Extending Merge Resolution to a Family of Proof Systems

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Abstract. Merge Resolution (MRes [4]) is a recently introduced proof system for false QBFs. Unlike other known QBF proof systems, it builds winning strategies for the universal player within the proofs. Every line of this proof system consists of existential clauses along with countermodels. MRes stores the countermodels as merge maps. Merge maps are deterministic branching programs in which isomorphism checking is efficient as a result MRes is a polynomial time verifiable proof system.

In this paper, we introduce a family of proof systems MRes-$\mathcal{R}$ in which, the information of countermodels are stored in any pre-fixed complete representation $\mathcal{R}$, instead of merge maps. Hence corresponding to each possible complete representation $\mathcal{R}$, we have a sound and refutationally complete QBF-proof system in MRes-$\mathcal{R}$. To handle arbitrary representations for the strategies, we introduce consistency checking rules in MRes-$\mathcal{R}$ instead of isomorphism checking in MRes. As a result these proof systems are not polynomial time verifiable. Consequently, the paper shows that using merge maps is too restrictive and can be replaced with arbitrary representations leading to several interesting proof systems.

The paper also studies proof theoretic properties of the family of new proof systems MRes-$\mathcal{R}$. We show that eFrege+$\forall$red simulates all valid refutations from proof systems in MRes-$\mathcal{R}$. Since proof systems in MRes-$\mathcal{R}$ may use arbitrary representations, in order to simulate them, we first represent the steps used by the proof systems as a new simple complete structure. As a consequence, the corresponding proof system belonging to MRes-$\mathcal{R}$ is able to simulate all proof systems in MRes-$\mathcal{R}$. Finally, we simulate this proof system via eFrege+$\forall$red using the ideas from [10].

On the lower bound side, we lift the lower bound result of regular MRes ([5]) for all regular proof systems in MRes-$\mathcal{R}$. To be precise, we show that the completion principle formulas from [17] which are shown to be hard for regular MRes in [5], are also hard for any regular proof system in MRes-$\mathcal{R}$. Thereby, the paper lifts the lower bound of regular MRes to an entire class of proof systems, which use some complete representation, including those undiscovered, instead of merge maps.

1 Introduction

Proof complexity is a sub-branch of computational complexity in which the main focus is to understand the complexity of proving (refuting) theorems (contradictions) in various proof systems. Informally, a proof system is a polynomial time

computable function which maps proofs to theorems. Several propositional proof systems like resolution [23], Cutting planes [13], and Frege [15] have been developed for proving (refuting) propositional formulas. The relative strength of these proof systems has been well studied [24]. In the literature, several proof systems which are not polynomial time computable (verifiable) have also been well studied. For example, semantic cutting planes [14].

Quantified Boolean formulas (QBFs) extend propositional logic by adding quantifications ∃ (there exists) and ∀ (for all) to the variables. Several QBF proof systems like Q-Res [19], QU-Res [26], LD-Q-Res [2], ∀Exp+Res [18], IR-calc, and IRM-calc [7] have been proposed in the literature. These are all either CDCL (Conflict Driven Clause Learning)-based, or expansion-based QBF proof systems. Cutting planes proof systems has also been extended for QBFs (CP+∀red) [8].

A new proof system Merge resolution (MRes) [4] has been developed recently. It follows a different QBF-solving approach. In MRes, winning strategies for the universal player are explicitly represented within the proof in the form of deterministic branching programs, known as merge maps [4]. MRes builds partial strategies at each line of the proof such that the strategy at the last line (corresponding to the empty clause) forms the complete countermodel for the input QBF. As a result, MRes admits strategy extraction by design. Before applying the refutation rules, MRes needs to check the partial strategies of the hypothesis to be isomorphic. Note that the isomorphism checking in 'merge maps' is efficient, hence MRes is a polynomial time verifiable proof system.

In this paper, we extend MRes to a family of sound and refutationally complete QBF proof systems MRes-$\mathcal{R}$. We observe that the representation of strategies in the proofs as merge maps is not relevant for the soundness and completeness of the proof system. Strategies can be depicted by any complete representation and by slightly modifying the refutation rules to include arbitrary complete representations, the soundness and completeness of the proof system remains intact. To be precise, we change the isomorphism checking rule in MRes to 'consistency' checking (Section 3.1) defined for Dependency Quantified Boolean Formulas (DQBFs) in [9]. This leads to the definition of a new proof system (Say $\mathcal{P}$) for each complete representation. All these new proof systems together form the family of proof systems denoted by MRes-$\mathcal{R}$. However, due to the consistency checking rules, the proof systems in MRes-$\mathcal{R}$ are not polynomial time verifiable. In literature, many interesting non-polynomial time verifiable proof systems have been studied, for example, semantic cutting planes for QBFs (SemCP+∀red) [8]. Because of the introduction of such powerful consistency checking rules, proof systems in MRes-$\mathcal{R}$ allow a few forbidden resolution steps of MRes (ref. Example 2)

The paper also studies in detail the strength of these new proof systems. We show that eFrege+∀red is powerful enough to simulate valid refutations of the proof systems in MRes-$\mathcal{R}$. Since these systems admit strategy extraction by design, we show the said simulations by using the ideas from [10]. Furthermore, the paper lifts the lower bound results from [5] of regular MRes to every regular proof system $\mathcal{P} \in$ MRes-$\mathcal{R}$. We explain our contributions in detail in the following section.

1.1 Our Contributions

1. **Introducing a new family of proof systems MRes-$\mathcal{R}$:** MRes [4] uses merge maps to store the countermodels within proofs. We observe that merge maps are not important for the soundness and completeness of the proof system. They just make the proof system polynomial time verifiable. However, at the same time they are too restrictive. In this paper, we propose a family of proof systems MRes-$\mathcal{R}$, one for each arbitrary complete representation of strategies into proofs (instead of merge maps). In order to make these proof systems sound and complete, we only need to modify the rules of MRes slightly. To be precise, we check the consistency relation instead of isomorphism among the strategies before applying the resolution rules (ref. Section 3).

   Although, this modification makes the proof systems not polynomial time verifiable; however, makes them very interesting, since the representations of strategies can be arbitrary. We only need that the representations be complete, in the sense that every finite function has at least one representation in the same. We need this for proving the completeness of our proposed proof systems (in Claim 7).

   To be precise, for proving completeness of MRes-$\mathcal{R}$, we consider the MRes-$\mathcal{M}$ proof system in MRes-$\mathcal{R}$ which uses merge maps as the representation for strategies. We then prove that MRes-$\mathcal{M}$ system p-simulates the MRes proof system (which is known to be complete) by showing that every rule of MRes is also valid in MRes-$\mathcal{M}$ (Theorem 6). We then show how any MRes-$\mathcal{M}$-proof can be (non-efficiently) converted to $\mathcal{P}$-proof for any $\mathcal{P} \in$ MRes-$\mathcal{R}$ (Claim 7).

   The soundness proof of MRes-$\mathcal{R}$ follows from proving that every line of the proof gives a partial falsifying strategy for the universal player.

2. **Proving a lower bound for Regular MRes-$\mathcal{R}$:** The Completion Formulas CR_n were first introduced in [17], to show that level-ordered Q-Res cannot p-simulate $\forall$Exp+Res. They were also used to show that level-ordered Q-Res cannot p-simulate tree-like Q-Res [22]. It has been shown recently in [5], that CR_n formulas are even hard for regular MRes. In this paper, we lift this lower bound of the Completion Formulas CR_n to all regular proof systems in MRes-$\mathcal{R}$. That is, we show that any regular proof system $\mathcal{P} \in$ MRes-$\mathcal{R}$, takes exponential time to refute the CR_n formulas.

   For this lower bound proof, we mostly follow the proof from [5, Theorem 9] where they have used the fact that most of the clauses in the MRes proof are going to be free of all the literals from right (in quantifier prefix) of the only universal variable z of CR_n. They showed that the number of such clauses are exponential in n proving the required lower bound.

   We established the similar argument for every regular proof system in MRes-$\mathcal{R}$. In [5, Theorem 9], the major part of the proof relied on the fact that MRes uses isomorphism so they could rule out the variables not in one hypothesis merge map of a resolution step, as also not to be present in the other. However, this is not the case in MRes-$\mathcal{R}$. So we provide a new Claim (ref. Claim 14) that even though MRes-$\mathcal{R}$ insists on consistency rather than isomorphism, the above property holds. That is, the clauses in CR_n make it such that the variables not in

one hypothesis strategy cannot be present in the other as well when consistency is maintained in the resolution steps.

3. **eFrege+∀red simulates MRes-$\mathcal{R}$:** We show that eFrege+∀red simulates valid refutations in every proof system belonging to MRes-$\mathcal{R}$. Since proof systems in MRes-$\mathcal{R}$ can use arbitrary representations, simulating the same is difficult even for powerful proof systems. However MRes-$\mathcal{R}$ family uses simple rules for refuting, which can even be detected by just observing the clauses of the lines, without exploring the representation parts. If one can come up with a complete representation which can represent the rules performed by any MRes-$\mathcal{R}$ proof system, then one can show the required simulation.
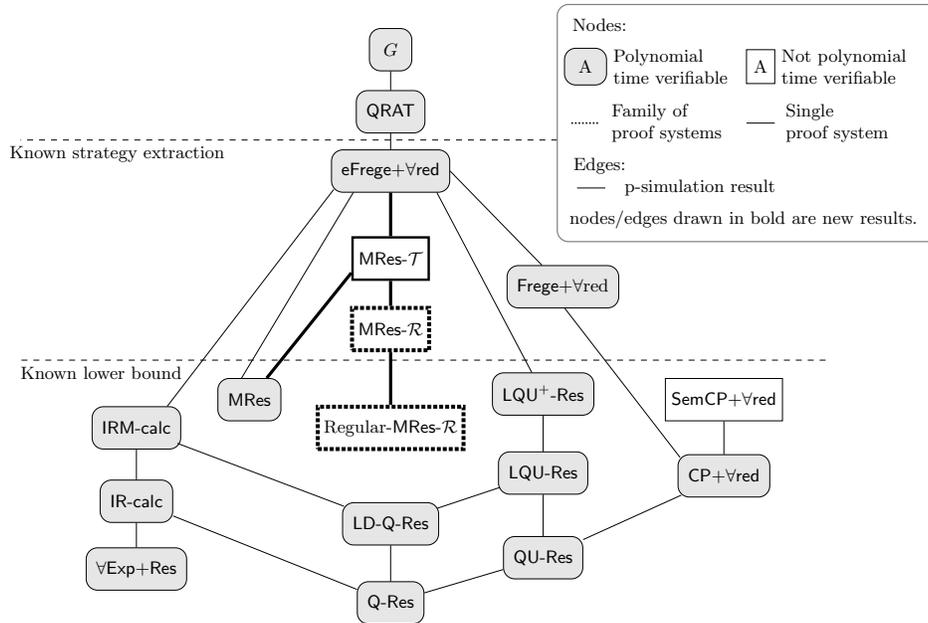
Fig. 1: *Various QBF proof systems and efficient simulations.* New results and proof systems are written in bold. MRes-$\mathcal{T}$ belongs to MRes-$\mathcal{R}$. Regular MRes-$\mathcal{R}$ are below the 'known lower bound' dashed line, due to Theorem 16. MRes-$\mathcal{T}$ p-simulates MRes due to Proposition 9. For the simulations of MRes, IRM-calc, and LQU+-Res by eFrege+∀red, and other known simulations refer [10, Fig. 1].

We use this observation and define a new complete representation, denoted as the T-representation (ref. Section 5.1). To handle all the MRes-$\mathcal{R}$ rules, we came up with this hybrid representation consisting of both circuits and branching programs. It is capable of representing all the refutation rules allowed in any MRes-$\mathcal{R}$ proof. To achieve this we introduce a new type of node, namely '#' node, which deals with the new consistency checking property (ref. Fig 2a). We also show that T representation is a complete representation, therefore the corresponding MRes-$\mathcal{T}$ proof system (5.1) belongs to MRes-$\mathcal{R}$.

The idea of the simulation is to convert every valid $\mathcal{P}$-proof ($\mathcal{P} \in$ MRes-$\mathcal{R}$) efficiently into an MRes-$\mathcal{T}$-proof as explained in Theorem 8. Then, as MRes-$\mathcal{T}$ admits strategy extraction by definition, we use the idea from [10] to simulate the valid MRes-$\mathcal{T}$-proof in eFrege+$\forall$red. Thereby proving that eFrege+$\forall$red simulates any valid refutations from proof system in MRes-$\mathcal{R}$.

This simulation result is a way forward towards uniform certification in QBF [10]. In [10], they used a very distinguished technique that allows eFrege+$\forall$red to simulate few proof systems that admit strategy extraction. Using this technique, they showed that eFrege+$\forall$red can simulate proof systems MRes, IRM-calc and LQU$^+$-Res. We use the same technique and prove that eFrege+$\forall$red even simulates the family of proof systems MRes-$\mathcal{R}$.

Observe that the proposed eFrege+$\forall$red simulation algorithm (Section 5) of the proof systems $\mathcal{P}$ in MRes-$\mathcal{R}$ assumes that the given $\mathcal{P}$-refutations are valid. As a result one cannot use this simulation algorithm to efficiently verify the correctness of the given $\mathcal{P}$-refutations. That is, even if the resulting eFrege+$\forall$red proof is valid (which is efficiently verifiable), one cannot infer if or not the initial $\mathcal{P}$-refutation is valid. However, the proposed simulation algorithm always produces a valid eFrege+$\forall$red proof for a given valid $\mathcal{P}$ proof.

Further, for the current simulation order among QBF proof systems refer Fig 1.

1.2   Organization of the paper

We present important notations and preliminaries used in this paper in Section 2. In Section 3, we present the new family of proof systems MRes-$\mathcal{R}$. We prove the soundness and refutational completeness of proof systems in MRes-$\mathcal{R}$ in Section 4. In Section 5, we show that eFrege+$\forall$red simulates proof systems in MRes-$\mathcal{R}$. We establish the lower bound results for every regular proof system in MRes-$\mathcal{R}$ in Section 6. Finally, we conclude and present a few open problems in Section 7.

2   Notations and Preliminaries

For a Boolean variable x, its literals can be x and $\overline{x}$. A clause C is a disjunction of literals and a conjunctive normal form (CNF) F is a conjunction of clauses. We denote the empty clause by $\perp$. $vars(C)$ is a set of all variables in C and width$(C) = |vars(C)|$.

A **proof system** [11] for a non-empty language $L \subseteq \{0,1\}^*$ is a polynomial time computable function $f : \{0,1\}^* \to \{0,1\}^*$ such that Range$(f) = L$. For string $x \in L$, we say a string $w \in \{0,1\}^*$ is an f-proof of x if $f(w) = x$. A proof system f for L is complete if and only if for every $x \in L$ we have a corresponding f-proof for x. A proof system f for L is sound if and only if the existence of an f-proof for x implies that $x \in L$. Informally, a proof system is a function f which maps proofs to theorems (or contradictions).

A proof system f p-simulates (polynomially simulates) another proof system g, if every g-proof of input x can be translated into an f-proof for the same input in polynomial time w.r.t the size of the g-proof. We denote this as $f \leq_p q$. Proof

systems for $L =$ FQBFs/ TQBFs are said to be QBF proof systems where, FQBFs (TQBFs) denote the set of all false (true) QBFs.

Quantified Boolean formulas: QBFs are an extension of the propositional Boolean formulas where each variable is quantified with one of $\{\exists, \forall\}$, the symbols having their general semantic definition of existential and universal quantifier respectively.

In this paper, we assume that QBFs are in closed prenex form with CNF matrix i.e., we consider the form $Q_1 X_1 ... Q_k X_k.\ \phi(X)$, where X_i are pairwise disjoint sets of variables; $Q_i \in \{\exists, \forall\}$ and $Q_i \neq Q_{i+1}$, and $\phi(X)$ is in CNF form over variables $X = X_1 \cup \cdots \cup X_k$, called the matrix of the QBF. We denote QBFs as $\mathcal{F} := Q.\phi$ in this paper, where Q is the quantifier prefix. If $x \in X_i$ then we denote $Q(x)$ to be equal to Q_i. For a variable x if $Q(x) = \exists$ (resp. $Q(x) = \forall$), we call x an existential (resp. universal) variable. If a variable x is in the set X_i, any $y \in X_j$ where $j < i$, we say that y occurs to the left of x in the quantifier prefix and write $y \leq_Q x$. On the other hand, if $j > i$ we say that y occurs to the right of x in the quantifier prefix and write $y \geq_Q x$. The set of existential variables to the left of a universal variable u will be denoted by $L_Q(u)$ in this paper.

Let $C \in \phi$ and $Q(u) = \forall$, then the 'falsifying u-literal' is defined to be 0 if $u \in C$, and 1 if $\overline{u} \in C$ and $*$ if $u \notin vars(C)$. Also, the existential subclause of C is the clause formed by only the existential literals from C.

If S is any set of variables, a complete assignment of S will be an assignment which assigns every variable in S to either 1 or 0. Similarly, a partial assignment is an assignment which assigns a subset of variables in S to either 1 or 0. Note that the $vars(S)$ that have not been assigned to 1 or 0 in a particular partial assignment of S are denoted as having an assignment of '$*$'. We denote $\langle S \rangle$ as the set of all possible complete assignments of S and $\langle\langle S \rangle\rangle$ as the set of all possible partial assignments of S.

QBFs as a game: [1] Given a QBF $\mathcal{F} = \exists X_1 \forall X_2 \ldots \exists X_n.\phi$ we may view it as game between universal and existential player. The rules of the game being that according to the quantification sequence the players assign values to the sets X_is alternatively. At the end, when substituting the complete assignment to all variables in ϕ, if ϕ evaluates to 1 (resp. 0) the existential (resp. universal) player wins.

For a QBF $Q.\phi$, a **strategy** of universal player is a decision function that returns the assignment to all universal variables of Q, where the decision for each u depends only on the variables in $L_Q(u)$. If H^u is the strategy for the universal variable u then, $vars(H^u)$ is the subset of existential variables from $L_Q(u)$ which are actually used in building the strategy H^u.

Winning strategy for the universal player is a strategy which gives an assignment to all universal variables of the given QBF for every possible assignment of existential variables, such that the substitution of this complete assignment falsifies the QBF. Winning strategy of the universal player is also called a countermodel in case of a false QBF. A QBF is false iff there exists a winning strategy for the universal player [1].

We say that a QBF proof system f admits **strategy extraction** if for any given valid f-proof of a false QBF $\mathcal{F}$, one can compute a winning strategy for the universal player in the time polynomial to the size of the f-proof.

As said earlier, strategies are basically decision functions. For the portrayal of the same, many representations can be used like truth tables, directed acyclic graphs (DAGs), merge maps, etc. A complete representation is the one in which every possible finite decision function can be represented.

Resolution [23] is the most studied redundancy rule in both SAT and QBF worlds, we define the same below:

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D},$$

where C, D are clauses and x is the pivot variable. The clause $(C \vee D)$ is called the resolvent. We denote this step as '$res(C \vee x, D \vee \overline{x}, x)$' throughout the paper.

Next, we define a few QBF proof systems that we require in this paper.

Q-Res [19] is one of a basic QBF proof system. It is an extension of the resolution proof system for QBFs. It allows the resolution rule defined above with the pivot variable being existential. For dealing with the universal variables, it defines a 'universal reduction' rule as follows:

The **Universal Reduction** (UR) rule of Q-Res allows dropping of a universal variable u from a clause C in the QBF, provided no existential variable $x \in C$ appears to the right of u in the quantifier prefix.

2.1   MRes

MRes is a proof system for false QBFs introduced in [4]. We describe MRes briefly in this section, please refer to [4] for its formal definition.

For a false QBF $Q.\phi$, an MRes refutation will be a sequence of lines of the form $L_i = (C_i, \{M_i^u\})$; where C_i is a clause consisting of only existential literals and $\{M_i^u\}$ is the set of merge maps of each universal variable $u \in Q$.

The **Merge maps** represent the partial strategies for each universal variable at any line. The merge map M_i^u is a decision branching graph with definite strategies $\{0, 1, *\}$ at the leaves nodes ($*$ is used when no strategy for u exists till that line). The intermediate nodes of merge map M_i^u branch on some existential variable (say $x) \in L_Q(u)$. That is, if $L_i = res(L_a, L_b, x)$ for some $a, b < i$, then M_i^u will get connected to M_a^u with an edge label of $\overline{x}$ and to M_b^u with an edge label of x.

An important property used in MRes refutation rules is defined below:

Isomorphism: Two merge maps are isomorphic if and only if there exists a bijection mapping from the nodes of one to that of another. In other words, two isomorphic merge maps represent exactly the same strategy.

Two operations needed for MRes refutations are defined below:

Select operation is defined on two merge maps. If they are isomorphic, then it outputs one of them. Or, if one of them is trivial (i.e $*$), then it outputs the other.

Merge(M_a^u, M_b^u, n, x) operation is defined when $a, b < n$, and it returns a new merge map where the new root node is connected to M_a^u with $\overline{x}$ and to M_b^u with x. Also, if any common line nodes exist in M_a^u, M_b^u, it merges them into a single node.

Now we define the MRes proof system:

For a false QBF $Q.\phi$, the MRes proof $\Pi := L_1, L_2, ..., L_m$ where every line $L_i := (C_i, \{M_i^u : \text{for every universal variable } u \text{ in } Q\})$ is derived using either an 'Axiom' step or a 'Resolution' step. In the axiom step, C_i will be the existential subclause of some $C \in \phi$ and every M_i^u will be a leaf node with the falsifying u-literal of C. In the resolution step, C_i is obtained from $res(C_a, C_b, x)$ where x is an existential variable and $a, b < i$. For this step to be valid, each M_i^u must either be equal to $select(M_a^u, M_b^u)$ or if $x <_Q u$ then it can be equal to $merge(M_a^u, M_b^u, i, x)$.

The final line L_m is the conclusion of Π, and Π is a refutation of $Q.\phi$ iff $C_m = \bot$. G_Π be the derivation graph corresponding to Π with edges directed from the hypothesis to the resolvent (i.e from the axioms to the final line). A refutation Π is said to be regular if no leaf-to-root path in G_Π has any existential variable x as a pivot more than once. For some given line L, Π_L is defined as the sub-derivation of Π deriving the line L.

2.2   eFrege+∀red

Frege systems are fundamental proof systems of propositional logic. Lines in a Frege proof are formulas inferred from the previous lines via few sound rules. The rules are not important as all Frege systems are p-equivalent, the only condition is that a Frege system needs to be sound and complete. So w.l.o.g, we can assume that 'modus ponens' is the only rule in a Frege system. The modus ponens is defined as: if $A \to B$ and A are present in the hypothesis then B can be logically implied by the hypothesis. For a detailed definition and explanation refer [20].

Extended Frege (eFrege) [12] is an extention of Frege systems which allows introduction of new variables not present in previous lines of the proof. This rule allows lines of the form $v \leftrightarrow f(S)$ where v is a new variable and f can be any formula on the set of variables S, where $v \notin S$.

For QBFs, eFrege is modified to be eFrege+∀red (Extended Frege + ∀ reduction) [6] which requires that the extention variable must be added in the prefix and quantified to the right of the variables used to define it. To deal with the universal variables, the universal reduction (UR) rule as defined in Q-Res is introduced into eFrege+∀red. The formal definition is as follows:

$$\frac{L_j}{L_i = L_j[u/B]} \ (\forall red)$$

where L_j is some previous line in the eFrege+∀red proof, u is a universal variable that is also rightmost in the prefix among all variables in L_j and B is the Herband function of u [6]. That is, a universal variable u in a formula can be replaced by 0 or 1 when no other variable in that formula are to the right of u in the prefix.

For the rest of known QBF-proof systems depicted in Fig 1, refer to [25]. We define the following formulas needed for proving lower bound later in the paper.

2.3 Completion Principle Formulas [17]

The QBFs CR_n are defined as follows:

$$\mathsf{CR}_n = \underset{i,j\in[n]}{\exists}\, x_{ij},\, \forall z,\, \underset{i\in[n]}{\exists}\, a_i,\, \underset{j\in[n]}{\exists}\, b_j\, \left(\bigwedge_{i,j\in[n]} (A_{ij} \wedge B_{ij}) \right) \wedge L_A \wedge L_B$$

where,

$$A_{ij} = x_{ij} \vee z \vee a_i \qquad\qquad B_{ij} = \overline{x_{ij}} \vee \overline{z} \vee b_j$$
$$L_A = \overline{a_1} \vee \cdots \vee \overline{a_n} \qquad\qquad L_B = \overline{b_1} \vee \cdots \vee \overline{b_n}$$

For any CR_n formula, we define the sets $A := \{a_1, a_2, ..., a_n\}$ and $B := \{b_1, b_2, ..., b_n\}$ as the set of all a, b variables respectively.

3 MRes-$\mathcal{R}$: A new family of proof systems for false QBFs

We define a family of proof systems MRes-$\mathcal{R}$, inspired from the MRes proof system. In MRes ([4]), strategies are built within the proof and are represented by merge maps. We observed that merge maps or any specific representations of strategies are not important for the soundness or completeness of the proof system. Since, isomorphism problem is efficient in merge maps, they make the proof systems polynomial time verifiable.

Based on this observations, we define a family of MRes-$\mathcal{R}$ where every proof system $\mathcal{P} \in$ MRes-$\mathcal{R}$ has it's own complete representation to represent the strategies. To allow the use of arbitrary representations in MRes-$\mathcal{R}$, we introduce consistency checking rules for strategies which are not as efficient as checking isomorphism for MRes. As a result, our proof systems are not polynomial time verifiable. However their soundness & completeness doesn't depend on their representations, which makes them interesting.

We use the idea of consistency checking from [9], which uses the same for DQBFs. For simplicity, we use the same notations from [9] whenever possible. We begin by defining some important notations and operations needed before actually defining the MRes-$\mathcal{R}$ systems.

3.1 Important notations used in MRes-$\mathcal{R}$

To begin, let us define what consistency means for any two assignments of a set of variables.

Definition 1 ([9]). *Let X be any set of variables and $\varepsilon, \delta \in \langle\langle X \rangle\rangle$. We say that ε and δ are consistent, denoted by $\varepsilon \simeq \delta$, if for every $x \in X$ for which $\varepsilon(x) \neq *$ and $\delta(x) \neq *$ we have $\varepsilon(x) = \delta(x)$.*

Let H_u and H'_u be individual strategy functions for the universal variable u, the consistency between two strategies is defined as follows:

We say that H_u and H'_u are consistent (written $H_u \simeq H'_u$) when $H_u(\varepsilon) \simeq H'_u(\varepsilon)$ for each $\varepsilon \in \langle\langle L_Q(u) \rangle\rangle$. Recall that $L_Q(u)$ are the existential variables to the left of u in the prefix. In other words H_u and H'_u are consistent, if the u-assignment given by $H_u(\varepsilon)$ and $H'_u(\varepsilon)$ should be consistent for every possible $L_Q(u)$-assignment ε.

By a change in notation, we can see (partial) assignments as both functions and sets of literals, i.e. an assignment ε corresponds to the set of literals it satisfies. For example, $\{x_1, x_2, \overline{x_3}, \overline{x_4}\}$ represents an assignment which sets 1 to the variables x_1 and x_2 and 0 to x_3 and x_4. In this notation as sets of literals, a union ($\cup$) of assignments ε, δ is defined when $\varepsilon \simeq \delta$ and it is equal to $\varepsilon \cup \delta$.

We now define a union operation ('$\circ$') on two consistent strategies H_u and H'_u.

Definition 2 ([9]). *Given two consistent strategies H_u and H'_u (i.e., $H_u \simeq H'_u$), we define the union strategy H''_u of H_u and H'_u, denoted by $H''_u = H_u \circ H'_u$, as follows:*

$$H''_u(\varepsilon) = H_u(\varepsilon) \cup H'_u(\varepsilon) \text{ for each } \varepsilon \in \langle L_Q(u) \rangle.$$

For example, if $H_u \& H'_u$ be defined as below, then $H''_u = H_u \circ H'_u$ will be:

$$H_u = \begin{cases} 1: & x \\ *: & \overline{x} \end{cases} \qquad H'_u = \begin{cases} *: & x \\ 0: & \overline{x} \end{cases}$$

$$H''_u = \begin{cases} 1 \cup * = 1: & x \\ * \cup 0 = 0: & \overline{x} \end{cases}$$

We now define a if-else operation ('$\bowtie$') on any two strategies H_u and H'_u.

Definition 3 ([9]). *Given any two strategies H_u and H'_u and an existential variable x, we define the if-else operation of H_u and H'_u on x to give the strategy H''_u, denoted by $H''_u = H_u \overset{x}{\bowtie} H'_u$, for every $\varepsilon \in \langle L_Q(u) \rangle$ as follows:*

$$H''_u(\varepsilon) = \begin{cases} H_u(\varepsilon): & \varepsilon(x) = 1 \\ H'_u(\varepsilon): & \varepsilon(x) = 0 \end{cases}$$

For example, if $H_u \& H'_u$ be defined as below, then $H''_u = H_u \overset{x}{\bowtie} H'_u$ will be:

$$H_u = \begin{cases} 1: & y \\ *: & \overline{y} \end{cases} \qquad H'_u = \begin{cases} *: & z \\ 0: & \overline{z} \end{cases} \quad ; \quad H''_u = \begin{cases} 1: & xy \\ *: & x\overline{y} \\ *: & \overline{x}z \\ 0: & \overline{x}\overline{z} \end{cases}$$

Note that the input strategies H_u, H'_u need not be consistent for an '$\bowtie$' operation, but they must be in case of an '$\circ$' operation.

3.2   Definition of MRes-$\mathcal{R}$

Let $\Phi = Q.\phi$ be a QBF with existential variables X and universal variables U. A MRes-$\mathcal{R}$ derivation of L_m from Φ is sequence $\pi = L_1, ..., L_m$ of lines where each $L_i = (C_i, \{H_i^u : u \in U\})$ in which at least one of the following holds for $i \in [m]$:

a. **Axiom**. There exists a clause in $C \in \phi$ such that C_i is the existential subclause of C, and for each $u \in U$, H_i^u is the strategy function for u mapping it to the falsifying u-literal for C or,
b. **Resolution**. There exist integers $a, b < i$ and an existential pivot $x \in X$ such that $C_i = res(C_a, C_b, x)$ and for each $u \in U$:
   i. if $x <_Q u$, then $H_i^u = H_b^u \overset{x}{\bowtie} H_a^u$
   ii. else if $x >_Q u$, then $H_i^u = H_a^u \circ H_b^u$.

π is a refutation of Φ iff $C_m = \bot$. Size of π is the number of lines i.e $|\pi| = m$. Observe that similar to MRes, proof systems in MRes-$\mathcal{R}$ have only existential literals in the clause part of the lines in a proof.

Let S be a subset of the existential variables X of a false QBF $\mathcal{F}$. We say that a $\mathcal{P}$-refutation π of $\mathcal{F}$ (where $\mathcal{P} \in$ MRes-$\mathcal{R}$) is S-regular if for each $x \in S$, there is no leaf-to-root path in G_π that uses x as pivot more than once. An X-regular proof is simply called a regular proof.

4   Soundness and Completeness of MRes-$\mathcal{R}$

In this section, we show that each proof system in MRes-$\mathcal{R}$ is sound and refutationally complete for false QBFs.

4.1   Soundness

The soundness proof of MRes-$\mathcal{R}$ follows closely to that of the MRes proof system. The following lemma proves the soundness of MRes-$\mathcal{R}$ family of proof systems.

Lemma 4. *Let $\mathcal{P} \in$ MRes-$\mathcal{R}$ be any proof system. Let $\pi = L_1, L_2, \ldots, L_m$ be a valid $\mathcal{P}$-proof of QBF $\Phi = Q.\phi$. Then, the strategy functions $\{H_m^u : u \in U\}$ in the conclusion line L_m of π, will form a countermodel for Φ.*

Proof. Given $\pi := L_1, ..., L_m$ be an $\mathcal{P}$-refutation of a QBF $\Phi = Q.\phi$. Let each $L_i = (C_i, \{H_i^u : u \in U\})$ and X, U are sets of all existential and universal variables in Q respectively. Further, for each $i \in [m]$,

- let $\alpha_i := \{\bar{l} : l \in C_i\}$ be the smallest assignment falsifying Ci ,
- let $A_i := \{\alpha \in \langle X \rangle : C_i \cap \alpha = \emptyset\}$ be all complete assignments to X consistent with α_i,
- for each $\alpha \in A_i$, let $l_i^u(\alpha) := H_i^u(\alpha)$ and $H_i(\alpha) := \{l_i^u(\alpha) : u \in U\} \setminus \{*\}$.

Induction statement:By induction on $i \in [m]$, we show, for each $\alpha \in A_i$, that the restriction of ϕ by $\alpha \cup H_i(\alpha)$ contains the empty clause.

Proof: For the base case $i = 1$, let $\alpha \in A_1$. As L_1 is introduced as an axiom, there exists a clause $C \in \phi$ such that C_1 is the existential subclause of C, and each H_1^u is the function outputting the falsifying u-literal for C. Hence, for each $u \in U$, $l_1^u(\alpha)$ is the falsifying u-literal for C, so $C[\alpha \cup H_1(\alpha)] = \emptyset$.

For the inductive step, let $i \geq 2$ and let $\alpha \in A_i$. The case where L_i is introduced as an axiom is identical to the base case, so we assume that L_i was derived by resolution. Then there exist integers $a, b < i$ and an existential pivot $x \in X$ such that $C_i = res(C_a, C_b, x)$

(1) Suppose that $\bar{x} \in \alpha$, each $u \in U$ has to satisfy either:
   (i) $x <_Q u$ and $H_i^u = H_b^u \overset{x}{\bowtie} H_a^u$: In which case, $l_i^u(\alpha) = l_a^u(\alpha)$.
   (ii) $x >_Q u$ and $H_i^u = H_a^u \circ H_b^u$: In which case, $l_i^u(\alpha) = \{l_a^u(\alpha) \cup l_b^u(\alpha)\}$.

It follows that $l_i^u \neq l_a^u$ only if $l_a^u = *$, and hence $H_a(\alpha) \subseteq H_i(\alpha)$. Since $C_a \setminus \{x\} \subseteq C_i$, we have $\alpha \in A_a$, so the restriction of ϕ by $\alpha \cup H_i(\alpha)$ contains the empty clause by the inductive hypothesis that $\alpha \cup H_a(\alpha)$ contains the empty clause.

(2) Suppose that $x \in \alpha$. A similar argument shows that $H_b(\alpha) \subseteq H_i(\alpha)$.

□

Since α_m is the empty assignment, we have $A_m = \langle X \rangle$ (i.e all complete assignments to X). We therefore prove the lemma at the final step $i = m$, as we show that $\{H_m^u : u \in U\}$ is a countermodel for Φ.   □

4.2   Completeness

One would notice that a major change of MRes-$\mathcal{R}$ from MRes is the usage of 'consistency' check instead of 'isomorphism' check. Note that the relation between them is as such: isomorphism $\Rightarrow$ consistency but not vice-versa. We use this in our proofs for completeness of MRes-$\mathcal{R}$.

   We show the completeness by first showing that MRes-$\mathcal{M}$ p-simulates MRes (Theorem 6). Here, MRes-$\mathcal{M}$ is a proof system in MRes-$\mathcal{R}$ which uses merge maps as the representation. Further, we show in Claim 7 that a MRes-$\mathcal{M}$-proof can be transformed into any MRes-$\mathcal{R}$-proof in exponential time. Nevertheless, completeness is guaranteed as MRes is complete and any MRes-proof can be transformed into a MRes-$\mathcal{M}$-proof which in-turn can be transformed as any MRes-$\mathcal{R}$-proof.

   We will need the following remark from the paper introducing MRes [4].

Remark 5. *[4, Proposition 10] Any two isomorphic merge maps compute the same function.*

Theorem 6. *MRes-$\mathcal{M}$ p-simulates MRes.*

Proof. Given a QBF Φ and its MRes-proof $\pi = L_1, ..., L_m$, where every line $L_i = \{C_i, \{M_i^u : u \in U\}\}$. We intend to build an MRes-$\mathcal{M}$-proof $\Pi = L_1', ..., L_m'$ for Φ, where each $L_i' = \{C_i', \{H_i^u : u \in U\}\}$.

   For every line L_i in π starting from $i = 1$ to m, if L_i is an axiom step then directly $C_i' = C_i$ and $H_i^u = M_i^u$ for all $u \in U$. Otherwise, if L_i is an resolution step i.e for some $a, b < i$, $C_i = res(C_a, C_b, x)$; then set $C_i' = C_i$ and for each $u \in U$ if $x <_Q u$ then set $H_i^u = H_b^u \overset{x}{\bowtie} H_a^u$ else set $H_i^u = H_a^u \circ H_b^u$. We see that these are sound steps as resolution in MRes can be of the following types:

(i) $x >_Q u$ and $M_i = \text{select}(M_a^u, M_b^u)$; in this case we set $H_i^u = H_a^u \circ H_b^u$ which holds given the Remark 5 and that isomorphism $\Rightarrow$ consistency.

(ii) $x <_Q u$ and $M_i = \text{merge}(M_a^u, M_b^u, i, x)$; in this case we set $H_i^u = H_b^u \overset{x}{\bowtie} H_a^u$ which is same as the merge function of MRes.

(iii) $x <_Q u$ and $M_i = \text{select}(M_a^u, M_b^u)$; in this case we set $H_i^u = H_b^u \overset{x}{\bowtie} H_a^u$ which is allowed as MRes did the isomorphism test on M_a^u and M_b^u, but we need no such check for $\bowtie$ in MRes-$\mathcal{R}$. (ref. the note just after Definition 3).

In case-(iii) above it remains to note that adding a $\bowtie$ to two isomorphic maps or when one of them is $*$, doesn't add any new strategy: it just dilutes the strategy represented by the corresponding merge map. That is, we are adding an if-else condition where both the outcomes are same or one of them is $*$. Hence doesn't affect future consistency checks which may arise in the proof. (For further clarity, one is suggested to look at Example 1 but it is not needed for the proof).

Therefore, one can clearly see that the proof Π constructed in this process is a valid MRes-$\mathcal{M}$-proof for Φ. Hence this proves the above theorem. $\square$

Claim 7. *Every MRes-$\mathcal{M}$-proof can be transformed into an MRes-$\mathcal{R}$-proof for any representation R in exponential time.*

Proof. Given a QBF Φ and its MRes-$\mathcal{M}$-proof $\pi = L_1, ..., L_m$, where every line $L_i = \{C_i, \{M_i^u : u \in U\}\}$. We intend to build an MRes-$\mathcal{R}$-proof $\Pi = L_1', ..., L_m'$ for Φ, where each $L_i' = \{C_i', \{H_i^u : u \in U\}\}$.

For every line L_i in π, we keep the clause part intact while we convert the merge maps into plain functions. Further as R is a complete representation, these functions should have a corresponding representation in R; we extensively search for the same. This search terminates at some point owing to R being a complete representation. (This is the place where we used the property that R is a complete representation). The result is an MRes-$\mathcal{R}$-proof for Φ. This process is not polynomial in time but regardless still proves completeness for the family of proof systems MRes-$\mathcal{R}$. $\square$

Let us consider example 1 (below) which was referred in Theorem 6. This example considers the situation corresponding to the case-(iii) of Theorem 6. That is, two isomorphic merge maps can be combined with an if-else and the resulting strategy will still output the same as input merge maps. Or when one of the input merge map being $*$, makes the resulting strategy diluted in the sense that for half the assignments it gives a $*$ and for others the same as the non-trivial input merge map.

Example 1. Let $M_1^u = M_2^u = 1$ be leaf nodes in MRes proof. It implies that corresponding $H_1^u = 1$ and $H_2^u = 1$ in MRes-$\mathcal{R}$ proof. Now say MRes performs a resolution on pivot variable x which is to the left of u, resulting in $M_3^u = select(M_1^u, M_2^u)$. Whereas the corresponding MRes-$\mathcal{R}$ rule needs to be a $H_3^u = H_1^u \overset{x}{\bowtie} H_2^u$ from case(iii) (ref. Theorem 6). That is, H_3^u in function form would be defined as follows:
$$H_3^u = \begin{cases} 1 : & x \\ 1 : & \overline{x} \end{cases}$$
Notice how this is just a diluted way of writing the strategy $H_3^u = 1$. Hence when in the next line of MRes if a $M_4^u = 1$ which is isomorphic to M_3^u is encountered; the corresponding $H_4^u = 1$ in MRes-$\mathcal{R}$ will still remain to be consistent with H_3^u (though they might seem to be structurally different).

In the same example if $M_2^u = *$ (i.e. trivial), the strategy H_3^u would have been:
$$H_3^u = \begin{cases} 1 : & x \\ * : & \overline{x} \end{cases}$$

Notice how this is another way of diluting the strategy and is still consistent with $H_4^u = 1$.

So far, we showed that each proof system in MRes-$\mathcal{R}$ is sound and refutationally complete for false QBFs. Next, we present an example of MRes-$\mathcal{R}$ allowing few resolution steps which are not allowed in MRes. Such examples may be useful for the separation results between the proof systems in MRes-$\mathcal{R}$ and the MRes proof system.

Example 2. Consider any proof system $\mathcal{P}$ in MRes-$\mathcal{R}$ which uses some complete R representation for strategies. The following Table 1 is a $\mathcal{P}$-refutation of the false QBF : $\exists x \forall u \exists y \ (y \vee x \vee u) \wedge (y \vee \overline{x}) \wedge (\overline{y} \vee x) \wedge (\overline{y} \vee \overline{x} \vee \overline{u})$

Line	Rule	C_i	H_i^u
L_1	axiom	$\{y, x\}$	0
L_2	axiom	$\{y, \overline{x}\}$	$*$
L_3	$res(L_1, L_2, x)$	$\{y\}$	$H_2^u \overset{x}{\bowtie} H_1^u$
L_4	axiom	$\{\overline{y}, x\}$	$*$
L_5	axiom	$\{\overline{y}, \overline{x}\}$	1
L_6	$res(L_4, L_5, x)$	$\{\overline{y}\}$	$H_5^u \overset{x}{\bowtie} H_4^u$
L_7	$res(L_3, L_6, y)$	$\{\}$	$H_3^u \circ H_6^u$

Table 1: $\mathcal{P}$-refutation, where $\mathcal{P} \in$ MRes-$\mathcal{R}$, of the false QBF in Example 2

The strategies H_3^u and H_6^u in function format are as follows:
$$H_3^u = \begin{cases} 0: & x = 0 \\ *: & x = 1 \end{cases} \qquad H_6^u = \begin{cases} *: & x = 0 \\ 1: & x = 1 \end{cases}$$
One can see that these strategies are consistent (but not isomorphic), hence the resolution of L_3, L_6 on y is allowed in the $\mathcal{P}$-refutation. But the analogous resolution would be blocked in MRes since the corresponding merge maps M_3^u, M_6^u will not be isomorphic.

5   eFrege+∀red simulates MRes-$\mathcal{R}$

In this section, we show that eFrege+∀red can efficiently simulate any valid refutation from proof system in MRes-$\mathcal{R}$. Therefore, the stronger proof systems like QRAT (Quantified Resolution Asymmetric Tautologies) [16] and G (Gentzen/Sequent Calculus) [21] can also simulate the same.

However, proof systems in MRes-$\mathcal{R}$ can have arbitrary representations. Simulating the same is a nightmare for even the strongest proof systems. But observe that the rules to construct the strategies in any representation are the same as defined in Section 3.2. We capture these rules as a new tree structure T. That is, given a valid proof π of any proof system $\mathcal{P} \in$ MRes-$\mathcal{R}$, $\pi = (C_1, R_1), (C_2, R_2), \ldots, (C_m, R_m)$, we construct a proof $\pi' := (C_1, T_1), (C_2, T_2), \ldots, (C_m, T_m)$ in such a way that the representation T_i captures the rules that have been used to construct the strategy R_i.

We show that the representation T is also a complete representation for finite functions. Therefore, MRes-$\mathcal{T}$ also belongs to the family MRes-$\mathcal{R}$. Further, we show that any valid $\mathcal{P}$-proof π (where $\mathcal{P} \in$ MRes-$\mathcal{R}$) can be efficiently converted to a MRes-$\mathcal{T}$-proof π'. Finally using the ideas from [10], we show that eFrege+∀red can efficiently simulate MRes-$\mathcal{T}$. This shows that eFrege+∀red can simulate any proof systems in MRes-$\mathcal{R}$. We now proceed and define the proof system MRes-$\mathcal{T}$.

5.1   MRes-$\mathcal{T}$ proof system

Given a false QBF $\mathcal{F}$, a MRes-$\mathcal{T}$ proof π of $\mathcal{F}$ is a sequence of lines

$$(C_1, T_1), (C_2, T_2), \ldots, (C_m, T_m)$$

where $C_m = \perp$ and each T_i is constructed as follows: if (C_i, T_i) is an axiom step, then T_i is constructed as in the MRes proof system. Otherwise if (C_i, T_i) is a Resolution step on a pivot left of the universal variable in question (i.e if-else step ('⋈') of MRes-$\mathcal{R}$), then T_i is constructed, as a merge node is constructed in MRes. Further, if (C_i, T_i) is constructed from a resolution step on (C_j, T_j) and (C_k, T_k) with pivot being right of the universal variable in question and both T_j and T_k are consistent (i.e union step ('∘') of MRes-$\mathcal{R}$), then T_i is constructed by adding a new type of node called the $\#$ node (defined below) with inputs T_j and T_k.

The $\#$ node is defined assuming both its inputs are consistent, and it outputs the result of a union operation on them; more clearly, it's truth table is shown in the Fig. 2a.

A	B	A $\#$ B
1	1	1
0	0	0
$*$	0/1	0/1
0/1	$*$	0/1
$*$	$*$	$*$

(a) Truth table for $\#$ operator.
(It assumes inputs to be consistent.)
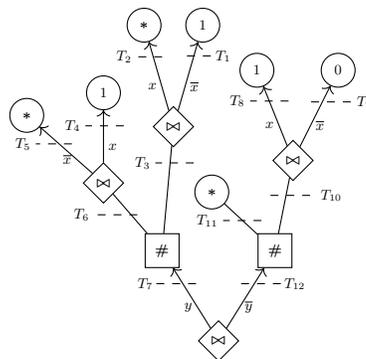
(b) T_{13}^u graph for Example 3

Fig. 2: Truth table of $\#$ operator is shown in Fig. 2a and its use in MRes-$\mathcal{T}$ depicted by an example QBF in Fig.2b

Note that $A = 1, B = 0$ and vice-versa cannot happen in a valid MRes-$\mathcal{R}$ proof owing to the definition of union('∘') which needs the input strategies to be consistent. Therefore, the corresponding rows are omitted from the $\#$ node truth table in Fig 2a. Let us illustrate a MRes-$\mathcal{T}$-proof below for an example QBF.

Example 3. Let $\Phi := \exists x, y, \forall u, \exists a, b \ (x, \overline{y}, \overline{u}, a) \wedge (\overline{x}, \overline{y}, a) \wedge (\overline{x}, \overline{y}, \overline{u}, \overline{a}) \wedge (x, \overline{y}, \overline{a}) \wedge (\overline{x}, y, \overline{u}, b) \wedge (x, y, u, b) \wedge (y, \overline{b})$. The MRes-$\mathcal{T}$ proof of Φ is shown below in Table 2:

Line	Rule	C_i	T_i^u	Type of node
L_1	axiom	$\{x, \overline{y}, a\}$	1	Leaf
L_2	axiom	$\{\overline{x}, \overline{y}, a\}$	$*$	Leaf
L_3	$res(L_1, L_2, x)$	$\{\overline{y}, a\}$	$T_2^u \overset{x}{\bowtie} T_1^u$	if-else
L_4	axiom	$\{\overline{x}, \overline{y}, \overline{a}\}$	1	Leaf
L_5	axiom	$\{x, \overline{y}, \overline{a}\}$	$*$	Leaf
L_6	$res(L_5, L_4, x)$	$\{\overline{y}, \overline{a}\}$	$T_4^u \overset{x}{\bowtie} T_5^u$	if-else
L_7	$res(L_3, L_6, a)$	$\{\overline{y}\}$	$T_3^u \circ T_6^u$	#
L_8	axiom	$\{\overline{x}, y, b\}$	1	Leaf
L_9	axiom	$\{x, y, b\}$	0	Leaf
L_{10}	$res(L_9, L_8, x)$	$\{y, b\}$	$T_8^u \overset{x}{\bowtie} T_9^u$	if-else
L_{11}	axiom	$\{y, \overline{b}\}$	$*$	Leaf
L_{12}	$res(L_{10}, L_{11}, b)$	$\{y\}$	$T_{10}^u \circ T_{11}^u$	#
L_{13}	$res(L_{12}, L_7, y)$	$\{\}$	$T_7^u \overset{y}{\bowtie} T_{12}^u$	if-else

Table 2: A MRes-$\mathcal{T}$ refutation of the false QBF in Example 3

The final T-graph of winning strategy for the only universal variable u from Example 3 is shown in Figure 2b. One can see that this graph is a hybrid structure of both branching programs and circuits. Since it has both 'branching' nodes ($\bowtie$ nodes) and 'circuit' nodes (# nodes).

Observe that the proposed T representation is complete. That is, any valid finite function can be represented by a T graph. This follows since, merge maps are a subset of T-graphs (i.e without # nodes) which are just branching programs, but known to be complete for all valid functions. Since T representations are complete, MRes-$\mathcal{T}$ is a member of MRes-$\mathcal{R}$ proof systems. Therefore this is a sound and complete proof system. Also note that MRes-$\mathcal{T}$ is not claimed to be polynomial time verifiable.

5.2   Conversion of MRes-$\mathcal{R}$ proofs into MRes-$\mathcal{T}$ proofs

In this section we show how to convert a valid $\mathcal{P}$-proof π into a valid MRes-$\mathcal{T}$-proof π', where $\mathcal{P}$ be any proof system in MRes-$\mathcal{R}$. Let $\pi = (C_1, R_1), (C_2, R_2), \ldots, (C_m, R_m)$ be a valid $\mathcal{P}$ proof of a QBF $\mathcal{F}$. We show how to convert π into a valid MRes-$\mathcal{T}$-proof $\pi' = (C_1, T_1), (C_2, T_2), \ldots, (C_m, T_m)$ of the same QBF $\mathcal{F}$. Note that here T_i is not the representation of R_i, but T_i is capturing how R_i has been constructed from some hypothesis R_j, R_k with $j, k < i$ using rules from Section 3.2. For this we do not need to interpret R_i's, but we can extract the required information from the clauses C_j, C_k and C_i of π.

It is also useful to note that, during this conversion, one doesn't need to check if the two strategies R_j, R_k are consistent or not. The conversion is smooth and simple as it assumes π to be a valid $\mathcal{P}$-proof of $\mathcal{F}$. We now proceed to give a detailed method for the same.

Theorem 8. *Any valid $\mathcal{P}$-proof ($\mathcal{P} \in$ MRes-$\mathcal{R}$) can be converted efficiently into an MRes-$\mathcal{T}$ proof.*

Proof. For a false QBF $\mathcal{F}$, proofs of proof systems belonging to MRes-$\mathcal{R}$ can have arbitrary representations for the strategies computed. However, the rules allowed to construct a strategy R_i using any strategies R_j and R_k (where $j, k < i$) are fixed. They must follow the rules mentioned in Section 3.2. MRes-$\mathcal{T}$ proof π' captures these rules only.

To be precise, given a $\mathcal{P}$-proof π of $\mathcal{F}$ where $\pi = (C_1, P_1), (C_2, P_2), ..., (C_m, P_m)$, we construct MRes-$\mathcal{T}$-proof π' as follows:

From the clause part of the proof π i.e $C_1, ..., C_m$ (in this sequence) based on what step is being followed (axiom, or resolution where pivot is on left, or resolution where pivot is on right), we build the corresponding T-maps as shown in the Figure 3.

(a) Axiom

(b) if-else node
(*res* when x is left of u in prefix)

(c) # node (i.e union node)
(*res* when x is right of u in prefix)
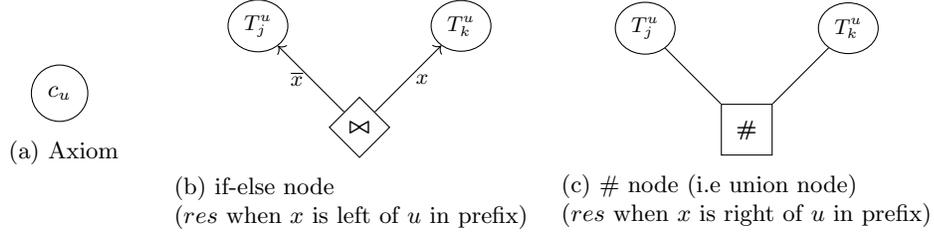
Fig. 3: Rules to construct T-graphs. In Figure 3a, c_u is the falsifying strategy of u for the axiom clause C_i. In Figure 3b, $C_i = res(C_j, C_k, x)$ and x is left of u in prefix i.e $T_i^u = T_k^u \overset{x}{\bowtie} T_j^u$. In Figure 3c, $C_i = res(C_j, C_k, x)$ and x is right of u in prefix i.e $T_i^u = T_j^u \circ T_k^u$. Note that the truth table of the '# gate' is defined in Figure 2a

After following this procedure for all lines in π, the sequence of lines so formed i.e $\pi' = (C_1, T_1), (C_2, T_2), \ldots, (C_m, T_m)$ is a valid MRes-$\mathcal{T}$ proof as the clauses $C_1, ..., C_m$ are the same as in the original MRes-$\mathcal{R}$ proof hence we know that C_m is definitely $\bot$ and that $T_1, ..., T_m$ are built using the same rules as used when building the valid $\mathcal{P}$-proof π. Therefore T_m is a countermodel as it is building the same strategy as in R_m. □

Now we proceed to show that MRes-$\mathcal{T}$ proof system can be efficiently simulated by eFrege+∀red. However before proving the same, observe that MRes-$\mathcal{T}$ efficiently simulates MRes proof system: due to Theorem 8, MRes-$\mathcal{T}$ simulates any MRes-$\mathcal{R}$ proof system, and therefore, it also simulates efficiently the MRes-$\mathcal{M} \in$ MRes-$\mathcal{R}$ proof system, which is known to simulate the MRes proof system efficiently (Theorem 6). Thus we have the following:

Proposition 9. *MRes-$\mathcal{T}$ efficiently simulates MRes.*

5.3   eFrege+∀red simulates MRes-$\mathcal{T}$

In this section, we show that eFrege+∀red efficiently simulates valid MRes-$\mathcal{T}$ refutations. We use the ideas from [10, Theorem 1] which shows how eFrege+∀red efficiently simulates MRes. Let us briefly explain the idea from [10, Theorem 1]: Given an MRes-proof $\pi = (C_1, M_1), (C_2, M_2), ..., (C_m, M_m)$ of a false QBF $\mathcal{F}$, we know that if π is valid then the merge map M_m in the last line gives a winning strategy S for the universal player of $\mathcal{F}$. That is, if we assign values of the universal variables based on S, it falsifies $\mathcal{F}$. In [10] they derived an eFrege+∀red proof π' from π efficiently in two phases: in the first phase, they derived $\mathcal{F} \vdash_{\text{eFrege}} (S \to \bot)$ using π. This is equivalent to $\mathcal{F} \vdash_{\text{eFrege}} \overline{S}$. This first phase was purely propositional. Later in the second phase, they used universal reduction to prove $\overline{S} \vdash_{\text{eFrege+∀red}} \bot$. Implying from both phases that $\mathcal{F} \vdash_{\text{eFrege+∀red}} \bot$. We also use the same tricks for simulating MRes-$\mathcal{T}$ with eFrege+∀red. Hence, we also simulate the same in two phases. However, in the first phase, they used a double induction in which the second induction depicted how to handle 'Select' and 'Merge' nodes of MRes. We extend this to MRes-$\mathcal{T}$ by introducing '# nodes' and giving a method to handle those in the second induction. We now prove this in detail.

Theorem 10. *eFrege+∀red efficiently simulates MRes-$\mathcal{T}$.*

Proof. **Phase-1**:
Given a valid MRes-$\mathcal{T}$ proof $\pi := (C_1, T_1), (C_2, T_2), ..., (C_m, T_m)$ of a false QBF $\mathcal{F}$, we create new extension variables for each node in every strategy appearing in the proof. That is, $s_{i,t}^u$ is created for the node t in the strategy T_i^u for the universal variable u.

We define $s_{i,t}^u$ based on whether the corresponding $T_i^u(t)$ is an axiom node, if-else node or # node as follows:

$$s_{i,t}^u := \begin{cases} \{1/0/*\} & T_i^u(t) = \{1/0/*\} \\ (y \wedge s_{i,b}^u) \vee (\overline{y} \wedge s_{i,c}^u) & T_i^u(t) = T_i^u(b) \overset{y}{\bowtie} T_i^u(c) \\ s_{i,b}^u \# s_{i,c}^u & T_i^u(t) = T_i^u(b) \circ T_i^u(c) \end{cases}$$

In the quantifier prefix, we place the newly created variables $s_{i,t}^u$ to the immediate left of u to maintain the soundness of the proof, as strategies for u depends on these variables.

We now prove the outer induction in 'Induction 1' which assumes that 'Induction 2' is valid and hence can derive the clause C_i by assigning local strategies to universal variables through a simple resolution for every line L_i.

Induction 1 *Consider the ith line of π, that is, (C_i, T_i). It is easy for eFrege+∀red to prove $\bigwedge_{u \in U_i}(u \leftrightarrow s_{i,r(u,i)}^u) \to C_i$, where $r(u,i)$ is the index of the root node of T_i^u. U_i is the subset of U for which T_i^u is non-trivial.*

Proof:
Base case: Axiom: Suppose C_i is derived by axiom download of some clause $C \in \mathcal{F}$. If u has a non-trivial strategy, it is because it appears in the clause C and

so $u \leftrightarrow s_{i,1}^u$, where $s_{i,1}^u \leftrightarrow c_u$ for $c_u \in \top, \bot$. The constant c_u is correctly chosen to oppose the literal in C so that C_i is just the simplified clause of C replacing all universal u with the corresponding constant c_u's. This is easy for eFrege+∀red to prove.

Inductive step: Resolution: If C_j is resolved with C_k to get C_i with pivots $\overline{x} \in C_j$ and $x \in C_k$, where $j, k < i$. From the induction hypothesis, we have $\bigwedge_{u \in U_j}(u \leftrightarrow s_{j,r(u,j)}^u) \to C_j$ and $\bigwedge_{u \in U_k}(u \leftrightarrow s_{k,r(u,k)}^u) \to C_k$. Observe that using these clauses, we cannot prove the required statement. However, note that if on the left hand side of theses clauses, one changes the j and k respectively to i, then using resolution we can derive C_i on the right hand side. We show in the Induction 2 (below) how to achieve the same. To be precise, from Induction 2 we prove that, $\bigwedge_{u \in U_i}(u \leftrightarrow s_{i,r(u,i)}^u) \to C_j$ and $\bigwedge_{u \in U_i}(u \leftrightarrow s_{i,r(u,i)}^u) \to C_k$ holds. We then resolve these together to derive C_i. This proves Induction 1.

□

Now in Induction 2 below, we prove what we claimed before in Induction 1 i.e, given $\bigwedge_{u \in U_j}(u \leftrightarrow s_{j,r(u,j)}^u) \to C_j$, we show $\bigwedge_{u \in U_i}(u \leftrightarrow s_{i,r(u,i)}^u) \to C_j$ holds. We proceed by handling each $u \in U_i$ one by one as follows:

Induction 2 U_i *is partitioned into W the set of adjusted variables and V the set of variables yet to be adjusted. For every such V, W, the following holds:*
$$(\bigwedge_{v \in V \cap U_j}(v \leftrightarrow s_{j,r(v,j)}^v)) \wedge (\bigwedge_{w \in W}(w \leftrightarrow s_{i,r(w,i)}^w)) \to C_j$$
Recall that U_i is the subset of U for which T_i^u is non-trivial.

Proof:
Base case: Initially W is empty and as strategies cannot go back to be trivial $U_j \subseteq U_i$. Hence the statement to prove is exactly the statement given above in the hypothesis. Therefore, base case is trivially true.
Inductive step:
Starting with $(\bigwedge_{v \in V \cap U_j}(v \leftrightarrow s_{j,r(v,j)}^v)) \wedge (\bigwedge_{w \in W}(w \leftrightarrow s_{i,r(w,i)}^w)) \to C_j$.
We pick a $u \in V$ to adjust into i-terms, i.e we show the following:
$(u \leftrightarrow s_{i,r(u,i)}^u) \wedge (\bigwedge_{v \in \{V \cap U_j\} \setminus \{u\}}(v \leftrightarrow s_{j,r(v,j)}^v)) \wedge (\bigwedge_{w \in W}(w \leftrightarrow s_{i,r(w,i)}^w)) \to C_j$.
We have three cases based on the rule used to derive the line $L_i = (C_i, T_i)$:

  i $T_j^u = *$
 ii $T_j^u \neq *$, $T_i^u = T_j^u \overset{x}{\bowtie} T_k^u$
iii $T_j^u \neq *$, $T_i^u = T_j^u \circ T_k^u$

In case (i) we can easily adjust the universal variable u. That is, we can simply add the following: $(u \leftrightarrow s_{i,r(u,i)}^u)$. This is sound because the clause $(u \leftrightarrow s_{j,r(u,j)}^u)$ has never appeared before in the left hand side of the hypothesis but still we were able to derive C_j. Therefore, adding $(u \leftrightarrow s_{i,r(u,i)}^u)$ to the left hand side of the hypothesis, will still be able to derive C_j.
In case (ii) we prove inductively that for each node t in T_j^u, $(s_{i,t}^u \leftrightarrow s_{j,t}^u)$ holds. This is true for all leaf and intermediate nodes of T_j^u as we are only going to connect two T-graphs(i.e T_j^u, T_k^u) by an extra if-else node in T_i^u, i.e. all nodes of T_j^u are present in T_i^u. Hence eventually at the root of T_j^u, we will have $s_{i,r(u,j)}^u \leftrightarrow s_{j,r(u,j)}^u$. However to prove the induction statement (Induction 2), we need to show this relation between

roots of T_i^u and T_j^u i.e, $s_{i,r(u,i)}^u \leftrightarrow s_{j,r(u,j)}^u$. For this we use the definition of merging that $x \rightarrow (s_{i,r(u,i)}^u \leftrightarrow s_{i,r(u,j)}^u)$ and so we have $(s_{i,r(u,i)}^u \leftrightarrow s_{i,r(u,j)}^u) \vee \overline{x}$. We almost got the relation we needed but only $\overline{x}$ is the extra literal. But note that $\overline{x}$ is already $\in C_j$. So, the $\overline{x}$ is absorbed by the C_j in right hand side of the implication.

In case (iii) using a similar induction as used in case (ii), we can derive $s_{i,r(u,j)}^u \leftrightarrow s_{j,r(u,j)}^u$: because we are not deleting any strategies just adding an $\#$ gate. By the definition of the $\#$ gate, $s_{i,r(u,i)}^u \neq s_{i,r(u,j)}^u$ only when $s_{i,r(u,j)}^u \leftrightarrow *$, in which case it is directly case-(i) above. That is, we can directly add $u \leftrightarrow s_{i,r(u,i)}^u$ to the given hypothesis and we are done. In the other case when $s_{i,r(u,i)}^u = s_{i,r(u,j)}^u$, we can simply add $s_{i,r(u,i)}^u \leftrightarrow s_{i,r(u,j)}^u$ which directly proves the induction step. $\square$

Phase-2:
At this point, from the Induction 1, we have derived:

$$\mathcal{F} \left|_{\overline{\text{eFrege}}} \left(\bigwedge_{u \in U_m} (u \leftrightarrow s_{m,r(u,m)}^u) \rightarrow \bot \right)\right.$$

In other words, we have derived the winning strategy (say S) for the universal player in the QBF $\mathcal{F}$ i.e $\mathcal{F} \left|_{\overline{\text{eFrege}}} (S \rightarrow \bot)\right.$. Equivalently, $\mathcal{F} \left|_{\overline{\text{eFrege}}} \overline{S}\right.$. Also, observe that so far we are only in the propositional world. Using the ideas from [10], now from the negation of the strategies for the universal player (i.e., $\overline{S}$), we can easily derive the empty clause using the universal reduction steps.

That is, we have the following $\overline{S} := \bigvee_{i=1}^n (u_i \oplus s_{m,r(u_i,m)}^{u_i})$, where $U_m = \{u_1, u_2, \ldots, u_n\}$ in this order in the prefix. Observe the following property for some $k = 1$ to $k = n$ in this order:

$$\bigvee_{i=1}^{n-k+1} (u_i \oplus s_{m,r(u_i,m)}^{u_i})$$

From the above formula, just pull out the last term and we have:

$$\bigvee_{i=1}^{n-k} (u_i \oplus s_{m,r(u_i,m)}^{u_i}) \vee (u_{n-k+1} \oplus s_{m,r(u_{n-k+1},m)}^{u_{n-k+1}})$$

Performing the universal reduction step on u_{n-k+1} is the same as:

$$\bigvee_{i=1}^{n-k} (u_i \oplus s_{m,r(u_i,m)}^{u_i}) \vee (0 \oplus s_{m,r(u_{n-k+1},m)}^{u_{n-k+1}}) \bigwedge \bigvee_{i=1}^{n-k} (u_i \oplus s_{m,r(u_i,m)}^{u_i}) \vee (1 \oplus s_{m,r(u_{n-k+1},m)}^{u_{n-k+1}})$$

Which is same as:

$$\bigvee_{i=1}^{n-k} (u_i \oplus s_{m,r(u_i,m)}^{u_i}) \vee (s_{m,r(u_{n-k+1},m)}^{u_{n-k+1}}) \bigwedge \bigvee_{i=1}^{n-k} (u_i \oplus s_{m,r(u_i,m)}^{u_i}) \vee (\overline{s_{m,r(u_{n-k+1},m)}^{u_{n-k+1}}})$$

Note that we can perform universal reduction on u_{n-k+1} as the only existential new variable appearing in the clause is to the left of it in the prefix.

We can resolve these two to get the following:

$$\bigvee_{i=1}^{n-k} \left(u_i \oplus s_{m,r(u_i,m)}^{u_i}\right)$$

Note that we used the following rule above: $0 \oplus x$ is x and $1 \oplus x$ is $\bar{x}$. We continue reducing all u_i's to derive the $\perp$ at the end.

The proof of Theorem 9 can be concluded by combining the results of Phase-1 and Phase-2 i.e, $\mathcal{F} \vdash_{\mathsf{eFrege}} \overline{S}$ and $\overline{S} \vdash_{\mathsf{eFrege}+\forall\mathsf{red}} \perp$. Implying from both phases that $\mathcal{F} \vdash_{\mathsf{eFrege}+\forall\mathsf{red}} \perp$.

$\square$

Note that given a valid MRes-$\mathcal{T}$ proof, by the simulation in Theorem 10 one can obtain a valid eFrege+$\forall$red proof. But the validness of the resultant eFrege+$\forall$red proof cannot be used to determine if the initial MRes-$\mathcal{T}$ proof was valid or not. That is, an invalid MRes-$\mathcal{T}$ proof may result into a valid eFrege+$\forall$red proof. Therefore as claimed before, these MRes-$\mathcal{R}$ proof systems are not polynomial time verifiable even after being simulated by the powerful eFrege+$\forall$red proof system. From the above discussions, Theorem 8 and Theorem 10 imply the following:

Theorem 11. *eFrege+$\forall$red efficiently simulates valid refutations from proof systems in MRes-$\mathcal{R}$.*

6 Lower Bound for Regular MRes-$\mathcal{R}$

In this section, we lift the lower bound of Completion Formulas (CR_n, Section 2.3) for Regular-MRes [5, Theorem 9], to Regular-MRes-$\mathcal{R}$. We state the CR_n formulas once again for ease of reference.

$$\mathsf{CR}_n = \underset{i,j\in[n]}{\exists} x_{ij}, \forall z, \underset{i\in[n]}{\exists} a_i, \underset{j\in[n]}{\exists} b_j \left(\bigwedge_{i,j\in[n]}(A_{ij} \wedge B_{ij})\right) \wedge L_A \wedge L_B$$

where,

$$A_{ij} = x_{ij} \vee z \vee a_i \qquad\qquad B_{ij} = \overline{x_{ij}} \vee \overline{z} \vee b_j$$
$$L_A = \overline{a_1} \vee \cdots \vee \overline{a_n} \qquad\qquad L_B = \overline{b_1} \vee \cdots \vee \overline{b_n}$$

The lower bound follows from a stronger result that we prove below in Theorem 12 that any (A ∪ B)-regular refutation of CR_n in any proof system belonging to MRes-$\mathcal{R}$ must have size $2^{O(n)}$. We use the ideas from [5] to prove the lower bound. We try to maintain the same notations wherever possible for simplicity.

Before presenting the lower bound proof in detail, we present the basic idea for the same. The proof setup is depicted in Figure 4. As every clause in CR_n has a variable from the set $A \cup B$, but the refutation should derive a $\perp$ at the final line; there must be a 'section' of the proof (See shaded region S' in Fig 4) which only has X variables in all its clauses. This section also includes the final line. The set of clauses at the 'border' (See the bold line S in Fig 4) of this section of the proof is shown to be wide (in terms of number of literals) in Lemma 13. Using this and the argument that the conjunction of clauses in S itself forms a false CNF formula,

we show in Theorem 12 that the number of clauses in S is large (exponential in n). This directly implies that the size of the MRes-$\mathcal{R}$-proof is also large.

To establish the width bound, we note that the pivots which are used while deriving clauses in S are variables from $A \cup B$ and that they are all to the right of z. Meaning that the corresponding resolutions must all be union steps i.e the incoming strategies must be consistent (not isomorphic as is the case in MRes). This especially makes it difficult to directly lift the lower bound proof of MRes from [5]. However we successfully overcome this issue in Claim 14 by arguing how L_A, L_B are the only clauses with trivial strategies and how any other clause which resolves with these will mask this trivial-ness with its own definitive strategy. Further, by analysing what axiom clauses cannot be used in the derivation of the clauses in S, we show that many variables cannot be resolved before these lines. Hence, these variables will still be present in the clause $\in S$, making it wide. We now clearly state and prove the theorem for the lower bound result.
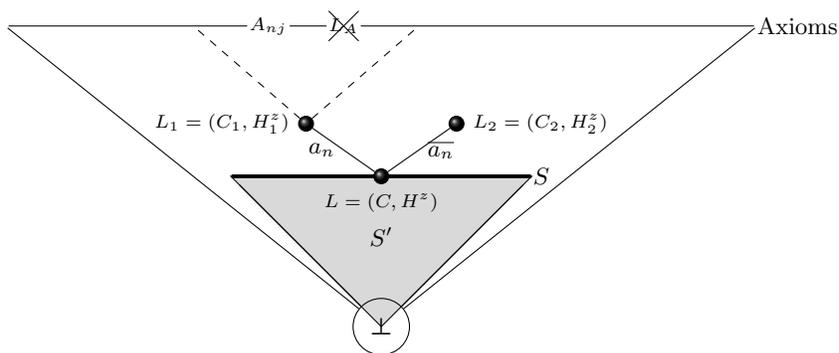
Fig. 4: *Lower bound proof illustration.* Given any $\mathcal{P} \in$ MRes-$\mathcal{R}$, a CR_n formula and it's $\mathcal{P}$-proof Π, this figure shows the graph G_Π. Claim 14 illustrates that $x_{ij} \notin var(H_2^z)$ for $i \in [n-1], j \in [n]$. Claim 15 illustrates that $|vars(C_2)| \geq n-1$. Lemma 13 shows that $|vars(C)| \geq n-1$. Theorem 12 proves that $|S| \geq 2^{n-1}$

Theorem 12. *Every $(A \cup B)$-regular refutation of CR_n in any proof system belonging to MRes-$\mathcal{R}$ has size $2^{\Omega(n)}$.*

Proof. For $\mathcal{P} \in$ MRes-$\mathcal{R}$, let Π be a $\mathcal{P}$-refutation of CR_n (for $n > 2$). Let the set of lines S, S' be defined as follows:

S': This set consists of all the lines $L = (C, H^z)$ from Π such that $vars(C) \cap \{A \cup B\} = \emptyset$ and there exists a path from L to $\perp$ in G_Π consisting of lines only from S'.

S: This set contains all the lines $L \in S'$ such that $L = Res(L_1, L_2, v)$ where $L_1, L_2 \notin S'$. Observe that the pivot variable v must belong to $\{A \cup B\}$.

Let $F = \bigwedge\limits_{(C,H^z)\in S} C$. Note that F is a false CNF formula because there exists a sub-derivation $\widehat{\Pi} = \{C | \exists L = (C, H^z) \in S'\}$ which derives a $\perp$ given F. The variables in F are only x_{ij}'s where $i, j \in [n]$, therefore it consists of n^2 variables. In Lemma 13 we prove that each clause in F has width $\geq n - 1$. That is each clause can be falsified by setting atleast n-1 variables to 0. Hence the number of complete assignments of X that can falsify a clause $\in F$ will be at most $2^{n^2-(n-1)}$. Since F is a false CNF formula, all assignments to X should falsify some clause of F. Therefore, the number of clauses in F should be $\geq 2^{n-1}$. This implies that the number of lines in S is at least 2^{n-1}. Therefore, the number of lines in Π must also be exponential in n. $\qquad\square$

Now it remains to prove Lemma 13 which we prove below.

Lemma 13 ([5]). *For all $L = (C, H^z) \in S$, $width(C) \geq n - 1$.*

Proof. Observe that L is not an axiom as all axioms of CR_n have a variable from $A \cup B$ and so they cannot belong to S. So, let $L = res(L_1, L_2, v)$ where $L_1, L_2 \notin S'$. Since two lines not belonging in S' resolve to make the resultant $\in S'$, the pivot (i.e v) should be from $A \cup B$. Assume $v \in A$, a similar argument can be made when $v \in B$. Without loss of generality, assume that $v = a_n$[1]; and $a_n \in C_1$ and $\overline{a_n} \in C_2$.

Since Π is $(A \cup B)$-regular, a_n does not occur as a pivot in the sub-derivation Π_{L_1}. It implies that the axiom clause L_A cannot be used in deriving L_1, because otherwise C_1 will have both a_n & $\overline{a_n}$ which makes it a tautology. That implies, axioms with other positive literals a_i's cannot be used in Π_{L_1} as the negated literals $\overline{a_i}$'s are only available in L_A which in-turn cannot be used in Π_{L_1}. Positive literals of a_i's only $\in A_{ij}$ for all $j \in [n]$. Hence, axioms A_{ij} for $i \in [n-1], j \in [n]$ also cannot be used in deriving the line L_1. Now, we know x_{ij} only occur in A_{ij}; so H_1^z has no x_{ij} variable for $i \in [n-1], j \in [n]$. Also, H_1^z is not a trivial strategy as some A_{nj} for $j \in [n]$ has been used because $a_n \in C_1$.

Since the pivot a_n at the resolution step obtaining line L is to the right of z,by the rules of $\mathsf{MRes}\text{-}\mathcal{R}$, H_1^z and H_2^z are consistent. In Claim 14, we prove that even though $\mathsf{MRes}\text{-}\mathcal{R}$ only insists on consistency, it still holds that for each $i \in [n-1]$, and each $j \in [n]$, $x_{ij} \notin \mathrm{var}(H_2^z)$. Using this result we prove in Claim 15 below, that C_2 will have at least $n - 1$ variables (including $\overline{a_n}$). Therefore, at least $n - 2$ variables from C_2 belong in C.

Also, observe that $x_{nj} \in C_1$ for some $j \in [n]$: Since some clause A_{nj} for $j \in [n]$ was used in Π_{L_1}, the literal x_{nj} is introduced into the proof and resolution of x_{nj} is not possible before L_1. This is because, the clause B_{nj} needed to resolve it, brings with it literal b_j which needs to be resolved before L_1 (as L_1 cannot have any $A \cup B$ literals other than a_n). To resolve this b_j, one needs to introduce the clause L_B, but L_B brings all $\overline{b}$'s into the resultant which cannot be further resolved as the B-clauses needed for the same do not have consistent strategies anymore. That is, because of the use of A_{nj} the resolvent has a 0 strategy for some assignment to X

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[1] Note that here a_n is used only for ease in dividing the set A into partitions. nowhere in the proof we use the fact that a_n is the last variable in A. Hence it is indeed w.l.o.g

variables, but B-clauses have a constant strategy of 1 hence these strategies will not be consistent to resolve further.

Hence, we know $x_{nj} \in C_1$ for some $j \in [n]$. It implies that $x_{nj} \in C$ as well. This x_{nj} cannot $\in C_2$ as the corresponding axiom clause needed for the same has a_n in it, which would make C_2 a tautology. Using the three results above, we can derive that width$(C) \geq n - 1$. $\qquad\square$

Claim 14. *For $i \in [n-1]$, and each $j \in [n]$, $x_{ij} \notin var(H_2^z)$.*

Proof. At the point of use of this claim in the proof of Lemma 13, we definitely know that for $i \in [n-1]$ & $j \in [n]$; $x_{ij} \notin H_1^z$. That is, if f_1 is the function representing the strategy H_1^z, then for any assignment σ of x_{nj}'s and $i \in [n-1], j \in [n]$, it implies that:

$$f_1(\sigma, x_{ij} = 0) = f_1(\sigma, x_{ij} = 1) \tag{1}$$

Let f_2 be the function representing the strategy H_2^z. Since a_n is to the right of z, we know that H_1^z and H_2^z are consistent, i.e for any assignment σ' (an extension of σ) and for $i \in [n-1], j \in [n]$, it implies that:

$$f_2(\sigma', x_{ij} = 0) \simeq f_1(\sigma', x_{ij} = 0) \tag{2}$$
$$f_2(\sigma', x_{ij} = 1) \simeq f_1(\sigma', x_{ij} = 1) \tag{3}$$

Only remaining question is if $f_2(\sigma', x_{ij} = 0) = f_2(\sigma', x_{ij} = 1)$? Observe that if this equality holds, then f_2 will be independent of x_{ij}'s, which implies that $x_{ij} \notin H_2^z$ for $i \in [n-1], j \in [n]$. Now, we are heading towards proving the equality holds. Note that if none of the terms in equation 2 and equation 3 give a '$*$' for any assignment of X, the equality in question definitely holds. So, now we prove that none of them can give a '$*$' for any given assignment.

The only axiom clauses of CR_n with trivial strategies are L_A, L_B and these axioms only contain variables of $A \cup B$, which are all to the right of z. Hence if any other clause is to be resolved with these clauses, the pivot has to be in $A \cup B$ i.e. a union step needs to be performed. At this point the trivial-ness of L_A (or L_B) is masked and does not show up in the final strategy of the resultant line; this is because union of any strategy with a trivial strategy will be the strategy itself. The only case by which a '$*$' can be in the resulting strategy is if L_A is resolved with L_B, which can clearly not happen as they have no common variable.

Since C_1, C_2 are definitely not the axiom clauses L_A (or L_B), using the above argument it is simply not possible for the functions f_1 (or f_2) to output a '$*$' for any input assignment provided. This means the equality in question above holds; meaning that H_2^z also doesn't depend on x_{ij}'s when $i \in [n-1], j \in [n]$ i.e $x_{ij} \notin vars(H_2^z)$. $\qquad\square$

Now we prove claim 15 which was used in Lemma 13.

Claim 15 ([5]). *Either for all $i \in [n-1]$, C_2 has a variable of the form x_{i*}, or for all $j \in [n]$, C_2 has a variable of the form x_{*j}*

Proof. At this point in the proof of Lemma 13, we definitely know that $\overline{a_n} \in C_2$, and for all $i \in [n-1]$, for all $j \in [n]$, $x_{ij} \notin \text{var}(H_2^z)$. We prove this claim by contradiction. Suppose the claim is wrong i.e, there exists some $u \in [n-1]$ where for all $l \in [n]$ $x_{ul} \notin var(C_2)$ and some $v \in [n]$ where for all $k \in [n]$ $x_{kv} \notin var(C_2)$.

Let ρ be the minimum partial assignment falsifying C_2. Then we know that :

  ▷ ρ sets $a_n = 1$, leaves all other variables in $A \cup B$ unset, since they $\notin C_2$.
  ▷ ρ does not set any x_{ul} or x_{kv}, since by our assumptions they all are not in C_2.

Now, extend ρ to assignment α by setting:

  ▷ $a_u = b_v = 0$ and rest all unset variables from $A \cup B$ to 1.
  ▷ Also except x_{uv}, set $x_{u*} = 1$ and $x_{*v} = 0$.

Observe that the assignment α satisfies all axiom clauses except A_{uv} and B_{uv} and does not falsify any axiom.
Now extend α to α_0 and α_1 by setting $x_{uv} = 0$ and 1 respectively.

The extension α_0 satisfies one more axiom i.e. B_{uv}; similarly α_1 satisfies one more axiom i.e. A_{uv}. Note that they still do not falsify the remaining axiom. That is, α_0 does not falsify A_{uv} and similarly, α_1 does not falsify B_{uv}.

α_0 and α_1 agree everywhere except on x_{ij} , and since $x_{ij} \notin \text{var}(H_2^z)$, it follows that $H_2^z(\alpha_0) = H_2^z(\alpha_1)$, say this value is equal d.

From the proved Induction in Lemma 4, the partial strategy of universal player at every line combined with the extension of the existential assignment falsifying it's clause part, should falsify some axiom of the QBF. Also, α_0 and α_1 falsify C_2, since they extend ρ. Hence, it is a contradiction that $(\alpha_{\overline{d}}, d)$ satisfies all axioms. Therefore, the claim needs to be true.

<div align="right">□</div>

From the above discussions and due to Theorem 12, we have the following:

Theorem 16. *Every MRes-$\mathcal{R}$-regular refutation of CR_n has size $2^{\Omega(n)}$.*

7 Conclusion and Future work

MRes proof system introduced recently in [4] builds strategies into proofs for false QBFs. We extend this proof system to a new family of sound, refutationally complete but not polynomial time verifiable proof systems MRes-$\mathcal{R}$. For each complete representation R, we have a proof system in MRes-$\mathcal{R}$. We also define a complete representation T, and it's proof system MRes-$\mathcal{T}$ belonging to MRes-$\mathcal{R}$. We show how this MRes-$\mathcal{T}$ proof system efficiently simulates the before-mentioned MRes proof system. We also prove that eFrege+∀red can simulate every valid refutation from proof systems belonging to MRes-$\mathcal{R}$. Further, we establish a lower bound of Completion Formulas (CR_n) for every regular-proof system in MRes-$\mathcal{R}$. Refer Fig 1 for the resulting landscape of QBF-proof systems with efficient simulations.

Still several open problems remain in the scope of this paper. We would like to end our discussions by pointing out a few of them.

The simulation relation between proof systems in MRes-$\mathcal{R}$ and MRes is still open. Since proof systems in MRes-$\mathcal{R}$ uses strong consistency checking rules as compared to the isomorphism rule in MRes, we believe that there exists a family of QBFs which are easy for proof systems in MRes-$\mathcal{R}$ but hard for MRes. For the motivation of the same refer Example 2. It presents the resolution steps forbidden in MRes but allowed in MRes-$\mathcal{R}$.

Another important open problem, is to establish a lower bound for proof systems in MRes-$\mathcal{R}$. Note that whether KBKF-lq formulas from [3], is hard or easy for proof systems in MRes-$\mathcal{R}$ is still open. These formulas have been shown to be hard for the MRes proof system in [5].

MRes proof system is inspired from the LD-Q-Res proof system. It allows some forbidden resolution steps of LD-Q-Res. It has already been shown that MRes efficiently simulates the reduction-less LD-Q-Res proof system [4]. However, it is still open whether MRes and LD-Q-Res are incomparable, or if one can simulate the other.

8   Acknowledgements

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