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On The "Majority is Least Stable" Conjecture

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Abstract

We show that the "majority is least stable" conjecture is true for $n = 1$ and 3 and false for all odd $n \geq 5$.

1 Introduction

A Boolean function $f : \{-1, 1\}^n \to \{-1, 1\}$ is said to be a linear threshold function (LTF) if there are real constants $w_0, w_1, \ldots, w_n$ such that for any $\mathbf{x} = (x_1, \ldots, x_n) \in \{-1, 1\}^n$, $f(\mathbf{x}) = \mathsf{sgn}(w_0 + w_1x_1 + \cdots + w_nx_n)$, where $\mathsf{sgn}(z) = 1$ if $z \geq 0$, and -1 if $z < 0$.

For $\mathbf{x} \in \{-1, 1\}^n$ and $\rho \in [0, 1]$, define a distribution $N_\rho(\mathbf{x})$ over $\{-1, 1\}^n$ in the following manner: $\mathbf{y} = (y_1, \ldots, y_n) \sim N_\rho(\mathbf{x})$ if for $i = 1, \ldots, n$, $y_i = x_i$ with probability ρ and $y_i = \pm 1$ with probability $(1 - \rho)/2$ each. The noise stability of a function $f : \{-1, 1\}^n \to \mathbb{R}$, denoted by $\mathsf{Stab}_\rho(f)$, is defined as follows.

$$\mathsf{Stab}_\rho(f) = \mathop{\mathbb{E}}_{\mathbf{x} \sim \{-1,1\}^n,\ \mathbf{y} \sim N_\rho(\mathbf{x})} \left[f(\mathbf{x}) f(\mathbf{y}) \right].$$

For odd n, the majority function $\mathsf{Maj}_n : \{-1, 1\}^n \to \{-1, 1\}$ is the following.

$$\mathsf{Maj}_n(x_1, \ldots, x_n) = \mathsf{sgn}(x_1 + x_2 + \ldots + x_n).$$

Benjamini, Kalai and Schramm in 1999 (see [1, 3]) put forward the following conjecture.

Conjecture 1 (*"Majority is Least Stable"*) : *Let n be odd and $f : \{-1, 1\}^n \to \{-1, 1\}$ be an LTF. Then for all $\rho \in [0, 1]$, $\mathsf{Stab}_\rho(f) \geq \mathsf{Stab}_\rho(\mathsf{Maj}_n)$.*

A counterexample to the conjecture for $n = 5$ has been reported in [5] by Vishesh Jain where it is also mentioned that there are other known counterexamples to this conjecture by Sivakanth Gopi (2013), and Steven Heilman and Daniel Kane (2017). We could not locate these other counterexamples.

In this note, we show that Conjecture 1 is true for $n = 1$ and 3 and false for odd $n \geq 5$. To show that the conjecture is false for odd $n \geq 5$, we define a sequence of Boolean functions g_n and show that $\mathsf{Stab}_\rho(g_n) < \mathsf{Stab}_\rho(\mathsf{Maj}_n)$. To show that the conjecture is true for $n = 3$, we employed a search over all locally monotone 3-variable Boolean functions f and obtained the expressions for $\mathsf{Stab}_\rho(f)$. It turns out that each of these expressions is greater than or equal to $\mathsf{Stab}_\rho(\mathsf{Maj}_n)$ for all $\rho \in [0, 1]$.

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2 Preliminaries

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$ and $2^{[n]}$ be the power set of $[n]$. The Fourier transform of $f : \{-1,1\}^n \to \{-1,1\}$ is a map $\hat{f} : 2^{[n]} \to [-1,1]$ defined as follows. For $S \subseteq [n]$,

$$\hat{f}(S) \quad = \quad \frac{1}{2^n} \sum_{\mathbf{x}=(x_1,\ldots,x_n)\in\{-1,1\}^n} f(\mathbf{x}) \prod_{i\in S} x_i. \tag{1}$$

For $f : \{-1,1\}^n \to \{-1,1\}$ and $k \in \{0, \ldots, n\}$, let $W^{(k)}[f] = \sum_{S\subseteq[n],|S|=k} \hat{f}^2(S)$ and $W^{\leq k}[f] = \sum_{i=0}^{k} W^{(i)}[f]$. We say that f is balanced if $\#\{\mathbf{x} : f(\mathbf{x}) = 1\} = \#\{\mathbf{x} : f(\mathbf{x}) = -1\}$. It follows that f is balanced if and only if $\hat{f}(\emptyset) = 0$.

The Fourier expression of $\mathsf{Stab}_\rho(f)$ is the following (see Page 56 of [6]).

$$\mathsf{Stab}_\rho(f) = \sum_{k=0}^{n} \rho^k \cdot W^{(k)}[f]. \tag{2}$$

It is easy to see that Maj_n is balanced and so $W^{(0)}[\mathsf{Maj}_n] = 0$. It is known that (see Page 62 of [6])

$$W^{(1)}[\mathsf{Maj}_n] = \left[\frac{\binom{n-1}{\frac{n-1}{2}}}{2^{n-1}}\right]^2 \cdot n. \tag{3}$$

It was observed in [5] that if f is a balanced linear threshold function, then showing $W^{(1)}[f] < W^{(1)}[\mathsf{Maj}_n]$ would disprove Conjecture 1. For the sake of completeness, we state a more general form of this observation as a lemma and provide a proof.

Lemma 1 *Let n be odd and $f : \{-1,1\}^n \to \{-1,1\}$ be a Boolean function such that $W^{(0)}[f] = 0$ and $W^{(1)}[f] < W^{(1)}[\mathsf{Maj}_n]$. Then there exists a $\delta > 0$ such that $\mathsf{Stab}_\rho(f) < \mathsf{Stab}_\rho(\mathsf{Maj}_n)$ for all $0 < \rho < \delta$. Consequently, the function f is a counter-example to Conjecture 1.*

Proof: For $k \geq 0$, let $a_k = W^{(k)}[f] - W^{(k)}[\mathsf{Maj}_n]$. Since by assumption, $W^{(0)}[f] = 0$, $W^{(1)}[f] < W^{(1)}[\mathsf{Maj}_n]$, and noting that Maj_n is balanced, it follows that $a_0 = 0$ and $-1 \leq a_1 < 0$. On the other hand, for $k \geq 2$, we have $-1 \leq a_k < 1$.

Now, $\mathsf{Stab}_\rho(f) - \mathsf{Stab}_\rho(\mathsf{Maj}_n) = \sum_{k=1}^{n} \rho^k \cdot a_k$. Therefore, $\mathsf{Stab}_\rho(f) - \mathsf{Stab}_\rho(\mathsf{Maj}_n) < 0$ if and only if $\rho(a_2 + \rho a_3 + \ldots + \rho^{n-2} a_n) < -a_1$. Since $a_k < 1$ for $k = 2, \ldots, n$, it follows that $\rho(a_2 + \rho a_3 + \ldots + \rho^{n-2} a_n)$ is upper bounded by $\rho(1 + \rho + \ldots + \rho^{n-2})$ whose limiting value is 0 as $\rho \to 0$. Therefore, there must exist some $\delta > 0$ such that for all $0 < \rho < \delta$, $\rho(a_2 + \rho a_3 + \ldots + \rho^{n-2} a_n) < -a_1$. Consequently, $\mathsf{Stab}_\rho(f) < \mathsf{Stab}_\rho(\mathsf{Maj}_n)$ for all $0 < \rho < \delta$. □

Next we introduce the notion of influence of a variable on a Boolean function. For $i \in [n]$ and $f : \{-1,1\}^n \to \{-1,1\}$, $\mathsf{Inf}_i(f)$ is defined as follows (see Page 46 of [6]).

$$\mathsf{Inf}_i(f) = \Pr_{\mathbf{x}\in\{-1,1\}^n}[f(\mathbf{x}) \neq f(\mathbf{x}^{\oplus \mathbf{i}})],$$

where $\mathbf{x}^{\oplus i}$ denotes the vector $(x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n)$.

An n-variable Boolean function f is said to be locally monotone if it is monotone increasing or decreasing in each variable. From [4] (see Lemma 2.2 and the comment following it), it follows that if f is a locally monotone function, then for all $i \in [n]$, $\mathsf{Inf}_i(f) = |\hat{f}(\{i\})|$. Since an LTF is locally monotone, we have the following result which has been used in the proof of Theorem 4.1 of [4].

Theorem 1 *[4] If $f : \{-1,1\}^n \to \{-1,1\}$ is an LTF. Then $\sum_{i=1}^{n} \mathsf{Inf}_i(f)^2 = W^{(1)}[f]$.*

3 Settling Conjecture 1

We state and prove some results from which the main theorem follows.

Lemma 2 *Let $n \geq 1$, w_0 be a non-negative integer and w_1, w_2 be positive integers. Let T be a subset of $[n]$ of cardinality $t \leq n/2$. Consider the following LTF:*

$$f(x_1,\ldots,x_n) = \mathsf{sgn}\left(w_0 + w_1 \cdot \sum_{u\in T} x_u + w_2 \cdot \sum_{v \in \overline{T}} x_v\right).$$

Then

$$W^{(0)}[f] = \frac{1}{2^{2n}} \cdot \left[\sum_{(i,j)\in\mathcal{S}_1} \binom{t}{i}\binom{n-t}{j}\right]^2, \quad (4)$$

$$W^{(1)}[f] = \frac{t}{2^{2n-2}} \cdot \left[\sum_{(i,j)\in\mathcal{S}_2} \binom{t-1}{i}\binom{n-t}{j}\right]^2 + \frac{n-t}{2^{2n-2}} \cdot \left[\sum_{(i,j)\in\mathcal{S}_3} \binom{t}{i}\binom{n-t-1}{j}\right]^2, \quad (5)$$

where $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ are defined as follows.

$\mathcal{S}_1 = \{(i,j) : 0 \leq i \leq t,\ 0 \leq j \leq n-t \text{ and } -w_0 \leq w_1(2i-t) + w_2(2j-(n-t)) \leq w_0\}$,
$\mathcal{S}_2 = \{(i,j) : 0 \leq i \leq t-1,\ 0 \leq j \leq n-t \text{ and } -w_1 \leq w_0 + w_1(2i-(t-1)) + w_2(2j-(n-t)) < w_1\}$,
$\mathcal{S}_3 = \{(i,j) : 0 \leq i \leq t,\ 0 \leq j \leq n-t-1 \text{ and } -w_2 \leq w_0 + w_1(2i-t) + w_2(2j-(n-t-1)) < w_2\}$.

Proof: For $\mathbf{x} \in \{-1,1\}^n$, let $A(\mathbf{x}) = w_1 \cdot \sum_{u\in T} x_u + w_2 \cdot \sum_{v\in\overline{T}} x_v$ so that $f(\mathbf{x}) = \mathsf{sgn}(w_0 + A(\mathbf{x}))$. Let N (resp. M) be the number of $\mathbf{x}$'s such that $w_0 + A(\mathbf{x}) \geq 0$ (resp. $w_0 + A(\mathbf{x}) < 0$). Then $\hat{f}(\emptyset) = (N-M)/2^n$. Further, let N_1 (resp. N_2) be the number of $\mathbf{x}$'s such that $A(\mathbf{x}) > w_0$ (resp. $-w_0 \leq A(\mathbf{x}) \leq w_0$). So, $N = N_1 + N_2$. Since $A(-\mathbf{x}) = -A(\mathbf{x})$, it follows that $N_1 = M$ and so $\hat{f}(\emptyset) = N_2/2^n$. Therefore to obtain $W^{(0)}[f] = \hat{f}^2(\emptyset)$ it is sufficient to obtain N_2.

For $\mathbf{x} \in \{-1,1\}^n$, let $i = \#\{u \in T : x_u = 1\}$ and $j = \#\{v \in \overline{T} : x_v = 1\}$. Then $A(\mathbf{x}) = w_1(2i-t) + w_2(2j-(n-t))$. For $0 \leq i \leq t$ and $0 \leq j \leq n-t$, the pair (i,j) is in $\mathcal{S}_1$ if and only if $-w_0 \leq A(\mathbf{x}) \leq w_0$. So, the number of $\mathbf{x}$'s for which $-w_0 \leq A(\mathbf{x}) \leq w_0$ holds is $\sum_{(i,j)\in\mathcal{S}_1}\binom{t}{i}\binom{n-t}{j}$ which is the value of N_2.

We next consider the proof of (5). Fix some $s \in T$ and some $r \in \overline{T}$. Due to symmetry, for any $i \in T$, we have $\mathsf{Inf}_i(f) = \mathsf{Inf}_s(f)$ and for any $j \in \overline{T}$, we have $\mathsf{Inf}_j(f) = \mathsf{Inf}_r(f)$ and so from Theorem 1,

$$W^{(1)}[f] = t \cdot \mathsf{Inf}_s(f)^2 + (n-t)\cdot \mathsf{Inf}_r(f)^2. \quad (6)$$

Let N_s (resp. N_r) be the number of $\mathbf{x} \in \{-1,1\}$ such that $f(\mathbf{x}) \neq f(\mathbf{x}^{\oplus s})$ (resp. $f(\mathbf{x}) \neq f(\mathbf{x}^{\oplus r})$). Then $\mathsf{Inf}_s(f) = N_s/2^{n-1}$ and $\mathsf{Inf}_r(f) = N_r/2^{n-1}$.

For $\mathbf{x} \in \{-1,1\}^n$, let $B(\mathbf{x}) = w_0 + w_1\sum_{u\in T\setminus\{s\}} x_u + w_2 \sum_{v\in\overline{T}} x_v$. From the definition of f, N_s is the number of $\mathbf{x}$'s such that either $(w_1x_s + B(\mathbf{x}) \geq 0$ and $-w_1x_s + B(\mathbf{x}) < 0)$ or $(w_1x_s + B(\mathbf{x}) < 0$ and $-w_1x_s + B(\mathbf{x}) \geq 0)$ holds. The two conditions are equivalent to $-w_1x_s \leq B(\mathbf{x}) < w_1x_s$ and $w_1x_s \leq B(\mathbf{x}) < -w_1x_s$ respectively. For the first condition, we must have $x_s = 1$, since for $x_s = -1$, we obtain $w_1 \leq B(\mathbf{x}) < -w_1$ which is a contradiction since $w_1 > 0$; similarly for the second condition, we must have $x_s = -1$. So, both the conditions boil down to $-w_1 \leq B(\mathbf{x}) < w_1$, and consequently, N_s is the number of $\mathbf{x}$'s such that $-w_1 \leq B(\mathbf{x}) < w_1$ holds.

For $\mathbf{x} \in \{-1,1\}^n$, let $i = \#\{u \in T \setminus \{s\} : x_u = 1\}$ and $j = \#\{v \in \overline{T} : x_v = 1\}$. Then $B(\mathbf{x}) = w_0 + w_1(2i-(t-1)) + w_2(2j-(n-t))$. For $0 \le i \le t-1$ and $0 \le j \le n-t$, the pair (i,j) is in $\mathcal{S}_1$ if and only if $-w_1 \le B(\mathbf{x}) < w_1$ holds. So, the number of $\mathbf{x}$'s for which $-w_1 \le B(\mathbf{x}) < w_1$ holds is $\sum_{(i,j)\in\mathcal{S}_2}\binom{t-1}{i}\binom{n-t}{j}$ which is the value of N_s.

A similar argument shows that N_r is equal to $\sum_{(i,j)\in\mathcal{S}_3}\binom{t}{i}\binom{n-t-1}{j}$. Using the values of N_s and N_r to obtain $\mathsf{Inf}_s(f)$ and $\mathsf{Inf}_r(f)$ respectively and substituting these in (6) gives the expression for $W^{(1)}[f]$ stated in (5). $\square$

Remark 1 *In Lemma 2, we have assumed both w_1 and w_2 to be positive. For other conditions on w_1 and w_2, it is possible to obtain similar (but not the same) expressions for $W^{(0)}[f]$ and $W^{(1)}[f]$.*

For odd $n \ge 3$, we define a sequence of functions $g_n : \{-1,1\}^n \to \{-1,1\}$ where

$$g_n(x_1,\ldots,x_n) = \mathsf{sgn}(2\cdot(x_1+\ldots+x_{n-3}) + x_{n-2} + x_{n-1} + x_n). \tag{7}$$

In [5], the function g_5 has been shown to be a counter-example to Conjecture 1.

Lemma 3 *For g_n defined in (7), we have*

$$\begin{aligned} W^{(0)}[g_n] &= 0, \\ W^{(1)}[g_n] &= (n-3)\cdot\left[\frac{\binom{n-4}{\frac{n-5}{2}}\cdot 8}{2^{n-1}}\right]^2 + 3\cdot\left[\frac{\binom{n-3}{\frac{n-3}{2}}\cdot 2}{2^{n-1}}\right]^2. \end{aligned} \tag{8}$$

Proof: We use Lemma 2. For g_n, we have $w_0 = 0$, $w_1 = 1$ and $w_2 = 2$. Also, $T = \{n-2, n-1, n\}$ so that $t = 3$. With these values, the sets $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ defined in Lemma 2 are the following.

$$\begin{aligned} \mathcal{S}_1 &= \{(i,j) : 0 \le i \le 3,\ 0 \le j \le n-3 \text{ and } (2i-3) + 2(2j-(n-3)) = 0\}, \\ \mathcal{S}_2 &= \{(i,j) : 0 \le i \le 2,\ 0 \le j \le n-3 \text{ and } -1 \le (2i-2) + 2(2j-(n-3)) < 1\}, \\ \mathcal{S}_3 &= \{(i,j) : 0 \le i \le 3,\ 0 \le j \le n-4 \text{ and } -2 \le (2i-3) + 2(2j-(n-4)) < 2\}. \end{aligned}$$

Since $(2i-3)+2(2j-(n-3))$ is odd, it cannot be zero and so $\mathcal{S}_1$ is empty showing that $W^{(0)}[g_n] = 0$.

Since $(2i-2)+2(2j-(n-3))$ is even it cannot be equal to -1 and so the only possible value it can take is 0. From this, we obtain $\mathcal{S}_2$ to be $\{(1,(n-3)/2)\}$.

Similarly, since $(2i-3)+2(2j-(n-4))$ is odd, the only possible values in the set $\{-2,-1,0,1\}$ that it can take are $\{-1,1\}$. Corresponding to these two values, we obtain $j = (n-3-i)/2$ and $j = (n-2-i)/2$ respectively. Since n is odd, in the first case i must be even, while in the second case i must be odd. So, $\mathcal{S}_3 = \{(0,(n-3)/2), (2,(n-5)/2), (1,(n-3)/2), (3,(n-5)/2)\}$.

Using the values of w_0, w_1, w_2, t as well as $\mathcal{S}_2$ and $\mathcal{S}_3$ in (5), we obtain

$$W^{(1)}[g_n] = \frac{3}{2^{2n-2}}\cdot\left[\binom{2}{1}\binom{n-3}{\frac{n-3}{2}}\right]^2 + \frac{n-3}{2^{2n-2}}\cdot\left[\binom{n-4}{\frac{n-3}{2}} + 3\binom{n-4}{\frac{n-5}{2}} + 3\binom{n-4}{\frac{n-3}{2}} + \binom{n-4}{\frac{n-5}{2}}\right]^2$$

Noting that $(n-3)/2 + (n-5)/2 = n-4$ leads to the expression for $W^{(1)}[g_n]$ given in (8). $\square$

Lemma 4 *Let g_n be defined as in (7). For odd $n \ge 5$, $W^{(1)}[g_n] < W^{(1)}[\mathsf{Maj}_n]$.*

Proof: The expression for $W^{(1)}[\mathsf{Maj}_n]$ is given by (3) and the expression for $W^{(1)}[g_n]$ is given by (8). Therefore

$$\frac{W^{(1)}[g_n]}{W^{(1)}[\mathsf{Maj}_n]} = \left[\frac{\binom{n-4}{\frac{n-5}{2}}\cdot 8}{\binom{n-1}{\frac{n-1}{2}}}\right]^2\left(\frac{n-3}{n}\right) + \left[\frac{\binom{n-3}{\frac{n-3}{2}}\cdot 2}{\binom{n-1}{\frac{n-1}{2}}}\right]^2\left(\frac{3}{n}\right) = \left[\frac{n-1}{n-2}\right]^2\left(\frac{4n-9}{4n}\right). \tag{9}$$

From (9), it follows that $W^{(1)}[g_n] < W^{(1)}[\mathsf{Maj}_n]$ if and only if $(n-3)^2 > 0$ i.e. $n \geq 5$. □
Note that from (9), for $n = 3$ we have $W^{(1)}[g_3] = W^{(1)}[\mathsf{Maj}_3]$.

Lemma 5 *Conjecture 1 is true for $n = 1$ and $n = 3$.*

Proof: For $n = 1$, the only LTF is the majority function and so Conjecture 1 is trivially true.

Using (2), for $n = 3$, it is easy to check that $\mathsf{Stab}_\rho(\mathsf{Maj}_3) = 0.75\rho + 0.25\rho^3$. We need to compare this expression with $\mathsf{Stab}_\rho(f)$ where f is an LTF. We used an exhaustive search. There is no easy way to determine whether a given function is an LTF. Instead we considered the set of all 3-variable locally monotone functions. Since an LTF is locally monotone, our search covered all LTFs. Let f be a 3-variable locally monotone function. We obtained the values of $W^{(k)}[f]$, for $k = 0, 1, 2, 3$ and using (2) obtained the expression for $\mathsf{Stab}_\rho(f)$. From the search, the possible expressions for $\mathsf{Stab}_\rho(f)$ were obtained to be one of the following: 1, ρ, $0.75\rho + 0.25\rho^3$, $0.0625 + 0.6875\rho + 0.1875\rho^2 + 0.0625\rho^3$, $0.25 + 0.5\rho + 0.25\rho^2$, $0.5625 + 0.1875\rho + 0.1875\rho^2 + 0.0625\rho^3$. For each of these expressions, it is easy to verify that $\mathsf{Stab}_\rho(f) \geq \mathsf{Stab}_\rho(\mathsf{Maj}_3)$ for all $\rho \in [0, 1]$. □

Based on Lemmas 1, 4 and 5, we obtain the main result of the paper, of which the case $n = 5$ was reported in [5].

Theorem 2 *Conjecture 1 is true for $n = 1$ and $n = 3$. For odd $n \geq 5$, Conjecture 1 is false.*

4 Limiting Value of $W^{\leq 1}[g_n]$

It is known (see Page 62 of [6]) that $W^{(\leq 1)}[\mathsf{Maj}_n]$ is a decreasing sequence which is lower bounded by $2/\pi$. It has been conjectured (see [1] and Page 115 of [6]) that if f is an LTF, then $W^{\leq 1}[f] \geq \frac{2}{\pi}$.

We have shown that for odd $n \geq 5$, the function g_n defined by (7) satisfies $W^{(1)}[g_n] < W^{(1)}[\mathsf{Maj}_n]$. This brings up the question of whether the sequence g_n also provides a counter-example to the $2/\pi$ lower bound conjecture for LTFs. In this section, we show that this is not the case.

From Lemma 3, $W^{(0)}[g_n] = 0$ and so, $W^{\leq 1}[g_n] = W^{(1)}[g_n]$. This shows that it is sufficient to consider $W^{(1)}[g_n]$. We show that $W^{(1)}[g_n]$ is a decreasing sequence which is lower bounded by $2/\pi$. The expression for $W^{(1)}[g_n]$ given by (8) involves binomial coefficients. We use the following bounds on factorial (see Page 54 of [2]).

$$\sqrt{2\pi m}\cdot\frac{m^m}{e^m}\exp\left(\frac{1}{12m+1}\right) \leq m! \leq \sqrt{2\pi m}\cdot\frac{m^m}{e^m}\exp\left(\frac{1}{12m}\right). \tag{10}$$

Let $p = \frac{k}{m}$ and $q = 1 - p$. Using 10, the following bounds on $\binom{m}{k}$ can be obtained.

$$\left.\begin{array}{lcl}\binom{m}{k} & \geq & \frac{1}{\sqrt{2\pi mpq}}(p^pq^q)^{-m}\exp\left(\frac{1}{12m+1} - \frac{1}{12k} - \frac{1}{12(m-k)}\right), \\ \binom{m}{k} & \leq & \frac{1}{\sqrt{2\pi mpq}}(p^pq^q)^{-m}\exp\left(\frac{1}{12m} - \frac{1}{12k+1} - \frac{1}{12(m-k)+1}\right). \end{array}\right\} \tag{11}$$

Lemma 6 *For g_n defined in (7), $W^{(1)}[g_n]$ is a decreasing sequence and $\lim_{n\to\infty} W^{(1)}[g_n] = \frac{2}{\pi}$. Consequently, for all odd n, $W^{(1)}[g_n] \geq 2/\pi$.*

Proof: Let $a_n = W^{(1)}[g_n]$ and $b_n = W^{(1)}[\mathsf{Maj}_n]$. We wish to show that a_n is a decreasing sequence. To do this, we compare a_{n+2}/b_n to a_n/b_n. The expression for a_n/b_n is given by (9). Using (3) and (8), we obtain $a_{n+2}/b_n = (4n-1)/(4n)$. We have $a_n \geq a_{n+2}$ if and only if $a_n/b_n \geq a_{n+2}/b_n$. Using the expressions for a_n/b_n and a_{n+2}/b_n, the last condition is equivalent to

$$\left[\frac{n-1}{n-2}\right]^2 \left(\frac{4n-9}{4n}\right) > \frac{4n-1}{4n}$$

which holds if and only if $n \geq 3$. So, a_n is a decreasing sequence for all odd $n \geq 3$.

Let $A_n = (n-3) \cdot \left[\frac{\binom{n-4}{\frac{n-5}{2}} \cdot 8}{2^{n-1}}\right]^2$ and $B_n = 3 \cdot \left[\frac{\binom{n-3}{\frac{n-3}{2}} \cdot 2}{2^{n-1}}\right]^2$ and so $a_n = A_n + B_n$. We show that A_n tends to $2/\pi$ and B_n tends to 0 and so a_n tends to $2/\pi$ as n goes to infinity.

First consider A_n. Letting $m = n-4$, $k = \frac{n-5}{2}$, $p = k/m$ and $q = 1-p$, from (11) and using some routine simplifications we obtain the following bounds on A_n.

$$\begin{aligned}
A_n &\geq \frac{2}{\pi} \left[\frac{n-4}{n-3}\right]^{(n-3)} \left[\frac{n-5}{n-4}\right]^{-(n-4)} \exp\left(\frac{2}{12n-47} - \frac{2}{6n-30} - \frac{2}{6n-18}\right), \\
A_n &\leq \frac{2}{\pi} \left[\frac{n-4}{n-3}\right]^{(n-3)} \left[\frac{n-5}{n-4}\right]^{-(n-4)} \exp\left(\frac{2}{12n-48} - \frac{2}{6n-29} - \frac{2}{6n-17}\right).
\end{aligned}$$

Since $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$ and $\lim_{x\to\infty}(1-\frac{1}{x})^x = \frac{1}{e}$, it follows that $\lim_{n\to\infty} A_n = 2/\pi$.

Now, consider B_n. Letting $m = n-3$, $k = \frac{n-3}{2}$ and $p = q = \frac{1}{2}$, from (11) and using some routine simplifications we obtain the following bounds on B_n.

$$\begin{aligned}
B_n &\geq \frac{3}{2\pi} \left(\frac{1}{n-3}\right) \exp\left(\frac{2}{12n-35} - \frac{2}{6n-18} - \frac{2}{6n-18}\right), \\
B_n &\leq \frac{3}{2\pi} \left(\frac{1}{n-3}\right) \exp\left(\frac{2}{12n-36} - \frac{2}{6n-17} - \frac{2}{6n-17}\right).
\end{aligned}$$

It follows that $\lim_{n\to\infty} B_n = 0$.

□

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