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Towards Uniform Certification in QBF

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Abstract. We pioneer a new technique that allows us to prove a multitude of previously open simulations in QBF proof complexity. In particular, we show that extended QBF Frege p-simulates clausal proof systems such as IR-Calculus, IRM-Calculus, Long-Distance Q-Resolution, and Merge Resolution. These results are obtained by taking a technique of Beyersdorff et al. (JACM 2020) that turns strategy extraction into simulation and combining it with new local strategy extraction arguments. This approach leads to simulations that are carried out mainly in propositional logic, with minimal use of the QBF rules. Our proofs therefore provide a new, largely propositional interpretation of the simulated systems. We argue that these results strengthen the case for uniform certification in QBF solving, since many QBF proof systems now fall into place underneath extended QBF Frege.

1 Introduction

The problem of evaluating Quantified Boolean Formulas (QBF), an extension of propositional satisfiability (SAT), is a canonical PSPACE-complete problem [1, 37]. Many tasks in verification, synthesis and reasoning have succinct QBF encodings [36], making QBF a natural target logic for automated reasoning. As such, QBF has seen considerable interest from the SAT community, leading to the development of a variety of QBF solvers (e.g., [20, 21, 30, 31, 33]). The underlying algorithms are often highly nontrivial, and their implementation can lead to subtle bugs [9]. While formal verification of solvers is typically impractical, trust in a solver's output can be established by having it generate a proof trace that can be externally validated. This is already standard in SAT solving with the DRAT proof system [40], for which even formally verified checkers are available [15]. A key requirement for standard proof formats like DRAT is that they *simulate* all current and emerging proof techniques.

Currently, there is no decided-upon checking format for QBF proofs (although there have been some suggestions [19, 23]). The main challenge of finding such an universal format, is that QBF solvers are so radically different in their proof techniques, that each solver basically works in its own proof system. For instance, solvers based on CDCL and

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(some) clausal abstraction solvers can generate proofs in Q-resolution (Q-Res) [26] or long-distance Q-resolution (LD-Q-Res) [2], while the proof system underlying expansion based solvers combines instantiation of universally quantified variables with resolution (∀Exp+Res) [22]. Variants of the latter system have been considered: IR-calc (**I**nstantiation **R**esolution) admits instantiation with partial assignments, and IRM-calc (**I**nstantiation **R**esolution **M**erge) additionally incorporates elements of long-distance Q-resolution [7].

A universal checking format for QBF ought to simulate all of these systems. A good candidate for such a proof system has been identified in extended QBF Frege (eFrege + ∀red): Beyersdorff et al. showed [6] that a lower bound for eFrege + ∀red would not be possible without a major breakthrough.

In this work, we show that eFrege + ∀red does indeed p-simulate IR-calc, IRM-calc, Merge Resolution (M-Res) and LQU^{+}-Res (a generalisation of LD-Q-Res), thereby establishing eFrege + ∀red and any stronger system (e.g., QRAT [19] or G [29]) as potential universal checking formats in QBF. As corollaries, we obtain (known) simulations of ∀Exp+Res [24] and LD-Q-Res [25] by QRAT, as well as a (new) simulation of IR-calc by QRAT, answering a question recently posed by Chede and Shukla [10]. A simulation structure with many of the known QBF proof systems and our new results is given in Figure 1.

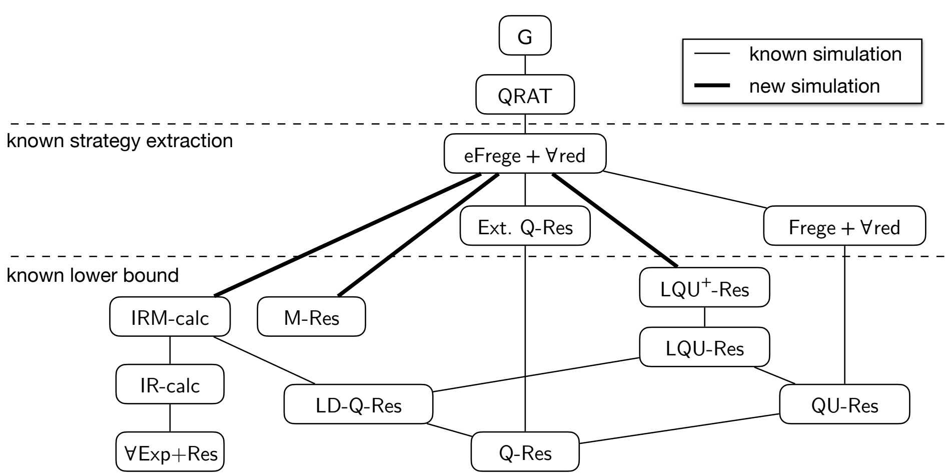

Fig. 1. Hasse diagram for polynomial simulation order of QBF calculi [2, 3, 5–7, 12, 13, 18, 39]. In this diagram all proof systems below the first line are known to have strategy extraction, and all below the second line have an exponential lower bound. G and QRAT have strategy extraction if and only if P = PSPACE.

Our proofs crucially rely on a property of QBF proof systems known as strategy extraction. Here, "strategy" refers to winning strategies of a set of PSPACE two-player games (see Section 2 for more details) each of which corresponds exactly to some QBF. A proof system is said to have strategy extraction if a strategy for the two-player game associated with a QBF can be computed from a proof of the formula in polynomial time. Balabanov and Jiang discovered [2] that Q-Resolution admitted a form of strategy extraction where a circuit computing a winning strategy could be extracted in linear time from the proofs. Strategy extraction was subsequently proven for many QBF proof systems (cf. Figure 1): the expansion based systems ∀Exp+Res [7], IR-calc [7] and IRM-calc [7], Long-Distance Q-Resolution [16], including with dependency schemes [16], Merge Resolution [5], Relaxing Stratex [11] and C-Frege + ∀red systems including eFrege + ∀red [6]. Strategy extraction also gained notoriety because it became a method to show Q-resolution lower bounds [7]. Beyersdorff et al. [6, 8] generalised this approach to more powerful proof systems, allowing them to establish a tight correspondence between lower bounds for eFrege + ∀red and two major open problems in circuit complexity and propositional proof complexity: they showed that proving a lower bound for eFrege + ∀red is equivalent to either proving a lower bound for P/poly or a lower bound for propositional eFrege. Chew conjectured [12] that this meant that all the aforementioned proof systems that had strategy extraction were very likely to be simulated by eFrege + ∀red and showed an outline of how to use strategy extraction to obtain the corresponding simulations.

We follow this outline in proving simulations for multiple systems by eFrege + ∀red. While the strategy extraction for expansion based systems [7] has been known for a while using the technique from Goultiaeva et. al [17], there currently is no intuitive way to formalise this strategy extraction into a simulation proof. Here we specifically studied a new strategy extraction technique given by Schlaipfer et al. [35], that creates local strategies for each ∀Exp+Res line. Inductively, we can affirm each of these local strategies and prove the full strategy extraction this way. This local strategy extraction technique is based on arguments of Suda and Gleiss [38], which allow it to be generalised to the expansion based system IRM-calc. We thus manage to prove a simulation for ∀Exp+Res and generalise it to IR-calc and then to IRM-calc. We also show a much more straight-forward simulation of M-Res and an adaptation of the IRM-calc argument to LQU^{+}-Res.

The remainder of the paper is structured as follows. In Section 2 we go over general preliminaries and the definition of $\mathsf{eFrege} + \forall\mathsf{red}$. The remaining sections are each dedicated to simulations of different calculi by $\mathsf{eFrege} + \forall\mathsf{red}$. In Section 3 we begin with a simulation of M-Res as a relatively easy example. In Section 4 we show for expansion based systems, how both an interpretation by in propositional logic and a local strategy is possible and why that leads to a simulation by $\mathsf{eFrege} + \forall\mathsf{red}$. For IR-calc we present an outline of the proof and for IRM-calc we detail which modifications are needed (full details are given in Appendix 7.1 and 7.2). In Section 5 we study the strongest CDCL proof system LQU^+-Res and show that it is also simulated by $\mathsf{eFrege} + \forall\mathsf{red}$, using a similar argument to IRM-calc.

2 Preliminaries

2.1 Quantified Boolean Formulas

A Quantified Boolean Formula (QBF) is a propositional formula augmented with Boolean quantifiers $\forall, \exists$ that range over the Boolean values $\bot, \top$ (the same as $0, 1$). Every propositional formula is already a QBF. Let ϕ be a QBF. The semantics of the quantifiers are that: $\forall x \phi(x) \equiv \phi(\bot) \wedge \phi(\top)$ and $\exists x \phi(x) \equiv \phi(\bot) \vee \phi(\top)$.

When investigating QBF in computer science we want to standardise the input formula. In a *prenex* QBF, all quantifiers appear outermost in a *(quantifier) prefix*, and are followed by a propositional formula, called the *matrix*. If every propositional variable of the matrix is bound by some quantifier in the prefix we say the QBF is a *closed* prenex QBF. We often want to standardise the propositional matrix, and so we can take the same approach as seen often in propositional logic. A *literal* is a propositional variable or its negation. A *clause* is a disjunction of literals. Since disjunction is idempotent, associative and commutative we can think of a clause simultaneously as a set of literals. The empty clause is just false. A *conjunctive normal form (CNF)* is a conjunction of clauses. Again, since conjunction is idempotent, associative and commutative a CNF can be seen as set of clauses. The empty CNF is true, and a CNF containing an empty clause is false. Every propositional formula has an equivalent formula in CNF, we therefore restrict our focus to closed *PCNF* QBFs, that is closed prenex QBFs with CNF matrices.

2.2 QBF Proof Systems

Proof Complexity A proof system [14] is a polynomial-time checking function that checks that every proof maps to a valid theorem. Different proof systems have varying strengths, in one system a theorem may require very long proofs, in another the proofs could be considerably shorter. We use *proof complexity* to analyse the strength of proof systems [27]. A proof system is said to have an $\Omega(f(n))$-lower bound, if there is a family of theorems such that shortest proof (in number of symbols) of the family are bounded below by $\Omega(f(n))$ where n is the size (in number of symbols) of the theorem. Proof system p is said to *simulate* proof system q if there is a fixed polynomial $P(x)$ such that for every q-proof π of every theorem y there is a p-proof of y no bigger than $P(|\pi|)$ where $|\pi|$ denotes the size of π. A stricter condition, proof system p is said to p-simulate proof system q if there is a polynomial-time algorithm that takes q-proofs to p-proofs preserving the theorem.

Extended Frege+∀-Red Frege systems are "text-book" style proof systems for propositional logic. They consist of a finite set of axioms and rules where any variable can be replaced by any formula (so each rule and axiom is actually a schema). A Frege system needs also to be sound and complete. Frege systems are incredibly powerful and can handle simple tautologies with ease. No lower bounds are known for Frege systems and all Frege systems are p-equivalent [14,34]. For these reasons we can assume all Frege-systems can handle simple tautologies and syllogisms without going into details.

Extended Frege (eFrege) takes a Frege system and allows the introduction of new variables that do not appear in any previous line of the proof. These variables abbreviate formulas. The rule works by introducing the axiom of $v \leftrightarrow f$ for new variable v (not in appearing in the formula f). Alternatively one can consider eFrege as the system where lines are circuits instead of formulas.

Extended Frege is a very powerful system, it was shown [4,28] that any propositional proof system f can be simulated by $\mathsf{eFrege} + ||\phi||$ where ϕ is a polynomially recognisable axiom scheme. The QBF analogue is eFrege + ∀red, which adds the reduction rule to all existing eFrege rules [6]. eFrege + ∀red is refutationally sound and complete for closed prenex QBFs. The reduction rules allows one to substitute a universal variable in a formula with 0 or with 1 as long as no other variable appearing in that formula is right of it in the prefix. Extension variables now must appear in the prefix and must be quantified right of the variables used to define it.

2.3 QBF Strategies

With a closed prenex QBF $\Pi\phi$, the semantics of a QBF has an alternative definition in games. The two-player QBF game has an $\exists$-player and a $\forall$-player. The game in played in order of the prefix Π left-to-right, whose quantifier appears gets to assign the quantified variable to $\bot$ or $\top$. The existential player is trying to make the matrix ϕ become true. The universal player is trying to make the matrix become false. $\Pi\phi$ is true if and only if there winning strategy for the $\exists$ player. $\Pi\phi$ is false if and only if there winning strategy for the $\forall$ player.

A *strategy* for a false QBF is a set of functions f_u for each universal variable u on variables left of u in the prefix. In a *winning strategy* the propositional matrix must evaluate to false when every u is replaced by f_u. A QBF proof system has *strategy extraction* if there is a polynomial time program that takes in a refutation π of some QBF Ψ and outputs circuits that represent the functions of a winning strategy.

A *policy* is similarly defined as a strategy but with partial functions for each universal variables instead of fully defined.

3 Extended Frege+$\forall$-Red p-simulates M-Res

In this section we show a first example of how the eFrege + $\forall$red simulation argument works in practice for systems that have strategy extraction. Merge resolution provides a straightforward example because the strategies themselves are very suitable to be managed in propositional logic. In later theorems where we simulate calculi like IR-calc and IRM-calc, representing strategies is much more of a challenge.

3.1 Merge Resolution

Merge resolution (M-Res) was first defined by Beyersdorff, Blinkhorn and Mahajan [5]. Its lines combines clausal information with a merge map, for each universal variable. Merge maps give a "local" strategy which when followed forces the clause to be true or the original CNF to be false.

Definition of Merge Resolution Each line of an M-Res proof consists of a clause on existential variables and partial universal strategy functions for universal variables. These functions are represented by *merge maps*, which are defined as follows. For universal variable u, let E_u be the set of existential variables left of u in the prefix. A non-trivial merge map

M_i^u is a collection of nodes in $[i]$, where the construction function $M_i^u(j)$ is either in $\{\bot, \top\}$ for leaf nodes or $E_u \times [j] \times [j]$ for internal nodes. The root $r(u,i)$ is the highest value of all the nodes M_i^u. The strategy function $h_{i,j}^u : \{0,1\}^{E_u} \to \{0,1\}$ maps assignments of existential variables E_u in the dependency set of u to a value for u. The function $h_{i,t}^u$ for leaf nodes t is simply the truth value $M_i^u(t)$. For internal nodes a with $M_i^u(a) = (y, b, c)$, we should interpret $h_{i,a}^u$ as "If y then $h_{i,b}^u$, else $h_{i,c}^u$" or $h_{i,a}^u = (y \wedge h_{i,b}^u) \vee (\neg y \wedge h_{i,c}^u)$. In summary the merge map $M_i^u(j)$ is a representation of the strategy given by function $h_{i,r(u,i)}^u$.

The merge resolution proof system inevitably has merge maps for the same universal variable interact, and we have two kinds of relations on pairs of merge maps.

Definition 1. *Merge maps M_j^u and M_k^u are said to be* consistent *if $M_j^u(i) = M_k^u(i)$ for each node i appearing in both M_j^u and M_k^u.*

Definition 2. *Merge maps M_j^u and M_k^u are said to be isomorphic if is there exists a bijection f from the nodes of M_j^u to the nodes of M_k^u such that if $M_j^u(a) = (y, b, c)$ then $M_k^u(f(a)) = (y, f(b), f(c))$ and if $M_j^u(t) = p \in \{\bot, \top\}$ then $M_k^u(f(t)) = p$.*

With two merge maps M_j^u and M_k^u, we define two operations as follows:

- `Select`(M_j^u, M_k^u) returns M_j^u if M_k^u is trivial (representing a "don't care"), or M_j^u and M_k^u are isomorphic and returns M_k^u if M_j^u is trivial and not isomorphic to M_j^u. If neither M_j^u or M_k^u is trivial and the two are not isomorphic then the operation fails.
- `Merge`(x, M_j^u, M_k^u) returns the map M_i^u with $i > j, i > k$ when M_j^u, M_k^u are consistent where if a is a node in M_j^u then $M_i^u(a) = M_j^u(a)$ and if a is a node in M_k^u then $M_i^u(a) = M_k^u(a)$. Merge map M_i^u has a new node $r(u,i)$ as a root node (which is greater than the maximum node in each of $M_i^u(a)$ or $M_j^u(a)$), and is defined as $M_i^u(r(u,i)) = (x, r(u,j), r(u,k))$.

Proofs in M-Res consist of lines, where every line is a pair $(C_i, \{M_i^u \mid u \in U\})$. Here, C_i is a purely existential clause (it contains only literals that are from existentially quantified variables). The other part is a set containing merge maps for each universal variable (some of the merge maps can be trivial, meaning they do not represent any function). Each line is derived by one of two rules:

Axiom: $C_i = \{l \mid l \in C, \mathrm{var}(l) \in E\}$ is the existential subset of some clause C where C is a clause in the matrix. If universal literals $u, \bar{u}$ do not appear in C, let M_i^u be trivial. If universal variable u appears in C then let i be the sole node of M_i^u with $M_i^u(i) = \bot$. Likewise if $\neg u$ appears in C then let i be the sole node of M_i^u with $M_i^u(i) = \top$.

Resolution: Two lines $(C_j, \{M_j^u \mid u \in U\})$ and $(C_k, \{M_k^u \mid u \in U\})$ can be resolved to obtain a line $(C_i, \mid \{M_i^u \mid u \in U\})$ if there is literal $\neg x \in C_j$ and $x \in C_k$ such that $C_i = C_j \cup C_k \setminus \{x, \neg x\}$, and every M_i^u can either be defined as $\texttt{Select}(M_j^u, M_k^u)$, when M_j^u and M_k^u are isomorphic or one is trivial, or as $\texttt{Merge}(x, M_j^u, M_k^u)$ when $x < u$ and M_j^u and M_k^u are consistent.

3.2 Simulation of Merge Resolution

We now state the main result of this section.

Theorem 1. eFrege + ∀red *simulates M-Res.*

For a false QBF $\Pi\phi$ refuted by M-Res, the final set of merge maps represent a falsifying strategy S for the universal player. It then should be the case that if ϕ is true, S must be false, a fact that can be proved propositionally, formally $\phi \vdash \neg S$.

To build up to this proof we can inductively find a local strategy S_i for each clause C_i that appears in an M-Res line $(C_i, \{M_i^u\})$ such that $\phi \vdash S_i \rightarrow C_i$. Elegantly, S_i is really just a circuit expressing that each $u \in U$ takes its value in M_i^u (if non-trivial). Extension variables are used to represent these local strategy circuits and so the proof ends up as a propositional extended Frege proof.

The final part of the proof is the technique suggested by Chew [12] which was originally used by Beyersdorff et al. [6]. That is, to use universal reduction starting from the negation of a universal strategy and arrive at the empty clause.

Proof. **Definition of extension variables.** We create new extension variables for each node in every non-trivial merge map appearing in a proof. $s_{i,j}^u$ is created for the node j in merge map M_i^u. $s_{i,t}^u$ is defined as a constant when t is leaf node in M_i^u. Otherwise $s_{i,a}^u$ is defined as $s_{i,a}^u := (y \wedge s_{i,b}^u) \vee (\neg y \wedge s_{i,c}^u)$, when $M_i^u(j) = (y, b, c)$. Because y has to be before u in the prefix, $s_{i,j}^u$ is always defined before universal variable u.
Induction Hypothesis: It is easy for eFrege to prove $\bigwedge_{u \in U_i}(u \leftrightarrow s_{i,r(u,i)}^u) \rightarrow C_i$, where $r(u,i)$ is the index of the root node of Merge map M_i^u. U_i is the subset of U for which M_i^u is non-trivial.
Base Case: Axiom: Suppose C_i is derived by axiom download of clause C. If u has a strategy, it is because it appears in a clause and so $u \leftrightarrow s_{i,i}^u$,

where $s^u_{i,i} \leftrightarrow c_u$ for $c_u \in \top, \bot$, c_u is correctly chosen to oppose the literal in C so that C_i is just the simplified clause of C replacing all universal u with their c_u. This is easy for `eFrege` to prove.

Inductive Step: Resolution: If C_j is resolved with C_k to get C_i with pivots $\neg x \in C_j$ and $x \in C_k$, we first show $\bigwedge_{u \in U_i}(u \leftrightarrow s^u_{i,r(u,i)}) \rightarrow C_j$ and $\bigwedge_{u \in U_r}(u \leftrightarrow s^u_{i,r(u,i)}) \rightarrow C_k$, where $r(u,i)$ is the root index of the Merge map for u on line i. We resolve these together.

To argue that $\bigwedge_{u \in U_i}(u \leftrightarrow s^u_{i,r(u,i)}) \rightarrow C_j$ we prove by induction that we can replace $u \leftrightarrow s^u_{j,r(u,j)}$ with $u \leftrightarrow s^u_{i,r(u,i)}$ one by one.

Induction Hypothesis: U_i is partitioned into W the set of adjusted variables and V the set of variables yet to be adjusted.

$(\bigwedge_{v \in V \cap U_j}(v \leftrightarrow s^v_{j,r(v,j)})) \wedge (\bigwedge_{v \in W}(v \leftrightarrow s^v_{i,r(v,i)})) \rightarrow C_j$

Base Case: $(\bigwedge_{v \in V \cap U_j}(v \leftrightarrow s^v_{j,r(v,j)})$ is the premise of the (outer) induction hypothesis

Inductive Step: Starting with $(\bigwedge_{v \in V \cap U_j}(v \leftrightarrow s^v_{j,r(v,j)})) \wedge (\bigwedge_{w \in W}(w \leftrightarrow s^w_{i,r(w,i)})) \rightarrow C_j$ We pick a $u \in V$ to show $(u \leftrightarrow s^w_{i,r(u,i)}) \wedge (\bigwedge_{v \in V \cap U_j}(v \leftrightarrow s^v_{j,r(v,j)})) \wedge (\bigwedge_{v \in W}(w \leftrightarrow s^w_{i,r(w,i)})) \rightarrow C_j$ We have four cases:

1. `Select` chooses $M^u_i = M^u_j$
2. `Select` chooses $M^u_i = M^u_k$ because M^u_j is trivial
3. `Select` chooses $M^u_i = M^u_k$ because there is an isomorphism f that maps M^u_j to M^u_k.
4. `Merge` so that M^u_i is the merge of M^u_j and M^u_k over pivot x

In (1) we prove inductively from the leaves to the root that $s^u_{i,t} \leftrightarrow s^u_{j,t}$. Eventually, we end up with $s^u_{i,r(u,i)} \leftrightarrow s^u_{j,r(u,j)}$. Then $(u \leftrightarrow s^u_{j,r(u,j)})$ can be replaced by $(u \leftrightarrow s^u_{i,r(u,i)})$.

In (2) we are simply weakening the implication as $(u \leftrightarrow s^u_{j,r(u,j)})$ never appeared before.

In (3) we prove inductively from the leaves to the root that $s^u_{i,f(t)} = s^u_{k,f(t)} = s^u_{j,t}$. Eventually, we end up with $s^u_{i,f(r(u,i))} = s^u_{k,f(r(u,i))} = s^u_{j,r(u,i)}$. Then $(u \leftrightarrow s^u_{j,r(u,j)})$ can be replaced by $(u \leftrightarrow s^u_{i,f(r(u,j))})$. As f is an isomorphism $f(r(u,j)) = r(u,k)$ and because `Select` is used $r(u,k) = r(u,i)$. Therefore we have $(u \leftrightarrow s^u_{i,r(u,i)})$.

In (4) we prove inductively that for each node t in M^u_j $(s^u_{i,t} \leftrightarrow s^u_{j,t})$. This is true in all leaf nodes as $s^u_{i,t}$ and $s^u_{j,t}$ have the same constant value. For intermediate nodes a, $s^u_{j,a} := (y \wedge s^u_{j,b}) \vee (\neg y \wedge s^u_{j,c})$ where b and c are other nodes. Since M^u_i is consistent with M^u_j then $s^u_{i,a} := (y \wedge s^u_{i,b}) \vee (\neg y \wedge s^u_{i,c})$ and since $s^u_{i,b} \leftrightarrow s^u_{j,b}$ and $s^u_{i,c} \leftrightarrow s^u_{j,c}$ by induction hypothesis, we have $s^u_{i,a} \leftrightarrow s^u_{j,a}$. eventually we have $s^u_{i,r(u,j)} \leftrightarrow s^u_{j,r(u,j)}$. However we need to replace $s^u_{j,r(u,j)}$ with $s^u_{i,r(u,i)}$, not $s^u_{i,r(u,j)}$. For this we use the definition of merging that $x \rightarrow (s^u_{i,r(u,i)} \leftrightarrow s^u_{i,r(u,j)})$ and so we have $(s^u_{i,r(u,i)} \leftrightarrow s^u_{j,r(u,j)}) \vee \neg x$ but the $\neg x$ is absorbed by the C_j in right hand side of the implication.

Finalise Inner Induction: At the end of this inner induction, we have $\bigwedge_{u\in U_i}(u \leftrightarrow s^u_{i,r(u,i)}) \rightarrow C_j$ and symmetrically $\bigwedge_{u\in U_i}(u \leftrightarrow s^u_{i,r(u,i)}) \rightarrow C_k$. We can then prove $\bigwedge_{u\in U_i}(u \leftrightarrow s^u_{i,r(u,i)}) \rightarrow C_i$.

Finalise Outer Induction: Note that we have done three nested inductions on the nodes in a merge maps, on the the universal variables, and then on the lines of an M-Res proof. Nonetheless, this gives a linear size eFrege proof in the number of nodes appearing in the proof. In M-Res the final line will be the empty clause and its merge maps. The induction gives us $\bigwedge_{u\in U_l}(u \leftrightarrow s^u_{l,r(u,l)}) \rightarrow \bot$. In other words, if $U_l = \{y_1, \dots y_n\}$, where y_i appears before y_{i+1} in the prefix, $\bigvee_{i=1}^{n}(y_i \oplus s^{y_i}_{l,r(y_i,l)})$

We derive $(0 \oplus s^{y_{n-k+1}}_{l,r(y_{n-k+1},l)}) \vee \bigvee_{i=1}^{n-k}(y_i \oplus s^{y_i}_{l,r(y_i,l)})$ and $(1 \oplus s^{y_{n-k+1}}_{l,r(y_{n-k+1},l)}) \vee \bigvee_{i=1}^{n-k}(y_i \oplus s^{y_i}_{l,r(y_i,l)})$ from reduction of $\bigvee_{i=1}^{n-k+1}(y_i \oplus s^{y_i}_{l,r(y_i,l)})$. We can resolve both with the easily proved tautology $\bigvee_{i=1}^{n-k}(y_i \oplus s^{y_i}_{l,r(y_i,l)})$. We continue this until we reach the empty disjunction.

4 Extended Frege+∀-Red p-simulates Expansion Based Systems

4.1 Expansion-Based Resolution Systems

The idea of an expansion based QBF proof system is to utilise the semantic identity: $\forall u \phi(u) = \phi(0) \wedge \phi(1)$, to replace universal quantifiers and their variables with propositional formulas. With $\forall u \exists x \phi(u) = \exists x \phi(0) \wedge \exists x \phi(1)$ the x from $\exists x \phi(0)$ and from $\exists x \phi(1)$ are actually different variables. The way to deal with this while maintaining prenex normal form is to introduce annotations that distinguish one x from another.

Definition 3.

1. *An* extended assignment *is a partial mapping from the universal variables to* $\{0, 1, *\}$*. We denote an extended assignment by a set or list of individual replacements i.e.* $0/u, */v$ *is an extended assignment.*
2. *An* annotated clause *is a clause where each literal is annotated by an extended assignment to universal variables.*
3. *For an extended assignment* σ *to universal variables we write* $l^{\mathsf{restrict}_l(\sigma)}$ *to denote an annotated literal where* $\mathsf{restrict}_l(\sigma) = \{c/u \in \sigma \mid \mathrm{lv}(u) < \mathrm{lv}(l)\}$.

4. *Two (extended) assignments* τ *and* μ *are called* contradictory *if there exists a variable* $x \in \mathsf{dom}(\tau) \cap \mathsf{dom}(\mu)$ *with* $\tau(x) \neq \mu(x)$.

Definitions The most simple way to use expansion would be to expand all universal quantifiers and list every annotated clause. The first expansion based system we consider, $\forall$Exp+Res, has a mechanism to avoid this potential exponential explosion in some (but not all) cases. An annotated clause is created and then checked to see if it could be obtained from expansion. This way a refutation can just use an unsatisfiable core rather than all clauses from a fully expanded matrix.

$$\frac{}{\left\{l^{\mathsf{restrict}_l(\tau)} \mid l \in C, l \text{ is existential}\right\} \cup \{\tau(l) \mid l \in C, l \text{ is universal}\}} \text{(Axiom)}$$

C is a clause from the matrix and τ is an assignment to all universal variables.

$$\frac{C_1 \cup \{x^\tau\} \qquad C_2 \cup \{\neg x^\tau\}}{C_1 \cup C_2} \text{(Res)}$$

Fig. 2. The rules of $\forall$Exp+Res (adapted from [22]).

The drawback of $\forall$Exp+Res is that one might end up repeating almost the same derivations over and over again if they vary only in changes in the annotation which make little difference in that part of the proof. This was used to find a lower bound to $\forall$Exp+Res for a family of formulas easy in system Q-Res [22]. To rectify this, IR-calc improved on $\forall$Exp+Res to allow a delay to the annotations in certain circumstances. Annotated clauses now have annotations with "gaps" where the value of the universal variable is yet to be set. When they are set there is the possibility of choosing both assignments without the need to rederive the annotated clauses with different annotations.

Definition 4. *Given two partial assignments (or partial annotations)* α *and* β*. The completion* $\alpha \circ \beta$*, is a new partial assignment, where*

$$\alpha \circ \beta(u) = \begin{cases} \alpha(u) & \text{if } u \in \mathsf{dom}(\alpha) \\ \beta(u) & \text{if } u \in \mathsf{dom}(\beta) \setminus \mathsf{dom}(\alpha) \\ \text{unassigned} & \text{otherwise} \end{cases}$$

For α an assignment of the universal variables and C an annotated clause we define $\mathsf{inst}(\alpha, C) := \bigvee_{l^\tau \in C} l^{\mathsf{restrict}_l(\tau \circ \alpha)}$. Annotation α here gives values to unset annotations where one is not already defined. Because the same α is used throughout the clause, the previously unset values gain consistent annotations, but mixed annotations can occur due to already existing annotations.

$$\frac{}{\left\{l^{\mathsf{restrict}_l(\tau)} \mid l \in C, l \text{ is existential}\right\}} \text{ (Axiom)}$$

C is a non-tautological clause from the matrix. $\tau = \{0/u \mid u \text{ is universal in } C\}$, where the notation $0/u$ for literals u is shorthand for $0/x$ if $u = x$ and $1/x$ if $u = \neg x$.

$$\frac{x^\tau \vee C_1 \qquad \neg x^\tau \vee C_2}{C_1 \cup C_2} \text{ (Resolution)} \qquad \frac{C}{\mathsf{inst}(\tau, C)} \text{ (Instantiation)}$$

τ is an assignment to universal variables with $\mathsf{rng}(\tau) \subseteq \{0, 1\}$.

Fig. 3. The rules of IR-calc [7].

The definition of IR-calc is given in Figure 3. Resolved variables have to match exactly, including that missing values are missing in both pivots. However, non-contradictory but different annotations may still be used for a later resolution step after the instantiation rule is used to make the annotations match the annotations of the pivot.

Local Strategies and Policies The work from Schlaipfer et al. [35] creates a conversion of each annotated clause C into a propositional formula $\mathrm{con}(C)$ defined in the original variables of ϕ (so without creating new annotated variables). C appearing in a proof asserts that there is some (not necessarily winning) strategy for the universal player to force $\mathrm{con}(C)$ to be true under ϕ. The idea is that for each line C in an ∀Exp+Res refutation of $\Pi\phi$ there is some local strategy S such that $S \wedge \phi \rightarrow \mathrm{con}(C)$.

The construction of the strategy is formed from the structure of the proof and follows the semantic ideas of Suda and Gleiss [38], in particular the `Combine` operation for resolution. The extra work by Schlaipfer et al. is that the strategy circuits (for each u) can be constructed in polynomial time, and can be defined in variables left of u_i in the prefix.

Let $u_1 \dots u_n$ be all universal variables in order. For each line in an ∀Exp+Res proof we have a strategy which we will here call S. For each u_i

there is an extension variable Val^i_S, before u_i, that represents the value assigned to u_i by S (under an assignment of existential variables). Using these variables, we obtain a propositional formula representing the strategy as $S = \bigwedge_{i=1}^{n} u_i \leftrightarrow \mathrm{Val}^i_S$. Additionally, we define a conversion of annotated logic in $\forall$Exp+Res to propositional logic as follows. For annotations τ let $\mathrm{anno}(\tau) = \bigwedge_{1/u_i \in \tau} u_i \wedge \bigwedge_{0/u_i \in \tau} \bar{u}_i$. We convert annotated literals as $\mathrm{con}(l^\tau) = l \wedge \mathrm{anno}(\tau)$ and clauses as $\mathrm{con}(C) = \bigvee_{l \in C} \mathrm{con}(l)$.

4.2 Simulating IR-calc

The conversion needs to be revised for IR-calc. In particular the variables not set in the annotations need to be understood. The solution is to basically treat unset as a third value, although in practice this requires new Set^i_S variables (left of u_i) which state that the ith universal variable is set by policy S. We include these variables in our encoding of policy S and let $S = \bigwedge_{i=1}^{n} \mathrm{Set}^i_S \rightarrow (u_i \leftrightarrow \mathrm{Val}^i_S)$. The conversion of annotations, literals and clauses also has to be changed. For annotations τ let

$$\underset{x,S}{\mathrm{anno}}(\tau) = \bigwedge_{1/u_i \in \tau} (\overset{i}{\underset{S}{\mathrm{Set}}} \wedge u_i) \wedge \bigwedge_{0/u_i \in \tau} (\overset{i}{\underset{S}{\mathrm{Set}}} \wedge \bar{u}_i) \wedge \bigwedge_{u_i <_\Pi x}^{u_i \notin \mathsf{dom}(\tau)} \neg \overset{i}{\underset{S}{\mathrm{Set}}}\,.$$

Let $\mathrm{con}_S(l^\tau) = l \wedge \mathrm{anno}_{x,S}(\tau)$ and $\mathrm{con}_S(C) = \bigvee_{l \in C} \mathrm{con}_S(l)$ similarly to before, we just reference a particular policy S. This means that we again want $S \wedge \phi \rightarrow \mathrm{con}_S(C)$ for each line, note that Set^i_S variables are defined in their own way.

The most crucial part of simulating IR-calc is that after each application of the resolution rule we can obtain a working policy.

Lemma 1. *Suppose, there are policies L and R such that $L \rightarrow \mathrm{con}_L(C_1 \vee \neg x^\tau)$ and $R \rightarrow \mathrm{con}_L(C_1 \vee x^\tau)$ then there is a policy B such that $B \rightarrow \mathrm{con}_B(C_1 \vee C_2)$ can be obtained in a short* eFrege *proof.*

The proof of the simulation of IR-calc relies on Lemma 1. To prove this we have to first give the precise definitions of the policy B based on policies L and R. Schlaipfer et al.'s work [35] is used to crucially make sure the strategy B, respects the prefix ordering.

Building the Strategy. We start to define Val^i_B and Set^i_B on lower i values first. In particular we will always start with $1 \leq i \leq m$ where u_m is the rightmost universal variable still before x in the prefix. Starting from $i = 0$, the initial segments of $\mathrm{con}_{x,L}(\tau)$ and $\mathrm{con}_{x,R}(\tau)$ may eventually reach such a point j where one is contradicted. Before this point L and R are detailing the same strategy (they may differ on Val^i but only when Set^i is false) so B can be played as both simultaneously as L and R. Without loss of generality, as soon as L contradicts $\mathrm{anno}_{x,L}(\tau)$, we know that $\mathrm{con}_L(x^\tau)$ is not satisfied by L and thus it makes sense for $B = B_L$, at this point and the rest of the strategy as it will satisfy $\mathrm{con}_B(C_1)$. It is entirely possible that we reach $i = m$ and not contradict either $\mathrm{con}_{x,L}(\tau)$ or $\mathrm{con}_{x,R}(\tau)$. Fortunately after this point in the game we now know the value the existential player has chosen for x. We can use the x value to decide whether to play B as L (if x is true) or R (if x is false).

To build the circuitry for Val^i_B and Set^i_B we will introduce other circuits that will act as intermediate. First we will use constants Set^i_τ and Val^i_τ that make $\mathrm{anno}_{x,S}(\tau)$ equivalent to $\bigwedge_{u_i <_\Pi x}(\mathrm{Set}^i_S \leftrightarrow \mathrm{Set}^i_\tau) \wedge \mathrm{Set}^i_\tau \rightarrow (u_i \leftrightarrow \mathrm{Val}^i_\tau)$. This mainly makes our notation easier. Next we will define circuits that represent two strategies being equivalent up to the ith universal variable. This is a generalisation of what was seen in the local strategy extraction for ∀Exp+Res [35].

$\mathrm{Eq}^0_{f=g} := 1$, $\mathrm{Eq}^i_{f=g} := \mathrm{Eq}^{i-1}_{f=g} \wedge (\mathrm{Set}^i_f \leftrightarrow \mathrm{Set}^i_g) \wedge (\mathrm{Set}^i_f \rightarrow (\mathrm{Val}^i_f \leftrightarrow \mathrm{Val}^i_g))$.

We specifically use this for a trigger variable that tells you which one of L and R differed from τ first.

$\mathrm{Dif}^0_L := 0$ and $\mathrm{Dif}^i_L := \mathrm{Dif}^{i-1}_L \vee (\mathrm{Eq}^{i-1}_{R=\tau} \wedge ((\mathrm{Set}^i_L \oplus \mathrm{Set}^i_\tau) \vee (\mathrm{Set}^i_\tau \wedge (\mathrm{Val}^i_L \oplus \mathrm{Val}^i_\tau))))$

$\mathrm{Dif}^0_R := 0$ and $\mathrm{Dif}^i_R := \mathrm{Dif}^{i-1}_R \vee (\mathrm{Eq}^{i-1}_{L=\tau} \wedge ((\mathrm{Set}^i_R \oplus \mathrm{Set}^i_\tau) \vee (\mathrm{Set}^i_\tau \wedge (\mathrm{Val}^i_R \oplus \mathrm{Val}^i_\tau))))$

Note that Dif^i_L and Dif^i_R can both be true but only if they start to differ at the same point.

Suda and Gleiss's `Combine` operation allows one to construct a bottom policy B that chooses between the left and right policies.

Definition 5 (Definition of resolvent policy for IR-calc).

For $0 \leq i \leq m$, *define* Val^i_B *and* Set^i_B *such* $\mathrm{Val}^i_B = \mathrm{Val}^i_R$ *and* $\mathrm{Set}^i_B = \mathrm{Set}^i_R$ *if*

$$\neg \underset{L}{\overset{i-1}{\mathrm{Dif}}} \wedge (\underset{R}{\overset{i-1}{\mathrm{Dif}}} \vee (\neg \underset{\tau}{\overset{i}{\mathrm{Set}}} \wedge \neg \underset{L}{\overset{i}{\mathrm{Set}}} \wedge \underset{R}{\overset{i}{\mathrm{Set}}}) \vee (\underset{\tau}{\overset{i}{\mathrm{Set}}} \wedge \underset{L}{\overset{i}{\mathrm{Set}}} \wedge (\underset{\tau}{\overset{i}{\mathrm{Val}}} \leftrightarrow \underset{L}{\overset{i}{\mathrm{Val}}})))$$

and $\mathrm{Val}^i_B = \mathrm{Val}^i_L$ *and* $\mathrm{Set}^i_B = \mathrm{Set}^i_L$, *otherwise.*

For $i > m$, *define* Val^i_B *and* Set^i_B *such* $\mathrm{Val}^i_B = \mathrm{Val}^i_R$ *and* $\mathrm{Set}^i_B = \mathrm{Set}^i_R$ *if*

$$\neg \underset{L}{\overset{m}{\mathrm{Dif}}} \wedge (\underset{R}{\overset{m}{\mathrm{Dif}}} \vee \bar{x})$$

and $\mathrm{Val}^i_B = \mathrm{Val}^i_L$ *and* $\mathrm{Set}^i_B = \mathrm{Set}^i_L$, *otherwise.*

We will now define variables B_L and B_R. These say that B is choosing L or R, respectively. These variables can appear rightmost in the prefix, as they will be removed before reduction takes place. The purpose of B_L (resp. B_R) is that con_B becomes the same as con_L (resp. con_R).

- $B_L := \bigwedge_{i=1}^{n}(\mathrm{Set}_B^i \leftrightarrow \mathrm{Set}_L^i) \wedge (\mathrm{Set}_B^i \rightarrow (\mathrm{Val}_B^i \leftrightarrow \mathrm{Val}_L^i))$
- $B_R := \bigwedge_{i=1}^{n}(\mathrm{Set}_B^i \leftrightarrow \mathrm{Set}_R^i) \wedge (\mathrm{Set}_B^i \rightarrow (\mathrm{Val}_B^i \leftrightarrow \mathrm{Val}_R^i))$

We have not fully defined B here (see Appendix 7.1 for details).

The important points are that B is set up so that it either takes values in L or R , i.e. $B \rightarrow B_L \vee B_R$, specifically we need that whenever the propositional formula $\mathrm{anno}_{x,B}(\tau)$ is satisfied,

$B = B_L$ when x, and $B = B_R$ when $\neg x$. The variables Set_B^i and Val_B^i that comprise the policy are carefully constructed to come before u_i.

Proof (Proof of Lemma 1). Since $B \wedge B_L \rightarrow L$ and $B \wedge B_R \rightarrow R$, $L \rightarrow \mathrm{con}_L(C_1 \vee \neg x^\tau)$ and $R \rightarrow \mathrm{con}_L(C_2 \vee x^\tau)$ imply $B \wedge B_L \rightarrow \mathrm{con}_B(C_1 \vee C_2) \vee \mathrm{anno}_{x,B}(\tau)$, $B \wedge B_R \rightarrow \mathrm{con}_B(C_1 \vee C_2) \vee \mathrm{anno}_{x,B}(\tau)$, $B \wedge B_L \rightarrow \mathrm{con}_B(C_1 \vee C_2) \vee \neg x$ and $B \wedge B_R \rightarrow \mathrm{con}_B(C_1 \vee C_2) \vee x$.

We combine $B \rightarrow B_L \vee B_R$ (proved in Lemma 10) with $B \wedge B_L \rightarrow \mathrm{con}_B(C_1 \vee C_2) \vee \mathrm{anno}_{x,B}(\tau)$ (removing B_L) and $B \wedge B_R \rightarrow \mathrm{con}_B(C_1 \vee C_2) \vee \mathrm{anno}_{x,B}(\tau)$ (removing B_R) to gain $B \rightarrow \mathrm{con}_B(C_1 \vee C_2) \vee \mathrm{anno}_{x,B}(\tau)$. Next, we derive $B \rightarrow \mathrm{con}_B(C_1 \vee C_2) \vee \neg\,\mathrm{anno}_{x,B}(\tau)$. Policy B is set up so that $B \wedge \mathrm{anno}_{x,B}(\tau) \wedge x \rightarrow B_L$ and $B \wedge \mathrm{anno}_{x,B}(\tau) \wedge \neg x \rightarrow B_R$ have short proofs (Lemma 11). We resolve these, respectively, with $B \wedge B_R \rightarrow \mathrm{con}_B(C_1 \vee C_2) \vee x$ (on x) to obtain $B \wedge \mathrm{anno}_{x,B}(\tau) \wedge B_R \rightarrow B_L \vee \mathrm{con}_B(C_1 \vee C_2)$, and with $B \wedge B_L \rightarrow \mathrm{con}_B(C_1 \vee C_2) \vee \neg x$ (on $\neg x$) to obtain $B \wedge \mathrm{anno}_{x,B}(\tau) \wedge B_L \rightarrow B_R \vee \mathrm{con}_B(C_1 \vee C_2)$. Putting these together allows us to remove B_L and B_R, deriving $B \wedge \mathrm{anno}_{x,B}(\tau) \rightarrow \mathrm{con}_B(C_1 \vee C_2)$, which can be rewritten as $B \rightarrow \mathrm{con}_B(C_1 \vee C_2) \vee \neg\,\mathrm{anno}_{x,B}(\tau)$.

We now have two formulas $B \rightarrow \mathrm{con}_B(C_1 \vee C_2) \vee \neg\,\mathrm{anno}_{x,B}(\tau)$ and $B \rightarrow \mathrm{con}_B(C_1 \vee C_2) \vee \mathrm{anno}_{x,B}(\tau)$, which resolve to get $B \rightarrow \mathrm{con}_B(C_1 \vee C_2)$.

Theorem 2. eFrege + ∀red *p-simulates* IR-calc.

Proof. We prove by induction that every annotated clause C appearing in an IR-calc proof has a local policy S such that $\phi \vdash_{\mathsf{eFrege}} S \rightarrow \mathrm{con}_S(C)$ and this can be done in a polynomial-size proof.

Axiom: Suppose $C \in \phi$ and $D = \mathsf{inst}(C, \tau)$ for partial annotation τ. We construct policy B such that $B \rightarrow \mathrm{con}_B(D)$.

$$\mathrm{Set}_B^j = \begin{cases} 1 & \text{if } u_j \in \mathsf{dom}(\tau) \\ 0 & u_j \notin \mathsf{dom}(\tau) \end{cases},\ \mathrm{Val}_B^j = \begin{cases} 1 & \text{if } 1/u_j \in \tau \\ 0 & \text{if } 0/u_j \in \tau \end{cases}$$

Instantiation: Suppose we have an instantiation step for C on a single universal variable u_i using instantiation $0/u_i$, so the new annotated clause is $D = \mathsf{inst}(C, 0/u_i)$. From the induction hypothesis $T \to \mathrm{con}_T(C)$ we will develop B such that $B \to \mathrm{con}_B(D)$.

$$\mathrm{Set}_B^j = \begin{cases} 1 & \text{if } j = i \\ \mathrm{Set}_T^j & \text{if } j \neq i \end{cases}, \ \mathrm{Val}_B^j = \begin{cases} \mathrm{Val}_T^j \wedge \mathrm{Set}_T^j & \text{if } j = i \\ \mathrm{Val}_T^j & \text{if } j \neq i \end{cases}$$

$\mathrm{Val}_T^j \wedge \mathrm{Set}_T^j$ becomes $\mathrm{Val}_T^j \vee \neg \mathrm{Set}_T^j$ for instantiation by $1/u_j$. Either case means B satisfies the same annotations anno as T appearing in our converted clauses $\mathrm{con}_B(C)$ and $\mathrm{con}_B(D)$, proving the rule as an inductive step.

Resolution: See Lemma 1.

Contradiction: At the end of the proof we have $T \to \mathrm{con}_T(\bot)$. T is a policy, so we turn it into a full strategy B by having for each i: $\mathrm{Val}_B^i \leftrightarrow (\mathrm{Val}_T^i \wedge \mathrm{Set}_T^i)$ and $\mathrm{Set}_B^i = 1$. Effectively this instantiates $\bot$ by the assignment that sets everything to 0 and we can argue that $B \to \mathrm{con}_B(\bot)$ although $\mathrm{con}_B(\bot)$ is just the empty clause. So we have $\neg B$. But $\neg B$ is just $\bigvee_{i=1}^{n}(u_i \oplus \mathrm{Val}_B^i)$. Furthermore, just as in Schlaipfer et al.'s work , we have been careful with the definitions of the extension variables Val_B^i so that they are left of u_i in the prefix. In $\mathsf{eFrege} + \forall\mathsf{red}$ we can use the reduction rule (this is the first time we use the reduction rule). We show an inductive proof of $\bigvee_{i=1}^{n-k}(u_i \oplus \mathrm{Val}_B^i)$ for increasing k eventually leaving us with the empty clause. This essentially is where we use the $\forall$-Red rule. Since we already have $\bigvee_{i=1}^{n}(u_i \oplus \mathrm{Val}_B^i)$ we have the base case and we only need to show the inductive step.

We derive from $\bigvee_{i=1}^{n+1-k}(u_i \oplus \mathrm{Val}_B^i)$ both $(0 \oplus \mathrm{Val}_B^{n-k+1}) \vee \bigvee_{i=1}^{n-k}(u_i \oplus \mathrm{Val}_B^i)$ and $(1 \oplus \mathrm{Val}_B^{n-k+1}) \vee \bigvee_{i=1}^{n-k}(u_i \oplus \mathrm{Val}_B^i)$ from reduction. We can resolve both with the easily proved tautology $(0 \leftrightarrow \mathrm{Val}_B^{n-k+1}) \vee (1 \leftrightarrow \mathrm{Val}_B^{n-k+1})$ which allows us to derive $\bigvee_{i=1}^{n-k}(u_i \oplus \mathrm{Val}_B^i)$.

We continue this until we reach the empty disjunction.

Corollary 1. $\mathsf{eFrege} + \forall\mathsf{red}$ *p-simulates* $\forall$*Exp+Res.*

While this can be proven as a corollary of the simulation of IR-calc, a more direct simulation can be achieved by defining the resolvent strategy by removing the Set^i variables (i.e. by considering them as always true).

4.3 Simulating IRM-calc

Definition IRM-calc was designed to compress annotated literals in clauses in order simulate LD-Q-Res. Like that system it uses the $*$ symbol, but since universal literals do not appear in an annotated clause, the $*$ value is added to the annotations, $0/u, 1/u, */u$ being the first three possibilities in an extended annotation (the fourth being when u does not appear in the annotation).

Axiom and instantiation rules as in IR-calc in Figure 3.

$$\frac{x^{\tau\cup\xi} \vee C_1 \qquad \neg x^{\tau\cup\sigma} \vee C_2}{\mathsf{inst}(\sigma, C_1) \cup \mathsf{inst}(\xi, C_2)} \text{ (Resolution)}$$

$\mathsf{dom}(\tau)$, $\mathsf{dom}(\xi)$ and $\mathsf{dom}(\sigma)$ are mutually disjoint. $\mathsf{rng}(\tau) = \{0, 1\}$

$$\frac{C \vee b^{\mu} \vee b^{\sigma}}{C \vee b^{\xi}} \text{ (Merging)}$$

$\mathsf{dom}(\mu) = \mathsf{dom}(\sigma)$. $\xi = \{c/u \mid c/u \in \mu, c/u \in \sigma\} \cup \{*/u \mid c/u \in \mu, d/u \in \sigma, c \neq d\}$

Fig. 4. The rules of IRM-calc.

The rules of IRM-calc as given in Figure 4, become more complicated as a result of the $*/u$. In particular resolution is no longer done between matching pivots but matching is done internally in the resolution steps. This is to prevent variables resolving with matching $*$ annotations. Allowing such resolution steps would be unsound in general, as these $*$ annotations show that the universal variables are set according to some function, but when appearing in two different literals the functions could be completely different. Resolutions where one pivot literal has a $*/u$ annotation means that the other pivot literal must not have u in its annotation's domain. The intuition is that the unset u is given a $*$ value during the resolution but it can be controlled to be exactly the same $*$ as in the other pivot. A $0/u, 1/u$ or $*/u$ value cannot be given a new $*$ value so cannot match the other $*/u$ annotation.

It is in IRM-calc where the positive Set literals introduced in the simulation of IR-calc become useful. In most ways Set^i_S asserts the same things as $*/u_i$, that u_i is given a value, but this value does not have to be specified.

Conversion The first major change from IR-calc is that while con_S worked on three values in IR-calc, in IRM-calc we effectively run in four values $\mathrm{Set}^i_S, \neg \mathrm{Set}^i_S, \mathrm{Set}^i_S \wedge u_i$ and $\mathrm{Set}^i_S \wedge \neg u_i$. Set^i_S is the new addition deliberately ambiguous as to whether u_i is true or false. Readers familiar with the $*$ used in IRM-calc may notice why Set^i_S works as a conversion of $*/u_i$, as Set^i_S is just saying our policy has given a value but it may be different values in different circumstances.

$$\mathrm{anno}_{x,S}(\tau) = \bigwedge_{1/u_i\in\tau}(\mathrm{Set}^i_S \wedge u_i) \wedge \bigwedge_{0/u_i\in\tau}(\mathrm{Set}^i_S \wedge \bar{u}_i) \wedge \bigwedge_{*/u_i\in\tau}(\mathrm{Set}^i_S) \wedge \bigwedge_{u_i\notin \mathsf{dom}(\tau)}(\neg \mathrm{Set}^i_S).$$
$$\mathrm{con}_S(x^\tau) = x \wedge \mathrm{anno}_{x,S}(\tau)$$
$$\mathrm{con}_S(C_1) = \bigvee_{x^\tau\in C_1} \mathrm{con}(x^\tau)$$

Like in the case of IR-calc, most work needs to be done in the IRM-calc resolution steps, although here it is even more complicated. A resolution step in IRM-calc is in two parts. Firstly $C_1 \vee \neg x^{\tau\sqcup\sigma}$, $C_2 \vee x^{\tau\sqcup\xi}$ are both instantiated (but by $*$ in some cases), secondly they are resolved on a matching pivot. We simplify the resolution steps so that σ and ξ only contain $*$ annotations, for the other constant annotations that would normally be found in these steps suppose we have already instantiated them in the other side so that they now appear in τ (this does not affect the resolvent).

Again we assume that there are policies L and R such that $L \rightarrow \mathrm{con}_L(C_1 \vee \neg x^{\tau\sqcup\sigma})$ and $R \rightarrow \mathrm{con}_R(C_2 \vee x^{\tau\sqcup\xi})$. We know that if L falsifies $\mathrm{anno}_{x,L}(\tau\sqcup\sigma)$ then $\mathrm{con}_L(C_1)$ and likewise if R falsifies $\mathrm{anno}_{x,R}(\tau\sqcup\xi)$ then $\mathrm{con}_R(C_2)$ is satisfied. However, this leaves cases when L satisfies $\mathrm{anno}_{x,L}(\tau\sqcup\sigma)$ and R satisfies $\mathrm{anno}_{x,R}(\tau\sqcup\xi)$ but L and R are not equal. This happens either when Set^i_L and $\neg\mathrm{Set}^i_R$ both occur for $*/u_i \in \sigma$ or when $\neg\mathrm{Set}^i_L$ and Set^i_R both occur for $*/u_i \in \xi$.

This would cause issues if B had to choose between L and R to satisfy $\mathrm{con}_B(C_1 \vee C_2)$. Fortunately, we are not trying to satisfy $\mathrm{con}_B(C_1 \vee C_2)$ but $\mathrm{con}_B(\mathsf{inst}(\xi, C_1) \vee \mathsf{inst}(\sigma, C_2))$, so we have to choose between a policy that will satisfy $\mathrm{con}_B(\mathsf{inst}(\xi, C_1))$ and a policy that will satisfy $\mathrm{con}_B(\mathsf{inst}(\sigma, C_2))$. By borrowing values from the opposite policy we obtain a working new policy that does not have to choose between left and right any earlier than we would have for IR-calc.

Policy We can once again use Dif and Eq notation but change the meanings of the variables.

Equivalence

$\mathrm{Eq}^0_{f=g} := 1$

$\mathrm{Eq}^i_{f=g} := \mathrm{Eq}^{i-1}_{f=g} \wedge (\mathrm{Set}^i_f \leftrightarrow \mathrm{Set}^i_g) \wedge (\mathrm{Set}^i_f \rightarrow (\mathrm{Val}^i_f \leftrightarrow \mathrm{Val}^i_g))$ when $*/u_i \notin g$

$\mathrm{Eq}^i_{f=g} := \mathrm{Eq}^{i-1}_{f=g} \wedge (\mathrm{Set}^i_f)$ when $*/u_i \in g$

Difference

$\mathrm{Dif}^0_L := 0$ and $\mathrm{Dif}^0_R := 0$

For $u_i \notin \mathsf{dom}(\tau \sqcup \sigma \sqcup \xi)$,
$\mathrm{Dif}^i_L := \mathrm{Dif}^{i-1}_L \vee (\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi} \wedge (\mathrm{Set}^i_L)$
$\mathrm{Dif}^i_R := \mathrm{Dif}^{i-1}_R \vee (\mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma} \wedge (\mathrm{Set}^i_R)$
For $u_i \in \mathsf{dom}(\tau)$,
$\mathrm{Dif}^i_L := \mathrm{Dif}^{i-1}_L \vee (\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi} \wedge (\neg \mathrm{Set}^i_L \vee (\mathrm{Set}^i_\tau \wedge (\mathrm{Val}^i_L \oplus \mathrm{Val}^i_\tau))))$
$\mathrm{Dif}^i_R := \mathrm{Dif}^{i-1}_R \vee (\mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma} \wedge (\neg \mathrm{Set}^i_R \vee (\mathrm{Set}^i_\tau \wedge (\mathrm{Val}^i_R \oplus \mathrm{Val}^i_\tau))))$
For $u_i \in \mathsf{dom}(\sigma)$,
$\mathrm{Dif}^i_L := \mathrm{Dif}^{i-1}_L \vee (\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi} \wedge (\neg \mathrm{Set}^i_L)$
$\mathrm{Dif}^i_R := \mathrm{Dif}^{i-1}_R \vee (\mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma} \wedge (\mathrm{Set}^i_R)$
For $u_i \in \mathsf{dom}(\xi)$,
$\mathrm{Dif}^i_L := \mathrm{Dif}^{i-1}_L \vee (\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi} \wedge (\mathrm{Set}^i_L)$
$\mathrm{Dif}^i_R := \mathrm{Dif}^{i-1}_R \vee (\mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma} \wedge (\neg \mathrm{Set}^i_R)$

Policy Variables

We define the policy variables Val^i_B and Set^i_B based on a number of cases, in all cases Val^i_B and Set^i_B are defined on variables left of u_i.

For $u_i \notin \mathsf{dom}(\tau \sqcup \sigma \sqcup \xi)$, $u_i < x$,

$$(\mathrm{Val}^i_B, \mathrm{Set}^i_B) = \begin{cases} (\mathrm{Val}^i_R, \mathrm{Set}^i_R) & \text{if } \neg \mathrm{Dif}^{i-1}_L \wedge (\mathrm{Dif}^{i-1}_R \vee \neg \mathrm{Set}^i_L) \\ (\mathrm{Val}^i_L, \mathrm{Set}^i_L) & \text{otherwise.} \end{cases}$$

For $u_i \in \mathsf{dom}(\tau)$,

$$(\mathrm{Val}^i_B, \mathrm{Set}^i_B) = \begin{cases} (\mathrm{Val}^i_R, \mathrm{Set}^i_R) & \text{if } \neg \mathrm{Dif}^{i-1}_L \wedge (\mathrm{Dif}^{i-1}_R \vee (\mathrm{Set}^i_L \wedge (\mathrm{Val}^i_L \leftrightarrow \mathrm{Val}^i_\tau))) \\ (\mathrm{Val}^i_L, \mathrm{Set}^i_L) & \text{otherwise.} \end{cases}$$

For $*/u_i \in \sigma$,

$$(\mathrm{Val}^i_B, \mathrm{Set}^i_B) = \begin{cases} (0,1) & \text{if } \neg \mathrm{Dif}^{i-1}_L \wedge \mathrm{Dif}^{i-1}_R \wedge \neg \mathrm{Set}^i_R \\ (\mathrm{Val}^i_R, \mathrm{Set}^i_R) & \text{if } \neg \mathrm{Dif}^{i-1}_L \wedge \mathrm{Set}^i_R \wedge (\mathrm{Dif}^{i-1}_R \vee \mathrm{Set}^i_L) \\ (\mathrm{Val}^i_L, \mathrm{Set}^i_L) & \text{otherwise.} \end{cases}$$

For $*/u_i \in \xi$,

$$(\mathrm{Val}^i_B, \mathrm{Set}^i_B) = \begin{cases} (0,1) & \text{if } \mathrm{Dif}^{i-1}_L \wedge \neg \mathrm{Set}^i_L \\ (\mathrm{Val}^i_R, \mathrm{Set}^i_R) & \text{if } \neg \mathrm{Dif}^{i-1}_L \wedge (\mathrm{Dif}^{i-1}_R \vee \neg \mathrm{Set}^i_L) \\ (\mathrm{Val}^i_L, \mathrm{Set}^i_L) & \text{otherwise.} \end{cases}$$

For $u_i > x$,

$$(\mathrm{Val}^i_B, \mathrm{Set}^i_B) = \begin{cases} (\mathrm{Val}^i_R, \mathrm{Set}^i_R) & \text{if } \neg \mathrm{Dif}^m_L \wedge (\mathrm{Dif}^m_R \vee \neg x) \\ (\mathrm{Val}^i_L, \mathrm{Set}^i_L) & \text{otherwise.} \end{cases}$$

Simulation

Theorem 3. $\mathsf{eFrege} + \forall\mathsf{red}$ *simulates IRM-calc.*

Proof. For each line C we create a policy S such that $S \to \mathrm{con}_S(C)$.

Axiom Suppose $C \in \phi$ and it is downloaded as $D = \mathsf{inst}(C, \tau)$ for partial annotation τ. We construct strategy B so that $B \to \mathrm{con}_B(D)$.

- $\mathrm{Set}^j_B = 1$ if $u_j \in \mathsf{dom}(\tau)$
- $\mathrm{Set}^j_B = 0$ if $u_j \notin \mathsf{dom}(\tau)$
- $\mathrm{Val}^j_B = 1$ if $1/u_j \in \tau$
- $\mathrm{Val}^j_B = 0$ if $0/u_j \in \tau$

Instantiation Suppose we have instantiation step on C on a single universal variable u_i using instantiation $0/u_i$. So the new annotated clause is $D = \mathsf{inst}(C, 0/u)$.

From the induction hypothesis $T \to \mathrm{con}_T(C)$ we will develop B such that $B \to \mathrm{con}_B(D)$.

- $\mathrm{Val}^i_B \leftrightarrow \mathrm{Val}^i_T \wedge \mathrm{Set}^i_T$ (for instantiation by 1 we use a disjunction instead)
- $\mathrm{Set}^i_B = 1$
- $\mathrm{Val}^j_B \leftrightarrow \mathrm{Val}^j_T$, for $j \neq i$
- $\mathrm{Set}^j_B \leftrightarrow \mathrm{Set}^j_T$, for $j \neq i$

Merge When merging the local strategy need not change. When literals y^α and y^β are merged the strategy only has to occasionally satisfy a Set^B_i variable instead of a $\mathrm{Set}^B_i \wedge u_i$ or $\mathrm{Set}^B_i \wedge \neg u_i$, so the condition that needs to be satisfied is weaker.
Resolution See the definition of B and Lemma 19.
Contradiction At the end of the proof we have $T \to \mathrm{con}_T(\bot)$ T is a policy, so we turn it into a strategy B by having for each i

- $\mathrm{Val}^i_B \leftrightarrow (\mathrm{Val}^i_T \wedge \mathrm{Set}^i_T)$
- $\mathrm{Set}^i_B = 1$

Effectively this instantiates $\bot$ by the assignment that sets everything to 0 and we can argue that $B \to \mathrm{con}_B(\bot)$ although $\mathrm{con}_B(\bot)$ is just the empty clause. so we have $\neg B$. But $\neg B$ is just $\bigvee_{i=1}^n (u_i \oplus \mathrm{Val}^i_B)$. In $\mathsf{eFrege} + \forall\mathsf{red}$ we can use the reduction rule (this is the first time we use the reduction rule). The proof follows from [12] We show an inductive proof of $\bigvee_{i=1}^{n-k} (y_i \oplus \mathrm{Val}^i_B)$ for increasing k eventually leaving us with the empty clause. This essentially is where we use the $\forall$-Red rule. Since we already have $\bigvee_{i=1}^n (y_i \oplus \mathrm{Val}^i_B)$ we have the base case and we only need to show the inductive step.

We derive from $\bigvee_{i=1}^{n+1-k} (y_i \oplus \mathrm{Val}^i_B)$ both $(0 \oplus \mathrm{Val}^{n-k+1}_B) \vee \bigvee_{i=1}^{n-k} (y_i \oplus \mathrm{Val}^i_B)$ and $(1 \oplus \mathrm{Val}^{n-k+1}_B) \vee \bigvee_{i=1}^{n-k} (y_i \oplus \mathrm{Val}^i_B)$ from reduction. We can resolve both with the easily proved tautology $(0 \leftrightarrow \mathrm{Val}_B^{n-k+1}) \vee (1 \leftrightarrow \mathrm{Val}_B^{n-k+1})$ which allows us to derive $\bigvee_{i=1}^{n-k}(y_i \oplus \mathrm{Val}_B^i)$.

We continue this until we reach the empty disjunction.

Corollary 2. eFrege + ∀red *simulates LD-Q-Res.*

5 Extended Frege+∀-Red p-simulates LQU$^+$-Res

5.1 QCDCL Resolution Systems

The most basic and important CDCL system is *Q-resolution (Q-Res)* by Kleine Büning et al. [26]. *Long-distance resolution (LD-Q-Res)* appears originally in the work of Zhang and Malik [41] and was formalized into a calculus by Balabanov and Jiang [2]. It merges complementary literals of a universal variable u into the special literal u^*. These special literals prohibit certain resolution steps. *QU-resolution (QU-Res)* [39] removes the restriction from Q-Res that the resolved variable must be an existential variable and allows resolution of universal variables. *LQU$^+$-Res* [3] extends LD-Q-Res by allowing short and long distance resolution pivots to be universal, however, the pivot is never a merged literal z^*. LQU$^+$-Res encapsulates Q-Res, LD-Q-Res and QU-Res. Figure 5 details the rules of LQU$^+$-Res.

5.2 Conversion to Propositional Logic and Simulation

LQU$^+$-Res and IRM-calc are mutually incomparable in terms of proof strength, however both share similarities. Once again we can use Set^i variables to represent an u_i^*, and a $\neg\,\mathrm{Set}_S^i$ variable to represent that policy S chooses not to issue a value to u_i.

For any set of universal variables U, let $\mathrm{anno}_{x,S}(U) = \bigwedge_{u_j<x}^{u_j\notin U} \neg\,\mathrm{Set}_S^j \wedge \bigwedge_{u_j<x}^{u_j\in U} \mathrm{Set}_S^j$. Note that we do not really need to add polarities to the annotations, these are taken into account by the clause literals. Literals u and $\bar{u}$ do not need to be assigned by the policy, they are now treated as a consequence of the the CNF. Because they can be resolved we treat them like existential variables in the conversion. For universal variable u_i, $\mathrm{con}_{S,C}(u_i) = u_i \wedge \neg\,\mathrm{Set}_S^i \wedge \mathrm{anno}_{x,S}(\{u \mid u^* \in C\})$ and $\mathrm{con}_{S,C}(\neg u_i) = \neg u_i \wedge \neg\,\mathrm{Set}_S^i \wedge \mathrm{anno}_{x,S}(\{v \mid v^* \in C\})$. We reserve Set_S^j for starred literals as they cannot be removed. For existential literal x, $\mathrm{con}_{S,C}(x) = x \wedge \mathrm{anno}_{x,S}(\{u \mid u^* \in C\})$. Finally, $\mathrm{con}_{S,C}(u^*) = \bot$, because we do not treat u^* as a literal but part of the "annotation" to literals right of it. Also, u^* cannot be resolved but it automatically reduced when

$$\frac{}{C}\ (\text{Axiom}) \qquad \frac{D \cup \{u\}}{D}\ (\forall\text{-Red})$$

$$\frac{D \cup \{u^*\}}{D}\ (\forall\text{-Red}^*)$$

C is a clause in the original matrix. Literal u is universal and $\text{lv}(u) \geq \text{lv}(l)$ for all $l \in D$.

$$\frac{C_1 \cup U_1 \cup \{\neg x\} \qquad C_2 \cup U_2 \cup \{x\}}{C_1 \cup C_2 \cup U}\ (\text{Res})$$

We consider two settings of the Res-rule:
SR: If $z \in C_1$, then $\neg z \notin C_2$. $U_1 = U_2 = U = \emptyset$.
LR: If $l_1 \in C_1, l_2 \in C_2$, and $\text{var}(l_1) = \text{var}(l_2) = z$ then $l_1 = l_2 \neq z^*$. U_1, U_2 contain only universal literals with $\text{var}(U_1) = \text{var}(U_2)$. $\text{ind}(x) < \text{ind}(u)$ for each $u \in \text{var}(U_1)$.
If $w_1 \in U_1, w_2 \in U_2$, $\text{var}(w_1) = \text{var}(w_2) = u$ then $w_1 = \neg w_2$ or $w_1 = u^*$ or $w_2 = u^*$.
$U = \{u^* \mid u \in \text{var}(U_1)\}$.

For $b = \{1, 2\}$, define $V_b = \{u^* \mid u^* \in C_b\}$. In other words V_b is the subclause of $C_b \vee U_b$ of starred literals left of x.

Fig. 5. The rules of LQU^+-Res.

no more literals are to the right of it. For clauses in LQU^+-Res, we let $\text{con}_S(C) = \bigvee_{l \in C} \text{con}_{S,C}(l)$. In summary, in comparison to $\mathsf{IRM\text{-}calc}$ the conversion now includes universal variables and gives them annotations, but removes polarities from the annotations. Policies still remain structured as they were for $\mathsf{IR\text{-}calc}$, with extension variables Val^i_S and Set^i_S, where $S = \bigwedge_{i=1}^n \text{Set}^i_S \rightarrow (u_i \leftrightarrow \text{Val}^i_S)$.

Observation 4 *$V_1 \cap V_2 = \varnothing$ by definition of resolution in LQU^+-Res (see Figure 5).*

Equivalence The notation for equivalence slightly changes due to the fact we are no longer working with annotations, but present starred literals. These work in much the same way.
$\text{Eq}^0_{f,V} := 1$
$\text{Eq}^i_{f,V} := \text{Eq}^{i-1}_{f=g} \wedge \text{Set}^i_f$ when $u^*_i \in V$
$\text{Eq}^i_{f=g} := \text{Eq}^{i-1}_{f=g} \wedge (\neg \text{Set}^i_f)$ when $u^*_i \notin V$

Difference $\text{Dif}^0_L := 0$ and $\text{Dif}^0_R := 0$
For $u^*_i \notin C_1 \cup C_2$,

$\text{Dif}_L^i := \text{Dif}_L^{i-1} \vee (\text{Eq}_{R,V_2}^{i-1} \wedge (\text{Set}_L^i)$
$\text{Dif}_R^i := \text{Dif}_R^{i-1} \vee (\text{Eq}_{L,V_2}^{i-1} \wedge (\text{Set}_R^i)$

For $u_i^* \in C_1$,
$\text{Dif}_L^i := \text{Dif}_L^{i-1} \vee (\text{Eq}_{R,V_2}^{i-1} \wedge (\neg \text{Set}_L^i)$
$\text{Dif}_R^i := \text{Dif}_R^{i-1} \vee (\text{Eq}_{L,V_1}^{i-1} \wedge (\text{Set}_R^i)$

For $u_i^* \in C_2$,
$\text{Dif}_L^i := \text{Dif}_L^{i-1} \vee (\text{Eq}_{R,V_2}^{i-1} \wedge (\text{Set}_L^i)$
$\text{Dif}_R^i := \text{Dif}_R^{i-1} \vee (\text{Eq}_{L,V_1}^{i-1} \wedge (\neg \text{Set}_R^i)$

Policy Variables For $u_i* \notin C_1 \cup C_2$, $i \leq m$

$$(\text{Val}_B^i, \text{Set}_B^i) = \begin{cases} (\text{Val}_R^i, \text{Set}_R^i) & \text{if } \neg \text{Dif}_L^{i-1} \wedge (\text{Dif}_R^{i-1} \vee \neg \text{Set}_L^i) \\ (\text{Val}_L^i, \text{Set}_L^i) & \text{otherwise.} \end{cases}$$

For $u_i^* \in C_1$, $i \leq m$

$$(\text{Val}_B^i, \text{Set}_B^i) = \begin{cases} (0,1) & \text{if } \neg \text{Dif}_L^{i-1} \wedge \text{Dif}_R^{i-1} \wedge \neg \text{Set}_R^i \\ (\text{Val}_R^i, \text{Set}_R^i) & \text{if } \neg \text{Dif}_L^{i-1} \wedge \text{Set}_R^i \wedge (\text{Dif}_R^{i-1} \vee \text{Set}_L^i) \\ (\text{Val}_L^i, \text{Set}_L^i) & \text{otherwise.} \end{cases}$$

For $u_i^* \in C_2$, $i \leq m$

$$(\text{Val}_B^i, \text{Set}_B^i) = \begin{cases} (0,1) & \text{if } \text{Dif}_L^{i-1} \wedge \neg \text{Set}_L^i \\ (\text{Val}_R^i, \text{Set}_R^i) & \text{if } \neg \text{Dif}_L^{i-1} \wedge (\text{Dif}_R^{i-1} \vee \neg \text{Set}_L^i) \\ (\text{Val}_L^i, \text{Set}_L^i) & \text{otherwise.} \end{cases}$$

For $u_i \in \mathsf{dom}(U)$, $i > m$

$$(\text{Val}_B^i, \text{Set}_B^i) = \begin{cases} (\text{Val}_R^i, \text{Set}_R^i) & \text{if } \text{Set}_R^i \wedge \neg \text{Dif}_L^m \wedge (\text{Dif}_R^m \vee \neg x) \\ (0,1) & \text{if } u_i \in U_2 \text{ and } \neg \text{Set}_R^i \wedge \neg \text{Dif}_L^m \wedge (\text{Dif}_R^m \vee \neg x) \\ (1,1) & \text{if } \neg u_i \in U_2 \text{ and } \neg \text{Set}_R^i \wedge \neg \text{Dif}_L^m \wedge (\text{Dif}_R^m \vee \neg x) \\ (\text{Val}_R^i, \text{Set}_R^i) & \text{if } u_i^* \in U_2 \text{ and } \neg \text{Set}_R^i \wedge \neg \text{Dif}_L^m \wedge (\text{Dif}_R^m \vee \neg x) \\ (\text{Val}_L^i, \text{Set}_L^i) & \text{Set}_L^i \wedge \text{Dif}_L^m \vee (\neg \text{Dif}_R^m \wedge x) \\ (0,1) & \text{if } u_i \in U_2 \text{ and } \neg \text{Set}_L^i \wedge \text{Dif}_L^m \vee (\neg \text{Dif}_R^m \wedge x)) \\ (1,1) & \text{if } \neg u_i \in U_2 \text{ and } \neg \text{Set}_L^i \wedge \text{Dif}_L^m \vee (\neg \text{Dif}_R^m \wedge x)) \\ (\text{Val}_R^i, \text{Set}_R^i) & \text{if } u_i^* \in U_2 \text{ and } \neg \text{Set}_L^i \wedge \text{Dif}_L^m \vee (\neg \text{Dif}_R^m \wedge x)) \end{cases}$$

For $u_i \notin \mathsf{dom}(U)$, $i > m$

$$(\text{Val}_B^i, \text{Set}_B^i) = \begin{cases} (\text{Val}_R^i, \text{Set}_R^i) & \text{if } \neg \text{Dif}_L^m \wedge (\text{Dif}_R^m \vee \neg x) \\ (\text{Val}_L^i, \text{Set}_L^i) & \text{otherwise.} \end{cases}$$

One may notice there are a larger number of cases for $i > m$ than in previous sections, this is because u and $\neg u$ become u^* and end up joining the annotation and policies.

Theorem 5. $\mathsf{eFrege} + \forall\mathsf{red}$ *simulates* LQU^+*-Res.*

Proof. We inductively build a strategy S such that $S \rightarrow \text{con}_S(C)$ can be proved from ϕ using eFrege, for every clause C in an LQU$^+$-Res proof. At the end we have the empty clause and a strategy and we can use reduction to remove the strategy and obtain the empty clause as in Theorems 1 and 2.

Axiom Each Axiom is treated with the empty strategy.

Reduction (u_i or $\neg u_i$) If the clause contains literal u_i, we know that $T \rightarrow \text{con}_T(C \vee u_i)$. We define S so that

$$(\text{Val}^j_S, \text{Set}^j_S) = (\text{Val}^j_T, \text{Set}^j_T)$$

$$(\text{Val}^i_S, \text{Set}^i_S) = \begin{cases} (\text{Val}^i_T, \text{Set}^i_T) & \text{if } \text{Set}^i_T \vee \text{con}_T(C) \text{ is satisfied,} \\ (0, 1) & \text{otherwise.} \end{cases}$$

We need to show that $S \rightarrow \text{con}_S(C)$. Note that $\text{con}_T(C \vee u_i) = \text{con}_T(C) \vee \text{con}_{T,C}(u_i)$. Therefore $T \rightarrow \text{con}_T(C)$ or $T \rightarrow \neg \text{Set}^i_T \wedge u_i$. If Set^i_T is true or $\text{con}_T(C)$ then $T \rightarrow \text{con}_T(C)$ is true and as S will match T, $S \rightarrow \text{con}_S(C)$. Suppose Set^i_T and $\text{con}_T(C)$ are both false. If S is true, then u_i is false by construction. Moreover, since S agrees with T on every variable except u_i, and T does not set u_i, T must be true as well. But since $\text{con}_T(C)$ is false, we must have $T \rightarrow \neg \text{Set}^i_T \wedge u_i$. In particular, u_i must be true, a contradiction. We conclude that the implication $S \rightarrow \text{con}_S(C)$ holds in this case.

Reduction (u^*_i) If $T \rightarrow \text{con}_T(C \vee u^*_i)$ and we reduce u^*_i we need to define the strategy S so that $S \rightarrow \text{con}_S(C)$. Since u^*_i is the rightmost literal in the clause $\text{con}_T(C \vee u^*_i) = \text{con}_T(C)$ so we define S the same way as T.

Resolution See Lemma 25.

Contradiction Just as in IR-calc we have to give a complete assignment to the missing values in the policy. We then have simply the negation of the strategy for which we can apply our same technique to reduce to the empty clause.

6 Conclusion

Our work reconciles many different QBF proof techniques under the single system eFrege+$\forall$red. This is also beneficial to QRAT, which inherits these simulations. QRAT's simulation of $\forall$Exp+Res is now upgraded to a simulation of IRM-calc, and we do not even have to use the extended universal reduction rule to do this. Existing QRAT checkers can be used to verify converted eFrege + $\forall$red proofs. Since our simulations split off propositional inference from a standardised reduction part at the end, another option is to use (highly efficient) propositional proof checkers instead. In either case there is at least one more hurdle to overcome, as our simulations use large amounts of extension variables which are known to negatively impact the checking time of existing tools such as DRAT-trim. One may hope that simulations presented in this paper can be refined to become more efficient in this regard.

While we proved a multitude of simulations in this work using a similar technique each time, it may yet be possible to subsume all the simulated proof systems under one class, and prove that eFrege + ∀red simulates all systems in this class. In addition there are other systems, particularly ones using dependency schemes, such as Q($\mathcal{D}^{\mathsf{rrs}}$)-Res and LD-Q($\mathcal{D}^{\mathsf{rrs}}$)-Res that have strategy extraction [32]. Local strategy extraction and ultimately a simulation seem likely for these systems, whether it can be proved directly or by generalising the simulation results from this paper.

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7 Appendix

7.1 Proof of Simulation of IR-calc

Lemma 2. *For $0 < j \leq m$ the following propositions have short derivations in Extended Frege:*

- $\mathrm{Dif}_L^j \rightarrow \bigvee_{i=1}^{j} \mathrm{Dif}_L^i \wedge \neg \mathrm{Dif}_L^{i-1}$
- $\mathrm{Dif}_R^j \rightarrow \bigvee_{i=1}^{j} \mathrm{Dif}_R^i \wedge \neg \mathrm{Dif}_R^{i-1}$
- $\neg \mathrm{Eq}_{f=g}^j \rightarrow \bigvee_{i=1}^{j} \neg \mathrm{Eq}_{f=g}^i \wedge \mathrm{Eq}_{f=g}^{i-1}$. *For $f, g \in \{L, R, \tau\}$.*

Proof. **Induction Hypothesis on j:** $\mathrm{Dif}_L^j \to \bigvee_{i=1}^{j} \mathrm{Dif}_L^i \wedge \neg \mathrm{Dif}_L^{i-1}$ has an $O(j)$-size proof
Base Case $j = 1$: $\mathrm{Dif}_L^1 \to \mathrm{Dif}_L^1$ is a basic tautology that Frege can handle, Dif_L^0 is false by definition so Frege can assemble $\mathrm{Dif}_L^1 \to \mathrm{Dif}_L^1 \wedge \neg \mathrm{Dif}_L^0$.
Inductive Step $j + 1$: $\neg \mathrm{Dif}_L^j \vee \mathrm{Dif}_L^j$ and $\mathrm{Dif}_L^{j+1} \to \mathrm{Dif}_L^{j+1}$ are tautologies that Frege can handle. Putting them together we get $\mathrm{Dif}_L^{j+1} \to \mathrm{Dif}_L^{j+1} \wedge (\neg \mathrm{Dif}_L^j \vee \mathrm{Dif}_L^j)$ and weaken to $\mathrm{Dif}_L^{j+1} \to (\mathrm{Dif}_L^{j+1} \wedge \neg \mathrm{Dif}_L^j) \vee \mathrm{Dif}_L^j$. Using the induction hypothesis, $\mathrm{Dif}_L^j \to \bigvee_{i=1}^{j} \mathrm{Dif}_L^i \wedge \neg \mathrm{Dif}_L^{i-1}$, we can change this tautology to

$$\mathrm{Dif}_L^{j+1} \to (\mathrm{Dif}_L^{j+1} \wedge \neg \mathrm{Dif}_L^j) \vee \bigvee_{i=1}^{j} \mathrm{Dif}_L^i \wedge \neg \mathrm{Dif}_L^{i-1}$$

Note that since $\neg \mathrm{Dif}_R^0, \mathrm{Eq}_{L=\tau\sqcup\xi}^0, \mathrm{Eq}_{L=\tau\sqcup\sigma}^0$ are all true . The proofs for Dif_R^j, $\neg \mathrm{Eq}_{L=\tau\sqcup\sigma}^j$ and $\neg \mathrm{Eq}_{R=\tau\sqcup\xi}^j$ are identical modulo the variable names.

Lemma 3. *For $0 \le i \le j \le m$ the following propositions that describe the monotonicity of* Dif *have short derivations in Extended Frege:*

- $\mathrm{Dif}_L^i \to \mathrm{Dif}_L^j$
- $\mathrm{Dif}_R^i \to \mathrm{Dif}_R^j$
- $\neg \mathrm{Eq}_{f=g}^i \to \neg \mathrm{Eq}_{f=g}^j$

Proof. For Dif_L and Dif_R,
Induction Hypothesis on j: $\mathrm{Dif}_L^i \to \mathrm{Dif}_L^j$ has an $O(j)$ proof.
Base Case $j = i$: $\mathrm{Dif}_L^i \to \mathrm{Dif}_L^i$ is a tautology that Frege can handle.
Inductive Step $j + 1$: $\mathrm{Dif}_L^{j+1} := \mathrm{Dif}_L^j \vee A$ where expression A depends on the domain of u_{j+1}. Therefore in all cases $\mathrm{Dif}_L^j \to \mathrm{Dif}_L^{j+1}$ is a straightforward corollary in Frege. Using the induction hypothesis $\mathrm{Dif}_L^i \to \mathrm{Dif}_L^j$ we can get $\mathrm{Dif}_L^i \to \mathrm{Dif}_L^{j+1}$. The proof is symmetric for R.

For $\neg \mathrm{Eq}_{f=g}$,
Induction Hypothesis on j: $\neg \mathrm{Eq}_{f=g}^i \to \neg \mathrm{Eq}_{f=g}^j$ has an $O(j)$ proof.
Base Case $j = i$: $\neg \mathrm{Eq}_{f=g}^i \to \neg \mathrm{Eq}_{f=g}^i$ is a tautology that Frege can handle.
Inductive Step $j + 1$: $\mathrm{Eq}_{f=g}^{j+1} := \mathrm{Eq}_{f=g}^j \wedge A$ where expression A depends on the domain of u_{j+1}. Therefore in all cases $\neg \mathrm{Eq}_{f=g}^j \to \neg \mathrm{Eq}_{f=g}^{j+1}$ is a straightforward corollary in Frege. Using the induction hypothesis $\neg \mathrm{Eq}_{f=g}^i \to \neg \mathrm{Eq}_{f=g}^j$ we can get $\neg \mathrm{Eq}_{f=g}^i \to \neg \mathrm{Eq}_{f=g}^{j+1}$.

Lemma 4. *For $0 \le i \le j \le m$ the following propositions describe the relationships between the different extension variables.*

- $\mathrm{Eq}^i_{L=\tau} \to \neg\,\mathrm{Dif}^i_L$
- $\mathrm{Dif}^i_L \wedge\neg\,\mathrm{Dif}^{i-1}_L \to \mathrm{Eq}^{i-1}_{R=\tau}$
- $\mathrm{Dif}^i_L \wedge\neg\,\mathrm{Dif}^{i-1}_L \to \neg\,\mathrm{Dif}^{i-1}_R$
- $\mathrm{Eq}^i_{R=\tau} \to \neg\,\mathrm{Dif}^i_R$
- $\mathrm{Dif}^i_R \wedge\neg\,\mathrm{Dif}^{i-1}_R \to \mathrm{Eq}^{i-1}_{L=\tau}$
- $\mathrm{Dif}^i_R \wedge\neg\,\mathrm{Dif}^{i-1}_R \to \neg\,\mathrm{Dif}^{i-1}_L$

Proof. **Induction Hypothesis on i:** $\mathrm{Eq}^i_{L=\tau} \to \neg\,\mathrm{Dif}^i_L$ in an $O(i)$-size eFrege proof.
Base Case $i = 0$: Dif^i_L is defined as 0 so $\neg\,\mathrm{Dif}^i_L$ is true and trivially implied by $\mathrm{Eq}^i_{L=\tau}$. Frege can manage this.
Inductive Step $i+1$: If Set^{i+1}_τ is false then $\mathrm{Eq}^{i+1}_{L=\tau}$ is equivalent to $\mathrm{Eq}^i_{L=\tau} \wedge\neg\,\mathrm{Set}^{i+1}_L$ and $\neg\,\mathrm{Dif}^{i+1}_L$ is equivalent to $\neg\,\mathrm{Dif}^i_L \wedge\neg\,\mathrm{Set}^{i+1}_L \vee\neg\,\mathrm{Eq}^i_{L=\tau}$. If Set^{i+1}_τ is true then $\mathrm{Eq}^{i+1}_{L=\tau}$ is equivalent to $\mathrm{Eq}^i_{L=\tau} \wedge \mathrm{Set}^{i+1}_L \wedge(\mathrm{Val}^{i+1}_L \leftrightarrow \mathrm{Val}^{i+1}_\tau)$ and $\neg\,\mathrm{Dif}^{i+1}_L$ is equivalent to $\neg\,\mathrm{Dif}^i_L \wedge \mathrm{Set}^{i+1}_L \wedge(\mathrm{Val}^{i+1}_L \leftrightarrow \mathrm{Val}^{i+1}_\tau)\vee \neg\,\mathrm{Eq}^i_{L=\tau}$. Therefore using the induction hypothesis $\mathrm{Eq}^i_{L=\tau} \to \neg\,\mathrm{Dif}^i_L$. Similarly for R.

The formulas $\mathrm{Dif}^i_L \wedge\neg\,\mathrm{Dif}^{i-1}_L \to \mathrm{Eq}^{i-1}_{R=\tau}$ are simple corollaries of the inductive definition of Dif^i_L, and combined with $\mathrm{Eq}^{i-1}_{R=\tau} \to \neg\,\mathrm{Dif}^{i-1}_R$ we get $\mathrm{Dif}^i_L \wedge\neg\,\mathrm{Dif}^{i-1}_L \to \neg\,\mathrm{Dif}^{i-1}_R$. Similarly if we swap L and R.

Lemma 5. *For any $0 \leq i \leq m$ the following propositions are true and have short Extended Frege proofs.*

- $L \wedge \mathrm{Dif}^i_L \to \neg\,\mathrm{anno}_{x,L}(\tau)$
- $R \wedge \mathrm{Dif}^i_R \to \neg\,\mathrm{anno}_{x,R}(\tau)$

Proof. We primarily use the disjunction in Lemma 2

$$\overset{i}{\underset{L}{\mathrm{Dif}}} \to \bigvee_{i=1}^{j} \overset{i}{\underset{L}{\mathrm{Dif}}} \wedge\neg \overset{i-1}{\underset{L}{\mathrm{Dif}}}$$

.

In each disjunct $\mathrm{Dif}^i_L \wedge\neg\,\mathrm{Dif}^{i-1}_L$ we can say that the difference triggers at that point. We can represent that in a proposition that can be proven in eFrege: $\mathrm{Dif}^i_L \wedge\neg\,\mathrm{Dif}^{i-1}_L \to ((\mathrm{Set}^i_L \oplus \mathrm{Set}^i_\tau) \vee (\mathrm{Set}^i_\tau \wedge(\mathrm{Val}^i_L \oplus \mathrm{Val}^i_\tau)))$ If L differs from τ on a Set^i_L value we contradict $\mathrm{anno}_{x,L}(\tau)$ in one of two ways: $L \wedge (\mathrm{Set}^i_L \oplus \mathrm{Set}^i_\tau) \wedge \mathrm{Set}^i_L \to \neg\,\mathrm{Set}^i_\tau$ or $L \wedge (\mathrm{Set}^i_L \oplus \mathrm{Set}^i_\tau) \wedge \neg\,\mathrm{Set}^i_L \to \mathrm{Set}^i_\tau$.

If L differs from τ on a Val^i_L value when $\mathrm{Set}^i_L = \mathrm{Set}^i_\tau = 1$ we contradict $\mathrm{anno}_{x,L}(\tau)$ in one of two ways:

- $L \wedge \mathrm{Set}^i_L \wedge \mathrm{Set}^i_\tau \wedge(\mathrm{Set}^i_\tau \to (\mathrm{Val}^i_L \oplus \mathrm{Val}^i_\tau)) \wedge \mathrm{Val}^i_L \to \neg\,\mathrm{Val}^i_\tau \wedge u_i$

- $L \wedge \mathrm{Set}^i_L \wedge \mathrm{Set}^i_\tau \wedge (\mathrm{Set}^i_\tau \rightarrow (\mathrm{Val}^i_L \oplus \mathrm{Val}^i_\tau)) \wedge \neg \mathrm{Val}^i_L \rightarrow \mathrm{Val}^i_\tau \wedge \neg u_i$.

When put together with the big disjunction this lends itself to a short eFrege proof which is also symmetric for R.

For a resolution step we want to define the strategy for the resolvent B based on the strategies L and R. We define the extension variables Val^i_B and Set^i_B based on $\mathrm{Val}^i_L, \mathrm{Set}^i_L, \mathrm{Val}^i_R, \mathrm{Set}^i_R$ and use the technical Dif variables to separate out the cases.

The idea is that B will be start off as both L and R while they are identical, and eventually pick one of them to commit to, depending on whether it will satisfy $\mathrm{con}(C_1)$ or $\mathrm{con}(C_2)$. The decision will be made by choosing the first L or R that falsifies $\neg x \wedge \mathrm{con}_{x,L}(\tau)$ or $x \wedge \mathrm{con}_{x,R}(\tau)$ (and given a draw prioritises L over R). As we have seen in Lemma 5, Dif^i_L means that L contradicts $\mathrm{con}(\tau)$. However we do not use Dif^i_L to decide the value of u_i under B since we want our Val^i_B and Set^i_B extension variables to appear before Dif variables. So instead we make the same decisions just with $\mathrm{Val}^i_L, \mathrm{Set}^i_L, \mathrm{Val}^i_R, \mathrm{Set}^i_R$. This is significantly more comprehensible in $\forall$Exp+Res where the Set variables play no role, but it works the same way in IR-calc just with more cases.

Lemma 6. *For any $1 \leq j \leq m$ the following propositions are true and have a short Extended Frege proof.*

- $\neg \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^j_R \rightarrow \mathrm{Eq}^j_L$
- $\neg \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^j_R \rightarrow \mathrm{Eq}^j_R$
- $\neg \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^j_R \rightarrow (\mathrm{Set}^j_B \leftrightarrow \mathrm{Set}^j_L)$
- $\neg \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^j_R \rightarrow \mathrm{Set}^i_B \rightarrow (\mathrm{Val}^i_B \leftrightarrow \mathrm{Val}^i_L)$
- $\neg \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^j_R \rightarrow (\mathrm{Set}^j_B \leftrightarrow \mathrm{Set}^j_R)$
- $\neg \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^j_R \rightarrow \mathrm{Set}^i_B \rightarrow (\mathrm{Val}^j_B \leftrightarrow \mathrm{Val}^j_R)$

Proof. We first show $\neg \mathrm{Eq}^j_{L=\tau} \rightarrow \neg \mathrm{Eq}^{j-1}_{R=\tau} \vee \mathrm{Dif}^j_L \vee \mathrm{Dif}^j_R$ and $\neg \mathrm{Eq}^j_{R=\tau} \rightarrow \neg \mathrm{Eq}^{j-1}_L \vee \mathrm{Dif}^j_R \vee \mathrm{Dif}^j_R$. $\neg \mathrm{Eq}^{j-1}_{R=\tau}$ and $\neg \mathrm{Eq}^{j-1}_{L=\tau}$ are the problems here respectively, but they can be removed via induction to eventually get $\neg \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^j_R \rightarrow \mathrm{Eq}^j_L$ and $\neg \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^j_R \rightarrow \mathrm{Eq}^j_{R=\tau}$. The remaining implications are corollaries of these and rely on the definition of Eq, Set_B and Val_B.

Induction Hypothesis on j: $\neg \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^j_R \rightarrow \mathrm{Eq}^j_L$ and $\neg \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^j_R \rightarrow \mathrm{Eq}^j_R$.

Base Case $j = 0$: $\mathrm{Eq}^j_{L=\tau}$ and $\mathrm{Eq}^j_{R=\tau}$ are both true by definition so the implications automatically hold.

Inductive Step j**:** $\neg \mathrm{Eq}^{j+1}_{L=\tau} \rightarrow \neg \mathrm{Eq}^{j-1}_{L=\tau} \vee (\mathrm{Set}^j_L \oplus \mathrm{Set}^j_\tau) \vee (\mathrm{Set}^j_L \wedge (\mathrm{Val}^j_L \oplus \mathrm{Val}^j_\tau))$, $(\mathrm{Set}^j_L \oplus \mathrm{Set}^j_\tau) \vee (\mathrm{Set}^j_L \wedge (\mathrm{Val}^j_L \oplus \mathrm{Val}^j_\tau)) \rightarrow \mathrm{Dif}^j_L \vee \neg \mathrm{Eq}^{j-1}_{R=\tau}$ so we get $\neg \mathrm{Eq}^j_{L=\tau} \rightarrow \neg \mathrm{Eq}^{j-1}_{L=\tau} \vee \mathrm{Dif}^j_L \vee \neg \mathrm{Eq}^{j-1}_{R=\tau}$, which using the induction hypothesis can be generalised to $\neg \mathrm{Eq}^j_{L=\tau} \rightarrow \mathrm{Dif}^j_R \vee \mathrm{Dif}^j_L$ which is equivalent to $\neg \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^j_R \rightarrow \mathrm{Eq}^j_L$. Similarly when swapping L and R.

We can obtain the remaining propositions as corollaries by using the definition of Eq.

Nonetheless, Dif^i_L and Dif^i_R still end up being relevant to the choice of Val^j_B.

Lemma 7. *For any* $0 \leq i \leq m$ *the following propositions are true and have short Extended Frege proofs.*

- $\mathrm{Dif}^i_L \rightarrow (\mathrm{Val}^i_B \leftrightarrow \mathrm{Val}^i_L) \wedge (\mathrm{Set}^i_B \leftrightarrow \mathrm{Set}^i_L)$
- $\neg \mathrm{Dif}^i_L \wedge \mathrm{Dif}^i_R \rightarrow (\mathrm{Val}^i_B \leftrightarrow \mathrm{Val}^i_R) \wedge (\mathrm{Set}^i_B \leftrightarrow \mathrm{Set}^i_R)$

Proof. Suppose we want to prove $\mathrm{Dif}^i_L \rightarrow (\mathrm{Val}^i_B \leftrightarrow \mathrm{Val}^i_L) \wedge (\mathrm{Set}^i_B \leftrightarrow \mathrm{Set}^i_L)$. We will assume the definition

$$\overset{i}{\underset{L}{\mathrm{Dif}}} := \overset{i-1}{\underset{L}{\mathrm{Dif}}} \vee (\overset{i-1}{\underset{R}{\mathrm{Eq}}} \wedge ((\overset{i}{\underset{L}{\mathrm{Set}}} \oplus \overset{i}{\underset{\tau}{\mathrm{Set}}}) \vee (\overset{i}{\underset{\tau}{\mathrm{Set}}} \wedge (\overset{i}{\underset{L}{\mathrm{Val}}} \oplus \overset{i}{\underset{\tau}{\mathrm{Val}}}))))$$

and show that the proposition

$$\neg \overset{i-1}{\underset{L}{\mathrm{Dif}}} \wedge (\overset{i-1}{\underset{R}{\mathrm{Dif}}} \vee (\neg \overset{i}{\underset{\tau}{\mathrm{Set}}} \wedge \neg \overset{i}{\underset{L}{\mathrm{Set}}} \wedge \overset{i}{\underset{R}{\mathrm{Set}}}) \vee (\overset{i}{\underset{\tau}{\mathrm{Set}}} \wedge \overset{i}{\underset{L}{\mathrm{Set}}} \wedge (\overset{i}{\underset{\tau}{\mathrm{Val}}} \leftrightarrow \overset{i}{\underset{L}{\mathrm{Val}}})))$$

is falsified.

The first thing is that we only need to consider $\mathrm{Dif}^i_L \wedge \neg \mathrm{Dif}^{i-1}_L$ as Dif^{i-1}_L already falsifies our proposition. Next we show $\neg \mathrm{Dif}^{i-1}_R$ is forced to be true in this situation. To do this we need Lemma 4 for $\mathrm{Dif}^i_L \wedge \neg \mathrm{Dif}^{i-1}_L \rightarrow \neg \mathrm{Dif}^{i-1}_R$.

Now we use $\mathrm{Dif}^i_L \wedge \neg \mathrm{Dif}^{i-1}_L \rightarrow ((\mathrm{Set}^i_L \oplus \mathrm{Set}^i_\tau) \vee (\mathrm{Set}^i_\tau \wedge (\mathrm{Val}^i_L \oplus \mathrm{Val}^i_\tau)))$, we break this down into three cases

1. $\mathrm{Dif}^i_L \wedge \neg \mathrm{Dif}^{i-1}_L \wedge \neg \mathrm{Set}^i_L \wedge \mathrm{Set}^i_\tau$
2. $\mathrm{Dif}^i_L \wedge \neg \mathrm{Dif}^{i-1}_L \wedge \mathrm{Set}^i_L \wedge \neg \mathrm{Set}^i_\tau$
3. $\mathrm{Dif}^i_L \wedge \neg \mathrm{Dif}^{i-1}_L \wedge (\mathrm{Set}^i_\tau \wedge (\mathrm{Val}^i_L \oplus \mathrm{Val}^i_\tau))$

1. $\mathrm{Dif}^i_L \wedge \neg \mathrm{Dif}^{i-1}_L$ contradicts Dif^{i-1}_R, Set^i_τ contradicts $(\neg \mathrm{Set}^i_\tau \wedge \neg \mathrm{Set}^i_L \wedge \mathrm{Set}^i_R)$, and $\neg \mathrm{Set}^i_L$ contradicts $(\mathrm{Set}^i_\tau \wedge \mathrm{Set}^i_L \wedge (\mathrm{Val}^i_\tau \leftrightarrow \mathrm{Val}^i_L))$.

2. $\mathrm{Dif}_L^i \wedge\neg \mathrm{Dif}_L^{i-1}$ contradicts Dif_R^{i-1}, Set_L^i contradicts $(\neg \mathrm{Set}_\tau^i \wedge\neg \mathrm{Set}_L^i \wedge \mathrm{Set}_R^i)$, and $\neg \mathrm{Set}_\tau^i$ contradicts $(\mathrm{Set}_\tau^i \wedge \mathrm{Set}_L^i \wedge(\mathrm{Val}_\tau^i \leftrightarrow \mathrm{Val}_L^i))$.
3. $\mathrm{Dif}_L^i \wedge\neg \mathrm{Dif}_L^{i-1}$ contradicts Dif_R^{i-1}, Set_τ^i contradicts $(\neg \mathrm{Set}_\tau^i \wedge\neg \mathrm{Set}_L^i \wedge \mathrm{Set}_R^i)$, $(\mathrm{Val}_L^i \oplus \mathrm{Val}_\tau^i)$ contradicts $(\mathrm{Set}_\tau^i \wedge \mathrm{Set}_L^i \wedge(\mathrm{Val}_\tau^i \leftrightarrow \mathrm{Val}_L^i))$

Since in all cases we contradict $\neg \mathrm{Dif}_L^{i-1} \wedge(\mathrm{Dif}_R^{i-1} \vee(\neg \mathrm{Set}_\tau^i \wedge\neg \mathrm{Set}_L^i \wedge \mathrm{Set}_R^i)\vee (\mathrm{Set}_\tau^i \wedge \mathrm{Set}_L^i \wedge(\mathrm{Val}_\tau^i \leftrightarrow \mathrm{Val}_L^i)))$ then as per definition $(\mathrm{Val}_B, \mathrm{Set}_B)=(\mathrm{Val}_L, \mathrm{Set}_L)$. Using $\mathrm{Dif}_L^i \rightarrow (\mathrm{Dif}_L^i \wedge\neg \mathrm{Dif}_L^{i-1}) \vee \mathrm{Dif}_L^{i-1}$ we get $\mathrm{Dif}_L^i \rightarrow (\mathrm{Val}_B^i \leftrightarrow \mathrm{Val}_L^i) \wedge (\mathrm{Set}_B^i \leftrightarrow \mathrm{Set}_L^i)$, in a polynomial number of Frege lines.

Now we suppose we want to prove the second proposition $\neg \mathrm{Dif}_L^i \wedge \mathrm{Dif}_R^i \rightarrow (\mathrm{Val}_B^i \leftrightarrow \mathrm{Val}_R^i) \wedge (\mathrm{Set}_B^i \leftrightarrow \mathrm{Set}_R^i)$. We need $\neg \mathrm{Dif}_L^i \wedge \mathrm{Dif}_R^i$ to satisfy $\neg \mathrm{Dif}_L^{i-1} \wedge(\mathrm{Dif}_R^{i-1} \vee(\neg \mathrm{Set}_\tau^i \wedge\neg \mathrm{Set}_L^i \wedge \mathrm{Set}_R^i) \vee (\mathrm{Set}_\tau^i \wedge \mathrm{Set}_L^i \wedge(\mathrm{Val}_\tau^i \leftrightarrow \mathrm{Val}_L^i)))$

Lemma gives us that $\neg \mathrm{Dif}_L^i \rightarrow \neg \mathrm{Dif}_L^{i-1}$. We can show that $\neg \mathrm{Dif}_L^{i-1} \wedge\neg \mathrm{Dif}_R^{i-1} \rightarrow \mathrm{Eq}_{L=\tau}^{i-1}$ using Lemma 9. This allows us to examine just the part where the difference is being triggered $\neg \mathrm{Dif}_L^i \wedge\neg \mathrm{Dif}_R^{i-1} \rightarrow (\mathrm{Set}_\tau^i \leftrightarrow \mathrm{Set}_L^i) \wedge (\mathrm{Set}_\tau^i \rightarrow (\mathrm{Val}_\tau^i \leftrightarrow \mathrm{Val}_L^i))$.

Suppose the term $(\neg \mathrm{Set}_\tau^i \wedge\neg \mathrm{Set}_L^i \wedge \mathrm{Set}_R^i)$ is false, assuming Dif_R^{i-1} is also false, we have to show that $(\mathrm{Set}_\tau^i \wedge \mathrm{Set}_L^i \wedge(\mathrm{Val}_\tau^i \leftrightarrow \mathrm{Val}_L^i)$ will be satisfied. We look at the three ways the term $(\neg \mathrm{Set}_\tau^i \wedge\neg \mathrm{Set}_L^i \wedge \mathrm{Set}_R^i)$ can be falsified and show that all the parts of the remaining term must be satisfied when assuming $\neg \mathrm{Dif}_L^i \wedge \mathrm{Dif}_R^i \wedge\neg \mathrm{Dif}_R^{i-1}$

1. Set_τ^i, in this case $(\mathrm{Val}_\tau^i \leftrightarrow \mathrm{Val}_L^i)$ is active and Set_L^i is implied by $(\mathrm{Set}_\tau^i \leftrightarrow \mathrm{Set}_L^i)$.
2. Set_L^i, Set_τ^i is implied by $(\mathrm{Set}_\tau^i \leftrightarrow \mathrm{Set}_L^i)$, then $(\mathrm{Val}_\tau^i \leftrightarrow \mathrm{Val}_L^i)$ is active.
3. $\neg \mathrm{Set}_R^i$, then using Dif_R^i and $\neg \mathrm{Dif}_R^{i-1}$ we must Set_τ^i (as this is the only allowed way Dif can trigger). Once again, $(\mathrm{Val}_\tau^i \leftrightarrow \mathrm{Val}_L^i)$ is active and Set_L^i is implied by $(\mathrm{Set}_\tau^i \leftrightarrow \mathrm{Set}_L^i)$

Since our trigger formula is always satisfied when $\neg \mathrm{Dif}_L^i \wedge \mathrm{Dif}_R^i \wedge\neg \mathrm{Dif}_R^{i-1}$. It means that $(\mathrm{Val}_B, \mathrm{Set}_B) = (\mathrm{Val}_R, \mathrm{Set}_R)$. Using $\mathrm{Dif}_R^i \rightarrow (\mathrm{Dif}_R^i \wedge\neg \mathrm{Dif}_R^{i-1})\vee \mathrm{Dif}_R^{i-1}$ we get $\neg \mathrm{Dif}_L^i \wedge \mathrm{Dif}_R^i \rightarrow (\mathrm{Val}_B^i \leftrightarrow \mathrm{Val}_R^i)\wedge(\mathrm{Set}_B^i \leftrightarrow \mathrm{Set}_R^i)$, in a polynomial number of Frege lines.

Lemma 8. *The following propositions are true and have short Extended Frege proofs.*

- $B \wedge \mathrm{Dif}_L^m \rightarrow B_L$
- $B \wedge \neg \mathrm{Dif}_L^m \wedge \mathrm{Dif}_R^m \rightarrow B_R$

Proof. We use the disjunction $\mathrm{Dif}_L^m \to \bigvee_{j=1}^m \mathrm{Dif}_L^j \vee \neg \mathrm{Dif}_L^{j-1}$ So there is some j where this is the case.

- For $1 \le i < j$ observe that $\mathrm{Dif}_L^j \vee \neg \mathrm{Dif}_L^{j-1} \to \neg \mathrm{Dif}_R^{j-1}$. Now these negative literals propagate downwards. $\neg \mathrm{Dif}_L^{j-1} \wedge \neg \mathrm{Dif}_R^{j-1} \to \neg \mathrm{Dif}_L^i \wedge \neg \mathrm{Dif}_R^i$ for $0 \le i < j$ and $\neg \mathrm{Dif}_L^i \wedge \neg \mathrm{Dif}_R^i$ means that B and L are consistent for those i as proven in Lemma 6.
- For $j \le k \le m$, $\mathrm{Dif}_L^j \to \mathrm{Dif}_L^k$ and Dif_L^k means B and L are consistent on those k as proven in Lemma 7.
- For indices greater than m, $B \wedge \mathrm{Dif}_L^m$ falsifies $\neg \mathrm{Dif}_L^m \wedge (\mathrm{Dif}_R^m \vee \bar{x})$, so B and L are consistent on those indices.

With the second proposition $\mathrm{Dif}_R^m \to \bigvee_{j=1}^m \mathrm{Dif}_R^j \vee \neg \mathrm{Dif}_R^{j-1}$ once again. So there is some j where this is the case. Note that $\neg \mathrm{Dif}_L^m \to \neg \mathrm{Dif}_L^k$ for $k \le m$.

- For $1 \le i < j$, both $\neg \mathrm{Dif}_L^i$ and $\neg \mathrm{Dif}_R^i$ occur so then B and R are consistent for these values.
- For $j \le k \le m$, $\mathrm{Dif}_R^j \to \mathrm{Dif}_R^k$ and $\mathrm{Dif}_R^k \wedge \neg \mathrm{Dif}_L^k$ means B and R are consistent on those k as proven in Lemma 7.
- For indices greater than m, $B \wedge \mathrm{Dif}_R^m \wedge \neg \mathrm{Dif}_L^m$ satisfies $\neg \mathrm{Dif}_L^m \wedge (\mathrm{Dif}_R^m \vee \bar{x})$, so B and R are consistent on those indices.

Lemma 9. *The following propositions are true and have short Extended Frege proofs.*

- $B \wedge \neg \mathrm{Dif}_L^m \wedge \neg \mathrm{Dif}_R^m \to B_L \vee \neg x$
- $B \wedge \neg \mathrm{Dif}_L^m \wedge \neg \mathrm{Dif}_R^m \to B_R \vee x$

Proof. For indices $1 \le i \le m$, but since $\neg \mathrm{Dif}_L^m \to \neg \mathrm{Dif}_L^i$ and $\neg \mathrm{Dif}_R^m \to \neg \mathrm{Dif}_R^i$, Lemma 6 can be used to show that $B \wedge \mathrm{Dif}_L^m \wedge \mathrm{Dif}_R^m$ leads to $\mathrm{Set}_B^i = \mathrm{Set}_L^i = \mathrm{Set}_R^i$ and $\mathrm{Val}_B^i = \mathrm{Val}_L^i = \mathrm{Val}_R^i$ whenever Set_B^i is also true. Extended Frege can prove $O(m)$ many propositions expressing as such.

For $i > m$, by definition $B \wedge \neg \mathrm{Dif}_L^m \wedge \neg \mathrm{Dif}_R^m \wedge x$ gives $\mathrm{Set}_B^i = \mathrm{Set}_L^i$ and $\mathrm{Val}_B^i = \mathrm{Val}_L^i$. And $B \wedge \neg \mathrm{Dif}_L^m \wedge \neg \mathrm{Dif}_R^m \wedge \neg x$ gives $\mathrm{Set}_B^i = \mathrm{Set}_R^i$ and $\mathrm{Val}_B^i = \mathrm{Val}_R^i$. The sum of this is that $B \wedge \mathrm{Dif}_L^m \wedge \mathrm{Dif}_R^m \wedge x \to B_L$ and $B \wedge \mathrm{Dif}_L^m \wedge \mathrm{Dif}_R^m \wedge \neg x \to B_R$.

Lemma 10. *The following proposition is true and has a short Extended Frege proof.* $B \to B_L \vee B_R$

Proof. This roughly says that B either is played entirely as L or is played as R. We can prove this by combining Lemmas 8 and 9, it essentially is a case analysis in formal form.

Lemma 11. *The following propositions are true and have short Extended Frege proofs.*

- $B \wedge \mathrm{anno}(\tau) \wedge x \to B_L$,
- $B \wedge \mathrm{anno}(\tau) \wedge \neg x \to B_R$

Proof. We start with $B \wedge \neg \mathrm{Dif}_L^m \wedge \neg \mathrm{Dif}_R^m \to B_L \vee \neg x$ and $B \wedge \neg \mathrm{Dif}_L^m \wedge \neg \mathrm{Dif}_R^m \to B_R \vee x$. It remains to remove $\neg \mathrm{Dif}_L^m \wedge \neg \mathrm{Dif}_R^m$ from the left hand side. This is where we use $L \wedge \mathrm{Dif}_L^i \to \neg \mathrm{anno}_L(\tau)$ and $R \wedge \mathrm{Dif}_R^i \to \neg \mathrm{anno}_R(\tau)$ from Lemma 5. These can be simplified to $B \wedge B_L \wedge \mathrm{Dif}_L^m \to \neg \mathrm{anno}_B(\tau)$ and $B \wedge B_R \wedge \mathrm{Dif}_R^m \to \neg \mathrm{anno}_B(\tau)$. The B_L and B_R can be removed by using $B \wedge \mathrm{Dif}_L^m \to B_L$ and $B \wedge \neg \mathrm{Dif}_L^m \wedge \mathrm{Dif}_R^m \to B_R$ and we can end up with $B \wedge B_R \to \neg \mathrm{anno}_B(\tau) \vee (\neg \mathrm{Dif}_R^m \wedge \neg \mathrm{Dif}_L^m)$ we can use this to resolve out $(\neg \mathrm{Dif}_R^m \wedge \neg \mathrm{Dif}_L^m)$ and get $B \wedge \mathrm{anno}(\tau) \wedge x \to B_L$ and $B \wedge \mathrm{anno}(\tau) \wedge \neg x \to B_R$.

7.2 Proof of Simulation of IRM-calc

Lemmas

Lemma 12. *For $0 < j \leq m$ the following propositions have short derivations in Extended Frege:*

- $\mathrm{Dif}_L^j \to \bigvee_{i=1}^j \mathrm{Dif}_L^i \wedge \neg \mathrm{Dif}_L^{i-1}$
- $\mathrm{Dif}_R^j \to \bigvee_{i=1}^j \mathrm{Dif}_R^i \wedge \neg \mathrm{Dif}_R^{i-1}$
- $\neg \mathrm{Eq}_{L=\tau\sqcup\sigma}^j \to \bigvee_{i=1}^j \neg \mathrm{Eq}_{L=\tau\sqcup\sigma}^i \wedge \mathrm{Eq}_{L=\tau\sqcup\sigma}^{i-1}$
- $\neg \mathrm{Eq}_{R=\tau\sqcup\xi}^j \to \bigvee_{i=1}^j \neg \mathrm{Eq}_{R=\tau\sqcup\xi}^i \wedge \mathrm{Eq}_{R=\tau\sqcup\xi}^{i-1}$

Proof. The proof of Lemma 2 still works despite the modifications to definition.

Lemma 13. *For $0 \leq i \leq j \leq m$ the following propositions that describe the monotonicity of* Dif *and* Eq *have short derivations in Extended Frege:*

- $\mathrm{Dif}_L^i \to \mathrm{Dif}_L^j$
- $\mathrm{Dif}_R^i \to \mathrm{Dif}_R^j$
- $\neg \mathrm{Eq}_{f=g}^i \to \neg \mathrm{Eq}_{f=g}^j$

Proof. The proofs of Lemma 3 still work despite the modifications to definition.

Lemma 14. *For $0 \leq i \leq j \leq m$ the following propositions describe the relationships between the different extension variables*

- $\mathrm{Eq}^i_{L=\tau\sqcup\sigma} \to \neg\,\mathrm{Dif}^i_L$
- $\mathrm{Dif}^i_L \wedge \neg\,\mathrm{Dif}^{i-1}_L \to \mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi}$
- $\mathrm{Dif}^i_L \wedge \neg\,\mathrm{Dif}^{i-1}_L \to \neg\,\mathrm{Dif}^{i-1}_R$
- $\mathrm{Eq}^i_{R=\tau\sqcup\xi} \to \neg\,\mathrm{Dif}^i_R$
- $\mathrm{Dif}^i_R \wedge \neg\,\mathrm{Dif}^{i-1}_R \to \mathrm{Eq}^{i-1}_{L=\tau\sqcup\xi}$
- $\mathrm{Dif}^i_R \wedge \neg\,\mathrm{Dif}^{i-1}_R \to \neg\,\mathrm{Dif}^{i-1}_L$

Proof. **Induction Hypothesis on i:** $\mathrm{Eq}^i_{L=\tau\sqcup\sigma} \to \neg\,\mathrm{Dif}^i_L$ in an $O(i)$-size eFrege proof.
Base Case $i = 0$: Dif^i_L is defined as 0 so $\neg\,\mathrm{Dif}^i_L$ is true and trivially implied by $\mathrm{Eq}^i_{L=\tau\sqcup\sigma}$. Frege can manage this.
Inductive Step $i+1$: This breaks into cases depending on the domains of u_{i+1}. If $u_{i+1} \notin \mathsf{dom}(\sigma)$ $\mathrm{Eq}^{i+1}_{L=\tau\sqcup\sigma} := \mathrm{Eq}^i_{L=\tau\sqcup\sigma} \wedge(\mathrm{Set}^{i+1}_L \leftrightarrow \mathrm{Set}^{i+1}_{\tau\sqcup\sigma}) \wedge (\mathrm{Set}^{i+1}_L \to (\mathrm{Val}^{i+1}_L \leftrightarrow \mathrm{Val}^{i+1}_{\tau\sqcup\sigma}))$ further if $u_{i+1} \notin \mathsf{dom}(\tau\sqcup\sigma)$ then $\mathrm{Dif}^{i+1}_L := \mathrm{Dif}^i_L \vee(\mathrm{Eq}^i_{R=\tau\sqcup\xi} \wedge(\mathrm{Set}^{i+1}_L)$ Note that here $\mathrm{Set}^{i+1}_{\tau\sqcup\sigma}$ is defined as 0 so $\mathrm{Eq}^{i+1}_{L=\tau\sqcup\sigma} \to (\mathrm{Eq}^i_{L=\tau\sqcup\sigma} \wedge(\neg\,\mathrm{Set}^{i+1}_L))$. The induction hypothesis gives $\mathrm{Eq}^{i+1}_{L=\tau\sqcup\sigma} \to \neg\,\mathrm{Dif}^i_L \wedge\neg\,\mathrm{Set}^{i+1}_L$. Note that because $\neg\,\mathrm{Dif}^i_L \wedge\neg\,\mathrm{Set}^{i+1}_L$ directly refutes $\mathrm{Dif}^i_L \vee(\mathrm{Eq}^i_{R=\tau\sqcup\xi} \wedge(\mathrm{Set}^{i+1}_L)$ we get $\mathrm{Eq}^{i+1}_{L=\tau\sqcup\sigma} \to \neg\,\mathrm{Dif}^{i+1}_L$. Now if $u_{i+1} \in \mathsf{dom}(\tau)$ then

$$\underset{L}{\overset{i}{\mathrm{Dif}}} := \underset{L}{\overset{i-1}{\mathrm{Dif}}} \vee(\underset{R=\tau\sqcup\xi}{\overset{i-1}{\mathrm{Eq}}} \wedge(\neg\,\underset{L}{\overset{i}{\mathrm{Set}}} \vee(\underset{\tau}{\overset{i}{\mathrm{Set}}} \wedge(\underset{L}{\overset{i}{\mathrm{Val}}} \oplus \underset{\tau}{\overset{i}{\mathrm{Val}}}))))$$

Now $\mathrm{Set}^{i+1}_{\tau\sqcup\sigma}$ is defined as 1. If $1/u_{i+1} \in \tau$ $\mathrm{Val}^{i+1}_{\tau\sqcup\sigma} := 1$ so $\mathrm{Dif}^{i+1}_L := \mathrm{Dif}^i_L \vee(\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi} \wedge(\neg\,\mathrm{Set}^{i+1}_L \vee \mathrm{Val}^{i+1}_L))$ and $\mathrm{Eq}^{i+1}_{L=\tau\sqcup\sigma} \to \mathrm{Eq}^i_{L=\tau\sqcup\sigma} \wedge(\mathrm{Set}^{i+1}_L) \wedge \mathrm{Val}^{i+1}_L)$. The induction hypothesis gives $\mathrm{Eq}^{i+1}_{L=\tau\sqcup\sigma} \to \mathrm{Dif}^i_L \wedge(\mathrm{Set}^{i+1}) \wedge \mathrm{Val}^{i+1}_L)$. But $\mathrm{Dif}^i_L \wedge \mathrm{Set}^{i+1}) \wedge \mathrm{Val}^{i+1}_L)$ falsifies $\mathrm{Dif}^i_L \vee(\mathrm{Eq}^i_{R=\tau\sqcup\xi} \wedge(\neg\,\mathrm{Set}^{i+1}_L \vee(\mathrm{Set}^{i+1}_L \wedge(\mathrm{Val}^{i+1}_L))))$. So $\mathrm{Eq}^{i+1}_{L=\tau\sqcup\sigma} \to \neg\,\mathrm{Dif}^{i+1}_L$. Similarly if $0/u_{i+1} \in \tau$ If $u_{i+1} \in \mathsf{dom}(\sigma)$, $\mathrm{Eq}^{i+1}_{L=\tau\sqcup\sigma} := \mathrm{Eq}^i_{L=\tau\sqcup\sigma} \wedge(\mathrm{Set}^{i+1}_L)$ and $\mathrm{Dif}^{i+1}_L := \mathrm{Dif}^i_L \vee(\mathrm{Eq}^i_{R=\tau\sqcup\xi} \wedge(\neg\,\mathrm{Set}^{i+1}_L))$ But from the induction hypothesis we can have $\mathrm{Eq}^{i+1}_{L=\tau\sqcup\sigma} \to \mathrm{Dif}^i_L \wedge \mathrm{Set}^{i+1}_L$ ' and $\mathrm{Dif}^i_L \wedge \mathrm{Set}^i_L$ directly contradicts $\mathrm{Dif}^i_L \vee(\mathrm{Eq}^i_{R=\tau\sqcup\xi} \wedge(\neg\,\mathrm{Set}^{i+1}_L))$ so then $\mathrm{Eq}^{i+1}_{L=\tau\sqcup\sigma} \to \neg\,\mathrm{Dif}^i_L$ Each case require a constant number of Frege steps

In every case $\mathrm{Dif}^i_L = \mathrm{Dif}^{i-1}_L \vee(\mathrm{Eq}^i_{R=\tau\sqcup\xi} \wedge A)$ where A is a formula dependent on the domain of u_i $\neg\,\mathrm{Dif}^{i-1}_L \wedge \mathrm{Dif}^i_L$ means that $\mathrm{Eq}^i_{R=\tau\sqcup\xi}$ must be true. So we have $\mathrm{Dif}^i_L \wedge\neg\,\mathrm{Dif}^{i-1}_L \to \mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi}$ in a constant size eFrege proof.

If we combine the above we have a linear size proof of $\mathrm{Dif}^i_L \wedge\neg\,\mathrm{Dif}^{i-1}_L \to \mathrm{Dif}^{i-1}_R$

The same proofs symmetrically work for R

Lemma 15. *For any $0 \leq i \leq m$ the following propositions are true and have short Extended Frege proofs.*

- $L \wedge \mathrm{Dif}^i_L \rightarrow \neg\, \mathrm{anno}_{x,L}(\tau \sqcup \sigma)$
- $R \wedge \mathrm{Dif}^i_R \rightarrow \neg\, \mathrm{anno}_{x,R}(\tau \sqcup \xi)$

Proof. If $u_i \notin \mathsf{dom}(\tau \sqcup \sigma)$, then $\mathrm{Dif}^i_L \wedge \neg\, \mathrm{Dif}^{i-1}_L \rightarrow \mathrm{Set}^i_L$ is a simple corollary of the definition line $\mathrm{Dif}^i_L \leftrightarrow \mathrm{Dif}^{i-1}_L \vee (\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi} \wedge \mathrm{Set}^i_L)$. But as $\mathrm{anno}_{x,L}(\tau\sqcup\sigma)$ insists on $\neg\, \mathrm{Set}^i_L$, we can get $\mathrm{Dif}^i_L \wedge \neg\, \mathrm{Dif}^{i-1}_L \rightarrow \neg\, \mathrm{anno}_{x,L}(\tau \sqcup \sigma)$

If $1/u_i \in \tau$, then $\mathrm{Dif}^i_L \wedge \neg\, \mathrm{Dif}^{i-1}_L \rightarrow \neg\, \mathrm{Set}^i_L \vee \neg\, \mathrm{Val}^i_L$ is a simple corollary of the definition lines $\mathrm{Dif}^i_L \leftrightarrow \mathrm{Dif}^{i-1}_L \vee (\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi} \wedge (\neg\, \mathrm{Set}^i_L \vee (\mathrm{Set}^i_\tau \wedge (\mathrm{Val}^i_L \oplus \mathrm{Val}^i_\tau))))$, Set^i_τ and Val^i_τ But as $\mathrm{anno}_{x,L}(\tau \sqcup \sigma)$ insists on $\mathrm{Set}^i_L \wedge u_i$, and L insists on $\mathrm{Val}^i_L \leftrightarrow u_i$ we get $L \wedge \mathrm{Dif}^i_L \wedge \neg\, \mathrm{Dif}^{i-1}_L \rightarrow \neg\, \mathrm{anno}_{x,L}(\tau \sqcup \sigma)$

Similarly, if $0/u_i \in \tau$, then $\mathrm{Dif}^i_L \wedge \neg\, \mathrm{Dif}^{i-1}_L \rightarrow \neg\, \mathrm{Set}^i_L \vee \mathrm{Val}^i_L$ is a simple corollary of the definition lines $\mathrm{Dif}^i_L \leftrightarrow \mathrm{Dif}^{i-1}_L \vee (\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi} \wedge (\neg\, \mathrm{Set}^i_L \vee (\mathrm{Set}^i_\tau \wedge (\mathrm{Val}^i_L \oplus \mathrm{Val}^i_\tau))))$, Set^i_τ and $\neg\, \mathrm{Val}^i_\tau$ But as $\mathrm{anno}_{x,L}(\tau \sqcup \sigma)$ insists on $\mathrm{Set}^i_L \wedge \neg u_i$, and L insists on $\mathrm{Val}^i_L \leftrightarrow u_i$ we get $\mathrm{Dif}^i_L \wedge \neg\, \mathrm{Dif}^{i-1}_L \rightarrow \neg\, \mathrm{anno}_{x,L}(\tau \sqcup \sigma)$

Finally if $*/u_i \in \sigma$, then $\mathrm{Dif}^i_L \wedge \neg\, \mathrm{Dif}^{i-1}_L \rightarrow \mathrm{Set}^i_L$ is a corollary of the definition line $\mathrm{Dif}^i_L \leftrightarrow \mathrm{Dif}^{i-1}_L \vee (\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi} \wedge (\neg\, \mathrm{Set}^i_L)$. But as $\mathrm{anno}_{x,L}(\tau \sqcup \sigma)$ insists on Set^i_L. we get $\mathrm{Dif}^i_L \wedge \neg\, \mathrm{Dif}^{i-1}_L \rightarrow \neg\, \mathrm{anno}_{x,L}(\tau \sqcup \sigma)$

$L \wedge \mathrm{Dif}^i_L \wedge \neg\, \mathrm{Dif}^{i-1}_L \rightarrow \neg\, \mathrm{anno}_{x,L}(\tau \sqcup \sigma)$ is not quite as strong as $L \wedge \mathrm{Dif}^i_L \wedge \rightarrow \neg\, \mathrm{con}_{x,L}(\tau\sqcup\sigma)$ However here we can use $\mathrm{Dif}^j_L \rightarrow \bigvee_{i=1}^j \mathrm{Dif}^i_L \wedge \neg\, \mathrm{Dif}^{i-1}_L$ which will give us $L \wedge \mathrm{Dif}^j_L \rightarrow \neg\, \mathrm{con}_{x,L}(\tau \sqcup \sigma)$ in a linear size proof which is also symmetric for R.

Lemma 16. *For any $0 \leq j \leq m$ the following propositions are true and have a short Extended Frege proof.*

- $\neg\, \mathrm{Dif}^j_L \wedge \neg\, \mathrm{Dif}^j_R \rightarrow \mathrm{Eq}^j_{L=\tau\sqcup\sigma}$
- $\neg\, \mathrm{Dif}^j_L \wedge \neg\, \mathrm{Dif}^j_R \rightarrow \mathrm{Eq}^j_{R=\tau\sqcup\xi}$
- $\neg\, \mathrm{Dif}^j_L \wedge \neg\, \mathrm{Dif}^j_R \rightarrow (\neg\, \mathrm{Set}^j_B \wedge \neg\, \mathrm{Set}^j_L \wedge \neg\, \mathrm{Set}^j_R)$ *when* $u_j \notin \mathsf{dom}(\tau \sqcup \sigma \sqcup \xi)$.
- $\neg\, \mathrm{Dif}^j_L \wedge \neg\, \mathrm{Dif}^j_R \rightarrow (\mathrm{Set}^j_B \wedge \mathrm{Set}^j_L \wedge \mathrm{Set}^j_R \wedge (\mathrm{Val}^j_B \leftrightarrow \mathrm{Val}^j_L) \wedge (\mathrm{Val}^j_B \leftrightarrow \mathrm{Val}^j_R))$ *when* $u_j \in \mathsf{dom}(\tau)$.
- $\neg\, \mathrm{Dif}^j_L \wedge \neg\, \mathrm{Dif}^j_R \rightarrow (\mathrm{Set}^j_B \wedge \mathrm{Set}^j_L \wedge \neg\, \mathrm{Set}^j_R \wedge (\mathrm{Val}^j_B \leftrightarrow \mathrm{Val}^j_L))$ *when* $*/u_j \in \sigma$.
- $\neg\, \mathrm{Dif}^j_L \wedge \neg\, \mathrm{Dif}^j_R \rightarrow (\mathrm{Set}^j_B \wedge \neg\, \mathrm{Set}^j_L \wedge \mathrm{Set}^j_R \wedge (\mathrm{Val}^j_B \leftrightarrow \mathrm{Val}^j_R))$ *when* $*/u_j \in \xi$.

Proof. We show that $\neg\, \mathrm{Eq}^i_{L=\tau\sqcup\sigma} \wedge \mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma} \rightarrow \mathrm{Dif}^i_L \vee \neg\, \mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi}$, and symmetrically that $\neg\, \mathrm{Eq}^i_{R=\tau\sqcup\xi} \wedge \mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi} \rightarrow \mathrm{Dif}^i_R \vee \neg\, \mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma}$. These will be useful ingredients in our proof by induction.

For $u_i \notin \mathsf{dom}(\tau \sqcup \sigma \sqcup \xi)$ or $u_i \in \mathsf{dom}(\xi)$ We use the definition formulas $\mathrm{Eq}^i_{L=\tau\sqcup\sigma} \leftrightarrow \mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma} \wedge (\mathrm{Set}^i_L \leftrightarrow \mathrm{Set}^i_{\tau\sqcup\sigma}) \wedge (\mathrm{Set}^i_{\tau\sqcup\sigma} \rightarrow (\mathrm{Val}^i_L \leftrightarrow \mathrm{Val}^i_{\tau\sqcup\sigma}))$ and $\neg\,\mathrm{Set}^i_{\tau\sqcup\sigma}$ to get $\neg\,\mathrm{Eq}^i_{L=\tau\sqcup\sigma} \wedge \mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma} \rightarrow \mathrm{Set}^i_L$. Likewise, we use $\mathrm{Dif}^i_L \leftrightarrow \mathrm{Dif}^{i-1}_L \vee (\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi} \wedge (\mathrm{Set}^i_L)$ to get $\mathrm{Set}^i_L \rightarrow \mathrm{Dif}^i_L \vee \neg\,\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi}$. We can combine the two to get $\neg\,\mathrm{Eq}^i_{L=\tau\sqcup\sigma} \wedge \mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma} \rightarrow \mathrm{Dif}^i_L \vee \neg\,\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi}$.

For $1/u_i \in \tau$, We use the definition formulas $\mathrm{Eq}^i_{L=\tau\sqcup\sigma} \leftrightarrow \mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma} \wedge (\mathrm{Set}^i_L \leftrightarrow \mathrm{Set}^i_{\tau\sqcup\sigma}) \wedge (\mathrm{Set}^i_{\tau\sqcup\sigma} \rightarrow (\mathrm{Val}^i_L \leftrightarrow \mathrm{Val}^i_{\tau\sqcup\sigma}))$, $\mathrm{Set}^i_{\tau\sqcup\sigma}$ and $\mathrm{Val}^i_{\tau\sqcup\sigma}$ to get $\neg\,\mathrm{Eq}^i_{L=\tau\sqcup\sigma} \wedge \mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma} \rightarrow \neg\,\mathrm{Set}^i_L \vee \neg\,\mathrm{Val}^i_L$. Likewise, we use $\mathrm{Dif}^i_L \leftrightarrow \mathrm{Dif}^{i-1}_L \vee (\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi} \wedge (\neg\,\mathrm{Set}^i_L \vee (\mathrm{Set}^i_\tau \wedge (\mathrm{Val}^i_L \oplus \mathrm{Val}^i_\tau))))$ and Val^i_τ to get $(\neg\,\mathrm{Set}^i_L \vee \neg\,\mathrm{Val}^i_L) \rightarrow \mathrm{Dif}^i_L \vee \neg\,\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi}$. We can combine the two to get $\neg\,\mathrm{Eq}^i_{L=\tau\sqcup\sigma} \wedge \mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma} \rightarrow \mathrm{Dif}^i_L \vee \neg\,\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi}$.

For $0/u_i \in \tau$, We use the definition formulas $\mathrm{Eq}^i_{L=\tau\sqcup\sigma} \leftrightarrow \mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma} \wedge (\mathrm{Set}^i_L \leftrightarrow \mathrm{Set}^i_{\tau\sqcup\sigma}) \wedge (\mathrm{Set}^i_{\tau\sqcup\sigma} \rightarrow (\mathrm{Val}^i_L \leftrightarrow \mathrm{Val}^i_{\tau\sqcup\sigma}))$, $\mathrm{Set}^i_{\tau\sqcup\sigma}$ and $\neg\,\mathrm{Val}^i_{\tau\sqcup\sigma}$ to get $\neg\,\mathrm{Eq}^i_{L=\tau\sqcup\sigma} \wedge \mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma} \rightarrow \neg\,\mathrm{Set}^i_L \vee \mathrm{Val}^i_L$. Likewise, we use $\mathrm{Dif}^i_L \leftrightarrow \mathrm{Dif}^{i-1}_L \vee (\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi} \wedge (\neg\,\mathrm{Set}^i_L \vee (\mathrm{Set}^i_\tau \wedge (\mathrm{Val}^i_L \oplus \mathrm{Val}^i_\tau))))$ and $\neg\,\mathrm{Val}^i_\tau$ to get $(\neg\,\mathrm{Set}^i_L \vee \mathrm{Val}^i_L) \rightarrow \mathrm{Dif}^i_L \vee \neg\,\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi}$. We can combine the two to get $\neg\,\mathrm{Eq}^i_{L=\tau\sqcup\sigma} \wedge \mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma} \rightarrow \mathrm{Dif}^i_L \vee \neg\,\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi}$.

For $*/u_i \in \sigma$, we use the definition formula $\mathrm{Eq}^i_{L=\tau\sqcup\sigma} \leftrightarrow \mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma} \wedge (\mathrm{Set}^i_L)$ to get $\neg\,\mathrm{Eq}^i_{L=\tau\sqcup\sigma} \wedge \mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma} \rightarrow \neg\,\mathrm{Set}^i_L$ Likewise, we use $\mathrm{Dif}^i_L \leftrightarrow \mathrm{Dif}^{i-1}_L \vee (\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi} \wedge (\neg\,\mathrm{Set}^i_L))$ to get $(\neg\,\mathrm{Set}^i_L) \rightarrow \mathrm{Dif}^i_L \vee \neg\,\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi}$. We can combine the two to get $\neg\,\mathrm{Eq}^i_{L=\tau\sqcup\sigma} \wedge \mathrm{Eq}^{i-1}_{L=\tau\sqcup\sigma} \rightarrow \mathrm{Dif}^i_L \vee \neg\,\mathrm{Eq}^{i-1}_{R=\tau\sqcup\xi}$.

Induction Hypothesis (on j): $(\neg\,\mathrm{Eq}^j_{L=\tau\sqcup\sigma} \vee \neg\,\mathrm{Eq}^j_{R=\tau\sqcup\sigma}) \rightarrow (\mathrm{Dif}^j_L \vee \mathrm{Dif}^j_R)$

Base Case: $\neg\,\mathrm{Eq}^1_{L=\tau\sqcup\sigma} \wedge \mathrm{Eq}^0_{L=\tau\sqcup\sigma} \rightarrow \mathrm{Dif}^1_L \vee \neg\,\mathrm{Eq}^0_{R=\tau\sqcup\xi}$, and $\neg\,\mathrm{Eq}^1_{R=\tau\sqcup\xi} \wedge \mathrm{Eq}^0_{R=\tau\sqcup\xi} \rightarrow \mathrm{Dif}^1_R \vee \neg\,\mathrm{Eq}^0_{L=\tau\sqcup\sigma}$

However since $\mathrm{Eq}^0_{L=\tau\sqcup\sigma}$ and $\mathrm{Eq}^0_{R=\tau\sqcup\xi}$ are both true it simplifies to $\neg\,\mathrm{Eq}^1_{L=\tau\sqcup\sigma} \rightarrow \mathrm{Dif}^1_L$ and $\neg\,\mathrm{Eq}^1_{R=\tau\sqcup\xi} \rightarrow \mathrm{Dif}^1_R$ which can be combined to get $(\neg\,\mathrm{Eq}^1_{L=\tau\sqcup\sigma} \vee \neg\,\mathrm{Eq}^1_{R=\tau\sqcup\sigma}) \rightarrow (\mathrm{Dif}^1_L \vee \mathrm{Dif}^1_R)$

Inductive Step: The Induction Hypothesis $(\neg\,\mathrm{Eq}^j_{L=\tau\sqcup\sigma} \vee \neg\,\mathrm{Eq}^j_{R=\tau\sqcup\sigma}) \rightarrow (\mathrm{Dif}^j_L \vee \mathrm{Dif}^j_R)$ can be weakened to $(\neg\,\mathrm{Eq}^j_{L=\tau\sqcup\sigma} \vee \neg\,\mathrm{Eq}^j_{R=\tau\sqcup\sigma}) \rightarrow (\mathrm{Dif}^{j+1}_L \vee \mathrm{Dif}^{j+1}_R)$, using $\mathrm{Dif}^j_L \rightarrow \mathrm{Dif}^{j+1}_L$ and $\mathrm{Dif}^j_R \rightarrow \mathrm{Dif}^{j+1}_R$. Now we need to replace $\neg\,\mathrm{Eq}^j_{L=\tau\sqcup\sigma}$ and $\neg\,\mathrm{Eq}^j_{R=\tau\sqcup\sigma}$. We can use $\neg\,\mathrm{Eq}^{j+1}_{L=\tau\sqcup\sigma} \wedge \mathrm{Eq}^j_{L=\tau\sqcup\sigma} \rightarrow \mathrm{Dif}^{j+1}_L \vee \neg\,\mathrm{Eq}^j_{R=\tau\sqcup\xi}$ and get $(\neg\,\mathrm{Eq}^{j+1}_{L=\tau\sqcup\sigma}) \rightarrow (\mathrm{Dif}^{j+1}_L \vee \mathrm{Dif}^{j+1}_R)$ or use $\neg\,\mathrm{Eq}^{j+1}_{R=\tau\sqcup\xi} \wedge \mathrm{Eq}^j_{R=\tau\sqcup\xi} \rightarrow \mathrm{Dif}^{j+1}_R \vee \neg\,\mathrm{Eq}^j_{L=\tau\sqcup\sigma}$ and get $(\neg \mathrm{Eq}^{j+1}_{R=\tau\sqcup\xi}) \to (\mathrm{Dif}^{j+1}_L \vee \mathrm{Dif}^{j+1}_R)$ and then putting them together we get $(\neg \mathrm{Eq}^{j+1}_{L=\tau\sqcup\sigma} \vee \neg \mathrm{Eq}^{j+1}_{R=\tau\sqcup\xi}) \to (\mathrm{Dif}^{j+1}_L \vee \mathrm{Dif}^{j+1}_R)$.

Once we are finished with the induction we have in $O(j)$-size proofs:

- $\neg \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^j_R \to \mathrm{Eq}^j_{L=\tau\sqcup\sigma}$
- $\neg \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^j_R \to \mathrm{Eq}^j_{R=\tau\sqcup\xi}$

If $u_j \notin \mathsf{dom}(\tau \sqcup \sigma \sqcup \xi)$, then $\mathrm{Set}^j_{\tau\sqcup\sigma}$ and $\mathrm{Set}^j_{\tau\sqcup\xi}$ are false. We therefore have $\mathrm{Eq}^j_{L=\tau\sqcup\sigma} \to \neg \mathrm{Set}^i_L$ and $\mathrm{Eq}^j_{R=\tau\sqcup\xi} \to \neg \mathrm{Set}^i_R$ then we need to work with the definition of Set_B to derive $(\mathrm{Set}_B \leftrightarrow \mathrm{Set}_L) \vee (\mathrm{Set}_B \leftrightarrow \mathrm{Set}_R)$, which gives $\neg \mathrm{Set}^i_L \wedge \neg \mathrm{Set}^i_R \to \neg \mathrm{Set}^i_B$ so therefore we can derive $\mathrm{Eq}^j_{L=\tau\sqcup\sigma} \wedge \mathrm{Eq}^j_{R=\tau\sqcup\xi} \to \neg \mathrm{Set}^i_L \wedge \neg \mathrm{Set}^i_R \wedge \neg \mathrm{Set}^i_B$.

Similarly if $u_j \in \mathsf{dom}(\tau)$ we can derive $\mathrm{Eq}^j_{L=\tau\sqcup\sigma} \wedge \mathrm{Eq}^j_{R=\tau\sqcup\xi} \to \mathrm{Set}^i_L \wedge \mathrm{Set}^i_R \wedge \mathrm{Set}^i_B$. However we can go even further as we can also derive $(\mathrm{Val}_B \leftrightarrow \mathrm{Set}_L) \vee (\mathrm{Val}_B \leftrightarrow \mathrm{Set}_R)$. But since we have $\mathrm{Eq}^j_{L=\tau\sqcup\sigma} \to (\mathrm{Set}^i_L \to (\mathrm{Val}_L \leftrightarrow \mathrm{Val}_{\tau\sqcup\sigma}))$ and $\mathrm{Eq}^j_{R=\tau\sqcup\xi} \to (\mathrm{Set}^i_R \to (\mathrm{Val}_R \leftrightarrow \mathrm{Val}_{\tau\sqcup\xi}))$ when Set^i_L and Set^i_R are true then $(\mathrm{Eq}^j_{L=\tau\sqcup\sigma} \wedge \mathrm{Eq}^j_{R=\tau\sqcup\xi}) \wedge ((\mathrm{Val}_B \leftrightarrow \mathrm{Val}_L) \vee (\mathrm{Val}_B \leftrightarrow \mathrm{Val}_R)) \to (\mathrm{Val}_B \leftrightarrow \mathrm{Val}_\tau)$ putting this all together we get $\neg \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^j_R \to (\mathrm{Set}^j_B \wedge \mathrm{Set}^j_L \wedge \mathrm{Set}^j_R \wedge (\mathrm{Val}^j_B \leftrightarrow \mathrm{Val}^j_L) \wedge (\mathrm{Val}^j_B \leftrightarrow \mathrm{Val}^j_R))$

Now we have $u_j \in \mathsf{dom}(\sigma)$ then $\mathrm{Set}^j_{\tau\sqcup\sigma}$ is true and $\mathrm{Set}^j_{\tau\sqcup\xi}$ is false. We therefore have $\mathrm{Eq}^j_{L=\tau\sqcup\sigma} \to \mathrm{Set}^j_L$ and $\mathrm{Eq}^j_{R=\tau\sqcup\xi} \to \neg \mathrm{Set}^j_R$. $\neg \mathrm{Dif}^j_R$ means that $\neg \mathrm{Dif}^{j-1}_R$ and so $\neg \mathrm{Dif}^{j-1}_R \wedge \neg \mathrm{Set}^j_R$ means $(\mathrm{Set}_B \leftrightarrow \mathrm{Set}_L)$ and $(\mathrm{Val}_B \leftrightarrow \mathrm{Val}_L)$. Therefore Set_B is true in this situation, so we have $\neg \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^j_R \to (\mathrm{Set}^j_B \wedge \mathrm{Set}^j_L \wedge \neg \mathrm{Set}^j_R \wedge (\mathrm{Val}^j_B \leftrightarrow \mathrm{Val}^j_L)$

Finally for $u_j \in \mathsf{dom}(\xi)$ we have $\mathrm{Set}^j_{\tau\sqcup\sigma}$ is false and $\mathrm{Set}^j_{\tau\sqcup\xi}$ is true. $\mathrm{Eq}^j_{L=\tau\sqcup\sigma} \to \neg \mathrm{Set}^j_L$ and $\mathrm{Eq}^j_{R=\tau\sqcup\xi} \to \mathrm{Set}^j_R$. $\neg \mathrm{Dif}^j_R$ means that $\neg \mathrm{Dif}^{j-1}_R$ and so $\mathrm{Dif}^{j-1}_L \wedge \neg \mathrm{Set}^j_L$ which satisfies $\neg \mathrm{Dif}^{i-1}_L \wedge (\mathrm{Dif}^{i-1}_R \vee \neg \mathrm{Set}^i_L)$ so $(\mathrm{Set}_B \leftrightarrow \mathrm{Set}_R)$ and $(\mathrm{Val}_B \leftrightarrow \mathrm{Val}_R)$ and thus Set_B is true. so we have $\neg \mathrm{Dif}^j_L \wedge \mathrm{Dif}^j_R \to (\mathrm{Set}^j_B \wedge \mathrm{Set}^j_L \wedge \neg \mathrm{Set}^j_R \wedge (\mathrm{Val}^j_B \leftrightarrow \mathrm{Val}^j_R)$.

Lemma 17. *Suppose* $L \to \mathrm{con}_L(C_1 \vee \neg x^{\tau\cup\sigma})$ *and* $R \to \mathrm{con}_L(C_1 \vee x^{\tau\cup\xi})$ *The following propositions are true and have short Extended Frege proofs.*

- $B \wedge \mathrm{Dif}^m_L \to L$
- $B \wedge \neg \mathrm{Dif}^m_L \wedge \mathrm{Dif}^m_R \to R$
- $B \wedge \mathrm{Dif}^m_L \to \mathrm{con}_B(\mathit{inst}(\xi, C_1))$
- $B \wedge \neg \mathrm{Dif}^m_L \wedge \mathrm{Dif}^m_R \to \mathrm{con}_B(\mathit{inst}(\sigma, C_2))$

Proof. Suppose we look at the L cases. In order to manage this proof we first break down the disjunction in C_1 into constituent literals. So we pick a particular literal $y^\alpha \in C_1$ and we argue that $(L \rightarrow \text{con}_L(y^\alpha)) \rightarrow (B \wedge \text{Dif}_L^m \rightarrow \text{con}_B(\mathsf{inst}(\xi, y^\alpha)))$.

For any i, such that $u_i < y$ in the prefix. We will show that $(\text{Dif}_L^m \wedge \text{Set}_B^i \rightarrow (u_i \leftrightarrow \text{Val}_B^i)) \rightarrow (\text{Set}_L^i \rightarrow (u_i \leftrightarrow \text{Val}_L^i))$. When we take a conjunction over all i, we get $B \wedge \text{Dif}_L^m \rightarrow L$. A maximum of one of $\neg \text{Set}_B^i$, Set_B^i, $\text{Set}_B^i \wedge u_i$ and $\text{Set}_B^i \wedge \neg u_i$ appears in $\text{anno}_{y,B}(\alpha \circ \xi)$, we treat $\text{anno}_{y,B}(\alpha[\xi])$ as a set containing these subformulas. We show that if formula $c_i \in \text{anno}_{y,B}(\alpha \circ \xi)$, when c_i is equal to $\neg \text{Set}_B^i$, Set_B^i, $\text{Set}_B^i \wedge u_i$ or $\text{Set}_B^i \wedge \neg u_i$ then $(L \rightarrow \text{anno}_{y,B}(\alpha)) \rightarrow (B \wedge L \wedge \text{Dif}_L^m \rightarrow c_i)$. We also have $(L \rightarrow y) \rightarrow (B \wedge L \rightarrow \wedge \text{Dif}_L^m \rightarrow y)$

Eventually we can put all these together and get $(L \rightarrow \text{con}_L(y^\alpha)) \rightarrow (B \wedge L \wedge \text{Dif}_L^m \rightarrow \text{con}_B(\mathsf{inst}(\xi, y^\alpha))$. We can cut out the L with $B \wedge \text{Dif}_L^m \rightarrow L$. If Dif_L^m then there is some $1 \leq j \leq m$ such that $\text{Dif}_L^j \wedge \neg \text{Dif}_L^{j-1} \wedge \neg \text{Dif}_R^{j-1}$ via Lemmas 12 and 14 For each $1 \leq i \leq m$ we have to argue for $j < i$, $j = i$ and $1 \leq i \leq j$, in order to cover all possibilities. For $i > m$ it is more simple.

The proof for each i adds a linear amount of lines in i for each proof , Once we have $L \rightarrow \text{con}_L(y^\alpha)) \rightarrow (B \wedge L \wedge \text{Dif}_L^m \rightarrow \text{con}_B(\mathsf{inst}(\xi, y^\alpha))$, for one literal we can have $(L \wedge \text{Dif}_L^m \rightarrow \text{con}_L(C_1)) \rightarrow (B \wedge L \wedge \text{Dif}_L^m \rightarrow \text{con}_B(\mathsf{inst}(\xi, C_1))$. However the premise is $(L \rightarrow \text{con}_L(C_1 \vee \neg x))$, so in order to remove the $\neg x$ we use Lemma 15. $L \wedge \text{Dif}_L^m \rightarrow \neg \text{anno}_{x,L}(\tau \sqcup \sigma)$, so $L \wedge \text{Dif}_L^m \rightarrow \neg \text{con}_L(x^\tau \sqcup \sigma)$, and thus $(L \wedge \text{Dif}_L^m \rightarrow \text{con}_L(C_1))$. We will detail all the cases here, note that we have to again do the same for R. The proof size will be $O(wn)$ where w is the width or number of literals in $\mathsf{inst}(\xi, C_1) \sqcup \mathsf{inst}(\sigma, C_2)$ and n is the number of universal variables in the prefix.

We detail the cases below:

Suppose $i > m$.

Dif_L^i refutes $\neg \text{Dif}_L^m \wedge (\text{Dif}_R^m \vee \neg \text{Set}_L^i)$ so whenever Dif_L^m is true, $(\text{Val}_B^i, \text{Set}_B^i) = (\text{Val}_L^i, \text{Set}_L^i)$, therefore $(\text{Set}_B^i \rightarrow (u_i \leftrightarrow \text{Val}_B^i)) \rightarrow (\text{Set}_L^i \rightarrow (u_i \leftrightarrow \text{Val}_L^i))$.

If $\neg \text{Set}_B^i \in \text{anno}_{y,B}(\alpha \circ \xi)$, then $u_i \notin \mathsf{dom}(\alpha \circ \xi)$. We know $u_i \notin \mathsf{dom}(\alpha)$ otherwise it would be in $\mathsf{dom}(\alpha \circ \xi)$. Therefore $\neg \text{Set}_L^i$ is in $\text{anno}_{y,L}(\alpha)$. And so if $L \rightarrow \text{anno}_{y,L}(\alpha)$ then $L \rightarrow \neg \text{Set}_L^i$, , therefore $B \wedge L \wedge \text{Dif}_L^m \rightarrow \neg \text{Set}_B^i$. We now look at all the cases of $c_i \in \text{anno}_{y,B}(\alpha \circ \xi)$ and show they can be satisfied with our strategy in B:

If $\text{Set}_B^i \in \text{anno}_{y,B}(\alpha \circ \xi)$, then $u_i \in \mathsf{dom}(\alpha \circ \xi)$ $u_i \notin \mathsf{dom}(\xi)$ because $\mathsf{dom}(\xi)$ only extends up to m hence $u_i \notin \mathsf{dom}(\alpha \circ \xi)$ and $\text{Set}_L^i \in \text{anno}_{y,L}(\alpha)$. And so if $L \rightarrow \text{anno}_{y,L}(\alpha)$ then $L \rightarrow \text{Set}_L^i$, therefore $B \wedge L \wedge \text{Dif}_L^m \rightarrow \text{Set}_B^i$.

If $\mathrm{Set}_B^i \wedge u_i \in \mathrm{anno}_{y,B}(\alpha \circ \xi)$ then $u_i \in \mathsf{dom}(\alpha \circ \xi)$. We know $u_i \notin \mathsf{dom}(\xi)$ because $\mathsf{dom}(\xi)$ only extends up to m hence $u_i \notin \mathsf{dom}(\alpha \circ \xi)$. Hence $u_i \in \mathsf{dom}(\alpha)$ and $\mathrm{Set}_L^i \wedge u_i \in \mathrm{anno}_{y,L}(\alpha)$ And so if $L \to \mathrm{anno}_{y,L}(\alpha)$ then $L \to \mathrm{Set}_L^i \wedge u_i$, therefore $B \wedge L \wedge \mathrm{Dif}_L^m \to \mathrm{Set}_B^i \wedge u_i$.

If $\mathrm{Set}_B^i \wedge \neg u_i \in \mathrm{anno}_{y,B}(\alpha \circ \xi)$ then $u_i \notin \mathsf{dom}(\alpha \circ \xi)$ $u_i \notin \mathsf{dom}(\xi)$ because $\mathsf{dom}(\xi)$ only extends up to m hence $u_i \notin \mathsf{dom}(\alpha \circ \xi)$. Hence $u_i \in \mathsf{dom}(\alpha)$ and $\mathrm{Set}_L^i \wedge \neg u_i \in \mathrm{anno}_{y,L}(\alpha)$ And so if $L \to \mathrm{anno}_{y,L}(\alpha)$ then $L \to \mathrm{Set}_L^i \wedge \neg \mathrm{Val}_L^i$, therefore $B \wedge L \wedge \mathrm{Dif}_L^m \to \mathrm{Set}_B^i \wedge \neg u_i$.

Suppose $j < i \leq m$**.**

We know $\mathrm{Dif}_L^j \to \mathrm{Dif}_L^{i-1}$ from Lemma 13, we will use that to get that when $\mathrm{Dif}_L^j \wedge \mathrm{Set}_L^i$ then $(\mathrm{Val}_B^i, \mathrm{Set}_B^i) = (\mathrm{Val}_L^i, \mathrm{Set}_L^i)$ which allows us to then show $(\mathrm{Set}_B^i \to (u_i \leftrightarrow \mathrm{Val}_B^i)) \to (\mathrm{Set}_L^i \to (u_i \leftrightarrow \mathrm{Val}_L^i))$. When Dif_L^{i-1} for $u_i \notin \mathsf{dom}(\xi)$ we refute $\neg \mathrm{Dif}_L^{i-1} \wedge (\mathrm{Dif}_R^{i-1} \vee \neg \mathrm{Set}_L^i)$, $\neg \mathrm{Dif}_L^{i-1} \wedge (\mathrm{Dif}_R^{i-1} \vee (\mathrm{Set}_L^i \wedge (\mathrm{Val}_L^i \leftrightarrow \mathrm{Val}_\tau^i)))$, $\neg \mathrm{Dif}_L^{i-1} \wedge \mathrm{Dif}_R^{i-1} \wedge \neg \mathrm{Set}_R^i$ and $\neg \mathrm{Dif}_L^{i-1} \wedge \mathrm{Set}_R^i \wedge (\mathrm{Dif}_R^{i-1} \vee \mathrm{Set}_L^i)$. When Dif_L^{i-1} for $u_i \in \mathsf{dom}(\xi)$ when Set_L^i is true we refute $\mathrm{Dif}_L^{i-1} \wedge \neg \mathrm{Set}_L^i$ and $\neg \mathrm{Dif}_L^{i-1} \wedge (\mathrm{Dif}_R^{i-1} \vee \neg \mathrm{Set}_L^i)$.

if $\neg \mathrm{Set}_B^i \in \mathrm{anno}_{y,B}(\alpha \circ \xi)$ then $u_i \notin \mathsf{dom}(\alpha \circ \xi)$, also $u_i \notin \mathsf{dom}(\alpha)$ and $u_i \notin \mathsf{dom}(\xi)$ so $\neg \mathrm{Set}_L^i \in \mathrm{anno}_{y,L}(\alpha)$ And so if $L \to \mathrm{anno}_{y,L}(\alpha)$ then $L \to \neg \mathrm{Set}_L^i$ when Dif_L^{i-1} and $u_i \notin \mathsf{dom}(\xi)$, $(\mathrm{Val}_B^i, \mathrm{Set}_B^i) = (\mathrm{Val}_L^i, \mathrm{Set}_L^i)$ and so $B \wedge L \wedge \mathrm{Dif}_L^{i-1} \wedge \mathrm{Set}_B^i$

If $\mathrm{Set}_B^i \in \mathrm{anno}_{y,B}(\alpha \circ \xi)$ then $*/u_i\, \mathsf{dom}(\alpha \circ \xi)$ so either $*/u_i \in \alpha$ or $u_i \notin \mathsf{dom}(\alpha)$ and $*/u_i \in \xi$. If $*/u_i \in \alpha$ then $\mathrm{Set}_L^i \in \mathrm{anno}_{y,L}(\alpha)$ and $L \to \mathrm{Set}_L^i$ so when $\mathrm{Dif}_L^{i-1} \wedge \mathrm{Set}_L^i$ no matter which domain u_i is in $(\mathrm{Val}_B^i, \mathrm{Set}_B^i) = (\mathrm{Val}_L^i, \mathrm{Set}_L^i)$ $B \wedge L \wedge \mathrm{Dif}_L^{i-1} \wedge \mathrm{Set}_B^i$. If $u_i \notin \mathsf{dom}(\alpha)$ and $*/u_i \in \xi$. $\neg \mathrm{Set}_L^i \in \mathrm{anno}_{y,L}(\alpha)$ so $L \to \neg \mathrm{Set}_L^i$. $u_i \in \mathsf{dom}(\xi)$ means that when Dif_L^{i-1} and $\neg \mathrm{Set}_L^i$ $(\mathrm{Val}_B^i, \mathrm{Set}_B^i) = (0, 1)$ so $B \wedge L \wedge \mathrm{Dif}_L^{i-1} \wedge \mathrm{Set}_B^i$

If $\mathrm{Set}_B^i \wedge u_i \in \mathrm{anno}_{y,B}(\alpha \circ \xi)$ then $1/u_i \in (\alpha \circ \xi)$ and it can only be that $1/u_i \in \alpha$ as ξ can only add $*/u_i$ So $\mathrm{Set}_L^i \wedge u_i \in \mathrm{anno}_{y,L}(\alpha)$ and $L \to \mathrm{Set}_L^i$. so when $\mathrm{Dif}_L^{i-1} \wedge \mathrm{Set}_L^i$ no matter which domain u_i is in $(\mathrm{Val}_B^i, \mathrm{Set}_B^i) = (\mathrm{Val}_L^i, \mathrm{Set}_L^i)$. $B \wedge L \wedge \mathrm{Dif}_L^{i-1} \wedge \mathrm{Set}_B^i \wedge u_i$.

Likewise, If $\mathrm{Set}_B^i \wedge \neg u_i \in \mathrm{con}_{y,B}(\alpha \circ \xi)$ then $0/u_i \in (\alpha \circ \xi)$ and it can only be that $0/u_i \in \alpha$ as ξ can only add $*/u_i$ So $\mathrm{Set}_L^i \wedge u_i \in \mathrm{anno}_{y,L}(\alpha)$ and $L \to \mathrm{Set}_L^i$. so when $\mathrm{Dif}_L^{i-1} \wedge \mathrm{Set}_L^i$ no matter which domain u_i is in $(\mathrm{Val}_B^i, \mathrm{Set}_B^i) = (\mathrm{Val}_L^i, \mathrm{Set}_L^i)$. $B \wedge L \wedge \mathrm{Dif}_L^{i-1} \wedge \mathrm{Set}_B^i \wedge \neg u_i$.

Suppose $i = j$**.**

$\neg \mathrm{Dif}_L^{j-1}$ by definition of j. $\neg \mathrm{Dif}_R^{j-1}$ is also true as Dif_R^{j-1} contradicts $\mathrm{Eq}_{R=\tau \vee \xi}^{j-1}$ which is necessary for Dif_L^j. With $\neg \mathrm{Dif}_R^{j-1}$, $(\mathrm{Val}_B^j, \mathrm{Set}_B^j)$ can only be defined as $(\mathrm{Val}_R^j, \mathrm{Set}_R^j)$ in a small selection of circumstances That is when: $\neg \mathrm{Set}_L^j$ and $u_i \notin \mathsf{dom}(\tau \sqcup \sigma \sqcup \xi)$ $\mathrm{Set}_L^j \wedge \mathrm{Val}_L^j$ and $1/u_j \in \tau$

$\mathrm{Set}^j_L \wedge \neg \mathrm{Val}^j_L$ and $0/u_j \in \tau$ $\mathrm{Set}^j_L \wedge \mathrm{Set}^j_R$ and $*/u_j \in \sigma$ $\neg \mathrm{Set}^j_L$ and $*/u_j \in \xi$ All but the latter contradict $\mathrm{Dif}^j_L \wedge \mathrm{Dif}^{j-1}_L$, but we can ignore whenever Set^j_L is false. So $\mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^{j-1}_L \wedge \mathrm{Set}^j_L \rightarrow \mathrm{Set}^j_B$ this means that $\mathrm{Set}^j_B \rightarrow (u_i \leftrightarrow \mathrm{Val}^j_B) \rightarrow \mathrm{Set}^j_L \rightarrow (u_i \leftrightarrow \mathrm{Val}^j_L)$.

if $\neg \mathrm{Set}^j_B \in \mathrm{anno}_{y,B}(\alpha \circ \xi)$ then $u_j \notin \mathsf{dom}(\alpha \circ \xi)$ and so $u_j \notin \mathsf{dom}(\alpha)$ $u_j \notin \mathsf{dom}(\xi)$. So $\neg \mathrm{Set}^j_L \in \mathrm{anno}_{y,L}(\alpha)$ and $L \rightarrow \neg \mathrm{Set}^j_L$ Since Dif^j_L is true then it can only be that $u_j \in \mathsf{dom}(\tau)$ or $u_j \in \mathsf{dom}(\sigma)$. If $u_j \in \mathsf{dom}(\tau)$ then $\neg \mathrm{Dif}^{j-1}_L \wedge(\neg \mathrm{Dif}^{j-1}_L \vee(\mathrm{Set}^j_L \wedge(\mathrm{Val}^j_L \leftrightarrow \mathrm{Val}^j_L)))$ is contradicted so $(\mathrm{Val}^j_B, \mathrm{Set}^j_B) = (\mathrm{Val}^j_L, \mathrm{Set}^j_L)$ and $B \wedge L \wedge \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^{j-1}_L \rightarrow \neg \mathrm{Set}^i_B$. If $u_j \in \mathsf{dom}(\sigma)$ then $\neg \mathrm{Dif}^{j-1}_L \wedge \mathrm{Dif}^{j-1}_R \wedge \neg \mathrm{Set}^j_R$ and $\neg \mathrm{Dif}^{j-1}_L \wedge \mathrm{Set}^j_R \wedge(\mathrm{Dif}^{j-1}_R \vee \mathrm{Set}^j_L)$ are contradicted so $(\mathrm{Val}^j_B, \mathrm{Set}^j_B) = (\mathrm{Val}^j_L, \mathrm{Set}^j_L)$ and $B \wedge L \wedge \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^{j-1}_L \rightarrow \neg \mathrm{Set}^i_B$. If $u_j \notin \mathsf{dom}(\tau \sqcup \sigma \sqcup \xi)$ Dif^j_L is false in this case So we can ignore it. $(\mathrm{Val}^j_B, \mathrm{Set}^j_B) = (\mathrm{Val}^j_L, \mathrm{Set}^j_L)$ means that $B \wedge L \wedge \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^{j-1}_L \wedge \neg \mathrm{Set}^j_B \wedge \neg u_j$.

If $\mathrm{Set}^j_B \in \mathrm{anno}_{y,B}(\alpha \circ \xi)$, $u_j \in \mathsf{dom}(\alpha \circ \xi)$. Either $*/u_j \in \alpha$ or $u_j \notin \mathsf{dom}(\alpha)$ and $*/u_j \in \xi$ If $*/u_j \in \alpha$, then $\mathrm{Set}^j_L \in \mathrm{anno}_{y,L}(\alpha)$ and $L \rightarrow \mathrm{Set}^j_L$. If $u_j \notin \mathsf{dom}(\tau \sqcup \sigma \sqcup \xi)$, $\neg \mathrm{Dif}^{j-1}_L \wedge(\mathrm{Dif}^{j-1}_R \vee \neg \mathrm{Set}^j_L$ is falsified so $(\mathrm{Val}^j_B, \mathrm{Set}^j_B) = (\mathrm{Val}^j_L, \mathrm{Set}^j_L)$ and $B \wedge L \wedge \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^{j-1}_L \rightarrow \mathrm{Set}^i_B$. If $u_j \in \mathsf{dom}(\tau)$, $\mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^{j-1}_L \wedge \mathrm{Set}^j_L$ means that $\mathrm{Val}^j_L \oplus \mathrm{Val}^j_\tau$ and so. $\neg \mathrm{Dif}^{j-1}_L \wedge(\mathrm{Dif}^{j-1}_R \vee(\mathrm{Set}^j_L \wedge(\mathrm{Val}^j_L \leftrightarrow \mathrm{Val}^j_\tau)))$ is falsified so $(\mathrm{Val}^j_B, \mathrm{Set}^j_B) = (\mathrm{Val}^j_L, \mathrm{Set}^j_L)$ and $B \wedge L \wedge \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^{j-1}_L \rightarrow \mathrm{Set}^i_B$. If $u_j \in \mathsf{dom}(\sigma)$ Set^j_L contradicts $\mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^{j-1}_L$, so this scenario does not occur. If $u_j \in \mathsf{dom}(\xi)$ $\mathrm{Dif}^{j-1}_L \wedge \neg \mathrm{Set}^j_L$ is falsified by $\neg \mathrm{Dif}^{j-1}_L$. $\neg \mathrm{Dif}^{j-1}_L \wedge(\mathrm{Dif}^{j-1}_R \vee \neg \mathrm{Set}^j_L)$ is falsified by Set^j_L so $(\mathrm{Val}^j_B, \mathrm{Set}^j_B) = (\mathrm{Val}^j_L, \mathrm{Set}^j_L)$ and $B \wedge L \wedge \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^{j-1}_L \rightarrow \mathrm{Set}^i_B$. If $u_j \notin \mathsf{dom}(\alpha)$ and $*/u_j \in \xi$ then $\neg \mathrm{Set}^j_L \in \mathrm{anno}_{y,L}(\alpha)$ and $L \rightarrow \neg \mathrm{Set}^j_L$. However this conflicts with $\mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^{j-1}_L$.

If $\mathrm{Set}^j_B \wedge \mathrm{Val}^j_B \in \mathrm{anno}_{y,B}(\alpha \circ \xi)$, $1/u_j \in (\alpha \circ \xi)$. As instantiate is only done by $*$ then $1/u_j \in (\alpha)$. So it follows $\mathrm{Set}^j_L \wedge \mathrm{Val}^j_L \in \mathrm{anno}_{y,L}(\alpha)$ If $u_j \notin \mathsf{dom}(\tau \sqcup \sigma \sqcup \xi)$, $\neg \mathrm{Dif}^{j-1}_L \wedge(\mathrm{Dif}^{j-1}_R \vee \neg \mathrm{Set}^j_L)$ is falsified so $(\mathrm{Val}^j_B, \mathrm{Set}^j_B) = (\mathrm{Val}^j_L, \mathrm{Set}^j_L)$ and $B \wedge L \wedge \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^{j-1}_L \rightarrow \mathrm{Set}^i_B \wedge \mathrm{Val}^i_B$. If $u_j \in \mathsf{dom}(\tau)$, $\mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^{j-1}_L \wedge \mathrm{Set}^j_L \wedge \mathrm{Val}^j_L$ means that $\neg \mathrm{Val}^j_\tau$ and so. $\neg \mathrm{Dif}^{j-1}_L \wedge(\mathrm{Dif}^{j-1}_R \vee(\mathrm{Set}^j_L \wedge(\mathrm{Val}^j_L \leftrightarrow \mathrm{Val}^j_\tau)))$ is falsified so $(\mathrm{Val}^j_B, \mathrm{Set}^j_B) = (\mathrm{Val}^j_L, \mathrm{Set}^j_L)$ and $B \wedge L \wedge \mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^{j-1}_L \rightarrow \mathrm{Set}^i_B \wedge \mathrm{Val}^i_B$. If $u_j \in \mathsf{dom}(\sigma)$ Set^j_L contradicts $\mathrm{Dif}^j_L \wedge \neg \mathrm{Dif}^{j-1}_L$, so this scenario does not occur. If $u_j \in \mathsf{dom}(\xi)$ $\mathrm{Dif}^{j-1}_L \wedge \neg \mathrm{Set}^j_L$ is falsified by

$\neg\,\mathrm{Dif}_L^{j-1}$. $\neg\,\mathrm{Dif}_L^{j-1} \wedge(\mathrm{Dif}_R^{j-1} \vee\neg\,\mathrm{Set}_L^j)$ is falsified by Set_L^j so $(\mathrm{Val}_B^j, \mathrm{Set}_B^j) = (\mathrm{Val}_L^j, \mathrm{Set}_L^j)$ and $B \wedge L \wedge \mathrm{Dif}_L^j \wedge\neg\,\mathrm{Dif}_L^{j-1} \rightarrow \mathrm{Set}_B^i \wedge \mathrm{Val}_B^i$.

If $\mathrm{Set}_B^j \wedge\neg\,\mathrm{Val}_B^j \in \mathrm{anno}_{y,B}(\alpha \circ \xi)$ $0/u_j \in (\alpha \circ \xi)$. As instantiate is only done by $*$ then $0/u_j \in (\alpha)$. So it follows $\mathrm{Set}_L^j \wedge\neg\,\mathrm{Val}_L^j \in \mathrm{anno}_{y,L}(\alpha)$ If $u_j \notin \mathsf{dom}(\tau \sqcup \sigma \sqcup \xi)$, $\neg\,\mathrm{Dif}_L^{j-1} \wedge(\mathrm{Dif}_R^{j-1} \vee\neg\,\mathrm{Set}_L^j$ is falsified so $(\mathrm{Val}_B^j, \mathrm{Set}_B^j) = (\mathrm{Val}_L^j, \mathrm{Set}_L^j)$ and $B \wedge L \wedge \mathrm{Dif}_L^j \wedge\neg\,\mathrm{Dif}_L^{j-1} \rightarrow \mathrm{Set}_B^i \wedge\neg\,\mathrm{Val}_B^i$. If $u_j \in \mathsf{dom}(\tau)$, $\mathrm{Dif}_L^j \wedge\neg\,\mathrm{Dif}_L^{j-1} \wedge \mathrm{Set}_L^j \wedge\neg\,\mathrm{Val}_L^j$ means that Val_τ^j and so $\neg\,\mathrm{Dif}_L^{j-1} \wedge(\mathrm{Dif}_R^{j-1} \vee(\mathrm{Set}_L^j \wedge(\mathrm{Val}_L^j \leftrightarrow \mathrm{Val}_\tau^j)))$ is falsified so $(\mathrm{Val}_B^j, \mathrm{Set}_B^j) = (\mathrm{Val}_L^j, \mathrm{Set}_L^j)$ and $B \wedge L \wedge \mathrm{Dif}_L^j \wedge\neg\,\mathrm{Dif}_L^{j-1} \rightarrow \mathrm{Set}_B^i \wedge\neg\,\mathrm{Val}_B^i$. If $u_j \in \mathsf{dom}(\sigma)$ Set_L^j contradicts $\mathrm{Dif}_L^j \wedge\neg\,\mathrm{Dif}_L^{j-1}$, so this scenario does not occur. If $u_j \in \mathsf{dom}(\xi)$ $\mathrm{Dif}_L^{j-1} \wedge\neg\,\mathrm{Set}_L^j$ is falsified by $\neg\,\mathrm{Dif}_L^{j-1}$. $\neg\,\mathrm{Dif}_L^{j-1} \wedge(\mathrm{Dif}_R^{j-1} \vee\neg\,\mathrm{Set}_L^j)$ is falsified by Set_L^j so $(\mathrm{Val}_B^j, \mathrm{Set}_B^j) = (\mathrm{Val}_L^j, \mathrm{Set}_L^j)$ and $B \wedge L \wedge \mathrm{Dif}_L^j \wedge\neg\,\mathrm{Dif}_L^{j-1} \rightarrow \mathrm{Set}_B^i \wedge\neg\,\mathrm{Val}_B^i$.

Suppose $i < j$.

In this case $\neg\,\mathrm{Dif}_L^i, \neg\,\mathrm{Dif}_L^{i-1}, \neg\,\mathrm{Dif}_R^i, \neg\,\mathrm{Dif}_R^{i-1}$ are all true. We can see from Lemma 16 that $\mathrm{Set}_L^i \rightarrow \mathrm{Set}_B^i$ in all cases. We observe all the cases when Set_L^i is true and Val_B^i is not defined as Val_L^i. For $u_i \in \mathsf{dom}(\tau)$, this happens if $(\mathrm{Val}_L^i \leftrightarrow \mathrm{Val}_\tau^i)$, but then also $(\mathrm{Val}_R^i \leftrightarrow \mathrm{Val}_\tau^i)$ if $\neg\,\mathrm{Dif}_R^i$. For $u_i \in \mathsf{dom}(\sigma)$ if $\neg\,\mathrm{Dif}_L^{i-1} \wedge \mathrm{Set}_R^i \wedge(\mathrm{Dif}_R^{i-1} \vee \mathrm{Set}_L^i)$ then $\mathrm{Val}_B^i = \mathrm{Val}_R^i$, but this cannot happen if $\neg\,\mathrm{Dif}_L^i \wedge\neg\,\mathrm{Dif}_L^{i-1}$. So in all cases of $\neg\,\mathrm{Dif}_L^i, \neg\,\mathrm{Dif}_L^{i-1}, \neg\,\mathrm{Dif}_R^i, \neg\,\mathrm{Dif}_R^{i-1}, \mathrm{Set}_L^i$ we have $\mathrm{Val}_B^i = \mathrm{Val}_L^i$. This means that $\mathrm{Set}_B^i \rightarrow (u_i \leftrightarrow \mathrm{Val}_B^i) \rightarrow \mathrm{Set}_L^i \rightarrow (u_i \leftrightarrow \mathrm{Val}_L^i)$.

If $\neg\,\mathrm{Set}_B^i \in \mathrm{anno}_{y,B}(\alpha \circ \xi)$ then $u_j \notin \mathsf{dom}(\alpha \circ \xi)$ and so $u_j \notin \mathsf{dom}(\alpha)$ $u_j \notin \mathsf{dom}(\xi)$. So $\neg\,\mathrm{Set}_L^i \in \mathrm{anno}_{y,L}(\alpha)$ and $L \rightarrow \neg\,\mathrm{Set}_L^i$ $\neg\,\mathrm{Dif}_L^i, \neg\,\mathrm{Dif}_L^{i-1}$ means that $u_i \notin \mathsf{dom}(\tau \sqcup \sigma \sqcup \xi)$ From Lemma 16 we know $\neg\,\mathrm{Dif}_L^i \wedge\neg\,\mathrm{Dif}_R^i \rightarrow \neg\,\mathrm{Set}_B^i$. So $B \wedge L \wedge \mathrm{Dif}_L^j \wedge\neg\,\mathrm{Dif}_L^i \rightarrow \neg\,\mathrm{Set}_B^i$

If $\mathrm{Set}_B^i \in \mathrm{anno}_{y,B}(\alpha \circ \xi)$ Either $*/u_i \in \alpha$ or $u_i \notin \mathsf{dom}(\alpha)$ and $*/u_i \in \xi$ If $*/u_i \in \alpha$, then $\mathrm{Set}_L^i \in \mathrm{anno}_{y,L}(\alpha)$ and $L \rightarrow \mathrm{Set}_L^i$. By Lemma 16, u_i must be in $\mathsf{dom}(\tau)$ or $\mathsf{dom}(\sigma)$. In either case Set_B^i is true. So $B \wedge L \wedge \mathrm{Dif}_L^j \wedge\neg\,\mathrm{Dif}_L^i \rightarrow \mathrm{Set}_B^i$ If $u_i \notin \mathsf{dom}(\alpha)$ and $*/u_i \in \xi$, then $\neg\,\mathrm{Set}_L^i \in \mathrm{anno}_{y,L}(\alpha)$ and $L \rightarrow \neg\,\mathrm{Set}_L^i$ By Lemma 16, Set_B^i is true. So $B \wedge L \wedge \mathrm{Dif}_L^j \wedge\neg\,\mathrm{Dif}_L^i \rightarrow \mathrm{Set}_B^i$.

If $\mathrm{Set}_B^i \wedge \mathrm{Val}_B^i \in \mathrm{anno}_{y,B}(\alpha \circ \xi)$ then $1/u_i \in \alpha \circ \xi$, so it must be that $1/u_i \in \alpha$. And so $\mathrm{Set}_L^i \wedge \mathrm{Val}_L^i \in \mathrm{anno}_{y,L}(\alpha)$ By Lemma 16, u_i must be in $\mathsf{dom}(\tau)$ or $\mathsf{dom}(\sigma)$ In either case $(\mathrm{Val}_B^i, \mathrm{Set}_B^i)$=$(\mathrm{Val}_L^i, \mathrm{Set}_L^i)$. So $B \wedge L \wedge \mathrm{Dif}_L^j \wedge\neg\,\mathrm{Dif}_L^i \rightarrow \mathrm{Set}_B^i \wedge \mathrm{Val}_L^i$, because $L \rightarrow \mathrm{Set}_L^i \wedge \mathrm{Val}_L^i$

Likewise, if $\mathrm{Set}_B^i \wedge\neg\,\mathrm{Val}_B^i \in \mathrm{anno}_{y,B}(\alpha \circ \xi)$ then $0/u_i \in \alpha \circ \xi$, so it must be that $0/u_i \in \alpha$. And so $\mathrm{Set}_L^i \wedge\neg\,\mathrm{Val}_L^i \in \mathrm{anno}_{y,L}(\alpha)$ By Lemma 16,

u_i must be in $\mathsf{dom}(\tau)$ or $\mathsf{dom}(\sigma)$ In either case $(\mathrm{Val}^i_B, \mathrm{Set}^i_B)$=$(\mathrm{Val}^i_L, \mathrm{Set}^i_L)$. So $B \wedge L \wedge \mathrm{Dif}^j_L \wedge \neg\,\mathrm{Dif}^i_L \rightarrow \mathrm{Set}^i_B \wedge \neg\,\mathrm{Val}^i_L$, because $L \rightarrow \mathrm{Set}^i_L \wedge \neg\,\mathrm{Val}^i_L$

In all Dif^m_L **cases** $\mathrm{Set}^i_B \rightarrow (u_i \leftrightarrow \mathrm{Val}^i_B) \rightarrow \mathrm{Set}^i_L \rightarrow (u_i \leftrightarrow \mathrm{Val}^i_L)$ so then $B \wedge \mathrm{Dif}^m_L \rightarrow L$. We also have $B \wedge \mathrm{Dif}^m_L \wedge L \rightarrow \mathrm{anno}_{y,B}(\alpha \circ \xi)$. We also get $B \wedge \mathrm{Dif}^m_L \wedge L \rightarrow \mathrm{con}_B(y)$, from $L \rightarrow y$ so we can get $B \wedge \mathrm{Dif}^m_L \wedge L \rightarrow \mathrm{anno}_B(y^{\alpha \circ \xi})$, this can be put in a disjunction $B \wedge \mathrm{Dif}^m_L \wedge L \rightarrow \mathrm{con}_B(\mathsf{inst}(\xi, C_1))$, when $L \rightarrow \mathrm{con}_L(C_1)$ instead of $L \rightarrow \mathrm{con}_L(y^\alpha)$. This is simplified to $B \wedge \mathrm{Dif}^m_L \rightarrow \mathrm{con}_B(\mathsf{inst}(\xi, C_1))$ as $B \wedge \mathrm{Dif}^m_L \rightarrow L$.

Now we argue that $(R \rightarrow \mathrm{con}_R(y^\alpha))$ implies $(B \wedge \neg\,\mathrm{Dif}_L \wedge \mathrm{Dif}^m_R) \rightarrow \mathrm{con}_R(y^\alpha \circ \sigma)$.

Suppose $i > m$

$\mathrm{Dif}_L \wedge \mathrm{Dif}^m_R$ satisfies $\neg\,\mathrm{Dif}^m_L \wedge (\mathrm{Dif}^m_R \vee \neg x)$ so $(\mathrm{Val}^i_B, \mathrm{Set}^i_B)$=$(\mathrm{Val}^i_R, \mathrm{Set}^i_R)$ in all cases. This means that $(\mathrm{Set}^i_B \rightarrow (\mathrm{Val}^i_B \leftrightarrow u_i)) \rightarrow (\mathrm{Set}^i_R \rightarrow (\mathrm{Val}^i_R \leftrightarrow u_i))$.

If $\neg\,\mathrm{Set}^i_B \in \mathrm{con}_{y,B}(\alpha \circ \sigma)$ then $u_i \notin \mathsf{dom}(\alpha)$ and $u_i \notin \mathsf{dom}(\sigma)$ then $\neg\,\mathrm{Set}^i_R \in \mathrm{con}_{y,R}(\alpha)$ so $R \rightarrow \neg\,\mathrm{Set}^i_R$. and so $B \wedge R \wedge \neg\,\mathrm{Set}^i_B \wedge \neg\,\mathrm{Dif}^m_L \wedge \mathrm{Dif}^m_R \rightarrow \neg\,\mathrm{Set}^i_B$

If $\mathrm{Set}^i_B \in \mathrm{con}_{y,B}(\alpha \circ \sigma)$ then $u_i \in \mathsf{dom}(\alpha \circ \sigma)$ Which means either $u_i \in \mathsf{dom}(\alpha)$ or $u_i \notin \mathsf{dom}(\alpha)$ and $u_i \in \sigma$. But $u_i \notin \sigma$ because $i > m$. Since $u_i \in \mathsf{dom}(\alpha)$ $\mathrm{Set}^i_R \in \mathrm{con}_{y,R}(\alpha)$ and so $B \wedge R \wedge \neg\,\mathrm{Set}^i_B \wedge \neg\,\mathrm{Dif}^m_L \wedge \mathrm{Dif}^m_R \rightarrow \mathrm{Set}^i_B$

If $\mathrm{Set}^i_B \wedge \mathrm{Val}^i_B \in \mathrm{con}_{y,B}(\alpha \circ \sigma)$ then $1/u_i \in \alpha \circ \sigma$ Which means $1/u_i \in \alpha$ $\mathrm{Set}^i_R \wedge \mathrm{Val}^i_R \in \mathrm{con}_{y,R}(\alpha)$ and so $B \wedge R \wedge \neg\,\mathrm{Set}^i_B \wedge \neg\,\mathrm{Dif}^m_L \wedge \mathrm{Dif}^m_R \rightarrow \mathrm{Set}^i_B \wedge \mathrm{Val}^i_B$

If $\mathrm{Set}^i_B \wedge \mathrm{Val}^i_B \in \mathrm{con}_{y,B}(\alpha \circ \sigma)$ then $0/u_i \in \alpha \circ \sigma$ Which means $0/u_i \in \alpha$ $\mathrm{Set}^i_R \wedge \neg\,\mathrm{Val}^i_R \in \mathrm{con}_{y,R}(\alpha)$ and so $B \wedge R \wedge \neg\,\mathrm{Set}^i_B \wedge \neg\,\mathrm{Dif}^m_L \wedge \mathrm{Dif}^m_R \rightarrow \mathrm{Set}^i_B \wedge \neg\,\mathrm{Val}^i_B$

Suppose $j < i \leq m$

In this case $\neg\,\mathrm{Dif}^{i-1}_L$, $\neg\,\mathrm{Dif}^i_L$, Dif^{i-1}_R and Dif^i_R are all true. If Set^i_R is true then

$\neg\,\mathrm{Dif}^{i-1}_L \wedge (\mathrm{Dif}^{i-1}_R \vee \neg\,\mathrm{Set}^i_L)$, $\neg\,\mathrm{Dif}^{i-1}_L \wedge (\mathrm{Dif}^{i-1}_R \vee (\mathrm{Set}^i_L \wedge (\mathrm{Val}^i_L \leftrightarrow \mathrm{Val}^i_\tau)))$, $\neg\,\mathrm{Dif}^{i-1}_L \wedge \mathrm{Set}^i_R \wedge (\mathrm{Dif}^{i-1}_R \vee \mathrm{Set}^i_L)$ and $\neg\,\mathrm{Dif}^{i-1}_L \wedge (\mathrm{Dif}^{i-1}_R \vee \neg\,\mathrm{Set}^i_L)$ are all satisfied. So $(\mathrm{Val}^i_B, \mathrm{Set}^i_B)$=$(\mathrm{Val}^i_R, \mathrm{Set}^i_R)$ whenever Set^i_R is true. This means that $(\mathrm{Set}^i_B \rightarrow (\mathrm{Val}^i_B \leftrightarrow u_i)) \rightarrow (\mathrm{Set}^i_R \rightarrow (\mathrm{Val}^i_R \leftrightarrow u_i))$.

If $\neg\,\mathrm{Set}^i_B \in \mathrm{con}_{y,B}(\alpha \circ \sigma)$ then $u_i \notin \mathsf{dom}(\alpha)$ and $u_i \notin \mathsf{dom}(\sigma)$ then $\neg\,\mathrm{Set}^i_R \in \mathrm{con}_{y,R}(\alpha)$ so $R \rightarrow \neg\,\mathrm{Set}^i_R$. When $\neg\,\mathrm{Dif}^{i-1}_L$ and Dif^{i-1}_R and $u_i \notin \mathsf{dom}(\sigma)$ then $(\mathrm{Val}^i_B, \mathrm{Set}^i_B)$=$(\mathrm{Val}^i_R, \mathrm{Set}^i_R)$, so $B \wedge \neg\,\mathrm{Dif}^j_L \wedge \neg\,\mathrm{Dif}^i_L \wedge \mathrm{Dif}^j_R \wedge \mathrm{Dif}^i_R \wedge R \rightarrow \neg\,\mathrm{Set}^i_B$.

If $\mathrm{Set}^i_B \in \mathrm{con}_{y,B}(\alpha \circ \sigma)$ then $*/u_i \in \alpha \circ \sigma$ So either $*/u_i \in \alpha$ or $*/u_i \in \sigma$ and $u_i \notin \mathsf{dom}(\alpha)$ If $*/u_i \in \alpha$ then $\mathrm{Set}^i_R \in \mathrm{con}_{y,R}(\alpha)$

and when Set^i_R is true then $(\mathrm{Val}^i_B, \mathrm{Set}^i_B)$=$(\mathrm{Val}^i_R, \mathrm{Set}^i_R)$ so $R \to \mathrm{Set}^i_R$ implies $B \wedge \neg\,\mathrm{Dif}^j_L \wedge\neg\,\mathrm{Dif}^i_L \wedge \mathrm{Dif}^j_R \wedge \mathrm{Dif}^i_R \wedge R \to \mathrm{Set}^i_B$ If $*/u_i \in \sigma$ and $u_i \notin \mathsf{dom}(\alpha)$ $\neg\,\mathrm{Set}^i_R \in \mathrm{con}_{y,R}(\alpha)$ $\neg\,\mathrm{Dif}^{i-1}_L \wedge \mathrm{Dif}^{i-1}_R \wedge\neg\,\mathrm{Set}^i_R$ is satisfied so$(\mathrm{Val}^i_B, \mathrm{Set}^i_B)$=$(0,1)$ therefore $B \wedge \neg\,\mathrm{Dif}^j_L \wedge\neg\,\mathrm{Dif}^i_L \wedge \mathrm{Dif}^j_R \wedge \mathrm{Dif}^i_R \wedge R \to \mathrm{Set}^i_B$

If $\mathrm{Set}^i_B \wedge \mathrm{Val}^i_B \in \mathrm{con}_{y,B}(\alpha \circ \sigma)$ then $1/u_i \in \alpha \circ \sigma$. and it must be that $1/u_i \in \alpha$ and so $\mathrm{Set}^i_R \wedge \mathrm{Val}^i_R \in \mathrm{con}_{y,R}(\alpha)$ and when Set^i_R is true then $(\mathrm{Val}^i_B, \mathrm{Set}^i_B)$=$(\mathrm{Val}^i_R, \mathrm{Set}^i_R)$ so $R \to \mathrm{Set}^i_R \wedge \mathrm{Val}^i_R$ implies $B \wedge \neg\,\mathrm{Dif}^j_L \wedge\neg\,\mathrm{Dif}^i_L \wedge \mathrm{Dif}^j_R \wedge \mathrm{Dif}^i_R \wedge R \to \mathrm{Set}^i_B \wedge \mathrm{Val}^i_B$

If $\mathrm{Set}^i_B \wedge\neg\,\mathrm{Val}^i_B \in \mathrm{con}_{y,B}(\alpha \circ \sigma)$ then $0/u_i \in \alpha \circ \sigma$. and it must be that $0/u_i \in \alpha$ and so $\mathrm{Set}^i_R \wedge \mathrm{Val}^i_R \in \mathrm{con}_{y,R}(\alpha)$ and when Set^i_R is true then $(\mathrm{Val}^i_B, \mathrm{Set}^i_B)$=$(\mathrm{Val}^i_R, \mathrm{Set}^i_R)$ so $R \to \mathrm{Set}^i_R \wedge\neg\,\mathrm{Val}^i_R$ implies $B \wedge \neg\,\mathrm{Dif}^j_L \wedge\neg\,\mathrm{Dif}^i_L \wedge \mathrm{Dif}^j_R \wedge \mathrm{Dif}^i_R \wedge R \to \mathrm{Set}^i_B \wedge\neg\,\mathrm{Val}^i_B$

Suppose $i = j$

In this case $\neg\,\mathrm{Dif}^{j-1}_L$, $\neg\,\mathrm{Dif}^j_L$, $\neg\,\mathrm{Dif}^{j-1}_R$ and Dif^j_R. If Set^j_R then either $(\mathrm{Val}^i_B, \mathrm{Set}^i_B)$=$(\mathrm{Val}^i_R, \mathrm{Set}^i_R)$ or $(\mathrm{Val}^i_B, \mathrm{Set}^i_B)$=$(\mathrm{Val}^i_L, \mathrm{Set}^i_L)$. We will argue that $(\mathrm{Val}^i_B, \mathrm{Set}^i_B)$=$(\mathrm{Val}^i_L, \mathrm{Set}^i_L)$ is not chosen because of $\neg\,\mathrm{Dif}^j_L$ and Eq_R $\neg\,\mathrm{Dif}^{i-1}_L \wedge(\mathrm{Dif}^{i-1}_R \vee\neg\,\mathrm{Set}^i_L)$ cannot be falsified because Set^i_L being true would contradict $\neg\,\mathrm{Dif}^j_L$. Likewise $\neg\,\mathrm{Dif}^{i-1}_L \wedge(\mathrm{Dif}^{i-1}_R \vee(\mathrm{Set}^i_L \wedge(\mathrm{Val}^i_L \leftrightarrow \mathrm{Val}^i_\tau)))(\mathrm{Val}^i_L, \mathrm{Set}^i_L)$ cannot be falsified as $(\mathrm{Set}^i_L \wedge(\mathrm{Val}^i_L \leftrightarrow \mathrm{Val}^i_\tau))$ being false would contradict $\neg\,\mathrm{Dif}^j_L$. If $u_i \in \mathsf{dom}(\sigma)$ then $\neg\,\mathrm{Dif}^{i-1}_L \wedge \mathrm{Dif}^{i-1}_R \wedge\neg\,\mathrm{Set}^i_R$ is false and $\neg\,\mathrm{Dif}^{i-1}_L \wedge \mathrm{Set}^i_R \wedge(\mathrm{Dif}^{i-1}_R \vee \mathrm{Set}^i_L$ is true. Likewise if $u_i \in \mathsf{dom}(\xi)$ then $\mathrm{Dif}^{i-1}_L \wedge\neg\,\mathrm{Set}^i_L$ is false and $\neg\,\mathrm{Dif}^{i-1}_L \wedge(\mathrm{Dif}^{i-1}_R \vee\neg\,\mathrm{Set}^i_L)$ is true. The result is that $(\mathrm{Set}^i_B \to (\mathrm{Val}^i_B \leftrightarrow u_i)) \to (\mathrm{Set}^i_R \to (\mathrm{Val}^i_R \leftrightarrow u_i))$.

If $\neg\,\mathrm{Set}^j_B \in \mathrm{con}_{y,B}(\alpha \circ \sigma)$ then $u_i \notin \mathsf{dom}(\alpha \circ \sigma)$, which means $u_i \notin \mathsf{dom}(\alpha)$ and $u_i \notin \mathsf{dom}(\sigma)$. So $\neg\,\mathrm{Set}^j_R \in \mathrm{con}_{y,R}(\alpha)$ and thus $R \to \neg\,\mathrm{Set}^j_R$ If $u_j \in \mathsf{dom}(\tau)$ We argue that $\neg\,\mathrm{Dif}^{j-1}_L \wedge(\mathrm{Dif}^{j-1}_R \vee(\mathrm{Set}^j_L \wedge(\mathrm{Val}^j_L \leftrightarrow \mathrm{Val}^{ij}_\tau)))$ is satisfied because of $\neg\,\mathrm{Dif}^i_L$. Hence $(\mathrm{Val}^i_B, \mathrm{Set}^i_B)$=$(\mathrm{Val}^i_R, \mathrm{Set}^i_R)$ and so $B \wedge \neg\,\mathrm{Dif}^{j-1}_L \wedge\neg\,\mathrm{Dif}^j_L \wedge\neg\,\mathrm{Dif}^{j-1}_R \wedge \mathrm{Dif}^j_R \wedge L \to \neg\,\mathrm{Set}^j_B$

If $u_j \in \mathsf{dom}(\xi)$ We argue that $\neg\,\mathrm{Dif}^{i-1}_L \wedge(\mathrm{Dif}^{i-1}_R \vee\neg\,\mathrm{Set}^i_L)$ is satisfied because of $\neg\,\mathrm{Dif}^i_L$ which insists on $\neg\,\mathrm{Set}^i_L$. Hence $(\mathrm{Val}^i_B, \mathrm{Set}^i_B)$=$(\mathrm{Val}^i_R, \mathrm{Set}^i_R)$ and so $B \wedge \neg\,\mathrm{Dif}^{j-1}_L \wedge\neg\,\mathrm{Dif}^j_L \wedge\neg\,\mathrm{Dif}^{j-1}_R \wedge \mathrm{Dif}^j_R \wedge L \to \neg\,\mathrm{Set}^j_B$.

' If $u_j \notin \mathsf{dom}(\tau \sqcup \sigma \sqcup \xi)$ We argue that $\neg\,\mathrm{Dif}^{i-1}_L \wedge(\mathrm{Dif}^{i-1}_R \vee\neg\,\mathrm{Set}^i_L)$ is satisfied because of $\neg\,\mathrm{Dif}^i_L$. Hence $(\mathrm{Val}^i_B, \mathrm{Set}^i_B)$=$(\mathrm{Val}^i_R, \mathrm{Set}^i_R)$ and so $B \wedge \neg\,\mathrm{Dif}^{j-1}_L \wedge\neg\,\mathrm{Dif}^j_L \wedge\neg\,\mathrm{Dif}^{j-1}_R \wedge \mathrm{Dif}^j_R \wedge L \to \neg\,\mathrm{Set}^j_B$.

If $\mathrm{Set}^j_B \in \mathrm{con}_{y,B}(\alpha \circ \sigma)$, so $*/u_j \in (\alpha \circ \sigma)$. So either $*/u_j \in \alpha$ or $*/u_j \notin \alpha$ and $*/u_j \in \sigma$. If $*/u_j \in \alpha$ then $\mathrm{Set}^j_R \in \mathrm{con}_{y,R}(\alpha)$

and $R \to \mathrm{Set}_R^j$ When Set_R^j is true we know $(\mathrm{Val}_B^j, \mathrm{Set}_B^j)$=$(\mathrm{Val}_R^j, \mathrm{Set}_R^j)$ and so $B \wedge \neg \mathrm{Dif}_L^{j-1} \wedge \neg \mathrm{Dif}_L^j \wedge \neg \mathrm{Dif}_R^{j-1} \wedge \mathrm{Dif}_R^j \wedge L \to \mathrm{Set}_B^j$ If $*/u_j \notin \alpha$ and $*/u_j \in \sigma$ So $\neg \mathrm{Set}_R^j \in \mathrm{con}_{y,R}(\alpha)$ and thus $R \to \neg \mathrm{Set}_R^j$ $\neg \mathrm{Dif}_L^{i-1} \wedge \mathrm{Set}_R^j \wedge (\mathrm{Dif}_R^{j-1} \vee \mathrm{Set}_L^j)$ is falsified. So $(\mathrm{Val}_B^j, \mathrm{Set}_B^j)$=$(\mathrm{Val}_L^j, \mathrm{Set}_L^i)$ But because $\neg \mathrm{Dif}_L^j$ we know that Set_L^i is true therefore $B \wedge \neg \mathrm{Dif}_L^{j-1} \wedge \neg \mathrm{Dif}_L^j \wedge \neg \mathrm{Dif}_R^{j-1} \wedge \mathrm{Dif}_R^j \wedge L \to \mathrm{Set}_B^j$

If $\mathrm{Set}_B^j \wedge \mathrm{Val}_B^j \in \mathrm{con}_{y,B}(\alpha \circ \sigma)$, so $1/u_j \in (\alpha \circ \sigma)$. So it must be that $1/u_j \in \alpha$ And so $\mathrm{Set}_R^j \wedge \mathrm{Val}_R^j \in \mathrm{con}_{y,R}(\alpha)$ and thus $R \to \neg \mathrm{Set}_R^j$ since Set_R^j is true we know that $(\mathrm{Val}_B^j, \mathrm{Set}_B^j)$=$(\mathrm{Val}_R^j, \mathrm{Set}_R^j)$ and so $B \wedge \neg \mathrm{Dif}_L^{j-1} \wedge \neg \mathrm{Dif}_L^j \wedge \neg \mathrm{Dif}_R^{j-1} \wedge \mathrm{Dif}_R^j \wedge L \to \mathrm{Set}_B^j \wedge \mathrm{Val}_B^j$

If $\mathrm{Set}_B^j \wedge \neg \mathrm{Val}_B^j \in \mathrm{con}_{y,B}(\alpha \circ \sigma)$, so $0/u_j \in (\alpha \circ \sigma)$. So it must be that $0/u_j \in \alpha$ And so $\mathrm{Set}_R^j \wedge \neg \mathrm{Val}_R^j \in \mathrm{con}_{y,R}(\alpha)$ and thus $R \to \neg \mathrm{Set}_R^j$ since Set_R^j is true we know that $(\mathrm{Val}_B^j, \mathrm{Set}_B^j)$=$(\mathrm{Val}_R^j, \mathrm{Set}_R^j)$ and so $B \wedge \neg \mathrm{Dif}_L^{j-1} \wedge \neg \mathrm{Dif}_L^j \wedge \neg \mathrm{Dif}_R^{j-1} \wedge \mathrm{Dif}_R^j \wedge L \to \mathrm{Set}_B^j \wedge \mathrm{Val}_B^j$

Suppose $i < j$.

In this case $\neg \mathrm{Dif}_L^i, \neg \mathrm{Dif}_L^{i-1}, \neg \mathrm{Dif}_R^i, \neg \mathrm{Dif}_R^{i-1}$ are all true. We can see from Lemma 16 that $\mathrm{Set}_R^i \to \mathrm{Set}_B^i$ in all cases. We observe all the cases when Set_R^i is true and Val_B^i is not defined as Val_R^i and show they cannot happen

For $u_i \notin \mathsf{dom}(\tau \sqcup \sigma \sqcup \xi)$, if $\neg \mathrm{Dif}_L^{i-1} \wedge (\mathrm{Dif}_R^{i-1} \vee \neg \mathrm{Set}_L^i)$ is false then Set_L^i must be true, but this conflicts with $\neg \mathrm{Dif}_L^i, \neg \mathrm{Dif}_L^{i-1}$. For $u_i \in \mathsf{dom}(\tau)$ if $\neg \mathrm{Dif}_L^{i-1} \wedge (\mathrm{Dif}_R^{i-1} \vee (\mathrm{Set}_L^i \wedge (\mathrm{Val}_L^i \leftrightarrow \mathrm{Val}_\tau^i)))$ is false then $\mathrm{Set}_L^i \to (\mathrm{Val}_L^i \oplus \mathrm{Val}_\tau^i)$ contradicting $\neg \mathrm{Dif}_L^i, \neg \mathrm{Dif}_L^{i-1}$ For $u_i \in \mathsf{dom}(\sigma)$ if $\neg \mathrm{Dif}_L^{i-1} \wedge \mathrm{Set}_R^i \wedge (\mathrm{Dif}_R^{i-1} \vee \mathrm{Set}_L^i)$ is false the then Set_L^i is false contradicting $\neg \mathrm{Dif}_L^i, \neg \mathrm{Dif}_L^{i-1}$. For $u_i \in \mathsf{dom}(\xi)$ if $\neg \mathrm{Dif}_L^{i-1} \wedge (\mathrm{Dif}_R^{i-1} \vee \neg \mathrm{Set}_L^i)$ is false then Set_L^i is true but in $\mathsf{dom}(\xi)$ this contradicts $\neg \mathrm{Dif}_L^i, \neg \mathrm{Dif}_L^{i-1}$. Therefore $(\mathrm{Set}_B^i \to (\mathrm{Val}_B^i \leftrightarrow u_i)) \to (\mathrm{Set}_R^i \to (\mathrm{Val}_R^i \leftrightarrow u_i))$

If $\neg \mathrm{Set}_B^i \in \mathrm{con}_{y,B}(\alpha \circ \sigma)$ then $u_j \notin \mathsf{dom}(\alpha \circ \sigma)$ and so $u_j \notin \mathsf{dom}(\alpha)$ $u_j \notin \mathsf{dom}(\sigma)$. So $\neg \mathrm{Set}_R^i \in \mathrm{con}_{y,R}(\alpha)$ and $R \to \neg \mathrm{Set}_R^i$ $\neg \mathrm{Dif}_R^i, \neg \mathrm{Dif}_R^{i-1}$ means that $u_i \notin \mathsf{dom}(\tau \sqcup \sigma \sqcup \xi)$ From Lemma 16 we know $\neg \mathrm{Dif}_L^i \wedge \neg \mathrm{Dif}_R^i \to \neg \mathrm{Set}_B^i$. So $B \wedge R \wedge \mathrm{Dif}_R^j \wedge \neg \mathrm{Dif}_L^i \to \neg \mathrm{Set}_B^i$

If $\mathrm{Set}_B^i \in \mathrm{con}_{y,B}(\alpha \circ \sigma)$ Either $*/u_i \in \alpha$ or $u_i \notin \mathsf{dom}(\alpha)$ and $*/u_i \in \sigma$ If $*/u_i \in \alpha$, then $\mathrm{Set}_R^i \in \mathrm{con}_{y,R}(\alpha)$ and $R \to \mathrm{Set}_R^i$. By Lemma 16, u_i must be in $\mathsf{dom}(\tau)$ or $\mathsf{dom}(\xi)$. In either case Set_B^i is true. So $B \wedge R \wedge \mathrm{Dif}_R^j \wedge \neg \mathrm{Dif}_L^i \to \mathrm{Set}_B^i$ If $u_i \notin \mathsf{dom}(\alpha)$ and $*/u_i \in \sigma$, then $\neg \mathrm{Set}_R^i \in \mathrm{con}_{y,R}(\alpha)$ and $R \to \neg \mathrm{Set}_R^i$ By Lemma 16, Set_R^i is true. So $B \wedge R \wedge \mathrm{Dif}_R^j \wedge \neg \mathrm{Dif}_L^i \to \mathrm{Set}_B^i$

If $\mathrm{Set}^i_B \wedge \mathrm{Val}^i_B \in \mathrm{con}_{y,B}(\alpha \circ \sigma)$ then $1/u_i \in \alpha \circ \sigma$, so it must be that $1/u_i \in \alpha$. And so $\mathrm{Set}^i_R \wedge \mathrm{Val}^i_R \in \mathrm{con}_{y,R}(\alpha)$ By Lemma 16, u_i must be in $\mathsf{dom}(\tau)$ or $\mathsf{dom}(\xi)$ In either case $(\mathrm{Val}^i_B, \mathrm{Set}^i_B)$=$(\mathrm{Val}^i_R, \mathrm{Set}^i_R)$. So $B \wedge R \wedge \mathrm{Dif}^j_R \wedge \neg \mathrm{Dif}^i_R \rightarrow \mathrm{Set}^i_B \wedge \mathrm{Val}^i_B$, because $R \rightarrow \mathrm{Set}^i_R \wedge \mathrm{Val}^i_R$

Likewise, if $\mathrm{Set}^i_B \wedge \neg u_i \in \mathrm{con}_{y,B}(\alpha \circ \sigma)$ then $0/u_i \in \alpha \circ \sigma$, so it must be that $0/u_i \in \alpha$. And so $\mathrm{Set}^i_R \wedge \mathrm{Val}^i_R \in \mathrm{con}_{y,R}(\alpha)$ By Lemma 16, u_i must be in $\mathsf{dom}(\tau)$ or $\mathsf{dom}(\xi)$ In either case $(\mathrm{Val}^i_B, \mathrm{Set}^i_B)$=$(\mathrm{Val}^i_R, \mathrm{Set}^i_R)$. So $B \wedge R \wedge \mathrm{Dif}^j_R \wedge \neg \mathrm{Dif}^i_R \rightarrow \mathrm{Set}^i_B \wedge \neg u_i$, because $R \rightarrow \mathrm{Set}^i_R \wedge \neg u_i$. With that we conclude all cases in R and argue similarly to L.

Lemma 18. *Suppose $L \rightarrow \mathrm{con}_L(C_1 \vee \neg x^\tau)$ and $R \rightarrow \mathrm{con}_L(C_1 \vee x^\tau)$. The following propositions are true and have short Extended Frege proofs.*

- $B \wedge \neg \mathrm{Dif}^m_L \wedge \neg \mathrm{Dif}^m_R \rightarrow \mathrm{con}_B(\mathit{inst}(\xi, C_1)) \vee \neg x$
- $B \wedge \neg \mathrm{Dif}^m_L \wedge \neg \mathrm{Dif}^m_R \rightarrow \mathrm{con}_B(\mathit{inst}(\sigma, C_2)) \vee x$

Proof. Suppose that $L \rightarrow \mathrm{con}_L(y^\alpha)$, we will show that $B \wedge \neg \mathrm{Dif}^m_L \wedge \neg \mathrm{Dif}^m_R \rightarrow \mathrm{con}_L(\mathsf{inst}(\xi, y^\alpha))$.

We show first that $\mathrm{Set}^i_L \wedge \neg \mathrm{Dif}^i_L \wedge \neg \mathrm{Dif}^i_R \rightarrow \mathrm{Set}^i_B \wedge (\mathrm{Val}^i_B \leftrightarrow \mathrm{Val}^i_L)$ this is true in each $i : 1 \leq i \leq m$ by observing each case in Lemma 16. For $i > m$, $\neg \mathrm{Dif}^m_L \wedge \neg \mathrm{Dif}^m_R \wedge x \rightarrow (\mathrm{Set}^i_L \leftrightarrow \mathrm{Set}^i_B) \wedge (\mathrm{Val}^i_B \leftrightarrow \mathrm{Val}^i_L))$. So for all $i$either $\neg \mathrm{Set}^i_L$ or $\mathrm{Set}^i_B \wedge (\mathrm{Val}^i_B \leftrightarrow \mathrm{Val}^i_L)$ when $\neg \mathrm{Dif}^i_L \wedge \neg \mathrm{Dif}^i_R$.

This we can use to show $B \wedge \neg \mathrm{Dif}^i_L \wedge \neg \mathrm{Dif}^i_R \wedge x \rightarrow L$ by taking a conjunction of all these. We then can derive $(L \rightarrow y) \rightarrow (B \wedge \neg \mathrm{Dif}^m_L \wedge \neg \mathrm{Dif}^m_R \wedge x \rightarrow y)$ for existential literal y.

We still have to show that $(L \rightarrow \mathrm{con}_{y,L}(\alpha)) \rightarrow (B \wedge \neg \mathrm{Dif}^m_L \wedge \neg \mathrm{Dif}^m_R \wedge x \rightarrow \mathrm{con}_{y,L}(\alpha \circ \xi))$ for y's annotation α. We next show that $\neg \mathrm{Set}^i_L \wedge \neg \mathrm{Dif}^i_L \wedge \neg \mathrm{Dif}^i_R \rightarrow \neg \mathrm{Set}^i_B$ when $u_i \notin \mathsf{dom}(\xi)$. We can do this by simply observing the lines in Lemma 16 when $\neg \mathrm{Set}^i_L$ is permitted.

And finally we show $\neg \mathrm{Set}^i_L \wedge \neg \mathrm{Dif}^i_L \wedge \neg \mathrm{Dif}^i_R \rightarrow \mathrm{Set}^i_B$ when $u_i \in \mathsf{dom}(\xi)$.

Remembering that $\neg \mathrm{Dif}^m_S \rightarrow \neg \mathrm{Dif}^i_S$ for $S \in \{L, R\}$ and $1 \leq i \leq m$. We can now know that if L satisfies $\mathrm{con}_{y,L}(\alpha)$ then $\neg \mathrm{Dif}^m_L \wedge \neg \mathrm{Dif}^m_L \wedge x$ will force B to satisfy $\mathrm{con}_{y,L}(\alpha \circ \xi)$ and we can prove this in eFrege as

$$(L \rightarrow \underset{y,L}{\mathrm{con}}(\alpha)) \rightarrow (B \wedge \neg \underset{L}{\overset{m}{\mathrm{Dif}}} \wedge \neg \underset{R}{\overset{m}{\mathrm{Dif}}} \wedge x \rightarrow \underset{y,B}{\mathrm{con}}(\alpha \circ \xi))$$

Adding $(L \rightarrow y) \rightarrow (B \wedge \neg \mathrm{Dif}^m_L \wedge \neg \mathrm{Dif}^m_R \wedge x \rightarrow y$ and for every literal $y^\alpha \in C_1$and annotation in C_1 we can assemble

$$(L \rightarrow \underset{y^L}{\mathrm{con}}(y^\alpha)) \rightarrow (B \wedge \neg \underset{L}{\overset{m}{\mathrm{Dif}}} \wedge \neg \underset{R}{\overset{m}{\mathrm{Dif}}} \wedge x \rightarrow \underset{B}{\mathrm{con}}(\mathsf{inst}(\xi, y^\alpha)))$$

Using $\text{con}_B(\neg x^{\tau \sqcup \sigma \sqcup \xi}) \rightarrow \neg x$ we can get

$$(L \rightarrow \underset{L}{\text{con}}(C_1 \vee x) \rightarrow (B \wedge \neg \underset{L}{\overset{m}{\text{Dif}}} \wedge \neg \underset{R}{\overset{m}{\text{Dif}}} \wedge x \rightarrow \underset{B}{\text{con}}(\mathsf{inst}(\xi, C_1)))$$

And symmetrically we can make a derivation of

$$(L \rightarrow \underset{R}{\text{con}}(C_2 \vee \neg x) \rightarrow (B \wedge \neg \underset{L}{\overset{m}{\text{Dif}}} \wedge \neg \underset{R}{\overset{m}{\text{Dif}}} \wedge \neg x \rightarrow \underset{B}{\text{con}}(\mathsf{inst}(\sigma, C_2)))$$

The proofs here are polynomial, in this proof section we argue for each literal in the clause, and for each universal variable, but also refer to Lemmas 16 and 13 which have linear proofs. So we have cubic size proofs in the worst case or more specifically $O(wn^2)$, where w is the number of literals in the derived clause $\mathsf{inst}(\sigma, C_2) \cup \mathsf{inst}(\xi, C_2)$.

Lemma 19. *Suppose* $L \rightarrow \text{con}_L(C_1 \vee \neg x^{\tau \sqcup \sigma})$ *and* $R \rightarrow \text{con}_L(C_1 \vee x^{\tau \sqcup \xi})$ *then* $B \rightarrow \text{con}_B(\mathsf{inst}(\xi, C_1) \vee \mathsf{inst}(\sigma, C_2))$ *has a short* eFrege *proof.*

Proof. $B \wedge \text{Dif}_L^m \rightarrow \text{con}_B(\mathsf{inst}(\xi, C_1))$, $B \wedge \neg \text{Dif}_L^m \wedge \text{Dif}_R \rightarrow \text{con}_B(\mathsf{inst}(\sigma, C_2))$, and $B \wedge \neg \text{Dif}_L^m \wedge \neg \text{Dif}_R \rightarrow \text{con}_B(\mathsf{inst}(\xi, C_1) \vee \mathsf{inst}(\sigma, C_2))$ and we can resolve on Dif_L^m and Dif_R^m

7.3 Proof of Simulation of LQU$^+$-Res

Lemmas

Lemma 20. *For* $0 < j \leq m$ *the following propositions have short derivations in Extended Frege:*

- $\text{Dif}_L^j \rightarrow \bigvee_{i=1}^{j} \text{Dif}_L^i \wedge \neg \text{Dif}_L^{i-1}$
- $\text{Dif}_R^j \rightarrow \bigvee_{i=1}^{j} \text{Dif}_R^i \wedge \neg \text{Dif}_R^{i-1}$
- $\neg \text{Eq}_{L,V_1}^j \rightarrow \bigvee_{i=1}^{j} \neg \text{Eq}_{L,V_1}^i \wedge \text{Eq}_{L,V_1}^{i-1}$
- $\neg \text{Eq}_{R,V_2}^j \rightarrow \bigvee_{i=1}^{j} \neg \text{Eq}_{R,V_2}^i \wedge \text{Eq}_{R,V_2}^{i-1}$

Proof. The proof of Lemma 2 still works despite the modifications to definition.

Lemma 21. *For* $0 \leq i \leq j \leq m$ *the following propositions that describe the monotonicity of* Dif *and* Eq *have short derivations in Extended Frege:*

- $\text{Dif}_L^i \rightarrow \text{Dif}_L^j$
- $\text{Dif}_R^i \rightarrow \text{Dif}_R^j$

– $\neg\,\mathrm{Eq}^i_{f=g} \to \neg\,\mathrm{Eq}^j_{f=g}$

Proof. The proofs of Lemma 3 still work despite the modifications to definition.

Lemma 22. *For any $0 \le j \le m$ the following propositions are true and have a short Extended Frege proof.*

– $\neg\,\mathrm{Dif}^j_L \wedge \neg\,\mathrm{Dif}^j_R \to \mathrm{Eq}^j_{L,V_1}$
– $\neg\,\mathrm{Dif}^j_L \wedge \neg\,\mathrm{Dif}^j_R \to \mathrm{Eq}^j_{R,V_2}$
– $\neg\,\mathrm{Dif}^j_L \wedge \neg\,\mathrm{Dif}^j_R \to (\neg\,\mathrm{Set}^j_B \wedge \neg\,\mathrm{Set}^j_L \wedge \neg\,\mathrm{Set}^j_R)$ *when* $u^*_j \notin C_1 \vee C_2$.
– $\neg\,\mathrm{Dif}^j_L \wedge \neg\,\mathrm{Dif}^j_R \to (\mathrm{Set}^j_B \wedge \mathrm{Set}^j_L \wedge \neg\,\mathrm{Set}^j_R \wedge (\mathrm{Val}^j_B \leftrightarrow \mathrm{Val}^j_L))$ *when* $u^*_j \in C_1$.
– $\neg\,\mathrm{Dif}^j_L \wedge \neg\,\mathrm{Dif}^j_R \to (\mathrm{Set}^j_B \wedge \neg\,\mathrm{Set}^j_L \wedge \mathrm{Set}^j_R \wedge (\mathrm{Val}^j_B \leftrightarrow \mathrm{Val}^j_R))$ *when* $u^*_j \in C_2$.

Proof. We show that $\neg\,\mathrm{Eq}^{j+1}_{L,V_1} \to \neg\,\mathrm{Eq}^j_{L,V_1} \vee \neg\,\mathrm{Eq}^j_{R,V_2} \vee \mathrm{Dif}^{j+1}_L$ and $\neg\,\mathrm{Eq}^{j+1}_{R,V_2} \to \neg\,\mathrm{Eq}^j_{R,V_2} \vee \neg\,\mathrm{Eq}^j_{L,V_2} \vee \mathrm{Dif}^{j+1}_R$. Suppose $u^*_{j+1} \in V_1$ then $\neg\,\mathrm{Eq}^{j+1}_{L,V_1} \wedge \mathrm{Eq}^j_{L,V_1} \to \mathrm{Set}^{j+1}_L$ and $\mathrm{Set}^{j+1}_L \to \neg\,\mathrm{Eq}^j_{R,V_2} \vee \mathrm{Dif}^{j+1}_L$, so we have $\neg\,\mathrm{Eq}^{j+1}_{L,V_1} \wedge \to \neg\,\mathrm{Eq}^j_{R,V_2} \vee \neg\,\mathrm{Eq}^j_{L,V_1} \vee \mathrm{Dif}^{j+1}_R$. This is symmetric for R and for $u^*_{j+1} \notin V_1$.

Induction Hypothesis (on j): $(\neg\,\mathrm{Eq}^j_{L,V_1} \vee \neg\,\mathrm{Eq}^j_{R,V_2}) \to (\mathrm{Dif}^j_L \vee \mathrm{Dif}^j_R)$

Base Case ($j = 1$): $\neg\,\mathrm{Eq}^1_{L,V_1} \wedge \mathrm{Eq}^0_{L,V_1} \to \mathrm{Dif}^1_L \vee \neg\,\mathrm{Eq}^0_{R,V_2}$, and $\neg\,\mathrm{Eq}^1_{R,V_2} \wedge \mathrm{Eq}^0_{R,V_2} \to \mathrm{Dif}^1_R \vee \neg\,\mathrm{Eq}^0_{L,V_1}$.

However since Eq^0_{L,V_1} and Eq^0_{R,V_2} are both true it simplifies to $\neg\,\mathrm{Eq}^1_{L,V_1} \to \mathrm{Dif}^1_L$ and $\neg\,\mathrm{Eq}^1_{R,V_2} \to \mathrm{Dif}^1_R$ which can be combined to get $(\neg\,\mathrm{Eq}^1_{L,V_1} \vee \neg\,\mathrm{Eq}^1_{R,V_2}) \to (\mathrm{Dif}^1_L \vee \mathrm{Dif}^1_R)$

Inductive Step ($j + 1$):

The Induction Hypothesis $(\neg\,\mathrm{Eq}^j_{L,V_1} \vee \neg\,\mathrm{Eq}^j_{R,V_2}) \to (\mathrm{Dif}^j_L \vee \mathrm{Dif}^j_R)$ can be weakened to $(\neg\,\mathrm{Eq}^j_{L,V_1} \vee \neg\,\mathrm{Eq}^j_{R,V_2}) \to (\mathrm{Dif}^{j+1}_L \vee \mathrm{Dif}^{j+1}_R)$, using $\mathrm{Dif}^j_L \to \mathrm{Dif}^{j+1}_L$ and $\mathrm{Dif}^j_R \to \mathrm{Dif}^{j+1}_R$.

We now need to replace $(\neg\,\mathrm{Eq}^j_{L,V_1} \vee \neg\,\mathrm{Eq}^j_{R,V_2})$ with $(\neg\,\mathrm{Eq}^{j+1}_{L,V_1} \vee \neg\,\mathrm{Eq}^{j+1}_{R,V_2})$. Suppose $u_{j+1} \in V_1$, note that $\neg\,\mathrm{Eq}^{j+1}_{L,V_1} \to \neg\,\mathrm{Eq}^j_{L,V_1} \vee \neg\,\mathrm{Set}^{j+1}_L$. $\neg\,\mathrm{Set}^{j+1}_L \wedge \mathrm{Eq}^j_{R,V_2} \to \mathrm{Dif}^{j+1}_R$

We show that $\neg\,\mathrm{Eq}^{j+1}_{L,V_1} \to \neg\,\mathrm{Eq}^j_{L,V_1} \vee \neg\,\mathrm{Eq}^j_{R,V_2} \vee \mathrm{Dif}^{j+1}_L$ and $\neg\,\mathrm{Eq}^{j+1}_{R,V_2} \to \neg\,\mathrm{Eq}^j_{R,V_2} \vee \neg\,\mathrm{Eq}^j_{L,V_2} \vee \mathrm{Dif}^{j+1}_R$.

Suppose $u^*_{j+1} \in V_1$ then $\neg\,\mathrm{Eq}^{j+1}_{L,V_1} \wedge \mathrm{Eq}^j_{L,V_1} \to \mathrm{Set}^{j+1}_L$ and $\mathrm{Set}^{j+1}_L \to \neg\,\mathrm{Eq}^j_{R,V_2} \vee \mathrm{Dif}^{j+1}_L$, so we have $\neg\,\mathrm{Eq}^{j+1}_{L,V_1} \wedge \to \neg\,\mathrm{Eq}^j_{R,V_2} \vee \neg\,\mathrm{Eq}^j_{L,V_1} \vee \mathrm{Dif}^{j+1}_R$. This is symmetric for R and for $u^*_{j+1} \notin V_1$.

We can use these formulas to show $\neg \mathrm{Eq}^{j+1}_{L,V_1} \wedge \neg \mathrm{Eq}^{j+1}_{R,V_2} \rightarrow \neg \mathrm{Eq}^{j}_{L,V_1} \vee \neg \mathrm{Eq}^{j}_{R,V_2} \vee \mathrm{Dif}^{j+1}_{L} \vee \mathrm{Dif}^{j+1}_{R}$ and we can simplify this to $\neg \mathrm{Eq}^{j+1}_{L,V_1} \wedge \neg \mathrm{Eq}^{j+1}_{R,V_2} \rightarrow \mathrm{Dif}^{j+1}_{L} \vee \mathrm{Dif}^{j+1}_{R}$.

$\neg \mathrm{Dif}^{j}_{L} \wedge \neg \mathrm{Dif}^{j}_{R} \rightarrow \mathrm{Eq}^{j}_{L,V_1}$, $\neg \mathrm{Dif}^{j}_{L} \wedge \neg \mathrm{Dif}^{j}_{R} \rightarrow \mathrm{Eq}^{j}_{R,V_2}$ are corollaries of this. $\neg \mathrm{Dif}^{j}_{L} \wedge \neg \mathrm{Dif}^{j}_{R}$ means $\neg \mathrm{Dif}^{j-1}_{L} \wedge \neg \mathrm{Dif}^{j-1}_{R}$. $u^*_j \in C_1$ implies $u^*_j \notin C_2$, so Set^{j}_{L} and $\neg \mathrm{Set}^{j}_{R}$, and that makes $(\mathrm{Val}^{i}_{B}, \mathrm{Set}^{i}_{B})$=$(\mathrm{Val}^{i}_{L}, \mathrm{Set}^{i}_{L})$.

$u^*_j \in C_2$ implies $u^*_j \notin C_1$ so $\neg \mathrm{Set}^{j}_{L}$ and Set^{j}_{R}, and that makes $(\mathrm{Val}^{i}_{B}, \mathrm{Set}^{i}_{B})$=$(\mathrm{Val}^{i}_{R}, \mathrm{Set}^{i}_{R})$.

$u^*_j \notin C_1 \cup C_2$ implies $\neg \mathrm{Set}^{j}_{L}$ and $\neg \mathrm{Set}^{j}_{L}$, therefore $(\mathrm{Val}^{i}_{B}, \mathrm{Set}^{i}_{B})$=$(\mathrm{Val}^{i}_{L}, \mathrm{Set}^{i}_{L})$.

Lemma 23. *The following propositions are true and have short Extended Frege proofs, given* $(L \rightarrow \mathrm{con}_L(C_1 \cup U_1 \vee \neg x))$ *and* $(R \rightarrow \mathrm{con}_R(C_2 \cup U_2 \vee x))$

- $B \wedge \mathrm{Dif}^{m}_{L} \rightarrow L$
- $B \wedge \neg \mathrm{Dif}^{m}_{L} \wedge \mathrm{Dif}^{m}_{R} \rightarrow R$
- $B \wedge \mathrm{Dif}^{m}_{L} \rightarrow \mathrm{con}_B(C_1 \vee V_2 \vee U)$
- $B \wedge \neg \mathrm{Dif}^{m}_{L} \wedge \mathrm{Dif}^{m}_{R} \rightarrow \mathrm{con}_B(C_2 \vee V_1 \vee U)$

Proof. Let us consider the L cases. Suppose we pick some non-starred literal $y \in C_1$ we will show that $(L \rightarrow \mathrm{con}_{L,C_1 \cup U_1}(y)) \rightarrow (B \wedge \mathrm{Dif}^{m}_{L} \rightarrow \mathrm{con}_{B,V_2 \cup C_1 \cup U_1}(y))$.

For any i, such that $u_i < y$, we will show that $\mathrm{Dif}^{m}_{L} \wedge \mathrm{Set}^{i}_{B} \rightarrow (u_i \leftrightarrow \mathrm{Val}^{i}_{B}) \rightarrow (\mathrm{Set}^{i}_{L} \rightarrow (u_i \leftrightarrow \mathrm{Val}^{i}_{L}))$ and when we take a conjunction over all i, we get $B \wedge \mathrm{Dif}^{m}_{L} \rightarrow L$. For $p \in \{1, 2\}$ let $W_p = \{u^* \mid u^* \in U_p\}$. For each i, either Set^{i}_{B} or $\neg \mathrm{Set}^{i}_{B}$ appears in $\mathrm{anno}_{y,B}(V_1 \cup V_2 \cup U^*)$, so we treat $\mathrm{anno}_{y,B}(V_1 \cup V_2 \cup U^*)$ as a set containing these subformulas. We show that if $c_i \in \mathrm{anno}_{y,B}(V_1 \cup V_2 \cup U^*)$ when $c_i = \mathrm{Set}^{i}_{B}$ or $c_i = \neg \mathrm{Set}^{i}_{B}$ then $L \rightarrow \mathrm{anno}_{y,L}(V_1 \cup W_1) \rightarrow B \wedge \mathrm{Dif}^{m}_{L} \rightarrow c_i$ and we also have $(L \rightarrow y) \rightarrow (B \wedge \mathrm{Dif}^{m}_{L} \rightarrow y)$. For existential y, we can put these all together to get $(L \rightarrow \mathrm{con}_{L,C_1 \cup U_1}(y)) \rightarrow (B \wedge L \wedge \mathrm{Dif}^{m}_{L} \rightarrow \mathrm{con}_{B,V_2 \cup C_1 \cup U_1}(y))$. L can be cut out when we show $B \wedge \mathrm{Dif}^{m}_{L} \rightarrow L$.

If Dif^{m}_{L} is true then there is some j such that $\mathrm{Dif}^{j}_{L} \wedge \neg \mathrm{Dif}^{j}_{L} \wedge \neg \mathrm{Dif}^{j}_{R}$ via Lemmas 20 and 21.

Suppose $i > m$.

Dif^{i}_{L} refutes $\neg \mathrm{Dif}^{m}_{L} \wedge (\mathrm{Dif}^{m}_{R} \vee \neg \mathrm{Set}^{i}_{L})$ so whenever Dif^{m}_{L} is true and Set^{i}_{R} is true, $(\mathrm{Val}^{i}_{B}, \mathrm{Set}^{i}_{B}) = (\mathrm{Val}^{i}_{L}, \mathrm{Set}^{i}_{L})$, therefore $(\mathrm{Set}^{i}_{B} \rightarrow (u_i \leftrightarrow \mathrm{Val}^{i}_{B})) \rightarrow (\mathrm{Set}^{i}_{L} \rightarrow (u_i \leftrightarrow \mathrm{Val}^{i}_{L}))$.

If $\mathrm{Set}^{i}_{B} \in \mathrm{anno}_{x,B}(V_1 \cup V_2 \cup U)$, then $u^*_i \in U$. Val^{i}_{B} depends on the polarity of variable u_i in the subclause U_1, but in every case Set^{i}_{B} is true when $\mathrm{anno}_{x,L}(V_1 \cup W_1)$ is affirmed by L.

If $\neg\,\mathrm{Set}^i_B \in \mathrm{anno}_{x,B}(V_1 \cup V_2 \cup U)$ then $u^*_i \notin U$, this means that $u^*_i \notin W_1$, so whenever $\mathrm{anno}_{x,L}(V_1 \cup W_1)$ is true, $\neg\,\mathrm{Set}^i_L$. But then $\neg\,\mathrm{Set}^i_B$ must be true because of Dif^m_L.

Suppose $j < i \leq m$.

We know $\mathrm{Dif}^j_L \to \mathrm{Dif}^{i-1}_L$ from Lemma 21, we will use that to get that when $\mathrm{Dif}^j_L \wedge \mathrm{Set}^i_L$ then $(\mathrm{Val}^i_B, \mathrm{Set}^i_B) = (\mathrm{Val}^i_L, \mathrm{Set}^i_L)$ which allows us to then show $(\mathrm{Set}^i_B \to (u_i \leftrightarrow \mathrm{Val}^i_B)) \to (\mathrm{Set}^i_L \to (u_i \leftrightarrow \mathrm{Val}^i_L))$.

Suppose $\neg\,\mathrm{Set}^i_B \in \mathrm{anno}_{x,B}(V_1 \cup V_2 \cup U)$, then $u^*_i \notin C_1 \cup C_2$ so $(\mathrm{Val}^i_B, \mathrm{Set}^i_B) = (\mathrm{Val}^i_L, \mathrm{Set}^i_L)$. But since Set^i_L will be false because $u^*_i \notin C_1$, Set^i_B will be false.

Now suppose $\mathrm{Set}^i_B \in \mathrm{anno}_{x,B}(V_1 \cup V_2 \cup U)$, either $u_i \in C_1$ in which case $(\mathrm{Val}^i_B, \mathrm{Set}^i_B) = (\mathrm{Val}^i_L, \mathrm{Set}^i_L)$, but since $u_i \in C_1$ Val^i_L must be true, or $u_i \in C_2$ in which $(\mathrm{Val}^i_B, \mathrm{Set}^i_B) = (\mathrm{Val}^i_L, \mathrm{Set}^i_L)$ or $\neg\,\mathrm{Set}^i_L$, but here we know Set^i_B will be forced to be true.

Suppose $i = j$.

Dif^i_L, $\neg\,\mathrm{Dif}^{i-1}_L$ and $\neg\,\mathrm{Dif}^{i-1}_R$ are all true. If $\mathrm{Set}^i_L \in \mathrm{anno}_{x,L}(V_1 \cup W_1)$ then $\neg\,\mathrm{Set}^i_L$, and if $\neg\,\mathrm{Set}^i_L \in \mathrm{anno}_{x,L}(V_1 \cup W_1)$ then Set^i_L. If $\mathrm{Set}^i_L \in \mathrm{anno}_{x,L}(V_1 \cup W_1)$ and $\neg\,\mathrm{Set}^i_L$ then $u^*_i \in C_1$ and so $(\mathrm{Val}^i_B, \mathrm{Set}^i_B) = (\mathrm{Val}^i_L, \mathrm{Set}^i_L)$. So if $\mathrm{anno}_{x,L}(V_1 \cup W_1)$ is satisfied by L the term $\mathrm{Set}^i_B \in \mathrm{anno}_{x,L}(V_1 \cup V_2 \cup U)$ is satisfied by B.

If $\neg\,\mathrm{Set}^i_L \in \mathrm{anno}_{x,L}(V_1 \cup W_1)$ and Set^i_L then if $u^*_i \in C_2$, we know $\mathrm{Set}^i_B \in \mathrm{anno}_{x,L}(V_1 \cup V_2 \cup U)$, since Set^i_L is true then Set^i_B is true.

If $u^*_i \notin C_1 \cup C_2$ then $\neg\,\mathrm{Set}^i_B \in \mathrm{anno}_{x,B}(V_1 \cup V_2 \cup U)$, but then $(\mathrm{Val}^i_B, \mathrm{Set}^i_B) = (\mathrm{Val}^i_L, \mathrm{Set}^i_L)$. So if $\mathrm{anno}_{x,L}(V_1 \cup W_1)$ is satisfied by L the term $\mathrm{Set}^i_B \in \mathrm{anno}_{x,L}(V_1 \cup V_2 \cup U)$ is satisfied by B.

Suppose $i < j$.

If $\neg\,\mathrm{Set}^i_B \in \mathrm{anno}_{x,B}(V_1 \cup V_2 \cup U)$ then $u^* \notin C_1 \cup C_2$ and so by Lemma 22 $\neg\,\mathrm{Set}^i_B$ is true. If $\mathrm{Set}^i_B \in \mathrm{anno}_{x,B}(V_1 \cup V_2 \cup U)$ then $u^* \in C_1 \cup C_2$ and so by Lemma 22, Set^i_B is true.

We can put this all together to show in eFrege that $B \wedge \mathrm{Dif}^m_L \to L$, $L \to \mathrm{con}_L\, C_1 \vee U_1 \vee \neg x(y) \to B \wedge L \wedge \mathrm{Dif}^m_L \to \mathrm{con}_{B, C_2 \vee V_2 \vee U}(y)$, for existential literal y. Note that Dif_L means that $\mathrm{con}_{R, C_2 \cup U_2 \vee x, R}(\neg x)$ is not satisfied by L to begin with.

Additional universal consideration.

If $y = u_k$, then when y does not become merged we also have to show that $\neg\,\mathrm{Set}^k_B$ is preserved when $\mathrm{con}_{L, C_1 \cup U_1 \vee x}(y)$ and Dif^m_L. Note that if Dif^k_L then the annotation is contradicted. If $u_k \in C_1 \vee C_2$ or $\neg u_k \in C_1 \vee C_2$, for $i \leq m$ then $\neg\,\mathrm{Set}^i_B$ is desired, but Set^i_B will only happen when forced by Set^i_R being true, but this would mean Dif^k_R and $\neg\,\mathrm{Dif}^k_L$, which would contradict Dif^m_L. If $u_k \in C_1 \vee C_2$ or $\neg u_k \in C_1 \vee C_2$ for $i > m$ then Dif^m_L

will contradict an annotation. $u_k \in U_1$ then the literal will not appear as such in $\mathrm{con}_B(C_1 \cup C_2 \cup U)$ because it will now only count as a starred literal.

In all Dif_L^m cases.

The sum of this for all literals is $(L \rightarrow \mathrm{con}_L(C_1 \cup U_1 \vee \neg x)) \rightarrow (B \wedge L \wedge \mathrm{Dif}_L^m \rightarrow \mathrm{con}_B(C_1 \vee V_2 \vee U))$. Using $B \wedge \mathrm{Dif}_L^m \rightarrow L$, this can be cut down to $(L \rightarrow \mathrm{con}_R(C_2 \cup U_2 \vee x)) \rightarrow (B \wedge \neg \mathrm{Dif}_L^m \wedge \mathrm{Dif}_R^m \rightarrow \mathrm{con}_B(C_2 \vee V_1 \vee U))$ which when combined with the premise $(L \rightarrow \mathrm{con}_R(C_1 \cup U_1 \vee \neg x))$ to get $(B \wedge \neg \mathrm{Dif}_L^m \wedge \mathrm{Dif}_R^m \rightarrow \mathrm{con}_B(C_L \vee V_2 \vee U))$.

Suppose $i > m$.

$\mathrm{Dif}_R^m \wedge \neg \mathrm{Dif}_L^m$ satisfies $\neg \mathrm{Dif}_L^m \wedge (\mathrm{Dif}_R^m \vee \neg \mathrm{Set}_L^i)$ so whenever $\mathrm{Dif}_R^m \wedge \neg \mathrm{Dif}_L^m$ is true and Set_R^i is true $(\mathrm{Val}_B^i, \mathrm{Set}_B^i) = (\mathrm{Val}_R^i, \mathrm{Set}_R^i)$, therefore $(\mathrm{Set}_B^i \rightarrow (u_i \leftrightarrow \mathrm{Val}_B^i)) \rightarrow (\mathrm{Set}_R^i \rightarrow (u_i \leftrightarrow \mathrm{Val}_R^i))$.

If $\mathrm{Set}_B^i \in \mathrm{anno}_{x,B}(V_1 \cup V_2 \cup U)$, then $u_i^* \in U$. Val_B^i depends on the polarity of variable u_i in the subclause U_2, but in every case Set_B^i is true when $\mathrm{anno}_{x,R}(V_2 \cup W_2)$ is affirmed by R and $\mathrm{Dif}_R^m \wedge \neg \mathrm{Dif}_L^m$ is true.

If $\neg \mathrm{Set}_B^i \in \mathrm{anno}_{x,B}(V_1 \cup V_2 \cup U)$ then $u_i^* \notin U$, this means that $u_i^* \notin W_2$, so whenever $\mathrm{anno}_{x,R}(V_2 \cup W_2)$ is true, $\neg \mathrm{Set}_R^i$. But then $\neg \mathrm{Set}_B^i$ must be true because of $\mathrm{Dif}_R^m \wedge \neg \mathrm{Dif}_L^m$.

Suppose $j < i \leq m$.

We know $\mathrm{Dif}_R^j \rightarrow \mathrm{Dif}_R^{i-1}$ and $\neg \mathrm{Dif}_R^m \rightarrow \neg \mathrm{Dif}_R^{i-1}$from Lemma 21, we will use that to get that when $\mathrm{Dif}_R^j \wedge \neg \mathrm{Dif}_L^m \, \mathrm{Set}_R^i$ then $(\mathrm{Val}_B^i, \mathrm{Set}_B^i) = (\mathrm{Val}_R^i, \mathrm{Set}_R^i)$ which allows us to then show $(\mathrm{Set}_B^i \rightarrow (u_i \leftrightarrow \mathrm{Val}_B^i)) \rightarrow (\mathrm{Set}_R^i \rightarrow (u_i \leftrightarrow \mathrm{Val}_R^i))$.

Suppose $\neg \mathrm{Set}_B^i \in \mathrm{anno}_{x,B}(V_1 \cup V_2 \cup U)$, then $u_i^* \notin C_1 \cup C_2$ so $(\mathrm{Val}_B^i, \mathrm{Set}_B^i) = (\mathrm{Val}_R^i, \mathrm{Set}_R^i)$. But since Set_R^i will be false because $u_i^* \notin C_2$, Set_B^i will be false.

Now suppose $\mathrm{Set}_B^i \in \mathrm{anno}_{x,B}(V_1 \cup V_2 \cup U)$, either $u_i \in C_2$ in which case $(\mathrm{Val}_B^i, \mathrm{Set}_B^i) = (\mathrm{Val}_R^i, \mathrm{Set}_R^i)$, but since $u_i \in C_2$ Val_R^i must be true, or $u_i \in C_1$ in which case $(\mathrm{Val}_B^i, \mathrm{Set}_B^i) = (\mathrm{Val}_R^i, \mathrm{Set}_R^i)$ or $\neg \mathrm{Set}_R^i$, but here we know Set_B^i will be forced to be true.

Suppose $i = j$.

$\mathrm{Dif}_R^i \, \neg \mathrm{Dif}_R^{i-1}$, $\neg \mathrm{Dif}_L^i$ and $\neg \mathrm{Dif}_L^{i-1}$ are all true. If $\mathrm{Set}_R^i \in \mathrm{anno}_{x,R}(V_2 \cup W_2)$ then $\neg \mathrm{Set}_R^i$, and if $\neg \mathrm{Set}_R^i \in \mathrm{anno}_{x,R}(V_2 \cup W_2)$ then Set_R^i. If $\mathrm{Set}_R^i \in \mathrm{anno}_{x,R}(V_2 \cup W_2)$ and $\neg \mathrm{Set}_R^i$ then $u_i^* \in C_2$ and $u_i \notin C_1$. $\neg \mathrm{Dif}_L^i$ and $\neg \mathrm{Dif}_L^{i-1}$ means that $\neg \mathrm{Set}_L^i$, so then $(\mathrm{Val}_B^i, \mathrm{Set}_B^i) = (\mathrm{Val}_R^i, \mathrm{Set}_R^i)$ So if $\mathrm{anno}_{x,L}(V_1 \cup W_1)$ is satisfied by R the term $\mathrm{Set}_B^i \in \mathrm{anno}_{x,L}(V_1 \cup V_2 \cup U)$ is satisfied by B.

If $\neg \mathrm{Set}_R^i \in \mathrm{anno}_{x,R}(V_R \cup W_R)$ and Set_R^i then if $u_i^* \in C_1$, we know $\mathrm{Set}_B^i \in \mathrm{anno}_{x,L}(V_1 \cup V_2 \cup U)$, $\neg \mathrm{Dif}_L^i$ and $\neg \mathrm{Dif}_L^{i-1}$ means that Set_L^i is true, since Set_R^i is also true then Set_B^i is true. If $u_i^* \notin C_1 \cup C_2$ then $\neg \mathrm{Set}_B^i \in \mathrm{anno}_{x,B}(V_1 \cup V_2 \cup U)$, $\neg \mathrm{Dif}_L^i$ and $\neg \mathrm{Dif}_L^{i-1}$ means that Set_L^i is true, so then $(\mathrm{Val}_B^i, \mathrm{Set}_B^i) = (\mathrm{Val}_R^i, \mathrm{Set}_R^i)$. So if $\mathrm{anno}_{x,R}(V_2 \cup W_2)$ is satisfied by R the term $\mathrm{Set}_B^i \in \mathrm{anno}_{x,B}(V_1 \cup V_2 \cup U)$ is satisfied by B.

Suppose $i < j$.

If $\neg \mathrm{Set}_B^i \in \mathrm{anno}_{x,B}(V_1 \cup V_2 \cup U)$ then $u^* \notin C_1 \cup C_2$ and so by Lemma 22 $\neg \mathrm{Set}_B^i$ is true. If $\mathrm{Set}_B^i \in \mathrm{anno}_{x,B}(V_1 \cup V_2 \cup U)$ then $u^* \in C_1 \cup C_2$ and so by Lemma 22, Set_B^i is true.

We can put this all together to show in eFrege that $B \wedge \mathrm{Dif}_R^m \wedge \neg \mathrm{Dif}_L^m \rightarrow R$ $R \rightarrow \mathrm{con}_{R,C_2 \vee U_2 \vee x}(y) \rightarrow B \wedge R \wedge \mathrm{Dif}_R^m \wedge \neg \mathrm{Dif}_L^m \rightarrow \mathrm{con}_{B,C_2 \vee V_2 \vee U}(y)$, for existential literal y. Note that Dif_R means that $\mathrm{con}_{R,C_2 \cup U_2 \vee x,R}(x)$ is not satisfied by R to begin with.

Additional universal consideration.

If $y = u_k$ then we also have to show that $\neg \mathrm{Set}_B^k$ is preserved when $\mathrm{con}_{R,C_2 \cup U_2 \vee x,L}(y)$ and $\mathrm{Dif}_R^m \wedge \neg \mathrm{Dif}_L^m$, Note that if Dif_R^k then the annotation is contradicted. If $u_k \in C_1 \vee C_2$ or $\neg u_k \in C_1 \vee C_2$, for $i \leq m$ then $\neg \mathrm{Set}_B^i$ is desired, but Set_B^i will only happen when forced by Set_L^i being true, but this would mean Dif_L^k contradicting $\neg \mathrm{Dif}_L^m$ If $u_k \in C_1 \vee C_2$ or $\neg u_k \in C_1 \vee C_2$ for $i > m$ then Dif_L^m will contradict an annotation. $u_k \in U_1$ then the literal will not appear as such in $\mathrm{con}_B(C_2 \vee V_2 \vee U)$ because it will now only count as a starred literal.

In all $\mathrm{Dif}_R^m \wedge \neg \mathrm{Dif}_L^m$ cases.

The sum of this for all literals is $(R \rightarrow \mathrm{con}_R(C_2 \cup U_2 \vee x)) \rightarrow (B \wedge R \wedge \neg \mathrm{Dif}_L^m \wedge \mathrm{Dif}_R^m \rightarrow \mathrm{con}_B(C_2 \vee V_1 \vee U))$. Using $B \wedge \mathrm{Dif}_R^m \wedge \neg \mathrm{Dif}_L^m \rightarrow R$, this can be cut down to $(R \rightarrow \mathrm{con}_R(C_2 \cup U_2 \vee x)) \rightarrow (B \wedge \neg \mathrm{Dif}_L^m \wedge \mathrm{Dif}_R^m \rightarrow \mathrm{con}_B(C_2 \vee V_1 \vee U))$ which when combined with the premise $(R \rightarrow \mathrm{con}_R(C_2 \cup U_2 \vee x))$ to get $(B \wedge \neg \mathrm{Dif}_L^m \wedge \mathrm{Dif}_R^m \rightarrow \mathrm{con}_B(C_2 \vee V_1 \vee U)$.

Lemma 24. *The following propositions are true and have short Extended Frege proofs, given* $(L \rightarrow \mathrm{con}_L(C_1 \cup U_1 \vee \neg x))$ *and* $(R \rightarrow \mathrm{con}_R(C_2 \cup U_2 \vee x))$.

- $B \wedge \neg \mathrm{Dif}_L^m \wedge \neg \mathrm{Dif}_R^m \rightarrow \mathrm{con}_B(C_1 \vee V_2 \vee U) \vee \neg x$
- $B \wedge \neg \mathrm{Dif}_L^m \wedge \neg \mathrm{Dif}_R^m \rightarrow \mathrm{con}_B(C_2 \vee V_1 \vee U) \vee x$

Proof. For indices $1 \leq i \leq m$, but since $\neg \mathrm{Dif}_L^m \rightarrow \neg \mathrm{Dif}_L^i$ and $\neg \mathrm{Dif}_R^m \rightarrow \neg \mathrm{Dif}_R^i$, Lemma 6 can be used to show that $B \wedge \mathrm{Dif}_L^m \wedge \mathrm{Dif}_R^m$ leads to Set_B^i taking the a value consistent with both $V_1 \cup V_2$, if L was consistent with V_1 and R was consistent with V_2.

For $i > m$, $\neg \mathrm{Dif}_R^m \wedge \neg \mathrm{Dif}_L^m$ will make the policy B pick between the left and right policy based on x. However in either case Set_B^i will be forced to update based on the new annotations.

Lemma 25. *Suppose, there are policies L and R such that $L \rightarrow \mathrm{con}_L(C_1 \vee \neg x \vee U_1)$ and $R \rightarrow \mathrm{con}_L(C_2 \vee x \vee U_2)$ then there is a policy B such that $B \rightarrow \mathrm{con}_B(C_1 \vee C_2 \vee U)$ can be obtained in a short* eFrege *proof, where C_1, C_2, U_1, U_2 and U follow the same definitions as in Figure 5.*

Proof. From Lemmas 24 and 23, $\mathrm{con}_B(C_1 \vee V_2 \vee U)$ and $\mathrm{con}_B(C_2 \vee V_1 \vee U)$ can be weakened to $\mathrm{con}_B(C_1 \vee C_2 \vee U)$. These can all be combined over the different possibilities to give $B \rightarrow \mathrm{con}_B(C_1 \vee C_2 \vee U)$.

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