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Influence of a Set of Variables on a Boolean Function

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Abstract

The influence of a set of variables on a Boolean function has three separate definitions in the literature, the first due to Ben-Or and Linial (1989), the second due to Fischer et al. (2002) and Blais (2009) and the third due to Tal (2017). The goal of the present work is to carry out a comprehensive study of the notion of influence of a set of variables on a Boolean function. To this end, we introduce a definition of this notion using the auto-correlation function. A modification of the definition leads to the notion of pseudo-influence. Somewhat surprisingly, it turns out that the auto-correlation based definition of influence is equivalent to the definition introduced by Fischer et al. (2002) and Blais (2009) and the notion of pseudo-influence is equivalent to the definition of influence considered by Tal (2017). Extensive analysis of influence and pseduo-influence as well as the Ben-Or and Linial notion of influence is carried out and the relations between these notions are established.
Keywords: Boolean function, influence, Fourier transform, resilient functions.

1 Introduction

The influence of a variable on a Boolean function is a well studied concept, particularly due to the Fourier Entropy/Influence (FEI) conjecture [6]. Extension from the influence of a single variable to the influence of a set of variables has been proposed in the literature. We have found three such extensions. The first definition was given by Ben-Or and Linial [1] in 1989. A different definition due to Fischer et al. [5] appeared in 2002 and the same definition was considered in 2009 by Blais [2]. A third definition appears in Tal [10] in 2017. All of these definitions agree with each other in the case of a single variable, but if the set contains more than one variable, then in general the values provided by the three notions of influence are different. This raises the questions of what is the correct notion of influence of a set of variables on a Boolean function, what should be the basic properties that one should expect from a notion of influence and also what are the relationships between the various definitions of influence. To the best of our knowledge, the literature does not contain any study along these lines.

We put forward a definition of influence of a set of variables on a Boolean function using the auto-correlation function. The notion is systematically studied and several characterisations are obtained in terms of the Fourier transform. The basic properties of influence are establised. In particular, it is shown that the influence is zero if and only if the function is degenerate on the set of variables, it satisfies sub-additivity, it is monotone increasing with the size of the set of variables, and the condition under which the influence attains its maximum value of one is characterised. For t in the range 1 to n, the total influence of all subsets of size t is considered. We prove that the total influence is maximum

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if and only if the function is $(n-t)$-resilient. Further, it is shown that the total influence increases monotonically with t.

A modification of the auto-correlation based definition of influence gives rise to another concept which we term as pseudo-influence. The pseudo-influence has a nice Fourier transform based characterisation. Our reasons for terming this concept pseudo-influence is that it is possible for the pseudo-influence to be zero even if the function is not degenerate on the set of variables in question; also, it is monotone decreasing on the size of the set of variables. We show that for any set of variables, the pseudo-influence is always at most the influence.

The auto-correlation based definitions of influence and pseudo-influence turns out to be same as the notions of influence introduced in [5, 2] and [10] respectively. This is demonstrated by the equivalence of the Fourier transform based characterisations of influence and pseudo-influence to the Fourier transform based characterisations obtained in [5, 2] and [10] respectively. Such equivalence is surprising, since the auto-correlation based definitions and the definitions given in [5, 2, 10] are quite different.

We make a systematic study of the Ben-Or and Linial (BL) notion of influence. We show that similar to the notion of influence, the BL-influence is zero if and only the function is degenerate on the set of variables and it is monotone increasing with the size of the set of variables. The BL-influence, however, does not satisfy sub-additivity. We prove that the value of the BL-influence is always at least the value of the influence and the condition under which equality holds is precisely characterised. Based on our analysis, we argue that the auto-correlation based definition may be considered to be the "correct" notion of influence.

Section 2 introduces the background and the notation. The notions of influence and pseudo-influence, their properties and equivalence to the notions of influence introduced by Fischer et al. and Tal are described in Section 3. The BL-influence is studied in Section 4.

2 Background and Notation

Let $\mathbb{F}_2 = \{0, 1\}$ denote the finite field consisting of two elements with addition represented by $\oplus$ and multiplication by $\cdot$; often, for $x, y \in \mathbb{F}_2$, the product $x \cdot y$ will be written as xy.

By $[n]$ we will denote the set $\{1, \ldots, n\}$. For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}_2^n$, the support of $\mathbf{x}$ will be denoted by $\mathsf{supp}(\mathbf{x})$ which is the set $\{i : x_i = 1\}$; the weight of $\mathbf{x}$ will be denoted as $\mathsf{wt}(\mathbf{x})$ and is equal to $\#\mathsf{supp}(\mathbf{x})$. For $i \in [n]$, $\mathbf{e}_i$ denotes the vector in $\mathbb{F}_2^n$ whose i-th component is 1 and all other components are 0. For $\mathbf{x} = (x_1, \ldots, x_2), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{F}_2^n$, we write $\mathbf{x} \leq \mathbf{y}$ if $x_i = 1$ implies $y_i = 1$ for $i = 1, \ldots, n$. For $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{F}_2^n$, the inner product $\langle \mathbf{x}, \mathbf{y} \rangle$ of $\mathbf{x}$ and $\mathbf{y}$ is defined to be $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 \oplus \cdots \oplus x_n y_n$. For $T \subseteq [n]$, χ_T denotes the vector in $\mathbb{F}_2^n$ where the i-th component of χ_T is 1 if and only if $i \in T$; further, $\overline{T}$ will denote the set $[n] \setminus T$.

An n-variable Boolean function f is a map $f : \mathbb{F}_2^n \to \mathbb{F}_2$. Variables will be written in upper case and vector of variables in bold upper case. For $\mathbf{X} = (X_1, \ldots, X_n)$, an n-variable Boolean function f will be written as $f(\mathbf{X})$. The support of a Boolean function f will be denoted by $\mathsf{supp}(f)$ which is the set $\{\mathbf{x} : f(\mathbf{x}) = 1\}$; the weight of f will be denoted as $\mathsf{wt}(f)$ and is equal to $\#\mathsf{supp}(f)$.

The Fourier transform of f is the map $\widehat{f} : \mathbb{F}_2^n \to [-1, 1]$ defined as follows. For $\boldsymbol{\alpha} \in \mathbb{F}_2^n$,

$$\widehat{f}(\boldsymbol{\alpha}) = \frac{1}{2^n} \sum_{\mathbf{x} \in \mathbb{F}_2^n} (-1)^{f(\mathbf{x}) \oplus \langle \mathbf{x}, \boldsymbol{\alpha} \rangle} \tag{1}$$

$$= 1 - \frac{\mathsf{wt}(f(\mathbf{X}) \oplus \langle \boldsymbol{\alpha}, \mathbf{X} \rangle)}{2^{n-1}}. \tag{2}$$

The function f is said to be balanced if $\mathsf{wt}(f) = 2^{n-1}$ and so $\widehat{f}(\boldsymbol{\alpha}) = 0$ if and only if the function $f(\mathbf{X}) \oplus \langle \boldsymbol{\alpha}, \mathbf{X} \rangle$ is balanced. From Parseval's theorem, it follows that

$$\sum_{\boldsymbol{\alpha} \in \mathbb{F}_2^n} \left(\widehat{f}(\boldsymbol{\alpha})\right)^2 = 1. \tag{3}$$

So, the values $\left\{\left(\widehat{f}(\boldsymbol{\alpha})\right)^2\right\}$ can be considered to be a probability distribution on $\mathbb{F}_2^n$, which assigns to $\boldsymbol{\alpha} \in \mathbb{F}_2^n$, the probability $\left(\widehat{f}(\boldsymbol{\alpha})\right)^2$. For $k \in \{0, \ldots, n\}$, let

$$\widehat{p}_f(k) = \sum_{\{\mathbf{u} \in \mathbb{F}_2^n : \mathsf{wt}(\mathbf{u}) = k\}} \left(\widehat{f}(\mathbf{u})\right)^2 \tag{4}$$

be the probability assigned by the Fourier transform of f to the integer k.

Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a vector of variables and suppose $\emptyset \neq T = \{i_1, \ldots, i_t\} \subseteq [n]$, where $i_1 \leq \cdots \leq i_t$. By $\mathbf{X}_T$ we denote the vector of variables $(X_{i_1}, \ldots, X_{i_t})$. Suppose $f(\mathbf{X})$ is an n-variable Boolean function. For $\boldsymbol{\alpha} \in \mathbb{F}_2^t$, by $f_{\mathbf{X}_T \leftarrow \boldsymbol{\alpha}}(\mathbf{X}_{\overline{T}})$ we denote the Boolean function on $n - t$ variables obtained by setting the variables in $\mathbf{X}_T$ to the respective values in $\boldsymbol{\alpha}$. The function f is said to be degenerate on the set of variables $\{X_{i_1}, \ldots, X_{i_t}\}$ if these variables do not influence the output of the function f, i.e., for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{F}_2^t$ if we set $f_{\boldsymbol{\alpha}} = f_{\mathbf{X}_T \leftarrow \boldsymbol{\alpha}}(\mathbf{X}_{\overline{T}})$ and $f_{\boldsymbol{\beta}} = f_{\mathbf{X}_T \leftarrow \boldsymbol{\beta}}(\mathbf{X}_{\overline{T}})$, then the functions $f_{\boldsymbol{\alpha}}$ and $f_{\boldsymbol{\beta}}$ are equal.

The (normalised) auto-correlation function of f is a map $C_f : \mathbb{F}_2^n \rightarrow [-1, 1]$ defined as follows. For $\boldsymbol{\alpha} \in \mathbb{F}_2^n$,

$$\begin{aligned} C_f(\boldsymbol{\alpha}) &= \frac{1}{2^n} \sum_{\mathbf{x} \in \mathbb{F}_2^n} (-1)^{f(\mathbf{x}) \oplus f(\mathbf{x} \oplus \boldsymbol{\alpha})} && (5) \\ &= 1 - \frac{\mathsf{wt}(f(\mathbf{X}) \oplus f(\mathbf{X} \oplus \boldsymbol{\alpha}))}{2^{n-1}}. && (6) \end{aligned}$$

Note that $C_f(\mathbf{0}) = 1$.

For a Boolean function f, the following connection between the Fourier transform of f and the auto-correlation function of f is well known (See Page 62 of [4]). For $\boldsymbol{\alpha} \in \mathbb{F}_2^n$,

$$\left(\widehat{f}(\boldsymbol{\alpha})\right)^2 = \frac{1}{2^n} \sum_{\mathbf{x} \in \mathbb{F}_2^n} (-1)^{\langle \boldsymbol{\alpha}, \mathbf{x} \rangle} C_f(\mathbf{x}). \tag{7}$$

Conversely, for $\mathbf{x} \in \mathbb{F}_2^n$,

$$C_f(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{F}_2^n} \left(\widehat{f}(\boldsymbol{\alpha})\right)^2 (-1)^{\langle \boldsymbol{\alpha}, \mathbf{x} \rangle}. \tag{8}$$

For a subspace E of $\mathbb{F}_2^n$, the following result has been proved in Proposition 5 of [3].

$$\sum_{\mathbf{w} \in E} \left(\widehat{f}(\mathbf{w})\right)^2 = \frac{\#E}{2^n} \sum_{\mathbf{u} \in E^{\perp}} C_f(\mathbf{u}). \tag{9}$$

Influence of a variable on a Boolean function. Let $f(\mathbf{X})$ be an n-variable Boolean function where $\mathbf{X} = (X_1, \ldots, X_n)$. For $i \in [n]$, the influence of X_i on f is denoted as $\mathsf{inf}_i(f)$ and is defined to be the probability (over a uniform random choice of $\mathbf{x} \in \mathbb{F}_2^n$) that $f(\mathbf{x})$ is not equal to $f(\mathbf{x} \oplus \mathbf{e}_i)$, i.e.,

$$\mathsf{inf}_i(f) \quad = \quad \Pr_{\mathbf{x} \in \mathbb{F}_2^n}[f(\mathbf{x}) \neq f(\mathbf{x} \oplus \mathbf{e}_i)]. \tag{10}$$

It is easy to see that

$$\mathsf{inf}_i(f) \quad = \quad \frac{1}{2}\left(1 - C_f(\mathbf{e}_i)\right) = 1 - \frac{1}{2}\left(C_f(\mathbf{0}) + C_f(\mathbf{e}_i)\right). \tag{11}$$

Some Boolean function classes. Let f be an n-variable Boolean function.

- The function f is said to be bent if $\widehat{f}(\boldsymbol{\alpha}) = 2^{-n/2}$ for all $\boldsymbol{\alpha} \in \mathbb{F}_2^n$ [7]. Bent functions exist only if n is even. It is known that if f is bent, then $C_f(\mathbf{u}) = 0$ for all non-zero $\mathbf{u} \in \mathbb{F}_2^n$.
- The function f is said to be m-resilient [9, 11] if $\widehat{f}(\boldsymbol{\alpha}) = 0$ for all $\boldsymbol{\alpha} \in \mathbb{F}_2^n$ with $0 \leq \mathsf{wt}(\boldsymbol{\alpha}) \leq m$.

3 Influence of a Set of Variables on a Boolean Function

Let $f(X_1, \ldots, X_n)$ be an n-variable Boolean function and $\emptyset \neq T = \{i_1, \ldots, i_t\} \subseteq [n]$. We denote the influence of the set of variables $\{X_{i_1}, \ldots, X_{i_n}\}$ corresponding to $T = \{i_1, \ldots, i_t\}$ on the Boolean function f by $\mathsf{inf}_f(T)$. Following the auto-correlation based characterisation of the influence of a single variable on a Boolean function given by (11), we put forward the following definition of $\mathsf{inf}_f(T)$.

$$\mathsf{inf}_f(T) \quad = \quad 1 - \frac{1}{2^{\#T}}\left(\sum_{\boldsymbol{\alpha} \leq \chi_T} C_f(\boldsymbol{\alpha})\right). \tag{12}$$

It is easy to note that for a singleton set $T = \{i\}$, $\mathsf{inf}_f(T) = \mathsf{inf}_i(f)$.

Let f be an n-variable function and t be an integer with $1 \leq t \leq n$. Then the t-influence of f is the total influence (scaled by $\binom{n}{t}$) obtained by summing the influence of every set of t variables on the function f, i.e.,

$$t\text{-}\mathsf{inf}(f) \quad = \quad \frac{\sum_{\{T \subseteq [n] : \#T = t\}} \mathsf{inf}_f(T)}{\binom{n}{t}}. \tag{13}$$

Note that if f is bent, then $\mathsf{inf}_f(T) = 1 - 2^{-\#T}$ for any non-empty subset T of $[n]$ and $t\text{-}\mathsf{inf}(f) = 1 - 2^{-t}$ for any $t \in [n]$.

The following result provides the characterisation of the influence in terms of the Fourier transform.

Theorem 1 *Let f be an n-variable Boolean function and $\emptyset \neq T \subseteq [n]$. Then*

$$\mathsf{inf}_f(T) \quad = \quad \sum_{\{\mathbf{u} \in \mathbb{F}_2^n : \mathsf{supp}(\mathbf{u}) \cap T \neq \emptyset\}} \left(\widehat{f}(\mathbf{u})\right)^2. \tag{14}$$

Consequently, for an integer t with $1 \leq t \leq n$,

$$t\text{-}\mathsf{inf}(f) \quad = \quad 1 - \sum_{k=0}^{n-t} \frac{\binom{n-k}{t}}{\binom{n}{t}} \widehat{p}_f(k). \tag{15}$$

Proof: Let $\#T = t$. Let E be the subspace $\{\mathbf{x} \in \mathbb{F}_2^n : \mathbf{x} \leq \chi_{\overline{T}}\}$. Then $\#E = 2^{n-t}$ and $E^{\perp} = \{\mathbf{y} \in \mathbb{F}_2^n : \mathbf{y} \leq \chi_T\}$. Using (9), we have

$$\sum_{\mathbf{x} \leq \chi_{\overline{T}}} \left(\widehat{f}(\mathbf{x})\right)^2 = \frac{2^{n-t}}{2^n} \sum_{\mathbf{y} \leq \chi_T} C_f(\mathbf{y}) = \frac{1}{2^{\#T}} \sum_{\mathbf{y} \leq \chi_T} C_f(\mathbf{y}).$$

So, from (12), we have

$$\begin{aligned}
\mathsf{inf}_f(T) &= 1 - \sum_{\mathbf{x} \leq \chi_{\overline{T}}} \left(\widehat{f}(\mathbf{x})\right)^2 \\
&= \sum_{\mathbf{w} \in \mathbb{F}_2^n} \left(\widehat{f}(\mathbf{w})\right)^2 - \sum_{\mathbf{x} \leq \chi_{\overline{T}}} \left(\widehat{f}(\mathbf{x})\right)^2 \quad \text{(from Parseval's theorem (3))} \\
&= \sum_{\mathbf{u} \not\leq \chi_{\overline{T}}} \left(\widehat{f}(\mathbf{u})\right)^2.
\end{aligned}$$

For $\mathbf{u} \in \mathbb{F}_2^n$, the condition $\mathbf{u} \not\leq \chi_{\overline{T}}$ is equivalent to $\mathsf{supp}(\mathbf{u}) \cap T \neq \emptyset$.

We next consider the expression for t-influence. Consider $\mathbf{u} \in \mathbb{F}_2^n$ with $\#\mathsf{supp}(\mathbf{u}) = k$. For $1 \leq i \leq \min(k,t)$, the number of subsets T of $[n]$ of cardinality t whose intersection with $\mathsf{supp}(\mathbf{u})$ is of size i is $\binom{k}{i}\binom{n-k}{t-i}$. Summing over i provides the number of subsets T of $[n]$ of cardinality t with which $\mathsf{supp}(\mathbf{u})$ has a non-empty intersection.

$$\begin{aligned}
t\text{-}\mathsf{inf}(f) &= \frac{1}{\binom{n}{t}} \sum_{k=1}^{n} \sum_{\{\mathbf{u} \in \mathbb{F}_2^n : \mathsf{wt}(\mathbf{u}) = k\}} \sum_{i=1}^{\min(k,t)} \binom{k}{i}\binom{n-k}{t-i} \left(\widehat{f}(\mathbf{u})\right)^2 \\
&= \frac{1}{\binom{n}{t}} \sum_{k=1}^{n} \sum_{i=1}^{\min(k,t)} \binom{k}{i}\binom{n-k}{t-i} \sum_{\{\mathbf{u} \in \mathbb{F}_2^n : \mathsf{wt}(\mathbf{u}) = k\}} \left(\widehat{f}(\mathbf{u})\right)^2 \\
&= \sum_{k=1}^{n} \left(\frac{\sum_{i=1}^{\min(k,t)} \binom{k}{i}\binom{n-k}{t-i}}{\binom{n}{t}} \right) \widehat{p}_f(k). \qquad (16)
\end{aligned}$$

Note that $\binom{k}{i}\binom{n-k}{t-i}/\binom{n}{t}$ is the probability that a random variable X takes the value i following the hypergeometric distribution. So, $\sum_{i=1}^{\min(k,t)} \binom{k}{i}\binom{n-k}{t-i}/\binom{n}{t} = 1 - \binom{n-k}{t}/\binom{n}{t}$. Also, using Parseval's theorem, $\sum_{k=1}^{n} \widehat{p}_f(k) = 1 - \widehat{p}_f(0)$. Using these two relations in (16), we obtain the desired expression for t-$\mathsf{inf}(f)$. $\square$

Suppose $t = 1$. Then using (15), we have

$$\begin{aligned}
1\text{-}\mathsf{inf}(f) &= \frac{1}{n}\left(n - \sum_{k=0}^{n-1} (n-k)\widehat{p}_f(k) \right) \\
&= \frac{1}{n}\left(n\left(1 - \sum_{k=0}^{n-1} \widehat{p}_f(k)\right) + \sum_{k=0}^{n-1} k\widehat{p}_f(k) \right) \\
&= \frac{1}{n}\left(n\widehat{p}_f(n) + \sum_{k=0}^{n-1} k\widehat{p}_f(k) \right) \quad \text{(using Parseval's theorem)} \\
&= \frac{1}{n}\left(\sum_{k=0}^{n} k\widehat{p}_f(k) \right). \qquad (17)
\end{aligned}$$

From (17), 1-$\mathsf{inf}(f)$ is the expected value (scaled by n) of a random variable which takes the value k with probability $\widehat{p}_f(k)$. Since $\mathsf{inf}_f(\{i\}) = \mathsf{inf}_i(f)$, for $i \in [n]$, it follows that $(\sum_{i\in[n]} \mathsf{inf}_i(f))/n =$ 1-inf.

An alternative expression for influence is given in the following result.

Theorem 2 *Let f be an n-variable function and $\emptyset \neq T \subseteq [n]$. Then*

$$\mathsf{inf}_f(T) \;=\; 1 - \frac{1}{2^{n-t}} \sum_{\boldsymbol{\alpha}\in\mathbb{F}_2^{n-t}} \left(\widehat{f_{\boldsymbol{\alpha}}}(\mathbf{0})\right)^2, \quad \textit{(here } \mathbf{0} \textit{ is in } \mathbb{F}_2^t\textit{)} \tag{18}$$

where $f_{\boldsymbol{\alpha}}$ denotes $f_{\mathbf{X}_{\overline{T}}\leftarrow\boldsymbol{\alpha}}$.

Proof: Let $T = \{i_1, \ldots, i_t\}$. Suppose π is a permutation of $[n]$ and define $g(\mathbf{X})$ to be the function $f(X_{\pi(1)}, \ldots, X_{\pi(n)})$ and $S = \{\pi(i_1), \ldots, \pi(i_t)\}$. Then $\mathsf{inf}_f(T) = \mathsf{inf}_g(S)$. In view of this, for the sake of notational convenience, we assume that $T = \{n-t+1, \ldots, n\}$ so that $\overline{T} = \{1, \ldots, n-t\}$. This is without loss of generality, since one may apply an appropriate permutation to the variables to ensure this condition.

For a Boolean condition Φ, let $[\![\Phi]\!]$ denote the value 1 if Φ is true and 0, otherwise. We perform an algebraic manipulation of the expression for the influence of T on f as follows.

$$\begin{aligned}
\mathsf{inf}_f(T) &= 1 - \frac{1}{2^t} \sum_{\{\mathbf{u}\in\mathbb{F}_2^n : \mathbf{u}\leq\chi_T\}} C_f(\mathbf{u}) \\
&= 1 - \frac{1}{2^t} \sum_{\mathbf{v}\in\mathbb{F}_2^t} C_f((\mathbf{0},\mathbf{v})) \quad \text{(here } \mathbf{0} \text{ is in } \mathbb{F}_2^{n-t}\text{)} \\
&= 1 - \frac{1}{2^{n+t}} \sum_{\mathbf{v}\in\mathbb{F}_2^t} \left(2^n - 2\mathsf{wt}(f(\mathbf{X}) \oplus f(\mathbf{X} \oplus (\mathbf{0},\mathbf{v})))\right) \quad \text{(from (6))} \\
&= \frac{1}{2^{n+t}} \sum_{\mathbf{v}\in\mathbb{F}_2^t} 2\mathsf{wt}(f(\mathbf{X}) \oplus f(\mathbf{X} \oplus (\mathbf{0},\mathbf{v}))) && (19) \\
&= \frac{1}{2^{n+t}} \sum_{\mathbf{v}\in\mathbb{F}_2^t} \sum_{\boldsymbol{\alpha}\in\mathbb{F}_2^{n-t}} \sum_{\boldsymbol{\sigma}\in\mathbb{F}_2^t} 2[\![f((\boldsymbol{\alpha},\boldsymbol{\sigma})) \neq f((\boldsymbol{\alpha},\boldsymbol{\sigma}\oplus\mathbf{v}))]\!] \\
&= \frac{1}{2^{n+t}} \sum_{\boldsymbol{\alpha}\in\mathbb{F}_2^{n-t}} \sum_{\boldsymbol{\sigma}\in\mathbb{F}_2^t} \sum_{\mathbf{v}\in\mathbb{F}_2^t} 2[\![f_{\boldsymbol{\alpha}}(\boldsymbol{\sigma}) \neq f_{\boldsymbol{\alpha}}(\boldsymbol{\sigma}\oplus\mathbf{v})]\!]. && (20)
\end{aligned}$$

Let $x_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\sigma}\in\mathbb{F}_2^t} \sum_{\mathbf{v}\in\mathbb{F}_2^t} [\![f_{\boldsymbol{\alpha}}(\boldsymbol{\sigma}) \neq f_{\boldsymbol{\alpha}}(\boldsymbol{\sigma}\oplus\mathbf{v})]\!]$. A pair $(\boldsymbol{\sigma},\mathbf{v})$ contributes 1 to $x_{\boldsymbol{\alpha}}$ if and only if exactly one of $\boldsymbol{\sigma}$ or $\boldsymbol{\sigma}\oplus\mathbf{v}$ is in $\mathsf{supp}(f_{\boldsymbol{\alpha}})$. So, if $\boldsymbol{\sigma} \in \mathsf{supp}(f_{\boldsymbol{\alpha}})$, then $\boldsymbol{\sigma}\oplus\mathbf{v} \notin \mathsf{supp}(f_{\boldsymbol{\alpha}})$; for each choice of $\boldsymbol{\sigma} \in \mathsf{supp}(f_{\boldsymbol{\alpha}})$, $\mathbf{v}$ can be chosen in $2^t - \mathsf{wt}(f_{\boldsymbol{\alpha}})$ ways and the number of such $(\boldsymbol{\sigma},\mathbf{v})$ pairs is $\mathsf{wt}(f_{\boldsymbol{\alpha}})(2^t - \mathsf{wt}(f_{\boldsymbol{\alpha}}))$. Similarly, if $\boldsymbol{\sigma} \notin \mathsf{supp}(f_{\boldsymbol{\alpha}})$, then $\boldsymbol{\sigma}\oplus\mathbf{v} \in \mathsf{supp}(f_{\boldsymbol{\alpha}})$ and the number of such $(\boldsymbol{\sigma},\mathbf{v})$ pairs is $(2^t - \mathsf{wt}(f_{\boldsymbol{\alpha}}))\mathsf{wt}(f_{\boldsymbol{\alpha}})$. So, $x_{\boldsymbol{\alpha}} = 2\mathsf{wt}(f_{\boldsymbol{\alpha}})(2^t - \mathsf{wt}(f_{\boldsymbol{\alpha}}))$. Using this in (20), we have

$$\begin{aligned}
\mathsf{inf}_f(T) &= \frac{1}{2^{n+t}} \sum_{\boldsymbol{\alpha}\in\mathbb{F}_2^{n-t}} 4\mathsf{wt}(f_{\boldsymbol{\alpha}})(2^t - \mathsf{wt}(f_{\boldsymbol{\alpha}})) \\
&= \frac{1}{2^{n+t}} \sum_{\boldsymbol{\alpha}\in\mathbb{F}_2^{n-t}} 4\left(2^{2t-2} - \left(2^{t-1} - \mathsf{wt}(f_{\boldsymbol{\alpha}})\right)^2\right)
\end{aligned}$$

$$= \frac{1}{2^{n-t}} \sum_{\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t}} \left(1 - \left(1 - \frac{\mathsf{wt}(f_{\boldsymbol{\alpha}})}{2^{t-1}}\right)^2\right)$$

$$= 1 - \frac{1}{2^{n-t}} \sum_{\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t}} \left(\widehat{f}_{\boldsymbol{\alpha}}(\mathbf{0})\right)^2.$$

□

Some basic properties of influence are given by the following result.

Theorem 3 *Let f be an n-variable Boolean function and $\emptyset \neq T, S \subseteq [n]$. Then*

1. $0 \leq \mathsf{inf}_f(T) \leq 1$.
2. $\mathsf{inf}_f(T) = 0$ *if and only if the function f is degenerate on the variables indexed by T.*
3. $\mathsf{inf}_f(T) = 1$ *if and only if $f_{\boldsymbol{\alpha}}$ is balanced for each $\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t}$, where $f_{\boldsymbol{\alpha}}$ denotes $f_{\mathbf{X}_{\overline{T}} \leftarrow \boldsymbol{\alpha}}$.*
4. $\mathsf{inf}_f(S \cup T) \geq \mathsf{inf}_f(T)$.
5. $\mathsf{inf}_f(S \cup T) = \mathsf{inf}_f(S) + \mathsf{inf}_f(T) - \sum_{\mathbf{u} \in \mathcal{U}} \left(\widehat{f}(\mathbf{u})\right)^2$, *where $\mathcal{U} = \{\mathbf{u} \in \mathbb{F}_2^n : \mathsf{supp}(\mathbf{u}) \cap S \neq \emptyset \neq \mathsf{supp}(\mathbf{u}) \cap T\}$. Consequently, $\mathsf{inf}_f(S \cup T) \leq \mathsf{inf}_f(S) + \mathsf{inf}_f(T)$.*

Proof: The first point follows from Theorem 1 and Parseval's theorem. The fourth and fifth points also follow from Theorem 1. The third point follows from Theorem 2.

Consider the second point. Suppose π is any permutation of $[n]$ and define $g(\mathbf{X})$ to be the function $f(X_{\pi(1)}, \ldots, X_{\pi(n)})$. Then f is degenerate on the variables indexed by a set $U = \{i_1, \ldots, i_t\}$ if and only if g is degenerate on the variables indexed by the set $V = \{\pi(i_1), \ldots, \pi(i_t)\}$. Also, $\mathsf{inf}_f(U) = \mathsf{inf}_g(V)$. In view of this, we consider the set T to be $\{1, \ldots, t\}$.

For $\boldsymbol{\alpha} \in \mathbb{F}_2^t$ and $\mathbf{Y} = (X_{t+1}, \ldots, X_n)$, let $f_{\boldsymbol{\alpha}}(\mathbf{Y}) = f(\boldsymbol{\alpha}, \mathbf{Y})$. The function f is degenerate on the variables indexed by T, if and only if $f_{\boldsymbol{\alpha}}(\mathbf{Y}) = f_{\boldsymbol{\beta}}(\mathbf{Y})$ for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{F}_2^t$. We show that the latter condition is equivalent to $f(\mathbf{X}) = f(\mathbf{X} \oplus \boldsymbol{\gamma})$ for any $\boldsymbol{\gamma} \leq \chi_T$. Note that by the choice of T, we have that for $\boldsymbol{\gamma} \leq \chi_T$, $\boldsymbol{\gamma} = (\boldsymbol{\delta}, \mathbf{0})$ for some $\boldsymbol{\delta} \in \mathbb{F}_2^t$. So, it is sufficient to show that $f(\boldsymbol{\alpha}, \mathbf{Y}) = f((\boldsymbol{\alpha}, \mathbf{Y}) \oplus (\boldsymbol{\delta}, \mathbf{0}))$ for all $\boldsymbol{\alpha} \in \mathbb{F}_2^t$. The last condition is equivalent to $f_{\boldsymbol{\alpha}}(\mathbf{Y}) = f_{\boldsymbol{\alpha} \oplus \boldsymbol{\delta}}(\mathbf{Y}) = f_{\boldsymbol{\beta}}(\mathbf{Y})$ where $\boldsymbol{\beta} = \boldsymbol{\alpha} \oplus \boldsymbol{\delta}$. This completes the proof that f is degenerate on the variables indexed by T if and only if $f(\mathbf{X}) = f(\mathbf{X} \oplus \boldsymbol{\gamma})$ for all $\boldsymbol{\gamma} \leq \chi_T$.

The condition $f(\mathbf{X}) = f(\mathbf{X} \oplus \boldsymbol{\gamma})$ for all $\boldsymbol{\gamma} \leq \chi_T$ is equivalent to $C_f(\boldsymbol{\gamma}) = 1$ for all $\boldsymbol{\gamma} \leq \chi_T$. So, f is degenerate on the set of variables indexed by T if and only if $C_f(\boldsymbol{\gamma}) = 1$ for all $\boldsymbol{\gamma} \leq \chi_T$. From (12), $\mathsf{inf}_f(T) = 0$ if and only if $\sum_{\mathbf{u} \leq \chi_T} C_f(\mathbf{u}) = 2^t$. Since $0 \leq C_f(\mathbf{u}) \leq 1$, the previous condition holds if and only if $C_f(\mathbf{u}) = 1$ for all $\mathbf{u} \leq \chi_T$. This shows that f is degenerate on the set of variables indexed by T if and only if $\mathsf{inf}_f(T) = 0$. □

Theorem 3 shows five properties of influence. The first point shows that influence is bounded between 0 and 1; the second point shows that the influence is 0 if and only if f is degenerate on the set of variables indexed by T; the third point characterises the condition under which the influence attains its maximum value; the fourth point shows that influence is monotone increasing with the size of the indexing set; and the fifth property shows that influence satisfies sub-additivity.

Theorem 4 *Let f be an n-variable Boolean function and t be an integer with $1 \leq t \leq n$.*

1. *t-$\mathsf{inf}(f)$ takes its maximum value of 1 if and only if f is $(n-t)$-resilient.*

2. t-$\mathsf{inf}(f)$ *takes its minimum value of 0 if and only if* f *is a constant function.*

Proof: From (15), t-$\mathsf{inf}(f)$ takes its maximum value of 1 if and only if

$$\sum_{k=0}^{n-t} \frac{\binom{n-k}{t}}{\binom{n}{t}} \widehat{p}_f(k) = 0$$

which holds if and only if $\widehat{p}_f(k) = 0$ for $k = 0, \ldots, n-t$, i.e., if and only if f is $(n-t)$-resilient. This shows the first point.

For the second point, from (15), t-$\mathsf{inf}(f) = 0$ if and only if

$$\binom{n}{t}\widehat{p}_f(0) + \binom{n-1}{t}\widehat{p}_f(1) + \cdots + \binom{t}{t}\widehat{p}_f(t) = \binom{n}{t}. \tag{21}$$

If f is a constant function, then $\widehat{p}_f(0) = 1$ and $\widehat{p}_f(k) = 0$ for $k \in [n]$. So, (21) holds. On the other hand, if f is not a constant function, then $\widehat{p}_f(0) < 1$. In this case,

$$\begin{aligned}
&\binom{n}{t}\widehat{p}_f(0) + \binom{n-1}{t}\widehat{p}_f(1) + \cdots + \binom{t}{t}\widehat{p}_f(t) \\
\leq\ & \binom{n}{t}\widehat{p}_f(0) + \binom{n-1}{t}(\widehat{p}_f(1) + \cdots + \widehat{p}_f(n)) \\
=\ & \binom{n}{t}\widehat{p}_f(0) + \binom{n-1}{t}(1 - \widehat{p}_f(0)) < \binom{n}{t}.
\end{aligned}$$

So, (21) holds if and only if f is a constant function. $\square$

The next result shows that as t increases, the value of t-$\mathsf{inf}(f)$ also increases.

Theorem 5 *Let* f *be an* n*-variable Boolean function. For* $t \in [n]$, t-$\mathsf{inf}(f)$ *increases monotonically with* t.

Proof: For $t \in [n-1]$, the following calculations show that t-$\mathsf{inf}(f)$ is at most $(t+1)$-$\mathsf{inf}(f)$.

$$\begin{aligned}
&t\text{-}\mathsf{inf}(f) \leq (t+1)\text{-}\mathsf{inf}(f) \\
\iff\ & 1 - \sum_{k=0}^{n-t} \frac{\binom{n-k}{t}}{\binom{n}{t}} \widehat{p}_f(k) \leq 1 - \sum_{k=0}^{n-t-1} \frac{\binom{n-k}{t+1}}{\binom{n}{t+1}} \widehat{p}_f(k) \\
\iff\ & \sum_{k=0}^{n-t} \frac{\binom{n-k}{t}}{\binom{n}{t}} \widehat{p}_f(k) \geq \sum_{k=0}^{n-t-1} \frac{\binom{n-k}{t+1}}{\binom{n}{t+1}} \widehat{p}_f(k) \\
\iff\ & \frac{1}{\binom{n}{t}}\widehat{p}_f(t) + \sum_{k=0}^{n-t-1} \left(\frac{\binom{n-k}{t}}{\binom{n}{t}} - \frac{\binom{n-k}{t+1}}{\binom{n}{t+1}} \right) \widehat{p}_f(k) \geq 0 \\
\iff\ & \frac{1}{\binom{n}{t}}\widehat{p}_f(t) + \sum_{k=0}^{n-t-1} \left(\frac{(n-k)!(n-t-1)!}{n!(n-k-t-1)!} \left(\frac{n-t}{n-t-k} - 1 \right) \right) \widehat{p}_f(k) \geq 0.
\end{aligned} \tag{22}$$

For k in the range 0 to $n-t-1$, it follows that $(n-t)/(n-t-k) \geq 1$. So, the relation in (22) holds showing that t-$\mathsf{inf}(f) \leq (t+1)$-$\mathsf{inf}(f)$.

$\square$

Recall that if f is a bent function, then $t\text{-}\mathsf{inf}(f) = 1 - 2^{-t}$ which provides an indication of how the values of $t\text{-}\mathsf{inf}(f)$ can increase with t. In general, however, it seems difficult to characterise the growth of the values of $t\text{-}\mathsf{inf}(f)$ with t.

The Fourier entropy $H(f)$ of f is defined to be the entropy of the probability distribution $\{\widehat{f}^2(\boldsymbol{\alpha})\}$ and is equal to

$$H(f) \quad = \quad -\sum_{\boldsymbol{\alpha}\in\mathbb{F}_2^n} \widehat{f}^2(\boldsymbol{\alpha}) \log \widehat{f}^2(\boldsymbol{\alpha}), \tag{23}$$

where log denotes $\log_2$ and the expressions $0\log 0$ and $0\log\frac{1}{0}$ are to be interpreted as 0. For $t \in [n]$, let

$$\rho_t(f) \quad = \quad \frac{H(f)/n}{t\text{-}\mathsf{inf}(f)}. \tag{24}$$

The Fourier entropy/influence conjecture [6] states that there is a universal constant C, such that for all Boolean functions f, $\rho_1(f) \leq C$. Since, $t\text{-}\mathsf{inf}(f)$ is monotonically increasing with t, it follows that $\rho_t(f)$ is monotonically decreasing in t. So, if the FEI conjecture holds, then $\rho_t(f) \leq C$ for $t \in [n]$, where C is the universal constant in the FEI conjecture. On the other hand, it is possible that the FEI conjecture does not hold, but for some $t_0 \in \{2,\ldots,n\}$, there is a constant C_0 such that $\rho_{t_0}(f) \leq C_0$ for all Boolean functions f.

3.1 Pseudo-Influence

Suppose $f(\mathbf{X})$ is an n-variable Boolean function where $\mathbf{X} = (X_1,\ldots,X_n)$ and $\emptyset \neq T = \{i_1,\ldots,i_t\} \subseteq [n]$. We define pseudo-influence $\mathsf{PI}_f(T)$ of the set of variables $\{X_{i_1},\ldots,X_{i_t}\}$ indexed by T on f in the following manner.

$$\mathsf{PI}_f(T) \quad = \quad \frac{1}{2^{\#T}}\left(\sum_{\boldsymbol{\alpha}\leq\chi_T} (-1)^{\mathsf{wt}(\alpha)} C_f(\boldsymbol{\alpha})\right). \tag{25}$$

For a singleton set $T = \{i\}$, $\mathsf{PI}_f(T) = \mathsf{inf}_f(T) = \mathsf{inf}_i(f)$.

Let f be an n-variable function and t be an integer with $1 \leq t \leq n$. Then the t-pseudo-influence of f is the total pseudo-influence (scaled by $\binom{n}{t}$) obtained by summing the pseudo-influence of every set of t variables on the function f, i.e.,

$$t\text{-}\mathsf{PI}(f) \quad = \quad \frac{\sum_{\{T\subseteq[n]:\#T=t\}} \mathsf{PI}_f(T)}{\binom{n}{t}}. \tag{26}$$

The characterisation of pseudo-influence in terms of the Fourier transform is given by the following result.

Theorem 6 *Let f be an n-variable Boolean function and $\emptyset \neq T \subseteq [n]$. Then*

$$\mathsf{PI}_f(T) \quad = \quad \sum_{\mathbf{u}\geq\chi_T} \left(\widehat{f}(\mathbf{u})\right)^2. \tag{27}$$

Consequently, for an integer t with $1 \leq t \leq n$,

$$t\text{-}\mathsf{PI}(f) \quad = \quad \frac{1}{\binom{n}{t}} \sum_{k=t}^{n} \binom{k}{t} \widehat{p}_f(k) \tag{28}$$

Proof: Let $t = \#T$. Using (8) in (25), we have the following.

$$\begin{aligned}
\mathsf{PI}_f(T) &= \frac{1}{2^t}\sum_{\boldsymbol{\alpha}\leq\chi_T}(-1)^{\mathsf{wt}(\boldsymbol{\alpha})}C_f(\boldsymbol{\alpha})\\
&= \frac{1}{2^t}\sum_{\boldsymbol{\alpha}\leq\chi_T}(-1)^{\mathsf{wt}(\boldsymbol{\alpha})}\sum_{\boldsymbol{\omega}}\left(\widehat{f}(\boldsymbol{\omega})\right)^2(-1)^{\langle\boldsymbol{\omega},\boldsymbol{\alpha}\rangle}\\
&= \frac{1}{2^t}\sum_{\boldsymbol{\omega}}\left(\widehat{f}(\boldsymbol{\omega})\right)^2\sum_{\boldsymbol{\alpha}\leq\chi_T}(-1)^{\mathsf{wt}(\boldsymbol{\alpha})}(-1)^{\langle\boldsymbol{\omega},\boldsymbol{\alpha}\rangle}\\
&= \frac{1}{2^t}\sum_{\boldsymbol{\omega}}\left(\widehat{f}(\boldsymbol{\omega})\right)^2\sum_{\boldsymbol{\alpha}\leq\chi_T}(-1)^{\langle\boldsymbol{\omega}\oplus\mathbf{1},\boldsymbol{\alpha}\rangle} \qquad (29)
\end{aligned}$$

Note that

$$\sum_{\boldsymbol{\alpha}\leq\chi_T}(-1)^{\langle\boldsymbol{\omega}\oplus\mathbf{1},\boldsymbol{\alpha}\rangle} = \begin{cases} 0 & \text{if } \mathsf{supp}(\boldsymbol{\omega}\oplus\mathbf{1})\cap T\neq\emptyset;\\ 2^t & \text{otherwise.}\end{cases}$$

Using this in (29), we obtain

$$\mathsf{PI}_f(T) = \sum_{\mathsf{supp}(\boldsymbol{\omega}\oplus\mathbf{1})\cap T=\emptyset}\left(\widehat{f}(\boldsymbol{\omega})\right)^2.$$

The proof follows by observing that the condition $\mathsf{supp}(\boldsymbol{\omega}\oplus\mathbf{1})\cap T=\emptyset$ is equivalent to the condition $\boldsymbol{\omega}\geq\chi_T$.

The expression for t-$\mathsf{PI}(f)$ can be seen as follows.

$$\begin{aligned}
t\text{-}\mathsf{PI}(f) &= \frac{1}{\binom{n}{t}}\sum_{k=1}^{n}\sum_{\{\mathbf{u}:\mathsf{wt}(\mathbf{u})=k\}}\binom{k}{t}\left(\widehat{f}(\mathbf{u})\right)^2\\
&= \frac{1}{\binom{n}{t}}\sum_{k=1}^{n}\binom{k}{t}\sum_{\{\mathbf{u}:\mathsf{wt}(\mathbf{u})=k\}}\left(\widehat{f}(\mathbf{u})\right)^2\\
&= \frac{1}{\binom{n}{t}}\sum_{k=1}^{n}\binom{k}{t}\widehat{p}_f(k)\\
&= \frac{1}{\binom{n}{t}}\sum_{k=t}^{n}\binom{k}{t}\widehat{p}_f(k). \qquad (30)
\end{aligned}$$

□

The following result states the basic properties of the pseudo-influence.

Theorem 7 *Let f be an n-variable Boolean function and $\emptyset\neq T\subseteq S\subseteq[n]$. Then*

1. $0\leq\mathsf{PI}_f(T)\leq 1$.
2. *If the function f is degenerate on the variables indexed by T, then* $\mathsf{PI}_f(T)=0$.
3. $\mathsf{PI}_f(S)\leq\mathsf{PI}_f(T)$.

Proof: The first point follows from Theorem 6 and Parseval's theorem. The third point also follows from Theorem 1.

Consider the second point. As in the proof of Theorem 3, if f is degenerate on the variables indexed by T, then $C_f(\boldsymbol{\alpha}) = 1$ for all $\boldsymbol{\alpha} \leq \chi_T$. Using this in the definition of pseudo-influence given by (25), we obtain the the second point. $\square$

Theorem 7 states that if f is degenerate on the variables indexed by T, then $\mathsf{PI}_f(T) = 0$. The converse, however, is not true. Suppose f is an n-variable function such that $\widehat{f}(\mathbf{1}) = 0$ and let $T = [n]$. Then from (27), $\mathsf{PI}_f(T) = 0$. This example can be generalised. Suppose g is an n-variable, m-resilient function and let $f(\mathbf{X}) = \langle \mathbf{1}, \mathbf{X} \rangle \oplus g(\mathbf{X})$. Using (2), we have $\widehat{f}(\boldsymbol{\alpha}) = \widehat{g}(\mathbf{1} \oplus \boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \in \mathbb{F}_2^n$. Since, g is m-resilient, $\widehat{g}(\boldsymbol{\omega}) = 0$ for all $\boldsymbol{\omega}$ with $\mathsf{wt}(\boldsymbol{\omega}) \leq m$. So, $\widehat{f}(\boldsymbol{\alpha}) = 0$ for all $\boldsymbol{\alpha}$ with $\mathsf{wt}(\boldsymbol{\alpha}) \geq n - m$. Consequently, for any $\emptyset \neq T \subseteq [n]$, with $\#T \geq n - m$, it follows that $\mathsf{PI}_f(T) = 0$. There are known examples of non-degenerate resilient functions. See for example [8].

Theorem 7 provides the reasons for naming the definition in (25) to be pseudo-influence. For one thing, as discussed above, it is possible that f is non-degenerate on the set of variables indexed by T and yet $\mathsf{PI}_f(T) = 0$. Secondly, as the cardinality of the indexing set T grows, the value of $\mathsf{PI}_f(T)$ decreases. These properties of $\mathsf{PI}_f(T)$ are in sharp contrast to the more intuitive properties of $\mathsf{inf}_f(T)$ stated in Theorem 3.

For $\mathbf{u} \in \mathbb{F}_2^n$ and $\emptyset \neq T \subseteq [n]$, $\mathbf{u} \geq \chi_T$ implies $\mathsf{supp}(u) \cap T \neq \emptyset$. So, from (14) and (27), we have the following result which states that influence is always at least as large as the pseudo-influence.

Proposition 1 *Let f be an n-variable Boolean function and $\emptyset \neq T \subseteq [n]$. Then $\mathsf{inf}_f(T) \geq \mathsf{PI}_f(T)$.*

Theorem 8 *Let $f(\mathbf{X})$ be an n-variable Boolean function where $\mathbf{X} = (X_1, \ldots, X_n)$ and t be an integer with $1 \leq t \leq n$.*

1. *t-$\mathsf{PI}(f)$ takes its maximum value of 1 if and only if f is of the form $f(\mathbf{X}) = \langle \mathbf{1}, \mathbf{X} \rangle$.*
2. *t-$\mathsf{PI}(f)$ takes its minimum value of 0 if and only if f is of the form $f(\mathbf{X}) = \langle \mathbf{1}, \mathbf{X} \rangle \oplus g(\mathbf{X})$, where $g(\mathbf{X})$ is $(n-t)$-resilient.*

Proof: From (28), t-$\mathsf{PI}(f)$ takes its maximum value of 1 if and only if

$$\sum_{k=t}^{n} \binom{k}{t} \widehat{p}_f(k) \quad = \quad \binom{n}{t}. \tag{31}$$

If $f(\mathbf{X}) = \langle \mathbf{1}, \mathbf{X} \rangle$, then $\widehat{p}_f(n) = 1$ and $\widehat{p}_f(k) = 0$ for $0 \leq k \leq n-1$. On the other hand, if $f(\mathbf{X}) \neq \langle \mathbf{1}, \mathbf{X} \rangle$, then $\widehat{p}_f(n) < 1$ and we have

$$\begin{aligned}
&\binom{t}{t}\widehat{p}_f(t) + \binom{t+1}{t}\widehat{p}_f(t+1) + \cdots + \binom{n}{t}\widehat{p}_f(n) \\
\leq\ & \binom{n-1}{t}(\widehat{p}_f(0) + \cdots + \widehat{p}_f(n-1)) + \binom{n}{t}\widehat{p}_f(n) \\
=\ & \binom{n-1}{t}(1 - \widehat{p}_f(n)) + \binom{n}{t}\widehat{p}_f(n) < \binom{n}{t}.
\end{aligned}$$

This completes the proof of the first point.

For the second point, from (28), one may note that the values $\widehat{p}_f(0), \ldots, \widehat{p}_f(t-1)$ do not affect the expression for t-$\mathsf{PI}(f)$. So, t-$\mathsf{PI}(f) = 0$ if and only if $\widehat{p}_f(t) = \cdots = \widehat{p}_f(n) = 0$. The last condition holds if and only if f is of the stated form. $\square$

From the second point of Theorem 8, it is possible to obtain examples of non-degenerate functions f such that t-$\mathsf{PI}(f)$ is 0.

3.2 Alternative Approach

We have defined influence in terms of the auto-correlation function. Previous work in the literature has defined influence using a different approach. In this section we briefly consider this approach.

A Boolean function can be represented as a map from $\{-1,1\}^n$ to $\{-1,1\}$ with the underlying intuition being that a bit b maps to $(-1)^b$. We will use the following convention to distinguish between the two representations. Boolean functions in the "bit representations" will be denoted in the usual font as f, g, h and so on, while Boolean functions in the "± 1" representations will be denoted using the 'mathfrak' font as $\mathfrak{f}, \mathfrak{g}, \mathfrak{h}$ and so on, respectively. The correspondence between $f : \mathbb{F}_2^n \to \mathbb{F}_2$ and $\mathfrak{f} : \{-1,1\}^n \to \{-1,1\}$ is the following: $\mathfrak{f}\left((-1)^{\mathbf{a}}\right) = (-1)^{f(\mathbf{a})}$, where for $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{F}_2^n$, $(-1)^{\mathbf{a}} = ((-1)^{a_1}, \ldots, (-1)^{a_n})$.

The Fourier transform of $\mathfrak{f} : \{-1,1\}^n \to \{-1,1\}$ is the map $\widehat{\mathfrak{f}} : \{-1,1\}^n \to [-1,1]$ where for $\mathbf{y} \in \{-1,1\}^n$,

$$\widehat{\mathfrak{f}}(\mathbf{y}) \quad = \quad \frac{1}{2^n} \sum_{\mathbf{x}=(x_1,\ldots,x_n)\in\{-1,1\}^n} \mathfrak{f}(\mathbf{x}) \prod_{\{i:y_i=-1\}} x_i. \tag{32}$$

It is easy to verify the following relation for a Boolean function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ and its corresponding "± 1" representation $\mathfrak{f} : \{-1,1\}^n \to \{-1,1\}$. For $\boldsymbol{\alpha} \in \mathbb{F}_2^n$, $\widehat{f}(\boldsymbol{\alpha}) = \widehat{\mathfrak{f}}\left((-1)^{\boldsymbol{\alpha}}\right)$.

For $\mathbf{x} = (x_1, \ldots, x_n), b \in \{-1,1\}$ and $T \subseteq [n]$, the following notation will be required.

- $\mathsf{wt}(\mathbf{x})$ denotes the number of -1's in $\mathbf{x}$.
- $\mathbf{x}^{(i \to b)} = (x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n)$.
- $\mathbf{x}_{\overline{T}} \star \mathbf{y}_T = (z_1, \ldots, z_n)$ is defined as follows: $z_i = x_i$ if $i \in \overline{T}$, and $z_i = y_i$, otherwise.

Definition of influence given by Fischer et al. [5] and Blais [2]. The same quantity has been defined in two different ways in Fischer et al. [5] and Blais [2]. In [5], this quantity has been called 'variation' and in [2], it was termed 'influence'. Here we provide the formulation as given in [2]. For $\mathfrak{f} : \{-1,1\}^n \to \{-1,1\}$ and $\emptyset \neq T \subseteq [n]$, the definition of influence given in [2] is the following.

$$I_{\mathfrak{f}}(T) \quad = \quad 2 \Pr_{\mathbf{x},\mathbf{y}\in\{-1,1\}^n} \left[\mathfrak{f}(\mathbf{x}) \neq \mathfrak{f}(\mathbf{x}_{\overline{T}} \star \mathbf{y}_T)\right]. \tag{33}$$

The following has been proved in [2, 5].

$$I_{\mathfrak{f}}(T) \quad = \quad \sum_{\mathbf{x}\in\{-1,1\}^n, \mathsf{wt}(\mathbf{x}_T)>0} \left(\widehat{\mathfrak{f}}(\mathbf{x})\right)^2. \tag{34}$$

It is easy to see that for $\mathbf{u} \in \mathbb{F}_2^n$, $\mathsf{supp}(\mathbf{u}) \cap T \neq \emptyset$ if and only if for $\mathbf{x} = (-1)^{\mathbf{u}}$, $\mathsf{wt}(\mathbf{x}_T) > 0$. Consequently, using (14) and (34), for an n-variable function $f : \mathbb{F}_2^n \to \mathbb{F}_2$, and $\emptyset \neq T \subseteq [n]$, we have the following.

$$\mathsf{inf}_f(T) \quad = I_{\mathfrak{f}}(T). \tag{35}$$

From (35), we obtain the result that the two definitions given by (12) and (33) are equivalent. This is quite surprising, since the actual definitions in (12) and (33) are very different.

In [2, 5], it has been proved that $I_{\mathfrak{f}}(T) \leq I_{\mathfrak{f}}(S \cup T) \leq I_{\mathfrak{f}}(S) + I_{\mathfrak{f}}(T)$, i.e., monotonicity and sub-additivity properties hold for $I_{\mathfrak{f}}$. These properties for $\mathsf{inf}_f(T)$ are covered by Points 4 and 5 of Theorem 3. Other than the monotonicity and the sub-additivity properties, the extensive analysis and results on influence obtained in this work do not appear in [2, 5] or in any other work in the literature.

Definition of influence given by Tal [10]. We next show that there is another definition of influence in the literature which turns out to be the same as the definition of pseudo-influence that we introduced using the auto-correlation function.

For $\mathfrak{f} : \{-1,1\}^n \to \{-1,1\}$ and $i \in [n]$, the i-th discrete derivative operator is defined to be the function $D_i\mathfrak{f} : \{-1,1\}^n \to \{-1,0,1\}$ defined as follows [10]:

$$D_i\mathfrak{f}(\mathbf{x}) = \frac{\mathfrak{f}(\mathbf{x}^{(i\to 1)}) - \mathfrak{f}(\mathbf{x}^{(i\to -1)})}{2}. \tag{36}$$

Let $\emptyset \neq T = \{i_1,\ldots,i_t\} \subseteq [n]$. The operator D_T acts on $\mathfrak{f}$ to define the function $D_T\mathfrak{f}$ defined as follows [10]:

$$D_T\mathfrak{f}(\mathbf{x}) = D_{i_1}D_{i_2}\cdots D_{i_t}\mathfrak{f}(\mathbf{x}). \tag{37}$$

For $\mathfrak{f} : \{-1,1\}^n \to \{-1,1\}$ and $\emptyset \neq T \subseteq [n]$, The notion of influence defined in [10] is the following.

$$J_{\mathfrak{f}}(T) = \mathbb{E}_{\mathbf{x}\in\{-1,1\}^n}\left[(D_T\mathfrak{f}(\mathbf{x}))^2\right]. \tag{38}$$

The following Fourier based characterisation of $J_{\mathfrak{f}}(T)$ has been obtained in [10].

$$J_{\mathfrak{f}}(T) = \sum_{\mathbf{x}\in\{-1,1\}^n, \mathsf{wt}(\mathbf{x}_T)=\#T} \left(\widehat{f}(\mathbf{x})\right)^2. \tag{39}$$

It is easy to see that for $\mathbf{u} \in \mathbb{F}_2^n$, $\mathbf{u} \geq \chi_T$ if and only if for $\mathbf{x} = (-1)^{\mathbf{u}}$, $\mathsf{wt}(\mathbf{x}_T) = \#T$. Consequently, using (27) and (39), for an n-variable function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ and $\emptyset \neq T \subseteq [n]$, we have

$$\mathsf{PI}_f(T) = J_{\mathfrak{f}}(T). \tag{40}$$

This again shows that though the two definitions given by (25) and (38) are different, through their respective Fourier characterisations, it can be seen that the two definitions coincide. The quantity $\sum_{\{T:\#T=t\}} J_{\mathfrak{f}}(T)$ was considered in [10] and the relevant expression corresponding to (28) was also obtained in [10]. Other than this, the analysis and results for the pseudo-influence presented in Section 3.1 do not appear in [10] or at any other place in the literature.

4 Ben-Or and Linial Definition of Influence

The first notion of influence of a set of variables on a Boolean function was proposed by Ben-Or and Linial in [1]. In this section, we introduce this notion, prove some of its basic properties and show its relationship with the notion of influence defined in Section 3.

For an n-variable function f and $\emptyset \neq T \subseteq [n]$, with $t = \#T$, the notion of influence introduced in [1] is the following.

$$\mathcal{I}_f(T) = \Pr_{\boldsymbol{\alpha}\in\mathbb{F}_2^{n-t}}\left[f_{\mathbf{X}_{\overline{T}}\leftarrow\boldsymbol{\alpha}}(\mathbf{X}_T) \text{ is not constant}\right]. \tag{41}$$

For $t \in [n]$, we define

$$t\text{-}\mathcal{I}(f) = \frac{\sum_{\{T\subseteq[n]:\#T=t\}} \mathcal{I}_f(T)}{\binom{n}{t}}. \tag{42}$$

The following result provides an alternative description of $\mathcal{I}_f(T)$.

Proposition 2 *For an n-variable function f and $\emptyset \neq T \subseteq [n]$, with $t = \#T$,*

$$\mathcal{I}_f(T) = 1 - \frac{\#\left\{\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t} : \left(\widehat{f}_{\boldsymbol{\alpha}}(\mathbf{0})\right)^2 = 1\right\}}{2^{n-t}} \tag{43}$$

$$= \frac{\#\left\{\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t} : \left(\widehat{f}_{\boldsymbol{\alpha}}(\mathbf{0})\right)^2 \neq 1\right\}}{2^{n-t}}, \tag{44}$$

where $f_{\boldsymbol{\alpha}}$ denotes $f_{\mathbf{X}_{\overline{T}} \leftarrow \boldsymbol{\alpha}}$.

Proof: From (41), it clearly follows that

$$\begin{aligned}\mathcal{I}_f(T) &= 1 - \frac{\#\{\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t} : f_{\boldsymbol{\alpha}} \text{ is constant}\}}{2^{n-t}} \\ &= 1 - \frac{\#\{\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t} : \mathsf{wt}(f_{\boldsymbol{\alpha}}) = 0, \text{ or } 2^t\}}{2^{n-t}} \\ &= 1 - \frac{\#\{\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t} : \widehat{f}_{\boldsymbol{\alpha}}(\mathbf{0}) = \pm 1\}}{2^{n-t}}.\end{aligned}$$

This shows (43), and (44) follows directly from (43). □

Some basic properties of $\mathcal{I}_f(T)$ are as follows.

Theorem 9 *Let f be an n-variable function and $\emptyset \neq T \subseteq S \subseteq [n]$. Let $\#T = t$.*

1. $0 \leq \mathcal{I}_f(T) \leq 1$.
2. $\mathcal{I}_f(T) = 0$ *if and only if f is degenerate on the variables indexed by T.*
3. $\mathcal{I}_f(T) = 1$ *if and only if $f_{\boldsymbol{\alpha}}$ is a non-constant function for every $\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t}$, where $f_{\boldsymbol{\alpha}}$ denotes $f_{\mathbf{X}_{\overline{T}} \leftarrow \boldsymbol{\alpha}}$. In particular, if $T = [n]$, then $\mathcal{I}_f(T) = 1$.*
4. $\mathcal{I}_f(T) \leq \mathcal{I}_f(S)$.

Proof: The first point is obvious.

For the second point, using (43) note that $\mathcal{I}_f(T) = 0$ if and only if for every $\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t}$, $\widehat{f}_{\boldsymbol{\alpha}}(\mathbf{0}) = \pm 1$, i.e., if and only if $\mathsf{wt}(f_{\boldsymbol{\alpha}}) = 0$, or 2^t, i.e., if and only if $f_{\boldsymbol{\alpha}}$ is constant. The last condition holds if and only if the variables indexed by T have no effect on the value of f, i.e., if and only if f is degenerate on the variables indexed by T.

To see the third point, note that $\mathcal{I}_f(T) = 1$ if and only if for every $\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t}$, $\left(\widehat{f}_{\boldsymbol{\alpha}}(\mathbf{0})\right)^2 \neq 1$, which holds if and only if $f_{\boldsymbol{\alpha}}$ is a non-constant function.

Let $\#S = s$. For the fourth point, it is sufficient to consider $s = t + 1$, since otherwise, we may define a sequence of sets $T \subset S_1 \subset S_2 \subset \cdots \subset S$, with $\#T + 1 = \#S_1$, $\#S_1 + 1 = \#S_2$, $\ldots$, and argue $\mathcal{I}_f(T) \leq \mathcal{I}_f(S_1) \leq \cdots \leq \mathcal{I}_f(S)$. Further, without loss of generality, we assume $T = \{n - t + 1, \ldots, n\}$ and $S = \{n - t, \ldots, n\}$ as otherwise, we may apply an appropriate permutation on the variables to ensure this condition. Then $\overline{T} = \{1, \ldots, n - t\}$ and $\overline{S} = \{1, \ldots, n - t - 1\}$.

Let $\mathcal{T} = \{\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t} : f_{\boldsymbol{\alpha}} \text{ is constant}\}$ and $\mathcal{S} = \{\boldsymbol{\beta} \in \mathbb{F}_2^{n-t-1} : f_{\boldsymbol{\beta}} \text{ is constant}\}$, where $f_{\boldsymbol{\beta}}$ is a shorthand for $f_{\mathbf{X}_{\overline{S}} \leftarrow \boldsymbol{\beta}}$. Note that if $\boldsymbol{\beta} \in \mathcal{S}$, then $(\boldsymbol{\beta}, 0), (\boldsymbol{\beta}, 1) \in \mathcal{T}$. So, $\#\mathcal{T} \geq 2\#\mathcal{S}$ which implies

$$\frac{\#\mathcal{T}}{2^{n-t}} \geq \frac{2\#\mathcal{S}}{2^{n-t}} \geq \frac{\#\mathcal{S}}{2^{n-t-1}}.$$

Consequently,

$$\mathcal{I}_f(T) = 1 - \frac{\#\mathcal{T}}{2^{n-t}} \leq 1 - \frac{\#\mathcal{S}}{2^{n-t-1}} = \mathcal{I}_f(S).$$

□

One may compare the properties of $\mathcal{I}_f(T)$ given by Theorem 9 to the properties of $\mathsf{inf}_f(T)$ given by Theorem 3. We note that the sub-additivity property does not hold for $\mathcal{I}_f(T)$. As an example, consider a 6-variable function f which maps $\mathbf{0}$ to 1 and all other elements of $\mathbb{F}_2^6$ to 0; let $S = \{4, 5, 6\}$ and $T = \{2, 3, 6\}$. Then $\mathcal{I}_f(S \cup T) = 1/2 > 1/8 + 1/8 = \mathcal{I}_f(S) + \mathcal{I}_f(T)$.

Next, we show that the Ben-Or and Linial notion of influence is always at least as much as the notion of influence defined in (12).

Theorem 10 *Let f be an n-variable function and $\emptyset \neq T \subseteq [n]$. Then $\mathsf{inf}_f(T) \leq \mathcal{I}_f(T)$. Further, equality holds if and only if $\left(\widehat{f}_{\boldsymbol{\alpha}}(\mathbf{0})\right)^2 = 0, 1$ for each $\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t}$, where $f_{\boldsymbol{\alpha}}$ denotes $f_{\mathbf{X}_{\overline{T}} \leftarrow \boldsymbol{\alpha}}$.*

Proof: We rewrite (18) in the following form.

$$\mathsf{inf}_f(T) \quad = \quad \frac{1}{2^{n-t}} \sum_{\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t}} \left(1 - \left(\widehat{f}_{\boldsymbol{\alpha}}(\mathbf{0})\right)^2\right). \tag{45}$$

Consider the expressions for $\mathsf{inf}_f(T)$ and $\mathcal{I}_f(T)$ given by (45) and (44) respectively. Both the expressions are sums over $\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t}$. Suppose $\boldsymbol{\alpha}$ is such that $\left(\widehat{f}_{\boldsymbol{\alpha}}(\mathbf{0})\right)^2 = 1$. The contribution of such an $\boldsymbol{\alpha}$ to both (45) and (44) is 0. Next suppose $\left(\widehat{f}_{\boldsymbol{\alpha}}(\mathbf{0})\right)^2 \neq 1$; the contribution of such an $\boldsymbol{\alpha}$ to (44) is 1 and the contribution to (45) is at most 1, and the value 1 is achieved if and only if $\widehat{f}_{\boldsymbol{\alpha}}(\mathbf{0}) = 0$. □

The measure $\mathcal{I}_f(T)$ satisfies some of the basic desirable properties expected of a measure of influence, namely, it is between 0 and 1; takes the value 0 if and only if f is degenerate on the variables indexed by T; and it is monotone increasing with the size of T. On the other hand, as noted above, it does not satisfy the sub-additivity property. Further, compared to $\mathsf{inf}_f(T)$, the value of $\mathcal{I}_f(T)$ rises quite sharply. To see this, it is useful to view the following expressions for the two quantities.

$$2^{n-t} \times \mathsf{inf}_f(T) \quad = \quad \sum_{\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t}} \left(1 - \left(\widehat{f}_{\boldsymbol{\alpha}}(\mathbf{0})\right)^2\right), \tag{46}$$

$$2^{n-t} \times \mathcal{I}_f(T) \quad = \quad \#\left\{\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t} : \left(\widehat{f}_{\boldsymbol{\alpha}}(\mathbf{0})\right)^2 \neq 1\right\}. \tag{47}$$

Suppose $\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t}$ is such that $f_{\boldsymbol{\alpha}}$ is a non-constant function, so that $\left(\widehat{f}_{\boldsymbol{\alpha}}(\mathbf{0})\right)^2 \neq 1$. Then such an $\boldsymbol{\alpha}$ contributes 1 to (47), while it contributes a value which is at most 1 to (46). More generally, $\boldsymbol{\alpha}$ contributes either 0 or 1 to (47) according as $f_{\boldsymbol{\alpha}}$ is constant or non-constant; on the other hand, the contribution of $\boldsymbol{\alpha}$ to (46) is more granular. Consequently, the value of $\mathcal{I}_f(T)$ rises more sharply than the value of $\mathsf{inf}_f(T)$. In particular, if f and g are two distinct functions such that for all $\boldsymbol{\alpha}$, both $f_{\boldsymbol{\alpha}}$ and $g_{\boldsymbol{\alpha}}$ are non-constant functions, then both $\mathcal{I}_f(T)$ and $\mathcal{I}_g(T)$ will be necessarily be equal to 1, whereas the values of $\mathsf{inf}_f(T)$ and $\mathsf{inf}_g(T)$ are neither necessarily 1 nor necessarily equal. So, while both $\mathsf{inf}_f(T)$ and $\mathcal{I}_f(T)$ share some intuitive basic properties expected of a definition of influence, in our opinion, $\mathsf{inf}_f(T)$ is a better measure compared to $\mathcal{I}_f(T)$.

The following result characterises the minimum and maximum values of t-$\mathcal{I}(f)$.

Theorem 11 *Let f be an n-variable Boolean function and t be an integer with $1 \leq t \leq n$.*

1. *t-$\mathcal{I}(f)$ takes its maximum value of 1 if and only if for every subset T of $[n]$ of size t, and for every $\boldsymbol{\alpha} \in \mathbb{F}_2^{n-t}$, the function $f_{\mathbf{X}_{\overline{T} \leftarrow \boldsymbol{\alpha}}}(\mathbf{X}_T)$ is constant.*

2. *t-$\mathcal{I}(f)$ takes its minimum value of 0 if and only if f is a constant function.*

Proof: The proof of the first point follows from the third point of Theorem 9.

For the second point, we note that if f is a constant function, then from (41), $\mathcal{I}_f(T) = 0$ for every subset T of $[n]$ and so t-$\mathcal{I}(f)$. On the other hand, if t-$\mathcal{I}(f) = 0$, then from Theorem 10, it follows that t-$\mathsf{inf}(f) = 0$ and so from the second point of Theorem 4 we have that f is a constant function. $\square$

5 Conclusion

In this paper, we have used the auto-correlation function to define the notions of influence and pseudo-influence of a set of variables on a Boolean function. Several properties of these notions have been derived and their characterisations in terms of the Fourier transform have been obtained. Through the Fourier transform based characterisation, it is seen that the definition of auto-correlation based influence is equivalent to the definition of influence introduced by Fischer et al. [5] and Blais [2] and the definition of pseudo-influence is equivalent to the definition of influence introduced by Tal [10]. The notion of influence introduced by Ben-Or and Linial [1] has been systematically studied and its relation to the auto-correlation based notion of influence precisely characterised.

References

[1] Michael Ben-Or and Nathan Linial. Collective coin flipping. *Adv. Comput. Res.*, 5:91–115, 1989.

[2] Eric Blais. Testing juntas nearly optimally. In Michael Mitzenmacher, editor, *Proceedings of the 41st Annual ACM Symposium on Theory of Computing, STOC 2009, Bethesda, MD, USA, May 31 - June 2, 2009*, pages 151–158. ACM, 2009.

[3] Anne Canteaut, Claude Carlet, Pascale Charpin, and Caroline Fontaine. Propagation characteristics and correlation-immunity of highly nonlinear boolean functions. In Bart Preneel, editor, *Advances in Cryptology - EUROCRYPT 2000, International Conference on the Theory and Application of Cryptographic Techniques, Bruges, Belgium, May 14-18, 2000, Proceeding*, volume 1807 of *Lecture Notes in Computer Science*, pages 507–522. Springer, 2000.

[4] Claude Carlet. *Boolean Functions for Cryptography and Coding Theory.* Cambridge University Press, October 2020.

[5] Eldar Fischer, Guy Kindler, Dana Ron, Shmuel Safra, and Alex Samorodnitsky. Testing juntas. In *43rd Symposium on Foundations of Computer Science (FOCS 2002), 16-19 November 2002, Vancouver, BC, Canada, Proceedings*, pages 103–112. IEEE Computer Society, 2002.

[6] Ehud Friedgut and Gil Kalai. Every monotone graph property has a sharp threshold. *Proceedings of the American mathematical Society*, 124(10):2993–3002, 1996.

[7] O. S. Rothaus. On “bent” functions. *J. Comb. Theory, Ser. A*, 20(3):300–305, 1976.

[8] Palash Sarkar and Subhamoy Maitra. Construction of nonlinear resilient boolean functions using "small" affine functions. *IEEE Trans. Inf. Theory*, 50(9):2185–2193, 2004.

[9] Thomas Siegenthaler. Correlation-immunity of nonlinear combining functions for cryptographic applications. *IEEE Trans. Inf. Theory*, 30(5):776–780, 1984.

[10] Avishay Tal. Tight bounds on the fourier spectrum of AC0. In Ryan O'Donnell, editor, *32nd Computational Complexity Conference, CCC 2017, July 6-9, 2017, Riga, Latvia*, volume 79 of *LIPIcs*, pages 15:1–15:31. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017.

[11] Guo-Zhen Xiao and James L. Massey. A spectral characterization of correlation-immune combining functions. *IEEE Trans. Inf. Theory*, 34(3):569–571, 1988.

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