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Superredundancy: A tool for Boolean formula minimization complexity analysis

Paolo Liberatore*

Abstract

A superredundant clause is a clause that is redundant in the resolution closure of a formula. The converse concept of superirredundancy ensures membership of the clause in all minimal CNF formulae that are equivalent to the given one. This allows for building formulae where some clauses are fixed when minimizing size. An example are proofs of complexity hardness of the problems of minimal formula size. Others are proofs of size when forgetting variables or revising a formula. Most clauses can be made superirredundant by splitting them over a new variable.

1 Introduction

Given a Boolean formula, the minimization problem is to find a formula of minimal size that is equivalent to it [24, 27, 5, 30, 6, 31]. The decision problem variant that is analyzed in computational complexity is to check whether a Boolean formula is equivalent to one that is bounded in size by a given number. This problem led to the creation of the polynomial hierarchy [29]. Yet, it eluded a precise complexity characterization for over twenty years [11, 4, 2, 14]. Framing it within a complexity class is easy, as it can be solved by a simple guess-and-check algorithm: guess a formula of the given size, check its equivalence with the given formula. The difficult part is proving hardness [11, 4, 2, 14].

An example is the proof of NP-hardness of checking whether a Horn formula can be reduced size within a certain bound [11, 4]. An hardness proof may start from an arbitrary CNF formula and produce a natural number and a Horn formula that is equivalent to a formula of size bounded by that number if and only if the CNF formula is satisfiable.

The Horn formula has to be related to the CNF formula by this condition, but can otherwise be chosen. A choice is to include in the Horn formula some essential prime implicates and some other clauses [11]. The first part is guaranteed to be in every minimal formula equivalent to the Horn one. The size of such minimal formulae are then given by how much the second part can be shrunk while maintaining equivalence. The equivalence between the non-essential clauses is helped by the essential ones: equivalence is on the whole formula, including the essential prime implicates.

In spite of its apparent simplicity, finding such a hardness proof turned out difficult [11, 4, 2, 14]. Ensuring essentiality is not easy when clauses are mixed. Making an example of an essential prime implicate is trivial: a non-tautological clause is always essential in a formula

*DIAG, Sapienza University of Rome. liberato@diag.uniroma1.it

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comprising it only. It may not be with other clauses. Other clauses are necessary, if not for making the formula reducible in size or not depending on the satisfiability of the CNF formula. Essentiality is complicated to establish because is not a local property. It does not depend on the clause only, but on the whole formula. A clause may be essential in a formula, but the addition of a single other clause makes it no longer so.

This article shows an alternative mechanism for existence proofs, which involve the construction of a formula, like hardness proofs. A clause of a formula is superirredundant if it is irredundant in the resolution closure of a formula. Such a clause is in all equivalent formulae of minimal size. Conceptually, superirredundancy can be established by iteratively resolving all clauses of the formula and then removing the given clause. If it follows from the others, it is superredundant. Otherwise, it is superirredundant.

The superirredundant clauses of a formula are in all equivalent formulae of minimal size, but not the other way around. Superirredundancy is sufficient, but is not necessary. When minimizing a formula, it is not as useful as essentiality as it may miss several clauses that are guaranteed in all minimal formulae. Yet, it compensates this drawback by simple conditions to prove it and simple ways to ensure it. A clause can often be proved superirredundant by replacing some variables with truth values in the formula. A clause can often be made superirredundant by splitting it on a new variable.

A hardness proof for a minimization problem can be built by assuming some clauses superirredundant. Assuming, not ensuring. They are assumed to be present in all minimal formulae, not proved so. The remaining clauses can be removed or otherwise replaced by others while maintaining equivalence. The clauses assumed superirredundant help in ensuring equivalence. They are targeted to this aim. To this aim only, they are not built to be superirredundant at the same time.

The Horn formula is built in such a way the other clauses can be reduced size if and only if the CNF formula is satisfiable. Only now the clauses that were assumed superirredundant are verified to be so. If they are not, they are split to make them superirredundant. Granted, this is not always possible. Yet, it often can. If it does, superirredundancy provides a simplification in finding such a proof. Instead of ensuring the essentiality of some clauses and the reducibility of the others at the same time, it allows concentrating on each aim at time. First, the other clauses are proved to be reducible in size when appropriate; second, the clauses that are supposed to be superirredundant are made so.

A following example illustrates the details of how to build a reduction for the problem of Horn minimization in a simple way—a way that did not take twenty years to be developed. It is just an example. This problem is already known to be NP-hard. Many other minimization problems have been framed exactly in the polynomial hierarchy. Yet, some related problems are still open. For example, given a formula, can forgetting a set of variables [22], or literals [17] or subformulae [8] be represented within a certain bound? With fixed symbols [25]? Can revising [26] or updating [16] a formula be represented within a certain bound? These are still minimization problems. They are still open. Another article employs superirredundancy in the complexity analysis of forgetting [21].

Superirredundancy differs from irredundancy, essentiality and membership in all minimal formulae. This is shown by the clause a in the formula $F = \{a, \neg a \vee b, \neg b \vee a\}$: it is not superirredundant, but is redundant, is an essential prime implicate and is in all minimal formulae that are equivalent to F.

- the resolution closure of F is $\{a, b, \neg a \vee b, \neg b \vee a\}$, where a is redundant; therefore, a is not superirredundant in F;
- the clause a is irredundant in F since $F \backslash \{a\}$ is $\{\neg a \vee b, \neg b \vee a\}$, which does not entail a;
- the prime implicates of F are a and b; the only CNF formula equivalent to F made only of prime implicates is $\{a, b\}$, which contains a; therefore, a is an essential prime implicate of F;
- the only minimal-size formula equivalent to F is $\{a, b\}$; as a result, a belongs to all minimal-size formulae equivalent to F.

Besides some preliminaries in Section 2, the technical content of the article is split in three parts: Section 3 defines superredundancy and gives a number of its necessary and sufficient conditions; Section 4 shows how to make a clause superirredundant in most cases; Section 5 shows the details of an example usage of superirredundancy in a hardness proof. Concluding remarks are in Section 6.

2 Preliminaries

2.1 Formulae

The formulae in this article are all propositional in conjunctive normal form (CNF): they are sets of clauses, a clause being the disjunction of some literals and a literal a propositional variable or its negation. This is not truly a restriction, as every formula can be turned into CNF without changing its semantics. A clause is sometimes identified with the set of literals it contains. For example, a subclause is a subset of a clause.

If l is a negative literal $\neg x$, its negation $\neg l$ is defined as x.

The variables a formula A contains are denoted $Var(A)$.

Definition 1 *The size $||A||$ of a formula A is the number of variable occurrences it contains.*

This is not the same as the cardinality of $Var(A)$ because a variable may occur multiple times in a formula. For example, $A = \{a, \neg a \vee b, a \vee \neg b\}$ has size five because it contains five literal occurrences even if its variables are only two. The size is obtained by removing from the formula all propositional connectives, commas and parentheses and counting the number of symbols left.

Other definitions are possible but are not considered in this article. An alternative measure of size is the total number of symbols a formula contains (including conjunctions, disjunctions, negations and parenthesis). Another is the number of clauses (regardless of their length).

The definition of size implies the definition of minimality: a formula is minimal if it is equivalent to no formula smaller than it. Given a formula, a minimal equivalent formula is a possibly different but equivalent formula that is minimal. As an example, $A = \{a, \neg a \vee b, a \vee \neg b\}$ has size five since it contains five literal occurrences; yet, it is equivalent to $B = \{a, b\}$, which only contains two literal occurrences. No formula equivalent to A or B is smaller than that: B is minimal. Minimizing a formula means obtaining a minimal equivalent formula. This problem has long been studied [14, 4].

Definition 2 *The clauses of a formula A that contain a literal l are denoted by $A \cap l = \{c \in A \mid l \in c\}$.*

This notation cannot cause confusion: when is between two sets, the symbol $\cap$ denotes their intersection; when is between a set and a literal, it denotes the clauses of the set that contain the literal. This is like seeing $A \cap l$ as the shortening of $A \cap \text{clauses}(l)$, where $\text{clauses}(l)$ is the set of all possible clauses that contain the literal l.

When a formula entails a clause but none of its strict subclauses, the clause is a prime implicate of the formula. Formally, $F \models c$ holds but $F \models c'$ does not for any clause $c' \subset c$. Prime implicates are a common tool in formula minimization [14, 4].

2.2 Resolution

Resolution is a syntactic derivation mechanism that produces a new clause that is a consequence of two clauses: $c_1 \vee l, c_2 \vee \neg l \vdash c_1 \vee c_2$. The result is implicitly removed repetitions.

Unless noted otherwise, tautologic clauses are excluded. Writing $c_1 \vee a, c_2 \vee \neg a \vdash c_1 \vee c_2$ implicitly assumes that none of the three clauses is a tautology unless explicitly stated. Two clauses that would resolve in a tautology are considered not to resolve, which is not a limitation [23]. Tautologic clauses are forbidden in formulae, which is not a limitation either since tautologies are always satisfied. This assumption has normally little importance, but is crucial to superredundancy, defined in the next section.

Lemma 1 *A clause that is the result of resolving two clauses does not contain the resolving variable and is different from both of them if none of them is a tautology.*

Proof. A clause c is the result of resolving two clauses only if they have the form $c_1 \vee a$ and $c_2 \vee \neg a$ for some variable a, and c is $c_1 \vee c_2$. If $c_1 \vee c_2$ is equal to $c_1 \vee a$, it contains a. Since clauses are sets of literals, they do not contain repeated elements. As a result, $a \notin c_1$ as otherwise $c_1 \vee a$ would contain a twice. Together with $a \in c_1 \vee c_2$ this implies $a \in c_2$, which makes $c_2 \vee \neg a$ a tautology. The case $c = c_2 \vee \neg a$ is similar. □

A resolution proof $F \vdash G$ is a binary forest where the roots are the clauses of G, the leaves the clauses of F and every parent is the result of resolving its two children.

Definition 3 *The resolution closure of a formula F is the set $ResCn(F) = \{c \mid F \vdash c\}$ of all clauses that result from applying resolution zero or more times from F.*

The clauses of F are derivable by zero-step resolutions from F. Therefore, $F \vdash c$ and $c \in \text{ResCn}(F)$ hold for every $c \in F$.

The resolution closure is similar to the deductive closure but not identical. For example, $a \vee b \vee c$ is in the deductive closure of $F = \{a \vee b\}$ but not in the resolution closure. It is a consequence of F but is not obtained by resolving clauses of F.

All clauses in the resolution closure $\text{ResCn}(F)$ are in the deductive closure but not the other way around. The closures differ because resolution does not expand clauses: $a \vee b \vee c$ is not a resolution consequence of $a \vee b$. Adding expansion kills the difference [18, 28].

$$F \models c \text{ if and only if } c' \in \text{ResCn}(F) \text{ for some } c' \subseteq c$$

That resolution does not include expansion may suggest that it cannot generate any non-minimal clause. That would be too good to be true, since a clause would be minimal just because it is obtained by resolution. In fact, it is not the case. Expansion is only one of the reasons clauses may not be minimal, as seen in the formula $\{a \vee b \vee c, a \vee b \vee e, \neg e \vee c \vee d\}$: the second and third clauses resolve to $a \vee c \vee b \vee d$, which is however not minimal: it contains the first clause of the formula, $a \vee b \vee c$.

What is the case is that resolution generates all prime implicates [18, 28], the minimal entailed clauses. The relation between $\text{ResCn}(F)$ and the deductive closure of F tells that if a clause is entailed, a subset of it is generated by resolution; since the only entailed subclause of a prime implicate is itself, it is the only one resolution may generate. Removing all clauses that contain others from $\text{ResCn}(F)$ results in the set of the prime implicates of F.

Minimal equivalent formulae are all made of minimal entailed clauses, as otherwise literals could be removed from them. Since resolution allows deriving all prime implicates of a formula [18, 28], it derives all clauses of all minimal equivalent formulae.

Property 1 *If B is a minimal CNF formula equivalent to A, then $B \subseteq ResCn(A)$.*

While $\text{ResCn}(F)$ contains all clauses generated by an arbitrary number of resolutions, some properties used in the following require the clauses obtained by a single resolution step.

Definition 4 *The resolution of two formulae is the set of clauses obtained by resolving each clause of the first formula with each clause of the second:*

$$resolve(A, B) = \{c \mid c', c'' \vdash c \text{ where } c' \in A \text{ and } c'' \in B\}$$

If either of the two formulae comprises a single clause, the abbreviations $resolve(A, c) = resolve(A, \{c\})$, $resolve(c, B) = resolve(\{c\}, B)$ and $resolve(c, c') = resolve(\{c\}, \{c'\})$ are used.

This set contains only the clauses that results from resolving a single clause of A with a single clause of B. Exactly one resolution of one clause with one clause. Not zero, not multiple ones. A clause of A is not by itself in $\text{resolve}(A, B)$ unless it is also the resolvent of another clause of A with a clause of B.

3 Superredundancy

The running example presented in the introduction is a prototypical minimization problem: given a Horn formula A and a number k, decide whether A is equivalent to a Horn formula of size k or less. Its NP membership is easy to prove: guess a Horn formula B of size at most k and verify its equivalence with A. Equivalence between Horn formulae can be checked in polynomial time. Therefore, the whole problem is in NP.

The difficult part of classing it in the polynomial hierarchy is to establish its hardness. This can be done by reducing the NP-hard problem of Boolean satisfiability into it. Given a CNF formula F, the task is to produce a Horn formula A and a number k such that A is equivalent to a Horn formula B of size bounded by k if and only if F is satisfiable. A similar reduction would prove the hardness of other minimization problems. For example, releasing the Horn constraint increases the complexity from NP to Σ_2^p. The translation is then required to produce a formula A that can be expressed in size k if and only the QBF $\forall X \exists Y.F$ is valid. For the sake of this exposition, A is assumed Horn and the condition is the satisfiability of F.

A reduction could translate a satisfiable F into $\{a \vee \neg b, a \vee c\}$ and an unsatifiable F into $\{a \vee \neg b, a \vee c, l\}$. The first formula has size 4, the second $4 + 1$. The presence of l provides the required increase in size.

While the addition of a literal like l works, it is not enough. A counterexample is a reduction that translates a satisfiable F into $\{a \vee \neg b, a \vee c\}$ and an unsatifiable F into $\{\neg a \vee c, l\}$. While l is necessary in all formulae equivalent to the second, it does not produce an increase in size: the first formula has size 4, the second $2 + 1$. The $+1$ is the second formula is overcome by the decrease of size of the other clauses from 4 to 2.

The mechanism of having or not having l in the formula only works if the rest of the formula is fixed. Otherwise, the presence of l may invalidate the construction like the last addition to a house of cards crushes its lower layers: the unsatisfiability of F may force l into the formula, but if it also allows removing a clause of two literals the total size change is -1, not the required $+1$.

The lower layers, the other clauses, are fixed in place by superirredundancy. Its formal definition is below, but what counts is that a superirredundant clause of a formula belongs to every minimal equivalent formula. The number of its literals is never subtracted from the overall size of the formula. The literals of the superirredundant clauses are fixed; the other clauses may provide the required $+1$ addition in size.

This is a plan of this section: Section 3.1 presents the formal definition of superredundancy and its converse, superirredundancy; Section 3.2 shows how superirredundancy relates to minimal formulae; Section 3.3 presents some equivalent conditions to superredundancy, Section 3.4 some necessary ones.

3.1 Definition of superredundancy

Superirredundancy is based on resolution. Summarizing the notation introduced in Section 2: $\text{resolve}(F, G)$ are the clauses obtained by resolving each clause of the first formula with each of the second; $F \vdash G$ means that the clauses of the second formula are obtained by repeatedly resolving the clauses of the first. This condition is formalized by $G \subset \text{ResCn}(F)$ since $\text{ResCn}(F)$ is the set of all clauses obtained by repeatedly resolving the clauses of a formula F.

Resolution does not just tell whether a clause is implied from a set. It also tells why: the implied clause is consequence of other two, in turns consequence of others and so on. This structure is necessary in many proofs below, and is the very reason why they employ resolution $\vdash$ instead of entailment $\models$.

A clause c of F may or may not be redundant in $\text{ResCn}(F)$. It is redundant if it is a consequence of $\text{ResCn}(F) \backslash \{c\}$, like in Figure 1.

An example is a in $\{a, \neg a \vee b, \neg b \vee a\}$. The resolution closure of this formula is $\text{ResCn}(\{a, \neg a \vee b, \neg b \vee a\}) = \{a, b, \neg a \vee b, \neg b \vee a\}$, where a is redundant since it is entailed

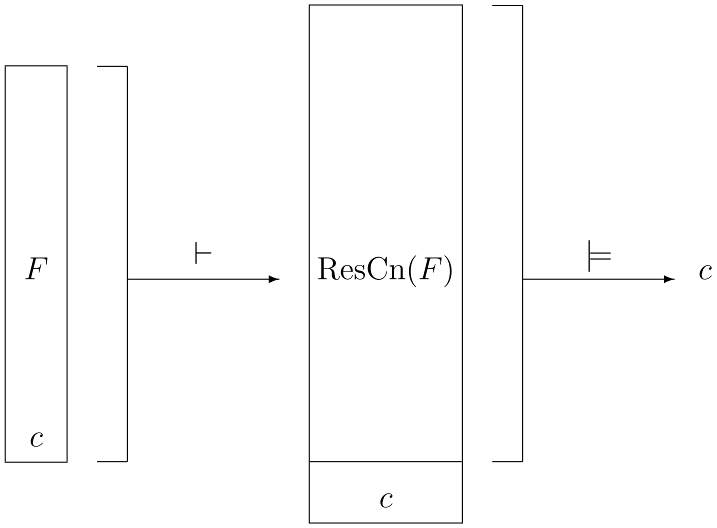


Figure 1: A superredundant clause

by b and $\neg b \vee a$.

Such a redundancy is not always the case. For example, the resolution closure of $\{a, b\}$ is itself, since its two clauses do not resolve. As a result, a is not redundant in $\mathrm{ResCn}(\{a, b\}) = \{a, b\}$.

If a clause is redundant [10, 20], it is also superredundant, but not the other way around. For example, a is superredundant in $\{a, \neg a \vee b, \neg b \vee a\}$ because it is entailed by the other clauses of the resolution closure of this formula but is not redundant because it is not entailed by the other clauses of the formula itself, which are $\neg a \vee b$ and $\neg b \vee a$.

When c is redundant in the resolution closure, $\mathrm{ResCn}(F)\backslash\{c\}$ is equivalent to $\mathrm{ResCn}(F)$. Even if c is not redundant in F, it is not truly necessary as a formula $\mathrm{ResCn}(F)\backslash\{c\}$ not containing it is equivalent to F. In the first example, a is not redundant in $\{a, \neg a \vee b, \neg b \vee a\}$, but this formula is equivalent to $\mathrm{ResCn}(\{a, \neg a \vee b, \neg b \vee a\})\backslash\{a\} = \{b, \neg a \vee b, \neg b \vee a\}$, which does not contain the clause a.

This is a weak version of redundancy: while c may not be removed from F, it can be replaced by other consequences of F. The converse is therefore a strong version of minimality: c cannot be removed even adding all other resolution consequences of F. This is why it is called superirredundant. A superirredundant clause cannot be removed even adding resolution consequences in its place. It is irredundant even expanding F this way, even switching to such supersets of F.

The opposite to superirredundancy is unsurprisingly called superredundancy: a clause is superredundant if it is redundant in the superset $\mathrm{ResCn}(F)$ of F. Such a clause can be replaced by other resolution consequences of F.

Definition 5 *A clause c of a formula F is superredundant if it is redundant in the resolution closure of the formula: $ResCn(F)\backslash\{c\} \models c$. It is superirredundant if it is not superredundant.*

A superirredundant clause of a formula will be proved to be in all minimal formulae equivalent to that formula. It is necessary in them. This is how fixing a part of the formula

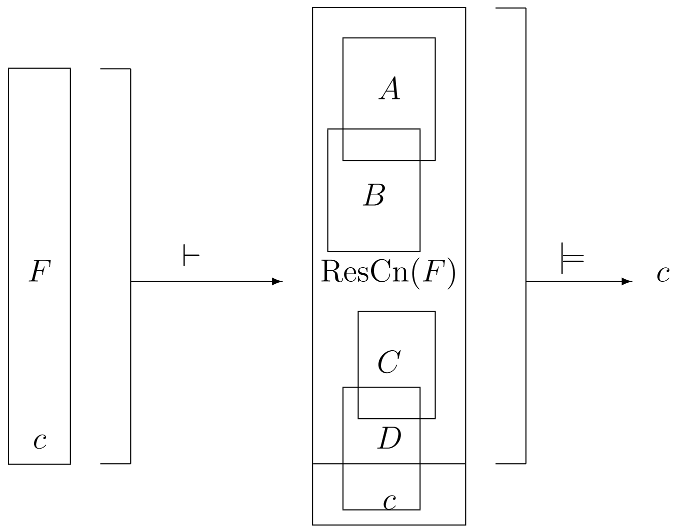


Figure 2: A superredundant clause and some minimal equivalent formulae

is achieved: by ensuring that its clauses are superirredundant. The opposite concept of superredundancy is introduced because it simplifies a number of technical results.

Contrary to what it may look, creating superirredundant clauses is not difficult.

A typical situation with a superredundant clause c is that F has some minimal equivalent formulae like D which contains c, but also has other minimal equivalent formulae A, B and C which do not contain c, like in Figure 2. This is possible because of $\mathrm{ResCn}(F)\backslash\{c\} \models c$, which allows some subsets of $\mathrm{ResCn}(F)\backslash\{c\}$ like D to entail c. Some other subsets like A, B and C may still be minimal even if they contain c.

Superredundancy is the same as redundancy in the resolution closure. It is not the same as redundancy in the deductive closure: such a redundancy is always the case unless the clause contains all variables in the alphabet. Otherwise, if a is a variable not in c, then $c \models c \vee a$ and $c \models c \vee \neg a$; as a result, if $c \in F$ then $Cn(F)\backslash\{c\}$ contains $c \vee a$ and $c \vee \neg a$, which imply c. The same argument does not apply to resolution because neither $c \vee a$ nor $c \vee \neg a$ follow from c by resolution. This is why superredundancy is defined in terms of resolution and not entailment.

Redundancy implies superredundancy: if c follows from $F\backslash\{c\}$ it also follows from $\mathrm{ResCn}(F)\backslash\{c\}$ by monotonicity of entailment. Not the other way around. For example, a is irredundant in $F = \{a, \neg a \vee b, \neg b \vee a\}$ but is superredundant: a and $\neg a \vee b$ resolve to b, which resolves with $\neg b \vee a$ back to a.

The formula in this example is not minimal, as it is equivalent to $\{a, b\}$. It shows that a non-minimal formula may contain a superredundant clause. This will be proved to be always the case. In reverse: a formula entirely made of superirredundant clauses is minimal.

3.2 Superirredundancy and minimality

The aim of superirredundancy is to prove that a clause belongs to all formulae that are equivalent to the given one. The following lemma proves this.

Lemma 2 *If c is a superirredundant clause of F, it is contained in every minimal CNF formula equivalent to F.*

Proof. Let B be a minimal formula that is equivalent to F. By Property 1, $B \subseteq \text{ResCn}(F)$. If B does not contain c, this containment strengthens to $B \subseteq \text{ResCn}(F) \backslash \{c\}$. A consequence of the equivalence between B and F is that B implies every clause of F, including c. Since $B \models c$ and $B \subseteq \text{ResCn}(F) \backslash \{c\}$, by monotonicity $\text{ResCn}(F) \backslash \{c\} \models c$ follows. This is the opposite of the assumed superirredundancy of c. □

This lemma provides a sufficient condition for a clause being in all minimal formulae equivalent to the given one. Not a necessary one, though. A clause that is not superirredundant may still be in all minimal formulae. A counterexample only requires three clauses.

Counterexample 1 *The first clause of $F = \{a, \neg a \vee b, a \vee \neg b\}$ is superredundant but is in all minimal formulae equivalent to F.*

Proof. Its first clause is a. It resolves with $\neg a \vee b$ into b. The resolution closure of F contains a, b and $a \vee \neg b$. The first is redundant as it is entailed by the second and the third. This proves it superredundant.

Yet, the only minimal formula equivalent to F is $F' = \{a, b\}$, which contains a in spite of its superredundancy. That $\{a, b\}$ is minimal is proved by the superirredundancy of its clauses: its resolution closure is $\{a, b\}$ itself since its clauses do not resolve; none of the two clauses is redundant in it. □

The proof of this counterexample relies on the syntactic dependency of superredundancy: a is superredundant in $\{a, \neg a \vee b, a \vee \neg b\}$ but superirredundant in its minimal equivalent formula $\{a, b\}$.

Lemma 3 *There exist two equivalent formulae such that: a. both formulae contain the same clause, and b. that clause is superredundant in one formula but not in the other.*

Proof. The formulae are $F = \{a, \neg a \vee b, a \vee \neg b\}$ and $F' = \{a, b\}$.

The resolution closure of the first is $\text{ResCn}(F) = \{a, b, \neg a \vee b, a \vee \neg b\}$ since the only possible resolutions in F are $a, \neg a \vee b \vdash b$ and $b, a \vee \neg b \vdash a$. While $\neg a \vee b$ and $a \vee \neg b$ have opposite literals, they would only resolve into a tautology. Since the clauses of F' do not resolve, its resolution closure is F' itself: $\text{ResCn}(F') = \{a, b\}$.

The clause a is redundant in $\text{ResCn}(F) = \{a, b, \neg a \vee b, a \vee \neg b\}$ since it is entailed by b and $a \vee \neg b$. It is not redundant in $\text{ResCn}(F') = \{a, b\}$ since it is not entailed by b. □

Being dependent on the syntax of the formula, superredundancy and superirredundancy differ from every condition that is independent of the syntax, such as:

- redundancy in the set of prime implicates, which is also employed in formula minimization [11];
- essentiality of prime implicates, defined as containment of a prime implicate in all prime CNFs equivalent to the formula [11]
- presence in all minimal CNF formulae equivalent to F.

These conditions are independent on the syntax since the prime implicates, the prime equivalent CNFs and the minimal equivalent formulae are the same for two equivalent formulae. Being independent on the syntax, they are not the same as superredundancy or superirredundancy, which are proved dependent on the syntax by Lemma 3.

Lemma 2 does not contradict the inequality of superirredundancy and presence in all minimal equivalent formulae. It only proves that a superirredundant clause of a formula is in all minimal formulae equivalent to it. Not the other way around. A clause may be in all minimal formulae while not being superirredundant.

Lemma 4 *There exists a formula that contains a superredundant clause that is in all its minimal CNF equivalent formulae.*

Proof. Let c be a clause that is superredundant in F and superirredundant in F' with $F \equiv F'$. Such a condition is possible according to Lemma 3.

Since c is superirredundant in F', it is contained in all minimal formulae equivalent to F'. Since this formula is equivalent to F, their minimal equivalent formulae are the same. □

Superirredundancy in a formula is a strictly stronger condition than membership in all its minimal equivalent formulae. It implies that, but is not implied.

This is to be kept in mind when superirredundancy is used as a precondition. Some lemmas below prove that if a clause is superirredundant in a formula then it is superirredundant in a similar formula. The precondition of such a result is the superirredundancy of the clause; its membership in all minimal equivalent formulae may not be enough. Superirredundancy is required. Membership to all minimal equivalent formulae is a consequence of both the premise and the conclusion, not a substitute of the premise.

An application of superirredundancy is to guarantee a formula to be minimal.

Lemma 5 *If a formula contains only superirredundant clauses, it is minimal.*

Proof. Let B be a minimal formula equivalent to A. By Lemma 2, the superirredundant clauses of A are in all minimal formulae that are equivalent to A. Therefore, B contains all of them. If A only comprises superirredundant clauses, B contains all of them: $A \subseteq B$. The only case where B could not be the same as A is when this containment is strict, but that would imply that B is not minimal since A is equivalent but smaller. □

If a formula is built so that all its clauses are superirredundant, it is guaranteed to be minimal. Not the other way around. Rather the opposite: a minimal formula may be made only of superredundant clauses. An example is $\{\neg a \vee b, \neg b \vee c, \neg c \vee a\}$. Resolving the first two clauses $\neg a \vee b$ and $\neg b \vee c$ generates $\neg a \vee c$. Resolving the other pairs produces the opposite cycle of clauses $\{\neg a \vee c, \neg c \vee b, \neg b \vee a\}$, which is equivalent to the original. The clauses of the original are therefore entailed by some of their resolution consequences. Yet, the original is minimal.

3.3 Equivalent conditions to superredundancy

Superirredundancy differs from membership in all minimal equivalent formula, but is easier than that to ensure. Simplicity is its motivation. Therefore, it makes sense to simplify it further. The following lemma gives a number of equivalent conditions, all based on F deriving

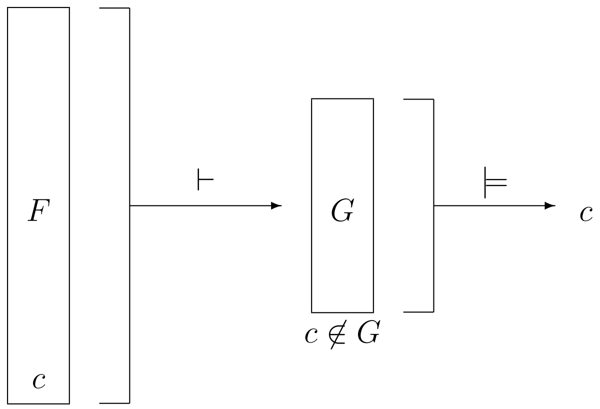


Figure 3: An example of a superredundant clause

another formula G that in turn derives c. The first equivalent condition is exemplified by Figure 3. Instead of proving that c is a consequence of $\mathrm{ResCn}(F) \backslash \{c\}$, these equivalent conditions allow proving the existence of a possibly smaller formula G that entails c.

Lemma 6 *A clause c of a formula F is superredundant if and only if a formula G satisfying either one of the following conditions exists:*

1. $F \vdash G$ *and* $G \models c$ *where* $c \not\in G$
2. $F \vdash G$ *and* $G \vdash c'$ *where* $c \not\in G$ *and* $c' \subseteq c$
3. $F \vdash G$ *and* $G \vdash c$ *with* $c \not\in G$ *or* $F \vdash c'$ *with* $c' \subset c$
4. $F \vdash G$ *and* $G \vdash c$ *with* $c \not\in G$ *or* $F \models c'$ *with* $c' \subset c$

Proof. Equivalence with the first condition is proved in the two directions. A clause c of F is superredundant if and only if $\mathrm{ResCn}(F) \backslash \{c\} \models c$. If this condition is true, the claim holds with $G = \mathrm{ResCn}(F) \backslash \{c\}$, since c is not in $\mathrm{ResCn}(F) \backslash \{c\}$ by construction and the definition of resolution closure implies $F \vdash c'$ for every $c' \in \mathrm{ResCn}(F)$. In the other direction, if $F \vdash G$ then $G \subseteq \mathrm{ResCn}(F)$. Since $c \not\in G$, this containment strengthens to $G \subseteq \mathrm{ResCn}(F) \backslash \{c\}$. Since $G \models c$, it follows $\mathrm{ResCn}(F) \backslash \{c\} \models c$ by monotonicity.

The first condition is equivalent to the second because $G \models c$ is the same as $G \vdash c'$ with $c' \subseteq c$.

The second condition is proved equivalent to the third considering the two directions separately. The second condition includes $c' \subseteq c$, which comprises two cases: $c' = c$ and $c' \subset c$. If $c' = c$, the second condition becomes $F \vdash G \vdash c$ with $c \not\in G$, the same as the first alternative of the third condition. If $c' \subset c$, the second condition is $F \vdash G \vdash c'$, which implies $F \vdash c'$ with $c' \subset c$; this is the second alternative of the third condition. In the other direction, the first alternative of the third condition is $F \vdash G \vdash c$ with $c \not\in G$, which is the same as the second condition with $c' = c$. The second alternative is $F \vdash c'$ with $c' \subset c$; the second condition holds with $G = \{c'\}$.

Equivalence with the fourth condition holds because $F \vdash c'$ implies $F \models c'$, and in the other direction $F \models c'$ implies $F \vdash c''$ with $c'' \subset c$, and the third equivalent condition holds with c'' in place of c'. $\square$

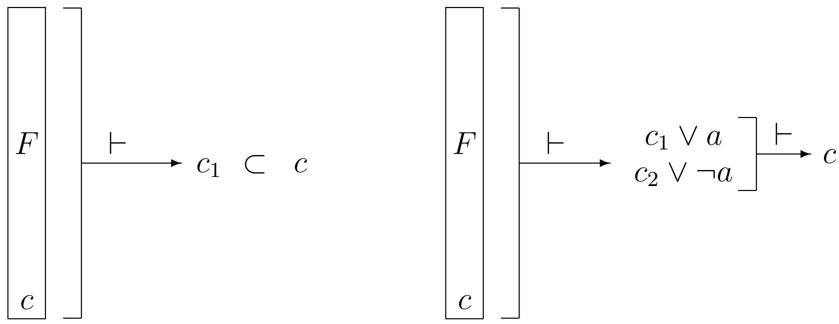


Figure 4: Superredundancy proved by the last step of resolution

A clause c is superredundant if it follows from F by a resolution proof that contains a set of clauses G sufficient to prove c. As such, G is a sort of "cut" in a resolution tree $F \vdash c$, separating c from F. This cut can be next to the root, next to the leaves, or somewhere in between. In practice, it is useful at the first resolution steps (next to the leaves) or at the last (next to the root).

The next equivalent condition to superredundancy cuts the resolution tree at its very last point, one step short of regenerating c. It comprises two alternatives, depicted in Figure 4. They are due to the two possibilities contemplated by the third condition of Lemma 6: either F implies a proper subset of c or c itself with a resolution proof cut by a set G.

Lemma 7 *A clause c of a formula F is superredundant if and only if either $F \vdash c_1$ where $c_1 \subset c$ or $F \vdash c_1 \vee a, c_2 \vee \neg a$ for some variable a not occurring in c and clauses c_1 and c_2 such that $c = c_1 \vee c_2$.*

Proof. By Lemma 6, superredundancy is equivalent to $F \vdash G \vdash c$ with $c \notin G$ or $F \vdash c'$ with $c' \subset c$.

The second part of this condition is the same as $F \vdash c_1$ and $c_1 \subset c$ with $c_1 = c'$, the first alternative in the statement of the lemma.

The first part $F \vdash G \vdash c$ with $c \notin G$ is now proved to be the same as the second alternative in the statement of the lemma: $F \vdash c_1 \vee a, c_2 \vee \neg a$ where $a \notin c$ and $c = c_1 \vee c_2$.

If $F \vdash G \vdash c$ with $c \notin G$, since c is not in G, the derivation $G \vdash c$ contains at least a resolution step. Let c' and c'' be the two clauses that resolve to c in this derivation. Since they resolve to c, they have the form $c' = c_1 \vee a$ and $c'' = c_2 \vee \neg a$ for some variable a. Their resolution $c = c_1 \vee c_2$ does not contain a by Lemma 1. These two clauses are obtained by resolution from G. Since $F \vdash G$, they also derive by resolution from F.

In the other direction, $F \vdash c_1 \vee a, c_2 \vee \neg a$ implies superredundancy. This is proved with $G = \{c_1 \vee a, c_2 \vee \neg a\}$. The first condition $G \vdash c$ holds because $c_1 \vee a$ and $c_2 \vee \neg a$ resolve into c. The second condition $c \notin G$ holds because c does not contain a by assumption while both clauses of G both do. □

This lemma says that looking at all possible sets of clauses G when checking superredundancy is a waste of time. The sets comprising pairs of clauses containing an opposite literal suffice. Their form provides an even further simplification: they are obtained by splitting the clause in two and adding an opposite literal to each. Superredundancy is the same as resolution deriving either such a pair or a proper subset of the clause.

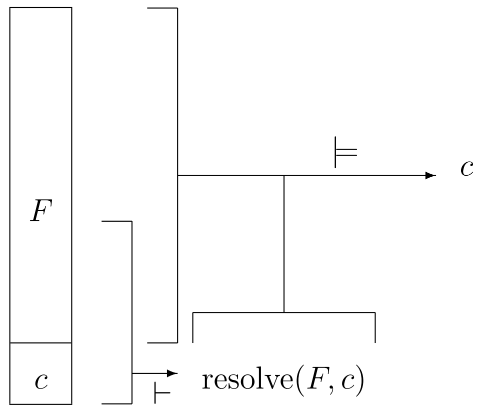


Figure 5: Superredundancy proved by immediate resolution consequences

A clause is proved superredundant by such a splitting. Yet, proving superredundancy is not the final goal. Proving the presence in all minimal equivalent formula is. It follows from superirredundancy, not superredundancy. The lemma helps in this. Instead of checking all possible sets of clauses G, it allows concentrating only on the pairs obtained by splitting the clause.

When the clauses of F do not resolve, the second alternative offered by Lemma 7 never materializes: a clause is superredundant if and only if it is a proper superset of a clause of F. This quite trivial specialization looks pointless, but turns essential when paired with the subsequent Lemma 12.

Lemma 8 *If no two clauses of F resolve, then a clause of F is superredundant if and only if F contains a clause that is a strict subset of it.*

Proof. By Lemma 7, $c \in F$ is superredundant if and only if $F \vdash c_1$ with $c_1 \subset c$ or $F \vdash c_1 \vee a, c_2 \vee \neg a$ with $c = c_1 \vee c_2$. The second condition implies $c_1 \vee a, c_2 \vee \neg a \in F$ since the clauses of F do not resolve; this contradicts the assumption since these two clauses resolve. As a result, the only actual possibility is the first: $F \vdash c_1$ with $c_1 \subset c$. Since the clauses of F do not resolve, c_1 cannot be the result of resolving clauses. Therefore, it is in F. $\square$

Formula G of Lemma 6 can be seen as a cut in a resolution tree from F to c. It separates all occurrences of $c \in F$ in the leaves from c in the root. Lemma 7 places the cut next to the root. The following places it next to the leaves. It proves that resolving c with clauses of F only is enough. The clauses obtained by these resolutions are resolve(c, F), according to Definition 4. This situation is shown by Figure 5.

Lemma 9 *A clause c of F is superredundant if and only if $F \backslash \{c\} \cup resolve(c, F) \models c$.*

Proof. The first equivalent condition to superredundancy offered by Lemma 6 is the existence of a set G such that $F \vdash G$, $G \models c$ and $c \not\in G$. The proof is composed of two parts: the first is that $F \backslash \{c\} \cup$ resolve(c, F) is such a set G if it entails c; the second is that if such a set G exists, the derivation of $F \vdash G$ can be rearranged so that c resolves only with other clauses of F. The first is almost trivial, the second is not because c may resolve with clauses obtained

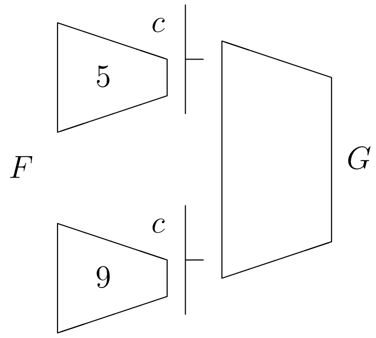


Figure 6: An example of the rise of a clause

by resolution in $F \vdash G$. The rearranged derivation begins with a batch of resolutions of c with other clauses of F, and c is then no longer used. The resolvents of these first resolutions and the rest of F makes the required set G.

If $F \backslash \{c\} \cup \text{resolve}(c, F) \models c$, then $G = F \backslash \{c\} \cup \text{resolve}(c, F)$ proves c superredundant: $F \vdash G$, $G \models c$ and $c \notin G$. The first condition $F \vdash G$ holds because the only clauses of G that are not in F are the result of resolving $c \in F$ with a clause of F; the second condition $G \models c$ holds by assumption; the third condition is that G does not contain c, and it holds because G is the union of $F \backslash \{c\}$ and resolve(c, F), where $F \backslash \{c\}$ does not contain c by construction and resolve(c, F) because resolving a clause does not generate the clause itself by Lemma 1.

The rest of the proof is devoted to proving the converse: $F \vdash G$, $G \models c$ and $c \notin G$ imply $F \backslash \{c\} \cup \text{resolve}(c, F) \models c$.

The claim is proved by repeatedly modifying G until it becomes a subset of $F \backslash \{c\} \cup \text{resolve}(c, F)$ while still maintaining its properties $F \vdash G$, $G \models c$ and $c \notin G$.

This process ends because a measure defined on the derivation $F \vdash G$ decreases until reaching zero. This measure is the almost-size of the derivation $F \vdash G$ plus the rise of c in it. Both are based on the size of the subtrees of $F \vdash G$: each clause in the derivation is generated independently of the others, and is therefore the root of its own tree.

The size of the derivation $F \vdash G$ is the number of clauses it contains. Its almost-size is the number of clauses except the roots.

The derivation $F \vdash G$ may contain some resolutions of c with other clauses. The rise of an individual resolution of c with a clause c'' in $F \vdash G$ is the number of nodes in the tree rooted at c'' minus one. The total rise of c in $F \vdash G$ is the sum of all resolutions of c in it. It measures the overall distance of c from the other leaves of the tree, its elevation from the ground. Figure 6 shows an example.

Both the almost-size and the rise of c are not negative. They are sums, each addend being the size of a nonempty tree minus one; since each tree is not empty, its size is at least one; the addend is at least zero. Their sums are at least zero.

If G is a subset of $F \cup \text{resolve}(c, F)$, then $c \notin G$ implies it is also a subset of $F \backslash \{c\} \cup \text{resolve}(c, F)$. Since G implies c, also does its superset $F \backslash \{c\} \cup \text{resolve}(c, F)$. This is the claim.

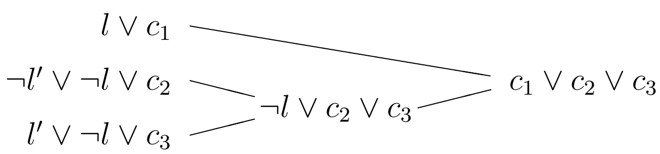
Otherwise, G is not a subset of $F \cup \text{resolve}(c, F)$. This means that G contains a clause $c' \in G$ that is not in F and is not the result of resolving c with a clause of F. Since c' is not in F, it is the result of resolving two clauses c'' and c'''. One of them may be c or not; if it is, the other one is not in F and is therefore the result of resolving two other clauses.

If neither c'' nor c''' is equal to c, the modified set $G' = G \backslash \{c'\} \cup \{c'', c'''\}$ has the same properties of G that prove c superredundant: $F \vdash G'$, $G' \models c$ and $c \not\in G'$. Let a and b be the size of the trees rooted in c'' and c''' in $F \vdash G$. Since $F \vdash G$ has c' as a root, its almost-size includes $a + b + 1 - 1 = a + b$. Instead, $F \vdash G'$ has c'' and c'' as roots in place of c'; therefore, its almost-size includes $(a - 1) + (b - 1) = a + b - 2$. The almost-size of $F \vdash G'$ is smaller than $F \vdash G$. The rise of c is the same, since the resolutions of c are the same in the two derivations. Summarizing, almost-size decreases while rise maintains its value.

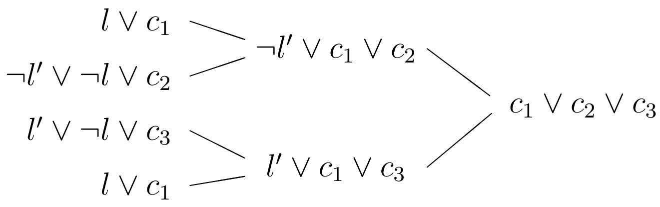
If either c'' or c''' is equal to c, the same set $G' = G \backslash \{c'\} \cup \{c'', c'''\}$ does not work because it does not maintain $c \not\in G'$. Since the two cases $c'' = c$ and $c''' = c$ are symmetric, only the second is analyzed: $c' \in G$ is generated by $c, c'' \vdash c'$ in $F \vdash G$. If c'' is in F, then c' is in $\text{resolve}(c, F)$ because it is the result of resolving c with a clause of F. Otherwise, c'' is a clause obtained by resolving two other clauses.

These two clauses resolve in c'', which resolves with c. Two resolutions, two pairs of opposite literals. Let l be the literal of c that is negated in c'' and l' the literal that is resolved upon in the resolution that generates c''. At least one of the two clauses that generate c'' contains $\neg l$ since c'' does. At least means either one or both.

The first case is that both clauses that resolve in c'' contain $\neg l$. Since they also contain l' and $\neg l'$, they can be written $\neg l' \vee \neg l \vee c_2$ and $l' \vee \neg l \vee c_3$. They resolve in $c'' = \neg l \vee c_2 \vee c_3$. Since c contains l, it can be written $l \vee c_1$. It resolves with $c'' = \neg l \vee c_2 \vee c_3$ to $c' = c_1 \vee c_2 \vee c_3$.



A different derivation from the same clauses resolves $l \vee c_1$ with $\neg l' \vee \neg l \vee c_2$, producing $c_1 \vee \neg l' \vee c_2$, and with $l' \vee \neg l \vee c_3$, producing $c_1 \vee l' \vee c_3$. The produced clauses $c_1 \vee \neg l' \vee c_2$ and $c_1 \vee l' \vee c_3$ resolve to $c_1 \vee c_2 \vee c_3$, the same conclusion of the original derivation. This is a valid derivation, with an exception discussed below.



This derivation proves $F \vdash G'$ where $G' = G \backslash \{c_1 \vee c_2 \vee c_3\} \cup \{c_1 \vee \neg l' \vee c_2, c_1 \vee l' \vee c_3\}$. The only clause of G that G' does not contain is $c_1 \vee c_2 \vee c_3$, which is implied by resolution from its clauses $c_1 \vee \neg l' \vee c_2$ and $c_1 \vee l' \vee c_3$. Therefore, $G' \models G$. This implies $G' \models c$ since $G \models c$. Since $c_1 \vee \neg l' \vee c_2$ is obtained by resolving two clauses over l, it does not contain l by

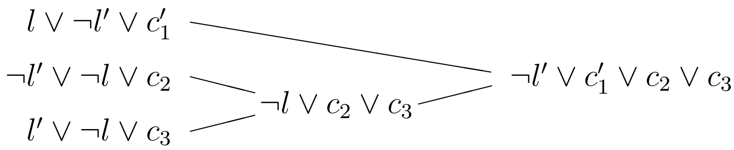
Lemma 1. The same applies to $c_1 \vee l' \vee c_3$. Since c contains l, it is not equal to either of these two clauses. It is not equal to any other clause of G' either, since these are also clauses of G and c is not in G. This proves that G' has the same properties that prove c superredundant: $F \vdash G'$, $G' \models c$ and $c \notin G'$.

Since $\neg l' \vee \neg l \vee c_2$ and $l' \vee \neg l \vee c_3$ are generated by resolution from F, each is the root of a resolution tree. Let a and b be their size. The almost-size of the original derivation $F \vdash G$ includes $a+b+2$, that of $F \vdash G'$ has $a+b+2$ in its place. Almost-size does not change.

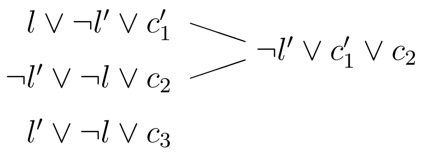
The rise of resolving $c = l \vee c_1$ with $c'' = \neg l \vee c_2 \vee c_3$ in the original derivation is one less the size of the tree rooted in c''. This tree comprises c'' and the trees rooted in $\neg l' \vee \neg l \vee c_2$ and $l' \vee \neg l \vee c_3$. Its size is therefore $a+b+1$. The rise of c is therefore $a+b$. The two resolutions of $c = l \vee c_1$ in the modified derivation are with $\neg l' \vee \neg l \vee c_2$ and $l' \vee \neg l \vee c_3$. The rise of c in the first is the size of tree rooted in $\neg l' \vee \neg l \vee c_2$ minus one: $a-1$; the rise of c in the second is $b-1$. Their sum is $a+b-2$, which is strictly less than $a+b$.

In summary, switching from $F \vdash G$ to $F \vdash G'$ maintains the almost-size and decreases the rise of c.

The exception mentioned above is that the new resolutions may generate tautologies, which are not allowed. Since $\neg l' \vee \neg l \vee c_2$, $l' \vee \neg l \vee c_3$ and $c_1 \vee c_2 \vee c_3$ are in the original derivation, they are not tautologies. As a result, c_1, c_2 and c_3 do not contain opposite literals, c_2 does not contain l' and c_3 does not contain $\neg l'$. The new clause $c_1 \vee \neg l' \vee c_2$ is tautological only if l' is in c_1, and $c_1 \vee l' \vee c_3$ only if $\neg l'$ is in c_1. By symmetry, only the second case is considered: $c_1 = \neg l' \vee c_1'$.



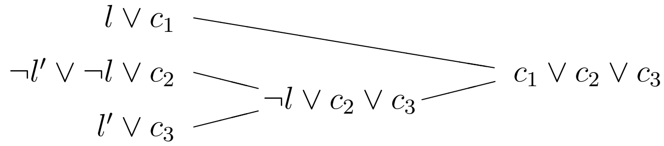
An alternative derivation resolves $l \vee \neg l' \vee c_1$ with $\neg l' \vee \neg l \vee c_2$, resulting in $\neg l' \vee c_1 \vee c_2$, a subset of the original result $\neg l' \vee c_1 \vee c_2 \vee c_2$.



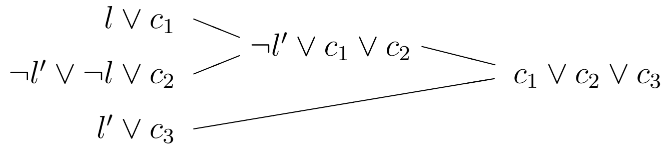
Since $\neg l' \vee c_1' \vee c_2 \subseteq \neg l' \vee c_1' \vee c_2 \vee c_3$, it holds $\neg l' \vee c_1' \vee c_2 \models \neg l' \vee c_1' \vee c_2 \vee c_3$, which implies $G' \models G$ where $G' = G \backslash \{\neg l' \vee c_1' \vee c_2 \vee c_3\} \cup \{\neg l' \vee c_1' \vee c_2\}$, which in turns implies $G' \models c$ since $G \models c$. This set G' is still obtained by F by resolution. It does not contain c because $c \notin G$ and the added clause $\neg l' \vee c_1' \vee c_2$ is not c. It is not c because it does not contain l while c does, and it does not contain l by Lemma 1 because it is the result of resolving two clauses over l. This proves that G' inherit from G all properties that prove c superredundant: $F \vdash G'$, $G' \models c$ and $c \notin G'$.

If the tree rooted in $\neg l' \vee \neg l \vee c_2$ has size a and the tree rooted in $l' \vee \neg l \vee c_3$ has size b, the derivation $F \vdash G$ includes $a+b+2$ in its almost-size. The derivation $F \vdash G'$ has $a+1$ in its place, a decrease in almost-size. The rise of this resolution of c in $F \vdash G$ is $a+b$. In $F \vdash G'$, it is $a-1$. Both almost-size and rise of c decrease.

The second case is that only one of the clauses that resolve into c'' contains $\neg l$. Their resolution literal is still denoted l'; therefore, they can be written $\neg l' \vee \neg l \vee c_2$ and $l' \vee c_3$. The result of resolving them is $c'' = \neg l \vee c_2 \vee c_3$, which resolves with $c = l \vee c_1$ to generate $c' = c_1 \vee c_2 \vee c_3$.



Since $l \vee c_1$ and $\neg l' \vee \neg l \vee c_2$ oppose on l, they resolve. The result is $c_1 \vee \neg l' \vee c_2$, which resolves with $l' \vee c_3$ into $c_1 \vee c_2 \vee c_3$. The same three clauses produce the same clause. This is a valid derivation with an exception discussed below.



This derivation proves $F \vdash G'$ where $G' = G \backslash \{c_1 \vee c_2 \vee c_3\} \cup \{c_1 \vee \neg l' \vee c_2, l' \vee c_3\}$. The only clause of G that G' does not contain is $c_1 \vee c_2 \vee c_3$, but this is the result of resolving $c_1 \vee \neg l' \vee c_2$ and $l' \vee c_3$, two clauses of G'. As a result, $G' \models G$. This proves $G' \models c$ since $G \models c$. Finally, c is not in G'. Since $c \not\in G$, suffices to prove that c is not any of the two clauses that G' contains while G does not. This is the case because $c = l \vee c_1$ contains l while the two clauses does not. The original derivation contains $c_1 \vee c_2 \vee c_3$ as the result of resolving two clauses over l; Lemma 1 tells that l is not in $c_1 \vee c_2 \vee c_3$. As a result, l is in $c_1 \vee \neg l' \vee c_2$ or $l' \vee c_3$ only if either $l = \neg l'$ or $l = l'$. That implies that $\neg l \vee c_2 \vee c_3$ contains l' while it is obtained in the original derivation by resolving two clauses over l', contradicting Lemma 1. This proves that G' inherits all properties that prove c superredundant: $F \vdash G'$, $G' \models c$ and $c \not\in G'$.

Since $\neg l' \vee \neg l \vee c_2$ and $l' \vee c_3$ are obtained by resolution from F, each is the root of a resolution tree. Let a and b be their size.

The almost-size of $F \vdash G$ includes a part for the tree rooted in $c_1 \vee c_2 \vee c_3$; that part is $a + b + 2$. The derivation $F \vdash G'$ is the same except that it has the parts for $c_1 \vee \neg l' \vee c_2$ and $l' \vee c_3$ instead: $a + 1$ and $b - 1$. The almost-size decreases from $a + b + 2$ to $a + b$.

The rise of the resolution of c with $\neg l \vee c_2 \vee c_3$ in $F \vdash G$ is the size of the tree rooted in $\neg l \vee c_2 \vee c_3$; this tree contains $a + b + 1$ nodes; the rise in $F \vdash G$ is therefore $a + b$. The rise of the resolution of c with $\neg l' \vee \neg l \vee c_2$ in $F \vdash G'$ is instead the size of tree rooted in $\neg l' \vee \neg l \vee c_2$ minus one: $a - 1$. This is smaller than $a + b$ since b is nonnegative.

Summarizing, the change in the derivation strictly decreases both its overall size and its rise of c.

The exception that makes the new derivation invalid is when the new clause $c_1 \vee \neg l' \vee c_2$ is a tautology. Valid resolution derivations do not contain tautologies. This also applies to the original one. Since $c_1 \vee c_2 \vee c_3$ is not a tautology, $c_1 \vee c_2$ is neither. Since $\neg l' \vee \neg l \vee c_2$ is not a tautology, c_2 does not contain l'. The new clause $c_1 \vee \neg l' \vee c_2$ is a tautology only if l' is in c_1. Equivalently, $c_1 = l' \vee c'_1$ for some c'_1.

$$
\begin{array}{lll}
l \vee l' \vee c'_1 & & \\
 & & l' \vee c'_1 \vee c_2 \vee c_3 \\
\neg l' \vee \neg l \vee c_2 & & \\
 & \neg l \vee c_2 \vee c_3 & \\
l' \vee c_3 & &
\end{array}
$$

The root of the derivation is $l' \vee c_1 \vee c_2 \vee c_3$ in this case. This is a superset of its grandchild $l' \vee c_3$. Since subclauses imply superclauses, $G' = G \backslash \{c_1 \vee c_2 \vee c_3\} \cup \{l' \vee c_3\}$ implies G. By transitivity, it implies c.

The clause $l' \vee c'_1 \vee c_2 \vee c_3$ does not contain l by Lemma 1 because it is the result of resolving two clauses upon l in the original derivation. As a result, its subset $l' \vee c_3$ does not contain l either. It therefore differs from c, which contains l. This implies $c \notin G'$ since $c \notin G$ and $c \neq l' \vee c_3$.

Since $F \vdash G$ includes the derivation of $l' \vee c_3$, also $F \vdash G'$ holds.

All three properties of G are inherited by G': $F \vdash G'$, $G' \models c$ and $c \notin G'$.

Let a and b be the size of the trees rooted at $\neg l' \vee \neg l \vee c_2$ and $l' \vee c_3$. The almost-size of $F \vdash G$ has a component for its root $l' \vee c_1 \vee c_2 \vee c_3$ of value $a + b + 2$. The derivation $F \vdash G'$ has the root $l' \vee c_3$ in its place, contributing only $a - 1$ to the almost-size. The almost-size decreases. The rise of c also decreases. In the original derivation c resolves with $\neg l \vee c_2 \vee c_3$, contributing $a + b$ to the overall rise. In $F \vdash G'$ this resolution is absent, contributing 0 to the overall rise. Since a is the size of a non-empty tree, it is greater than zero. The same holds for b. The rise of c decreases by $a + b$, which is at least 2.

All of this proves that if G is not a subset of $F \cup \text{resolve}(c, F)$ it can be changed to decrease its overall measure while still proving c superredundant. This change preserves $F \vdash G$, $G \models c$ and $c \notin G$ and strictly decreases the measure of $F \vdash G$, defined as the sum of its almost-size and rise of c.

This change can be iterated as long as G is not a subset of $F \cup \text{resolve}(c, F)$. It terminates because the measure strictly decreases at each step but is never negative as proved above. When it terminates, G is a subset of $F \cup \text{resolve}(c, F)$ because otherwise it could be iterated. Since $c \notin G$ is one of the preserved properties, G is also a subset of $F \backslash \{c\} \cup \text{resolve}(c, F)$. Since G implies c, its superset $F \backslash \{c\} \cup \text{resolve}(c, F)$ implies c too. This is the claim. □

Lemma 9 allows for a simple algorithm for checking superredundancy: resolve c with all other clauses of F and then remove it. If c is still entailed, it is superredundant. This proves that checking superredundancy is polynomial-time for example in the Horn and Krom cases (the latter is that all clauses contain two literals at most). More generally, it is polynomial in all restrictions that are closed under resolution and where inference can be checked in polynomial time.

Theorem 1 *Checking superredundancy is polynomial-time in the Horn and Krom cases.*

Proof. The clauses in $\text{resolve}(F, c)$ can be generated in polynomial time by resolving each clause of F with c. The result is still Horn or Krom because resolving two Horn clauses or two Krom clauses respectively generates a Horn and Krom clause. Checking $F \backslash \{c\} \cup \text{resolve}(F, c) \models c$ therefore only takes polynomial time. □

If c is a single literal l, it only resolves with clauses containing $\neg l$. The condition is very simple in this case.

Theorem 2 *A single-literal clause l of F is superredundant if and only if*

$$\{c \in F \mid \neg l \notin c,\ c \neq l\} \cup \{c \mid c \vee \neg l \in F\} \models l$$

Proof. When a clause c comprises a single literal l, Lemma 9 equates its superredundancy to $F \backslash \{l\} \cup \text{resolve}(F, l) \models l$. The first part $F \backslash \{l\}$ of the formula comprises clauses that resolve with l and clauses that do not:

$$F \backslash \{l\} = \{c \in F \mid \neg l \notin c,\ c \neq l\} \cup \{c \in F \mid \neg l \in c\}$$

If c is in the second set, then $\text{resolve}(F, l)$ contains $\text{resolve}(c, l) = c \backslash \{\neg l\}$, which implies c. Therefore, $F \backslash \{l\} \cup \text{resolve}(F, l)$ is equivalent to $\{c \in F \mid \neg l \notin c,\ c \neq l\} \cup \text{resolve}(F, l)$. The clauses that resolve with l are all of the form $c \vee \neg l$, and the result of the resolution is c. Therefore, $\text{resolve}(F, l)$ can be rewritten as $\{c \mid c \vee \neg l \in F\}$. This proves the claim. □

This theorem shows how to check the superredundancy of a single-literal clause of a formula. All it takes is a simple transformation of the formula: the unit clause l is removed, and the literal $\neg l$ is deleted from all clauses containing it. If what remains imply l, then l is superredundant.

This condition can be further simplified when not only the clause to be checked is a literal, but its converse does not even occur in the formula. Such literals are usually called pure in the automated reasoning field [15].

Theorem 3 *If $\neg l$ does not occur in F, the single-literal clause l of $F \cup \{l\}$ is superredundant if and only if $F \models l$.*

Proof. A clause l of $F \cup \{l\}$ is superredundant if and only if $F' \models l$ where $F' = \{c \in F \cup \{l\} \mid \neg l \notin c,\ c \neq l\} \cup \{c \mid c \vee \neg l \in F \cup \{l\}\} \models l$ thanks to Theorem 2 applied to $F \cup \{l\}$. If $\neg l$ does not occur in F, then F' simplifies as follows.

$$\begin{aligned} F' &= \{c \in F \cup \{l\} \mid \neg l \notin c,\ c \neq l\} \cup \{c \mid c \vee \neg l \in F \cup \{l\}\} \\ &= \{c \in F \cup \{l\} \mid c \neq l\} \\ &= F \end{aligned}$$

This proves that l is superredundant in $F \cup \{l\}$ if and only if $F \models l$ in this case. □

The formula where l is superredundant is $F \cup \{l\}$. Alternatively, l is superredundant in F if and only if $F \backslash \{l\} \models l$, provided that $\neg l$ does not occur in F.

Talking about pure literals, another equivalent condition exists. A variable may always occur with the same sign in a formula; if so, the clauses containing it are irrelevant to the superredundancy of the others.

Lemma 10 *If $\neg l$ does not occur in F, $l \notin c$ and $l \in c'$ for some clause c', the clause c of F is superredundant if and only if it is superredundant in $F \backslash \{c'\}$.*

Proof. No derivation $F \vdash c$ involves c'. This is proved by contradiction. Clause c' may resolve with other clauses of F, but the resolving variable cannot be l since no clause of F contains $\neg l$. Therefore, the resulting clauses all contain l. The same applies to them: they may resolve, but the result contains l. Inductively, this proves that l is also in the root of the derivation tree. The root is c, which does not contain l by assumption. This contradiction proves that c' is not involved in $F \vdash c$.

A derivation $F \vdash G \vdash c$ is a resolution tree with leaves F and root c. It is a particular case of $F \vdash c$. Therefore, it does not contain c'. As a result, it can be rewritten $F \backslash \{c'\} \vdash G \vdash c$.

A derivation $F \vdash c''$ with $c'' \subset c$ does not contain c' because c'' does not contain l. Therefore, $F \vdash c''$ is the same as $F \backslash \{c\} \vdash c''$ for every subset c'' of c.

The third equivalent condition to superredundancy in Lemma 6 is: $F \vdash G \vdash c$ with $c \notin G$ or $F \vdash c''$ with $c'' \subset c$. These conditions are respectively equivalent to $F \backslash \{c'\} \vdash G \vdash c$ and $F \backslash \{c'\} \vdash c''$, the third equivalent condition to the superredundancy of c in $F \backslash \{c'\}$ of Lemma 6. □

A particular case meeting the assumption of the lemma is when a variable only occurs in one clause. The lemma specializes as follows.

Corollary 1 *If a variable occurs in F only in the clause c, then $c' \neq c$ is superredundant in F if and only if it is superredundant in $F \backslash \{c\}$.*

3.4 Sufficient conditions to superredundancy

A number of equivalent conditions have been provided. Time to turn to sufficient conditions. The next corollary shows a sufficient condition to superredundancy. The following results are about superirredundancy.

Corollary 2 *If $F \models c'$ and $c' \subset c \in F$, then c is superredundant in F.*

Proof. Immediate consequence of the fourth equivalent condition of Lemma 6. □

While proving superredundancy is sometimes useful, its main aim is to prove a clause in all minimal equivalent formulae, which is the case if it is superirredundant. This is why much effort is devoted to proving superirredundancy.

A way to prove superirredundancy is by first simplifying the formula and then proving superirredundancy on the result. Of course, not all simplifications work. Proving the superirredundancy of $a \vee b \vee c$ in $F = \{a \vee b, a \vee b \vee c\}$ is a counterexample. Removing the first clause from F makes $a \vee b \vee c$ superirredundant in what remains, $F' = \{a \vee b \vee c\}$. Yet, $a \vee b \vee c$ is not superirredundant in F.

A simplification works only if the superirredundancy of the clause in the simplified formula implies its superirredundancy in the original formula. In the other way around, superredundancy in the original implies superredundancy in the simplification.

What is required is that "c superredundant in F" implies "c superredundant in the simplified F". One direction is enough. "Implies", not "if and only if".

The following lemmas are formulated in the direction where superredundancy implies superredundancy. This simplifies their formulation and their proofs, but they are mostly used in reverse: superirredundancy implies superirredundancy.

Lemma 11 *If a clause c of F is superredundant, it is also superredundant in $F \cup \{c'\}$.*

Proof. The assumed superredundancy of a clause c of F is by definition $\mathrm{ResCn}(F) \backslash \{c\} \models c$. The derivations by resolution from F are also valid from $F \cup \{c'\}$, where c' is just not used. Therefore, $\mathrm{ResCn}(F) \subseteq \mathrm{ResCn}(F \cup \{c'\})$. This implies $\mathrm{ResCn}(F) \backslash \{c\} \subseteq \mathrm{ResCn}(F \cup \{c'\}) \backslash \{c\}$, which implies $\mathrm{ResCn}(F \cup \{c'\}) \backslash \{c\} \models \mathrm{ResCn}(F) \backslash \{c\}$. By transitivity of entailment, the claim follows: $\mathrm{ResCn}(F \cup \{c'\}) \backslash \{c\} \models c$. □

How this lemma is used: a formula F may simplify when a clause is added to it, making superirredundancy easy to prove. For example, adding the single-literal clause x allows removing from F all clauses that contain x. If all clauses but c contain x, the superirredundancy of c in F follows from the superirredundancy of c in $\{c, x\}$.

This example of adding a single-literal clause extends to a full-fledged sufficient condition to superredundancy. Adding x to F has the same effect of replacing x with true and simplifying the formula. This transformation is defined as follows.

$$
\begin{aligned}
F[\mathsf{true}/x] &= \{c[\mathsf{true}/x] \mid c \in F,\ c[\mathsf{true}/x] \neq \top\} \\
c[\mathsf{true}/x] &= \begin{cases} c \backslash \{\neg x\} & \text{if } x \notin c \\ \top & \text{otherwise} \end{cases}
\end{aligned}
$$

The symbol $\top$ used in the definition does not occur in the final formula since clauses that are turned into $\top$ are removed from F. In other words, $F[\mathsf{true}/x]$ is a formula built over variables and propositional connectives; it does not contain any special symbol for true or false.

Swapping x and $\neg x$ turns the definition of $F[\mathsf{true}/x]$ into $F[\mathsf{false}/x]$.

The next lemma shows that such substitutions often preserve superredundancy. This would be obvious if superirredundancy were the same as clause primality or a similar semantical notion, but it has been proved not to be by Lemma 3.

Lemma 12 *A clause c of $F[\mathsf{true}/x]$ is superredundant if it is superredundant in F, it contains neither x nor $\neg x$ and F does not contain $c \vee \neg x$. The same holds for $F[\mathsf{false}/x]$ if F does not contain $c \vee x$.*

Proof. The claim is proved for $x = \mathsf{true}$. It holds for $x = \mathsf{false}$ by symmetry.

The assumption that c is superredundant in F is equivalent to $F \backslash \{c\} \cup \mathrm{resolve}(F, c) \models c$ thanks to Lemma 9. The claim is the superredundancy of c in $F[\mathsf{true}/x]$, which is equivalent to $F[\mathsf{true}/x] \backslash \{c\} \cup \mathrm{resolve}(F[\mathsf{true}/x], c)) \models c$ still thanks to Lemma 9.

The claim is the last of a chain of properties that follow from the assumption $F \backslash \{c\} \cup \mathrm{resolve}(F, c) \models c$.

1. $(F \backslash \{c\} \cup \mathrm{resolve}(F, c))[\mathsf{true}/x] \models c$

   Let $H = F \backslash \{c\} \cup \mathrm{resolve}(F, c)$. The assumption is $H \models c$, the claim $H[\mathsf{true}/x] \models c$. By Boole's expansion theorem [1], H is equivalent to $x \wedge H[\mathsf{true}/x] \vee \neg x \wedge H[\mathsf{false}/x]$. The assumption $H \models c$ is therefore the same as $x \wedge H[\mathsf{true}/x] \vee \neg x \wedge H[\mathsf{false}/x] \models c$. Since a disjunction is implied by any of its disjuncts, $x \wedge H[\mathsf{true}/x] \models c$ follows. Since neither c nor $H[\mathsf{true}/x]$ contain x, this is the same as $H[\mathsf{true}/x] \models c$.

2. $(F \backslash \{c\} \cup \text{resolve}(F, c))[\mathsf{true}/x] = (F \backslash \{c\})[\mathsf{true}/x] \cup \text{resolve}(F, c)[\mathsf{true}/x]$

   Formula $F \backslash \{c\} \cup \text{resolve}(F, c)$ is a union. The substitution therefore applies to each of its sets.

   The two parts of the formula are considered separately.

3. $(F \backslash \{c\})[\mathsf{true}/x] = F[\mathsf{true}/x] \backslash \{c\}$ if $c \vee \neg x \notin F$

   Both formulae are made of some clauses of F with the substitution $[\mathsf{true}/x]$ applied to them. They differ on whether c is subtracted before or after the substitution. The claim is proved by showing that for every $c' \in F$, the clause $c'[\mathsf{true}/x]$ is in $(F \backslash \{c\})[\mathsf{true}/x]$ if and only if it is in $F[\mathsf{true}/x] \backslash \{c\}$. The two cases $x \in c'$ and $x \notin c'$ are considered separately.

   If c' contains x, then $c'[\mathsf{true}/x] = \top$. As a result, $F[\mathsf{true}/x]$ does not contain $c'[\mathsf{true}/x]$; its subset $F[\mathsf{true}/x] \backslash \{c\}$ does not either. Neither does $(F \backslash \{c\})[\mathsf{true}/x]$; indeed, $c' \in F \backslash \{c\}$ since c' contains x while c does not, but still $c'[\mathsf{true}/x]$ is equal to $\top$ because it contains x, and is not therefore in $(F \backslash \{c\})[\mathsf{true}/x]$.

   If c' does not contain x, then $c'[\mathsf{true}/x] = c' \backslash \{\neg x\}$. This clause is equal to c if and only if c' is either c or $c \vee \neg x$; the second cannot be the case since F by assumption contains c' but not $c \vee \neg x$. As a result, $c'[\mathsf{true}/x] = c$ if and only if $c' = c$. If $c' = c$, then $c'[\mathsf{true}/x] = c$ is removed from $F[\mathsf{true}/x]$ when subtracting c and c' is removed from F when subtracting c. If $c' \neq c$, then $c'[\mathsf{true}/x] = c$ is not removed from $F[\mathsf{true}/x]$ and is therefore in $F[\mathsf{true}/x] \backslash \{c\}$; also c' is not removed from F and is therefore in $F \backslash \{c\}$, which means that $c'[\mathsf{true}/x]$ is in $(F \backslash \{c\})[\mathsf{true}/x]$. In both cases, either $c'[\mathsf{true}/x]$ is in both sets or in none.

4. $\text{resolve}(F, c)[\mathsf{true}/x] = \text{resolve}(F[\mathsf{true}/x], c)$

   Expanding the definitions of $\text{resolve}(F, c)$ and $\text{resolve}(F[\mathsf{true}/x], c)$ shows that the claim is:

   $$(\bigcup_{c' \in F} \text{resolve}(c', c))[\mathsf{true}/x] = \bigcup_{c'' \in F[\mathsf{true}/x]} \text{resolve}(c'', c)$$

   The first substitution is applied to a union, and can therefore equivalently be applied to each of its members:

   $$\bigcup_{c' \in F} (\text{resolve}(c', c)[\mathsf{true}/x]) = \bigcup_{c'' \in F[\mathsf{true}/x]} \text{resolve}(c'', c)$$

   If x is in c', it is also in the result of resolving c' with c since c does not contain $\neg x$ by assumption. As a result, $\text{resolve}(c', c)[\mathsf{true}/x] = \top$: the clauses c' that contain x do not contribute to the first union. The claim therefore becomes:

   $$\bigcup_{c' \in F,\ x \notin x} (\text{resolve}(c', c)[\mathsf{true}/x]) = \bigcup_{c'' \in F[\mathsf{true}/x]} \text{resolve}(c'', c)$$

The formula $F[\mathsf{true}/x]$ comprises by definition the clauses $c'' = c'[\mathsf{true}/x]$ such that $c' \in F$ and $c'[\mathsf{true}/x] \neq \top$. The second condition $c'[\mathsf{true}/x] \neq \top$ is false if $x \in c'$. The clauses containing x do not contribute to the second union either.

$$\bigcup_{c' \in F,\ x \not\in x} (\text{resolve}(c', c)[\mathsf{true}/x]) = \bigcup_{c' \in F,\ x \not\in x} \text{resolve}(c'[\mathsf{true}], c)$$

This equality is proved as a consequence of the pairwise equality of the elements of the unions.

$$\text{resolve}(c', c)[\mathsf{true}/x] = \text{resolve}(c'[\mathsf{true}/x], c) \text{ for every } c' \in F \text{ such that } x \not\in c'$$

Neither c nor c' contain x: the first by the assumption of the lemma, the second because of the restriction in the above equality. Since $\text{resolve}(c', c)$ only contains literals of c and c', it does not contain x either. Replacing x with true in a clause that does not contain x is the same as removing $\neg x$.

$$\text{resolve}(c', c)\backslash\{\neg x\} = \text{resolve}(c'\backslash\{\neg x\}, c) \text{ for every } c' \in F \text{ such that } x \not\in c'$$

If l is a literal of c' such that $\neg l \in c$, then $l \neq \neg x$ because c does not contain x. As a result, $l \in c'$ implies $l \in c'\backslash\{\neg x\}$. The converse also holds because of set containment. This proves that c resolves with c' on a literal if and only it resolves with $c'\backslash\{\neg x\}$ on the same literal.

If these clauses do not resolve, both sides of the equality are empty and therefore equal. Otherwise, both c' and $c'\backslash\{\neg x\}$ resolve with c on the same literal l. The result of resolving c' with c is $\text{resolve}(c', c) = c \cup c'\backslash\{l, \neg l\}$; as a result, the left-hand side of the equality is $\text{resolve}(c', c)\backslash\{\neg x\} = c \cup c'\backslash\{l, \neg l\}\backslash\{\neg x\}$. This is the same as the right hand side $\text{resolve}(c'\backslash\{x\}, c) = c \cup (c'\backslash\{\neg x\})\backslash\{l, \neg l\}$ since c does not contain $\neg x$ by assumption.

Summing up, the superredundancy of c in F expressed as $F\backslash\{c\} \cup \text{resolve}(F, c) \models c$ thanks to Lemma 9 implies $(F\backslash\{c\} \cup \text{resolve}(F, c))[\mathsf{true}/x] \models c$, and the formula in this entailment is the same as $F[\mathsf{true}/x]\backslash\{c\} \cup \text{resolve}(F[\mathsf{true}/x], c)$. The conclusion $F[\mathsf{true}/x]\backslash\{c\} \cup \text{resolve}(F[\mathsf{true}/x], c) \models c$ is equivalent to the superredundancy of c in $F[\mathsf{true}/x]$ thanks to Lemma 9. □

Is the assumption $c \vee \neg x \not\in F$ necessary? A counterexample disproves the claim of lemma without this assumption: the clause a is superredundant in $F = \{a \vee \neg x, a, x\}$ because it is redundant, but is superirredundant in $F[\mathsf{true}/x] = \{a\}$.

A way to prove superirredundancy is by applying Lemma 12 coupled with Lemma 8. A suitable evaluation of the variables not in c removes or simplifies the other clauses of F to the point they do not resolve, where Lemma 8 shows that c is superirredundant. Lemma 12 proves that c is also superirredundant in F.

An example is $F = \{a \vee b, b \vee c, \neg b \vee \neg d, \neg c \vee d \vee e\}$. Replacing c with true and d with false deletes the second and third clause and simplifies the fourth, leaving $F[\mathsf{true}/c][\mathsf{false}/d] =$

$\{a \vee b, e\}$, where $a \vee b$ is superirredundant because no clause resolve in this formula. This proves that $a \vee b$ is also superirredundant in F.

A substitution may not prevent all resolutions, but still breaks the formula in small unlinked parts. Such parts can be worked on separately.

Lemma 13 *If F' does not share variables with F and F' is satisfiable, a clause c of F is superredundant if and only if it is superredundant in $F \cup F'$.*

Proof. Since c is in F, it is also in $F \cup F'$. By Lemma 9, the superredundancy of c in $F \cup F'$ is equivalent to $(F \cup F' \cup \text{resolve}(F \cup F', c)) \backslash \{c\} \models c$. Since c is in F and F does not share variables with F', the clause c does not share variables with F' and therefore does not resolve with any clause in F'. This proves $\text{resolve}(F \cup F', c) = \text{resolve}(F, c)$. The entailment becomes $(F \cup F' \cup \text{resolve}(F, c)) \backslash \{c\} \models c$, and also $(F \cup \text{resolve}(F, c)) \backslash \{c\} \cup F' \models c$ since $c \notin F'$. This is the same as $(F \cup \text{resolve}(F, c)) \backslash \{c\} \models c$ because F' is satisfiable and because of the separation of the variables. This is equivalent to the superredundancy of c in F by Lemma 9. □

Short guide on using Lemma 12 and Lemma 13: to prove c superirredundant in F, all clauses c' that resolve with it are found and all their variables not in c collected; these variables are set to values that satisfy as many clauses c' as possible. All these clauses link c with the rest of F, and removing them makes c isolated and therefore superirredundant.

For example, the clause $c = a \vee b$ is proved superirredundant in $F = \{a \vee b, \neg a \vee c \vee d, \neg b \vee \neg c \vee \neg f, \neg d \vee f \vee g, d \vee h\}$ by a substitution that removes the clauses of F that share variables with c.

The clauses of F that share variables with $c = a \vee b$ are $\neg a \vee c \vee d$ and $\neg b \vee \neg c \vee \neg f$. They are to be removed by substituting variables other than a and b. For example, $c = \mathsf{true}$ removes the first and simplifies the second into $\neg b \vee \neg f$, which is removed by $f = \mathsf{false}$. This substitution turns F into $F[c/\mathsf{true}][f/\mathsf{false}] = \{a \vee b, \neg d \vee g, d \vee h\}$. Since its clause $c = a \vee b$ does not share variables with the other two clauses, it is superirredundant. As a result, it is also superredundant in F.

A final sufficient condition to superirredundancy is given by the following lemma. It is specular to Theorem 3: that result applies when $\neg l$ is not in F, this one when l is not in F.

Lemma 14 *If l does not occur in F, then l is superirredundant in $F \cup \{l\}$ if this formula is satisfiable.*

Proof. By Theorem 2, the superredundancy of l in $F \cup \{l\}$ is the same as $F' \models l$, where F' is:

$$
\begin{aligned}
F' &= \{c \in F \cup \{l\} \mid \neg l \notin c,\ c \neq l\} \cup \{c \mid c \vee \neg l \in F \cup \{l\}\} \\
&= \{c \in F \mid \neg l \notin c\} \cup \{c \mid c \vee \neg l \in F\}
\end{aligned}
$$

The first part of the union is a subset of F; the second comprises only subclauses of F. Since F does not contain l, this union F' does not contain l either. Therefore, F' entails l only if it is unsatisfiable.

The unsatisfiability of F' is proved to contradict the assumption of the lemma. The first part of F' is a subset of F; each clause c of its second part is a consequence of $c \vee \neg l \in F$ and l, and is therefore entailed by $F \cup \{l\}$. Therefore, this union is entailed by $F \cup \{l\}$. It unsatisfiability implies the unsatisfiability of $F \cup \{l\}$, which is contrary to an assumption of the lemma. □

4 Ensuring superirredundancy

The intended application of superirredundancy is in existence proofs: produce a formula satisfying certain conditions, some involving its minimal equivalent formulae. Superirredundancy fixes a part of these minimal equivalent formulae. The other conditions are ensured separately. The key is "separately". The formula can be built to meet these other conditions and then some of its clauses turned superirredundant.

The example in the next section shows how to create a reduction from Boolean satisfiability to the problem of Horn minimality. It does not attempt to build a Horn formula that can be shrunk over a certain limit if and only if a given CNF formula is satisfiable. Rather, it builds the Horn formula so that some of its clauses can be removed under the same condition. The other clauses are then fixed by turning them superirredundant. This simplifies the process of creating the reduction, as the target formula needs not to satisfy all required conditions right from the beginning.

If $a \vee b$ is not superirredundant but is required in all minimal equivalent formulae, it is split into $a \vee x$ and $b \vee \neg x$, where x is a new variable. The two resulting clauses are superirredundant in most cases.

The replacing clauses $a \vee x$ and $b \vee \neg x$ imply the original clause $a \vee b$ by resolution. They are however not exactly equivalent to it because of the new variable. They are only when restricting to all variables but x. This restriction defines the concept of forgetting [22, 17].

Definition 6 *A formula B expresses forgetting all variables from A except Y if and only if $Var(B) \subseteq Y$ and $B \models C$ is the same as $A \models C$ for all formulae C such that $Var(C) \subseteq Y$.*

The formula $B = A \backslash \{a \vee b\} \cup \{a \vee x, b \vee \neg x\}$ expresses forgetting x from A. It is equivalent to it when disregarding the new variable x. An alternative definition is that the models of A and B are the same when neglecting the evaluation of x. The following theorem proves it, where $A \cap l$ are the clauses of A that contain the literal l.

Theorem 4 ([32, Theorem 6],[7, Theorem 6]) *The formula $A \backslash (A \cap x) \backslash (A \cap \neg x) \cup resolve(A \cap x, A \cap \neg x)$ expresses forgetting x from A.*

In the example, A is the formula after splitting $a \vee b$ into $a \vee x$ and $b \vee \neg x$. Therefore, $A \cap x$ is $\{a \vee x\}$ and $A \cap \neg x$ is $\{b \vee \neg x\}$. As a result, $A \backslash (A \cap x) \backslash (A \cap \neg x) \cup \mathrm{resolve}(A \cap x, A \cap \neg x)$ is exactly the formula before splitting.

This is the first requisite on turning a clause superirredundant: the change does not alter the semantics of the formula. Introducing a new variable makes exact equivalence impossible, but forgetting is the close enough.

Another requirement is that turning $a \vee b$ superirredundant does not make other clauses superredundant. That would make the change work only for a single clause, not for all clauses that are required to be superirredundant.

In summary, splitting a clause works if:

- the resulting formula is similar enough to the original;
- the split clause is superirredundant;
- the other clauses remain superirredundant.

The first point is formalized as: the original formula expresses forgetting the new variable from the generated formula. The second and the third points have additional requirements, they are not always the case. They are proved in reverse, by showing the consequences of superredundancy.

The following section shows an example of the mechanism, the subsequent ones illustrate each of the points above.

4.1 An example of making a clause superirredundant

An example illustrates the method. The first clause is superredundant in the following formula, as can be proved by computing the resolution closure.

$$\{a \vee b \vee c, \neg a \vee d, \neg c \vee d, \neg d \vee a \vee c\}$$

The last three clauses are the same as $a \vee c \equiv d$. They make $a \vee c$ equivalent to d. Consequently, the first clause $a \vee b \vee c$ is replaceable by $d \vee b$ and therefore superredundant.

If it is required to be superirredundant, it can be made so by splitting it into $a \vee x$ and $\neg x \vee b \vee c$.

$$\{a \vee x, \neg x \vee b \vee c, \neg a \vee d, \neg c \vee d, \neg d \vee a \vee c\}$$

The last three clauses still make $a \vee c$ equivalent to d, but this is no longer a problem because a and c are now separated: a is in $a \vee x$ and c is in $\neg x \vee b \vee c$. The only way to join them back to apply their equivalence to d is to resolve the two parts into $a \vee b \vee c$. This removes x and $\neg x$, which are necessary to derive the two clauses $a \vee x$ and $\neg x \vee b \vee c$ back. All clauses in this formula are superirredundant, which can be checked by computing the resolution closure.

4.2 Preserving the semantics of the formula

The first point to prove is that splitting a clause does not change the meaning of the formula. The semantics changes slightly since the original formula does not mention x at all while the modified one does. In the example, $a = \mathsf{false}$, $b = \mathsf{true}$ and $x = \mathsf{false}$ satisfy the original clause $a \vee b \vee c$ but not its part $a \vee x$. The modified formula cannot be equivalent to the original since it contains the new variable. Yet, it is equivalent apart from it. This is what forgetting does: it removes a variable while semantically preserving everything else.

Lemma 15 *Every CNF formula F that contains a clause $c_1 \vee c_2$, where c_1 and c_2 are two clauses, and does not mention x expresses forgetting x from $F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee x, c_2 \vee \neg x\}$.*

Proof. The formula $F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee x, c_2 \vee \neg x\}$ in the statement of the lemma is denoted F''. Theorem 4 proves that a formula expresses forgetting x from it is $F'' \backslash (F'' \cap x) \backslash (F'' \cap \neg x) \cup \text{resolve}(F'' \cap x, F'' \cap \neg x)$. The claim is proved by showing that this formula is F.

The only clause of F'' containing x is $c_1 \vee x$ and the only clause containing $\neg x$ is $c_2 \vee \neg x$. Therefore, $F'' \cap x$ is $\{c_1 \vee x\}$ and $F'' \cap \neg x$ is $\{c_2 \vee \neg x\}$. The formula that expresses forgetting is therefore $F'' \backslash \{c_1 \vee x\} \backslash \{c_2 \vee \neg x\} \cup \text{resolve}(c_1 \vee x, c_2 \vee \neg x)$, which is equal to $F'' \backslash \{c_1 \vee x\} \backslash \{c_2 \vee \neg x\} \cup \{c_1 \vee c_2\}$ since $\text{resolve}(c_1 \vee x, c_2 \vee \neg x) = \{c_1 \vee c_2\}$. Replacing F'' with its definition turns this formula into $F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee x, c_2 \vee \neg x\} \backslash \{c_1 \vee x\} \backslash \{c_2 \vee \neg x\} \cup \{c_1 \vee c_2\}$. Computing unions and set subtractions shows that this formula is F. □

This lemma tells that $F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee x, c_2 \vee \neg x\}$ is like F apart from x. This is the basic requirement for the split: it preserves the semantics as much as possible. The modified formula has the same consequences of the original that do not involve x.

4.3 Making a clause superirredundant

The aim of the split is not just to preserve the semantics but also to make the two pieces of the split clause superirredundant.

The addition of x and $\neg x$, more than the split itself, is what creates superirredundancy. The two parts contain x and $\neg x$, and are the only clauses containing them. They are necessary to derive every other clause containing them, including themselves.

An exception is when the part containing x derives another containing x which derives it back. The presence of x in the whole derivation sequence ensures that removing x everywhere does not invalidate the derivation. The result is a derivation from a part of the original clause (without x added) to other clauses and back. It proves the superredundancy of that part. This explains the exception: superirredundancy is only obtained if none of the two parts of the clause is superredundant by itself.

Lemma 16 *If $c_1 \vee x$ is superredundant in $F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee x, c_2 \vee \neg x\}$, then c_1 is superredundant in $F \cup \{c_1\}$, provided that:*

- *$c_1 \vee c_2$ is in F;*
- *c_1 is not in F; and*
- *x does not occur in F.*

Proof. Lemma 9 reformulates the superredundancy in the assumption and in the claim as entailments.

$$
\begin{aligned}
& F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee x, c_2 \vee \neg x\} \backslash \{c_1 \vee x\} \cup \text{resolve}(c_1 \vee x, F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee x, c_2 \vee \neg x\}) \\
& \qquad \models c_1 \vee x \\
& F \cup \{c_1\} \backslash \{c_1\} \cup \text{resolve}(c_1, F \cup \{c_1\}) \\
& \qquad \models c_1
\end{aligned}
$$

The claim is proved if the first entailment implies the second. Since F contains $c_1 \vee c_2$, it is the same as $F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee c_2\}$. This allows reformulating the second entailment in terms of $F' = F \backslash \{c_1 \vee c_2\}$; the first can be as well.

$$
\begin{aligned}
F' \cup \{c_1 \vee x, c_2 \vee \neg x\} \backslash \{c_1 \vee x\} \cup \text{resolve}(c_1 \vee x, F' \cup \{c_1 \vee x, c_2 \vee \neg x\}) &\models c_1 \vee x \\
F' \cup \{c_1 \vee c_2\} \cup \{c_1\} \backslash \{c_1\} \cup \text{resolve}(c_1, F' \cup \{c_1 \vee c_2\} \cup \{c_1\}) &\models c_1
\end{aligned}
$$

The set subtractions can be computed immediately.

In the first formula, $F' \cup \{c_1 \vee x, c_2 \vee \neg x\} \backslash \{c_1 \vee x\}$ is equal to $F' \cup \{c_2 \vee \neg x\}$ since neither $c_2 \vee \neg x$ nor any clause in F' is equal to $c_1 \vee x$. The former is not because it contains $\neg x$, the latter are not because F' is a subset of F, which does not mention x.

In the second formula, $F' \cup \{c_1 \vee c_2\} \cup \{c_1\} \backslash \{c_1\}$ is equal to $F' \cup \{c_1 \vee c_2\}$ since neither $c_1 \vee c_2$ nor any clause in F' is equal to c_1. The first is not because it is in F while c_1 is not, the second are not because F' is a subset of F, which does not contain c_1.

$$
\begin{aligned}
F' \cup \{c_2 \vee \neg x\} \cup \text{resolve}(c_1 \vee x, F' \cup \{c_1 \vee x, c_2 \vee \neg x\}) &\models c_1 \vee x \\
F' \cup \{c_1 \vee c_2\} \cup \text{resolve}(c_1, F' \cup \{c_1 \vee c_2\} \cup \{c_1\}) &\models c_1
\end{aligned}
$$

Both entailments contain the resolution of a clause with a union. This is the same as the resolution of the clause with each component of the union.

$$
\begin{aligned}
F' \cup \{c_2 \vee \neg x\} \cup \text{resolve}(c_1 \vee x, F') \cup \text{resolve}(c_1 \vee x, \{c_1 \vee x\}) \cup \text{resolve}(c_1 \vee x, \{c_2 \vee \neg x\}) &\models c_1 \vee x \\
F' \cup \{c_1 \vee c_2\} \cup \text{resolve}(c_1, F') \cup \text{resolve}(c_1, \{c_1 \vee c_2\}) \cup \text{resolve}(c_1, \{c_1\}) &\models c_1
\end{aligned}
$$

Some parts of these formulae are empty because clauses do not resolve with themselves or with their superclauses.

$$
\begin{aligned}
F' \cup \{c_2 \vee \neg x\} \cup \text{resolve}(c_1 \vee x, F') \cup \text{resolve}(c_1 \vee x, \{c_2 \vee \neg x\}) &\models c_1 \vee x \\
F' \cup \{c_1 \vee c_2\} \cup \text{resolve}(c_1, F') &\models c_1
\end{aligned}
$$

Since $c_1 \vee c_2$ is in F and formulae are assumed not to contain tautologies, the two subclauses c_1 and c_2 do not contain opposite literals. Therefore, resolving $c_1 \vee x$ and $c_2 \vee \neg x$ only generates $c_1 \vee c_2$, which is not a tautology. This simplifies $\text{resolve}(c_1 \vee x, \{c_2 \vee \neg x\})$ into $\{c_1 \vee c_2\}$.

$$
\begin{aligned}
F' \cup \{c_2 \vee \neg x\} \cup \text{resolve}(c_1 \vee x, F') \cup \{c_1 \vee c_2\} &\models c_1 \vee x \\
F' \cup \{c_1 \vee c_2\} \cup \text{resolve}(c_1, F') &\models c_1
\end{aligned}
$$

The set $\text{resolve}(c_1 \vee x, F')$ contains the result of resolving $c_1 \vee x$ with the clauses of F'. Since F' is a subset of F, it does not contain x. Therefore, the resolving literal of $c_1 \vee x$ with a clause of $c'' \in F'$ is not x if any. If $c_1 \vee x$ resolves with $c'' \in F$, then c_1 does as well. Adding x to the resolvent generates the resolvent of $c_1 \vee x$ and c''. Formally, $\text{resolve}(c_1 \vee x, F') = \{c' \vee x \mid c' \in \text{resolve}(c_1, F')\}$.

$$
\begin{aligned}
F' \cup \{c_2 \vee \neg x\} \cup \{c' \vee x \mid c' \in \text{resolve}(c_1, F')\} \cup \{c_1 \vee c_2\} &\models c_1 \vee x \\
F' \cup \{c_1 \vee c_2\} \cup \text{resolve}(c_1, F') &\models c_1
\end{aligned}
$$

Replacing x with false in the first entailment results in $F' \cup \{c_2 \vee \neg \mathsf{false}\} \cup \{c' \vee \mathsf{false} \mid c' \in \mathrm{resolve}(c_1, F')\} \cup \{c_1 \vee c_2\} \models c_1 \vee \mathsf{false}$. Simplifying according to the rules of propositional logic $\mathsf{true} \vee G = \mathsf{true}$ and $\mathsf{false} \vee G = G$ turns this entailment into $F' \cup \{\mathsf{true}\} \cup \{c' \mid c' \in \mathrm{resolve}(c_1, F')\} \cup \{c_1 \vee c_2\} \models c_1$, which is the same as $F' \cup \mathrm{resolve}(c_1, F') \cup \{c_1 \vee c_2\} \models c_1$, the second entailment.

This proves that the first entailment implies the second: the assumption implies the claim. $\square$

The intended usage of the lemma is to split a clause $c_1 \vee c_2$ of F into $c_1 \vee x$ and $c_2 \vee \neg x$, where x is a new variable. Being new, x does not occur in the rest of the formula. If the lemma is used this way, its first and last assumptions are met. The second may not, and the claim may not hold if F contains c_1. Actually, the claim may not hold if any of its three assumptions does not hold.

If $c_1 \vee c_2$ is not in F, the claim may not hold. A counterexample is $c_1 = a \vee b$, $c_2 = a$ and $F = \emptyset$. The other preconditions of the lemma are satisfied: c_1 is not in F, where x does not occur; $c_1 \vee x$ is superredundant in $F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee x, c_2 \vee \neg x\}$ since this formula is $\{a \vee b \vee x, a \vee \neg x\}$, whose two clauses resolve in $a \vee b$, which entails $c_1 \vee x = a \vee b \vee x$. The conclusion that $c_1 = a \vee b$ is superredundant in $F \cup \{c_1\} = \emptyset \cup \{a \vee b\} = \{a \vee b\}$ is false since this formula allows no resolution.

If c_1 is in F, the claim may not hold. A counterexample is $c_1 = a \vee b$, $c_2 = a$ and $F = \{a \vee b\}$. The other preconditions of the lemma are satisfied: $c_1 \vee c_2$ is in F, where x does not occur; $c_1 \vee x$ is superredundant in $F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee x, c_2 \vee \neg x\}$ since this formula is $\{a \vee b \vee x, a \vee \neg x\}$, whose two clauses resolve in $a \vee b$, which entails $c_1 \vee x = a \vee b \vee x$. The conclusion that $c_1 = a \vee b$ is superredundant in $F \cup \{c_1\} = \{a \vee b\} \cup \{a \vee b\} = \{a \vee b\}$ is false since this formula allows no resolution.

If F mentions x, the claim may not hold. A counterexample is $c_1 = a$, $c_2 = b$ and $F = \{a \vee b, x\}$. The other preconditions of the lemma are satisfied: $c_1 \vee c_2 = a \vee b$ is in F, while c_1 is not; $c_1 \vee x$ is superredundant in $F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee x, c_2 \vee \neg x\}$, which is $\{a \vee b, x\} \backslash \{a \vee b\} \cup \{a \vee x, b \vee \neg x\}$, which is the same as $\{x\} \cup \{a \vee x, b \vee \neg x\}$, where $c_1 \vee x = a \vee x$ is superredundant because it is entailed by x. The conclusion that $c_1 = a$ is superredundant in $F \cup \{c_1\}$ is false since this formula is $\{a \vee b, x\} \cup \{a\} = \{a \vee b, x, a\}$, where no clauses resolve.

The three assumptions do not hinder the intended usage of the lemma: make a clause $c_1 \vee c_2$ of F superirredundant by splitting it into $c_1 \vee x$ and $c_2 \vee \neg x$ on a new variable x. The first assumption is met because $c_1 \vee c_2$ is a clause of F to be made superirredundant. The third is met because x is new. The second is met in the sense that $c_1 \vee c_2$ can just be removed if c_1 is also in the formula.

When the three assumptions are met, the lemma tells that c_1 is superredundant in $F \cup \{c_1\}$ if $c_1 \vee x$ is superredundant in $F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee x, c_2 \vee \neg x\}$. This implication is useful in reverse: $c_1 \vee x$ is superirredundant in $F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee x, c_2 \vee \neg x\}$ unless c_1 is superredundant in $F \cup \{c_1\}$. The goal of making $c_1 \vee x$ superirredundant is hit, but only if c_1 is superirredundant in $F \cup \{c_1\}$.

This condition is necessary. The following example shows it cannot be lifted.

$$F = \{a \vee b, \neg a \vee c, a \vee \neg c\}$$

The last two clauses are equivalent to $a \equiv c$. They make the first clause superredundant because $a \vee b$ derives $c \vee b$ which derives $a \vee b$ back. Splitting does not make the clause superirredundant: $a \vee x$ still derives $c \vee x$, which derives $a \vee x$ back. Removing x from this derivation results in a that derives c that derives a back. Splitting $a \vee b$ does not work because a alone is already superredundant. Adding a new variable x does not change the situation.

4.4 Maintaining the superirredundancy of the other clauses

Preserving the semantics of the formula and making a clause superirredundant is not enough. The other clauses must remain superirredundant. Otherwise, the process may go on forever. Even attempting to have two clauses superirredundant would fail if making one so makes the other not. The final requirement of clause splitting is that the other clauses remain superirredundant. This is mostly the case, with an exception that is discussed after the proof of the lemma.

The lemma is formulated in reverse. Instead of "superirredundancy is maintained except in this condition", it states "superredundancy is generated only in this condition".

Lemma 17 *If c and $c_1 \vee c_2$ are two different clauses of F and c is superredundant in $F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee x, c_2 \vee \neg x\}$ and x does not occur in F then either:*

- *c resolves with both c_1 and c_2; or*
- *c is superredundant in F.*

Proof. Lemma 9 reformulates the superredundancy in the assumption and in the claim.

$$\begin{aligned} F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee x, c_2 \vee \neg x\} \backslash \{c\} \cup \text{resolve}(c, F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee x, c_2 \vee \neg x\}) &\models c \\ F \backslash \{c\} \cup \text{resolve}(c, F) &\models c \end{aligned}$$

Since F contains $c_1 \vee c_2$, it is the same as $F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee c_2\}$. Both this expression and $F \backslash \{c_1 \vee c_2\} \cup \{c_1 \vee x, c_2 \vee \neg x\}$ contain $F' = F \backslash \{c_1 \vee c_2\}$.

$$\begin{aligned} F' \cup \{c_1 \vee x, c_2 \vee \neg x\} \backslash \{c\} \cup \text{resolve}(c, F' \cup \{c_1 \vee x, c_2 \vee \neg x\}) &\models c \\ F' \cup \{c_1 \vee c_2\} \backslash \{c\} \cup \text{resolve}(c, F' \cup \{c_1 \vee c_2\}) &\models c \end{aligned}$$

Resolving a clause with a set is the resolution of the clause with each clause in the set.

$$\begin{aligned} F' \cup \{c_1 \vee x, c_2 \vee \neg x\} \backslash \{c\} \cup \text{resolve}(c, F') \cup \text{resolve}(c, \{c_1 \vee x\}) \cup \text{resolve}(c, \{c_2 \vee \neg x\}) &\models c \\ F' \cup \{c_1 \vee c_2\} \backslash \{c\} \cup \text{resolve}(c, F') \cup \text{resolve}(c, \{c_1 \vee c_2\}) &\models c \end{aligned}$$

The clauses in these entailments are the same except for:

- the first formula contains $c_1 \vee x$, $c_2 \vee \neg x$, $\text{resolve}(c, c_1 \vee x)$ and $\text{resolve}(c, c_2 \vee \neg x)$
- the second formula contains $c_1 \vee c_2$ and $\text{resolve}(c, c_1 \vee c_2)$

The difference depends on whether c resolves with $c_1 \vee x$, $c_2 \vee \neg x$ and $c_1 \vee c_2$. Since c does not contain x, it resolves with $c_1 \vee x$ if and only if it resolves with c_1, and the same for c_2. It resolves with $c_1 \vee c_2$ if it resolves with either c_1 or c_2. All depends on whether c resolves with c_1 or with c_2.

Four cases are possible. Apart from the last case, the claim is proved by removing all clauses containing x from the first formula and adding their resolution, which produces the second formula. Theorem 4 proves that this procedure generates a formula that expresses forgetting x and therefore entails the same consequences that do not contain x, such as c.

1. c resolves with neither c_1 nor c_2

   The three sets resolve$(c, c_1 \vee c_2)$, resolve$(c, c_1 \vee x)$ and resolve$(c, c_2 \vee \neg x)$ are empty. The only other differing clauses are $c_1 \vee x$ and $c_2 \vee \neg x$ in the first formula and $c_1 \vee c_2$ in the second. The first two are the only clauses containing x. Resolving them results in the third. This proves the claim by Theorem 4.

2. c resolves with c_1 but not with c_2

   The two sets resolve$(c, c_1 \vee c_2)$ and resolve$(c, c_1 \vee x)$ contain a clause, but resolve$(c, c_2 \vee \neg x)$ does not since c does not contain x. The only differing clauses between the two formulae are $c_1 \vee x$, $c_2 \vee \neg x$ and resolve$(c, c_1 \vee x)$ in the first and $c_1 \vee c_2$ and resolve$(c, c_1 \vee c_2)$ in the second.

   Only two pairs of clauses contain x with opposite sign: the first is $c_1 \vee x$ and $c_2 \vee \neg x$, the second is resolve$(c, c_1 \vee x)$ and $c_2 \vee \neg x$.

   The first pair resolves into $c_1 \vee c_2$, the first differing clause in the second formula.

   The second pair is shown to resolve in the second differing clause, resolve$(c, c_1 \vee c_2)$. If l is the resolving literal l between c and $c_1 \vee x$, then resolve$(c, c_1 \vee x)$ is $c \vee c_1 \vee x \backslash \{l, \neg l\}$. Since c does not contain x, the resolving literal l cannot be x. As a result, this clause contains x. It therefore resolves with $c_2 \vee \neg x$ into $c \vee c_1 \vee x \backslash \{l, \neg l\} \vee c_2 \vee \neg x \backslash \{x, \neg x\} = c \vee c_1 \backslash \{l, \neg l\} \vee c_2$. Since c_2 does not resolve with c, it does not contain $\neg l$. It does not contain l either since otherwise $c_1 \vee c_2$ would be tautological. The clause is therefore the same as $c \vee c_1 \vee c_2 \backslash \{l, \neg l\}$. This is resolve$(c, c_1 \vee c_2)$, the second differing clause in the second formula.

   This proves that replacing all clauses containing x in the first formula with their resolution produces the second. This proves the claim by Theorem 4.

3. c resolves with c_2 but not with c_1

   Same as the previous case by symmetry.

4. c resolves with both c_1 and c_2

   The claim is proved because its first alternative is exactly that c resolves with both c_1 and c_2.

All of this proves the claim in all four cases. In the first three, replacing all clauses containing x with their resolution in the first formula produces the second; this implies that the two formulae have the same consequences that do not contain x, such as c. This is the first alternative of the claim. The fourth case coincides with the second alternative of the claim. □

Ideally, all clauses would maintain their superirredundancy. This is the case for most but not all. The exception is the clauses that resolve with both parts of the clause that is split. Such clauses invalidate the proof. That raises the question: could the proof be improved to include them? Or do they falsify the statement of the lemma instead? The following example proves the latter.

$$
\begin{array}{rcl}
F & = & \{d, c_1 \vee c_2, a \vee e, \neg e \vee \neg a \vee \neg d\} \\
F'' & = & \{d, c_1 \vee x, \neg x \vee c_2, a \vee e, \neg e \vee \neg a \vee \neg d\} \\
c & = & a \vee b \vee d \vee e \\
c_1 & = & \neg a \vee b \\
c_2 & = & \neg d \vee e
\end{array}
$$

The formula obtained by splitting $c_1 \vee c_2$ is denoted F''. The clause $c_1 \vee c_2 = \neg a \vee b \vee \neg d \vee e$ is superredundant in F because it resolves with $a \vee e$ into $e \vee b \vee \neg d \vee e$, which resolves with $\neg e \vee \neg a \vee \neg d$ back into $\neg a \vee b \vee \neg d \vee e$. To make this clause superirredundant, it is split. However, that makes the first clause $c = a \vee b \vee d \vee e$ superredundant.

That c is superirredundant in F is proved replacing e with true and simplifying the formula. That removes $a \vee e$ and turns $\neg e \vee \neg a \vee \neg d$ into $\neg a \vee \neg d$. What remains is $F[\mathsf{true}/e] = \{a \vee b \vee d \vee e, \neg a \vee b \vee \neg d \vee e, \neg a \vee \neg d\}$. The first clause resolves both with the second and the third, but the result is a tautology in both cases: $F[\mathsf{true}/e] \backslash \{c\} \cup \mathrm{resolve}(c, F)$ is equivalent to $F[\mathsf{true}/e] \backslash \{c\}$, which does not entail c. Lemma 9 proves that c is not superredundant in $F[\mathsf{true}/e]$. Since c contains neither x nor $\neg x$ and F does not contain $c \vee \neg e$, Lemma 12 ensures that c would be superredundant in $F[\mathsf{true}/e]$ if it were in F. But c is not superredundant in $F[\mathsf{true}/e]$. As a result, it is not superredundant in F.

Yet, c is superredundant in F'', the formula after the split: $c = a \vee b \vee d \vee e$ resolves with $c_1 \vee x = \neg a \vee b \vee x$ into $x \vee b \vee d \vee e$; it also resolves with $\neg x \vee c_2 = \neg x \vee \neg d \vee e$ into $\neg x \vee a \vee b \vee e$; the resulting two clauses resolve into c, and therefore imply it. They are the set G that proves c superredundant according to Lemma 6, since they are obtained from F'' by resolution, none of them is c, and they imply c.

If the target was to make both c and $c_1 \vee c_2$ superirredundant, Lemma 17 misses it. Yet, a second shot gets it: $c_1 \vee x$ and $\neg x \vee c_2$ are now superirredundant, but c no longer is; splitting it makes it so:

$$F''' = \{a \vee b \vee y, \neg y \vee d \vee e, \neg a \vee b \vee x, \neg x \vee \neg d \vee e, a \vee e, \neg e \vee \neg a \vee \neg d\}$$

The split separates c into $a \vee b \vee y$ and $\neg y \vee d \vee e$. Both parts resolve with $c_1 \vee c_2$, but this is not a problem because $c_1 \vee c_2$ is no longer in the formula. It has already been split into $c_1 \vee x$ and $\neg x \vee c_2$. The first resolves with $a \vee b$ but not with $d \vee e$, the second with $d \vee e$ but not with $a \vee b$. The original clause $c_1 \vee c_2$ would be made superredundant by this splitting, but its two parts $c_1 \vee x$ and $\neg x \vee c_2$ are not. By first splitting a clause and then the other, both are made superirredundant.

Mission accomplished: if a clause is not superirredundant but should be, splitting it on a new variable makes it so. Subclauses and clauses that resolve with both parts are to be watched out, but the mechanism mostly works.

Making clauses superirredundant nails them to the formula. It forces them in all minimal equivalent CNF formulae. Every CNF formula F is $F' \cup F''$, where F' are its superirredundant clauses; every minimal formula equivalent to F is $F' \cup F'''$. The superirredundant clauses F' are always there. They provide the basement over which the other clauses build upon. They are the skeleton, with its hard bones but also its flexible joins. The muscles, the other clauses, may move it not by bending the bones but by rotating them at the joins. The superirredundant clauses are fixed but may still leave space for other clauses to change. Minimizing $F' \cup F''$ is altering F'' while keeping F'. Is finding a minimal version of F'' that is equivalent to the original formula when F' is always present.

A way to ensure superirredundancy is to make clauses superirredundant. The three lemmas in this section do this:

Lemma 15 proves that splitting a clause $c_1 \vee c_2$ into $c_1 \vee x$ and $c_2 \vee \neg x$ does not change the meaning of the formula except for the new variable x;

Lemma 16 proves that the two parts $c_1 \vee x$ and $c_2 \vee \neg x$ are superirredundant unless c_1 or c_2 are superredundant when added to the formula;

Lemma 17 proves that the other clauses remain superirredundant after the split; the exception are the clauses that resolve with both parts of the split clause; these are made superredundant, but can themselves be split.

5 Example

Superirredundancy is applied to finding a proof of NP-hardness of deciding whether a Horn formula can be compressed in a given size. This problem is known to be NP-complete [11, 4, 12]. The new proof shows that a reduction can be found progressively, by first building a simplified version where some clauses are fixed and then making them superirredundant.

Technically, an instance of the problem comprises a Horn formula A and an integer k; the question is whether a formula B equivalent to A exists with $||B|| \leq k$.

It is proved NP-hard by a reduction from propositional satisfiability: given a CNF formula F, the reduction builds an instance comprising A and k such that A is equivalent to another formula B of size bounded by k if and only if F is satisfiable.

The proof based on superirredundancy simplifies the task of finding such a reduction by assuming that a part A' of A is fixed, that is, is also in every equivalent B. This way, the question turns from the compressibility of A into the compressibility of $A'' = A \backslash A'$.

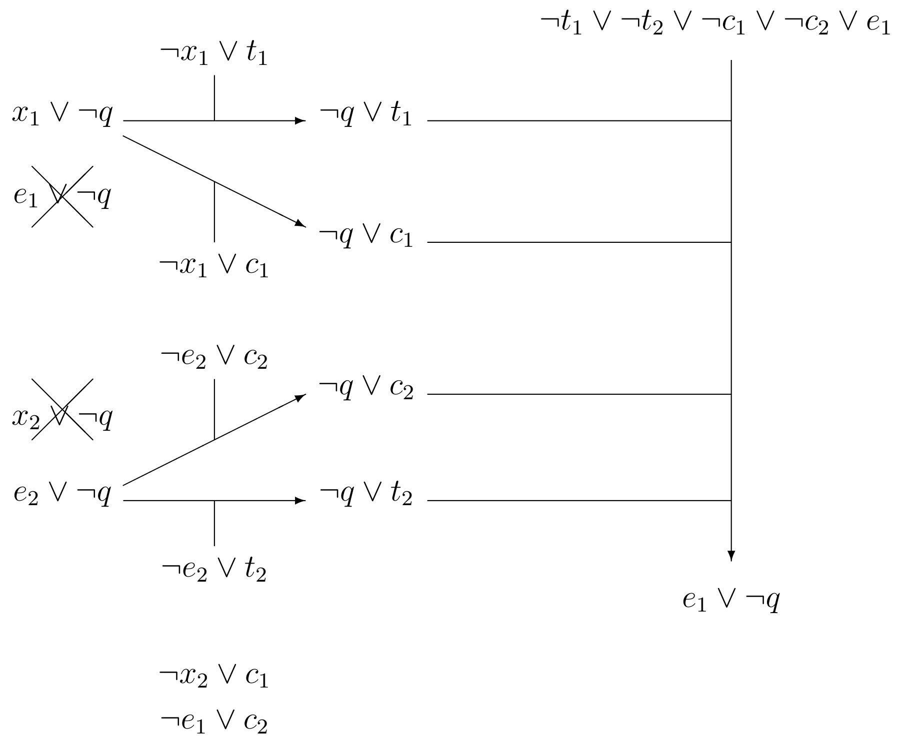
- A'' comprises a clause $x_i \vee \neg q$ and a clause $e_i \vee \neg q$ for every variable in F; this way, every propositional interpretation over the alphabet of F corresponds to a subset of A'', the one containing $x_i \vee \neg q$ if x_i is true and $e_i \vee \neg q$ if false;

- A' ensures that such a subset of A'' entails the rest of A'' if and only if the propositional interpretation satisfies F.

From this roadmap, finding the reduction itself is almost trivial: a clause $t_i \vee \neg q$ is entailed if and only if the subset of A'' includes either $x_i \vee \neg q$ or $e_i \vee q$; another $c_j \vee \neg q$ is entailed if and only if the j-th clause of F is satisfied, which means that the subset of A'' includes the clause that corresponds to a literal of the clause; if all these clauses are entailed, all clauses $x_i \vee \neg q$ and $e_i \vee \neg q$ are entailed.

The clauses allowing these entailments are assumed superirredundant. An example clause of F may be $x_1 \vee x_2$. It is satisfied by setting either x_1 or x_2 to true. The subsets of A'' corresponding to these evaluations respectively include $x_1 \vee \neg q$ and $x_2 \vee \neg q$. The superirredundant clauses $\neg x_1 \vee c_1$ and $\neg x_2 \vee c_1$ allow the derivation of $c_1 \vee \neg q$ by resolution from them.

The following figure shows how the missing clause $e_1 \vee \neg q$ is derived when the formula also contains a second clause $\neg x_1 \vee \neg x_2$. The clauses translate into $\{\neg x_1 \vee c_1, \neg x_2 \vee c_1, \neg e_1 \vee c_2, \neg e_2 \vee c_2\}$. The clauses not included in the subset of A'' are crossed.



The clauses $\neg x_1 \vee t_1$ and $\neg e_2 \vee t_1$ ensure that either $x_1 \vee \neg q$ or $e_1 \vee \neg q$ is included for each index i. Otherwise, a subset of the same size could contain neither while including both $x_2 \vee \neg q$ and $e_2 \vee \neg q$, which is invalid because it corresponds to evaluating x_2 to both true and false.

The clause $\neg t_1 \vee \neg t_2 \vee \neg c_1 \vee \neg c_2 \vee e_1$ completes the derivation. If the subset of A'' corresponds to a propositional interpretation that satisfies both clauses of F, then all clauses $t_i \vee \neg q$ and $c_j \vee \neg q$ are derived. All these clauses resolve into $e_2 \vee \neg q$, which was missing in the subset of A''.

The complete reduction from F to A and k is as follows, where the formula is $F = \{f_1, \ldots, f_m\}$ and $X = \{x_1, \ldots, x_n\}$ are its variables. The formula A is built over an extended alphabet comprising X and the additional variables $E = \{e_1, \ldots, e_n\}$, $T = \{t_1, \ldots, t_n\}$, $C = \{c_1, \ldots, c_m\}$ and q.

$$
\begin{array}{rcl}
A & = & A_F \cup A_T \cup A_C \cup A_B \\
A_F & = & \{x_i \vee \neg q \mid x_i \in X\} \cup \{e_i \vee \neg q \mid x_i \in X\} \\
A_T & = & \{\neg x_i \vee t_i, \neg e_i \vee t_i \mid x_i \in X\} \\
A_C & = & \{\neg x_i \vee c_j \mid x_i \in f_j,\ f_j \in F\} \cup \{\neg e_i \vee c_j \mid \neg x_i \in f_j,\ f_j \in F\} \\
A_B & = & \{\neg t_1 \vee \cdots \vee \neg t_n \vee \neg c_1 \vee \cdots \vee \neg c_m \vee x_i \vee \neg q \mid x_i \in X\} \cup \\
& & \{\neg t_1 \vee \cdots \vee \neg t_n \vee \neg c_1 \vee \cdots \vee \neg c_m \vee e_i \vee \neg q \mid x_i \in X\} \\
k & = & 2 \times n + ||A_T|| + ||A_C|| + ||A_B||
\end{array}
$$

The fixed clauses are all of them but A_F. This way, they are in all formulae equivalent to A. For equivalence, these need to entail all clauses of A_F they do not contain. They can do in size k only by including a clause of $x_i \vee \neg q$ and a clause $e_i \vee \neg q$ for every index i, and they do only if this choice corresponds to a model of F.

This argument assumes that the clauses of A_F are fixed. Superirredundancy ensures that. Lemma 12 ensures superirredundancy. For example, replacing q with false and simplifying the result removes all clauses but $A_T \cup A_C$. These clauses contain only the literals $\neg x_i$, $\neg e_i$, t_i and c_j. They do not contain their negation. Therefore, they do not resolve. Since they are not contained in each other, Lemma 8 proves them superirredundant.

The clauses of A_B are not superirredundant, but can be turned so using the technique of Section 4: each clause $\neg t_1 \vee \cdots \vee \neg t_n \vee \neg c_1 \vee \cdots \vee \neg c_m \vee x_i \vee \neg q$ is split by a new variable r_i. The result is the pair of clauses $\neg t_1 \vee \cdots \vee \neg t_n \vee \neg c_1 \vee \cdots \vee \neg c_m \vee x_i \vee \neg r_i$ and $r_i \vee \neg q$.

This completes the proof. Its construction was incremental. Superirredundancy is initially assumed so that some clauses are considered fixed. They allow deriving the clauses that are not included in the minimal Horn formula if and only if F is satisfiable. Only when the reduction is completed, superirredundancy is actually ensured by splitting the clauses that are not so.

6 Conclusions

Superirredundancy helps to build formulae including clauses that resist minimization in size: a superirredundant clause is in all minimal-size versions of the formula. An application is hardness proofs of minimization problems, like checking whether a formula can be compressed within a certain size. Superirredundancy is not aimed at the minimization itself, but at building formulae that have certain properties, like the targets of hardness reductions. An example shown in this article is an alternative proof of the NP-hardness of the problem of checking whether a Horn formula can be squeezed within a certain bound. Another application is proving the hardness of checking minimal size after forgetting some variables from a Horn or CNF formula [21].

Superredundancy is defined in terms of resolution, not in terms of minimization. The presence of a superirredundant clause in all formulae that are minimal among the equivalent ones is a consequence, not its definition. Superirredundancy is sufficient to that, not necessary. Yet, it is easier to achieve than that. Some conditions that are equivalent to superirredundancy and others that are necessary and still others that are sufficient are presented. A mechanism that often make a clause superirredundant while preserving the superirredundancy of the others is also shown. It allows building a formula incrementally: first its semantics is established, then the clauses that have to be superirredundant and made so.

The example application is an alternative proof of hardness. The claim is already known via prime implicate essentiality instead of superirredundancy [11]. Yet, while this proof surfaced some twenty years after the problem was open, the one based on superirredundancy was very simple to come up with. Its proof of correctness is not much shorter that the previous one, but neither was this its aim. Building the reduction was, not proving it correct.

Why bothering introducing a new notion just for proving again something that was already known? Boolean minimization has been computationally framed in many variants depending on the restriction on the formula and the definition of minimality. Yet, it is not closed. An example open problem is the complexity of checking whether forgetting some variables from a formula is expressed by a formula of a certain size; four hardness proofs are obtained by applying superirredundancy in a separate article [21]. Another example where superirredundancy could be applied is formula revision or update [26, 16]: these transformations are known to potentially increase the size of the changed formula [3]; minimizing it [19] is a problem where superirredundancy could be applied. In general, every mechanism that transforms a formula in whichever way (update, summarize, expand, etc.) is subject to minimizing, and superirredundancy applies. Finally, given that some sufficient conditions to superirredundancy are computationally easy (like replacing variables with values and checking the resulting formula for separation of variables), they may also be used as a simple preliminary test when performing formula minimization.

Superredundancy is a derivation property. As such, it depends on the syntax of the formula. Therefore, it is not the same as any semantical property like implication, prime implication, redundancy in the set of prime implicates or essentiality. It depends on the syntax because it is based on resolution, and resolution is a restricted form of entailment: it does not allow adding arbitrary literals to clauses. In the other way around, entailment is resolution plus expansion. The large corpus of research on automated reasoning [9, 13] offers numerous alternative forms of derivation that work even when formulae are not in clausal form, like natural deduction and Frege systems. Some variant of superredundancy may be defined for them.

Complete proof

The following is the complete proof of correctness of the reduction presented in Section 5.

Theorem 5 *The problem of establishing the existence of a formula B such that $||B|| \leq k$ and $B \equiv A$ is* NP*-hard.*

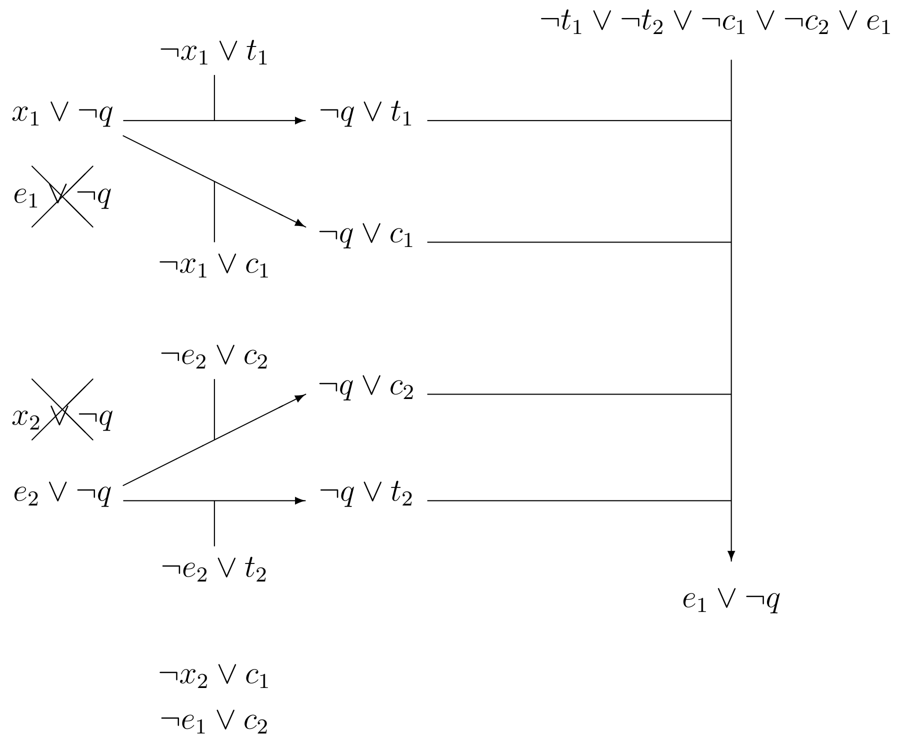
Proof. Proof is by reduction from propositional satisfiability. An arbitrary CNF formula F is shown satisfiable if and only if a Horn formula A is equivalent to one of size bounded by k.

Let the CNF formula be $F = \{f_1, \ldots, f_m\}$ and $X = \{x_1, \ldots, x_n\}$ its variables. The formula A is built over an extended alphabet comprising the variables X and the additional variables $E = \{e_1, \ldots, e_n\}$, $T = \{t_1, \ldots, t_n\}$, $C = \{c_1, \ldots, c_m\}$, $R = \{r_1, \ldots, r_n\}$ and $S = \{s_1, \ldots, s_n\}$, q. The formula A and the integer k are as follows.

$$
\begin{array}{rcl}
A & = & A_F \cup A_T \cup A_C \cup A_B \\
A_F & = & \{x_i \vee \neg q \mid x_i \in X\} \cup \{e_i \vee \neg q \mid x_i \in X\} \\
A_T & = & \{\neg x_i \vee t_i, \neg e_i \vee t_i \mid x_i \in X\} \\
A_C & = & \{\neg x_i \vee c_j \mid x_i \in f_j,\ f_j \in F\} \cup \{\neg e_i \vee c_j \mid \neg x_i \in f_j,\ f_j \in F\} \\
A'_B & = & \{\neg t_1 \vee \cdots \vee \neg t_n \vee \neg c_1 \vee \cdots \vee \neg c_m \vee x_i \vee \neg r_i, r_i \vee \neg q \mid x_i \in X\} \cup \\
 & & \{\neg t_1 \vee \cdots \vee \neg t_n \vee \neg c_1 \vee \cdots \vee \neg c_m \vee e_i \vee \neg s_i, s_i \vee \neg q \mid x_i \in X\}
\end{array}
$$

Before formally proving that the reduction works, a short summary of why it works is given. All clauses of A but A_F are superirredundant: all minimal equivalent formulae contain them. The bound k allows only one clause of A_F for each i. Combined with the clauses of A_T they entail $t_i \vee \neg q$. If F is satisfiable, they also combine with the clauses A_C to imply all clauses $c_j \vee \neg q$. Resolving these clauses with A_B produces all clauses $x_i \vee \neg q$ and $e_i \vee \neg q$, including the ones not in the selection. This way, a formula that contains one clause of A_F for each index i implies all of A_F, but only if F is satisfiable.

The following figure shows how $e_1 \vee q$ is derived from $x_1 \vee q$ and $e_2 \vee q$, when the formula is $F = \{f_1, f_2\}$ where $f_1 = x_1 \vee x_2$ and $f_2 = \neg x_1 \vee \neg x_2$. These clauses translate into $A_C = \{\neg x_1 \vee c_1, \neg x_2 \vee c_1, \neg e_1 \vee c_2, \neg e_2 \vee c_2\}$.



For each index i, at least one among $x_i \vee \neg q$ and $e_i \vee \neg q$ is necessary for deriving $\neg q \vee t_i$, which is entailed by A. Alternatively, $\neg q \vee t_i$ itself is necessary for the formula to be equivalent. Either way, for each index i at least a two-literal clause is necessary.

The claim is formally proved in four steps: first, the superirredundant clauses are identified; second, an equivalent formula of size k is built if F is satisfiable; third, the necessary clauses in every equivalent formula are identified; fourth, if F is unsatisfiable every equivalent formula is proved to have size greater than k.

Superirredundancy.

The claim requires A to be minimal, which follows from all its clauses being superirredundant by Lemma 5. Most of them survive forgetting; the reduction is based on these being superirredundant. Instead of proving superirredundancy in two different but similar formulae, it is proved in their union.

In particular, the clauses $A_T \cup A_C \cup A'_B$ are shown superirredundant in $A_F \cup A_T \cup A_C \cup A'_B$.

Superirredundancy is proved via Lemma 12: a substitution simplify $A_F \cup A_T \cup A_C \cup A'_B$ enough to prove superirredundancy easily, for example because its clauses do not resolve and Lemma 8 applies.

- Replacing all variables x_i, e_i, t_i and c_j with true removes from $A_F \cup A_T \cup A_C \cup A'_B$ all clauses of A_F, A_T, A_C and all clauses of A'_B but $r_i \vee \neg q$ and $s_i \vee \neg q$. The remaining clauses contain only the literals r_i, s_i and $\neg q$. Therefore, they do not resolve. Since none is contained in another, they are all superirredundant by Lemma 8. This proves the superirredundancy of all clauses $r_i \vee \neg q$ and $s_i \vee \neg q$.

- Replacing all variables q, r_i and s_i with false removes from $A_F \cup A_T \cup A_C \cup A'_B$ all clauses but $A_T \cup A_C$. These clauses contain only the literals $\neg x_i$, $\neg e_i$, t_i and c_j. Therefore, they do not resolve. Since they are not contained in each other, Lemma 8 proves them superirredundant.

- Replacing all variables with false except for all variables t_i and c_j and the single variable x_h removes all clauses from $A_F \cup A_T \cup A_C \cup A'_B$ but $\neg x_h \vee t_h$, $\neg t_1 \vee \cdots \vee \neg t_n \vee \neg c_1 \vee \cdots \vee \neg c_m \vee x_h \vee \neg r_h$ and all clauses $\neg x_h \vee c_j$ with $x_h \in f_j$. They only resolve in tautologies. Therefore, their resolution closure only contains them. Removing $\neg t_1 \vee \cdots \vee \neg t_n \vee \neg c_1 \vee \cdots \vee \neg c_m \vee x_h \vee \neg r_h$ from the resolution closure leaves only $\neg x_h \vee t_h$ and all clauses $\neg x_h \vee c_j$ with $x_h \in f_j$. They do not resolve since they do not contain opposite literals. Since $\neg t_1 \vee \cdots \vee \neg t_n \vee \neg c_1 \vee \cdots \vee \neg c_m \vee x_h \vee \neg r_h$ is not contained in them, it is not entailed by them. This proves it superirredundant. A similar replacement proves the superirredundancy of each $\neg t_1 \vee \cdots \vee \neg t_n \vee \neg c_1 \vee \cdots \vee \neg c_m \vee e_h \vee \neg s_h$.

These points prove that the clauses in $A_T \cup A_C \cup A'_B$ are superirredundant in A. The only clauses that may be superredundant are A_F.

Formula F is satisfiable.

Let M be a model satisfying F. The set A'_R is defined as comprising the clauses $x_i \vee \neg q$ such that $M \models x_i$ and the clauses $e_i \vee \neg q$ such that $M \models \neg x_i$. The Horn formula $A'_R \cup A_C \cup A_T \cup A'_B$ has size k. It is equivalent to $A_R \cup A_T \cup A_C \cup A_B$. This is proved by showing that it entails every clause in A_R, including the only clauses of A it does not contain.

Since M satisfies every clause $f_j \in F$, it satisfies at least a literal of f_j: for some x_i, either $x_i \in f_j$ and $M \models x_i$ or $\neg x_i \in f_j$ and $M \models \neg x_i$. By construction, $x_i \in f_j$ implies $\neg x_i \vee c_j \in A_C$ and $\neg x_i \in f_j$ implies $\neg e_i \vee c_j \in A_C$. Again by construction, $M \models x_i$ implies $x_i \vee \neg q \in A'_R$ and $M \models \neg x_i$ implies $e_i \vee \neg q \in A'_R$. As a result, either $x_i \vee \neg q \in A'_R$ and

$\neg x_i \vee c_j \in A_C$ or $e_i \vee \neg q \in A'_R$ and $\neg e_i \vee c_j \in A_C$. In both cases, the two clause resolve in $c_j \vee q$.

Since M satisfies either x_i or $\neg x_i$, either $x_i \vee \neg q \in A'_R$ or $e_i \vee \neg q \in A'_R$. The first clause resolve with $\neg x_i \vee t_i$ and the second with $\neg e_i \vee t_i$. The result is $t_i \vee \neg q$ in both cases.

Resolving all these clauses $t_i \vee \neg q$ and $c_j \vee q$ with $\neg t_1 \vee \cdots \vee \neg t_n \vee \neg c_1 \vee \cdots \vee \neg c_m \vee x_i \vee \neg r_i$ and then with $r_i \vee \neg q$, the result is $x_i \vee \neg q$. In the same way, resolving these clauses with $\neg t_1 \vee \cdots \vee \neg t_n \vee \neg c_1 \vee \cdots \vee \neg c_m \vee e_i \vee \neg s_i$ and $s_i \vee \neg q$ produces $e_i \vee \neg q$. This proves that all clauses of A_R are entailed.

Necessary clauses

All CNF formulae that are equivalent to $A_F \cup A_T \cup A_C \cup A'_B$ and have minimal size contain $A_T \cup A_C \cup A'_B$ because these clauses are superirredundant. Therefore, these formulae are $A_N \cup A_T \cup A_C \cup A'_B$ for some set of clauses A_N. This set A_N is now proved to contain either $x_h \vee \neg q$, $x_h \vee \neg r_i$, $e_h \vee \neg q$, $e_h \vee \neg s_i$ or $t_h \vee \neg q$ for each index h. Let M and M' be the following models.

$$
\begin{aligned}
M &= \{x_i = e_i = t_i = \mathsf{true} \mid i \neq h\} \cup \{x_h = e_h = t_h = \mathsf{false}\} \cup \\
&\quad \{c_j = \mathsf{true}\} \cup \{q = \mathsf{true}\} \cup \{r_i = \mathsf{true}, s_i = \mathsf{true}\} \\
M' &= \{x_i = e_i = t_i = \mathsf{true} \mid i \neq h\} \cup \{x_h = e_h = t_h = \mathsf{true}\} \cup \\
&\quad \{c_j = \mathsf{true}\} \cup \{q = \mathsf{true}\} \cup \{r_i = \mathsf{true}, s_i = \mathsf{true}\}
\end{aligned}
$$

The three clauses are falsified by M. Since the two of them $xh \vee \neg q$ and $e_h \vee \neg q$ are in A_F, this set is also falsified by M. As a result, M is not a model of $A_F \cup A_T \cup A_C \cup A'_B$. This formula is equivalent to $A_N \cup A_T \cup A_C \cup A'_B$, which is therefore falsified by M. In formulae, $M \not\models A_N \cup A_T \cup A_C \cup A'_B$.

The formula $A_N \cup A_T \cup A_C \cup A'_B$ contains a clause falsified by M. Since $M \models A_T \cup A_C \cup A'_B$, this clause is in A_N but not in $A_T \cup A_C \cup A'_B$. In formulae, $M \not\models c$ for some $c \in A_N$ and $c \notin A_T \cup A_C \cup A'_B$. This clause is entailed by $A_F \cup A_T \cup A_C \cup A'_B$ because this formula entails all of $A_N \cup A_T \cup A_C \cup A'_B$, and c is in A_N. In formulae, $A_F \cup A_T \cup A_C \cup A'_B \models c$.

This clause c contains either x_h, e_h or t_h. This is proved by deriving a contradiction from the assumption that c does not contain any of these three literals. Since $M \not\models c$, the clause c contains only literals that are falsified by M. Not all of them: it does not contain x_h, e_h and t_h by assumption. It does not contain $\neg x_h$, $\neg e_h$ and $\neg t_h$ either because it would otherwise be satisfied by M. As a result, c is also falsified by M', which is the same as M but for the values of x_h, e_h and t_h. At the same time, M' satisfies $A_F \cup A_T \cup A_C \cup A'_B$, contradicting $A_F \cup A_T \cup A_C \cup A'_B \models c$. This contradiction proves that c contains either x_h, e_h or t_h.

From the fact that c contains either x_h, e_h or t_h, that is a consequence of $A_F \cup A_T \cup A_C \cup A'_B$, and that is in a minimal-size formula, it is now possible to prove that c contains either $x_h \vee \neg q$, $x_h \vee \neg r_i$, $e_h \vee \neg q$, $e_h \vee \neg s_i$ or $t_h \vee \neg q$.

Since c is entailed by $A_F \cup A_T \cup A_C \cup A'_B$, a subset of c follows from resolution from it: $A_F \cup A_T \cup A_C \cup A'_B \vdash c'$ with $c' \subseteq c$. This implies $A_N \cup A_T \cup A_C \cup A'_B \models c'$ by equivalence. If $c' \subset c$, then $A_N \cup A_T \cup A_C \cup A'_B$ would not be minimal because it contained a non-minimal clause $c \in A_N$. Therefore, $A_F \cup A_T \cup A_C \cup A'_B \vdash c$.

The only two clauses of $A_F \cup A_T \cup A_C \cup A'_B$ that contain x_h are $x_h \vee \neg q$ and $\neg t_1 \vee \cdots \vee \neg t_n \vee \neg c_1 \vee \cdots \vee \neg c_m \vee x_h \vee \neg r_h$. They contain either $\neg q$ or $\neg r_h$. These literals are only resolved out by clauses containing their negations q and r_h. No clause contains q and the only clause that contains r_h is $r_h \vee \neg q$, which contains $\neg q$. If a result of resolution contains x_h, it also contains either $\neg q$ or $\neg r_h$. This applies to c because it is a result of resolution.

The same applies if c contains e_h: it also contains either $\neg q$ or $\neg s_i$.

The case of $t_h \in c$ is a bit different. The only two clauses of $A_F \cup A_T \cup A_C \cup A'_B$ that contain t_h are $\neg x_h \vee t_h$ and $\neg e_h \vee t_h$. Since both are in A_T and $c \notin A_T$, they are not c. The first clause $\neg x_h \vee t_h$ only resolves with $x_i \vee \neg q$ or $\neg t_1 \vee \cdots \vee \neg t_n \vee \neg c_1 \vee \cdots \vee \neg c_m \vee x_h \vee \neg r_h$, but resolving with the latter generates a tautology. The result of resolving $\neg x_h \vee t_h$ with $x_i \vee \neg q$ is $t_h \vee \neg q$; no clause contains q. Therefore, c can only be $t_h \vee \neg q$. The second clause $\neg e_h \vee t_h$ leads to the same conclusion.

In summary, c contains either $x_h \vee \neg q$, $x_h \vee \neg r_i$, $e_h \vee \neg q$, $e_h \vee \neg s_i$ or $t_h \vee \neg q$. In all these cases it contains at least two literals. This is the case for every index h; therefore, A_N contains at least n clauses of two literals. Every minimal CNF formula equivalent to $A_R \cup A_T \cup A_C \cup A_B$ has size at least $2 \times n$ plus the size of $A_T \cup A_C \cup A_B$. This sum is exactly k. This proves that every minimal CNF formula expressing forgetting contains at least k literal occurrences. Worded differently, every CNF formula expressing forgetting has size at least k.

Formula F is unsatisfiable

The claim is that no CNF formula of size k expresses forgetting if F is unsatisfiable. This is proved by deriving a contradiction from the assumption that such a formula exists.

It has been proved that the minimal CNF formulae equivalent to $A_F \cup A_T \cup A_C \cup A'_B$ are $A_N \cup A_T \cup A_C \cup A'_B$ for some set A_N that contains clauses that include either $x_h \vee \neg q$, $x_h \vee \neg r_i$, $e_h \vee \neg q$, $e_h \vee \neg s_i$ or $t_h \vee \neg q$ for each index h.

If A_N contains other clauses, or more than one clause for each h, or these clauses contain other literals, the size of $A_N \cup A_T \cup A_C \cup A'_B$ is larger than $k = 2 \times n + ||A_T|| + ||A_C|| + ||A'_B||$, contradicting the assumption. This proves that every formula of size k that is equivalent to $A_F \cup A_T \cup A_C \cup A'_B$ is equal to $A_N \cup A_T \cup A_C \cup A'_B$ where A_N contains exactly one clause among $x_h \vee \neg q$, $x_h \vee \neg r_i$, $e_h \vee \neg q$, $e_h \vee \neg s_i$ or $t_h \vee \neg q$ for each index h.

The case $x_h \vee \neg r_h \in A_N$ is excluded. It would imply $A_F \cup A_T \cup A_C \cup A'_B \models x_h \vee \neg r_h$, which implies the redundancy of $\neg t_1 \vee \cdots \vee \neg t_n \vee \neg c_1 \vee \cdots \vee \neg c_m \vee x_h \vee \neg r_h \in A_B$ contrary to its previously proved superirredundancy. A similar argument proves $e_h \vee \neg s_h \notin A_N$.

The conclusion is that every formula of size k that is equivalent to $A_F \cup A_T \cup A_C \cup A'_B$ is equal to $A_N \cup A_T \cup A_C \cup A'_B$ where A_N contains exactly one clause among $x_h \vee \neg q$, $e_h \vee \neg q$, $t_h \vee \neg q$ for each index h.

If F is unsatisfiable, all such formulae are proved to be satisfied by a model that falsifies $A_F \cup A_T \cup A_C \cup A'_B$, contrary to the assumed equivalence.

Let M be the model that assigns $q = \mathsf{true}$ and $t_i = \mathsf{true}$, and assigns $x_i = \mathsf{true}$ and $e_i = \mathsf{false}$ if $x_i \vee \neg q \in A_N$ and $x_i = \mathsf{false}$ and $e_i = \mathsf{true}$ if $e_i \vee \neg q \in A_N$ or $t_i \vee \neg q \in A_N$. All clauses of A_N and A_T are satisfied by M.

This model M can be extended to satisfy all clauses of $A_C \cup A'_B$. Since F is unsatisfiable, M falsifies at least a clause $f_j \in F$. Let M' be the model obtained by extending M with the assignments of c_j to false, all other variables in C to true and all variables r_i and s_i to true. This extension satisfies all clauses of A_B either because it sets c_j to false or because it sets

r_i and s_i to true. It also satisfies all clauses of A_C that do not contain c_j because it sets all variables of C but c_j to true.

The only clauses that remain to be proved satisfied are the clauses of A_C that contain c_j. They are $\neg x_i \vee c_j$ for all $x_i \in f_j$ and $\neg e_i \vee c_j$ for all $\neg x_i \in f_j$. Since M' falsifies f_j, it falsifies every $x_i \in f_j$; therefore, it satisfies $\neg x_i \vee c_j$. Since M' falsifies f_j, it falsifies every $\neg x_i \in f_j$; since by construction it assigns e_i opposite to x_i, it falsifies e_i and therefore satisfies $\neg e_i \vee c_j$.

This proves that M' satisfies $A_N \cup A_T \cup A_C \cup A'_B$. It does not satisfy $A_F \cup A_T \cup A_C \cup A'_B$. If $x_1 \vee \neg q \in A_N$, then M' sets x_1 to true and e_1 to false; therefore, it does not satisfy $e_1 \vee \neg q \in A_R$. Otherwise, M' sets x_1 to false and e_1 to true; therefore, it does not satisfy $x_1 \vee \neg q$.

This contradicts the assumption that $A_N \cup A_T \cup A_C \cup A'_B$ is equivalent to $A_F \cup A_T \cup A_C \cup A'_B$. The assumption that it has size k is therefore false. □

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