On the Gaussian surface area of spectrahedra

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Abstract

We show that for sufficiently large \( n \geq 1 \) and \( d = Cn^{3/4} \) for some universal constant \( C > 0 \), a random spectrahedron with matrices drawn from Gaussian orthogonal ensemble has Gaussian surface area \( \Theta(n^{1/8}) \) with high probability.

1 Introduction

A spectrahedron \( S \subseteq \mathbb{R}^n \) is a set of the form

\[
S = \left\{ x \in \mathbb{R}^n : \sum_i x_i A^{(i)} \preceq B \right\},
\]

for some \( d \times d \) symmetric matrices \( A^{(1)}, \ldots, A^{(n)}, B \in \text{Sym}_d \). Here we will be concerned with the Gaussian surface area of \( S \), defined as

\[
\text{GSA}(S) = \liminf_{\delta \to 0} \frac{G^n(S_{\delta}^{\text{out}})}{\delta}, \tag{1}
\]

where \( S_{\delta}^{\text{out}} = \{ x \notin S : \text{dist}(x, S) \leq \delta \} \) denotes the outer \( \delta \)-neighborhood of \( S \) under Euclidean distance and \( G^n(\cdot) \) denotes the standard Gaussian measure on \( \mathbb{R}^n \). Ball showed that the GSA of any convex body in \( \mathbb{R}^n \) is \( O(n^{1/4}) \) [Bal93], which was later shown to be tight by Nazarov [Naz03]. Moreover, Nazarov [KOS08] showed that the GSA of a \( d \)-facet polytope\(^1\) in \( \mathbb{R}^n \) is \( O(\sqrt{\log d}) \) and this fact has found application in constructing pseudorandom generators for polytopes [HKM13, ST17, CDS19]. Motivated by recent work [AY21], this raises the question of whether the GSA of spectrahedra is also small. In this note we answer this question in the negative.

Theorem 1. For a universal constant \( C > 0 \) and any integers \( n, d \geq 1 \) satisfying \( d \leq n/C \) the following hold. If \( A^{(1)}, \ldots, A^{(n)} \) are i.i.d. drawn from the \( d \times d \) Gaussian orthogonal ensemble (see below for the definition), then the spectrahedron

\[
T = \left\{ x \in \mathbb{R}^n : \sum_i x_i A^{(i)} \leq 2\sqrt{nd} \cdot I \right\} \tag{2}
\]

satisfies \( \text{GSA}(T) = \Omega(\sqrt{n/d}) \) with probability at least \( 1 - C \exp(-dn^{-3/4}/C) \). Moreover, for any integer \( d \) satisfying \( d \leq n/C \), \( \text{GSA}(T) = O(\sqrt{n/d}) \) holds with probability at least \( 1 - \exp(-n/50) \).

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\(^1\)A \( d \)-facet polytope is the special case of a spectrahedron when the matrices, \( A^{(1)}, \ldots, A^{(n)}, B \) are diagonal.
The theorem shows the existence of spectrahedra with GSA of $\Omega(n^{1/8})$. (In fact, a random spectrahedron as above satisfies this with constant probability). This lower bound is close to the GSA upper bound of Ball [Bal93] of $O(n^{1/4})$ for arbitrary convex bodies. Moreover, the lower bound shows that in contrast to the case of polytopes, the GSA of spectrahedra can depend polynomially on $d$. A natural open question is how large the GSA of arbitrary spectrahedra can be; can spectrahedra achieve a GSA of $\Theta(n^{1/4})$?

## 2 Preliminaries

We use $g$, $x$, $A$ to denote random variables. We let $G(0, \sigma^2)$ be the normal distribution with mean 0 and variance $\sigma^2$. We denote by $\mathcal{H}_d$ the $d \times d$ Gaussian orthogonal ensemble (GOE). Namely, $A \sim \mathcal{H}_d$ if it is a symmetric matrix with $\{A_{ij}\}_{i \leq j}$ independently distributed satisfying $A_{ij} \sim G(0,1)$ for $i < j$ and $A_{i,i} \sim G(0,2)$. To keep notations short, for $b \geq 0$ we use $[a + b]$ to represent the interval $[a - b, a + b]$. For every $c \geq 0$, we use $c \cdot [a + b]$ to represent the interval $[ac \pm bc]$. We denote the set of $n$-dimensional unit vectors by $S^{n-1}$. Finally, we let $\chi_n$ be the $\chi$ distribution with $n$ degrees of freedom, which is the square root of the sum of the squares of $n$ independent standard normal variables. The following are some simple facts about the $\chi$ distribution.

**Fact 2.** Let $n \in \mathbb{Z}_{>0}$ and $h(\cdot)$ be the pdf of $\chi_n$. Then $h(x) \geq c$ for $x \in [\sqrt{n} \pm c]$, where $c > 0$ is an absolute constant.

**Fact 3.** Let $n \in \mathbb{Z}_{>0}$ and $h(\cdot)$ be the pdf of $\chi_n$. Then $h(x) \leq O(\sqrt{n}/|x|)$ for $x \in \mathbb{R}$.

**Proof.** Recall that by definition

$$h(x) = \frac{1}{2^{n-1} \Gamma(n/2)} x^{n-1} e^{-x^2/2}$$

for $x \geq 0$, and $h(x) = 0$ otherwise. Hence the fact is trivial for $x \leq 0$. For $x > 0$, the fact follows from the inequalities $x^n e^{-x^2/2} \leq n^{n/2} e^{-n/2}$ and $\Gamma(z) \geq \sqrt{2\pi} z^{z-1/2}e^{-z}$ for all $z > 0$ [AAR99, Jam15].

**Lemma 4** ([LM00, comment below Lemma 1]). For $n \geq 1$, let $r$ be a random variable distributed according to $\chi_n$. Then for every $x > 0$, we have

$$\Pr \left[ n - 2\sqrt{nx} \leq r^2 \leq n + 2\sqrt{nx} + 2x \right] \geq 1 - 2e^{-x}. $$

For our purposes, it will be convenient to use an alternative definition of Gaussian surface area in terms of the inner surface area. Namely, for $S^n_\delta = \{x \in S : \text{dist}(x,S^c) \leq \delta\}$ where $S^c$ is the complement of the body $S$, we define,

$$\text{GSA}(S) = \lim_{\delta \to 0} \frac{\mathcal{G}(S^n_\delta)}{\delta}. \quad (3)$$

It follows from Huang et al. [HXZ21, Theorem 3.3] that this definition is equal to the one in Eq. (1) when $S$ is a convex body that contains the origin, which is sufficient for our purposes.

To prove our main theorem, we use the following facts, starting with a well known bound on the size of an $\varepsilon$-net of the $n$-dimensional sphere.

**Fact 5** ([Tao12, Lemma 2.3.4]). For every $d \geq 1$ and any $0 < \varepsilon < 1$ there exists an $\varepsilon$-net of the sphere $S^{d-1}$ of cardinality at most $(C/\varepsilon)^d$ for some universal constant $C > 0$. 

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Claim 6 ([RS15, Page 134, Theorem 3]). Let \( x, y \) be two real-valued random variables and \( f \) be the pdf of \((x, y)\). Then the pdf of \( z = x \cdot y \) is given by

\[
g(z) = \int_{-\infty}^{\infty} f\left( x, \frac{z}{x} \right) \frac{1}{|x|} dx.
\]

Theorem 7 ([LR10, Theorem 1]). Let \( A \sim \mathcal{H}_d \). For every \( 0 < \eta < 1 \), it holds that

\[
\Pr\left[ \lambda_{\text{max}}(A) \in 2\sqrt{d}[1 \pm \eta] \right] \geq 1 - C \cdot e^{-d\eta^{3/4}/C},
\]

for some absolute constant \( C > 0 \).

3 Proof of main theorem

The core of the argument is in the following lemma, bounding \( q(2\sqrt{nd}) \) where \( q \) is the pdf of the largest eigenvalue of the matrix showing up in Eq. (2). We will later show that this value is essentially the same as \( \text{GSA}(\mathcal{T}) \), where \( \mathcal{T} \) is the spectrahedron in the statement of the theorem.

Lemma 8. For \( n, d \geq 1 \) and \( A^{(1)}, \ldots, A^{(n)} \in \text{Sym}_d \), let \( q(\cdot) \) be the probability density function of

\[
\lambda_{\text{max}}\left( \sum_{i} x_i A^{(i)} \right),
\]

where \( x = (x_1, \ldots, x_n) \) is a random vector and each entry is i.i.d. drawn from \( \mathcal{G}(0, 1) \). If \( A^{(1)}, \ldots, A^{(n)} \) are i.i.d. drawn from the \( d \times d \) Gaussian orthogonal ensemble, then \( q(2\sqrt{nd}) = \Omega(\sqrt{1/d}) \) with probability at least \( 1-C\exp(-dn^{-3/4}/C) \) (over the choice of \( A^{(1)}, \ldots, A^{(n)} \)) where \( C > 0 \) is a universal constant. Moreover, for any integer \( d \) and any \( d \times d \) matrices \( A^{(1)}, \ldots, A^{(n)} \), \( q(2\sqrt{nd}) = O(\sqrt{1/d}) \).

Proof. Let \( y \sim S^{n-1} \) be chosen uniformly from the unit sphere and for matrices \( A^{(1)}, \ldots, A^{(n)} \), denote by \( p \) the pdf of \( \lambda_{\text{max}}(\sum_i y_i A^{(i)}) \). Let \( r \sim \chi_n \) and notice that \( ry \) is distributed like \( x \). Denote by \( h \) the pdf of \( r \). By Claim 6, we have

\[
q(2\sqrt{nd}) = \int_{-\infty}^{\infty} h(2\sqrt{nd}/z) p(z) \frac{1}{|z|} dz. \tag{4}
\]

Using Fact 3, \( h(2\sqrt{nd}/z)/|z| = O(1/\sqrt{d}) \) for all \( z \). Hence Eq. (4) can be bounded as \( O(1/\sqrt{d}) \int_{-\infty}^{\infty} p(z) dz = O(1/\sqrt{d}) \), establishing the claimed upper bound on \( q \).

To prove the lower bound on \( q \), let \( A^{(1)}, \ldots, A^{(n)} \sim \mathcal{H}_d \) be \( n \) matrices chosen i.i.d. from the GOE. Observe that by Theorem 7, we have

\[
\Pr\left[ \lambda_{\text{max}}\left( \sum_{i=1}^{n} y_i A^{(i)} \right) \in I \right] \geq 1 - C\exp(-dn^{-3/4}/C), \tag{5}
\]

where

\[
I = 2\sqrt{d} \cdot [1 \pm c/\sqrt{n}],
\]

for some universal constants \( C, c > 0 \). Define the set of matrices

\[
G = \left\{ (A^{(1)}, \ldots, A^{(n)}) : \Pr\left[ \lambda_{\text{max}}\left( \sum_{i=1}^{n} y_i A^{(i)} \right) \in I \right] \geq \frac{1}{2} \right\}.
\]
Then, using the definition of $G$ and Eq. (5), we have

$$\Pr \left[ \left( A^{(1)}, \ldots, A^{(n)} \right) \in G \right] \geq 1 - 2C \exp(-dn^{-3/4}/C).$$

Now fix any $\left( A^{(1)}, \ldots, A^{(n)} \right) \in G$. By definition of $G$, $\int_I p(z)dz \geq 1/2$, and therefore the right-hand side of Eq. (4) is at least

$$\int_I h\left(2\sqrt{nd}/z\right)p(z)\frac{1}{z}dz \geq \Omega(1) \cdot \int_I p(z)\frac{1}{z}dz \geq \Omega(1/\sqrt{d}),$$

where we used Fact 2 to conclude that $h(2\sqrt{nd}/z) \geq \Omega(1)$ for all $z \in I$.

We next relate $q(2\sqrt{nd})$ to $GSA(T)$. For a vector $v \in S^{d-1}$, and $d \times d$ symmetric matrices $A^{(1)}, \ldots, A^{(n)}$, define the vector

$$W_v = (v^T A^{(1)} v, v^T A^{(2)} v, \ldots, v^T A^{(n)} v) \in \mathbb{R}^n. \tag{7}$$

Notice that $T$ can be written as

$$T = \left\{ x \in \mathbb{R}^n : \sum_i x_i A^{(i)} \preceq 2\sqrt{nd} \cdot I \right\} = \left\{ x \in \mathbb{R}^n : \forall v \in S^{d-1}, \langle x, W_v \rangle \leq 2\sqrt{nd} \right\}.$$ 

We say that $A^{(1)}, \ldots, A^{(n)}$ are good if

$$\forall v \in S^{d-1}, \frac{1}{2} \sqrt{n} \leq \|W_v\| \leq 2\sqrt{n}.$$ 

**Lemma 9.** There exists a constant $C \geq 1$ such that for all integers $n$ and $d \leq n/C$, random matrices $A^{(1)}, \ldots, A^{(n)}$ drawn i.i.d. from $H_d$ are good with probability at least $1 - \exp\left(-n/50\right)$.

**Proof.** For a fixed $v \in S^{d-1}$, we claim that

$$\Pr[n \leq \|W_v\|^2 \leq 3n] \geq 1 - 2 \exp\left(-n/40\right). \tag{8}$$

To see this, observe that for $A \sim H_d$ and unit vector $v \in \mathbb{R}^d$, $v^T A v$ is distributed according to

$$\left(4 \sum_{i<j} v_i^2 v_j^2 + 2 \sum_i v_i^4 \right)^{1/2} \cdot G(0, 1) = \sqrt{2} \cdot G(0, 1).$$

Therefore, each entry in $W_v$ is distributed according to $G(0, 2)$, and Lemma 4 implies Eq. (8). We next prove that with high probability (over the $A^{(i)}$s), for every unit vector $z$, $\|W_z\|$ is large. First, by Fact 5, there exists a set $V = \{v_1, \ldots, v_{(10^4M)^d}\} \subseteq \mathbb{R}^d$ of unit vectors that form a $10^{-4}$-net of the unit Euclidean sphere where $M$ is a constant. Applying a union bound on $V$, we have

$$\Pr[\forall v \in V : n \leq \|W_v\|^2 \leq 3n] \geq 1 - 2 \exp\left(-n/40\right) \cdot (10^4M)^d \geq 1 - \exp\left(-n/50\right), \tag{9}$$

here we used that $M$ is a constant and $d \leq n/C$ for a sufficiently large $C$.

To conclude the proof, it suffices to show that if $A^{(1)}, \ldots, A^{(n)}$ are such that

$$\forall v \in V, n \leq \|W_v\|^2 \leq 3n,$$
then also
\[ \forall z \in S^{d-1}, \| W_z \| \geq \frac{1}{2} \sqrt{n}. \]

Let \( b_{\text{max}} = \max_{z \in S^{d-1}} \| W_z \| \) and \( b_{\text{min}} = \min_{z \in S^{d-1}} \| W_z \| \). Let \( z_{\text{max}} \) and \( z_{\text{min}} \) be the vectors achieving the maximum and the minimum respectively. Let \( v_{\text{max}} \) and \( v_{\text{min}} \) be the vectors in \( V \) that are closest to \( z_{\text{max}} \) and \( z_{\text{min}} \), respectively. For any vectors \( z, v \in S^{d-1} \) with \( \| z - v \| \leq 10^{-4} \), applying the spectral decomposition of \( z z^T - vv^T \), there exist unit vectors \( u_1, u_2 \) and \( 0 \leq \lambda \leq \frac{1}{100} \) such that
\[ zz^T - vv^T = \lambda \left( u_1 u_1^T - u_2 u_2^T \right). \] (10)

Hence
\[
\| W_z - W_v \|^2 = \sum_{i=1}^{n} \left( z^T A^{(i)} z - v^T A^{(i)} v \right) = \sum_{i=1}^{n} \left( \text{Tr} \left( A^{(i)} (z z^T - vv^T) \right) \right)^2
\leq \frac{1}{10^4} \sum_{i=1}^{n} \left( u_1^T A^{(i)} u_1 - u_2^T A^{(i)} u_2 \right)^2
\leq \frac{1}{5000} \sum_{i=1}^{n} \left( u_1^T A^{(i)} u_1 \right)^2 + \left( u_2^T A^{(i)} u_2 \right)^2
\leq \frac{b_{\text{max}}^2}{2500}.
\]

Choosing \( z = z_{\text{max}} \) and \( v = v_{\text{max}} \), we have
\[
\| W_{z_{\text{max}}} \| \leq \| W_{v_{\text{max}}} \| + \frac{b_{\text{max}}}{50}.
\]

Now, since \( \| W_{z_{\text{max}}} \| = b_{\text{max}} \), we have
\[
b_{\text{max}} \leq \frac{50}{49} \| W_{v_{\text{max}}} \| \leq \frac{50}{49} \sqrt{3n} \leq 2 \sqrt{n}.
\]

Similarly, we set \( z = z_{\text{min}} \) and \( v = v_{\text{min}} \) and obtain
\[
b_{\text{min}} \geq \| W_{v_{\text{min}}} \| - \frac{b_{\text{max}}}{50} \geq \sqrt{n} - \frac{1}{25} \sqrt{n} > \frac{1}{2} \sqrt{n}.
\]

This concludes the result. \( \square \)

For the following claim, we define the inner and outer shells of \( \mathcal{T} \) as
\[
D_{\delta}^{\text{in}} = \left\{ x : \lambda_{\text{max}} \left( \sum_i x_i A^{(i)} \right) \in \sqrt{n} \cdot [2 \sqrt{d} - \delta, 2 \sqrt{d}] \right\},
\]
\[
D_{\delta}^{\text{out}} = \left\{ x : \lambda_{\text{max}} \left( \sum_i x_i A^{(i)} \right) \in \sqrt{n} \cdot [2 \sqrt{d}, 2 \sqrt{d} + \delta] \right\}.
\]

Also recall the inner and outer neighborhoods of \( \mathcal{T} \), defined as
\[
\mathcal{T}_{\delta}^{\text{in}} = \left\{ x \in \mathcal{T} : \exists y \notin \mathcal{T} : \| x - y \| \leq \delta \right\},
\]
\[
\mathcal{T}_{\delta}^{\text{out}} = \left\{ x \notin \mathcal{T} : \exists y \in \mathcal{T} : \| x - y \| \leq \delta \right\}.
\]

**Claim 10.** For sufficiently small \( \delta > 0 \) and any good \( A^{(1)}, \ldots, A^{(n)} \), we have \( D_{\delta}^{\text{in}} \subseteq \mathcal{T}_{4\delta}^{\text{in}} \) and \( \mathcal{T}_{\delta}^{\text{out}} \subseteq D_{2\delta}^{\text{out}} \).
Proof. For every $x \in D^{\text{in}}_d$, let $v$ be a unit eigenvector of $\sum_i x_i A^{(i)}$ with the eigenvalue $\lambda_{\text{max}}(\sum_i x_i A^{(i)})$. Therefore,

$$\langle x, W_v \rangle = v^T \left( \sum_i x_i A^{(i)} \right) v \geq (2\sqrt{d} - \delta)\sqrt{n}.$$ 

Setting $y = 2\delta \sqrt{n} W_v / \|W_v\|^2$, we have

$$\langle x + y, W_v \rangle = \langle x, W_v \rangle + 2\delta \sqrt{n} \geq \left( 2\sqrt{d} - \delta \right)\sqrt{n} + 2\delta \sqrt{n} = \left( 2\sqrt{d} + \delta \right)\sqrt{n},$$

and so $x + y \notin T$. Moreover, since $A^{(1)}, \ldots, A^{(n)}$ are good, $\|y\| = 2\delta \sqrt{n} / \|W_v\| \leq 4\delta$ and therefore $x \in T^{\text{in}}_{4\delta}$, as desired. For the other containment, let $x \in T^{\text{out}}_\delta$. Then for any unit vector $v$, by Cauchy-Schwarz and using $\|W_v\| \leq 2\sqrt{n}$,

$$\langle x, W_v \rangle \leq 2\sqrt{nd} + 2\delta \sqrt{n},$$

implying that $x \in D^{\text{out}}_{2\delta}$, as desired.

We now prove our main theorem.

Proof of Theorem 1. First observe that since $q(\cdot)$ is continuous, the lower bound on the pdf in Lemma 8 implies that $G^n(D^{\text{in}}_{\delta}) \geq \Omega(\delta \sqrt{n/d})$ for sufficiently small $\delta > 0$. Thus, $G^n(T^{\text{in}}_{\delta}) = \Omega(\delta \sqrt{n/d})$ by Claim 10. By definition of $\text{GSA}(S) = \lim_{\delta \to 0} G^n(S^{\text{in}}_{\delta})/\delta$, we obtain the desired lower bound on GSA. Similarly, using the upper bound on the pdf in Lemma 8, $G^n(D^{\text{out}}_{\delta}) = O(\delta \sqrt{n/d})$ for sufficiently small $\delta > 0$. Thus, $G^n(T^{\text{out}}_{\delta/2}) = O(\delta \sqrt{n/d})$ by Claim 10. We complete the proof using $\text{GSA}(S) = \lim_{\delta \to 0} G^n(S^{\text{out}}_{\delta})/\delta$.

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