol{\mu}(\mathcal{D})$ , this measure is uniquely maximized by  $\mathcal{D}_{\boldsymbol{\mu}(\mathcal{D})}$ . We show that for every distribution  $\mathcal{D}$  that is not canonical one can take a basic step that increases  $\boldsymbol{\mu}(\mathcal{D})$ . Unfortunately the measure  $\Phi$  is real-valued and the increases per step can be by arbitrarily small amounts, so we are not done.
3. We give a combinatorial proof that there is a path of finite length (some function of  $k$ ) that takes us from an arbitrary distribution to the canonical one.

Putting the three ingredients together, along with a proof that a “basic step” is indistinguishable gives us the final theorem.

We start with the definition of the chain and the canonical distribution. For a distribution  $\mathcal{D} \in \Delta(\{-1, 1\}^k)$ , its support is the set  $\text{supp}(\mathcal{D}) = \{\mathbf{a} \in \{-1, 1\}^k \mid \mathcal{D}(\mathbf{a}) > 0\}$ . Next, we consider the following partial order on  $\{-1, 1\}^k$ . For vectors  $\mathbf{a}, \mathbf{b} \in \{-1, 1\}^k$  we use the notation  $\mathbf{a} \leq \mathbf{b}$  if  $a_i \leq b_i$  for every  $i \in [k]$ . Further we use  $\mathbf{a} < \mathbf{b}$  if  $\mathbf{a} \leq \mathbf{b}$  and  $\mathbf{a} \neq \mathbf{b}$ .

**Definition 7.1** (Chain). *We refer to a sequence  $\mathbf{a}(0) < \mathbf{a}(1) < \dots < \mathbf{a}(\ell)$ ,  $\mathbf{a}(i) \in \{-1, 1\}^k$  for every  $i \in \{0, \dots, \ell\}$ , as a chain of length  $\ell$ . Note that chains in  $\{-1, 1\}^k$  have length at most  $k$ .*

**Definition 7.2** (Canonical distribution). *Given a vector of marginals  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k) \in [-1, 1]^k$ , the canonical distribution associated with  $\boldsymbol{\mu}$ , denoted  $\mathcal{D}_{\boldsymbol{\mu}}$ , is defined as follows: Let  $\rho: [k] \rightarrow [k]$  be a permutation such that  $-1 \leq \mu_{\rho(1)} \leq \dots \leq \mu_{\rho(k)} \leq 1$ . For  $i \in \{0, \dots, k\}$ , let  $\mathbf{a}(i) \in \{-1, 1\}^k$  be given by  $\mathbf{a}(i)_j = -1$  if  $j \in \{\rho(1), \dots, \rho(k-i)\}$  and  $\mathbf{a}(i)_j = 1$  otherwise. (Note that  $\mathbf{a}(0) < \dots < \mathbf{a}(k)$ .) Then  $\mathcal{D}_{\boldsymbol{\mu}}(\mathbf{a}(i)) = \frac{1}{2}(\mu_{\rho(k-i+1)} - \mu_{\rho(k-i)})$ , where we define  $\mu_{\rho(0)} = -1$  and  $\mu_{\rho(k+1)} = 1$ . Finally,  $\mathcal{D}_{\boldsymbol{\mu}}(\mathbf{a}) = 0$  for all  $\mathbf{a} \notin \{\mathbf{a}(0), \dots, \mathbf{a}(k)\}$ .*

It is easy to verify that  $\mathcal{D}_\mu$  is indeed a distribution, and that it has the desired marginals, i.e.,  $\mu(\mathcal{D}_\mu) = \mu$ . Note that a distribution is a canonical distribution if and only if its support is a chain. Furthermore, the canonical distribution is uniquely determined even though  $\rho$ , and hence the chain  $\mathbf{a}(0), \dots, \mathbf{a}(k)$ , may not be uniquely determined. This is so since  $\rho$  is non-unique only if  $\mu_{\rho(i)} = \mu_{\rho(i+1)}$  for some  $i$ , and in this case  $\mathcal{D}_\mu(\mathbf{a}(i)) = 0$  so the “non-uniqueness of  $\mathbf{a}(i)$ ” does not affect  $\mathcal{D}_\mu$ .

Next we define a potential associated with distributions. For a distribution  $\mathcal{D} \in \Delta(\{-1, 1\}^k)$  define its potential to be

$$\Phi(\mathcal{D}) = \mathbb{E}_{\mathbf{b} \sim \mathcal{D}} \left[ \left( \sum_{j \in [k]} b_j \right)^2 \right].$$

We will show shortly that  $\mathcal{D}_\mu$  is the distribution with maximum potential among all distributions with marginal  $\mu$ . In the process of showing this we will introduce a “polarization operator” which maps a distribution  $\mathcal{D}$  to a new one that increases the potential for typical distributions. Since this operator is useful also for further steps, we start with defining this operator and analyzing its effect on the potential.

## 7.1 Polarization

Briefly, suppose the support of a distribution contains both  $(-1)^i(1)^{k-i}$  and  $1^i(-1)^{k-i}$ . Then the polarization operator moves some of this mass (as much as possible while maintaining the property that the result is a distribution) to the more “polarized” points  $(-1)^k$  and  $1^k$ . The operator is defined more generally to allow the two starting points to agree on some coordinates. To define this operator, the following notation will be useful.

For  $\mathbf{u}, \mathbf{v} \in \{-1, 1\}^k$ , let  $\mathbf{u} \wedge \mathbf{v} = (\min\{u_1, v_1\}, \dots, \min\{u_k, v_k\})$  and let  $\mathbf{u} \vee \mathbf{v} = (\max\{u_1, v_1\}, \dots, \max\{u_k, v_k\})$ . We say  $\mathbf{u}$  and  $\mathbf{v}$  are incomparable if  $\mathbf{u} \not\leq \mathbf{v}$  and  $\mathbf{v} \not\leq \mathbf{u}$ . Note that if  $\mathbf{u}$  and  $\mathbf{v}$  are incomparable then  $\{\mathbf{u}, \mathbf{v}\}$  and  $\{\mathbf{u} \vee \mathbf{v}, \mathbf{u} \wedge \mathbf{v}\}$  are disjoint<sup>9</sup>.

**Definition 7.3** (Polarization (update) operator). *Given a distribution  $\mathcal{D} \in \Delta(\{-1, 1\}^k)$  and incomparable elements  $\mathbf{u}, \mathbf{v} \in \{-1, 1\}^k$ , we define the  $(\mathbf{u}, \mathbf{v})$ -polarization of  $\mathcal{D}$ , denoted  $\mathcal{D}_{\mathbf{u}, \mathbf{v}}$ , to be the distribution as given below. Let  $\varepsilon = \min\{\mathcal{D}(\mathbf{u}), \mathcal{D}(\mathbf{v})\}$ .*

$$\mathcal{D}_{\mathbf{u}, \mathbf{v}}(\mathbf{b}) = \begin{cases} \mathcal{D}(\mathbf{b}) - \varepsilon & , \mathbf{b} \in \{\mathbf{u}, \mathbf{v}\} \\ \mathcal{D}(\mathbf{b}) + \varepsilon & , \mathbf{b} \in \{\mathbf{u} \vee \mathbf{v}, \mathbf{u} \wedge \mathbf{v}\} \\ \mathcal{D}(\mathbf{b}) & , \text{otherwise.} \end{cases}$$

We refer to  $\varepsilon(\mathcal{D}, \mathbf{u}, \mathbf{v}) = \min\{\mathcal{D}(\mathbf{u}), \mathcal{D}(\mathbf{v})\}$  as the polarization amount.

In what follows we sometimes refer to this operator as

It can be verified that the polarization operator preserves the marginals, i.e.,  $\mu(\mathcal{D}) = \mu(\mathcal{D}_{\mathbf{u}, \mathbf{v}})$ . Note also that this operator is non-trivial, i.e.,  $\mathcal{D}_{\mathbf{u}, \mathbf{v}} = \mathcal{D}$ , if  $\{\mathbf{u}, \mathbf{v}\} \not\subseteq \text{supp}(\mathcal{D})$ . By correlating the “+1”s and “-1”s, the polarization operator makes the support of  $\mathcal{D}$  more polarized in the sense quantified in the following lemma.

<sup>9</sup>To see this, suppose  $\mathbf{u} = \mathbf{u} \wedge \mathbf{v}$ , then we have  $u_j = \min\{u_j, v_j\}$  for all  $j \in [k]$  and hence  $\mathbf{u} \leq \mathbf{v}$ , which is a contradiction. The same analysis works for the other cases.

**Lemma 7.4** (Polarization increases potential). *Let  $\mathcal{D} \in \Delta(\{-1, 1\}^k)$  be a distribution with marginal vector  $\boldsymbol{\mu} = \boldsymbol{\mu}(\mathcal{D})$  and let  $\mathbf{u}, \mathbf{v} \in \text{supp}(\mathcal{D})$  be incomparable. Then we have*

$$\Phi(\mathcal{D}_{\mathbf{u}, \mathbf{v}}) = \Phi(\mathcal{D}) + 8 \cdot \varepsilon \cdot s \cdot t$$

where  $\varepsilon = \varepsilon(\mathcal{D}, \mathbf{u}, \mathbf{v})$  is the polarization amount, and  $s = |\{j \in [k] \mid u_j = -v_j = 1\}|$  and  $t = |\{j \in [k] \mid u_j = -v_j = -1\}|$ . In particular  $\Phi(\mathcal{D}_{\mathbf{u}, \mathbf{v}}) > \Phi(\mathcal{D})$ .

*Proof.* We look at the difference  $\Phi(\mathcal{D}_{\mathbf{u}, \mathbf{v}}) - \Phi(\mathcal{D})$ . Let  $\ell = \sum_{j \in [k]: u_j = v_j} u_j$ . We have:

$$\begin{aligned} \Phi(\mathcal{D}_{\mathbf{u}, \mathbf{v}}) - \Phi(\mathcal{D}) &= \sum_{\mathbf{b} \in \{-1, 1\}^k} (\mathcal{D}_{\mathbf{u}, \mathbf{v}}(\mathbf{b}) - \mathcal{D}(\mathbf{b})) \cdot \Phi(\mathbf{b}) \\ &= \varepsilon \cdot (\Phi(\mathbf{u} \wedge \mathbf{v}) + \Phi(\mathbf{u} \vee \mathbf{v}) - \Phi(\mathbf{u}) - \Phi(\mathbf{v})) \\ &= \varepsilon \cdot ((\ell + s + t)^2 + (\ell - s - t)^2 - (\ell + s - t)^2 - (\ell - s + t)^2) \\ &= 8 \cdot \varepsilon \cdot s \cdot t. \end{aligned}$$

Finally note that  $s, t > 0$  since  $\mathbf{u}$  and  $\mathbf{v}$  are incomparable, and  $\varepsilon > 0$  since  $\mathbf{u}, \mathbf{v} \in \text{supp}(\mathcal{D})$ , thus yielding  $\Phi(\mathcal{D}_{\mathbf{u}, \mathbf{v}}) > \Phi(\mathcal{D})$ .  $\square$

**Lemma 7.5** ( $\mathcal{D}_{\boldsymbol{\mu}}$  maximizes potential). *For every distribution  $\mathcal{D} \in \Delta(\{-1, 1\}^k)$  with  $\boldsymbol{\mu} = \boldsymbol{\mu}(\mathcal{D})$  we have  $\Phi(\mathcal{D}) \leq \Phi(\mathcal{D}_{\boldsymbol{\mu}})$ . Furthermore the inequality is strict if  $\mathcal{D} \neq \mathcal{D}_{\boldsymbol{\mu}}$ .*

*Proof.* Let  $\mathcal{D}^*$  be a distribution with marginal  $\boldsymbol{\mu}$  that maximized  $\Phi(\mathcal{D})$ . Suppose there exist incomparable  $\mathbf{u}, \mathbf{v} \in \text{supp}(\mathcal{D}^*)$ , then by Lemma 7.4 we have that  $\Phi(\mathcal{D}^*) < \Phi(\mathcal{D}_{\mathbf{u}, \mathbf{v}}^*)$  contradicting the maximality of  $\mathcal{D}^*$ . It follows that there are no incomparable elements in  $\text{supp}(\mathcal{D}^*)$ , or in other words,  $\text{supp}(\mathcal{D}^*)$  is a chain. We now show that this implies  $\mathcal{D}^* = \mathcal{D}_{\boldsymbol{\mu}}$ .

More specifically we show that any distribution  $\mathcal{D}$  supported on a chain is uniquely determined by its marginal  $\boldsymbol{\mu}$ . To see this, let  $\rho : [k] \rightarrow [k]$  be a bijection such that  $\mu_{\rho(j)} \leq \mu_{\rho(j+1)}$  for all  $j$ . Let  $\tau_0 < \tau_1 < \dots < \tau_\ell$  be the attainable values of  $\boldsymbol{\mu}$ , i.e.,  $\{\tau \mid \exists j \in [k] \text{ s.t. } \mu_j = \tau\} = \{\tau_0, \dots, \tau_\ell\}$ . For  $0 \leq i \leq \ell$ , let  $\mathbf{a}(i)$  be given by  $\mathbf{a}(i)_j = -1$  if  $\mu_j \leq \tau_{\ell-i}$  and  $\mathbf{a}(i)_j = 1$  otherwise. Note that  $\mathbf{a}(0) < \dots < \mathbf{a}(\ell)$ . It can be verified that  $\text{supp}(\mathcal{D}^*) = \{\mathbf{a}(0), \dots, \mathbf{a}(\ell)\}$ , and  $\mathcal{D}^*(\mathbf{a}(i))$  is uniquely defined for all  $i$ .

**Claim 7.6.**  *$\text{supp}(\mathcal{D}^*) = \{\mathbf{a}(0), \dots, \mathbf{a}(\ell)\}$ , and  $\mathcal{D}^*(\mathbf{a}(i)) = (\tau_{\ell-i+1} - \tau_{\ell-i})/2$ , where  $\tau_{-1} = -1$  and  $\tau_{\ell+1} = 1$ .*

*Proof.* For the sake of contradiction, assume  $\text{supp}(\mathcal{D}^*) = \{\mathbf{a}'(0), \dots, \mathbf{a}'(\ell')\} \neq \{\mathbf{a}(0), \dots, \mathbf{a}(\ell)\}$  where  $\mathbf{a}'(0) < \mathbf{a}'(1) < \dots < \mathbf{a}'(\ell')$  is a chain. Let  $0 \leq i \leq \min\{\ell, \ell'\}$  be the smallest  $i$  such that  $\mathbf{a}(i) \neq \mathbf{a}'(i)$ . Consider the following three situations: (i)  $\mathbf{a}(i) < \mathbf{a}'(i)$ , (ii)  $\mathbf{a}(i) > \mathbf{a}'(i)$ , and (iii)  $\mathbf{a}(i)$  and  $\mathbf{a}'(i)$  are incomparable.

For (i) and (iii), due to the construction of  $\{\mathbf{a}(0), \dots, \mathbf{a}(\ell)\}$  and the fact that  $\{\mathbf{a}'(0), \dots, \mathbf{a}'(\ell')\}$  is a chain, we have that for each  $j, j' \in [k]$  with  $\tau_{i-2} < \mu_j, \mu_{j'} \leq \tau_i$ ,  $\mathbf{a}'(i')_j = \mathbf{a}'(i')_{j'}$  for all  $0 \leq i' \leq \ell'$ . This implies that  $\mu_j = \mu_{j'}$  which is a contradiction because there are two attainable values  $\tau_i$  and  $\tau_{i-1}$  lie in the interval  $(\tau_{i-2}, \tau_i]$ . Similar argument also works for situation (ii).

So now we know that  $\text{supp}(\mathcal{D}^*) = \{\mathbf{a}(0), \dots, \mathbf{a}(\ell)\}$ . It is immediate to see that  $\mathcal{D}^*(\mathbf{a}(i))$  is uniquely defined for all  $i$  by solving the following linear system.

$$\boldsymbol{\mu} = \begin{bmatrix} | & | & \cdots & | \\ \mathbf{a}(0) & \mathbf{a}(1) & \cdots & \mathbf{a}(\ell) \\ | & | & & | \end{bmatrix} \begin{bmatrix} \mathcal{D}^*(\mathbf{a}(0)) \\ \mathcal{D}^*(\mathbf{a}(1)) \\ \cdots \\ \mathcal{D}^*(\mathbf{a}(\ell)) \end{bmatrix}.$$

Note that by the construction of  $\{\mathbf{a}(0), \dots, \mathbf{a}(\ell)\}$ , the matrix has full rank, and, hence, a unique solution. It can be verified that the solution is given by  $\mathcal{D}^*(\mathbf{a}(i)) = (\tau_{\ell-i+1} - \tau_{\ell-i})/2$ , where  $\tau_{-1} = -1$  and  $\tau_{\ell+1} = 1$ .  $\square$

In summary,  $\mathcal{D}^*$  is uniquely determined by  $\boldsymbol{\mu}(\mathcal{D})$  and its support is a chain. This implies  $\mathcal{D}^* = \mathcal{D}_{\boldsymbol{\mu}}$ , so  $\mathcal{D}_{\boldsymbol{\mu}}$  is the unique distribution that maximizes the potential.  $\square$

## 7.2 Indistinguishability of a polarization update

Our next observation is that for every distribution  $\mathcal{D}$  with incomparable elements  $\mathbf{u}, \mathbf{v}$  in their support,  $\mathcal{D}$  is indistinguishable, in the RMD problem, from its  $(\mathbf{u}, \mathbf{v})$ -polarization  $\mathcal{D}_{\mathbf{u}, \mathbf{v}}$ .

**Lemma 7.7** (Polarization update preserves indistinguishability). *Let  $\alpha_0(k)$  be as given in [Theorem 6.1](#). Let  $k \in \mathbb{N}$ ,  $\alpha \in (0, \alpha_0)$ ,  $\delta \in (0, 1/2)$ . Then for every distribution  $\mathcal{D} \in \Delta(\{-1, 1\}^k)$  and incomparable  $\mathbf{u}, \mathbf{v} \in \text{supp}(\mathcal{D})$  there exists  $\tau > 0$  and  $n_0$  such that for every  $n \geq n_0$  every protocol for  $(\mathcal{D}, \mathcal{D}_{\mathbf{u}, \mathbf{v}})$ -RMD achieving advantage  $\delta$  on instances of length  $n$  requires  $\tau\sqrt{n}$  bits of communication.*

We prove [Lemma 7.7](#) by a reduction. We show that there exists a pair of distributions  $\mathcal{D}_Y$  and  $\mathcal{D}_N$  with marginals being zero such that given a protocol  $\Pi$  for  $(\mathcal{D}, \mathcal{D}_{\mathbf{u}, \mathbf{v}})$ -RMD we can get a protocol  $\Pi'$  for  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD. We then use [Theorem 6.1](#) to get a lower bound on the communication of  $\Pi'$  and thus of  $\Pi$ . Specifically, we divide the proof into three steps. In step one, we define  $\mathcal{D}_Y$  and  $\mathcal{D}_N$  and provide intuition on the reduction. Next, we formally describe the reduction by designing a protocol for  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD from a protocol for  $(\mathcal{D}, \mathcal{D}_{\mathbf{u}, \mathbf{v}})$ -RMD. Finally, we prove the correctness of the reduction and wrap up the proof of [Lemma 7.7](#).

**Step 1: The auxiliary distributions  $\mathcal{D}_Y$  and  $\mathcal{D}_N$ .** We start by defining  $\mathcal{D}_Y$  and  $\mathcal{D}_N$ . Let  $S = \{i \in [k] \mid u_i \neq v_i\}$ . Let  $k' = |S|$ . Without loss of generality, we re-index the coordinates and assume  $S = \{1, 2, \dots, k'\}$ . Let  $\mathbf{a} = \mathbf{u}|_S$  so that  $\mathbf{v}|_S = -\mathbf{a}$ . We also let  $\tilde{\mathbf{u}} = \mathbf{u}|_{\bar{S}}$  denote the common parts of  $\mathbf{u}$  and  $\mathbf{v}$ . Let  $\mathcal{D}_Y$  be the uniform distribution over  $\{\mathbf{a}, -\mathbf{a}\}$ , and  $\mathcal{D}_N$  be the uniform distribution over  $\{1^{k'}, (-1)^{k'}\}$ . Note that  $\boldsymbol{\mu}(\mathcal{D}_Y) = \boldsymbol{\mu}(\mathcal{D}_N) = 0^{k'}$ . Let  $\mathcal{D}_1 = \text{Unif}(\{\mathbf{u}, \mathbf{v}\})$  and  $\mathcal{D}_2 = \text{Unif}(\{\mathbf{u} \vee \mathbf{v}, \mathbf{u} \wedge \mathbf{v}\})$ . Let  $\varepsilon = \varepsilon(\mathcal{D}, \mathbf{u}, \mathbf{v})$  be the polarization amount. Let  $\mathcal{D}_0 \in \Delta(\{-1, 1\}^k)$  be such that  $\mathcal{D} = (1 - 2\varepsilon)\mathcal{D}_0 + 2\varepsilon\mathcal{D}_1$ . Note that  $\mathcal{D}_{\mathbf{u}, \mathbf{v}} = (1 - 2\varepsilon)\mathcal{D}_0 + 2\varepsilon\mathcal{D}_2$ .

We give an informal idea now, before giving the (potentially notationally complex) details. The rough idea is that Alice and Bob first pad their inputs with lots of dummy variables (whose values are known to both) and expand the masks from  $\mathcal{D}_Y$  (or  $\mathcal{D}_N$ ) into masks that are from  $\mathcal{D}_1$  (respectively  $\mathcal{D}_2$ ). They then augment the sequence of masks from  $\alpha n'$  to  $\alpha n = \Omega(\alpha n' / \varepsilon)$ , injecting many random masks from  $\mathcal{D}_0$ . This gives them an instance of  $(\mathcal{D}, \mathcal{D}_{\mathbf{u}, \mathbf{v}})$ -RMD to solve for which they use the protocol  $\Pi$ . It is not too hard to see all this can be done locally by Alice and Bob; and this is proved formally below.

**Step 2: A reduction from  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD to  $(\mathcal{D}, \mathcal{D}_{\mathbf{u}, \mathbf{v}})$ -RMD.** Consider a protocol  $\Pi = (\Pi_A, \Pi_B)$  for  $(\mathcal{D}, \mathcal{D}_{\mathbf{u}, \mathbf{v}})$ -RMD with parameter  $\alpha \leq 1/(200k)$  using  $C(n)$  bits of communication to achieve an advantage of  $\delta$  on instances of length  $n$ . We let  $n' = (k'\varepsilon/k)n$  where  $k'$  was chosen in the previous step. We also let  $\alpha' = (2k/k')\alpha$  so that  $\alpha' \leq 1/(100k')$ . We use  $\Pi$  to design a protocol  $\Pi'$  for  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD with parameter  $\alpha'$  achieving advantage of at least  $\delta/2$  on instances of length  $n'$  with communication  $C'(n') = C(n)$ . We conclude by [Theorem 6.1](#) that there exists a constant  $\tau'$  such that  $C(n) \geq \tau'\sqrt{n'} = \tau\sqrt{n}$ , where  $\tau = \tau'\sqrt{\varepsilon k'/k} > 0$  as desired.

Our protocol  $\Pi'$  uses shared randomness between Alice and Bob (while we assume  $\Pi$  is deterministic). Let  $n'' = kn'/k'$  so that  $n = n''/(2\varepsilon)$ . Let  $\alpha'' = \alpha'n'/n'' = k\alpha/k'$ . Recall that an instance of  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD is determined by a four tuple  $(\mathbf{x}', M', \mathbf{z}', \mathbf{b}')$  with  $\mathbf{x}' \in \{-1, 1\}^{n'}$ ,  $M' \in \{0, 1\}^{k'\alpha'n' \times n'}$  and  $\mathbf{z}', \mathbf{b}' \in \{-1, 1\}^{k'\alpha'n'}$  with  $\mathbf{z}' = M'\mathbf{x}' \odot \mathbf{b}'$ . See [Figure 2](#) for a pictorial description.

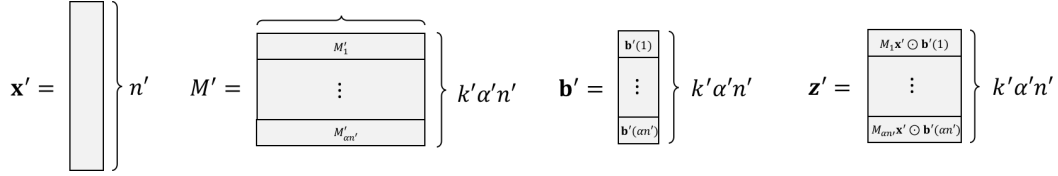


Figure 2: Pictorial description of  $(\mathbf{x}', M', \mathbf{b}', \mathbf{z}')$ .

We give two maps using shared randomness  $R'$  and  $R''$ :

- (i) **From  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD to  $(\mathcal{D}_1, \mathcal{D}_2)$ -RMD:**  $(\mathbf{x}', M', \mathbf{b}', \mathbf{z}', R') \mapsto (\mathbf{x}'', M'', \mathbf{b}'', \mathbf{z}'')$  where  $\mathbf{x}'' \in \{0, 1\}^{n''}$ ,  $M'' \in \{0, 1\}^{k\alpha''n'' \times n''}$  and  $\mathbf{b}'', \mathbf{z}'' \in \{-1, 1\}^{k\alpha''n''}$ .
- (ii) **From  $(\mathcal{D}_1, \mathcal{D}_2)$ -RMD to  $(\mathcal{D}, \mathcal{D}_{\mathbf{u}, \mathbf{v}})$ -RMD:**  $(\mathbf{x}'', M'', \mathbf{b}'', \mathbf{z}'', R'') \mapsto (\mathbf{x}, M, \mathbf{b}, \mathbf{z})$ , where  $\mathbf{x} \in \{0, 1\}^n$ ,  $M \in \{0, 1\}^{k\alpha n \times n}$  and  $\mathbf{b}, \mathbf{z} \in \{-1, 1\}^{k\alpha n}$ .

Before describing the two maps, let us first state the desired conditions.

#### Success conditions for the reduction

- (1) **The reduction is locally well-defined.** Namely, there exist random strings  $R'$  and  $R''$  so that (i) Alice can get  $\mathbf{x}$  through the maps  $(\mathbf{x}', R') \mapsto \mathbf{x}''$  and  $(\mathbf{x}'', R'') \mapsto \mathbf{x}$  while Bob can get  $(M, \mathbf{z})$  through the maps  $(M', \mathbf{z}', R') \mapsto (M'', \mathbf{z}'')$  and  $(M'', \mathbf{z}'', R'') \mapsto (M, \mathbf{z})$ .
- (2) **The reduction is sound and complete.** Namely, (i)  $\mathbf{z}'' = M''\mathbf{x}'' \odot \mathbf{b}''$  and  $\mathbf{z} = M\mathbf{x} \odot \mathbf{b}$ . (ii) If  $\mathbf{b}' \sim \mathcal{D}_Y^{\alpha'n'}$  then  $\mathbf{b}'' \sim \mathcal{D}_1^{\alpha''n''}$  and  $\mathbf{b} \sim \mathcal{D}^{\alpha n}$ . Similarly if  $\mathbf{b}' \sim \mathcal{D}_N^{\alpha'n'}$  then  $\mathbf{b}'' \sim \mathcal{D}_2^{\alpha''n''}$  and  $\mathbf{b} \sim \mathcal{D}_{\mathbf{u}, \mathbf{v}}^{\alpha n}$ . (iii)  $\mathbf{x}'' \sim \text{Unif}(\{-1, 1\}^{n''})$ ,  $\mathbf{x} \sim \text{Unif}(\{-1, 1\}^n)$  and  $M$  is a uniformly random matrix conditioned on having exactly one “1” per row and at most one “1” per column.

In [Claim 7.8](#) and [Claim 7.9](#) we show that the above conditions hold except for an error event that occurs with tiny ( $\exp(-n)$ ) probability. For now, let us show that these conditions imply the success of the reduction. Assuming conditions (1) and (2) the rest is simple. Alice computes  $\mathbf{x}$  from  $\mathbf{x}'$ ,  $R'$  and  $R''$  and sends  $m = \Pi_A(\mathbf{x})$  to Bob, who computes  $(M, \mathbf{z})$  from  $M', \mathbf{z}', R'$  and  $R''$  and

outputs  $\Pi_B(m, M, \mathbf{z})$ . Conditions (1)-(2) combined with the bound on the error event imply that if  $\Pi$  has advantage  $\delta$  then  $\Pi'$  has advantage at least  $\delta - \exp(-n) \geq \delta/2$  as desired.

In the rest of this subsection, we describe the two maps and show that they satisfy the described success conditions. We wrap up the reduction and the proof of [Lemma 7.7](#) in the end.

**Step 3: Specify and analyze the first map.** We now turn to specifying the maps mentioned above and proving that they satisfy conditions (1)-(2). We start with  $(\mathbf{x}', M', \mathbf{b}', \mathbf{z}', R')$   $\mapsto$   $(\mathbf{x}'', M'', \mathbf{b}'', \mathbf{z}'')$ . For this part, we let  $R' \sim \text{Unif}(\{-1, 1\}^{n''-n'})$ . We set  $\mathbf{x}'' = (\mathbf{x}', R')$ . To get  $M''$ ,  $\mathbf{z}''$  and  $\mathbf{b}''$  we need some more notations. First, note that  $\alpha'n' = \alpha''n''$  due to the choice of parameters. Next, note that  $M''$  can be viewed as the stacking of matrices  $M'_1, \dots, M'_{\alpha'n'}$   $\in \{0, 1\}^{k' \times n'}$ . We first extend  $M'_i$  by adding all-zero columns at the end to get  $N''_i \in \{0, 1\}^{k' \times n''}$ . We then stack  $N''_i$  on top of  $P''_i \in \{0, 1\}^{(k-k') \times n''}$  to get  $M''_i$ , where  $(P''_i)_{j\ell} = 1$  if and only if  $\ell = n' + (i-1)k + j$ . See [Figure 4](#) for a pictorial description of  $N''_i$  and  $P''_i$ . We let  $M''$  be the stacking of  $M''_1, \dots, M''_{\alpha''n''}$ . Next we turn to  $\mathbf{b}''$ . Let  $\mathbf{b}' = (\mathbf{b}'(1), \dots, \mathbf{b}'(\alpha'n'))$ . Let  $\tilde{\mathbf{u}} = (u_{k'+1}, \dots, u_k)$  denote the common parts of  $\mathbf{u}$  and  $\mathbf{v}$ . We let  $\mathbf{b}''(i) = (\mathbf{b}'(i), \tilde{\mathbf{u}})$  and  $\mathbf{b}'' = (\mathbf{b}''(1), \dots, \mathbf{b}''(\alpha''n''))$ . Finally we let  $\mathbf{z}'' = M''\mathbf{x}'' \odot \mathbf{b}''$  as required. See [Figure 3](#) for a pictorial description.

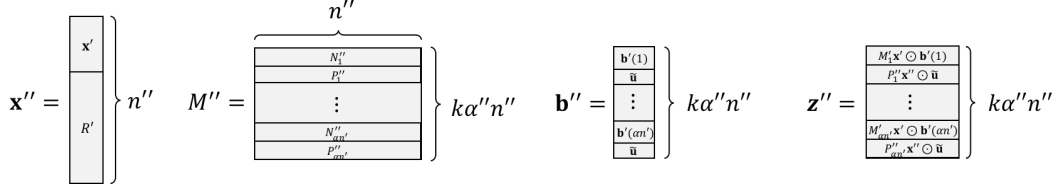


Figure 3: Pictorial description of  $(\mathbf{x}'', M'', \mathbf{b}'', \mathbf{z}'')$ .

Now, we verify that the first map satisfies the success conditions mentioned above.

**Claim 7.8.** *The first map in the reduction is locally well-defined, sound, and complete.*

*Proof.* To see the first map is locally well-defined, note that Alice can compute  $\mathbf{x}'' = (\mathbf{x}', R')$  locally. Similarly, Bob can compute  $M''$  locally by construction. As for  $\mathbf{z}''$ , note that  $\mathbf{z}''$  interleaves (in a predetermined order) the bits of  $\mathbf{z}'$  and those of  $(P_i \mathbf{x}'' \odot \tilde{\mathbf{u}})_{i \in [\alpha'n']}$ . Furthermore  $P_i \mathbf{x}''$  depends only on  $R'$  (since the first  $n'$  columns of all  $P_i$ s are zero). Thus Bob can locally compute  $P_i \mathbf{x}''$  for every  $i$ , and since  $\tilde{\mathbf{u}}$  is also known Bob can compute  $\mathbf{z}''$  locally.

To see the first map is sound and complete, (i)  $\mathbf{z}'' = M''\mathbf{x}'' \odot \mathbf{b}''$  follows from the construction. As for (ii), for each  $i \in [\alpha'n'] = [\alpha''n'']$ , if  $\mathbf{b}'_i \sim \mathcal{D}_Y = \text{Unif}(\{\mathbf{a}, -\mathbf{a}\})$ , then  $\mathbf{b}''_i \sim \text{Unif}(\{\mathbf{a}, \tilde{\mathbf{u}}\}, \{-\mathbf{a}, \tilde{\mathbf{u}}\})$ . Note that  $\mathbf{a}$  is chosen to be the uncommon part of  $\mathbf{u}$  and  $\mathbf{v}$  and hence  $(\mathbf{a}, \tilde{\mathbf{u}}) = \mathbf{u}$  and  $(-\mathbf{a}, \tilde{\mathbf{u}}) = \mathbf{v}$ . Thus,  $\mathbf{b}''_i \sim \text{Unif}(\{\mathbf{u}, \mathbf{v}\}) = \mathcal{D}_1$  as desired. Similarly, one can show that if  $\mathbf{b}'_i \sim \mathcal{D}_N$ , then  $\mathbf{b}''_i \sim \mathcal{D}_2$ . Finally, we have  $\mathbf{x}'' \sim \text{Unif}(\{-1, 1\}^{n''})$  by construction and hence (iii) holds.

This completes the proof of conditions (1)-(2) for the first step of the reduction.  $\square$

**Step 4: Specify and analyze the second map.** We now turn to the second map. Here  $R''$  will be composed of many smaller parts which we introduce now. Let  $\mathbf{y} \sim \text{Unif}(\{-1, 1\}^{n-n''})$ ,  $\mathbf{w} \sim \text{Bern}(2\varepsilon)^{\alpha n}$ . Let  $\Gamma \in \{0, 1\}^{n \times n}$  be a uniform permutation matrix. Let  $\mathbf{c} = (\mathbf{c}(1), \dots, \mathbf{c}((n-n'')/k))$  where  $\mathbf{c}(i) \sim \mathcal{D}_0$  are chosen independently. We let  $R'' = (\mathbf{y}, \mathbf{w}, \Gamma, \mathbf{c})$ . Let  $\#_w(i) = |\{j \in [i] \mid w_j = 1\}|$  denote the number 1's among the first  $i$  coordinates of  $\mathbf{w}$ . If  $\#_w(\alpha n) \geq \alpha'n''$  or if  $\alpha n - \#_w(\alpha n) \geq$



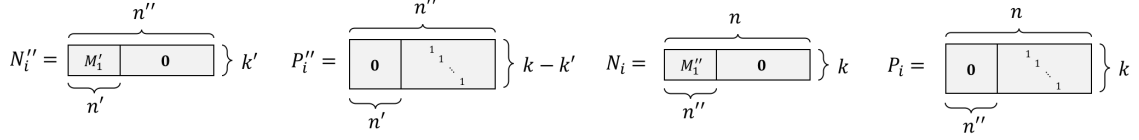


Figure 4: Pictorial description of  $N_i'', P_i'', N_i, P_i$ .

$(n - n'')/k$  we declare an error, Note  $\mathbb{E}[\#_w(n)] = \alpha''n''/2$  so the probability of error is negligible (specifically it is  $\exp(-n)$ ).

We now define the elements of  $(\mathbf{x}, M, \mathbf{b}, \mathbf{z})$ . We set  $\mathbf{x} = \Gamma(\mathbf{x}'', \mathbf{y})$  so  $\mathbf{x}$  is a random permutation of the concatenation of  $\mathbf{x}''$  and  $\mathbf{y}$ . Next, let  $M'' = (M_1'', \dots, M_{\alpha''n''}'')$  where  $M_i'' \in \{0, 1\}^{k \times n''}$ . We extend  $M_i''$  to  $N_i \in \{0, 1\}^{k \times n}$  by adding all-zero columns to the right. For  $i \in \{1, \dots, (n - n'')/k\}$ , let  $P_i \in \{0, 1\}^{k \times n}$  be given by  $(P_i)_{j\ell} = 1$  if and only if  $\ell = n'' + (i - 1)k + j$ . See Figure 4 for a pictorial description of  $N_i$  and  $P_i$ . Next we define a matrix  $\tilde{M} \in \{0, 1\}^{k\alpha n \times n} = (\tilde{M}_1, \dots, \tilde{M}_{\alpha n})$  where  $\tilde{M}_i \in \{0, 1\}^{k \times n}$  is defined as follows: If  $w_i = 1$  then we let  $\tilde{M}_i = N_{\#_w(i)}$  else we let  $\tilde{M}_i = P_{i - \#_w(i)}$ . Finally we let  $M = \tilde{M} \cdot \Gamma^{-1}$ . Next we turn to  $\mathbf{b}$ . Again let  $\mathbf{b}'' = (\mathbf{b}''(1), \dots, \mathbf{b}''(\alpha''n''))$ . We let  $\mathbf{b} = (\mathbf{b}(1), \dots, \mathbf{b}(\alpha n))$  where  $\mathbf{b}(i)$  is defined as follows: If  $w_i = 1$  then  $\mathbf{b}(i) = \mathbf{b}''(\#_w(i))$ , else  $\mathbf{b}(i) = \mathbf{c}(i - \#_w(i))$ . Finally,  $\mathbf{z} = M\mathbf{x} \odot \mathbf{b}$ . See Figure 5 for a pictorial description. This concludes the description of the map and we turn to analyzing its properties.

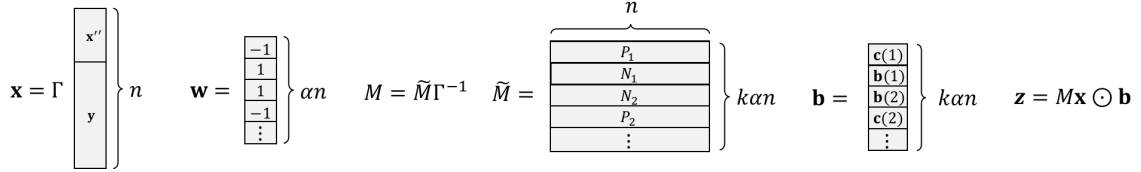


Figure 5: Pictorial description of  $\mathbf{x}, \mathbf{w}, M, \mathbf{b}, \mathbf{z}$ .

Now, we verify that the first map satisfies the success conditions mentioned above.

**Claim 7.9.** *If  $\#_w(\alpha n) \leq \alpha''n''$  and  $\alpha n - \#_w(\alpha n) \leq (n - n'')/k$ , then the second map in the reduction is locally well-defined, sound, and complete. In particular, the error event happens with probability at most  $\exp(-\Omega(n))$  over the randomness of  $R''$ .*

*Proof.* To see the second map is locally well-defined, first note that Alice can compute  $\mathbf{x} = \Gamma(\mathbf{x}'', \mathbf{y})$  from  $\mathbf{x}''$  and the shared randomness  $R''$  locally. As for Bob, note that the maximum index needed for  $N$  and  $\mathbf{b}''$  (resp.  $P$  and  $\mathbf{c}$ ) is at most  $\#_w(\alpha n)$  (resp.  $\alpha n - \#_w(i)$ ). Namely, if  $\#_w(\alpha n) \leq \alpha''n''$  and  $\alpha n - \#_w(\alpha n) \leq (n - n'')/k$ , then  $M$  and  $\mathbf{b}$  are well-defined. Also, using similar argument as in the proof of Claim 7.8, one can verify that  $M$  and  $\mathbf{b}$  can be locally computed by  $M''$ ,  $\mathbf{b}''$ , and the shared randomness  $R''$ .

To see the second map is sound and complete, (i)  $\mathbf{z} = M\mathbf{x} \odot \mathbf{b}$  directly follows from the construction. As for (ii), if  $\mathbf{b}' \sim \mathcal{D}_Y^{\alpha' n'}$ , from Claim 7.8 we know that  $\mathbf{b}'' \sim \mathcal{D}_1^{\alpha' n'} = \text{Unif}(\{\mathbf{u}, \mathbf{v}\})^{\alpha' n'}$ . Now, for each  $i \in [\alpha n]$ ,  $\mathbf{b}(i) = \mathbf{b}''(\#_w(i))$  with probability  $2\varepsilon$  and  $\mathbf{b}(i) = \mathbf{c}(i - \#_w(i))$  with probability  $1 - 2\varepsilon$ . As  $\mathbf{b}''(i') \sim \mathcal{D}_1$  for every  $i' \in [\alpha' n']$  and  $\mathbf{c}(i'') \sim \mathcal{D}_0$  for every  $i'' \in [(n - n'')/k]$ , we have  $\mathbf{b}(i) \sim (1 - 2\varepsilon)\mathcal{D}_0 + 2\varepsilon\mathcal{D}_1 = \mathcal{D}$  as desired. Similarly, one can show that for every  $i \in [\alpha' n'] = [\alpha''n'']$ , if  $\mathbf{b}'(i) \sim \mathcal{D}_N^{\alpha' n'}$ , then  $\mathbf{b}(i) \sim \mathcal{D}_{\mathbf{u}, \mathbf{v}}$ . Finally, we have  $\mathbf{x} \sim \text{Unif}(\{-1, 1\}^n)$  and  $M$  is

a uniformly random matrix with exactly one “1” per row and at most one “1” per column (due to the application of a random permutation  $\Gamma$ ) by construction.

This completes the proof of conditions (1)-(2) for the second step of the reduction.  $\square$

### Step 5: Wrap up the proof of Lemma 7.7.

*Proof of Lemma 7.7.* Let us start with setting up the parameters. Given  $k, \alpha \in (0, \alpha_0), n, \mathcal{D}$ , and incomparable pair  $(\mathbf{u}, \mathbf{v}) \in \text{supp}(\mathcal{D})$  and polarization amount  $\varepsilon = \varepsilon(\mathcal{D}, \mathbf{u}, \mathbf{v})$ , let  $k' = |\{i \in [k] \mid u_i \neq v_i\}|$ ,  $n' = (k'\varepsilon/k)n$ ,  $\alpha' = (2k/k')\alpha$ ,  $n'' = kn'/k'$ ,  $\alpha'' = \alpha'n'/n''$ , and  $\delta' = \delta/2$ .

Now, for the sake of contradiction, we assume that there exists a protocol  $\Pi = (\Pi_A, \Pi_B)$  for  $(\mathcal{D}, \mathcal{D}_{\mathbf{u}, \mathbf{v}})$ -RMD with advantage  $\delta$  and at most  $\tau\sqrt{n}$  bits of communication.

First, observe that  $n - n'' = (1 - \varepsilon)n$  and  $\alpha''n'' = 2\varepsilon\alpha n$ . As  $\mathbf{w} \sim \text{Bern}(\varepsilon)^{\alpha n}$ , we have  $\#_w(\alpha n) \leq \alpha''n''$  and  $\alpha n - \#_w(\alpha n) \leq (n - n'')/k$  with probability at least  $1 - \exp(-\Omega(n))$ . Thus, combine with Claim 7.8 and Claim 7.9, if  $(\mathbf{x}', M', \mathbf{z}')$  is a Yes (resp. No) instance of  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD, then the output of the reduction, i.e.,  $(\mathbf{x}, M, \mathbf{z})$ , is a Yes (resp. No) instance of  $(\mathcal{D}, \mathcal{D}_{\mathbf{u}, \mathbf{v}})$ -RMD with probability at least  $1 - \exp(-\Omega(n))$ . Moreover, Claim 7.8 and Claim 7.9 also show that the reduction can be implemented locally and hence Alice and Bob can run the protocol  $\Pi$  on  $(\mathbf{x}, M, \mathbf{z})$ . In particular, Alice and Bob computes  $\mathbf{x}$  and  $(M, \mathbf{z})$  using their inputs and shared randomness respectively. Then, Alice sends  $m = \Pi_A(\mathbf{x})$  to Bob and Bob outputs  $\Pi_B(m, M, \mathbf{z})$ . By the correctness of the reduction as well as that of the protocol, we know that Alice and Bob have advantage at least  $\delta - \exp(-\Omega(n)) \geq \delta/2 = \delta'$  in solving  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD with at most  $\tau\sqrt{n} = \tau\sqrt{(k/(k'\varepsilon))n'}$  bits of communication.

Finally, by Theorem 6.1, we know that there exists a constant  $\tau_0 > 0$  such that any protocol for  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD with advantage  $\delta'$  requires at least  $\tau_0\sqrt{n'}$  bits of communication. This implies that  $\tau \geq \tau_0\sqrt{k'\varepsilon/k}$ . We conclude that any protocol for  $(\mathcal{D}, \mathcal{D}_{\mathbf{u}, \mathbf{v}})$ -RMD with advantage  $\delta$  requires at least  $\tau\sqrt{n}$  bits of communication.  $\square$

## 7.3 Finite upper bound on the number of polarization steps

In this section we prove that there is a finite upper bound on the number of polarization steps needed to move from a distribution  $\mathcal{D} \in \Delta(\{-1, 1\}^k)$  to the canonical distribution with marginal  $\boldsymbol{\mu}(\mathcal{D})$ , i.e.,  $\mathcal{D}_{\boldsymbol{\mu}(\mathcal{D})}$ . Together with the indistinguishability result from Lemma 7.7 this allows us to complete the proof of Theorem 5.3 by going from  $\mathcal{D}_Y$  to  $\mathcal{D}_{\boldsymbol{\mu}(\mathcal{D}_Y)} = \mathcal{D}_{\boldsymbol{\mu}(\mathcal{D}_N)}$  and then to  $\mathcal{D}_N$  by using the triangle inequality for indistinguishability.

In this section we extend our considerations to functions  $A : \{-1, 1\}^k \rightarrow \mathbb{R}^{\geq 0}$ . Let  $\mathcal{F}(\{-1, 1\}^k) = \{A : \{-1, 1\}^k \rightarrow \mathbb{R}^{\geq 0}\}$ . For  $A \in \mathcal{F}(\{-1, 1\}^k)$ , let  $\mu_0(A) = \sum_{\mathbf{a} \in \{-1, 1\}^k} A(\mathbf{a})$ . Note  $\Delta(\{-1, 1\}^k) \subseteq \mathcal{F}(\{-1, 1\}^k)$  and  $A \in \Delta(\{-1, 1\}^k)$  if and only if  $A \in \mathcal{F}(\{-1, 1\}^k)$  and  $\mu_0(A) = \sum_{\mathbf{a} \in \{-1, 1\}^k} A(\mathbf{a}) = 1$ . We extend the definition of marginals, support, canonical distribution, potential and polarization operators to  $\mathcal{F}(\{-1, 1\}^k)$ . In particular we let  $\boldsymbol{\mu}(A) = (\mu_0, \mu_1, \dots, \mu_k)$  where  $\mu_0 = \mu_0(A)$  and  $\mu_j = \sum_{\mathbf{a} \in \{-1, 1\}^k} a_j A(\mathbf{a})$  for  $j \in [k]$ . We also define canonical function and polarization operators so as to preserve  $\boldsymbol{\mu}(A)$ . So given arbitrary  $A$ , let  $\mathcal{D} = \frac{1}{\mu_0(A)} \cdot A$ . Note  $\mathcal{D} \in \Delta(\{-1, 1\}^k)$ . For  $\boldsymbol{\mu} = (\mu_0, \mu_1, \dots, \mu_k) \in \mathbb{R}^{k+1}$ , we define  $A_{\boldsymbol{\mu}} = \mu_0 \cdot \mathcal{D}_{\boldsymbol{\mu}'}$  where  $\boldsymbol{\mu}' = (\mu_1/\mu_0, \dots, \mu_k/\mu_0)$  to be the canonical function associated with  $\boldsymbol{\mu}$ . We remark that by Lemma 7.4 and Lemma 7.5,  $A_{\boldsymbol{\mu}(A)}$  is the unique function such that (i) it has the same marginals as  $A$  and (ii) it supports a chain.

**Definition 7.10** (Polarization length). For distribution  $A \in \mathcal{F}(\{-1, 1\}^k)$ , let  $N(A)$  be the smallest  $t$  such that there exists a sequence  $\mathbf{A} = A_0, A_1, \dots, A_t$  such that  $A_0 = A$ ,  $A_t = A_{\boldsymbol{\mu}(A)}$  is canonical and for every  $i \in [t]$  it holds that there exists incomparable  $\mathbf{u}_i, \mathbf{v}_i \in \text{supp}(A_{i-1})$  such that  $A_i = (A_{i-1})_{\mathbf{u}_i, \mathbf{v}_i}$ . If no such finite sequence exists then let  $N(A)$  be infinite. Let  $N(k) = \sup_{A \in \mathcal{F}(\{-1, 1\}^k)} \{N(A)\}$ . Again, if  $N(A) = \infty$  for some  $A$  or if no finite upper bound exists,  $N(k)$  is defined to be  $\infty$ .

Note that if  $\mathcal{D} \in \Delta(\{-1, 1\}^k)$  so is every element in the sequence, so the polarization length bound below applies also to distributions. Our main lemma in this subsection is the following:

**Lemma 7.11** (A finite upper bound on  $N(k)$ ).  $N(k)$  is finite for every finite  $k$ . Specifically  $N(k) \leq (k^2 + 3)(1 + N(k - 1))$ .

We prove [Lemma 7.11](#) constructively in the following four steps.

**Step 1: Description of the algorithm POLARIZE.** Let us start with some notations. For  $A \in \mathcal{F}(\{-1, 1\}^k)$  we let  $A|_{x_\ell=b}$  denote the function  $A$  restricted to the subcube  $\{-1, 1\}^{\ell-1} \times \{b\} \times \{-1, 1\}^{k-\ell}$ . Note that  $A$  restricted to subcubes is effectively a  $(k - 1)$ -dimensional function and we will use this reduction in dimension in our recursive algorithm.

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**Algorithm 2** POLARIZE( $\cdot$ )

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**Input:**  $A \in \mathcal{F}(\{-1, 1\}^k)$ .

- 1: **if**  $k=2$  **then**
  - 2:     **Output:**  $A_{(-1,1),(1,-1)}$ .
  - 3:  $(A_0)|_{x_k=-1} \leftarrow \text{POLARIZE}(A|_{x_k=-1})$ ;  $(A_0)|_{x_k=1} \leftarrow \text{POLARIZE}(A|_{x_k=1})$ ;  $t \leftarrow 0$ .
  - 4: Let  $(-1)^k = \mathbf{a}_t(0) < \dots < \mathbf{a}_t(k-1) = (1^{k-1}, -1)$  be a chain supporting  $(A_t)|_{x_k=-1}$ .
  - 5: Let  $((-1)^{k-1}, 1) = \mathbf{b}_t(0) < \dots < \mathbf{b}_t(k-1) = 1^k$  be a chain supporting  $(A_t)|_{x_k=1}$ .
  - 6: **while**  $\exists(i, j)$  with  $j < k-1$  such that  $\mathbf{a}_t(i) \vee \mathbf{b}_t(j) = 1^k$  and  $A_t(\mathbf{a}_t(i)), A_t(\mathbf{b}_t(j)) > 0$  **do**
  - 7:     Let  $(i_t, j_t)$  be the lexicographically smallest such pair  $(i, j)$ .
  - 8:      $B_t \leftarrow (A_t)_{\mathbf{a}_t(i_t), \mathbf{b}_t(j_t)}$ .
  - 9:      $(A_{t+1})|_{x_k=-1} \leftarrow \text{POLARIZE}(B_t|_{x_k=-1})$ ;  $(A_{t+1})|_{x_k=1} \leftarrow (B_t)|_{x_k=1}$ .
  - 10:      $t \leftarrow t + 1$ .
  - 11:     Let  $(-1)^k = \mathbf{a}_t(0) < \dots < \mathbf{a}_t(k-1) = (1^{k-1}, -1)$  be a chain supporting  $(A_t)|_{x_k=-1}$ .
  - 12:     Let  $((-1)^{k-1}, 1) = \mathbf{b}_t(0) < \dots < \mathbf{b}_t(k-1) = 1^k$  be a chain supporting  $(A_t)|_{x_k=1}$ .
  - 13: Let  $\ell \in [k]$  be such that for every  $\mathbf{a} \in \{-1, 1\}^k \setminus \{1^k\}$  we have  $A_t(\mathbf{a}) > 0 \Rightarrow a_\ell = -1$ .
  - 14:  $(A_{t+1})|_{x_\ell=-1} \leftarrow \text{POLARIZE}(A_t)|_{x_\ell=-1}$ .  $(A_{t+1})|_{x_\ell=1} \leftarrow (A_t)|_{x_\ell=1}$ .
  - 15: **Output:**  $A_{t+1}$ .
- 

The goal of the rest of the proof is to show that [Algorithm 2](#) terminates after a finite number of steps and outputs  $A_{\boldsymbol{\mu}(A)}$ .

**Step 2: Correctness assuming POLARIZE terminates.**

**Claim 7.12** (Correctness condition of POLARIZE). For every  $A \in \mathcal{F}(\{-1, 1\}^k)$ , if POLARIZE terminates, then  $\text{POLARIZE}(A) = A_{\boldsymbol{\mu}(A)}$ . In particular,  $\text{POLARIZE}(A)$  has the same marginals as  $A$  and is supported on a chain.

*Proof.* First, by the definition of the polarization operator (Definition 7.3), the marginals of  $A_t$  are the same for every  $t$ . So in the rest of the proof, we focus on inductively showing that if POLARIZE terminates, then POLARIZE( $A$ ) is supported on a chain.

For the base case where  $k = 2$ , we always have POLARIZE( $A$ ) =  $A_{(-1,1),(1,-1)}$  supported on a chain as desired.

When  $k > 2$ , note that when the algorithm enters the Clean-up stage, if we let  $m$  and  $n$  denote the largest indices such that  $A_t(\mathbf{a}_t(m)), A_t(\mathbf{b}_t(n)) > 0$  and  $A_t(\mathbf{b}_t(n)) \neq 1^k$ , then the condition that  $\mathbf{a}_t(m) \vee \mathbf{b}_t(n) \neq 1^k$  implies that there is a coordinate  $\ell$  such that  $\mathbf{a}_t(m)_\ell = \mathbf{b}_t(n)_\ell = -1$ . Since every  $\mathbf{c}$  such that  $A_t(\mathbf{c}) > 0$  and  $c_k = -1$  satisfies  $\mathbf{c} \leq \mathbf{a}_t(m)$ , we have  $A_t(\mathbf{c}) > 0$  implies  $c_\ell = -1$ . Similarly for every  $\mathbf{c} \neq 1^k$  such that  $c_k = 1$ , we have  $A_t(\mathbf{c}) > 0$  implies  $c_\ell = -1$ . We conclude that  $A_t$  is supported on  $\{1^k\} \cup \{\mathbf{c} \mid c_\ell = -1\}$ . Thus, by the induction hypothesis, after polarizing the subcube  $x_\ell = -1$  and leaving the subcube  $x_\ell = 1$  unchanged, we get that the resulting function  $A_{t+1}$  is supported on a chain as desired and complete the induction. We conclude that if POLARIZE terminates, we have POLARIZE( $A$ ) =  $A_{\mu(A)}$ .  $\square$

**Step 3: Invariant in POLARIZE.** Now, in the rest of the proof of Lemma 7.11, the goal is to show that for every input  $A$ , the number of iterations of the while loop in Algorithm 2 is finite. The key claim (Claim 7.16) here asserts that the sequence of pairs  $(i_t, j_t)$  is monotonically increasing in lexicographic order. Once we establish this claim, it follows that there are at most  $k^2$  iterations of the while loop and so  $N(k) \leq (k^2 + 3) \cdot (1 + N(k - 1))$ , proving Lemma 7.11. Before proving Claim 7.16, we establish the following properties that remain invariant after every iteration of the while loop.

**Claim 7.13.** *For every  $t \geq 0$ , we have  $\forall b \in \{-1, 1\}$ ,  $(A_t)|_{x_k=b}$  is supported on a chain.*

*Proof.* For  $b = -1$ , the claim follows from the correctness of the recursive call to POLARIZE. For  $b = 1$ , we claim by induction on  $t$  that the supporting chain  $\mathbf{b}_t(0) < \dots < \mathbf{b}_t(k - 1)$  never changes (with  $t$ ). To see this, note that  $\mathbf{b}_t(k - 1) = 1^k$  is the only point in the subcube  $\{x_k = 1\}$  that increases in value compared to  $A_t$ , and this is already in the supporting chain. Thus  $\mathbf{b}_t(0) < \dots < \mathbf{b}_t(k - 1)$  continues to be a supporting chain for  $(A_{t+1})|_{x_k=1}$ .  $\square$

For  $\mathbf{c} \in \{-1, 1\}^k$ , we say that a function  $A : \{-1, 1\}^k \rightarrow \mathbb{R}^{\geq 0}$  is  **$\mathbf{c}$ -subcube-respecting** ( **$\mathbf{c}$ -respecting**, for short) if for every  $\mathbf{c}'$  such that  $A(\mathbf{c}') > 0$ , we have  $\mathbf{c}' \geq \mathbf{c}$  or  $\mathbf{c}' \leq \mathbf{c}$ . We say that  $A$  is  **$\mathbf{c}$ -downward-respecting** if  $A$  is  $\mathbf{c}$ -respecting and the points in the support of  $A$  above  $\mathbf{c}$  form a partial chain, specifically, if  $\mathbf{u}, \mathbf{v} > \mathbf{c}$  have  $A(\mathbf{u}), A(\mathbf{v}) > 0$  then either  $\mathbf{u} \geq \mathbf{v}$  or  $\mathbf{v} \geq \mathbf{u}$ .

Note that if  $A$  is supported on a chain then  $A$  is  $\mathbf{c}$ -respecting for every point  $\mathbf{c}$  in the chain. Conversely, if  $A$  is supported on a chain and  $A$  is  $\mathbf{c}$ -respecting, then  $A$  is supported on a chain that includes  $\mathbf{c}$ .

**Claim 7.14** (Polarization on subcubes). *Let  $A$  be a  $\mathbf{c}$ -respecting function and let  $\tilde{A}$  be obtained from  $A$  by a finite sequence of polarization updates, as in Definition 7.3. Then  $\tilde{A}$  is also  $\mathbf{c}$ -respecting. Furthermore if  $A$  is  $\mathbf{c}$ -downward-respecting and  $\mathbf{w} > \mathbf{c}$  then  $\tilde{A}$  is also  $\mathbf{c}$ -downward-respecting and  $A(\mathbf{w}) = \tilde{A}(\mathbf{w})$ .*

*Proof.* Note that it suffices to prove the claim for a single update by a polarization operator since the rest follows by induction. So let  $\tilde{A} = A_{\mathbf{u}, \mathbf{v}}$  for incomparable  $\mathbf{u}, \mathbf{v} \in \text{supp}(A)$ .

Since  $A$  is  $\mathbf{c}$ -respecting, and  $\mathbf{u}, \mathbf{v}$  are incomparable, either  $\mathbf{u} \leq \mathbf{c}, \mathbf{v} \leq \mathbf{c}$  or  $\mathbf{u} \geq \mathbf{c}, \mathbf{v} \geq \mathbf{c}$ . Suppose the former is true, then  $\mathbf{u} \vee \mathbf{v} \leq \mathbf{c}$  and  $\mathbf{u} \wedge \mathbf{v} \leq \mathbf{c}$ , and hence,  $\tilde{A}$  is  $\mathbf{c}$ -respecting. Similarly, in the

case when  $\mathbf{u} \geq \mathbf{c}, \mathbf{v} \geq \mathbf{c}$ , we can show that  $\tilde{A}$  is  $\mathbf{c}$ -respecting. The furthermore part follows by noticing that for  $\mathbf{u}$  and  $\mathbf{v}$  to be incomparable if  $A$  is  $\mathbf{c}$ -downward-respecting and  $A(\mathbf{u}), A(\mathbf{v}) > 0$ , then  $\mathbf{u}, \mathbf{v} \leq \mathbf{c}$ , and so the update changes  $A$  only at points below  $\mathbf{c}$ .  $\square$

The following claim asserts that in every iteration of the while loop, by the lexicographically minimal choice of  $(i_t, j_t)$ , there exists a coordinate  $h$  such that every vector  $\mathbf{c} < a_t(i_t)$  in the support of  $A_t, B_t$ , or  $A_{t+1}$  has  $c_h = -1$ , and every vector  $\mathbf{c} \neq 1^k$  in the support of  $(A_t)|_{x_k=1}$  has  $c_h = -1$ .

**Claim 7.15.** *For every  $t \geq 0$ ,  $\exists h \in [k-1]$  such that  $\forall \mathbf{c} \in \{-1, 1\}^k$ , if  $\mathbf{c} \in \text{supp}(A_t) \cup \text{supp}(B_t) \cup \text{supp}(A_{t+1})$ , then the following hold:*

- If  $\mathbf{c} < \mathbf{a}_t(i_t)$ , then  $c_h = -1$ .
- If  $c_k = 1$  and  $\mathbf{c} \neq 1^k$ , then  $c_h = -1$ .

*Proof.* Since  $(i_t, j_t)$  is the lexicographically smallest incomparable pair in the support of  $A_t$ , for  $i < i_t, j < k-1$ , and  $A_t(\mathbf{a}(i)), A_t(\mathbf{b}(j)) > 0$ , we have  $\mathbf{a}(i) \vee \mathbf{b}(j) \neq 1^k$ . Let  $m$  be the largest index smaller than  $i_t$  such that  $A_t(\mathbf{a}_t(m)) > 0$ . Similarly, let  $n < k-1$  be the largest index such that  $A_t(\mathbf{b}_t(n)) > 0$ . Then the fact that  $\mathbf{a}_t(m) \vee \mathbf{b}_t(n) \neq 1^k$  implies that there exists  $h \in [k-1]$  such that  $\mathbf{a}_t(m)_h = \mathbf{b}_t(n)_h = -1$ . Now, using the fact (from Claim 7.13) that  $(A_t)|_{x_k=-1}$  is supported on a chain, we conclude that for every  $\mathbf{c} < \mathbf{a}_t(i_t)$ ,  $A_t(\mathbf{c}) > 0$  implies that  $\mathbf{c} \leq \mathbf{a}_t(m)$  and hence,  $c_h = -1$ . Similarly, for every vector  $\mathbf{c} \neq 1^k$  in the support of  $(A_t)|_{x_k=1}$ , by the maximality of  $n$ , we have  $c_h = -1$ .

We now assert that the same holds for  $B_t$ . First, recall that  $\text{supp}(B_t) \subset \text{supp}(A_t) \cup \{1^k, \mathbf{a}_t(i_t) \wedge \mathbf{b}_t(j_t)\}$  since  $B_t = (A_t)_{\mathbf{a}_t(i_t), \mathbf{b}_t(j_t)}$ . Next, note that the only point (other than  $1^k$ ) where  $B_t$  is larger than  $A_t$  is  $\mathbf{a}_t(i_t) \wedge \mathbf{b}_t(j_t)$ . It suffices to show that  $(\mathbf{a}_t(i_t) \wedge \mathbf{b}_t(j_t))_h = -1$ . We have  $\mathbf{a}_t(i_t) \wedge \mathbf{b}_t(j_t) \leq \mathbf{b}_t(j_t) \leq \mathbf{b}_t(n)$  and hence  $(\mathbf{a}_t(i_t) \wedge \mathbf{b}_t(j_t))_h = -1$ .

Finally, we assert that same holds also for  $A_{t+1}$ . Since  $A_{t+1}|_{x_k=1} = B_t|_{x_k=1}$ , the second item in the claim follows trivially. To prove the first item, let us consider  $\mathbf{a}' \in \{-1, 1\}^k$  defined as follows:  $\mathbf{a}'_h = -1$  and  $\mathbf{a}'_r = \mathbf{a}_t(i_t)_r$  for  $r \neq h$ . Note that  $B_t|_{x_k=-1}$  is  $\mathbf{a}_t(i_t)$ -respecting since potentially the only new point in its support (compared to  $A_t|_{x_k=-1}$ ) is  $\mathbf{a}_t(i_t) \wedge \mathbf{b}_t(j_t) \leq \mathbf{a}_t(i_t)$ . From the previous paragraph we also have that if  $B_t(\mathbf{c}) > 0$  and  $\mathbf{c} < \mathbf{a}_t(i_t)$ , then  $c_h = -1$  and hence,  $\mathbf{c} \leq \mathbf{a}'$ . On the other hand, if  $B_t(\mathbf{c}) > 0$  and  $\mathbf{c} \geq \mathbf{a}_t(i_t)$ , then  $\mathbf{c} \geq \mathbf{a}'$ . Therefore,  $B_t|_{x_k=-1}$  is  $\mathbf{a}'$ -respecting. By applying Claim 7.14, we conclude that  $(A_{t+1})|_{x_k=-1}$  is also  $\mathbf{a}'$ -respecting. It follows that if  $\mathbf{c} < \mathbf{a}(i_t)$  and  $A_{t+1}(\mathbf{c}) > 0$ , then  $\mathbf{c} \leq \mathbf{a}'$  and so  $c_h = -1$ .  $\square$

**Step 4: Proof of Lemma 7.11.** The following claim establishes that the while loop in the POLARIZE algorithm terminates after a finite number of iterations.

**Claim 7.16.** *For every  $t \geq 0$ ,  $(i_t, j_t) < (i_{t+1}, j_{t+1})$  in lexicographic ordering.*

*Proof.* Consider the chain  $\mathbf{a}_{t+1}(0) < \dots < \mathbf{a}_{t+1}(k-1)$  supporting  $A_{t+1}|_{x_k=-1}$ . Note that for  $i \geq i_t$ ,  $A_{t+1}|_{x_k=-1}$  is  $\mathbf{a}_t(i)$ -respecting (since  $A_t|_{x_k=-1}$  and  $B_t|_{x_k=-1}$  were also so). It follows that we can use  $\mathbf{a}_{t+1}(i) = \mathbf{a}_t(i)$  for  $i \geq i_t$ . (In particular,  $A_t|_{x_k=-1}$  is  $\mathbf{a}_t(i)$ -respecting because it is supported on a chain containing  $a_t(i)$ . Next  $B_t|_{x_k=-1}$  is  $\mathbf{a}_t(i)$ -respecting since potentially the only new point in its support is  $\mathbf{a}_t(i_t) \wedge \mathbf{b}_t(j_t) \leq \mathbf{a}_t(i)$ . Finally,  $A_{t+1}|_{x_k=-1}$  is also  $\mathbf{a}_t(i)$ -respecting using Claim 7.14. Thus we can build a chain containing  $\mathbf{a}_t(i)$  that supports  $A_{t+1}|_{x_k=-1}$ .) Now consider  $i < i_t$ . We

must have  $\mathbf{a}_{t+1}(i) < \mathbf{a}_{t+1}(i_t) = \mathbf{a}_t(i_t)$ . By [Claim 7.15](#), there exists  $h \in [k-1]$  such that for  $i < i_t$ ,  $\mathbf{a}_{t+1}(i)_h = -1$ .

We now turn to analyzing  $(i_{t+1}, j_{t+1})$ . Note that by definition,  $A_{t+1}(\mathbf{a}_{t+1}(i_{t+1})) > 0$  and  $A_{t+1}(\mathbf{b}_{t+1}(j_{t+1})) > 0$ . First, let us show that  $i_t \leq i_{t+1}$ . On the contrary, let us assume that  $i_{t+1} < i_t$ . It follows from the above paragraph that  $\mathbf{a}_{t+1}(i_{t+1})_h = -1$ . Also, for every  $\mathbf{b}_{t+1}(j)$  with  $j < k-1$  and  $A_{t+1}(\mathbf{b}_{t+1}(j)) > 0$ , we have  $\mathbf{b}_{t+1}(j)_h = -1$ . Therefore,  $\mathbf{a}(i_{t+1}) \vee \mathbf{b}(j_{t+1}) \neq 1^k$  (in particular  $(\mathbf{a}(i_{t+1}) \vee \mathbf{b}(j_{t+1}))_h = -1$ ), which is a contradiction.

Next, we show that if  $i_{t+1} = i_t$ , then  $j_{t+1} \geq j_t$ . By the minimality of  $(i_t, j_t)$  in the  $t$ -th round, for  $j < j_t$  such that  $A_t(b_t(j)) > 0$ , we have  $\mathbf{a}_t(i_t) \vee \mathbf{b}_t(j) \neq 1^k$ . Since  $i_{t+1} = i_t$ ,  $\mathbf{a}_{t+1}(i_{t+1}) = \mathbf{a}_{t+1}(i_t) = \mathbf{a}_t(i_t)$ . We already noted in the proof of [Claim 7.13](#) that  $\mathbf{b}_t(0) < \dots < \mathbf{b}_t(k-1)$  is also a supporting chain for  $(A_{t+1})|_{x_k=1}$ . The only point where the function  $A_{t+1}|_{x_k=1}$  has greater value than  $A_t|_{x_k=1}$  is  $1^k$ . Therefore, for  $j < j_t$  such that  $A_{t+1}(b_{t+1}(j)) > 0$ , we have  $\mathbf{a}_{t+1}(i_{t+1}) \vee \mathbf{b}_{t+1}(j) \neq 1^k$  and hence,  $j_{t+1} \geq j_t$ .

So far, we have established that  $(i_{t+1}, j_{t+1}) \geq (i_t, j_t)$  in lexicographic ordering. Finally, we will show that  $(i_{t+1}, j_{t+1}) \neq (i_t, j_t)$  by proving that at least one of  $A_{t+1}(\mathbf{a}_{t+1}(i_t))$  and  $A_{t+1}(\mathbf{b}_{t+1}(j_t))$  is zero. The polarization update ensures that at least one of  $B_t(\mathbf{a}_t(i_t))$  and  $B_t(\mathbf{b}_t(j_t))$  is zero. If  $B_t(\mathbf{b}_t(j_t)) = 0$ , then by definition, we have  $A_{t+1}(\mathbf{b}_{t+1}(j_t)) = A_{t+1}(\mathbf{b}_t(j_t)) = 0$ . Finally to handle the case  $B_t(\mathbf{a}_t(i_t)) = 0$ , let us again define  $\mathbf{a}'$  as:  $\mathbf{a}'_h = -1$  and  $\mathbf{a}'_r = \mathbf{a}_t(i_t)_r$  for  $r \neq h$ , where  $h$  is as given by [Claim 7.15](#). We assert that  $B_t|_{x_k=-1}$  is  $\mathbf{a}'$ -downward-respecting. As shown in the proof of [Claim 7.15](#), we have  $B_t|_{x_k=-1}$  is  $\mathbf{a}'$ -respecting. The support of  $B_t|_{x_k=-1}$  is contained in  $\{\mathbf{a}_t(0), \dots, \mathbf{a}_t(k-1)\} \cup \{\mathbf{a}_t(i_t) \wedge \mathbf{b}_t(j_t)\}$  and  $\mathbf{a}_t(i_t) \wedge \mathbf{b}_t(j_t) < \mathbf{a}_t(i_t)$ , and by [Claim 7.15](#),  $\mathbf{a}_t(i_t) \wedge \mathbf{b}_t(j_t) \leq \mathbf{a}'$ . It follows that  $B_t|_{x_k=-1}$  is  $\mathbf{a}'$ -downward-respecting. Finally, by the furthermore part of [Claim 7.14](#) applied to  $B_t|_{x_k=-1}$  and  $\mathbf{w} = \mathbf{a}_t(i_t)$ , we get that  $A_{t+1}(\mathbf{a}_{t+1}(i_t)) = A_{t+1}(\mathbf{a}_t(i_t)) = B_t(\mathbf{a}_t(i_t)) = 0$ . It follows that  $(i_{t+1}, j_{t+1}) \neq (i_t, j_t)$ .  $\square$

*Proof of Lemma 7.11.* By [Claim 7.12](#), we know that if [Algorithm 2](#) terminates, we have  $\text{POLARIZE}(A) = A_{\mu(A)}$ . Hence, the maximum number of polarization updates used in  $\text{POLARIZE}$  (on input from  $\mathcal{F}(\{-1, 1\}^k)$ ) serves as an upper bound for  $N(k)$ . By [Claim 7.16](#), we know that there are at most  $k^2$  iterations of the while loop and so  $N(k) \leq (k^2 + 3) \cdot (1 + N(k-1))$  as desired.  $\square$

## 7.4 Putting it together

We now have the ingredients in place to prove [Theorem 5.3](#).

*Proof of Theorem 5.3.* Given distribution  $\mathcal{D}_Y, \mathcal{D}_N$  with  $\mu = \mu(\mathcal{D}_Y) = \mu(\mathcal{D}_N)$ . Applying [Lemma 7.11](#) to  $\mathcal{D}_Y$  we get there exist  $\mathcal{D}_0 = \mathcal{D}_Y, \mathcal{D}_1, \dots, \mathcal{D}_t = \mathcal{D}_\mu$  such that  $\mathcal{D}_{i+1} = (\mathcal{D}_i)_{\mathbf{u}(i), \mathbf{v}(i)}$ , i.e.,  $\mathcal{D}_i$  is an update of  $\mathcal{D}_i$ , with  $t \leq N(k) < \infty$ . Similarly applying [Lemma 7.11](#) to  $\mathcal{D}_N$  we get there exist  $\mathcal{D}'_0 = \mathcal{D}_N, \mathcal{D}'_1, \dots, \mathcal{D}'_{t'} = \mathcal{D}_\mu$  such that  $\mathcal{D}'_{i+1} = (\mathcal{D}'_i)_{\mathbf{u}'(i), \mathbf{v}'(i)}$  with  $t' \leq N(k) < \infty$ . Applying [Lemma 7.7](#) with  $\delta' = \delta/(2N(k))$  to the pairs  $\mathcal{D}_i$  and  $\mathcal{D}_{i+1}$  we get there exists  $\tau_i$  such that every protocol for  $(\mathcal{D}_i, \mathcal{D}_{i+1})$ -RMD requires  $\tau_i \sqrt{n}$  bits of communication to achieve advantage  $\delta'$ . Similarly applying [Lemma 7.7](#) again with  $\delta' = \delta/(2N(k))$  to the pairs  $\mathcal{D}'_i$  and  $\mathcal{D}'_{i+1}$  we get there exists  $\tau'_i$  such that every protocol for  $(\mathcal{D}'_i, \mathcal{D}'_{i+1})$ -RMD requires  $\tau'_i \sqrt{n}$  bits of communication to achieve advantage  $\delta'$ . Letting  $\tau = \min\{\min_{i \in [t]}\{\tau_i\}, \min_{i \in [t']}\{\tau'_i\}\}$ , we get, using the triangle inequality for indistinguishability, that every protocol  $\Pi$  for  $(\mathcal{D}_Y, \mathcal{D}_N)$ -RMD achieving advantage  $\delta \geq (t + t')\delta'$  requires  $\tau \sqrt{n}$  communication.  $\square$

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