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HYPERCONTRACTIVITY ON HIGH DIMENSIONAL EXPANDERS: APPROXIMATE EFRON-STEIN DECOMPOSITIONS FOR ε-PRODUCT SPACES

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Abstract. We prove hypercontractive inequalities on high dimensional expanders. As in the settings of the p-biased hypercube, the symmetric group, and the Grassmann scheme, our inequalities are effective for global functions, which are functions that are not significantly affected by a restriction of a small set of coordinates. As applications, we obtain Fourier concentration, small-set expansion, and Kruskal–Katona theorems for high dimensional expanders. Our techniques rely on a new approximate Efron–Stein decomposition for high dimensional link expanders.

1. Introduction

High-dimensional expanders (HDX) are sparse simplicial complexes with strong structural properties. More accurately, a simplicial complex X is a ϵ-HDX (or an ϵ-link expander) if the 1-skeleton of each link of the complex X is a spectral expander graph whose second-largest eigenvalue is bounded by ϵ. In recent years, HDX have received much attention in theoretical computer science [33, 45, 34, 42, 41, 35, 27], finding applications in property testing [32, 14, 28], coding theory [13, 10], statistical physics [3, 7, 2], complexity theory [1, 4, 12, 29], and beyond. Notably, very recently the study of HDX led to a breakthrough in quantum computing, breaking the $\sqrt{n}$ distance in quantum LDPC [19], as well as to a resolution of one of the most important questions in coding theory, namely, the first construction of $O(1)$-query asymptotically good locally testable codes [11].

In this work, we focus on analysis of Boolean functions on high dimensional expanders, whose systematic study was recently initiated by Dikstein et al. [9]. This continues a long line of investigation of Fourier analysis of Boolean functions on extended domains beyond the Boolean hypercube, such as the Boolean slice [44, 20, 25, 24], the Grassmann scheme [15, 39, 18], the symmetric group [23, 21, 8], the p-biased cube [17, 40, 22], and the multi-slice [26, 6]. The foregoing extended domains arise naturally throughout theoretical computer science, and indeed, the study of analysis of Boolean functions on extended domains has recently led to a breakthrough regarding the unique games conjecture [38, 16, 15, 39].

Hypercontractive inequalities are amongst the most powerful technical tools in Fourier analysis, yielding a plethora of applications in algorithms, complexity, learning theory, statistical physics, social choice, and beyond (see [43] and references therein). Loosely speaking, such statements assert that functions of low Fourier degree are "well behaved" in terms of their distribution around their mean. Concretely, in the Boolean hypercube, the simplest example of a hypercontractive inequality is Bonami's lemma, which states that for every function $f\colon \{0,1\}^n \to \mathbb{R}$ of Fourier degree at most d, it holds that $\|f\|_4 \le \sqrt{3^d}\|f\|_2$.

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Alas, in the setting of high dimensional expanders, where the domain is not a product space and the induced measure is biased, general strong hypercontractivity cannot hold. The heart of the problem is that some highly local functions, such as dictators (i.e., $f(x) = x_i$), provide strong counterexamples to hypercontractivity. A similar phenomenon also occurs in several prominent extended domains, such as the p-biased cube and the Grassmann scheme.

Fortunately, as observed in the setting of the p-biased cube [36], all of the aforementioned examples are local, in the sense that a small number of coordinates can significantly influence the output of the function. This led to the definition of 'global' functions. For Boolean valued functions, these are functions wherein a small number of coordinates can change the output of the function only with a negligible probability. For real valued functions, this is captured by the 2-norm remaining roughly the same when restricting O(1) coordinates of the input. More precisely, consider the setting of a general product measure. Let (V_i, μ_i) be probability spaces, let $V_S = \prod_{i\in S} V_i$ and equip V_S with the product measure, which we denote by μ_S. Every function $f \in L^2\left(V_{[k]}, \mu\right)$ is equipped with an orthogonal decomposition $\sum_{S\subseteq[n]} f^{=S}$ known as the Efron–Stein decomposition. The function $f^{=S}$ in the Efron–Stein decomposition plays a similar role to the function $\hat{f}(S)\chi_S$ in the Boolean cube. Using that analogy we write

$$f^{\leq d} = \sum_{|S|\leq d} f^{=S},$$

and f is said to be of degree d if $f = f^{\leq d}$. Keevash et al. [37] introduced the following notions. The *Laplacians of* f are given by

$$L_S[f] = \sum_{T\supseteq S} (-1)^{|T|} f^{=T}.$$

For $x \in V_S$ the *derivatives* are given by restricting the laplacians

$$D_{S,x}f = L_S[f](x,\cdot),$$

and the (S,x)-*influence* of f is defined as

$$I_{S,x}[f] = \|D_{S,x}[f]\|_2^2.$$

In this setting, a function f is (r,δ)-*global* if $\|f(x,\cdot)\|^2_{2,\mu_{[n]\setminus S}} \leq \delta$ for each $|S| \leq r$. We remark that here, being (r,δ)-global for a small $\delta > 0$ is, in a sense, equivalent to having $I_{S,x}[f] \leq \delta'$ for a small δ' for all $|S| \leq r$ and all x. In fact, δ, δ' can be taken to be within a factor of 2^r of one another.

In [37], it was shown that if $f \in L^2(V,\mu)$ is of degree d, then the following hypercontractive inequality holds:

$$\|f\|_4^4 \leq 1000^d \sum_S \mathbb{E}_{x\sim\mu_S} I_{S,x}[f]^2. \tag{1.1}$$

This allowed them to deduce if a function f of degree d is (d,δ)-*global*, then

$$\|f\|_4^4 \leq \delta 8000^d \|f\|_2^2.$$

Here when setting $\delta = 100\|f^{\leq d}\|_2^2$ one gets the statement $\|f\|_4 \leq C^d\|f\|_2$, which replicates the behavior in the Boolean cube. Moreover, the statement is useful even for larger values of δ.

In this work, we raise the question we raise the following question.

Does hypercontractivity hold for high dimensional expanders?

1.1. **Main results.** We answer the question above in the affirmative. Namely, our main contribution is a hypercontractive inequality for functions on the k-faces of an ϵ-HDX. We denote by $X(k)$ the k-faces of a simplicial complex X, and denote by μ the uniform measure on its k-faces. We define the influences $I_{S,x}^{\leq d}$ and the degree restriction operator $(\cdot)^{\leq d}$ analogously to their definition on the p-biased cube (see Section 4 for precise definition). We then prove the following hypercontractive statement for high dimensional expanders in the spirit of (1.1).

Theorem 1.1. *Let X be an ϵ-HDX, and let $f \in L^2(X(k), \mu)$. We have*

$$\|f^{\leq d}\|_4^4 \leq 20^d \sum_{|S|\leq d} (4d)^{|S|} \mathbb{E}_{x\sim\mu_S} I_{S,x}^{\leq d}[f]^2 + O_k\left(\epsilon^2\right) \|f\|_2^2 \|f\|_\infty^2.$$

In the setting of ϵ-HDX, we say that a function f is (d,δ)-global if for each $|S| \leq d$, we have $\|f(x,\cdot)\|_{L^2(V_x,\mu_x)} \leq \delta$. We show that we can bound the infinity norm of global functions and obtain the following strong hypercontractive inequality for global functions on ϵ-HDX.

Corollary 1.2. *For each $\zeta, d, k > 0$, there exists $\epsilon_0 = \epsilon_0(\zeta, k, d), \delta_0 = \delta_0(\zeta, d)$, such that the following holds. Let $\epsilon \leq \epsilon_0, \delta \leq \delta_0$, let X be an ϵ-HDX, and let $f \in L^2(X(k), \mu)$. If f is (d,δ)-global, then we have*

$$\|f^{\leq d}\|_4^4 \leq \zeta \|f\|_2^2.$$

We remark that, in fact, we prove our results in a slightly more general setting, to which we refer as ϵ-product measures. See Section 7 for details.

1.2. **Applications.** As corollaries of our hypercontractive inequality for high dimensional expanders, we obtain several applications, which we discuss below. See Section 8 for more details.

1.2.1. *Fourier spectrum concentration theorem.* Fourier concentration results are widely useful in complexity theory and learning theory. Our first application is a Fourier concentration theorem for HDX. Namely, the following theorem shows that global Boolean functions on ϵ-HDX are concentrated on the high degrees, in the sense that the 2-norm of the restriction of a function to its low-degree coefficients only constitutes a tiny fraction of its total 2-norm.

Theorem 1.3. *For each $\zeta, d, k > 0$, there exists $\epsilon_0 = \epsilon_0(\zeta, k, d), \delta_0 = \delta_0(\zeta, d)$, such that the following holds. Let $\epsilon \leq \epsilon_0, \delta \leq \delta_0$, let X be an ϵ-HDX, and let $f\colon X(k) \to \{0,1\}$ be (d,δ)-global. Then*

$$\|f^{\leq d}\|_2^2 \leq \zeta \|f\|_2^2.$$

1.2.2. *Small set expansion theorem.* Small set expansion is a fundamental property that is prevalent in combinatorics and complexity theory. In the setting of the ρ-noisy Boolean hypercube, the small set expansion theorem of Kahn, Kalai, and Linial [31] gives an upper bound on $\mathrm{Stab}_\rho(1_A) = \langle 1_A, T_\rho 1_A\rangle$ for indicators 1_A of small sets A. The noise stability $\mathrm{Stab}_\rho(1_A)$ captures the probability that a random edge (x,y) of the ρ-noisy hypercube has both its endpoints in A. Hence, an inequality of the form $\mathrm{Stab}_\rho(1_A) \leq \zeta\|1_A\|_2^2$ for an arbitrarily small ζ and sufficiently small A implies that that small sets are expanding in the sense that the random walk makes you leave them with probability $\geq 1-\zeta$. Our second application is a small set expansion theorem for global functions on ϵ-HDX, captured via bounding the natural noise operator in this setting. Let $\rho \in (0,1)$ be a noise-rate parameter. The noise operator is given by

$$\mathrm{T}_\rho f(x) := \sum_{S\subseteq[k]} \rho^{|S|} (1-\rho)^{k-|S|} \mathbb{E}_{y\sim\mu}\left[f(y) \,|y_S = x_S\right].$$

In other words, T_ρ corresponds to the random walk that starts with x chooses a ρ-biased random $S \subseteq [k]$, keeps x_S, and re-randomises x given x_S. Our small set expansion theorem tells us that if we start with a small subset $A \subseteq X(k)$ and we apply one step of the random walk, then we leave A with probability 0.99.

Theorem 1.4. *For each $\zeta, d, k > 0$, there exists $\epsilon_0 = \epsilon_0(\zeta, k, d), \delta_0 = \delta_0(\zeta, d)$, such that the following holds. Let $\epsilon \le \epsilon_0, \delta \le \delta_0$, and let X be an ϵ-HDX. If $f\colon V_{[k]} \to \{0,1\}$ is (d,δ)-global, then*

$$\|\mathrm{T}_\rho f\|_2^2 \le \zeta \|f\|_2^2.$$

1.2.3. *Kruskal–Katona theorem.* Our last application is an analogue of the Kruskal–Katona theorem in the setting of high dimensional expanders. The Kruskal-Katona theorem is a fundamental and widely-applied result in extremal combinatorics, which gives a lower bound on the size of the lower shadow $\partial(A)$ of a k-uniform hypergraph A on n vertices. The *lower shadow* is defined to be the family of all $(k-1)$-sets that are contained in an edge of A. More generally, if $A \subseteq X(k)$, then we similarly let $\partial(A)$ be the family of all $k-1$-faces that are contained in a k-face of A.

Filmus et al. [23] used their hypercontractivity theorem to prove a stability result for the Kruskal–Katona theorem. We prove a similar stability result for ϵ-HDX.

Theorem 1.5. *Let X be an ϵ-HDX, for a sufficiently small $\epsilon > 0$. Let $\delta \le (200d)^{-d}$, and let $A \subseteq X(k-1)$ be (d,δ)-global. Then*

$$\mu(\partial(A)) \ge \mu(A)\left(1 + \frac{d}{2k}\right).$$

1.3. **Techniques.** Conceptually, one can view the theory of expanders and pseudorandom graphs in the following perspective: Given a pseudorandom regular graph $G = (V,E)$ and $(x,y) \sim E$, the goal is to show that x, y behave similarly to independent random variables $x, y \sim V$, i.e., as an approximation of a product space.

In the theory of high dimensional expanders, we are given a distribution μ on $(k+1)$-tuples by choosing a random k-face $(x_1, \ldots, x_{k+1})$ of a sparse simplicial complex, and the goal is again to show that the variables $\{x_i\}$ approximately behave as though they were independent. Thus, our main objective is to generalise results from the product space setting, where the x_i's are independent, to the setting of HDX, where we only have local spectral information about the links. However, such a generalisation yields significant challenges.

One of the fundamental tools for studying the product space setting is the aforementioned Efron–Stein decomposition. Its role in the analysis of product spaces is that it allows us to easily generalise techniques from the Boolean cube by replacing the Fourier expression $\hat{f}(S)\chi_S$ with the function $f^{=S}$.

Our high-level proof strategy is to develop new Efron–Stein decompositions for HDX. We show that despite the more involved setting, and despite the fact that we only have mere local spectral information, we can still obtain similar structural properties as in product spaces. We now list a few of the challenges that we are facing, which require fundamentally new ideas and techniques.

Dikstein et al. [9] gave a decomposition of the form $f = \sum_{d=0}^{k} f^{=d}$. We provide a new decomposition $\{f^{=S}\}_{S \subseteq [k]}$ such that $f = \sum_{S \subseteq [k]} f^{=S}$, and despite not having orthogonality, we can still show that the inner product $\langle f^{=S}, f^{=T}\rangle$ is negligible compared to $\|f\|_2^2$. This allows us to generalise the Laplacians, derivatives and influences, but we have to deal with the following problems:

- Let $\mathcal{F} \subseteq [k]$ be a small set. We would like to say that $g = \sum_{S\in\mathcal{F}} f^{=S}$ is supported on $\mathcal{F}$, but we have no way of knowing that looking at $\left\{g^{=S}\right\}_{S\subseteq[k]}$, as $g^{=S}$ may be nonzero even for $S \notin \mathcal{F}$. This leads to the problem of how to even define the degree of a function. We would like to say that $f^{\leq d} := \sum_{|S|\leq d} f^{=S}$ is of degree at most d, and that f is of degree d if $f = f^{\leq d}$. Alas, according to this definition the function $f^{\leq d}$ is not of degree d.
- We can and do define the derivatives $D_{S,x}$ to be the restrictions of the Laplacians. In the product case the derivatives decrease the degree by $|S|$, and this is a very desirable property as our proof goes by induction on d. However, this is no longer true in the HDX setting.
- We may define the influences by taking 2-norms of the derivatives. However, now it is no longer true that having small influences is equivalent to being global. This leads us to the following problem which is the source for all of the difficulty.
- The spectral information tells us that HDX should behave similarly to product spaces with respect to the L^2-norm. However, we care about L_4 information when bounding $\|f\|_4$, and we deal with L_∞-hypothesis as the globalness notion is about **all** the restrictions. There is no reason for HDX to behave well with respect to L_4 and even more so for L_∞.

At first, the above, and especially the last point, seem as fundamental barriers to this approach.

Nevertheless, we overcome this barrier by developing an alternative notion, which we call the *approximate Efron–Stein decomposition*. Our new notion has the following properties that fix all of the above problems.

- If $\{f_S\}_{S\subseteq[k]}$ is an approximate Efron–Stein decomposition, then crucially, $\{f_S\}_{S\in\mathcal{F}}$ is an approximate Efron–Stein decomposition for $\sum_{S\in\mathcal{F}} f^{=S}$.
- If f is approximately of degree d, in the sense that $\{f_S\}$ is an approximate Efron-Stein decomposition for f, then the derivative $D_{S,x}[f]$ may be L_4-approximated by $D_{S,x}[f]^{\leq d-|S|}$.
- We find a way of proving an inequality of the form

$$\mathbb{E}_{x\sim\mu_S} I^2_{S,x}[f] \leq \delta \mathbb{E}_{x\sim\mu_S}[I_{S,x}],$$

  without having the traditional hypothesis $\max_x I_{S,x}[f] \leq \delta$ at our disposal.
- We show that we may move freely between different approximate Efron–Stein decomposition up to a small L_4-norm error term.

We believe that our approximate Efron–Stein decomposition provides the desired comfortable platform for analysing functions on HDX in the same way one would analyze a product space.

See Section 4 for a detailed exposition of our approximate Efron–Stein decomposition, and see Section 5 for a more detailed proof overview of our main hypercontractivity results, which build on the aforementioned decomposition.

1.4. **Related work.** Simultaneously and independently to this work, Bafna, Hopkins, Kaufman, and Lovett [5] also obtained hypercontractive inequalities for high dimensional expanders. We remark that while the main hypercontractive inequalities in both papers achieve essentially the same parameters, the techniques are completely different. Namely, in [5] the proof strategy follows the approach of analogous results in the setting of the Grassmann graph, whereas our approach generalises Efron–Stein decompositions and hypercontractivity for general product spaces. We further note that our approximate Efron–Stein decomposition extends approximate Fourier decompositions that appeared in several recent works [34, 35, 9, 1, 30].

1.5. **Organisation.** The rest of the paper is organised as follows. We start in Section 2, where we recall the notions of hypercontractivity and globalness in general product spaces, as well as provide an alternative proof of a slightly weaker hypercontractive inequality that is more amenable for generalisation to non-product spaces. In Section 3, we present the framework of ϵ-product spaces, of which high dimensional expanders are a special case, and we also define key operators in this setting and show some basic properties they satisfy. Next, in Section 4, which is introducing a new approximate Efron–Stein decomposition and developing a framework for proving hypercontractivity results using this decomposition. Then, in Section 5, we give a detailed proof overview of our hypercontractive inequalities for high dimensional expanders, which build on the foregoing framework. In Section 6, we define the notions of laplacians, derivatives and influences in the setting of ϵ-measures, give bounded approximated Efron–Stein decompositions related to the Laplacians, define globalness, and show that it implies small influences.. Then, we provide the full proof of our main hypercontractivity results in Section 7. Finally, in Section 8, we show how to derive the applications from our hypercontractive inequalities.

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2. Recalling globalness and hypercontractivity in the product space setting

We begin by recalling the Efron–Stein decomposition, as well as derivatives and Laplacians in the setting of general product spaces, and state the hypercontractivity inequalities for product spaces that were shown in [37]. We then give a proof, inspired by [18], of a slightly weaker hypercontractivity inequality that we will later generalise to approximate product spaces.

2.1. **Efron-Stein decomposition.** Let $(V_1,\mu_1),\ldots,(V_k,\mu_k)$ be a probability space. Let μ be the corresponding product measure $\mu_1\otimes\cdots\otimes\mu_k$. For a set $S\subseteq[k]$, we write $V_S=\prod_{i\in S}V_i$, and we write μ_S for the product measure $\mu_S=\bigotimes_{i\in S}\mu_i$. The *Efron–Stein decomposition* is a decomposition of $L^2\left(V_{[k]},\mu\right)$ into 2^k orthogonal spaces $\{W_S\}_{S\subseteq[k]}$. Every function $f\in L^2\left(V_{[k]},\mu\right)$ can then be decomposed as $f=\sum_{S\subseteq[k]}f^{=S}$, where $f^{=S}$ is the projection of f to W_S. The Efron–Stein decomposition is characterised by the orthogonality of $\{W_S\}$, the fact that $\sum_S W_S=L^2\left(V,\mu\right)$, and the fact that the space W_S is composed of functions depending only on S.

The functions $f^{=S}$ also have an explicit formula for $x\in V_S$, where we denote

$$A_Sf\left(x\right)=\mathbb{E}_{y\sim\left(V_{\overline{S}},\mu_{\overline{S}}\right)}\left[f\left(x,y\right)\right],$$

where $\bar{S}=[k]\setminus S$. We then write

$$f^{=S}=\sum_{T\subseteq S}\left(-1\right)^{|S\setminus T|}A_Tf.$$

The function $A_S\left[f\right]$ then has the following neat Efron–Stein decomposition

$$A_S\left[f\right]=\sum_{T\subseteq S}f^{=T}.$$

See [43, Chapter 8] for more details.

2.2. **Notations.** We write $a = b \pm \epsilon$ to indicate that $a \in (b-\epsilon, b+\epsilon)$. We use $a \le O(b)$ to denote that the inequality holds up to an absolute constant, and $a \le O_k(b)$ to denote that the inequality holds up to a constant only depending on k.

2.3. **Derivatives and Laplacians.** Let $\mu = \mu_1 \otimes \cdots \otimes \mu_k$ be a product measure. Let $f \in L^2\left(V_{[k]}, \mu\right)$, $S \subseteq [n]$. The *Laplacian* is given by the formula

$$L_S[f] = \sum_{T \supseteq S} f^{=T} = \sum_{T \subseteq S} (-1)^{|T|} A_{[k]\setminus T} f.$$

For $S \subseteq [n]$ and $x \in V_S$ the *derivative* $D_{S,x} \in L^2(V_{\overline{S}}, \mu_{\overline{S}})$ is defined by

$$D_{S,x} f = L_S[f](x, \cdot).$$

For convenience, we also write $D_\varnothing f = f$. The (S,x)-*influence* of f is defined as

$$I_{S,x}[f] = \|D_{S,x}[f]\|_2^2.$$

This includes the case $S = \varnothing$, where we have $I_\varnothing[f] := \|f\|_2^2$.

We now state a few facts from [37] that we generalise. The following lemma, which appears in [37], shows that the notion of small influences corresponds to small 2-norms of the restriction of f.

Lemma 2.1. *Suppose that $I_{S,x}[f] \le \delta$ for each set S of size at most r. Then $\|f(x,\cdot)\|_{2,\mu_{[n]\setminus S}}^2 \le \delta 4^r$ for each S and $x \in V_S$. Conversely, if $\|f(x,\cdot)\|_{2,\mu_{[n]\setminus S}}^2 \le \delta$ for each $|S| \le r$ and $x \in V_S$, then $I_{S,x}[f] \le \delta 4^r$ for each S of size at most r and $x \in V_S$.*

For the above reason they gave the following definition.

Definition 2.2. A function f is said to be (r,δ)-*global* if $I_{S,x}[f] \le \delta$ for each $|S| \le r$.

The *degree* of a function is the largest S, such that $f^{=S} \neq 0$. The derivatives decrease the degrees for the following reason.

Lemma 2.3. *$D_{S,x}\left[f^{=T}\right]$ is 0 unless $S \subseteq T$, and if $S \subseteq T$, then*

$$D_{S,x}\left[f^{=T}\right] \in W^{T \setminus S}.$$

Consequently, if $f = \sum_{|S| \le d} f^{=S}$ is of degree d, then $D_{S,x}[f]$ is of degree $d - |S|$.

2.4. **Hypercontractivity.** The following result is by [37].

Theorem 2.4. *If $f \in L^2(V, \mu)$ is of degree d, then*

$$\|f\|_4^4 \le 1000^d \sum_S \mathbb{E}_x I_{S,x}[f]^2.$$

To show the implication of the theorem for global functions they use the following inequality.

Lemma 2.5.

$$\sum_S \mathbb{E}_x I_{S,x}[f] \le 2^d \|f\|_2^2.$$

Proof. The right hand side is equal to

$$\sum_S \|L_S[f]\|_2^2 = \sum_S \sum_{T \supseteq S, |T| \le d} \|f^{=T}\|_2^2 \le 2^d \sum_T \|f^{=T}\|_2^2 = 2^d \|f\|_2^2.$$

□

Combining Theorem 2.4 and Lemma 2.5, we obtain the following corollary.

Corollary 2.6. *If f of degree d is (d,δ)-global. Then* $\|f^{\leq d}\|_4^4 \leq \delta 2000^d \|f\|_2^2$.

Proof. We have

$$\begin{aligned}\|f^{\leq d}\|_4^4 &\leq 1000^d \sum_{S\subseteq[n]} \mathbb{E}_{x\sim\mu_S} \|I_{S,x}[f]\|_2^4\\ &\leq \delta 1000^d \sum_S \mathbb{E}_{x\sim\mu_S}\|I_{S,x}[f]\|_2^2\\ &\leq \delta 2000^d \|f\|_2^2.\end{aligned}$$

□

2.5. **An alternative proof of hypercontractivity on product spaces.** We give an alternative proof of the following slightly weaker version of Theorem 2.4. The proof is inspired by a future work by Ellis, Kindler, and the second author [18], who show that the same idea works in the Grassmann setting. In this paper we show that it generalises to HDX as well.

Theorem 2.7. *Let $f \in L^2(V,\mu)$ be of degree d. Then*

$$\|f\|_4^4 \leq 2\cdot 9^d \sum_{|T|\leq d} (9d)^{|T|}\, \mathbb{E}_x\left[I_{S,x}[f]^2\right].$$

2.5.1. *Proof overview.* Before providing the full proof, we first describe the high-level approach for proving Theorem 2.7. The strategy is to first show a lemma that gives the following bound

$$\|f\|_4^4 \leq C^d \|f\|_2^4 + \sum_{S\subseteq[n]} (4d)^{|S|}\,\|L_S[f]\|_4^4, \tag{2.1}$$

for a constant C. Using this lemma, we can give an inductive proof by first noting that $\|L_S[f]\|_4^4 = \mathbb{E}_x\|D_{S,x}\|_4^4$, and then applying induction using the fact that $D_{S,x}$ is of degree $d-|S|$. Finally, using the fact that $D_{S,x}D_{T,y} = D_{S\cup T,(x,y)}$, we can get our desired hypercontractive statement.

Hence, the key step is to prove the aforementioned lemma. To this end, we first use the fact that

$$\mathbb{E}\left[f^4\right] = \sum_S \|\left(f^2\right)^{=S}\|_2^2.$$

We then expand the summands of $\left(f^2\right)^{=S}$ as sums of terms of the form $\left(f^{=T_1}f^{=T_2}\right)^{=S}$. Next, we note that the nonzero terms either satisfy $T_1\cap T_2\cap S\neq\varnothing$ or satisfy $T_1\Delta T_2 = S$. Terms of the first kind are cancelled out by $L_i[f]^4$ for an $i\in T_1\cap T_2\cap S$ on the right hand side of (2.1). (The terms $\|L_S[f]\|_4^4$ appear because of over counting, which we resolve by inclusion exclusion.) Terms of the latter kind correspond to the situation in the Boolean cube where $f^{=T} = \hat{f}(T)\chi_T$ and $\chi_T\chi_S = \chi_{T\Delta S}$. We then upper bound $\|\left(f^{=T_1}f^{=T_2}\right)^{=S}\|_2$ by $\|f^{=T_1}\|_2\|f^{=T_2}\|_2$. This allows us to translate the problem of upper bounding the terms of the first kind to the problem of upper bounding the 4-norm of a low degree function on the Boolean cube. Namely, the function

$$\sum_{|T|\leq d}\|f^{=T}\|_2\chi_T.$$

Finally, we use hypercontractivity to upper bound the 4-norm by its 2-norm, which is equal to the 2-norm of f. This concludes the proof overview.

2.5.2. *Proof of hypercontractivity on product spaces.* We now give a formal proof of Theorem 2.7. We shall first need the following key lemma, which admits the inductive approach.

Lemma 2.8. *Let $f \in L^2(V, \mu)$ be of degree d. Then*

$$\frac{1}{2}\|f\|_4^4 \leq 9^d \|f\|_2^4 + \sum_{T \neq \varnothing} (4d)^{|T|} \|L_T[f]\|_4^4.$$

We are now ready to prove the lemma.

Proof of Lemma 2.8. By Parseval we have

$$\|f\|_4^4 = \sum \| \left(f^2\right)^{=S} \|_2^2.$$

We bound each term $\| \left(f^2\right)^{=S} \|_2^2$ individually. By expanding and using the linearity of the $\cdot^{=S}$ operator we have

$$\left(f^2\right)^{=S} = \sum_{T_1, T_2} \left(f^{=T_1} f^{=T_2}\right)^{=S}.$$

We now divide the pairs (T_1, T_2) into three sums.

(1) We let I_1 be the set of pairs (T_1, T_2) such that $T_1 \cap T_2 \cap S \neq \varnothing$. If i is in $T_1 \cap T_2 \cap S$, then the summand $\left(f^{=T_1} f^{=T_2}\right)^{=S}$ appears as a summand when expanding $\left(L_i[f]^2\right)^{=S}$. This explains the role of the Laplacians in the right hand side.
(2) We let I_2 be the set of pairs such that $T_1 \Delta T_2 = S$. These kind of pairs have a similar behavior to the one in the Boolean cube. There $f^{=S} = \hat{f}(S) \chi_S$ and
$$f^{=S} f^{=T} = \hat{f}(S) \hat{f}(T) \chi_{S \Delta T}.$$
We show that the contribution from the pairs in I_2 is $\leq C^d \|f\|_2^2$.
(3) We let $I_3 = (T_1, T_2)$ such that either $(T_1 \Delta T_2) \setminus S \neq \varnothing$ or $S \setminus (T_1 \cup T_2) \neq \varnothing$. We show that in this case $\left(f^{=T_1} f^{=T_2}\right)^{=S} = 0$.

It is easy to verify that each pair (T_1, T_2) belongs to at least one of the sets I_1, I_2, I_3. We additionally have $I_1 \cap I_2 = \varnothing$.

Upper bounding the contribution from I_1. Let us start by upper bounding the contribution from pairs corresponding to I_1. For a nonempty $T \subseteq S$ write $I_1(T)$ for the pairs (T_1, T_2), such that $T_1 \cap T_2 \supseteq T$. Then

$$\left(L_T[f]^2\right)^{=S} = \sum_{(T_1, T_2) \in I_1(T)} \left(f^{=T_1} f^{=T_2}\right)^{=S}.$$

Now $I_1 = \bigcup_{i \in S} I_1(i)$, so as a multiset inclusion-exclusion shows that we have

$$I_1 = \sum_{T \subseteq S} (-1)^{|T|-1} \bigcap_{i \in T} I_1(i) = \sum_{T \subseteq S} (-1)^{|T|-1} I_1(T).$$

We therefore have the equality:

$$\sum_{(T_1, T_2) \in I_1} \left(f^{=T_1} f^{=T_2}\right)^{=S} = \sum_{T \subseteq S, T \neq \varnothing} (-1)^{|T|-1} \left(L_T[f]^2\right)^{=S}.$$

By the triangle inequality and Cauchy–Schwarz, we obtain that

$$\|\sum_{(T_1,T_2)\in I_1}\left(f^{=T_1}f^{=T_2}\right)^{=S}\|_2^2 \le \left(\sum_{i=1}^{|S|}\binom{|S|}{i}(4|S|)^{-i}\right)\left(\sum_{T\subseteq S}(4|S|)^{|T|}\|\left(L_T[f]^2\right)^{=S}\|_2^2\right)$$
$$\le \sum_{T\subseteq S}(4|S|)^{|T|}\|\left(L_T[f]^2\right)^{=S}\|_2^2.$$

Summing over all S we have

$$\sum_S\|\sum_{(T_1,T_2)\in I_1}\left(f^{=T_1}f^{=T_2}\right)^{=S}\|_2^2 \le \sum_T (4d)^{|T|}\|L_T[f]\|_4^4.$$

Upper bounding the contribution from I_2. We now upper bound the contribution from I_2. Let $T_1\Delta T_2 = S$. Then for each $S' \subsetneq S$, we assert that $A_{S'}\left(f^{=T_1}f^{=T_2}\right) = 0$. Let $i\in S\setminus S'$. Then $i\in T_1\Delta T_2$. Assume without loss of generality that $i\in T_1$. Then

$$A_{S'\cup T_2}\left(f^{=T_1}f^{=T_2}\right) = f^{=T_2}A_{S'\cup T_2}f^{=T_1} = 0.$$

This shows that $A_{S'}\left[f^{=T_1}f^{=T_2}\right] = 0$. Hence,

$$\left(f^{=T_1}f^{=T_2}\right)^{=S} = A_S\left(f^{=T_1}f^{=T_2}\right) = \left\langle f^{=T_1}(x,\cdot), f^{=T_2}(x,\cdot)\right\rangle_{L^2\left(\mu_{\overline{S}}\right)}.$$

By Cauchy–Schwarz we have

$$\|\sum_{(T_1,T_2)\in I_2}\left(f^{=T_1}f^{=T_2}\right)^{=S}\|_2^2 = \sum_{T_1\Delta T_2=T_3\Delta T_4=S}\left\langle\left(f^{=T_1}f^{=T_2}\right)^{=S},\left(f^{=T_3}f^{=T_4}\right)^{=S}\right\rangle$$
$$\le \sum_{T_1\Delta T_2=T_3\Delta T_4=S}\left\|\left(f^{=T_1}f^{=T_2}\right)^{=S}\right\|_2\left\|\left(f^{=T_3}f^{=T_4}\right)^{=S}\right\|_2.$$

Now, for each $(T_1,T_2)\in I_2$ we have

$$\left\|\left(f^{=T_1}f^{=T_2}\right)^{=S}\right\|_2^2 = \mathbb{E}_{x\sim\mu_S}\left\langle f^{=T_1}(x,\cdot), f^{=T_2}(x,\cdot)\right\rangle_{L^2\left(\mu_{\overline{S}}\right)}^2$$
$$\le \mathbb{E}_{x\sim\mu_S}\left[\|f^{=T_1}(x,\cdot)\|_{L^2\left(\mu_{\overline{S}}\right)}^2\|f_x^{=T_2}\|_{L^2\left(\mu_{\overline{S}}\right)}^2\right]$$
$$= \mathbb{E}_{x\sim\mu_S}\|f_x^{=T_1}\|_2^2\mathbb{E}_{x\sim\mu_S}\|f_x^{=T_2}\|_2^2$$
$$= \|f^{=T_1}\|_2^2\|f^{=T_2}\|_2^2,$$

where in the second equality we used the fact that $\|f^{=T}(x,\cdot)\|_{L^2\left(\mu_{\overline{S}}\right)}^2$ depends only on $x_{T\cap S}$, so these are independent for $T=T_1$ and $T=T_2$. This establishes

$$\mathbb{E}\left[\left(f^{=T_1}f^{=T_2}\right)^{=S}\left(f^{=T_3}f^{=T_4}\right)^{=S}\right] \le \|f^{=T_1}\|_2\|f^{=T_2}\|_2\|f^{=T_3}\|_2\|f^{=T_4}\|_2$$

Summing over all S, we obtain

$$\sum_S\|\sum_{(T_1,T_2)\in I_2}\left(f^{=T_1}f^{=T_2}\right)^{=S}\|_2^2 \le \mathbb{E}_{\left(\{0,1\}^n,\mu_{\frac{1}{2}}\right)}\left(\sum_{S\subseteq[n]}\|f^{=S}\|_2\chi_S\right)^4$$
$$\le 9^d\mathbb{E}\left[\left(\sum\|f^{=S}\|_2\chi_S\right)^2\right]^2$$
$$= 9^d\|f\|_2^4.$$

Here the first inequality follows by expanding both terms and the second is a well known consequence of hypercontractivity in the uniform cube.

Showing that there is no contribution from I_3. We recall that I_3 consist of the pairs with either $(T_1 \Delta T_2) \setminus S \neq \varnothing$ or $S \setminus (T_1 \cup T_2)$. Then we claim that $\left(f^{=T_1} f^{=T_2}\right)^{=S} = 0$. If $T_1 \cup T_2$ does not contain S, then

$$f^{=T_1} f^{=T_2} = A_{T_1 \cup T_2}\left(f^{=T_1} f^{=T_2}\right) = \sum_{S' \subseteq S} \left(f^{=T_1} f^{=T_2}\right)^{=S'}.$$

The uniqueness of the Efron–Stein decomposition shows that $\left(f^{=T_1} f^{=T_2}\right)^{=S} = 0$. Suppose now that there exists $i \in (T_1 \Delta T_2) \setminus S$. Without loss of generality $i \in T_1$. We then have

$$A_{[k]\setminus\{i\}}\left(f^{=T_1} f^{=T_2}\right) = f^{=T_2} \cdot A_{[k]\setminus\{i\}} f^{=T_1} = 0.$$

In particular, $\left(f^{=T_1} f^{=T_2}\right)^{=S} = 0$ as for each $S \subseteq [k] \setminus \{i\}$ we have

$$\left(f^{=T_1} f^{=T_2}\right)^{=S} \left[A_{[k]\setminus\{i\}}\left(f^{=T_1} f^{=T_2}\right)\right]^{=S} = 0$$

Combining the contributions from I_1 and I_2. The lemma now follows by Cauchy–Schwarz. We have

$$\begin{aligned}
\|f\|_4^4 &\leq \sum_S \| \left(f^2\right)^{=S} \|_2^2 \\
&\leq \sum_S \left(2 \sum_{(T_1,T_2)\in I_1} \| \left(f^2\right)^{=S} \|_2^2 + 2 \sum_{(T_1,T_2)\in I_2} \| \left(f^2\right)^{=S} \|_2^2\right) \\
&\leq 2 \sum_T (4d)^{|T|} \|L_T[f]\|_4^4 + 2 \cdot 9^d \|f\|_2^4.
\end{aligned}$$

□

Finally, using Lemma 2.8, we can derive Theorem 2.7 as follows.

Proof of Theorem 2.7. The proof is by induction on d. Since $D_{T,x}[f]$ is of degree $d - |T|$, we have

$$\begin{aligned}
\frac{1}{2}\|f\|_4^4 &\leq 9^d \|f\|_2^4 + \sum_{T \neq \varnothing} (4d)^{|T|} \|L_T[f]\|_4^4 \\
&= 9^d \|f\|_2^4 + \sum_{T \neq \varnothing} (4d)^{|T|} \mathbb{E}_{x\sim\mu_T} \|D_{T,x}[f]\|_4^4 \\
&\leq 9^d \|f\|_2^4 + \sum_{T \neq \varnothing} 2 \cdot 9^{d-|T|} (4d)^{|T|} \sum_{T' \subseteq [n]\setminus T} (8d)^{T'} \mathbb{E}_{x\sim\mu_{T\cup T'}} I_{T\cup T',x}^2 \\
&= 9^d \|f\|_2^4 + \sum_{T \cap T' = \varnothing} 2^{|T'|+1} 9^{d-|T|} (4d)^{|T\cup T'|} \mathbb{E}_{x\sim\mu_{T\cup T'}} \|D_{T'\cup T,x}[f]\|_2^4 \\
&\leq 9^d \sum_{T\subseteq S} (9d)^{|T|} \mathbb{E}_{x\sim\mu_T} I_{T,x}^2.
\end{aligned}$$

□

3. ϵ-PRODUCT SPACES AND THE OPERATORS $A_{S,T}$

In this section, we present the framework of ϵ-product spaces, of which high dimensional expanders are a special case. We also define key operators in this setting and show some basic properties that they satisfy.

3.1. **Complexes having ϵ-pseudorandom links.**

It is useful for us to consider measures on $V_1\times\cdots\times V_k$ rather than pure $(k-1)$-dimensional complexes, which can be identified with subsets $S\subseteq V^k$. Instead we identify a set with the uniform measure over it.

Projected complexes. Let μ be a probability measure on $V_1\times\cdots\times V_k$. We say that μ is a, weighted k-partite, $(k-1)$-dimensional complex. Let $S\subseteq[k]$ we write μ_S for the projection of μ on S. We write μ_i rather than $\mu_{\{i\}}$. We write V_S for the support of μ_S inside $\prod_{i\in S}V_i$. We write $\overline{S}$ for the complement of S.

Restricted complexes. Let $x\in V_S$. We write μ_x for the measure on $V_{\overline{S}}$ given by

$$\mu_x\left(y\right)=\frac{\mu\left(x,y\right)}{\mu_S\left(x\right)}.$$

We write V_x for the support of μ_x. We refer to (V_x,μ_x) as the *link* of μ on x.

ϵ-pseudorandom weighted graphs. Let V_1,V_2 be finite sets. A measure μ on $V_1\times V_2$ can be thought of as a weighted bipartite graph. We say that μ is ϵ-pseudorandom if for each $f_1\colon V_1\to\mathbb{R}$, $f_2\colon V_2\to\mathbb{R}$ we have

$$\begin{aligned}\left|\mathbb{E}_{(x_1,x_2)\sim\mu}\left[f_1\left(x_1\right)f_2\left(x_2\right)\right]-\mathbb{E}_{x_1\sim\mu_1}\left[f_1\left(x_1\right)\right]\mathbb{E}_{x_2\sim\mu_2}\left[f_2\left(x_2\right)\right]\right|\leq\\ \epsilon\sqrt{\mathrm{Var}_{x_1\sim\mu_1}\left[f_1\left(x_1\right)\right]\mathrm{Var}_{x_2\sim\mu_2}\left[f_2\left(x_2\right)\right]}.\end{aligned}$$

We let A_{12} be the operator from $L^2\left(V_1,\mu_1\right)$ to $L^2\left(V_2,\mu_2\right)$ given by

$$A_{12}f\left(x\right)=\mathbb{E}_{y\sim\mu_x}\left[f\left(y\right)\right].$$

We have the following standard lemma.

Lemma 3.1. *The following are equivalent.*

1. μ *is ϵ-pseudorandom*
2. $\|A_{12}-\mathbb{E}\|_{2\to2}\leq\epsilon$.
3. *The second eigenvalue of* $\mathrm{A}_{12}^*\mathrm{A}_{12}$ *is* $\leq\epsilon^2$.

ϵ-pseudorandom links. Now let μ on $V_1\times\cdots\times V_k$. We say that μ has *ϵ-pseudorandom skeletons* if for each S of size 2 the measure μ_S is ϵ-pseudorandom.

We say that μ is *ϵ-product* if for each $S\subseteq[k]$ of size $\leq k-2$ and each $x\in V_S$ the link μ_x has ϵ-pseudorandom skeletons.

In all that follows we assume that μ is an ϵ-product measure on $V_1\times\cdots\times V_k$.

Inheritance. The definition of ϵ-product makes it easy for inductive type argument for the following reason.

Lemma 3.2. *Let μ on $\prod_{i=1}^k V_i$ be ϵ-product. Let $S,T\subseteq[k]$ be disjoint. Then for each $x\in V_S$, the probability measure $\left(\mu_x\right)_T=\left(\mu_{S\cup T}\right)_x$ is ϵ-product.*

Proof. All the skeletons of links of $(\mu_{S\to x})_T$ are also skeletons of links of μ. $\square$

Pseudorandomness as a measure of independence. Let $S,T\subseteq[n]$. Then we have an operator $A_{S,T}:L^2(V_S,\mu_S)\to L^2(V_T,\mu_T)$. The operator is given by

$$A_{S,T}f(y)=\mathbb{E}_{x\sim\mu}\left[f(x_S)\,|x_T=y\right].$$

We write $A^{\mu}_{S,T}$ to stress that the operator is taken with respect to μ. We write A_S for $A_{[k],S}$, the operator given by restricting S and taking expectation.

When S,T are disjoint we expect $A_{S,T}f$ to be close to $\mathbb{E}[f]$, as in the product case $A_{S,T}$ is equal to the expectation. In fact, we do have the following.

Lemma 3.3. *Let μ be ϵ-product. Let $S,T\subseteq[k]$ be disjoint, and let $f\in L^2(V_S,\mu_S)$. We have*

$$\|A_{S,T}f-\mathbb{E}[f]\|_2^2\le|S|\,|T|\,\epsilon^2\|f\|_2^2$$

Proof. We prove it by induction on k. The case where $k=2$ is Lemma 3.1, so we assume $k>2$. Given a probability space (Ω,μ) we write $1^{\perp}$ for the subspace of $L^2(\Omega,\mu)$ consisting of functions that are orthogonal to the constant function 1. We write $\tilde{\|}A_{S,T}\tilde{\|}$ for the L^2 operator norm of $A_{S,T}$ as an operator from $1^{\perp}$ to $1^{\perp}$. I.e.

$$\tilde{\|}A_{S,T}\tilde{\|}=\max_{f\in1^{\perp}}\frac{\|A_{S,T}f\|_2}{\|f\|_2}.$$

Our goal is to show that

$$\tilde{\|}A_{S,T}\tilde{\|}\le\sqrt{|S|\,|T|}\epsilon.$$

Discarding the trivial cases. If $T=\varnothing$, then $A_{S,T}=\mathbb{E}$ and the result is trivial. If $S\cup T\neq[k]$ the result follows by inducting by working with the space $(V_{S\cup T},\mu_{S\cup T})$ rather then $\left(V_{[k]},\mu_{[k]}\right)$. We also have $\tilde{\|}A_{S,T}\tilde{\|}=\tilde{\|}A^*_{S,T}\tilde{\|}$ as $A_{S,T}1=1$. As $A^*_{S,T}=A_{T,S}$ we may assume that $|T|\le|S|$. As $k>2$ we may therefore assume that $|T|\ge2$.

Completing the proof in the case where $|T|>1$. Assume without loss of generality that $1\in T$.

Let $f\in1^{\perp}$. Using the fact that the equality

$$\|X\|_2^2=\mathbb{E}[X]^2+\|X-\mathbb{E}[X]\|_2^2$$

holds for every random variable X we have

$$\begin{aligned}\|A_{S,T}f\|_2^2&=\mathbb{E}_{y\sim\mu_T}\mathbb{E}^2_{x\sim\mu_y}\left[f(x_S)\right]\\&=\mathbb{E}_{a\sim\mu_1}\|A^{\mu_a}_{S,T\setminus\{1\}}f\|_2^2\\&=\mathbb{E}_{a\sim\mu_1}\left[\mathbb{E}^2_{\mu_a}f+\|A^{\mu_a}_{S,T\setminus\{1\}}f-\mathbb{E}_{\mu_a}f\|_2^2\right].\\&=\mathbb{E}_{a\sim\mu_1}\left[A_{S,1}f^2(a)+\|A^{\mu_a}_{S,T\setminus\{1\}}f-\mathbb{E}_{\mu_a}f\|_2^2\right]\end{aligned}$$

By induction we may upper bound the right hand side we have

$$\begin{aligned}RHS&\le\mathbb{E}_{a\sim\mu_1}\left[A_{S,1}f^2(a)+|S|\,|T-1|\,\epsilon^2\|f\|^2_{L^2(\mu_a)}\right].\\&=\|A_{S,1}f\|_2^2+|S|\,|T-1|\,\epsilon^2\|f\|_2^2.\\&\le|S|+|S|\,|T-1|\,\epsilon^2\|f\|_2^2\\&=|S|\,|T|\,\epsilon^2\|f\|_2^2\end{aligned}$$

□

Understanding the operators $A_{S,T}$ and their compositions. We now deduce that we have a similar upper bound of the form

$$\|A_{S,T}-A_{S,S\cap T}\|_{2\to 2}\leq\sqrt{|S|\,|T|}\epsilon.$$

Corollary 3.4. *Let $S,T\subseteq[k]$, and let $f\in L^2(\mu_S)$. Then*

$$\|A_{S,T}f-A_{S,S\cap T}f\|_2^2\leq|S|\,|T|\,\epsilon^2\|f\|_2^2.$$

Proof. Lemma 3.3 covers the case $S\cap T=\varnothing$. This shows that the corollary is true in μ_x for each $x\in V_{S\cap T}$. Therefore

$$\begin{aligned}\|A_{S,T}f-A_{S,S\cap T}f\|_2^2 &= \mathbb{E}_{x\sim\mu_{S\cap T}}\|A^{\mu_x}_{S\setminus T,T\setminus S}f-A^{\mu_x}_{S\setminus T,\varnothing}f\|^2_{L^2(\mu_x)}\leq\\ &\leq|S|\,|T|\,\epsilon^2\mathbb{E}_x\|f\|^2_{L^2(\mu_x)}\\ &=|S|\,|T|\,\epsilon^2\|f\|_2^2.\end{aligned}$$

□

We now show that compositions behave similarly to the product space setting.

Lemma 3.5. *We have*

$$\|A_{T_2}A_{T_1}-A_{T_1\cap T_2}\|_{2\to 2}\leq|T_1|\,|T_2|\,\epsilon.$$

Proof. We may assume that $T_1\cap T_2=\varnothing$. Indeed, if the lemma holds for $T_1\cap T_2=\varnothing$ then it holds in general. Indeed, write

$$\tilde{T_1}=T_1\setminus T_2,\tilde{T_2}=T_2\setminus T_1,A=[k]\setminus(T_1\cap T_2).$$

Let $x\in V_{T_1\cap T_2}$. Then we have

$$(A_{T_2}A_{T_1}f)(x,\cdot)=\left(A^{\mu_x}_{\tilde{T}_2}A^{\mu_x}_{\tilde{T}_1}\right)(f(x,\cdot))$$

and

$$A_{T_1\cap T_2}f(x)=\mathbb{E}_{y\sim\mu_x}[f(x,y)].$$

Therefore once we prove the case $T_1\cap T_2=\varnothing$ it would imply that for each x

$$\mathbb{E}_{y\sim\mu_x}(A_{T_2}A_{T_1}f(x,y)-A_{T_1\cap T_2}f(x,y))^2\leq|T_1|\,|T_2|\,\epsilon^2\mathbb{E}_{y\sim\mu_x}f(x,y)^2.$$

The lemma will then follow by taking expectations over x.

Let us now settle the case $T_1\cap T_2=\varnothing$. Write $\mathrm{T}=A_{T_2}A_{T_1}$. Then

$$\mathrm{T}=A_{T_1,T_2}A_{T_1}.$$

Write $g=A_{T_1}f$. We have $\|g\|_2\leq\|f\|_2$ by Cauchy–Schwarz. By Lemma 3.3 we have

$$\begin{aligned}\|\mathrm{T}f-\mathbb{E}[f]\|_2^2 &= \|A_{T_1,T_2}g-\mathbb{E}g\|_2^2\\ &\leq|T_1|\,|T_2|\,\epsilon^2\|g\|_2^2\\ &\leq|T_1|\,|T_2|\,\epsilon^2\|f\|_2^2.\end{aligned}$$

□

4. Efron–Stein decompositions for link expanders

In this section, we introduce a new approximate Efron–Stein decomposition for high dimensional expanders. In fact, it is more convenient to state and prove our results in the more general setting of ϵ-product spaces, of which high dimensional expanders are a special case. We proceed to discuss this setting below.

We first define the Efron–Stein decomposition via the usual formula for it.

Definition 4.1. Let $f \in L^2(V,\mu)$ and $S \subseteq [n]$. We write

$$f^{=S} = \sum_{T \subseteq [S]} (-1)^{|S \setminus T|} A_T f.$$

The functions $f^{=S}$ are defined in terms of the operators A_T. L^2-wise the composition of the operators $\{A_T\}_{T \subseteq [k]}$ behave similarly to the compositions in the product case setting. We satrt this section by making use of that and showing that may known facts from the product setting generalize to the ϵ-product setting up to a small error.

4.1. L^2-approximations for the Efron–Stein decomposition.

Thinking of ϵ as tending to 0 in a much quicker pace than $\frac{1}{k}$. Our goal is now to show that if μ is ϵ-product, then we have:

(1)
$$\left| \|f\|_2^2 - \sum_{S \subseteq [k]} \|f^{=S}\|_2^2 \right| = o\left(\|f\|_2^2\right),$$

(2) and more generally
$$\left| \langle f, g \rangle - \sum_S \left\langle f^{=S}, g^{=S} \right\rangle \right| = o\left(\|f\|_2 \|g\|_2\right).$$

One main tool involves the notion of a junta. We say that $g \colon V \to \mathbb{R}$ is a *T-junta* if $g(x)$ depends only on x_T. Equivalently, g is a T-junta if $A_T g = g$.

Our first step towards the proof is a near orthogonality result between $f^{=T}$ and $g^{=S}$ for $T \neq S$.

We start by a Fourier formula that holds exactly, this is unlike most of the results in this section that only generalize the situation from the product space setting up to a small error term.

Lemma 4.2. *We have*

$$A_S[f] = \sum_{T \subseteq S} f^{=T}(x).$$

In particular $f = \sum_{S \subseteq [k]} f^{=S}$.

Proof. We have

$$\begin{aligned} \sum_{T \subseteq S} f^{=T} &= \sum_{T \subseteq S} \sum_{T' \subseteq T} (-1)^{|T \setminus T'|} A_{T'} f \\ &= \sum_{T' \subseteq S} A_{T'} f \sum_{T' \subseteq T \subseteq S} (-1)^{|T \setminus T'|} \\ &= A_S f, \end{aligned}$$

where the last equality follows from the fact that whenever $T' \neq S$ and $i \in S \setminus T'$ the pairs

$$(T, T \Delta \{i\})$$

contribute opposing signs to the sum $\sum_{T'\subseteq T\subseteq S}(-1)^{|T\setminus T'|}$. The 'in particular' part follows by taking $S=[k]$. □

The following lemma holds even without assuming that μ is ϵ-product.

Lemma 4.3. *We have* $\|A_{S,T}\|_{2\to 2}\le 1$ *and*

$$\|f^{=S}\|_2 \le 2^{|S|}\|f\|_2.$$

Proof. The triangle inequality implies that it suffices to prove the former claim. Now by Cauchy–Schwarz we have

$$\begin{aligned}\|A_{S,T}f\|_2^2 &= \mathbb{E}_{x\sim\mu_T}A_{S,T}f(x)^2\\ &= \mathbb{E}_{x\sim\mu_T}\left(\mathbb{E}_{y\sim(\mu_x)_T}f(y)\right)^2\\ &\le \mathbb{E}_{x\sim\mu_T}\mathbb{E}_{y\sim(\mu_x)_T}f(y)^2\\ &= \|f\|_2^2.\end{aligned}$$

□

Lemma 4.4. *Let $f\colon V\to\mathbb{R}$ let T be a set not containing S and let g be a T-junta. Then*

$$\left\langle f^{=S},g\right\rangle \le \epsilon\sqrt{|S|\,|T|}2^{|S|}\|f\|_2\|g\|_2.$$

Proof. As A_T is the dual to the inclusion operator $L^2(V_T)\to L^2\left(V_{[k]}\right)$ we have

$$\left\langle f^{=S},g\right\rangle = \left\langle A_T f^{=S},g\right\rangle.$$

By Cauchy–Schwarz it is sufficient to show that

$$\|A_T f^{=S}\|_2 \le \epsilon\,|S|\,|T|\,2^{|S|}\|f\|_2.$$

Now

$$A_T f^{=S} = \sum_{S'\subseteq S}(-1)^{|S\setminus S'|}A_T A_{S'}f.$$

Roughly speaking, we rely on Lemma 4.2, which says that $\|A_T A_{S'}-A_{T\cap S'}\|_{2\to 2}$ is small together with the fact that

$$\sum_{S'\subseteq S}(-1)^{|S\setminus S'|}A_{T\cap S'}f = 0. \tag{4.1}$$

The equality follows by choosing an arbitrary $i\in S\setminus T$ and noting that the sets $(S',S'\Delta\{i\})_{S'\subseteq S}$ correspond to the same term $A_{[k],T\cap S'}$, while appearing with opposite signs. This shows that we have

$$A_T f^{=S} = \sum_{S'\subseteq S}(-1)^{|S\setminus S'|}\left(A_T A_{S'}f - A_{T\cap S'}f\right).$$

By Lemma 3.5 we have

$$\|A_T A_{S'}f - A_{T\cap S'}f\|_2 \le \sqrt{|T|\,|S|}\epsilon\|f\|_2 \le \sqrt{|S|\,|T|}\epsilon\|f\|_2.$$

Hence,

$$\|A_T f^{=S}\|_2 \le \sqrt{|S|\,|T|}2^{|S|}\epsilon.$$

□

Proof of our near orhogonality result.

Corollary 4.5. *Let $T \neq S$. Then* $\left\langle f^{=S}, g^{=T} \right\rangle \leq 2^{2|S|+2|T|}\epsilon\|f\|_2\|g\|_2$.

Proof. The function $g^{=T}$ is a T-junta. By Lemmas 4.4 and 4.3 we therefore have the following chain of inequalities if T does not contain S.

$$\left\langle f^{=S}, g^{=T} \right\rangle \leq \epsilon\sqrt{|S|\,|T|}2^{|S|}\|f\|_2\|g^{=T}\|_2 \leq \epsilon 2^{2|S|+2|T|}\|f\|_2\|g\|_2.$$

A similar chain of inequalities holds when S does not contain T. □

Parseval holds approximately for the Efron–Stein decomposition.

Lemma 4.6. *We have*

$$\left| \langle f, g\rangle - \sum_{S\subseteq[k]} \left\langle f^{=S}, g^{=S} \right\rangle \right| \leq 2^{4k}\epsilon\|f\|_2\|g\|_2.$$

Moreover, if f is a T-junta, then

$$\left| \langle f, g\rangle - \sum_{S\subseteq T} \left\langle f^{=S}, g^{=S} \right\rangle \right| = 2^{4|T|}\epsilon\|f\|_2\|g\|_2.$$

Proof. We have $\langle f, g\rangle = \sum_{S\subseteq[k]} \left\langle f^{=S}, g^{=S} \right\rangle + \sum_{S\neq T} \left\langle f^{=S}, g^{=T} \right\rangle$. By corollary 4.5 we have

$$\sum_{S\neq T} \left\langle f^{=S}, g^{=T} \right\rangle \leq 2^{4k}\epsilon\|f\|_2\|g\|_2.$$

For the ‘moreover’ part note that if f is a T-junta, then

$$\langle f, g\rangle = \langle f, A_T g\rangle_{L^2(\mu_T)}.$$

We may then apply the first part of the lemma in μ_T noting that $(A_T g)^{=T'} = g^{=T'}$ for each $T' \subseteq T$. □

$\left(f^{=S}\right)^{=S}$ **is L^2-close to $f^{=S}$..** In the product space setting we have $\left(f^{=S}\right)^{=T} = \begin{cases} f^{=S} & T = S \\ 0 & T \neq S \end{cases}$. Here we have the following instead:

Lemma 4.7. *Let $g = f^{=S}$. Then:*

(1) *If $S \neq T$, then*

$$\|g^{=T}\|_2^2 \leq 2^{8k}\epsilon^2\|f\|_2^2$$

(2)

$$\|g^{=S} - g\|_2^2 \leq 2^{10k}\epsilon^2\|f\|_2^2.$$

Proof. We have

$$g^{=T} = \sum_{T'\subseteq T, S'\subseteq S} (-1)^{|S\setminus S'|+|T\setminus T'|} A_{T'}A_{S'}f.$$

Write

$$h = \sum_{T'\subseteq T, S'\subseteq S} (-1)^{|S\setminus S'|+|T\setminus T'|} A_{T'\cap S'}f.$$

By Lemma 3.4 we therefore have

$$\|h - g^{=T}\|_2 \leq 2^{2k}\max_{T',S'}\|A_{T'}A_{S'} - A_{T'\cap S'}\|_{2\to 2}\|f\|_2 \leq 2^{4k}\epsilon\|f\|_2.$$

Now we claim that $h = 0$. Indeed, assume without loss of generality that T is not contained in S and let $i \in T \setminus S$. Then the terms $A_{T' \cap S'}$ appears with opposing sums for the pairs T' and $T' \Delta \{i\}$.

(2)-follows by the fact that

$$\|g^{=S} - g\|_2 = \left\| \sum_{T \neq S} g^{=T} \right\|_2 \leq \sum_{T \neq S} \|g^{=T}\|_2 \leq 2^{5k} \epsilon \|f\|_2.$$

□

4.2. Approximate Efron-Stein decomposition.

Again think of ϵ as tending to 0 much more quickly than $\frac{1}{k}$. We now define a notion of (α, ϵ')-approximate Efron–Stein decomposition. We show that a version of Lemma 4.6 still holds for these approximate Efron–Stein decompositions.

Motivation. One reason that demonstrates our need for an approximate Efron–Stein decomposition is as follows. Let $f^{\leq d} = \sum_{|S| < d} f^{=S}$. Then we do not have

$$\left(f^{\leq d}\right)^{=S} = \begin{cases} f^{=S} & |S| \leq d \\ 0 & |S| > d \end{cases},$$

but we would nevertheless like to work with the decomposition $\left\{f^{=S}\right\}_{|S| \leq d}$ as an approximate Efron–Stein decomposition for f. We capture that notion as follows.

Defining the (α, ϵ')-approximate Efron–Stein decomposition.

Definition 4.8. We say that $\{f_S\}_{S \subseteq [k]}$ is an (α, ϵ')-approximate Efron–Stein decomposition if

(1) $\|f\|_2 \leq \alpha$.
(2)
$$\|f - \sum_S f_S\|_2 < \epsilon',$$
(3) For each S there exists h_S with $\|h_S\|_2 \leq \alpha$ and
$$\|h_S^{=S} - f_S\|_2 \leq \epsilon'.$$

It turns out that we have an approximate Parseval theorem for every approximate Efron–Stein decomposition.

Lemma 4.9. *Let $\alpha_1, \alpha_2, \epsilon_1, \epsilon_2 > 0$. Suppose that f has an (α_1, ϵ_1)-bounded approximate Efron–Stein decomposition $\{f_S\}$ and g has an (α_2, ϵ_2)-bounded Efron–Stein decomposition $\{g_S\}$. Then*

$$\left| \langle f, g \rangle - \sum_S \langle f_S, g_S \rangle \right| \leq 2^{6k} \left(\epsilon_1 \alpha_2 + \epsilon_2 \alpha_1 + \epsilon \alpha_1 \alpha_2\right).$$

Proof. For each $S \subseteq [k]$ let $\tilde{f}_S, \tilde{g}_S$ be with $\|\tilde{f}_S\|_2 \leq \alpha_1, \|\tilde{g}_S\|_2 \leq \alpha_2$

$$\|\tilde{f}_S^{=S} - f_S\|_2 \leq \epsilon_1,$$

and

$$\|\tilde{g}_S^{=S} - g_S\|_2 \leq \epsilon_2.$$

Let

$$f'_S = \tilde{f}_S^{=S}, g'_S = \tilde{g}_S^{=S},$$

$$f' = \sum_{S\subseteq[k]} f'{}_S$$

and

$$g' = \sum_{S\subseteq[k]} g'_S.$$

By Lemma 4.5 we have

$$\begin{aligned}\langle f', g'\rangle &= \sum_S \langle f'_S, g'_S\rangle + \sum_{S\neq T\subseteq[k]} \langle f'_S, g'_T\rangle \\ &= \sum_S \langle f'_S, g'_S\rangle \pm 2^{6k}\epsilon\alpha_1\alpha_2.\end{aligned}$$

Now by Cauchy–Schwarz

$$\begin{aligned}\langle f, g\rangle &= \langle f', g'\rangle + \langle f', g - g'\rangle + \langle f - f', g\rangle \\ &= \sum_S \langle f'_S, g'_S\rangle \pm \left(2^{6k}\epsilon\alpha_1\alpha_2 + \|f'\|_2\|g - g'\|_2 + \|f - f'\|_2\|g\|_2\right) \\ &= \sum_S \langle f'_S, g'_S\rangle \pm \left(2^{6k}\epsilon\alpha_1\alpha_2 + 2^{2k}\alpha_1\epsilon_2 + \epsilon_1\alpha_2\right),\end{aligned}$$

where the last equality used

$$\|f'\|_2 \le \sum \|f'_S\|_2 \le 2^{k+|S|}\alpha_1 \le 2^{2k}\alpha_1,$$

which follows from Lemma 4.3.

To complete the proof we note that we similarly have

$$\begin{aligned}\langle f'_S, g'_S\rangle &= \langle f_S, g_S\rangle \pm \|f_S\|_2\|g_S - g'_S\|_2 + \|f'_S - f_S\|_2\|g'_S\|_2 \\ &= \langle f_S, g_S\rangle \pm \alpha\epsilon_2 + 2^k\epsilon_1\alpha_2.\end{aligned}$$

□

The above approximate Efron–Stein decomposition works well when we care about L_2-norms. We actually care about closeness in higher norms specifically 4-norms. Our strategy when wishing to upper bound $\|f - f'\|_4$ is to use the inequality

$$\|f - f'\|_4^4 \le \|f - f'\|_2^2 \left(\|f\|_\infty + \|f'\|_\infty\right).$$

Where we hope that the L^2-closeness is sufficient to overcome the loss of using infinity norms. We would therefore like everything to have a relatively small infinity norm.

Definition 4.10. We say that $\{f_S\}$ is a $(\beta, \alpha, \epsilon')$-*bounded approximate Efron-Stein* decomposition if it is an (α, ϵ')-approximate Efron–Stein decomposition and moreover for each S:

$$\|h_S^{=S}\|_\infty, \|f_S\|_\infty, \|f\|_\infty$$

are all $\le \beta$. Here $h_S^{=S}$ is as in Definition 4.8.

We now show that the different Efron–Stein decompositions of a function f are all close in L_4.

Lemma 4.11. *Suppose that $\{f_S\}, \{f'_S\}$ are $(\beta, \alpha, \epsilon')$-bounded approximate Efron–Stein decompositions for f. Then*

(1)
$$\|f_S - f'_S\|_2^2 \le O_k\left(\epsilon'\right)^2 + O_k\left(\epsilon\alpha^2\right),$$

(2)
$$\|f_S - f'_S\|_4^4 \le O_k\left(\epsilon'^2\beta^2\right) + O_k\left(\epsilon\alpha^2\beta^2\right),$$

(3)
$$\|\sum_S \left(f_S - f'_S\right)\|_4^4 \le O_k\left(\epsilon'^2\beta^2\right) + O_k\left(\epsilon^2\alpha^2\beta^2\right),$$

(4) *and*
$$\|f - \sum_{S\subseteq[k]} f_S\|_4^4 \le O_k\left(\epsilon^2\beta^2\right)\left(\alpha^2 + \|f\|_2^2\right).$$

Proof. (3) is an immediate corollary of (2). (4) also follows immediately from (3) by setting $f'_S = f^{=S}$ while applying it with $2^k\beta$ rather than β. Indeed, $\|f^{=S}\|_\infty \le 2^k\|f\|_\infty \le 2^k\beta$. Therefore $\left\{f^{=S}\right\}_{S\subseteq[k]}$ is a $\left(2^k\beta, \alpha, 0\right)$-approximate Efron–Stein decomposition for f. (2) follows immediately from (1) as we have

$$\|f_S - f'_S\|_4^4 \le \|f_S - f'_S\|_2^2\|f_S - f'_S\|_\infty^2$$

and $\|f_S - f'_S\|_\infty^2 \le 4\beta^2$.

We now prove (1).

Reducing to the case that $f'_S = f^{=S}$. First we assert that we may assume that $f'_S = f^{=S}$ for each S. Indeed, $\left\{f^{=S}\right\}$ is a $(\beta, \alpha, 0)$-Efron–Stein decomposition. By the triangle inequality we have

$$\|f_S - f'_S\|_2 \le \|f_S - f^{=S}\|_2 + \|f^{=S} - f'_S\|_2,$$

which implies (by Hólder) that

$$\|f_S - f'_S\|_2^2 \le 2\|f_S - f^{=S}\|_2^2 + 2\|f^{=S} - f'_S\|_2^2.$$

This shows that it is sufficient to prove the theorem when $\{f_S\} = \left\{f^{=S}\right\}$ and when $\{f'_S\} = \left\{f^{=S}\right\}$. Without loss of generality we may assume that $f'_S = f^{=S}$.

Reducing to the case that $f_S = h_S^{\overline{=}S}$. Let h_S be with $\|h_S\|_2 \le \alpha$ and $\|f_S - h_S^{\overline{=}S}\|_2 < \epsilon'$. Setting $\tilde{f}_S = h_S^{\overline{=}S}$ we obtain by the triangle inequality that $\left\{\tilde{f}_S\right\}_{S\subseteq[k]}$ is a $\left(\beta, \alpha, \left(2^k+1\right)\epsilon'\right)$-bounded approximate Efron–Stein decomposition for f. We have

$$\|f_S - f^{=S}\|_2^2 \le 2\|\tilde{f}_S - f_S\|_2^2 + 2\|\tilde{f}_S - f^{=S}\|_2 \le 2\epsilon' + 2\|\tilde{f}_S - f^{=S}\|_2.$$

Therefore it is sufficient to prove (1) when f_S is replaced by $\tilde{f}_S$.

Proving the lemma when $f_S = h_S^{\overline{=}S}$ and $f'_S = f^{=S}$. By Cauchy–Schwarz and Corollary 4.5 we have:

$$\begin{aligned}
\left\langle f_S - f^{=S}, f\right\rangle &= \sum_{T\subseteq[k]}\left\langle f_S - f^{=S}, f^{=T}\right\rangle \\
&= \left\langle f_S - f^{=S}, f^{=S}\right\rangle + \sum_{T\neq S}\left\langle f^{=S} - h_S^{\overline{=}S}, f^{=T}\right\rangle \\
&= \left\langle f_S - f^{=S}, f^{=S}\right\rangle + O_k\left(\epsilon\alpha^2\right).
\end{aligned}$$

Again by Corollary 4.5 and Cauchy–Schwarz we have:

$$\begin{aligned}\langle f_S - f^{=S}, f\rangle &= \left\langle f_S - f^{=S}, \sum_T f_T \right\rangle + \left\langle f_S - f^{=S}, f - \sum_T f_T \right\rangle \\ &= \langle f_S - f^{=S}, f_S\rangle + O_k\left(\epsilon\alpha^2\right) + \|f_S - f^{=S}\|_2\epsilon'.\end{aligned}$$

Rearranging we obtain,

$$\|f_S - f^{=S}\|_2^2 \leq O_k\left(\epsilon'\right)\left(\|f^{=S} - f_S\|_2\right) + O_k\left(\epsilon\alpha^2\right).$$

This shows that

$$\|f_S - f^{=S}\|_2^2 \leq O_k\left(\epsilon'\right)^2 + O_k\left(\epsilon\alpha^2\right).$$

□

5. Proof overview

Building on the framework we established in Section 4, we can now give a proof overview for our hypercontactive inequality on high dimensional expanders. Recall that in the setting of direct products, we first prove a key lemma, (Lemma 2.8) and then use it to derive the theorem via an inductive argument. We now give a sketch of how to generalise this approach to the ϵ-product setting.

5.1. **Generalising Lemma 2.8.**

Recall that we would like to show a lemma of the form

$$\|f\|_4^4 \leq C^d\|f\|_2^4 + \sum_S (4d)^{|S|}\|L_S[f]\|_4^4.$$

We instead show a similar lemma that holds up to a small error term of $O_k\left(\epsilon\|f\|_2^2\|f\|_\infty^2\right)$:

$$\frac{1}{2}\|f^{\leq d}\|_4^4 \leq 9^d\|f^{\leq d}\|_2^4 + 4\sum_{0<|T|\leq d}(4d)^{|T|}\|L_T^{\leq d}[f]\|_4^4 + O_k(\epsilon)\|f\|_2^2\|f\|_\infty^2. \tag{5.1}$$

However, first note that we do not have a useful notion of a low degree function. Instead we work with

$$f^{\leq d} = \sum_{|S|\leq d} f^{=S}.$$

In turn, instead of $L_S[f]$ we have

$$L_S^{\leq d}[f] = \sum_{T\supseteq S, |T|\leq d} f^{=T}.$$

We show that when expanding

$$\left(\left(f^{\leq d}\right)^2\right)^{=S} = \sum_{T_1,T_2}\left(f^{=T_1}f^{=T_2}\right)^{=S},$$

there are three kinds of terms: (1) terms that vanish in the product space setting, but here they do not; (2) terms with $T_1\cap T_2\cap S\neq\varnothing$; and (3) terms with $T_1\Delta T_2 = S$.

Our high-level approach is to show that the same proof as in the setting of product spaces works up to an error term. We accomplish that by expressing everything in terms of our operators $\{A_S\}$, and we then replace equalities that hold in the product space by L_2-approximation of the form

$$\|A_SA_T - A_{S\cap T}\|_{2\to 2} \leq O_k(\epsilon).$$

At first glance, it might appear that this approach would not suffice, as we eventually would like to upper bound 4-norms of terms, or 2-norms of expressions involving the product of two functions such as $\left(f^{=T_1}f^{=T_2}\right)^S$. Nevertheless, we are able to accomplish that via inequalities of the form

$$\|f\|_4^4 \le \|f\|_2^2\|f\|_\infty^2.$$

We then use the fact that all our terms are bounded by $O_k\left(\|f\|_\infty\right)$, and our L_2-approximations involve ϵ, and therefore beat the $O_k\left(1\right)$-terms. This allows us to generalise Lemma 2.8 and prove (5.1).

5.2. **Applying induction.** After having an inequality of the form

$$\|f^{\le d}\|_4^4 \le C^d\|f^{\le d}\|_2^2 + \sum_S \left(4d\right)^{|S|}\|L_S^{\le d}\left[f\right]\|_4^4,$$

we would like to use a similar idea to the one we used in the product space setting; that is, restrict S to some $x\in V_S$, and then apply induction for the function $L_S^{\le d}\left[f\right]\left(x,\cdot\right)$. The problem is that the restricted function $L_S^{\le d}\left[f\right]\left(x,\cdot\right)$ is no longer of degree $d-|S|$, and hence we can no longer use induction.

We overcome this problem by using the notion of our approximate Efron–Stein decompositions. Namely, we show that $L_S^{\le d}$ has two different approximate Efron–Stein decomposition. The first one is

$$\left\{f^{=T}\right\}_{T\supseteq S,|T|\le d},$$

and the other one replaces $f^{=T}$ by the function f_T

$$\left(x,y\right)\mapsto\left(L_S\left[f\right]\left(x,\cdot\right)\right)^{=T\setminus S}\left(y\right).$$

We then obtain that $\sum_{|T|\supseteq S,|T|\le d} f_T\left(x,\cdot\right)$ is of the form $D_{S,x}^{\le d-|S|}$, which allows us to use induction similarly as in the product space setting.

After applying induction we get the compositions of two derivatives, and we are again able to translate them back to expressions of the form $\mathbb{E}_{x\sim\mu_S}I_{S,x}^2$ by showing that $D_{S,x}D_{T,y}$ and $D_{S\cup T,(x,y)}$ are both approximate Efron–Stein decompositions of the same expression.

The remaining step is to upper bound the influences. We achieve that by generalising the inequality

$$\mathbb{E}_{x\sim\mu_S}\left[I_{S,x}^2\right] \le \delta\|L_S\left[f\right]\|_2^2$$

from the product space setting, where crucially, we obtain that without upper bounding $\|I_{S,x}\|_\infty$.

6. Laplacians, influences, and globalness on ϵ-measures

In this section, we define the notions of laplacians, derivatives and influences in the setting of ϵ-measures, give bounded approximated Efron–Stein decompositions related to the Laplacians, define globalness, and show that it implies small influences.

6.1. Defining the Laplacians, derivatives and influences.

Definition 6.1. We define the Laplacians via the formula

$$L_i\left[f\right] = f - A_{[k]\setminus\{i\}}f.$$

Lemma 6.2. *We have*

$$L_i\left[f\right] = \sum_{S\ni i} f^{=S}.$$

Proof. This follows immediately from Lemma 4.2, which shows that

$$A_{[k]\setminus\{i\}}[f] = \sum_{S\subseteq[k]\setminus\{i\}} f^{=S}.$$

□

Definition 6.3. We define $L_S[f] = \sum_{T\supseteq S} f^{=T}$. Alternatively,

$$L_S[f] = \sum_{T\subseteq S} (-1)^{|T|} A_{[k]\setminus T} f.$$

Let $x \in V_S$. We let $D_{S,x} = L_S[f](x,\cdot)$, i.e. the function in $L^2(V_x, \mu_x)$ obtained by plugging in x in the S coordinates. We let

$$I_{S,x}[f] = \|D_{S,x}[f]\|_{L^2(V_x,\mu_x)}.$$

6.2. **Bounded approximated Efron–Stein decompositions related to the Laplacians.**

Lemma 6.4. . *There exists $C = O_k(1)$, such that $\left\{f^{=T}\right\}_{T\supseteq S}$ is a $(C\|f\|_\infty, C\|f\|_2, 0)$-bounded approximate Efron–Stein decomposition for $L_S[f]$.*

Proof. We have

$$\|f^{=S}\|_\infty \le \sum_{T\subseteq S} \|A_T f\|_\infty \le 2^{|S|}\|f\|_\infty$$

and

$$\|L_S[f]\|_\infty \le \sum_{T\subseteq S} \|A_{\overline{T}} f\|_\infty \le 2^{|S|}\|f\|_\infty.$$

The other properties are easy to verify. □

Let $f \in L^2\left(V_{[k]}, \mu_{[k]}\right)$ and let $g_T \in L^2(V_T, \mu_T)$ be given by

$$g_T(x) = I_{T,x}[f].$$

Then g_T can be interpreted interms of the Laplacians and the averaging operators as

$$g_T = A_T\left(L_T[f]^2\right).$$

Suppose that $\{f_S\}_{S\subseteq[k]}$ is a $(\beta, \alpha, \epsilon')$-bounded approximate Efron–Stein decompositions for f and set $\widetilde{L_T[f]} = \sum_{S\subseteq T} f_S$. The following lemma essentially shows that the function $A_T\left(\widetilde{L_T[f]}^2\right)$ is a good L_2-approximation for the function g_T. This can be interpreted by saying that the generalised influences could be computed via any $(\beta, \alpha, \epsilon')$-bounded approximate Efron–Stein decomposition for f.

Lemma 6.5. *Let $\{f_S\}$ and $\{f'_S\}$ be $(\beta, \alpha, \epsilon')$-bounded Efron-Stein decompositions for f. Then*

$$\|A_T\left(\sum_{S\supseteq T} f_S\right)^2\|_2^2 \le 2\|A_T\left(\sum_{S'\supseteq T} f'_S\right)^2\|_2^2 + O_k\left(\epsilon'^2\beta^2\right) + O_k\left(\epsilon\alpha^2\beta^2\right).$$

Proof. By Cauchy–Schwarz we have

$$\left(\sum_{S\supseteq T} f_S\right)^2 \le 2\left(\sum_{S\supseteq T} f'_S\right)^2 + 2\left(\sum_{S\supseteq T} f'_S - f_S\right)^2.$$

Therefore

$$\|A_T\left(\sum_{S\supseteq T} f_S\right)^2\|_2^2 \le 2\|A_T\left(\sum_{S'\supseteq T} f'_S\right)^2\|_2^2 + 2\|A_T\left(\sum_{S\supseteq T} f'_S - f_S\right)^2\|_2^2.$$

Now since A_T contracts 2-norms (Lemma 4.3). We have

$$2\|A_T\left(\sum_{S\supseteq T} f'_S - f_S\right)^2\|_2^2 \le 2\|\sum_{S\supseteq T} f'_S - f_S\|_4^4.$$

Lemma 4.11 now completes the proof. □

We now show that Lemma 6.4 is a special case of a more general phenomenon. Whenever $\{f_S\}_{S\subseteq[k]}$ is a (β,α,ϵ')-bounded approximate Efron–Stein decomposition for f, we obtain that $\{f_T\}_{T\supseteq S}$ is a $\left(\tilde{\beta},\tilde{\alpha},\tilde{\epsilon}\right)$-bounded approximate Efron–Stein decomposition for suitable values of $\tilde{\beta},\tilde{\alpha},\tilde{\epsilon}$. We show the following.

Lemma 6.6. *There exists $C = O_k(1)$, such that the following holds. Suppose that $\{f_T\}_{T\subseteq[k]}$ is a (β,α,ϵ')-Approximate Efron–Stein decomposition for f. Then $\{f_T\}_{T\supseteq S}$ is a $(C\beta,\alpha,C(\epsilon'+\alpha\sqrt{\epsilon}))$-Approximate Efron–Stein decomposition for $L_S[f]$.*

Proof. The only requirements that are not automatically inherited from f are the upper bounds on $\|L_S[f]\|_\infty$, and on $\|L_S f - \sum_{T\supseteq S} f_T\|_2$. The former inequality follows from the inequality

$$\|L_S[f]\|_\infty \le 2^{|S|}\|f\|_\infty \le 2^k\beta.$$

While the latter follows from Lemma 4.11 and the triangle inequality:

$$\begin{aligned}\|L_S f - \sum_{T\supseteq S} f_T\|_2 &= \|\sum_{T\supseteq S}\left(f^{=T} - f_T\right)\|_2.\\ &\le \sum_{T\supseteq S}\|f^{=T} - f_T\|_2\\ &\le O_k(\epsilon') + O_k\left(\alpha\sqrt{\epsilon}\right).\end{aligned}$$

□

6.3. **Low degree functions and truncations.**

Definition 6.7. We define the *low degree part* of f by setting

$$f^{\le d} = \sum_{|S|\le d} f^{=S}$$

we define the *low degree Laplacians* of f by setting

$$L_T^{\le d}[f] = \sum_{S\supseteq T,|S|\le d} f^{=T}.$$

We now show that if $\{f_T\}_{T\subseteq[k]}$ is a (β,α,ϵ')bounded approxiate Efron–Stein decomposition for f, then we may turn it into an Efron–Stein decomposition for $f^{\le d}$ and $L_S^{\le d}[f]$ in the obvious way.

Lemma 6.8. *There exists $C = O_k(1)$, such that the following holds. Suppose that $\{f_T\}_{T\subseteq[k]}$ is a (β,α,ϵ')-Approximate Efron–Stein decomposition for f. Then*

(1) *The functions $\{f_T\}_{|T|\le d}$ are a $(C\beta,\alpha,C\epsilon+C\epsilon')$-approximate Efron–Stein decomposition for $f^{\le d}$.*

(2) *the functions* $\{f_T\}_{|T|\supseteq S,|T|\leq d}$ *are a* $(C\beta,\alpha,C\epsilon+C\epsilon')$*-approximate Efron–Stein decomposition for* $L_S^{\leq d}[f]$.

Proof. It is sufficient to prove (2) as (1) is the special case where $S=\varnothing$. By Lemma 4.11, we have

$$\|f_T-f^{=T}\|_2\leq O_k(\epsilon')+O_k\left(\sqrt{\epsilon}\alpha\right).$$

Hence, by the triangle inequality we have

$$\begin{aligned}\|L_S^{\leq d}[f]-\sum_{T\supseteq S,|T|\leq d}f_T\|_2&\leq\sum_{T\supseteq S,|T|\leq d}\|f^{=T}-f_T\|_2\\&=O_k(\epsilon')+O_k\left(\sqrt{\epsilon}\alpha\right).\end{aligned}$$

Moreover,

$$\begin{aligned}\|L_S^{\leq d}[f]\|_\infty&=\left\|\sum_{T\supseteq S,|T|\leq d}f^{=T}\right\|_\infty\\&\leq 2^k\max_S\|f^{=S}\|_\infty\\&\leq 4^k\|f\|_\infty\leq 4^k\beta.\end{aligned}$$

□

6.4. **Globalness.** Unlike the product space setting the two possible definitions of globalness are not equivalent. It turns out to be more convenient to work with the notion concerning the restrictions.

Definition 6.9. We say that f is (d,δ)-global if for each $|S|\leq d$ we have

$$\|f(x,\cdot)\|_{L^2(V_x,\mu_x)}\leq\delta.$$

Claim 6.10. If f is (d,δ)-global and ϵ is sufficiently small, then for each T of size $\leq d$ we have

$$\|f^{=T}\|_\infty\leq 2^{|T|}\delta.$$

Proof. This follows from the triangle inequality once we show that $\|A_{T'}f\|_\infty\leq\delta$ for each $T'\subseteq T$. Indeed, for each x we have

$$A_{T'}f(x)=\mathbb{E}_{(V_x,\mu_x)}f(x,\cdot)\leq\|f(x,\cdot)\|_{L^2(V_x,\mu_x)}\leq\delta.$$

□

Lemma 6.11. *Suppose that* f *is* (d,δ)*-global. Then* $\left\{f^{=S}\right\}_{|S|\leq d}$ *is a* $\left(k^d\delta,\|f\|_2,0\right)$*-bounded Efron–Stein decomposition for* $f^{\leq d}$.

Proof. We have $\|f^{=S}\|_\infty\leq 2^d\delta$ by Claim 6.10. We also have

$$\|f\|_\infty\leq\sum\|f^{=S}\|_\infty\leq k^d\delta.$$

The rest of the conditions hold automatically. □

Definition 6.12. We say that f is of (β,α)-degree d if $f=\sum_{|S|\leq d}f_S$ and $f_S=h_S^{=S}$ where $\|h_S\|_2\leq\alpha$, $\|h_S^{=S}\|_\infty\leq\beta$ and $\|f\|_2\leq\alpha$, $\|f\|_\infty\leq\beta$.

If $f=\sum_{|S|\leq d}f_S$ is of (β,α)-degree d as above, then $\{f_S\}$ is one $(\beta,\alpha,0)$-bounded Efron–Stein decomposition for f. We now show that in this case the canonical $\left\{f^{=S}\right\}_{|S|\leq d}$ is also $(\beta',\alpha',\epsilon')$-bounded Efron–Stein decomposition for the right parameters.

6.5. Other approximate Efron-Stein decompositions for $L_T[f], L_T^{\leq d}[f]$.

Definition 6.13. We define the low degree derivatives for $T \subseteq [k]$ and $x \in V_T$

$$D_{T,x}^{\leq d}\colon L^2(\mu) \to L^2(V_x, \mu_x)$$

via

$$D_{T,x}^{\leq d}[f] = L_T^{\leq d}[f](x, \cdot)$$

The low degree influences for $T \subseteq [n]$ and $x \in V_T$ are defined by

$$I_{T,x}^{\leq d}[f] = \|L_T^{\leq d}[f](x, \cdot)\|_{L^2(V_x,\mu_x)}^2.$$

We now move on to the critical lemma for our inductive approach. In the product space setting our inductive approach relied on the fact that $D_{T,x}f$ is of degree $\leq d - |T|$ whenever f is of degree d. Here we show that $L_T[f]$ has an alternative $(\beta, \alpha, \epsilon)$-bounded approximate Efron–Stein decompositions $\{f_S\}_{S \supseteq T}$ that gives rise to a function

$$\widetilde{L_T^{\leq d}}[f] = \sum_{S \supseteq T, |S| \leq d} f^{=S}$$

with the property that for each x $\widetilde{L_T^{\leq d}}[f](x, \cdot)$ is of degree $d - |T|$.

Lemma 6.14. *Let $f_S(x, y) = D_{T,x}^{=S \setminus T}$. Then for each f:*

(1) The set $\{f_S\}_{S \supseteq T}$ is a $(C\|f\|_\infty, C\|f\|_2, C\epsilon\|f\|_2)$-bounded approximate Efron–Stein decomposition for $L_T[f]$.
(2) The set $\{f_S\}_{S \supseteq T, |S| \leq d}$ is a $(C\|f\|_\infty, C\|f\|_2, C\epsilon\|f\|_2)$-bounded approximate Efron–Stein decomposition for $L_T^{\leq d}[f]$.
(3) If $T' \supseteq T$, then the set $\{f_S\}_{S \supseteq T'}$ is a $(C\|f\|_\infty, C\|f\|_2, C\epsilon\|f\|_2)$-bounded approximate Efron Stein decomposition for $L_{T'}[f]$.
(4) The set $\{f_S\}_{S \supseteq T', |S| \leq d}$ is a $(C\|f\|_\infty, C\|f\|_2, C\epsilon\|f\|_2)$-bounded approximate Efron–Stein decomposition for $L_{T'}^{\leq d}[f]$.
(5) If f is (d, δ)-global. Then $\{f_S\}_{S \supseteq T', |S| \leq d}$ is a $(C\delta, C\|f\|_2, C\epsilon\|f\|_2)$-bounded approximate Efron–Stein decomposition for $L_{T'}^{\leq d}[f]$.

Proof. Due to Lemma 6.8 (1) implies (2)-(4). By Lemma 4.3 all the operators A_I contract ∞-norms. We therefore have

$$\|f_S\|_\infty \leq \max_x 2^{|S \setminus T|}\|D_{T,x}f\|_\infty = 2^{|S \setminus T|}\|L_T f\|_\infty \leq 2^{|S|}\|f\|_\infty.$$

To complete the proof it is sufficient to show that

$$\|f_S - f^{=S}\|_2 \leq O_k(\epsilon\|f\|_2)$$

as this will also imply that

$$\begin{aligned}\|\sum_{S \supseteq T} f_S - L_T[f]\|_2 &= \|\sum_{S \supseteq T} f_S - \sum_{S \supseteq T} f^{=S}\|_2 \\ &= O_k(\epsilon\|f\|_2).\end{aligned}$$

We have

$$\begin{aligned} f_S &= \sum_{T\subseteq S'\subseteq S} (-1)^{\left|S\setminus S'\right|} A_{S'} L_T f. \\ &= \sum_{T\subseteq S'\subseteq S} \sum_{T'\subseteq T} (-1)^{\left|S\setminus S'\right|+\left|S\setminus S''\right|} A_{S'} A_{T'} f. \end{aligned}$$

Write

$$h = \sum_{T\subseteq S'\subseteq S} \sum_{T'\subseteq T} (-1)^{\left|S\setminus S'\right|+\left|S\setminus T'\right|} A_{S'\cap T'} f.$$

Then by Lemma 3.4 we have

$$\|f_S - h\|_2 \leq O_k(\epsilon)\,\|f\|_2.$$

We now assert that whenever $S' \neq S$ the sum corresponding to it is 0. Indeed otherwise there is some $i \in S \setminus S'$ and $T', T' \cup \{i\}$ appear with alternating signs and correspond to the same term $A_{S'\cap T'}$. We therefore have

$$h = \sum_{T'\subseteq S} (-1)^{\left|S\setminus T'\right|} A_{T'} f = f^{=S}.$$

This shows that $\|f_S - f^{=S}\|_2 \leq O_k(\epsilon)\,\|f\|_2$, which completes the proof. □

6.6. **Globalness implies small influences.** In the product space setting we had $\|I_{T,x}\|_\infty \leq 4^d\delta^2$ and we used it via the inequality

$$\mathbb{E}_{x\sim\mu_T}\left[I_{T,x}^2\right] \leq \mathbb{E}_{x\sim\mu_T}\left[I_{T,x}\right] 4^d\delta^2 = 4^d\delta^2 \|L_T[f]\|_2^2. \tag{6.1}$$

See the proof of Corollary 2.6. Here we find a convoluted way of proving an analogue of (6.1) without having any upper bound on $\|I_{T,x}\|_\infty$ at our disposal.

Lemma 6.15. *Suppose that $f\colon V_{[k]} \to \mathbb{R}$ is (d,δ)-global, and let $|T| \leq d$. Then*

$$\mathbb{E}_{x\sim\mu_T}\left[\left(I_{T,x}[f]\right)^2\right] \leq 2^{d+1}\delta^2\mathbb{E}_{x\sim\mu_T} I_{T,x} + O_k\left(\epsilon^2\|f\|_4^4\right).$$

Proof. Write $g(x) = I_{T,x}[f]$. Then $g = A_T\left[\left(L_T[f]\right)^2\right]$. We would like to upper bound $\|g\|_2^2$. We accomplish that by upper bounding $\|g\|_2^2$ by $\mathbb{E}[gg'']$ for a function g'' with a small ∞-norm.

By Cauchy–Schwarz we have

$$L_T[f]^2 \leq 2^{|T|} \sum_{T'\supseteq \overline{T}} A_{T'}[f]^2.$$

This shows that

$$g \leq 2^{|T|} \sum_{T'\subseteq T} A_T\left[\left(A_{T'} f\right)^2\right], \tag{6.2}$$

on all x. Let us denote by g' the right hand side of 6.2. Also let

$$g'' = 2^{|T|} \sum_{T'\subseteq T} A_{T\cap T'}\left[\left(A_{T'} f\right)^2\right].$$

Then in the product space setting the functions g', g'' would have been equal. Here we have an L^2-approximation between them.

Claim 6.16. $\|g' - g''\|_2 \leq O_k(\epsilon)\,\|f\|_4^2$.

Proof. As $(A_{T'}f)^2$ is a T'-junta we have

$$(A_{T'}f)^2 = A_{T'}\left[(A_{T'}f)^2\right].$$

By Lemma 3.4 we have

$$\|A_T A_{T'} - A_{T\cap T'}\|_{2\to 2} \leq O_k(\epsilon).$$

We therefore have

$$\begin{aligned}\|A_T\left[(A_{T'}f)^2\right] - A_{T\cap T'}\left[(A_{T'}f)^2\right]\|_2^2 &\leq O_k\left(\epsilon^2\right)\|\left(A_{T'}f\right)^2\|_2^2 \\ &\leq O_k\left(\epsilon^2\right)\|f\|_4^4,\end{aligned}$$

as $A_{T'}$ contracts 4-norms. Therefore,

$$\|g' - g''\|_2 \leq O_k(\epsilon)\|f\|_4^2.$$

□

As $0 \leq g \leq g'$ we have

$$\mathbb{E}\left[g^2\right] \leq \mathbb{E}\left[g'g\right] \leq \mathbb{E}\left[g''g\right] + \mathbb{E}\left[\left(g' - g''\right)g\right].$$

By Cauchy–Schwarz we have

$$\mathbb{E}\left[\left(g' - g''\right)g\right] \leq \|g' - g''\|_2\|g\|_2 \leq O_k(\epsilon)\|f\|_4^2\|g\|_2.$$

Now either

$$\mathbb{E}\left[g^2\right] \leq 2\mathbb{E}\left[g''g\right] \tag{6.3}$$

or

$$\mathbb{E}\left[g^2\right] \leq 2\mathbb{E}\left[\left(g' - g''\right)g\right] \leq O_k(\epsilon)\|f\|_4^2\|g\|_2.$$

In the latter case we have

$$\|g\|_2^2 \leq O_k\left(\epsilon^2\right)\|f\|_4^4. \tag{6.4}$$

after rearranging. We can now sum the upper bounds of (6.3) and (6.4) corresponding to each of the cases to obtain the upper bound

$$\mathbb{E}\left[g^2\right] \leq 2\mathbb{E}\left[g''g\right] + O_k\left(\epsilon^4\right)\|f\|_4^4.$$

that is true in both cases. The following claim completes the proof.

Claim 6.17. $\|g''\|_\infty \leq 2^d\delta^2$.

Proof. By Cauchy–Schwarz we point-wise have $(A_{T'}f)^2 \leq A_{T'}\left(f^2\right)$. We therefore have

$$A_{T\cap T'}\left[(A_{T'}f)^2\right] \leq A_{T\cap T'}A_{T'}\left(f^2\right) = A_{T\cap T'}\left(f^2\right) \leq \delta^2.$$

This shows that $\|g''\|_\infty \leq 2^d\delta^2$. □

□

The same proof works for the truncated influences.

Lemma 6.18. *Suppose that $f\colon V_{[k]} \to \mathbb{R}$ is (d,δ)-global. Suppose additionally that $\epsilon \leq \epsilon_0(k)$. Then we have*

$$\mathbb{E}_{x\sim\mu_T}\left[\left(I_{T,x}^{\leq d}[f]\right)^2\right] \leq 2^{d+4}\delta^2\mathbb{E}_{x\sim\mu_T}\left[I_{T,x}^{\leq d}\right] + O_k\left(\epsilon^2\|f\|_\infty^2\|f\|_2^2\right).$$

Proof. Write

$$g_1(x) = \|\left(D_{T,x}[f]\right)^{\leq d-|T|}\|_2^2.$$

We now proceed with the following steps.

Upper bounding $\mathbb{E}_{x\sim\mu_T}\left[\left(I_{T,x}^{\leq d}[f]\right)^2\right]$ in terms of g_1. By Lemma 6.14 the functions

$$\left\{D_{T,x}[f]^{=S}\right\}_{|S|\leq d-|T|},$$

is an alternative $(O_k\|f\|_\infty, O_k\|f\|_2, O_k(\epsilon\|f\|_2))$-bounded approximate Efron–Stein decomposition for $L_T^{\leq d}[f]$. Therefore by Lemma 6.5 we have

$$\mathbb{E}_{x\sim\mu_T}\left[I_{T,x}^{\leq d}[f]^2\right] \leq 2\mathbb{E}_{x\sim\mu_T}\left[g_1(x)^2\right] + O_k\left(\epsilon\|f\|_2^2\|f\|_\infty^2\right). \tag{6.5}$$

Repeating the proof of Lemma 6.18. Now by Lemma 4.9 we have $g_1(x) \leq 2I_{T,x}$ for each x, provided that ϵ is sufficiently small. Write $g_2(x) = I_{T,x}[f]$. Similarly to the proof of Lemma 6.18 we let

$$g_2' = 2^{|T|}\sum_{T'\subseteq T} A_T\left[\left(A_{T'}f\right)^2\right]$$

and let

$$g_2'' = 2^{|T|}\sum_{T'\subseteq T} A_{T\cap T'}\left[\left(A_{T'}f\right)^2\right].$$

By Cauchy–Schwarz we have

$$\|g_1\|_2^2 \leq 2\mathbb{E}[g_1g_2] \leq 2\mathbb{E}[g_1g_2'] \leq 2\mathbb{E}[g_1g_2''] + 2\mathbb{E}[g_1(g_2'-g_2'')].$$

Now either

$$\|g_1\|_2^2 \leq 4\mathbb{E}[g_2''g_1],$$

which would imply

$$\|g_1\|_2^2 \leq 4\mathbb{E}[g_2''g_1] \leq 2^{d+2}\delta\mathbb{E}[g_1]$$

by Claim 6.17, or

$$\|g_1\|_2^2 \leq 4\mathbb{E}[g_1(g_2'-g_2'')] \leq 4\|g_1\|_2\|g_2'-g_2''\|_2$$

and rearranging, we obtain

$$\|g_1\|_2^2 \leq 16\|g_2'-g_2''\|_2^2 \leq O_k\left(\epsilon^2\|f\|_4^4\right)$$

by Claim 6.16. This shows that

$$\|g_1\|_2^2 \leq 2^{d+2}\delta\mathbb{E}[g_1] + O_k\left(\epsilon^2\|f\|_4^4\right). \tag{6.6}$$

Moving back from g_1 to $I_{T,x}^{\leq d}$. By Lemmas 6.14 we have

$$\|\left(D_{T,x}[f]\right)^{\leq d-|T|} - D_{T,x}^{\leq d}[f]\|_2^2 \leq O_k\left(\epsilon^2\|f\|_2^2\right)$$

yielding

$$\mathbb{E}\left[g_1\right] \leq 2\|D_{T,x}^{\leq d}\left[f\right]\|_2^2 + O_k\left(\epsilon^2\|f\|_2^2\right) \tag{6.7}$$
$$= 2\mathbb{E}_{x\sim\mu_T} I_{T,x}^{\leq d}\left[f\right] + O_k\left(\epsilon^2\right)\|f\|_2^2$$

by the triangle inequality and Cauchy–Schwarz. By combining (6.5), (6.6) with (6.7) we obtain

$$\mathbb{E}_{x\sim\mu_T}\left[I_{T,x}^{\leq d}\left[f\right]^2\right] \leq 2\|g_1\|_2^2 + O_k\left(\epsilon\|f\|_2^2\|f\|_\infty^2\right).$$
$$\leq 2^{d+3}\delta^2\mathbb{E}\left[g_1\right] + O_k\left(\epsilon^2\|f\|_2^2\|f\|_\infty^2\right) + O_k\left(\epsilon^2\|f\|_4^4\right)$$
$$\leq 2^{d+3}\delta^2\mathbb{E}\left[g_1\right] + O_k\left(\epsilon^2\|f\|_2^2\|f\|_\infty^2\right).$$

The lemma now follows by putting everything together. □

7. Proving hypercontractivity for ϵ-product measures

We suggest revisiting Section 2 before reading this section. Our strategy is the same as in the product case, and we deal with the differences by appealing to the tools developed in Sections 3-6.

7.1. **Upper bounding $\|f^{\leq d}\|_4^4$ by 4-norms of non-trivial Laplacians and $\|f^{\leq d}\|_2^4$.** We now move on to preparing the ground for the proof of our hypercontractive inequality.

Lemma 7.1. *Let f be (d,δ)-global. Suppose that $\epsilon \leq \epsilon_0(k)$. Then we have*

$$\frac{1}{2}\|f^{\leq d}\|_4^4 \leq 9^d\|f^{\leq d}\|_2^4 + 4\sum_{0<|T|\leq d}(4d)^{|T|}\|L_T^{\leq d}\left[f\right]\|_4^4 + O_k\left(\epsilon\right)\|f\|_2^2\|f\|_\infty^2.$$

Proof. Let $g = f^{\leq d}$. By Lemma 4.9 we have

$$\|g\|_4^4 \leq 2\sum_S\|\left(g^2\right)^{=S}\|_2^2.$$

We now upper bound $\|\left(g^2\right)^{=S}\|_2^2$. We have

$$\left(g^2\right)^{=S} = \sum_{|T_1|\leq d,|T_2|\leq d}\left(f^{=T_1}f^{=T_2}\right)^{=S}.$$

Let

(1) $I_1 = \{(T_1,T_2) : T_1\cap T_2\cap S \neq \varnothing\}$.
(2) $I_2 = ((T_1,T_2) : T_1\Delta T_2 = S)$
(3) $I_3 = (T_1\Delta T_2)\setminus S \neq \varnothing$ or $S\setminus(T_1\cup T_2)\neq\varnothing$.

Our first step is to show that the contribution from I_3 is negligible. This is to be expected as in the product space setting we were able to show that the contribution from I_3 is 0.

Claim 7.2. Let $(T_1,T_2)\in I_3$. Then

$$\|\left(f^{=T_1}f^{=T_2}\right)^{=S}\|_2 \leq O_k\left(\epsilon^2\right)\|f\|_2^2\|f\|_\infty^2.$$

Proof. Suppose first that $(T_1\Delta T_2)\setminus S\neq\varnothing$. Then without loss of generality we may assume that there is some $i\in T_1\setminus(T_2\cup S)$. By Lemma 4.9 we have

$$\|\left(f^{=T_1}f^{=T_2}\right)^{=S}\|_2^2 \leq 2\|A_{[k]\setminus\{i\}}\left(f^{=T_1}f^{=T_2}\right)\|_2^2.$$

Now

$$A_{[k]\setminus\{i\}}\left(f^{=T_1}f^{=T_2}\right)=\left(A_{[k]\setminus\{i\}}\left(f^{=T_1}\right)\right)f^{=T_2}.$$

By Lemma 4.7 we have

$$\|A_{[k]\setminus\{i\}}f^{=T_1}\|_2\le\sum_{T'\not\ni i}\|\left(f^{=T_1}\right)^{=T'}\|_2\le O_k\left(\epsilon\right)\|f\|_2.$$

This shows that

$$\begin{aligned}\|A_{[k]\setminus\{i\}}\left(f^{=T_1}f^{=T_2}\right)\|_2^2&\le\|A_{[k]\setminus\{i\}}f^{=T_1}\|_2^2\|f^{=T_2}\|_\infty^2\\&\le O_k\left(\epsilon^2\right)\|f\|_\infty^2\|f\|_2^2.\end{aligned}$$

Suppose now that $S\setminus(T_1\cup T_2)\neq\varnothing$. Let $i\in S\setminus(T_1\cup T_2)$. Then the function $gf^{=T_1}f^{=T_2}$ is a $T_1\cup T_2$-junta. This shows that $g=A_{T_1\cup T_2}g$.

Hence by Lemma 4.7 and the triangle inequality we have

$$\begin{aligned}\|g^{=S}\|_2&=\|\left(A_{T_1\cup T_2}g\right)^{=S}\|_2\\&\le\|\sum_{T\subseteq T_1\cup T_2}\left(g^{=T}\right)^{=S}\|_2\\&\le O_k\left(\epsilon\right)\|g\|_2.\end{aligned}$$

It now remains to note that $\|g\|_2^2\le\|f^{=T_1}\|_2\|f^{=T_2}\|_\infty\le 2^{2k}\|f\|_2\|f\|_\infty$. □

We now move on to our next step of upper bounding the contribution from the pairs in I_1.

Claim 7.3. $\sum_{(T_1,T_2)\in I_1}\left(f^{=T_1}f^{=T_2}\right)^{=S}=\sum_{T\subseteq S}(-1)^{|T|+1}\|\left(L_T^{\le d}[f]^2\right)^{=S}\|_2^2$.

Proof. The proof is exactly the same as in the product case so we omit it. □

It now remains to consider the contribution from I_2, i.e. the case $T_1\Delta T_2=S$. Here just like the product case it is sufficient to show the following claim

Claim 7.4. Let $T_1\Delta T_2=S$. Then we have

$$\|\left(f^{=T_1}f^{=T_2}\right)^{=S}\|_2\le 2\|f^{=T_1}\|_2\|f^{=T_2}\|_2+O_k\left(\epsilon\right)\|f\|_2\|f\|_\infty,$$

provided that ϵ is sufficiently small.

Proof. First let $S'\subsetneq S$. As $T_1\Delta T_2=S$, there exists $i\in(T_1\Delta T_2)\setminus S'$. Without loss of generality $i\in T_1$. By Lemmas 4.9, 4.7, and 4.2 we have

$$\begin{aligned}\|A_{S'}\left(f^{=T_1}f^{=T_2}\right)\|_2&\le 2\|A_{S'\cup T_2}\left(f^{=T_1}f^{=T_2}\right)\|_2\\&\le 2\|A_{S'\cup T_2}f^{=T_1}\|_2\|f^{=T_2}\|_\infty\\&\le O_k\left(\epsilon\right)\|f\|_2\|f\|_\infty.\end{aligned}$$

By the triangle inequality this shows that

$$\begin{aligned}\|\left(f^{=T_1}f^{=T_2}\right)^{=S}\|_2&\le\sum_{S'}(-1)^{\left|S\setminus S'\right|}A_{S'}\left(f^{=T_1}f^{=T_2}\right).\\&\le\|A_S\left(f^{=T_1}f^{=T_2}\right)\|_2+O_k\left(\epsilon\right)\|f\|_2\|f\|_\infty\end{aligned}$$

We now upper bound $\|A_S\left(f^{=T_1}f^{=T_2}\right)\|_2$.

By Cauchy–Schwarz for $x \in V_S$ we have

$$\begin{aligned} A_S\left(f^{=T_1} f^{=T_2}\right)(x) &= \left\langle f^{=T_1}(x,\cdot), f^{=T_2}(x,\cdot)\right\rangle_{L^2(V_x,\mu_x)} \\ &\le \|f^{=T_1}(x,\cdot)\|_{L^2(V_x,\mu_x)} \|f^{=T_2}(x,\cdot)\|_{L^2(V_x,\mu_x)}. \end{aligned}$$

This shows that

$$\|A_S\left(f^{=T_1} f^{=T_2}\right)\|_2^2 \le \mathbb{E}_{x\sim\mu_S}\left[\|f^{=T_1}(x,\cdot)\|_{L^2(V_x,\mu_x)}^2 \|f^{=T_2}(x,\cdot)\|_{L^2(V_x,\mu_x)}^2\right]. \tag{7.1}$$

We have

$$\|f^{=T_1}(x,\cdot)\|_{L^2(V_x,\mu_x)}^2 = A_S\left[\left(f^{=T_1}\right)^2\right] = A_S A_{T_1}\left(f^{=T_1}\right)^2$$

By Lemma 3.4

$$\|A_S A_{T_1} - A_{S\cap T_1}\|_{2\to 2} \le O_k(\epsilon).$$

Hence,

$$\begin{aligned} \|A_S A_{T_1}\left(f^{=T_1}\right)^2 - A_{S\cap T_1}\left[\left(f^{=T_1}\right)^2\right]\|_2^2 &\le O_k\left(\epsilon^2 \|f^{=T_1}\|_4^2\right) \\ &\le O_k\left(\epsilon^2 \|f\|_2 \|f\|_\infty\right). \end{aligned}$$

By Cauchy–Schwarz this shows that

$$\begin{aligned} \text{RHS of (7.1)} &= \left\langle A_S A_{T_1}\left(f^{=T_1}\right)^2, A_S\left(f^{=T_2}\right)^2\right\rangle_{L^2(V_S,\mu_S)} \\ &= \left\langle A_{S\cap T_1}\left(f^{=T_1}\right)^2, A_S\left(f^{=T_2}\right)^2\right\rangle_{L^2(V_S,\mu_S)} \\ &\quad + O_k\left(\epsilon \|f\|_2 \|f\|_\infty\right) \|A_S\left(f^{=T_2}\right)^2\|_2^2. \end{aligned}$$

As we have

$$\|A_S\left(f^{=T_2}\right)^2\|_2^2 \le O_k\left(\|f\|_4^2 \le \|f\|_2 \|f\|_\infty\right).$$

This shows that

$$\|A_S\left(f^{=T_1} f^{=T_2}\right)\|_2^2 \le O_k\left(\epsilon \|f\|_2^2 \|f\|_\infty^2\right).$$

Therefore,

$$\begin{aligned} \|A_S\left(f^{=T_1} f^{=T_2}\right)\|_2^2 &\le \left\langle A_{S\cap T_1}\left(f^{=T_1}\right)^2, A_S\left(f^{=T_2}\right)^2\right\rangle_{L^2(V_S,\mu_S)} + O_k\left(\epsilon \|f\|_2^2 \|f\|_\infty^2\right) \\ &= \left\langle A_{S\cap T_1}\left(f^{=T_1}\right)^2, A_{S\cap T_1}\left(f^{=T_2}\right)^2\right\rangle_{L^2(\mu)} + O_k\left(\epsilon \|f\|_2^2 \|f\|_\infty^2\right). \end{aligned}$$

Now

$$A_{S\cap T_1}\left(f^{=T_2}\right)^2 = A_{S\cap T_1} A_{T_2}\left(f^{=T_2}\right)^2$$

and $\|A_{S\cap T_1} A_{T_2} - \mathbb{E}\|_{2\to 2} \le \epsilon$ by Lemma 3.4. Therefore we similarly have

$$\begin{aligned} \left\langle A_{S\cap T_1}\left(f^{=T_1}\right)^2, A_{S\cap T_1}\left(f^{=T_2}\right)^2\right\rangle_{L^2(\mu)} &= \left\langle A_{S\cap T_1}\left(f^{=T_1}\right)^2, \|f^{=T_2}\|_2^2\right\rangle + O_k\left(\epsilon \|f\|_2^2 \|f\|_\infty^2\right). \\ &= \|f^{=T_1}\|_2^2 \|f^{=T_2}\|_2^2 + O_k\left(\epsilon \|f\|_2^2 \|f\|_\infty^2\right). \end{aligned}$$

This completes the proof of the claim. □

The rest of the proof is the exactly the same as in the product case setting. □

Now the only thing to remains is to apply the inductive hypothesis.

Theorem 7.5. *We have* $\|f^{\leq d}\|_4^4 \leq 20^d \sum_{|S|\leq d} (4d)^{|S|}\, \mathbb{E}_{x\sim\mu_S} I_{S,x}^{\leq d}[f]^2 + O_k(\epsilon^2)\, \|f\|_2^2 \|f\|_\infty^2$.

Proof. The proof is by induction on d. By Lemma 7.1 we have

$$\|f^{\leq d}\|_4^4 \leq 2\cdot 9^d \|f^{\leq d}\|_2^4 + 2\cdot \sum_{S\neq\varnothing} (4d)^{|S|}\, \|L_S^{\leq d}[f]\|_4^4 + O_k(\epsilon)\, \|f\|_2^2\|f\|_\infty^2. \tag{7.2}$$

Write $g_{S,x}(y) = (D_{S,x}[f])^{\leq d-|T|}(y)$. Then by Lemma 6.14 and Lemma 4.11 we have:

$$\mathbb{E}_x \|D_{S,x}^{\leq d}[f]\|_4^4 \leq 2\mathbb{E}_x \|g_{S,x}\|_4^4 + O_k(\epsilon^2)\, \|f\|_2^2\|f\|_\infty^2.$$

By induction, we have

$$\|g_{S,x}\|_4^4 \leq 20^{d-|S|} \sum_{T\cap S=\varnothing, |T|\leq d-|S|} (4d)^{|T|}\, \mathbb{E}_{y\sim\mu_T} I_{T,y}^2[g_{S,x}] + O_k(\epsilon)\, \|g_{S,x}\|_2^2 \|g_{S,x}\|_\infty^2. \tag{7.3}$$

By Lemma 6.14 we have $\|g_{S,x}\|_\infty = O_k(\|f\|_\infty)$. By Lemmas 6.14, 4.11, and 4.9 we have

$$\begin{aligned} \mathbb{E}_{x\sim\mu_S} \|g_{S,x}\|_2^2 &\leq 2\mathbb{E}_{x\sim\mu_S} \|D_{S,x}^{\leq d} f\|_2^2 + O_k(\epsilon^2)\, \|f\|_2^2 \\ &\leq O_k\left(\|f\|_2^2\right) \end{aligned} \tag{7.4}$$

Taking expectations over (7.3), and plugging in (7.4) we obtain:

$$\mathbb{E}_x \|g_{S,x}\|_4^4 \leq 2 \sum_{T\cap S=\varnothing, |T|\leq d-|S|} (4d)^{|T|}\, \mathbb{E}_{(x,y)\sim\mu_{S\cup T}} I_{T,y}^2[g_{S,x}] + O_k(\epsilon)\, \|f\|_2^2\|f\|_\infty^2.$$

By Lemmas 6.14 and 4.11 we have

$$\mathbb{E}_{(x,y)\sim\mu_{S\cup T}} I_{T,y}^2[g_{S,x}] = \mathbb{E}_{z\sim\mu_{T\cup S}} I_{T\cup S,z}^2[g] + O_k(\epsilon^2)\, \|f\|_2^2\|f\|_\infty^2.$$

Hence,

$$\mathbb{E}_x \|g_{S,x}\|_4^4 \leq 20^{d-|S|} \sum_{S'\supseteq S |S'|\leq d} (4d)^{|S'\setminus S|}\, \mathbb{E}_{z\sim\mu_{S'}} \left(I_{S',z}^{\leq d}[f]\right)^2 + O_k(\epsilon)\, \|f\|_2^2\|f\|_\infty^2.$$

This gives

$$\mathbb{E}_x \|D_{S,x}^{\leq d}[f]\|_4^4 \leq 2\cdot 20^{d-|S|} \sum_{S'\supseteq S |S'|\leq d} (4d)^{|S'\setminus S|}\, \mathbb{E}_{z\sim\mu_{S'}} \left(I_{S',z}^{\leq d}[f]\right)^2 + O_k(\epsilon)\, \|f\|_2^2\|f\|_\infty^2$$

The proof is now completed by plugging this inequality in (7.2). Indeed, we have

$$\begin{aligned} \|f^{\leq d}\|_4^4 &\leq 2\cdot 9^d \|f\|_2^4 + O_k\left(\epsilon\|f\|_2^2\|f\|_\infty^2\right) \\ &\quad + \sum_{0<|S|\leq d} (4d)^{|S|}\cdot 2\cdot 20^{d-|S|} \sum_{S'\supseteq S |S'|\leq d} (4d)^{|S'\setminus S|}\, \mathbb{E}_{z\sim\mu_{S'}} \left(I_{S',z}^{\leq d}[f]\right)^2 \\ &\leq 20^d \sum_{|S'|\leq d} (4d)^{|S|}\, \mathbb{E}_{z\sim\mu_{S'}} \left(I_{S',z}^{\leq d}[f]\right)^2 + O_k\left(\epsilon\|f\|_2^2\|f\|_\infty^2\right). \end{aligned}$$

□

7.2. **The case where $\|f\|_\infty$ is large.** Here we show a hypercontractive inequality whose error term does not include the factor $\|f\|_\infty$. This may be useful when $\|f\|_\infty$ is significantly larger than δ.

Theorem 7.6. *Suppose that f is (d,δ)-global, then*

$$\|f^{\leq d}\|_4^4 \leq 20^{d+1} \sum_{|S|\leq d} (4d)^{|S|}\, \mathbb{E}_{x\sim\mu_S} I_{S,x}^{\leq d}[f]^2 + O_k(\epsilon^2\delta^2)\, \|f\|_2^2.$$

Proof. By applying Theorem 7.7 with $f^{\leq d}$ rather then f and using $\|f^{\leq d}\|_\infty \leq \delta$ we obtain

$$\|\left(f^{\leq d}\right)^{\leq d}\|_4^4 \leq 20^d \sum_{|S|\leq d} (4d)^{|S|}\, \mathbb{E}_{x\sim\mu_S} I_{S,x}^{\leq d}\left[f^{\leq d}\right]^2 + O_k\left(\epsilon\delta^2\right)\|f\|_2^2.$$

The theorem now follows from Lemmas 6.11, 4.11 and 6.5. □

Theorem 7.7. *Let $\epsilon \leq \epsilon_0(k)$ be sufficiently small. Suppose that f is (d,δ)-global. Then we have*

$$\|f^{\leq d}\|_4^4 \leq (100d)^d\,\delta^2\|f^{\leq d}\|_2^2 + O_k\left(\delta^2\epsilon^2\|f\|_2^2\right).$$

Proof. By Theorem 7.6, Lemma 6.18, and 4.9 we have

$$\begin{aligned}
\|f^{\leq d}\|_4^4 &\leq 20^d \sum_{|S|\leq d} (4d)^d\, \mathbb{E}_{x\sim\mu_S}\left(I_{S,x}^{\leq d}[f]\right)^2 + O_k\left(\epsilon^2\|f\|_2^2\|f\|_\infty^2\right)\\
&\leq 20^d \sum_{|S|\leq d} (8d)^{d+2}\,\delta^2\mathbb{E}_x\left[I_{S,x}^{\leq d}[f]\right] + O_k\left(\epsilon\|f\|_2^2\|f\|_\infty^2\right)\\
&\leq 20^d \sum_{|S|\leq d} (8d)^{d+2}\,\delta^2 \sum_{T\supseteq S,|T|\leq d} \|f^{=T}\|_2^2 + O_k\left(\epsilon\|f\|_2^2\|f\|_\infty^2\right)\\
&\leq (40d)^d\,\delta^2 \sum_{|T|\leq d}\|f^{=T}\|_2^2 + O_k\left(\epsilon\|f\|_2^2\|f\|_\infty^2\right)\\
&\leq 2\,(40d)^d\,\delta^2\|f^{\leq d}\|_2^2 + O_k\left(\epsilon\|f\|_2^2\|f\|_\infty^2\right).
\end{aligned}$$

□

8. Applications

In this section, we show our applications of the hypercontractive inequality on high dimensional expanders, which we have shown in the previous section. The applications follow in a fairly straightforward way, and hence we present them with brevity.

8.1. **Global Boolean functions are concentrated on the high degrees.** Fourier concentration results are widely useful in complexity theory and learning theory. Our first application is a Fourier concentration theorem for HDX. Namely, the following theorem shows that global Boolean functions on ϵ-HDX are concentrated on the high degrees, in the sense that the 2-norm of the restriction of a function to its low-degree coefficients only constitutes a tiny fraction of its total 2-norm.

Corollary 8.1. *If $f\colon V_{[k]} \to \{0,1\}$ is (d,δ)-global and ϵ is sufficiently small. Then*

$$\|f^{\leq d}\|_2^2 \leq \left(O_k\left(\sqrt{\epsilon}\right) + (200d)^d\,\delta^{\frac{1}{2}}\right)\|f\|_2^2.$$

Proof. By Lemma 4.9 we have

$$\|f^{\leq d}\|_2^2 = \left\langle f^{\leq d}, f\right\rangle - O_k(\epsilon)\|f\|_2^2.$$

We also have by Theorem 7.7

$$\begin{aligned}
\left\langle f^{\leq d}, f\right\rangle &\leq \|f^{\leq d}\|_4\|f\|_{\frac{4}{3}}\\
&\leq (100d)^d\,\delta^{\frac{1}{2}}\sqrt{\|f^{\leq d}\|_2}\|f\|_{\frac{4}{3}} + O_k\left(\sqrt{\epsilon}\|f\|_2^{\frac{1}{2}}\|f\|_\infty^{\frac{1}{2}}\|f\|_{\frac{4}{3}}\right).\\
&\leq 2\,(100d)^d\,\delta^{\frac{1}{2}}\sqrt{\|f\|_2}\|f\|_{\frac{4}{3}} + O_k\left(\sqrt{\epsilon}\|f\|_2^2\right)\\
&\leq (200d)^d\,\delta^{\frac{1}{2}}\|f\|_2^2 + O_k\left(\sqrt{\epsilon}\|f\|_2^2\right)
\end{aligned}$$

□

The Corollary completes the proof of Theorem 1.3.

8.2. **Small-set expansion theorem.** Small set expansion is a fundamental property that is prevalent in combinatorics and complexity theory. In the setting of the ρ-noisy Boolean hypercube, the small set expansion theorem gives an upper bound on $\mathrm{Stab}_\rho(1_A) = \langle 1_A, T_\rho 1_A\rangle = E[1_A(x)1_A(y)]$ for indicators 1_A of small sets A, which captures the probability that a random walk starting at a point $x \in A$ remains in A, hence showing that small sets are expanding. Our second application is a small set expansion theorem for global functions on ϵ-HDX, captured via bounding the natural noise operator in this setting.

Definition 8.2. Let $\rho \in (0,1)$. Given $x \in V_{[k]}$ we let $N_\rho(x)$ be the distribution where $y \sim N_\rho(x)$ is chosen by choosing a random set S where each i is in S independently with probability ρ, then choosing $z \sim \mu_{x_S}$ and setting $y = (x_S, z)$. We then set

$$\mathrm{T}_\rho f(x) = \mathbb{E}_{y\sim N_\rho(x)} f.$$

Alternatively we can use the averaging operators to give the following equivalent definition:

$$\mathrm{T}_\rho := \sum_{S\subseteq[k]} \rho^{|S|}(1-\rho)^{k-|S|} A_S[f].$$

We have the following formula for the noise operator, which is similar to the one in the product space setting.

Claim 8.3. We have $\mathrm{T}_\rho f = \sum_S \rho^{|S|} f^{=S}$.

Proof. We have

$$\begin{aligned}
\mathrm{T}_\rho f &= \sum_{S\subseteq[k]} \rho^{|S|}(1-\rho)^{k-|S|} A_S[f] \\
&= \sum_{S\subseteq[k]} \rho^{|S|}(1-\rho)^{k-|S|} \sum_{T\subseteq S} f^{=T} \\
&= \sum_{T\subseteq[k]} \sum_{S\supseteq T} \rho^{|S|}(1-\rho)^{k-|S|} \\
&= \sum_{T\subseteq[k]} \rho^{|T|} f^{=T}.
\end{aligned}$$

□

Via a standard argument we have the following bound on the noise operator.

Lemma 8.4. *We have*

$$\|\mathrm{T}_\rho f\|_2^2 \le \|f^{\le d}\|_2^2 + \left(\rho^d + O_k(\epsilon)\right)\|f\|_2^2.$$

Proof. This is immediate from Lemmas 8.3 and 4.9. □

Our small set expansion applications are as follows.

Corollary 8.5 (Small set expansion theorem)**.** *If $f\colon V_{[k]} \to \{0,1\}$ is (d,δ)-global. Then*

$$\|\mathrm{T}_\rho f\|_2^2 \le \left(\rho^d + (100d)^d \delta^2 + O_k\left(\sqrt{\epsilon}\right)\right)\|f\|_2^2.$$

Proof. This follows immediately from Lemma 8.4 and Corollary 8.1. □

8.3. **Kruskal–Katona theorem.** Our last application is an analogue of the Kruskal–Katona theorem in the setting of high dimensional expanders. The Kruskal-Katona theorem is a fundamental and widely-applied result in algebraic combinatorics, which gives a lower bound on the size of the lower shadow of a set A, denoted $\partial(A) = \{x\colon y \prec x, \text{ for some } y \in A\}$.

We first consider the natural up-down walk in our setting.

Definition 8.6. The operator corresponding to up-down random walk is

$$\mathrm{T} = \frac{1}{k}\sum_{i=1}^{k} A_{[k]\setminus\{i\}}\left[f\right] = \sum_{S} \frac{k-|S|}{k} f^{=S}.$$

By applying the approximate Parseval inequality (Lemma 4.9), we obtain the following claim.

Claim 8.7. We have

$$\langle f - \mathrm{T}f, f\rangle \geq \frac{d}{k}\|f^{\geq d}\|_2^2 - O_k\left(\epsilon\right)\|f\|_2^2.$$

By our Fourier concentration theorem, (Corollary 8.1), we have the following lower bound on the 2-norm of the high degree part of f.

Claim 8.8. Let $\delta \leq (200d)^{-2d}$, and $\epsilon \leq \epsilon_0\left(k\right)$ be sufficiently small. If $f\colon V_{[k]} \to \{0,1\}$ is (d,δ)-global. Then

$$\|f^{\geq d}\|_2^2 \geq \frac{1}{2}\|f\|_2^2.$$

Combining the above claims we get the following.

Claim 8.9. Let $\delta \leq (200d)^{-2d}$. We have

$$\langle f - \mathrm{T}f, f\rangle \geq \frac{d}{2k}\|f\|_2^2$$

We are now ready to prove the Kruskal–Katona theorem in the setting of high dimensional expanders.

Corollary 8.10. *Let X be an ϵ-HDX, for a sufficiently small $\epsilon > 0$. Let $\delta \leq (200d)^{-d}$, and let $A \subseteq X\left(k-1\right)$ be (d,δ)-global. Then*

$$\mu\left(\partial(A)\right) \geq \mu\left(A\right)\left(1 + \frac{d}{2k}\right).$$

Proof. Let $f = 1_A$. We have

$$\begin{aligned}\langle f - \mathrm{T}f, f\rangle &= \Pr_{\sigma\sim X(k-1)}\Pr_{\tau_1,\tau_2\supset\sigma}\left[\tau_1 \in A, \tau_2 \notin A\right].\\ &\leq \Pr\left[\sigma \in \partial(A), f\left(\tau_2\right) \notin A\right]\\ &= \mu\left(\partial(A)\right) - \mu\left(A\right).\end{aligned}$$

□

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