



# ON PUBLIC-COIN ZERO-ERROR RANDOMIZED COMMUNICATION COMPLEXITY

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**ABSTRACT.** We prove the largest possible gap between the deterministic communication complexity and the public-coin zero-error randomized communication complexity is essentially quadratic. This was previously known only in the private-coin model.

## 1. INTRODUCTION

The field of communication complexity studies the amount of communication required to solve the problem of computing discrete functions when the input is split between two or more parties. In the most commonly studied framework, there are two parties, often called Alice and Bob, and a communication problem is defined by a Boolean function  $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$ . An input  $x \in \mathcal{X}$  is given to Alice, and an input  $y \in \mathcal{Y}$  is given to Bob. Together, they should both compute the entry  $f(x, y)$  by exchanging bits of information in turn, according to a previously agreed-on protocol. There is no restriction on their computational power; the only measure we care to minimize is the number of exchanged bits.

A deterministic protocol  $\pi$  specifies for each of the two players, the bit to send next, as a function of their input and history of the communication so far. It naturally corresponds to a binary tree as follows. Every internal node is associated with either Alice or Bob. If an internal node  $v$  is associated with Alice, then it is labeled with a function  $a_v : \mathcal{X} \rightarrow \{0, 1\}$ , which prescribes the bit sent by Alice at this node as a function of her input. After this bit was sent, the players move to the corresponding child of  $v$ : they move to the left child if the bit is 0, and to the right child if the bit is 1. Similarly, Bob's nodes are labeled with Boolean functions on  $\mathcal{Y}$ . Each leaf is labeled by 0 or 1, which corresponds to the output of the protocol. The *cost* of the protocol on an input  $(x, y)$ , denoted by  $\text{cost}(\pi(x, y))$ , is the number of bits exchanged on this input. The cost of the protocol is the maximum of  $\text{cost}(\pi(x, y))$  over all inputs  $(x, y)$ . The *deterministic communication complexity* of  $f$ , denoted by  $D(f)$ , is the smallest cost of a protocol that computes  $f$  correctly on all inputs.

Next we discuss randomized communication complexity. The randomness can be introduced in two different ways: private randomness or public randomness.

A *private-coin randomized protocol* assumes that each player has access to his or her independent random bits, and can use them to decide which bit to send next. More precisely, Alice and Bob have access to random strings  $R_A$  and  $R_B$ , respectively. These two strings are chosen independently, each according to some probability distribution described by the protocol. The bit sent by Alice at a node  $v$  is now determined as a function of both  $x$  and  $R_A$ . Similarly the bits sent by Bob are determined as functions of  $y$  and  $R_B$ .

On the other hand, in the *public-coin* model, it is assumed that the players have access to a shared source of randomness. In other words Alice and Bob are both given the same random string  $R$ . The public-coin model is stronger than the private-coin model as the former can simulate the latter by using  $R = (R_A, R_B)$  as the public random string.

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In this article, we are interested in *zero-error* randomized communication protocols. These protocols, which are also called Las Vegas protocols, always give correct answer, and the cost of such a protocol is defined as the *expected number* of bits exchanged over the worst input  $(x, y)$ . The private-coin zero-error randomized communication complexity of a function  $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$ , denoted by  $R_0^{\text{prv}}(f)$ , is the infimum cost over all private-coin randomized protocols that compute  $f$  with zero error:

$$R_0^{\text{prv}}(f) = \inf_{\pi_{R_A, R_B}} \max_{(x, y)} \mathbb{E}_{R_A, R_B} [\text{cost}(\pi_{R_A, R_B}(x, y))],$$

where the infimum is over all zero-error *private-coin* protocols  $\pi_{R_A, R_B}$ . The public-coin zero-error randomized communication complexity is defined similarly using public randomness and is denoted by  $R_0^{\text{pub}}(f)$ . That is

$$R_0^{\text{pub}}(f) = \inf_{\pi_R} \max_{(x, y)} \mathbb{E}_R [\text{cost}(\pi_R(x, y))],$$

where the infimum is over all zero-error *public-coin* protocols  $\pi_R$ .

We have the trivial relations

$$R_0^{\text{pub}}(f) \leq R_0^{\text{prv}}(f) \leq D(f).$$

A fundamental fact in communication complexity is that the leaves of every deterministic protocol computing  $f$  partitions  $f$  into rectangles (a rectangle is a set  $S \times T \subseteq \mathcal{X} \times \mathcal{Y}$ ) such that the value of  $f$  is constant on each rectangle. Such rectangles are called *monochromatic*. Hence every deterministic protocol of cost  $c$  provides a partition of  $\mathcal{X} \times \mathcal{Y}$  into at most  $2^c$  monochromatic rectangles. Since the rank (as a real matrix) of a monochromatic rectangle is at most one, we obtain the classical lower-bound

$$(1) \quad D(f) \geq \log \text{rk}(f),$$

where  $\text{rk}(f)$  denotes the rank of  $f$  as a matrix over the reals, and here and throughout the paper logarithms are in base 2.

Let  $N(f)$  denote the logarithm of the minimum number of monochromatic rectangles required to *cover*  $\mathcal{X} \times \mathcal{Y}$ . It follows from the above discussion that  $N(f) \leq D(f)$ , and furthermore as it is shown in [AUY83] (see also the proof of [KN97, Theorem 2.11]) the gap can be at most quadratic:

$$(2) \quad D(f) \leq 4N(f)^2.$$

The quantity  $N(f)$  can also be used to lower-bound  $R_0^{\text{prv}}(f)$ . Indeed, a private-coin protocol can be interpreted as a deterministic protocol where Alice's input is  $(x, R_A)$  and Bob's input is  $(y, R_B)$ . Hence the leaves of such a protocol provide a *partition* of  $(\mathcal{X} \times \Omega_A) \times (\mathcal{Y} \times \Omega_B)$  into combinatorial rectangles, where  $\Omega_A$  and  $\Omega_B$  are the supports of the random strings  $R_A$  and  $R_B$ , respectively. If the protocol is of zero-error, then every rectangle  $S \times T \subseteq (\mathcal{X} \times \Omega_A) \times (\mathcal{Y} \times \Omega_B)$  in this partition corresponds to a *monochromatic* rectangle  $\hat{S} \times \hat{T} \subseteq \mathcal{X} \times \mathcal{Y}$  defined as

$$\hat{S} = \{x \in \mathcal{X} : \exists r_A \in \Omega_A, (x, r_A) \in S\}, \quad \hat{T} = \{y \in \mathcal{Y} : \exists r_B \in \Omega_B, (y, r_B) \in T\}.$$

Furthermore, if the protocol has cost  $c$ , then for every  $(x, y)$ , we have  $\mathbb{E}[\text{cost}(\pi_{R_A, R_B}(x, y))] \leq c$ , and thus for every  $(x, y)$ , there exists a choice of  $r_A$  and  $r_B$  such that  $(x, r_A, y, r_B)$  leads to a leaf of depth at most  $c$ . Hence the set of monochromatic rectangles  $\hat{S} \times \hat{T}$  that correspond to the leaves of depth at most  $c$  provide a cover of  $\mathcal{X} \times \mathcal{Y}$ . There are at most  $2^c$  such leaves, and thus  $N(f) \leq \log 2^c = R_0^{\text{prv}}(f)$ . Combining this with Equation (2) yields

$$R_0^{\text{prv}}(f) = \Omega(\sqrt{D(f)}).$$

Furthermore, as it is shown by Fürer [Für87] there are examples for which this quadratic gap is necessary.

Next we turn our attention to the public-coin model, which is the main focus of this article. Perhaps the most famous result relating public-coin and private-coin models is Newman’s lemma [New91], which in the zero-error case (see [KN97, Exercise 3.15]) states

$$(3) \quad R_0^{\text{prv}}(f) \leq O(R_0^{\text{pub}}(f) + \log \log |\mathcal{X} \times \mathcal{Y}|).$$

Note that since  $\log \log |\mathcal{X} \times \mathcal{Y}|$  can be the dominant term in Equation (3), Newman’s lemma does not provide a polynomial relation between the public-coin and private-coin zero-error communication complexities. To the best of our knowledge, it was unknown whether  $R_0^{\text{pub}}(f)$  is polynomially equivalent to  $R_0^{\text{prv}}(f)$  and  $D(f)$ . The aim of this article is to establish such a relation.

**Theorem 1.1** (Main Theorem). *For every  $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$  we have*

$$R_0^{\text{pub}}(f) = \Omega \left( \frac{\sqrt{D(f)}}{\log D(f)} \right).$$

## 2. PROOF OF THEOREM 1.1

The key insight for proving Theorem 1.1 is to consider the relation between  $\text{rk}(f)$  and  $D(f)$ . If  $\log \text{rk}(f)$  is small, then by applying a classical result of Nisan and Wigderson [NW95], one can deduce that  $D(f)$  cannot be much larger than  $R_0^{\text{pub}}(f)$ .

On the other hand, if  $\log \text{rk}(f)$  is large, then by Equation (1) one can consider a full-rank submatrix  $A$  of  $f$  whose deterministic communication complexity is large as well. Since this submatrix is of full-rank, its dimensions are equal to its rank. In this case, we prove a variant of Newman’s lemma, which bounds the deterministic communication complexity by the public-coin zero-error randomized communication complexity up to a multiplicative  $\log \log |A|$  term. The key observation is that, in this case, the  $\log \log |A|$  term is bounded by  $\log D(f)$ , allowing us to obtain a strong lower-bound on the public-coin zero-error communication complexity.

We start by stating the result of Nisan and Wigderson [NW95]. In the following statement, for a rectangle  $A \subseteq \mathcal{X} \times \mathcal{Y}$ ,  $|A|$  denotes its set-theoretic cardinality.

**Lemma 2.1.** [NW95] *Let  $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$  be a Boolean function, and  $\delta > 0$  a real number. Assume that for every rectangle  $A_1 \subseteq \mathcal{X} \times \mathcal{Y}$ , there exists a sub-rectangle  $A_2 \subseteq A_1$  such that  $A_2$  is monochromatic and  $|A_2| \geq \delta |A_1|$ . Then*

$$D(f) \leq O(\log(1/\delta) \log \text{rk}(f) + (\log^2 \text{rk}(f))).$$

In the following simple corollary, we observe that  $f$  satisfies the assumption of Lemma 2.1 with  $\delta = 2^{-2R_0^{\text{pub}}(f)-1}$ .

**Corollary 2.2.** *There exists a universal constant  $c \geq 1$  such that every  $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$  satisfies*

$$D(f) \leq c R_0^{\text{pub}}(f) \log \text{rk}(f) + c \log^2 \text{rk}(f).$$

*Proof.* Let  $\pi_R$  be a zero-error public-coin communication protocol of cost  $R_0^{\text{pub}}(f)$  for  $f$ . Here  $R$  denotes the public randomness, and any fixation of  $R$  to a string  $r$  leads to a deterministic protocol  $\pi_r$ . Consider a rectangle  $A_1 \subseteq \mathcal{X} \times \mathcal{Y}$ . We have

$$\mathbb{E}_R \mathbb{E}_{(x,y) \in A_1} \text{cost}(\pi_R(x, y)) \leq R_0^{\text{pub}}(f),$$

which shows that there exists a fixation of  $R$  to  $r$  such that

$$\mathbb{E}_{(x,y) \in A_1} \text{cost}(\pi_r(x, y)) \leq R_0^{\text{pub}}(f).$$

Consequently,

$$\Pr_{(x,y) \in A_1} \left[ \text{cost}(\pi_r(x, y)) > 2 R_0^{\text{pub}}(f) \right] \leq \frac{1}{2}.$$

This shows that the leaves of the protocol  $\pi_r$  that are in depth at most  $2R_0^{\text{pub}}(f)$  cover at least half of the points in  $A_1$ . Since each leaf corresponds to a monochromatic sub-rectangle, one of these leaves must correspond to a monochromatic sub-rectangle of size at least  $2^{-2R_0^{\text{pub}}(f)-1}|A_1|$ .  $\square$

To optimize our bound, instead of applying Newman's lemma directly, we show how to directly bound  $D(f)$  in terms of  $R_0^{\text{pub}}(f)$ . We also show how this proof gives a bound on  $N(f)$ , although we do not use this latter bound.

**Lemma 2.3.** *Every Boolean function  $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$  satisfies*

$$D(f) \leq 2R_0^{\text{pub}}(f) \cdot \log \log |\mathcal{X}||\mathcal{Y}|,$$

and

$$N(f) \leq 2R_0^{\text{pub}}(f) + \log \log |\mathcal{X}||\mathcal{Y}| + 1.$$

*Proof.* Let  $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$  be a Boolean function and  $R_0^{\text{pub}}(f) = t$ , witnessed by a zero-error randomized protocol  $\pi_R$ . Let  $\pi'_R$  denote the truncation of  $\pi_R$  by restricting to computational paths of length at most  $2t$ , that is we trim off all branches at the  $2t^{\text{th}}$  node and output  $\perp$  indicating failure to compute  $f(x, y)$  in those cases. By Markov's bound, for every  $(x, y) \in \mathcal{X} \times \mathcal{Y}$ , we have

$$\Pr_R[\pi'_R(x, y) \neq \perp] \geq \frac{1}{2}.$$

Let  $\ell = 2 \log(|\mathcal{X}||\mathcal{Y}|)$  so that  $2^\ell > |\mathcal{X} \times \mathcal{Y}|$ , and consider  $\ell$  independent executions of  $\pi'_R$  by considering  $\ell$  independent copies  $R_1, \dots, R_\ell$  of  $R$ . For every  $(x, y)$ , we have

$$\Pr_{R_1, \dots, R_\ell} [\forall i, \pi'_{R_i}(x, y) = \perp] \leq \frac{1}{2^\ell} < \frac{1}{|\mathcal{X} \times \mathcal{Y}|}.$$

In particular there exists a choice of  $r_1, \dots, r_\ell$  such that for every  $(x, y) \in \mathcal{X} \times \mathcal{Y}$ , there exists at least one  $i \in \{1, \dots, \ell\}$  with  $\pi'_{r_i}(x, y) \neq \perp$ , and note that, since the protocol does not make errors, in this case  $\pi'_{r_i}(x, y) = f(x, y)$ . This suggests a deterministic protocol, where on input  $(x, y)$ , the parties simulate  $\pi'_{r_1}(x, y), \dots, \pi'_{r_\ell}(x, y)$ , and output the first  $\pi'_{r_i}(x, y) \neq \perp$ . The total cost of this protocol is at most  $2t\ell = 2R_0^{\text{pub}}(f) \cdot \log \log |\mathcal{X} \times \mathcal{Y}|$  as desired.

Now, to obtain the desired improved bound on  $N(f)$ , observe that each  $\pi'_{r_i}$  is a deterministic protocol with communication cost at most  $2t$ , and thus provides a partition of the points  $(x, y)$  with  $\pi_{r_i}(x, y) \neq \perp$  into at most  $2^{2t}$  monochromatic rectangles. Consequently, there exists a collection of  $\ell 2^{2t}$  monochromatic rectangles whose union is all of  $\mathcal{X} \times \mathcal{Y}$ , or in other words

$$N(f) \leq \log(\ell 2^{2t}) = 2R_0^{\text{pub}}(f) + \log \log |\mathcal{X} \times \mathcal{Y}| + 1.$$

$\square$

*Proof of Theorem 1.1.* Let  $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$  be a Boolean function, and let  $c \geq 1$  be the constant from Corollary 2.2. We consider two cases.

**Case I:** In this case we assume

$$\log \text{rk}(f) \leq \frac{1}{2c} \sqrt{D(f)}.$$

By Corollary 2.2, we have

$$D(f) \leq \frac{1}{2} R_0^{\text{pub}}(f) \sqrt{D(f)} + \frac{D(f)}{4},$$

which simplifies to

$$\sqrt{D(f)} \leq R_0^{\text{pub}}(f).$$

**Case II:** In this case we assume

$$\log \text{rk}(f) > \frac{1}{2c} \sqrt{D(f)}.$$

Denote  $m = 2^{\frac{1}{2c}} \sqrt{D(f)}$ . By the assumption,  $f$  contains a full-rank  $m \times m$  submatrix  $A$ , which by [Lemma 2.3](#) satisfies

$$D(A) \leq 2 R_0^{\text{pub}}(f) \cdot \log \log(m^2) \leq 4 R_0^{\text{pub}}(f) \cdot \log \log m.$$

Combining this with the rank lower-bound (see [Equation \(1\)](#))

$$D(A) \geq \log \text{rk}(A) \geq \log m,$$

yields

$$\frac{\log m}{\log \log m} \leq 4 R_0^{\text{pub}}(f).$$

Substituting  $m = 2^{\frac{1}{2c}} \sqrt{D(f)}$ , we obtain

$$R_0^{\text{pub}}(f) = \Omega\left(\frac{\sqrt{D(f)}}{\log D(f)}\right).$$

□

### 3. CONCLUDING REMARKS AND OPEN PROBLEMS

- We suspect that the  $\log D(f)$  term in [Theorem 1.1](#) can be removed. Is it true that similar to private-coin model, for every function  $f : \mathcal{X} \times \mathcal{Y} \rightarrow \{0, 1\}$ ,

$$(4) \quad R_0^{\text{pub}}(f) = \Omega(\sqrt{D(f)})?$$

Note that by Fürer's result [[Für87](#)] there are functions  $f$  for which  $R_0^{\text{pub}}(f) = \Theta(\sqrt{D(f)})$ , and thus [Equation \(4\)](#) is the strongest bound one can hope for.

- We are not aware of any examples separating  $N(f)$ ,  $R_0^{\text{prv}}(f)$ , and  $R_0^{\text{pub}}(f)$ . To the best of our knowledge, it could be the case that these parameters are within constant factors of each other. See also [[KN97](#), Problem 3.11].
- In the case where protocols are allowed to make error, the gap between the public-coin and private-coin randomized communication complexities can be arbitrarily large. For example, for the  $N \times N$  identity matrix  $\mathbf{I}_N$ , allowing an error probability of  $\frac{1}{3}$ , we have

$$R_{1/3}^{\text{pub}}(\mathbf{I}_N) = O(1), \quad R_{1/3}^{\text{prv}}(\mathbf{I}_N) = \Theta(\log N), \quad D(\mathbf{I}_N) = \Theta(\log N).$$

- [Corollary 2.2](#) shows that if  $\text{rk}(f)$  is small, then  $D(f)$  can be bounded as a function of  $R_0^{\text{pub}}(f)$ . Gavinsky and Lovett [[GL14](#)] show that this can be generalized to a similar bound for  $R_{1/3}^{\text{pub}}(f)$ , albeit with slightly worse parameters:

$$D(f) = O(R_{1/3}^{\text{pub}}(f) \log^2 \text{rk}(f)).$$

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