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On the Gaussian surface area of spectrahedra

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Abstract

We show that for sufficiently large $n \geq 1$ and $d = Cn^{3/4}$ for some universal constant $C > 0$, a random spectrahedron with matrices drawn from Gaussian orthogonal ensemble has Gaussian surface area $\Theta(n^{1/8})$ with high probability.

1 Introduction

A *spectrahedron* $S \subseteq \mathbb{R}^n$ is a set of the form

$$S = \left\{ x \in \mathbb{R}^n : \sum_i x_i A^{(i)} \preceq B \right\},$$

for some $d \times d$ symmetric matrices $A^{(1)}, \ldots, A^{(n)}, B \in \mathrm{Sym}_d$. Here we will be concerned with the *Gaussian surface area* of S, defined as

$$\mathsf{GSA}(S) = \liminf_{\delta \to 0} \frac{\mathcal{G}^n(S_\delta^{\mathrm{out}})}{\delta}, \tag{1}$$

where $S_\delta^{\mathrm{out}} = \{x \notin S : \mathrm{dist}(x, S) \leq \delta\}$ denotes the outer δ-neighborhood of S under Euclidean distance and $\mathcal{G}^n(\cdot)$ denotes the standard Gaussian measure on $\mathbb{R}^n$. Ball showed that the GSA of any convex body in $\mathbb{R}^n$ is $O(n^{1/4})$ [Bal93], which was later shown to be tight by Nazarov [Naz03]. Moreover, Nazarov [KOS08] showed that the GSA of a d-facet polytope[1] in $\mathbb{R}^n$ is $O(\sqrt{\log d})$ and this fact has found application in constructing pseudorandom generators for polytopes [HKM13, ST17, CDS19]. Motivated by recent work [AY21], this raises the question of whether the GSA of spectrahedra is also small. In this note we answer this question in the negative.

Theorem 1. *For a universal constant $C > 0$ and any integers $n, d \geq 1$ satisfying $d \leq n/C$ the following hold. If $\boldsymbol{A}^{(1)}, \ldots, \boldsymbol{A}^{(n)}$ are i.i.d. drawn from the $d \times d$ Gaussian orthogonal ensemble (see below for the definition), then the spectrahedron*

$$\mathcal{T} = \left\{ x \in \mathbb{R}^n : \sum_i x_i \boldsymbol{A}^{(i)} \preceq 2\sqrt{nd} \cdot \mathbb{I} \right\} \tag{2}$$

satisfies $\mathsf{GSA}(\mathcal{T}) = \Omega(\sqrt{n/d})$ with probability at least $1 - C\exp(-dn^{-3/4}/C)$. Moreover, for any integer d satisfying $d \leq n/C$, $\mathsf{GSA}(\mathcal{T}) = O(\sqrt{n/d})$ holds with probability at least $1 - \exp(-n/50)$.

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[1] A d-facet polytope is the special case of a spectrahedron when the matrices, $A^{(1)}, \ldots, A^{(n)}, B$ are *diagonal*.

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The theorem shows the existence of spectrahedra with GSA of $\Omega(n^{1/8})$. (In fact, a random spectrahedron as above satisfies this with constant probability). This lower bound is close to the GSA upper bound of Ball [Bal93] of $O(n^{1/4})$ for *arbitrary* convex bodies. Moreover, the lower bound shows that in contrast to the case of polytopes, the GSA of spectrahedra can depend polynomially on d. A natural open question is how large the GSA of arbitrary spectrahedra can be; can spectrahedra achieve a GSA of $\Theta(n^{1/4})$?

2 Preliminaries

We use $\boldsymbol{g}, \boldsymbol{x}, \boldsymbol{A}$ to denote random variables. We let $\mathcal{G}(0, \sigma^2)$ be the normal distribution with mean 0 and variance σ^2. We denote by $\mathcal{H}_d$ the $d \times d$ Gaussian orthogonal ensemble (GOE). Namely, $\boldsymbol{A} \sim \mathcal{H}_d$ if it is a symmetric matrix with $\{\boldsymbol{A}_{i,j}\}_{i \le j}$ independently distributed satisfying $\boldsymbol{A}_{i,j} \sim \mathcal{G}(0,1)$ for $i < j$ and $\boldsymbol{A}_{i,i} \sim \mathcal{G}(0,2)$. To keep notations short, for $b \ge 0$ we use $[a \pm b]$ to represent the interval $[a-b, a+b]$. For every $c \ge 0$, we use $c \cdot [a \pm b]$ to represent the interval $[ac \pm bc]$. We denote the set of n-dimensional unit vectors by S^{n-1}. Finally, we let χ_n be the χ distribution with n degrees of freedom, which is the square root of the sum of the squares of n independent standard normal variables. The following are some simple facts about the χ distribution.

Fact 2. *Let $n \in \mathbb{Z}_{>0}$ and $h(\cdot)$ be the pdf of χ_n. Then $h(x) \ge c$ for $x \in [\sqrt{n} \pm c]$, where $c > 0$ is an absolute constant.*

Fact 3. *Let $n \in \mathbb{Z}_{>0}$ and $h(\cdot)$ be the pdf of χ_n. Then $h(x) \le O(\sqrt{n}/|x|)$ for $x \in \mathbb{R}$.*

Proof. Recall that by definition

$$h(x) = \frac{1}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})} x^{n-1} e^{-x^2/2}$$

for $x \ge 0$, and $h(x) = 0$ otherwise. Hence the fact is trivial for $x \le 0$. For $x > 0$, the fact follows from the inequalities $x^n e^{-x^2/2} \le n^{n/2} e^{-n/2}$ and $\Gamma(z) \ge \sqrt{2\pi} z^{z-1/2} e^{-z}$ for all $z > 0$ [AAR99, Jam15]. □

Lemma 4 ([LM00, comment below Lemma 1])**.** *For $n \ge 1$, let $\boldsymbol{r}$ be a random variable distributed according to χ_n. Then for every $x > 0$, we have*

$$\Pr\left[n - 2\sqrt{nx} \le \boldsymbol{r}^2 \le n + 2\sqrt{nx} + 2x\right] \ge 1 - 2e^{-x}.$$

For our purposes, it will be convenient to use an alternative definition of Gaussian surface area in terms of the *inner* surface area. Namely, for $S_\delta^{\text{in}} = \{x \in S : \text{dist}(x, S^c\} \le \delta)$ where S^c is the complement of the body S, we define,

$$\mathsf{GSA}(S) = \lim_{\delta \to 0} \frac{\mathcal{G}^n\left(S_\delta^{\text{in}}\right)}{\delta}. \tag{3}$$

It follows from Huang et al. [HXZ21, Theorem 3.3] that this definition is equal to the one in Eq. (1) when S is a convex body that contains the origin, which is sufficient for our purposes.

To prove our main theorem, we use the following facts, starting with a well known bound on the size of an ε-net of the n-dimensional sphere.

Fact 5 ([Tao12, Lemma 2.3.4])**.** *For every $d \ge 1$ and any $0 < \varepsilon < 1$ there exists an ε-net of the sphere S^{d-1} of cardinality at most $(C/\varepsilon)^d$ for some universal constant $C > 0$.*

Claim 6 ([RS15, Page 134, Theorem 3])**.** *Let $\boldsymbol{x}, \boldsymbol{y}$ be two real-valued random variables and f be the pdf of $(\boldsymbol{x}, \boldsymbol{y})$. Then the pdf of $\boldsymbol{z} = \boldsymbol{x} \cdot \boldsymbol{y}$ is given by*

$$g(z) = \int_{-\infty}^{\infty} f\left(x, \frac{z}{x}\right) \cdot \frac{1}{|x|} dx.$$

Theorem 7 ([LR10, Theorem 1])**.** *Let $\boldsymbol{A} \sim \mathcal{H}_d$. For every $0 < \eta < 1$, it holds that*

$$\Pr\left[\lambda_{\max}(\boldsymbol{A}) \in 2\sqrt{d}\big[1 \pm \eta\big]\right] \geq 1 - C \cdot e^{-d\eta^{3/2}/C},$$

for some absolute constant $C > 0$.

3 Proof of main theorem

The core of the argument is in the following lemma, bounding $q(2\sqrt{nd})$ where q is the pdf of the largest eigenvalue of the matrix showing up in Eq. (2). We will later show that this value is essentially the same as $\mathsf{GSA}(\mathcal{T})$, where $\mathcal{T}$ is the spectrahedron in the statement of the theorem.

Lemma 8. *For $n, d \geq 1$ and $A^{(1)}, \ldots, A^{(n)} \in \mathrm{Sym}_d$, let $q(\cdot)$ be the probability density function of*

$$\lambda_{\max}\left(\sum_i \boldsymbol{x}_i A^{(i)}\right),$$

where $\boldsymbol{x} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ is a random vector and each entry is i.i.d. drawn from $\mathcal{G}(0,1)$. If $\boldsymbol{A}^{(1)}, \ldots, \boldsymbol{A}^{(n)}$ are i.i.d. drawn from the $d \times d$ Gaussian orthogonal ensemble, then $q(2\sqrt{nd}) = \Omega(\sqrt{1/d})$ with probability at least $1 - C\exp(-dn^{-3/4}/C)$ (over the choice of $\boldsymbol{A}^{(1)}, \ldots, \boldsymbol{A}^{(n)}$) where $C > 0$ is a universal constant. Moreover, for any integer d and any $d \times d$ matrices $A^{(1)}, \ldots, A^{(n)}$, $q(2\sqrt{nd}) = O(\sqrt{1/d})$.

Proof. Let $\boldsymbol{y} \sim S^{n-1}$ be chosen uniformly from the unit sphere and for matrices $A^{(1)}, \ldots, A^{(n)}$, denote by p the pdf of $\lambda_{\max}\big(\sum_i \boldsymbol{y}_i A^{(i)}\big)$. Let $\boldsymbol{r} \sim \chi_n$ and notice that $\boldsymbol{r}\boldsymbol{y}$ is distributed like $\boldsymbol{x}$. Denote by h the pdf of $\boldsymbol{r}$. By Claim 6, we have

$$q\left(2\sqrt{nd}\right) = \int_{-\infty}^{\infty} h\left(2\sqrt{nd}/z\right) p(z) \frac{1}{|z|} dz \,. \tag{4}$$

Using Fact 3, $h(2\sqrt{nd}/z)/|z| = O(1/\sqrt{d})$ for all z. Hence Eq. (4) can be bounded as $O(1/\sqrt{d}) \int_{-\infty}^{\infty} p(z) dz = O(1/\sqrt{d})$, establishing the claimed upper bound on q.

To prove the lower bound on q, let $\boldsymbol{A}^{(1)}, \ldots, \boldsymbol{A}^{(n)} \sim \mathcal{H}_d$ be n matrices chosen i.i.d. from the GOE. Observe that by Theorem 7, we have

$$\Pr\left[\lambda_{\max}\left(\sum_{i=1}^{n} \boldsymbol{y}_i \boldsymbol{A}^{(i)}\right) \in I\right] \geq 1 - C\exp(-dn^{-3/4}/C) \,, \tag{5}$$

where

$$I = 2\sqrt{d} \cdot [1 \pm c/\sqrt{n}] \,,$$

for some universal constants $C, c > 0$. Define the set of matrices

$$G = \left\{\left(A^{(1)}, \ldots, A^{(n)}\right) : \Pr\left[\lambda_{\max}\left(\sum_{i=1}^{n} \boldsymbol{y}_i A^{(i)}\right) \in I\right] \geq \frac{1}{2}\right\}.$$

Then, using the definition of G and Eq. (5), we have

$$\Pr\left[\left(\boldsymbol{A}^{(1)},\ldots,\boldsymbol{A}^{(n)}\right)\in G\right]\geq 1-2C\exp(-dn^{-3/4}/C)\ .$$

Now fix any $(A^{(1)},\ldots,A^{(n)})\in G$. By definition of G, $\int_I p(z)dz\geq 1/2$, and therefore the right-hand side of Eq. (4) is at least

$$\int_I h\left(2\sqrt{nd}/z\right)p(z)\frac{1}{z}dz\geq\Omega(1)\cdot\int_I p(z)\frac{1}{z}dz\geq\Omega(1/\sqrt{d})\ , \tag{6}$$

where we used Fact 2 to conclude that $h(2\sqrt{nd}/z)\geq\Omega(1)$ for all $z\in I$. □

We next relate $q(2\sqrt{nd})$ to $\mathsf{GSA}(\mathcal{T})$. For a vector $v\in S^{d-1}$, and $d\times d$ symmetric matrices $A^{(1)},\ldots,A^{(n)}$, define the vector

$$W_v=\left(v^TA^{(1)}v,v^TA^{(2)}v,\ldots,v^TA^{(n)}v\right)\in\mathbb{R}^n\ . \tag{7}$$

Notice that $\mathcal{T}$ can be written as

$$\mathcal{T}=\left\{x\in\mathbb{R}^n:\sum_i x_iA^{(i)}\preceq 2\sqrt{nd}\cdot\mathbb{I}\right\}=\left\{x\in\mathbb{R}^n:\forall v\in S^{d-1},\ \langle x,W_v\rangle\leq 2\sqrt{nd}\right\}\ .$$

We say that $A^{(1)},\ldots,A^{(n)}$ are *good* if

$$\forall v\in S^{d-1},\ \frac{1}{2}\sqrt{n}\leq\|W_v\|\leq 2\sqrt{n}\ .$$

Lemma 9. *There exists a constant $C\geq 1$ such that for all integers n and $d\leq n/C$, random matrices $\boldsymbol{A}^{(1)},\ldots,\boldsymbol{A}^{(n)}$ drawn i.i.d. from $\mathcal{H}_d$ are good with probability at least $1-\exp(-n/50)$.*

Proof. For a fixed $v\in S^{d-1}$, we claim that

$$\Pr[n\leq\|\boldsymbol{W}_v\|^2\leq 3n]\geq 1-2\exp(-n/40). \tag{8}$$

To see this, observe that for $\boldsymbol{A}\sim\mathcal{H}_d$ and unit vector $v\in\mathbb{R}^d$, $v^T\boldsymbol{A}v$ is distributed according to

$$\left(4\sum_{i<j}v_i^2v_j^2+2\sum_i v_i^4\right)^{1/2}\cdot\mathcal{G}(0,1)=\sqrt{2}\cdot\mathcal{G}(0,1).$$

Therefore, each entry in $\boldsymbol{W}_v$ is distributed according to $\mathcal{G}(0,2)$, and Lemma 4 implies Eq. (8). We next prove that with high probability (over the $\boldsymbol{A}^{(i)}$s), for *every* unit vector z, $\|\boldsymbol{W}_z\|$ is large. First, by Fact 5, there exists a set $\mathcal{V}=\{v_1,\ldots,v_{(10^4M)^d}\}\subseteq\mathbb{R}^d$ of unit vectors that form a 10^{-4}-net of the unit Euclidean sphere where M is a constant. Applying a union bound on $\mathcal{V}$, we have

$$\Pr[\forall v\in\mathcal{V}:n\leq\|\boldsymbol{W}_v\|^2\leq 3n]\geq 1-2\exp(-n/40)\cdot(10^4M)^d\geq 1-\exp(-n/50)\ , \tag{9}$$

here we used that M is a constant and $d\leq n/C$ for a sufficiently large C.

To conclude the proof, it suffices to show that if $\boldsymbol{A}^{(1)},\ldots,\boldsymbol{A}^{(n)}$ are such that

$$\forall v\in\mathcal{V},\ n\leq\|\boldsymbol{W}_v\|^2\leq 3n\ ,$$

then also

$$\forall z \in S^{d-1},\ \|\boldsymbol{W}_z\| \geq \frac{1}{2}\sqrt{n}\ .$$

Let $\boldsymbol{b}_{\max} = \max_{z \in S^{d-1}} \|\boldsymbol{W}_z\|$ and $\boldsymbol{b}_{\min} = \min_{z \in S^{d-1}} \|\boldsymbol{W}_z\|$. Let $\boldsymbol{z}_{\max}$ and $\boldsymbol{z}_{\min}$ be the vectors achieving the maximum and the minimum respectively. Let $\boldsymbol{v}_{\max}$ and $\boldsymbol{v}_{\min}$ be the vectors in $\mathcal{V}$ that are closest to $\boldsymbol{z}_{\max}$ and $\boldsymbol{z}_{\min}$, respectively. For any vectors $z, v \in S^{d-1}$ with $\|z - v\| \leq 10^{-4}$, applying the spectral decomposition of $zz^T - vv^T$, there exist unit vectors u_1, u_2 and $0 \leq \lambda \leq \frac{1}{100}$ such that

$$zz^T - vv^T = \lambda \cdot \left(u_1 u_1^T - u_2 u_2^T\right)\ . \tag{10}$$

Hence

$$\begin{aligned}\|\boldsymbol{W}_z - \boldsymbol{W}_v\|^2 = \sum_{i=1}^{n} \left(z^T \boldsymbol{A}^{(i)} z - v^T \boldsymbol{A}^{(i)} v\right)^2 &= \sum_{i=1}^{n} \left(\mathrm{Tr}\left(\boldsymbol{A}^{(i)}\left(zz^T - vv^T\right)\right)\right)^2 \\ &\leq \frac{1}{10^4} \sum_{i=1}^{n} \left(u_1^T \boldsymbol{A}^{(i)} u_1 - u_2^T \boldsymbol{A}^{(i)} u_2\right)^2 \\ &\leq \frac{1}{5000} \sum_{i=1}^{n} \left(\left(u_1^T \boldsymbol{A}^{(i)} u_1\right)^2 + \left(u_2^T \boldsymbol{A}^{(i)} u_2\right)^2\right) \\ &\leq \frac{\boldsymbol{b}_{\max}^2}{2500}\ .\end{aligned}$$

Choosing $z = \boldsymbol{z}_{\max}$ and $v = \boldsymbol{v}_{\max}$, we have

$$\|\boldsymbol{W}_{\boldsymbol{z}_{\max}}\| \leq \|\boldsymbol{W}_{\boldsymbol{v}_{\max}}\| + \frac{\boldsymbol{b}_{\max}}{50}\ .$$

Now, since $\|\boldsymbol{W}_{\boldsymbol{z}_{\max}}\| = \boldsymbol{b}_{\max}$, we have

$$\boldsymbol{b}_{\max} \leq \frac{50}{49}\|\boldsymbol{W}_{\boldsymbol{v}_{\max}}\| \leq \frac{50}{49}\sqrt{3n} \leq 2\sqrt{n}\ .$$

Similarly, we set $z = \boldsymbol{z}_{\min}$ and $v = \boldsymbol{v}_{\min}$ and obtain

$$\boldsymbol{b}_{\min} \geq \|\boldsymbol{W}_{\boldsymbol{v}_{\min}}\| - \frac{\boldsymbol{b}_{\max}}{50} \geq \sqrt{n} - \frac{1}{25}\sqrt{n} > \frac{1}{2}\sqrt{n}\ .$$

This concludes the result. □

For the following claim, we define the inner and outer shells of $\mathcal{T}$ as

$$\begin{aligned}\mathcal{D}_\delta^{\mathrm{in}} &= \left\{x : \lambda_{\max}\left(\sum_i x_i A^{(i)}\right) \in \sqrt{n} \cdot \left[2\sqrt{d} - \delta, 2\sqrt{d}\right]\right\}\ , \\ \mathcal{D}_\delta^{\mathrm{out}} &= \left\{x : \lambda_{\max}\left(\sum_i x_i A^{(i)}\right) \in \sqrt{n} \cdot \left[2\sqrt{d}, 2\sqrt{d} + \delta\right]\right\}\ .\end{aligned}$$

Also recall the inner and outer neighborhoods of $\mathcal{T}$, defined as

$$\begin{aligned}\mathcal{T}_\delta^{\mathrm{in}} &= \{x \in \mathcal{T} : \exists y \notin \mathcal{T} : \|x - y\| \leq \delta\}\ , \\ \mathcal{T}_\delta^{\mathrm{out}} &= \{x \notin \mathcal{T} : \exists y \in \mathcal{T} : \|x - y\| \leq \delta\}\ .\end{aligned}$$

Claim 10. *For sufficiently small $\delta > 0$ and any good $A^{(1)}, \ldots, A^{(n)}$, we have $\mathcal{D}_\delta^{\mathrm{in}} \subseteq \mathcal{T}_{4\delta}^{\mathrm{in}}$ and $\mathcal{T}_\delta^{\mathrm{out}} \subseteq \mathcal{D}_{2\delta}^{\mathrm{out}}$.*

Proof. For every $x \in \mathcal{D}_\delta^{\text{in}}$, let v be a unit eigenvector of $\sum_i x_i A^{(i)}$ with the eigenvalue $\lambda_{\max}(\sum_i x_i A^{(i)})$. Therefore,

$$\langle x, W_v \rangle = v^T \Big(\sum x_i A^{(i)} \Big) v \geq (2\sqrt{d} - \delta)\sqrt{n} \; .$$

Setting $y = 2\delta\sqrt{n} W_v / \|W_v\|^2$, we have

$$\langle x + y, W_v \rangle = \langle x, W_v \rangle + 2\delta\sqrt{n} \geq \Big(2\sqrt{d} - \delta\Big)\sqrt{n} + 2\delta\sqrt{n} = \Big(2\sqrt{d} + \delta\Big)\sqrt{n} \; ,$$

and so $x + y \notin \mathcal{T}$. Moreover, since $A^{(1)}, \ldots, A^{(n)}$ are good, $\|y\| = 2\delta\sqrt{n}/\|W_v\| \leq 4\delta$ and therefore $x \in \mathcal{T}_{4\delta}^{\text{in}}$, as desired. For the other containment, let $x \in \mathcal{T}_\delta^{\text{out}}$. Then for any unit vector v, by Cauchy-Schwarz and using $\|W_v\| \leq 2\sqrt{n}$,

$$\langle x, W_v \rangle \leq 2\sqrt{nd} + 2\delta\sqrt{n} \; ,$$

implying that $x \in \mathcal{D}_{2\delta}^{\text{out}}$, as desired. □

We now prove our main theorem.

Proof of Theorem 1. First observe that since $q(\cdot)$ is continuous, the lower bound on the pdf in Lemma 8 implies that $\mathcal{G}^n(\mathcal{D}_\delta^{\text{in}}) \geq \Omega(\delta\sqrt{n/d})$ for sufficiently small $\delta > 0$. Thus, $\mathcal{G}^n(\mathcal{T}_{4\delta}) = \Omega(\delta\sqrt{n/d})$ by Claim 10. By definition of $\mathsf{GSA}(S) = \lim_{\delta \to 0} \mathcal{G}^n(S_\delta^{\text{in}})/\delta$, we obtain the desired lower bound on GSA. Similarly, using the upper bound on the pdf in Lemma 8, $\mathcal{G}^n(\mathcal{D}_\delta^{\text{out}}) = O(\delta\sqrt{n/d})$ for sufficiently small $\delta > 0$. Thus, $\mathcal{G}^n(\mathcal{T}_{\delta/2}^{\text{out}}) = O(\delta\sqrt{n/d})$ by Claim 10. We complete the proof using $\mathsf{GSA}(S) = \lim_{\delta \to 0} \mathcal{G}^n(S_\delta^{\text{out}})/\delta$. □

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