

Efficient decoding up to a constant fraction of the code length for asymptotically good quantum codes

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Abstract

We introduce and analyse an efficient decoder for the quantum Tanner codes of [LZ22] that can correct adversarial errors of linear weight. Previous decoders for quantum low-density parity-check codes could only handle adversarial errors of weight $O(\sqrt{n \log n})$. We also work on the link between quantum Tanner codes and the Lifted Product codes of Panteleev and Kalachev, and show that our decoder can be adapted to the latter. The decoding algorithm alternates between sequential and parallel procedures and converges in linear time.

1 Introduction

1.1 Contributions

Historically, a major motivation behind the study of classical low-density parity-check (LDPC) codes was the possibility of efficient decoding. In his thesis, Gallager showed that random LDPC codes were asymptotically good codes with high probability and proposed a first decoding algorithm based on message passing [Gal62]. Much later, Sipser and Spielman introduced expander codes which are explicit LDPC codes with a minimum distance linear in their length n , together with an efficient decoder that provably corrects adversarial errors of linear weight [SS96]. These expander codes are a special instance of Tanner codes [Tan81] defined on a Δ -regular expander graph: bits are associated to the edges of the graph while parity-check constraints are enforced at each vertex *via* small linear codes of length Δ .

The LDPC property may be of even greater interest in the quantum case because such codes can significantly reduce the required overhead for fault-tolerant quantum computing [Got14]. For this, it is enough to find a code family encoding $\Theta(n)$ logical qubits within n physical qubits, with a sufficient minimum distance $d = \Omega(n^\alpha)$ for some $\alpha > 0$ and an efficient decoding algorithm. Quantum expander codes, which are obtained by taking the hypergraph product [TZ14] of two expander codes, form one such family

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when combined with the small-set-flip decoder [LTZ15, FGL18]. This decoder corrects adversarial errors of weight proportional to the minimum distance, that is $O(\sqrt{n})$. Very recently, Panteleev and Kalachev discovered a first family of asymptotically good quantum LDPC codes [PK21a] and suggested that a decoder similar to that of quantum expander codes might be able to correct errors of large weight in linear time.

In the present paper, we describe such a linear-time decoder for quantum Tanner codes [LZ22], a family of good quantum LDPC codes obtained by applying the Tanner code construction to a Left-Right Cayley complex [DEL⁺21, BE21] rather than to a graph. This decoder can correct adversarial errors of linear weight. Previously, the best decoder available was able to correct error of weight $O(\sqrt{n \log n})$ [EKZ20].

1.2 Context and history

Quantum LDPC codes. Quantum error correcting codes (QECC), initially devised to fight decoherence with the goal of building large-scale quantum computers, have become a central object of study in fields as diverse as computer science, topological phases in physics, quantum information and even quantum gravity. Probably the best-known QECC is the toric code of Kitaev which corresponds to the degenerate ground-space of a simple local Hamiltonian defined on a torus [Kit03]. It is a 2^2 -dimensional subspace encoding 2 qubits within the Hilbert space $(\mathbb{C}^2)^{\otimes n}$ of n physical qubits, defined as the common eigenspace with eigenvalue 1 of a set of $n-2$ commuting Pauli operators, called *generators*. The *minimum distance* of a code is the minimal weight¹ of a Pauli operator that commutes with all the generators but that cannot be written as a product of generators. The toric code achieves a remarkable distance of $\Theta(\sqrt{n})$, which remains the current record for quantum codes with generators acting locally in a topological space of constant dimension. In general, the parameters of such local codes are rather severely constrained by the ambient space [BT09, BPT10, Del13, BK22], and it is useful to relax the locality constraint and only require that the generators of the code have constant weight, but are otherwise arbitrary². In addition, we ask that any qubit is involved in at most a constant number of generators. Such QECC are called LDPC.

An important stepping stone in the study of quantum LDPC codes was the constant-rate³ generalisation of the toric code called hypergraph (or homological) product construction that forms a quantum LDPC code from two arbitrary classical LDPC codes [TZ14]. In particular, if the classical codes are asymptotically good, then the resulting QECC has constant rate and a minimum distance $d = \Theta(\sqrt{n})$. Ideas from higher-dimensional expansion were useful to break the so-called *square-root barrier* for the minimum distance, but only by polylogarithmic factors [EKZ20, KT21]. A much more impressive improvement came from the idea of adding a twist to the homological product construction [HHO20] to obtain $d = \tilde{\Theta}(n^{3/5})$. Further generalizations, either balanced product codes or lifted product codes, were developed in [BE21] and [PK20] and finally

¹The weight of a Pauli operator is the number of qubits on which the operator acts nontrivially.

²We do not require the generators to be geometrically local anymore, and they can therefore act on an arbitrary set of qubits. This set should simply be of constant size.

³The rate of a QECC is the ratio k/n of the number k of logical qubits per physical qubit.

led to the asymptotically good quantum LDPC codes of Panteleev and Kalachev [PK21a]. Hidden behind this final construction lies a higher-dimensional generalization of a graph called a Left-Right Cayley complex introduced in [DEL⁺21] to construct (classical) locally testable codes. Combining this complex, which is a balanced product of Cayley graphs, with the Tanner code construction of [Tan81] yields an alternative family of good quantum LDPC codes called quantum Tanner codes [LZ22]. It should be emphasized that before these recent works, it was completely unclear whether quantum LDPC codes with a minimum distance significantly above \sqrt{n} could exist at all.

Decoders for quantum LDPC codes. Topological codes, such as the toric code and its various generalizations in higher dimensions, are by far the most studied quantum LDPC codes. In particular, they now come with relatively efficient decoders that solve the following problem: given the syndrome of a physical Pauli error e , *i.e.* the list of generators that do not commute with the error, return a guess \hat{e} for the error such that e and \hat{e} differ by a product of generators⁴, which is sufficient since generators act trivially on the codespace. This is a crucial relaxation compared to the classical decoding problem which requires to recover the exact error. While this could suggest at first sight that the decoding problem is simpler in the quantum case, this is in fact far from clear. A typical issue arises when the error corresponds to half a generator. In that case, it might not be clear for the decoder how to break the symmetry between the two equivalent errors corresponding to each half of the generator. A related observation is that the value of the error on a given qubit is never well defined since it is straightforward to find equivalent errors with a distinct action on that qubit. These issues are by now relatively well understood for topological codes and the minimum weight perfect matching decoder [Edm65] yields a decoder with very good performance and a reasonable complexity of $O(n^3)$. Faster decoders exist for topological codes, for instance Union-Find has essentially a linear complexity in the worst case and performs optimally for errors of weight less than $(d-1)/2$ [DN21].

Decoding more general, non topological LDPC codes, appears to be more challenging. While some solutions behave reasonably well against random errors, most notably the combination of Belief Propagation and Ordered Statistics Decoding proposed in [PK21b], an important challenge is to understand how well one can correct adversarial errors. Inspired by a decoder of Hastings [Has14] for 4-dimensional hyperbolic codes [GL14], the small-set-flip (SSF) decoder of quantum expander codes can correct $\Theta(\sqrt{n})$ errors and works in a greedy fashion by trying to find small local patterns that can decrease the syndrome weight [LTZ15]. The decoder of [EKZ20] exploits the homological product structure in a more global way to (slightly) beat the \sqrt{n} bound. The recent invention of good quantum LDPC codes raises the natural question of whether one can indeed correct adversarial errors of linear weight in polynomial, or even linear time⁵. We answer this question in the affirmative, but note that our decoder is somewhat

⁴In this case, e and \hat{e} are called *equivalent*.

⁵We note that [LH22] studies such a decoder for a possible construction of asymptotically good quantum LDPC codes that relies on a conjecture about the existence of 2-sided lossless expander graphs

more complicated than SSF. In particular, while the *normal* mode of the decoder is very similar to SSF, we also need to consider an *exceptional* mode to take care of potentially problematic error patterns.

The paper is organised as follows. Section 2 gives an overview of the paper, recalls the construction of the quantum Tanner codes, describes the decoding algorithm and states our main result showing that the decoder corrects all error patterns of weight below a constant fraction of the code length. Section 3 is a preliminary to the detailed part of the paper and introduces the required technical material. Section 4 gives a detailed description of the quantum Tanner codes which rely here on a quadripartite version of the left-right Cayley complex and not a bipartite version as in [LZ22], since this would complicate the exposition. Similarly, Section 5 discusses the decoding algorithm and the tools used in its analysis. Section 6 is the core of the paper, giving the precise description of the decoder and its detailed analysis, and establishing the main theorem. Finally, Section 7 explores the links between the quantum Tanner codes and the lifted product codes of Pantelev and Kalachev, and explains how our decoding algorithm yields an efficient decoder for the lifted product codes as well, thereby solving an open problem of [PK21a].

2 Overview

The left-right Cayley complex. We recall the construction of the square complex of Dinur *et al.* [DEL⁺21]. It is an incidence structure X between a set V of vertices, two sets of edges E_A and E_B , that we will refer to as A -edges and B -edges, and a set Q of squares (or quadrangles). The vertex-set V is defined from a group G . While we considered a bipartite complex in [LZ22], we take here a quadripartite version as it will simplify the exposition. The vertex set is therefore partitioned as $V = V_{00} \cup V_{01} \cup V_{10} \cup V_{11}$, with each part identified as a copy of the group G . Formally, we set $V_{ij} = G \times \{ij\}$ for $i, j \in \{0, 1\}$. We also have two self-inverse subsets $A = A^{-1}$ and $B = B^{-1}$ of the group G : for $i \in \{0, 1\}$, a vertex $v = (g, i0) \in V_{i0}$ and a vertex $v' = (g', i1) \in V_{i1}$ are said to be related by an A -edge if $g' = ag$ for some $a \in A$. Similarly, for $j \in \{0, 1\}$, vertices $v = (g, 0j)$ and $v' = (gb, 1j)$ are said to be related by a B -edge if $g' = gb$ for some $b \in B$. The sets E_A and E_B make up the set of A -edges and B -edges respectively, and define graphs \mathcal{G}_A and \mathcal{G}_B where \mathcal{G}_A consists of two copies of the double cover of the *left* Cayley graph $\text{Cay}(G, A)$ and \mathcal{G}_B consists of two copies of the double cover of the *right* Cayley graph $\text{Cay}(G, B)$.

Next, the set Q of squares is defined as the set of 4-subsets of vertices of the form

$$\{(g, 00), (ag, 01), (gb, 10), (agb, 11)\}.$$

The four vertices of a square therefore belong to the four distinct copies of G , as depicted on Figure 1.

with free group action.

An advantage of the quadripartite version we consider here is that we do not need to enforce any additional constraint on G, A, B such as the Total No-Conjugacy condition defined in [DEL⁺21] requiring that $ag \neq gb$ for all choices of g, a, b .

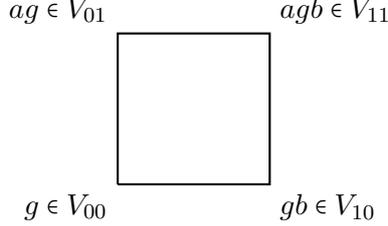


Figure 1: Square $\{(g, 00), (ag, 01), (agb, 11), (gb, 10)\}$ of the complex.

If we restrict the vertex set to $V_0 := V_{00} \cup V_{11}$, every square is now incident to only two vertices: one in V_0 and one in V_{11} . The set of squares can then be seen as a set of edges on V_0 , and it therefore defines a bipartite graph that we denote by $\mathcal{G}_0^\square = (V_0, Q)$. Similarly, the restriction to the vertices of $V_1 := V_{01} \cup V_{10}$ defines the graph \mathcal{G}_1^\square , which is an exact replica of \mathcal{G}_0^\square : both graphs are defined over two copies of the group G , with $g, g' \in G$ being related by an edge whenever $g' = agb$ for some $a \in A, b \in B$. We assume for simplicity that A and B are of the same cardinality Δ . For any vertex v , we denote by $Q(v)$ the Q -neighbourhood (or link) of v which is defined as the set of squares incident to v . The Q -neighbourhood $Q(v)$ has cardinality Δ^2 and is isomorphic to the product set $A \times B$: the situation is illustrated on Figure 2 and discussed in detail in Section 4.

Quantum Tanner codes on the complex X . A *Tanner code*, or expander code, on a Δ -regular graph $\mathcal{G} = (V, E)$ is the set of binary vectors indexed by E (functions from E to \mathbb{F}_2), such that on the edge neighbourhood of every vertex $v \in V$, we see a codeword of a small code C of length Δ [Tan81, SS96]. We denote the resulting code by $\text{Tan}(\mathcal{G}, C) \subset \mathbb{F}_2^E$.

Following [LZ22], we consider *quantum Tanner codes* which are quantum CSS codes formed by two classical Tanner codes \mathcal{C}_0 and \mathcal{C}_1 with support on the set Q of squares of a Left-Right Cayley complex. The CSS construction requires both codes to satisfy the orthogonality condition $\mathcal{C}_0^\perp \subset \mathcal{C}_1$. Enforcing this condition requires some care for the choice of the local codes of \mathcal{C}_0 and \mathcal{C}_1 . We will define local codes on the space $\mathbb{F}_2^{A \times B}$ that we may think of as the space of matrices whose rows (columns) are indexed by A (by B). If $C_A \subset \mathbb{F}_2^A$ and $C_B \subset \mathbb{F}_2^B$ are two linear codes, we define the *tensor* (or product) code $C_A \otimes C_B$ as the space of matrices x such that for every $b \in B$ the column vector $(x_{ab})_{a \in A}$ belongs to C_A and for every $a \in A$ the row vector $(x_{ab})_{b \in B}$ belongs to C_B . Recall that the dual C^\perp of a code C is the set of words orthogonal to all words in C :

$$C^\perp := \{x \in \mathbb{F}_2^{|C|} : \langle x, y \rangle = 0 \ \forall y \in C\}.$$

We finally define \mathcal{C}_0 and \mathcal{C}_1 to be the following classical Tanner codes:

$$\mathcal{C}_0 = \text{Tan}(\mathcal{G}_0^\square, (C_A \otimes C_B)^\perp), \quad \mathcal{C}_1 = \text{Tan}(\mathcal{G}_1^\square, (C_A^\perp \otimes C_B^\perp)^\perp),$$

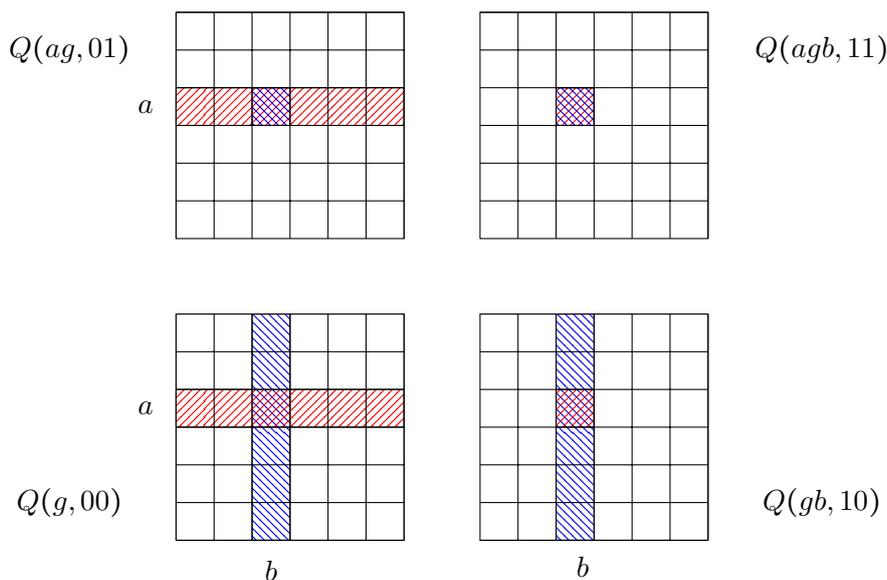


Figure 2: The four local views $Q(v)$ that contain the square $\{(g,00), (ag,01), (agh,11), (gb,10)\}$. The views of two vertices connected by an A -edge (resp. a B -edge) share a row depicted in red (resp. a column in blue). The labeling is chosen to ensure that a given square, such as the one in red and blue, is indexed similarly, by (a,b) here, in the four local views. The σ_X -type generators are codewords of $C_A \otimes C_B$ in the local views of $V_{00} \cup V_{11}$; the σ_Z -type generators are codewords of $C_A^\perp \otimes C_B^\perp$ in the local views of $V_{01} \cup V_{10}$. They automatically commute since their support can only intersect on a shared row or column (as depicted), and the orthogonality of the local codes ensure that they commute on this row or column.

with bits associated to each square of Q and local constraints enforced at the vertices of V_0 and V_1 , respectively. Let us denote $C_0 := C_A \otimes C_B$ and $C_1 := C_A^\perp \otimes C_B^\perp$, so that $\mathcal{C}_i = \text{Tan}(\mathcal{G}_i^\square, C_i^\perp)$ for $i \in \{0,1\}$. To check the orthogonality condition between the two codes, it is convenient to look at their generators (or parity-checks). We define a C_0 -generator for \mathcal{C}_0 (resp. a C_1 -generator for \mathcal{C}_1) as a vector of \mathbb{F}_2^Q whose support lies entirely in the Q -neighborhood $Q(v)$ of V_0 (resp. V_1), and which is equal to a codeword of C_0 (resp. C_1) on $Q(v)$. The Tanner code \mathcal{C}_0 (resp. \mathcal{C}_1) is defined as the set of vectors orthogonal to all C_0 -generators (resp. C_1 -generators). The condition $\mathcal{C}_0^\perp \subset \mathcal{C}_1$ simply says that all C_0 -generators are orthogonal to all C_1 -generators: this follows from the fact that if a C_0 -generator on $v_0 \in V_0$ and a C_1 -generator on $v_1 \in V_1$ have intersecting supports, then v_0 and v_1 must be neighbours in the left-right Cayley complex and their local views must intersect on either a column or a row, on which the two generators equal codewords of C_A and C_A^\perp , or of C_B and C_B^\perp (see Fig. 2).

We denote by $\mathcal{Q} = (\mathcal{C}_0, \mathcal{C}_1)$ the *quantum Tanner code* obtained in this way from \mathcal{C}_0 and \mathcal{C}_1 , and recall the main result from [LZ22].

Theorem 1 (Quantum Tanner codes are asymptotically good [LZ22]). *There exists an infinite family of square complexes X such that the following holds. For any $\rho \in (0, 1/2)$, $\varepsilon \in (0, 1/2)$ and $\delta > 0$ satisfying $-\delta \log_2 \delta - (1 - \delta) \log_2(1 - \delta) < \rho$, randomly choosing C_A and C_B of rates ρ and $1 - \rho$ yields, with probability > 0 for Δ large enough, an infinite sequence of quantum codes $\mathcal{Q} = (\mathcal{C}_0, \mathcal{C}_1)$ of rate $(2\rho - 1)^2$, length n and minimum distance $\geq \delta n / 4\Delta^{3/2+\varepsilon}$.*

Concretely, the complexes X that work in Theorem 1 are all complexes such that $\text{Cay}(G, A)$ and $\text{Cay}(G, B)$ are Ramanujan graphs and for which Δ is fixed, independent of the complex size, and large enough.

The proof of Theorem 1 in [LZ22] went along the following lines: If one considers a codeword \mathbf{x} of \mathcal{C}_1^\square and identifies it with its support, a set of squares that reduces to a set of edges in the graph \mathcal{G}_1^\square , then this is a subgraph of minimum degree at least $\delta\Delta$, a lower bound on the minimum distance of the component dual tensor code. Expansion in \mathcal{G}_1^\square is not quite enough to deduce that $|\mathbf{x}|$ must be large, but if $|\mathbf{x}|$ is small, expansion does tell us that its local views on vertices of V_1 must have a small weight (close to Δ) on average. If we decompose these local views, which have the structure of dual tensor codewords on $A \times B$, as sums of column vectors, *i.e.* elements of $C_A \otimes \mathbb{F}_2^B$, and row vectors in $\mathbb{F}_2^A \otimes C_B$, then, the definition of the complex X tells us that the individual column and row vectors from these local views also exist in the Q -neighbourhoods of vertices of V_0 . Switching to expansion in the graphs \mathcal{G}_A and \mathcal{G}_B , we obtain that most individual column C_A -codewords and row C_B -codewords from the Q -neighbourhoods of vertices of V_1 , must cluster around local views of vertices of V_0 . A local analysis of these clustered local view then shows that adding some non-zero tensor codeword must reduce its Hamming weight, and in this way one obtains iteratively that the global codeword \mathbf{x} of \mathcal{C}_1 can be expressed as a sum of generators, which yields the lower bound on the minimum distance.

Robustness of the component codes C_A and C_B . Crucial to the analysis sketched above is the ability to claim that if a dual tensor codeword $x = (x_{ab}) \in \mathbb{F}_2^{A \times B}$ has sufficiently small weight, then it can be expressed as a sum $x = r + c$, where the union of row codewords $r \in \mathbb{F}_2^A \otimes C_B$ and the union of column codewords $c \in C_A \otimes \mathbb{F}_2^B$ are both of small weight. This is a *robustness property*. More precisely, we say that a dual tensor code $(C_A^\perp \otimes C_B^\perp)^\perp = C_A \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B$ is w -robust if any codeword x of weight $\leq w$ has its support included in the union of $|x|/d_A$ columns and $|x|/d_B$ rows, where d_A and d_B are the minimum distances of C_A and C_B . A similar notion is used both in [DEL⁺21] and [PK21a]. What was shown in [LZ22] is that for any $\varepsilon > 0$ and large enough Δ , when C_A and C_B are chosen at random, then the dual tensor codes $(C_A \otimes C_B)^\perp$ and $(C_A^\perp \otimes C_B^\perp)^\perp$ are both $\Delta^{3/2-\varepsilon}$ -robust with high probability. This property will again be crucial when considering decoding issues.

The decoding problem. A standard property of QECC is that the ability to correct all Pauli errors of weight up to t implies the ability to correct *arbitrary* errors of weight less than t . For a CSS code, bit flips and phase flips can be decoded independently, and

it is therefore sufficient to consider only one type of errors. Since we chose C_A and C_B of rate ρ and $1 - \rho$, respectively, we see that the classical Tanner codes \mathcal{C}_0 and \mathcal{C}_1 have the same parameters and that the resulting quantum Tanner codes will protect equally well against bit flips and phase flips. Without loss of generality, we therefore consider a phase-flip error with support on $e \in \mathbb{F}_2^Q$. This error is detected by the classical Tanner code \mathcal{C}_1 , and the goal of the decoder is to output an error candidate \hat{e} such that $e + \hat{e} \in \mathcal{C}_0^\perp$. Recall indeed that an element of \mathcal{C}_0^\perp is a sum of generators, and therefore acts trivially on the codespace. For this reason, it is not necessary to recover the exact error e , and any *equivalent* error in the coset $e + \mathcal{C}_0^\perp$ is equally good.

An approach mentioned in [LZ22] for decoding (classical) Tanner codes is to define a *mismatch* vector that summarises how the local decoders associated to each local code may disagree about the error, and then try to locally modify this mismatch in order to reduce its weight. It is natural to see the error e as a collection of local views on the vertices of V_1 : abusing notation slightly, we can write $e = \{e_v\}_{v \in V_1}$, where we see the local views e_v both as small length vectors in $\mathbb{F}_2^{Q(v)}$ and as vectors of \mathbb{F}_2^Q with 0 coordinate values outside $Q(v)$. Since each square of Q belongs both to a local view of V_{01} and to a local view of V_{10} , we have that $\sum_{v \in V_{01}} e_v = \sum_{v \in V_{10}} e_v$. For each vertex $v \in V_1$, one can compute a local error ε_v with support on $Q(v)$ of minimal Hamming weight yielding the corresponding local syndrome. This gives a decomposition of the local views of the error

$$e_v = \varepsilon_v + c_v + r_v,$$

with $c_v \in C_A \otimes \mathbb{F}_2^B$, $r_v = \mathbb{F}_2^A \otimes C_B$, and ε_v of minimal Hamming weight. Here, we have that $c_v + r_v$ is a codeword of the dual tensor code $C_1^\perp = C_A \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B$. The issue is that the local views $\{\varepsilon_v\}_{v \in V_1}$ are in general not consistent and do not define a global error candidate. We measure this inconsistency by defining the mismatch vector

$$Z := \sum_{v \in V_1} \varepsilon_v \in \mathbb{F}_2^Q. \quad (1)$$

If it is equal to zero, it means that each square/qubit is affected the same value for the two views it belong to, and the decoder is able to define a global error. Otherwise, the support of Z corresponds to the set of squares for which the local views disagree. Exploiting the previous remark that $\sum_{v \in V_1} e_v = 0$, we can rewrite the mismatch as

$$Z = \sum_{v \in V_1} r_v + c_v = C_0 + R_0 + C_1 + R_1,$$

where we defined

$$C_0 = \sum_{v \in V_{10}} c_v, \quad R_0 = \sum_{v \in V_{01}} r_v, \quad C_1 = \sum_{v \in V_{01}} c_v, \quad R_1 = \sum_{v \in V_{10}} r_v.$$

The idea behind our decoder is to find a decomposition $\{\hat{r}_v, \hat{c}_v\}_{v \in V_1}$ such that $Z = \sum_{v \in V_1} \hat{r}_v + \hat{c}_v$. In that case, the decoder will output the error candidate $\hat{e} = \{\hat{e}_v\}_{v \in V_1}$ with

$$\hat{e}_v := \varepsilon_v + \hat{r}_v + \hat{c}_v. \quad (2)$$

In particular, the vectors \mathbf{e} and $\hat{\mathbf{e}}$ differ by an element of \mathcal{C}_1 (since they have the same syndrome on V_1), and a sufficient condition to guarantee the success of the decoder is that $|\mathbf{e} + \hat{\mathbf{e}}|$ is less than the minimum distance. In that case, it means that \mathbf{e} and $\hat{\mathbf{e}}$ necessarily differ by an element of \mathcal{C}_0^\perp , that is a sum of generators. It is not difficult to see that $|Z| = O(|\mathbf{e}|)$. Therefore if $|\mathbf{e}| \leq \kappa n$ for some sufficiently small $\kappa > 0$ and if the algorithm can find $\hat{\mathbf{e}}$ of weight at most $O(|Z|)$, then the decoder will return a correct solution since the minimum distance is linear in n .

We now describe the main subroutine of decoding algorithm, which aims at finding a decomposition $Z = \sum_{v \in V_1} \hat{r}_v + \hat{c}_v$. We will keep track of 5 vectors initialized as follows

$$\hat{Z} := Z, \quad \hat{C}_0 := 0, \quad \hat{R}_0 := 0, \quad \hat{C}_1 := 0, \quad \hat{R}_1 := 0.$$

It will alternate between two procedures:

Sequential procedure in V_0 . While there exists some $v \in V_{0i}$ and $c_v \in C_A \otimes \mathbb{F}_2^B, r_v \in \mathbb{F}_2^A \otimes C_B$ such that $|\hat{Z} + c_v + r_v| < |\hat{Z}|$, perform the update:

$$\hat{Z} \leftarrow \hat{Z} + c_v + r_v, \quad \hat{C}_i \leftarrow \hat{C}_i + c_v, \quad \hat{R}_i \leftarrow \hat{R}_i + r_v.$$

Now it may happen that this sequential decoder will get stalled at some point, and not be able to decrease the Hamming weight $|\hat{Z}|$ by a local modification on a Q -neighbourhood of a vertex of V_0 . When this happens, we can consider the subgraph of \mathcal{G}_0^\square induced by the vector $(R_0 + C_0) \cap (R_1 + C_1)$ and realise that its minimum degree must be at least $\delta\Delta/2$ (provided the initial error \mathbf{e} is of sufficiently small weight). Conceptually, this subgraph does not look very different from a Tanner codeword, and if we follow the blueprint of the proof of the lower bound on the minimum distance (sketched just after Theorem 1), then it is natural to expect that the stalled decoder will be unlocked simply by switching the sequential decoding procedure to vertices of V_1 . However, we cannot quite make this work. The issue is that we again need robustness of the component codes C_A, C_B , and this time the robustness parameter $w = \Delta^{1/2-\varepsilon}$ guaranteed us by random choice falls just short. To circumvent this problem, we use a more complicated decoding procedure to unlock the stalled sequential decoder. It consists of two rounds of parallel decoding.

Parallel decoding procedure.

- **First parallel decoding step, on vertices of V_1 .** Identify vertices of V_1 for which there exists $c_v \in C_A \otimes \mathbb{F}_2^B, r_v \in \mathbb{F}_2^A \otimes C_B$ and subsets $A_0 \subset A$ and $B_0 \subset B$ of the local coordinate sets that are sufficiently large and such that the Hamming weight of $\hat{Z} + c_v + r_v$ decreases sufficiently *on the reduced coordinate set* $A_0 \times B_0$. For all vertices v where this is satisfied, update as before:

$$\hat{Z} \leftarrow \hat{Z} + c_v + r_v, \quad \hat{C}_j \leftarrow \hat{C}_j + c_v, \quad \hat{R}_i \leftarrow \hat{R}_i + r_v.$$

See Section 6.1 for a precise description of the criteria for updating.

- **Second parallel decoding step, on vertices of V_0 .** Identify vertices of V_0 for which there exists $c_v \in C_A \otimes \mathbb{F}_2^B, r_v \in \mathbb{F}_2^A \otimes C_B$ such that $|\hat{Z} + c_v + r_v| < |\hat{Z}|$. Among all possible choices, maximise the difference $|\hat{Z}| - |\hat{Z} + c_v + r_v|$. For all such vertices perform the update

$$\hat{Z} \leftarrow \hat{Z} + c_v + r_v, \quad \hat{C}_j \leftarrow \hat{C}_j + c_v, \quad \hat{R}_i \leftarrow \hat{R}_i + r_v.$$

What the decoder does by default is apply the sequential decoding procedure. Whenever this is not possible, it applies the parallel decoding procedure. It continues until $\hat{Z} = 0$, at which point it stops and outputs $(\hat{R}_0, \hat{C}_0, \hat{R}_1, \hat{C}_1)$.

A word of comment is in order here. As mentioned above, what happens is that when the sequential decoder is stalled, we cannot guarantee the existence of a Q -neighbourhood $Q(v)$, for $v \in V_1$, on which we can decrease $|\hat{Z}|$. But this is almost the case: what we can guarantee is the existence of vertices v for which one can decrease $|\hat{Z}|$ on a large subset of $Q(v)$. Furthermore, this will be the case for most vertices of V_1 on whose Q -neighbourhoods \hat{Z} has a large Hamming weight: this is why we apply this tweaked local decoding procedure in parallel on all possible vertices of V_1 . After this first parallel decoding step we cannot guarantee that $|\hat{Z}|$ has decreased, however we can guarantee a substantial decrease of $|\hat{Z}|$ after the *second* parallel decoding step described above.

Finally, the output of the decoder gives us a decomposition $Z = \hat{C}_0 + \hat{R}_0 + \hat{C}_1 + \hat{R}_1$ of the original mismatch (1). For $v \in V_{01}$, the local view \hat{r}_v of \hat{R}_0 and the local view \hat{c}_v of \hat{C}_1 give the required local view (2) of a candidate global error vector \hat{e} , while for $v \in V_{10}$ the local view \hat{e}_v of \hat{e} stems from the local views of \hat{R}_0 and \hat{C}_1 .

Our main result states that the above decoder will always succeed in producing an adequate error vector \hat{e} , provided the initial error weight $|e|$ is a sufficiently small fraction of the code length.

Theorem 2. *There exists a constant κ , depending only on δ , a lower bound for the minimum distances of both component codes C_A and C_B , such that for large enough fixed Δ , the above decoding algorithm corrects all error patterns of weight less than $\kappa n / \Delta^4$ for the quantum Tanner code of length $n = |Q|$.*

Since the decoder needs only to look at Q -neighbourhoods on which the mismatch $|Z|$ is nonzero, by carefully keeping track of this set of vertices, we obtain a decoder that runs in linear time.

A direct consequence of the analysis is the soundness of the code, a weaker property than local testability that asserts that sufficiently small errors have a syndrome with a weight proportional to that of the error (local testability would require this to hold for arbitrary errors). This result was previously established for quantum Tanner codes in [HL22].

Interestingly, our decoder can be adapted to work with the asymptotically good codes of Panteleev and Kalachev [PK21a]. In particular, we show that the theorem above still holds for these codes, provided the error patterns have weight at most $\kappa n / \Delta^6$. Along the way, we point out the hidden relation between the quantum Tanner codes and the lifted product codes of [PK21a].

Additional comments. So as to not let the analysis of the decoder become overly burdensome, we have not tried to make the constant κ explicit in Theorem 2, nor have we tried to optimise the dependency in Δ of the number of correctable errors. For a similar reason, we have left out possible variations on the decoder. In particular, since one has to resort to parallel decoding when the sequential decoder is stalled, a natural temptation is to make the decoder fully parallel and replace the sequential decoding step by a parallel one. We do not anticipate difficulties of a novel nature in doing so, but our analysis stems rather naturally from a sequential approach to decoding, and we have tried to limit the flow of technicalities by avoiding these variations.

As mentioned in [PK21a] and [LZ22], the question of alternatives to random choice for the component codes C_A, C_B remains open. Additionally, finding component codes C_A, C_B with better robustness would simplify our analysis and yield a fully sequential decoder. On the other hand, showing that decoding a linear number of adversarial errors is possible with reduced robustness potentially simplifies the search for constructions of adequate component codes.

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3 Preliminaries

3.1 Expander Graphs

Let $\mathcal{G} = (V, E)$ be a graph. Graphs will be undirected but may have multiple edges. For $S, T \subset V$, let $E(S, T)$ denote the multiset of edges with one endpoint in S and one endpoint in T . Let \mathcal{G} be a Δ -regular graph on n vertices, and let $\Delta = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of the adjacency matrix of \mathcal{G} . For $n \geq 3$, we define $\lambda(\mathcal{G}) := \max\{|\lambda_i|, \lambda_i \neq \pm\Delta\}$. The graph \mathcal{G} is said to be *Ramanujan* if $\lambda(\mathcal{G}) \leq 2\sqrt{\Delta - 1}$.

We recall the following version of the expander mixing lemma (see *e.g.* [HLW06]) for bipartite graphs.

Lemma 3 (Expander mixing lemma). *Let \mathcal{G} be a connected Δ -regular bipartite graph on the vertex set $V_0 \cup V_1$. For any pair of sets $S \subset V_0, T \subset V_1$, it holds that*

$$|E(S, T)| \leq \frac{\Delta}{|V_0|} |S||T| + \lambda(\mathcal{G}) \sqrt{|S||T|}.$$

3.2 Tanner codes

A binary linear code of length n is an \mathbb{F}_2 -linear subspace of \mathbb{F}_2^n . For sets E of cardinality $|E| = n$, it will be convenient for us to identify \mathbb{F}_2^n with \mathbb{F}_2^E , which we can think of as the space of functions from E to \mathbb{F}_2 . Identification with \mathbb{F}_2^n amounts to defining a one-to-one map between E and $[n] = \{1, 2, \dots, n\}$, *i.e.* a numbering of the elements of E .

Let $\mathcal{G} = (V, E)$ be a regular graph of degree Δ , and for any vertex v denote by $E(v)$ the set of edges incident to v . Assume an identification of $\mathbb{F}_2^{E(v)}$ with \mathbb{F}_2^Δ for every $v \in V$. Let $x \in \mathbb{F}_2^E$ be a vector indexed by (or a function defined on) the set E . Let us define the *local view* of x at vertex v as the subvector $x_v := (x_e)_{e \in E(v)}$, *i.e.* x restricted to the edge-neighbourhood $E(v)$ of v .

Let C_0 be a linear code of length Δ , dimension $k_0 = \rho_0 \Delta$, and minimum distance $d_0 = \delta_0 \Delta$. We define the Tanner code [Tan81] associated to \mathcal{G} and C_0 as

$$\text{Tan}(\mathcal{G}, C_0) := \{x \in \mathbb{F}_2^E : x_v \in C_0 \text{ for all } v \in V\}.$$

In words, the Tanner code is the set of vectors over E all of whose local views lie in C_0 . By counting the number of linear equations satisfied by the Tanner code, we obtain

$$\dim \text{Tan}(\mathcal{G}, C_0) \geq (2\rho_0 - 1)n. \tag{3}$$

We also have the bound [SS96, Gur10] on the minimum distance d of the Tanner code:

$$d \geq \delta_0(\delta_0 - \lambda(\mathcal{G})/\Delta)n.$$

Therefore, if (\mathcal{G}_i) is a family of Δ -regular expander graphs with $\lambda(\mathcal{G}_i) \leq \lambda < d_0$, and if $\rho_0 > 1/2$, then the associated family of Tanner codes has rate and minimum distance which are both $\Omega(n)$, meaning we have an asymptotically good family of codes, as was first shown in [SS96].

3.3 Quantum CSS codes

A quantum CSS code is specific instance of a stabilizer code [Got97] that can be defined by two classical codes \mathcal{C}_0 and \mathcal{C}_1 in the ambient space \mathbb{F}_2^n , with the property that $\mathcal{C}_0^\perp \subset \mathcal{C}_1$ [CS96, Ste96]. It is a *low-density parity-check* (LDPC) code whenever both \mathcal{C}_0 and \mathcal{C}_1 are the kernels of sparse parity-check matrices. The resulting quantum code $\mathcal{Q} = (\mathcal{C}_0, \mathcal{C}_1)$ is a subspace of $(\mathbb{C}_2)^{\otimes n}$, the space of n qubits:

$$\mathcal{Q} := \text{Span} \left\{ \sum_{z \in \mathcal{C}_1^\perp} |x+z\rangle : x \in \mathcal{C}_0 \right\},$$

where $\{|x\rangle : x \in \mathbb{F}_2^n\}$ is the canonical basis of $(\mathbb{C}_2)^{\otimes n}$. The dimension k of the code counts the number of logical qubits and is given by

$$k = \dim(\mathcal{C}_0/\mathcal{C}_1^\perp) = \dim \mathcal{C}_0 + \dim \mathcal{C}_1 - n.$$

Its minimum distance is $d = \min(d_X, d_Z)$ with

$$d_X = \min_{w \in \mathcal{C}_0 \setminus \mathcal{C}_1^\perp} |w|, \quad d_Z = \min_{w \in \mathcal{C}_1 \setminus \mathcal{C}_0^\perp} |w|.$$

We denote the resulting code parameters by $\llbracket n, k, d \rrbracket$ and say that a code family $(Q_n)_n$ is *asymptotically good* if its parameters are of the form

$$\llbracket n, k = \Theta(n), d = \Theta(n) \rrbracket.$$

An n -qubit Pauli error $E_1 \otimes \dots \otimes E_n$ with $E_i \in \{\mathbb{1}, \sigma_X, \sigma_Y, \sigma_Z\}$ ⁶ is conveniently described by two n -bit strings $(e_0, e_1) \in \mathbb{F}_2^n \times \mathbb{F}_2^n$ via the mapping

$$\mathbb{1} \mapsto (0, 0), \quad \sigma_X \mapsto (1, 0), \quad \sigma_Y \mapsto (1, 1), \quad \sigma_Z \mapsto (0, 1),$$

which forgets global phases. The parity-check matrices of \mathcal{C}_0 and \mathcal{C}_1 give rise to syndrome maps $\sigma_0, \sigma_1 : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m$ that associate a pair of syndromes $(\sigma_0(\mathbf{e}_0), \sigma_1(\mathbf{e}_1)) \in \mathbb{F}_2^m \times \mathbb{F}_2^m$ to any n -qubit Pauli error $(\mathbf{e}_0, \mathbf{e}_1) \in \mathbb{F}_2^n \times \mathbb{F}_2^n$. The decoding problem for a stabilizer code is as follows: given a syndrome $(\sigma_0(\mathbf{e}_0), \sigma_1(\mathbf{e}_1))$, recover the error up to an element of the stabilizer group, that is return $(\hat{\mathbf{e}}_0, \hat{\mathbf{e}}_1)$ such that $\mathbf{e}_0 + \hat{\mathbf{e}}_0 \in \mathcal{C}_1^\perp$ and $\mathbf{e}_1 + \hat{\mathbf{e}}_1 \in \mathcal{C}_0^\perp$.

While an optimal decoding of *random* errors would typically exploit possible correlations between \mathbf{e}_0 and \mathbf{e}_1 , it is always possible to correct both errors independently. Here, we will be concerned with the adversarial setting where \mathbf{e}_0 and \mathbf{e}_1 are of sufficiently low weight, but otherwise arbitrary. In that case, both errors should be decoded independently, and we will focus on the case where $(\mathbf{e}_0 = 0, \mathbf{e}_1 = e)$ in the following.

⁶The 1-qubit Pauli matrices are defined by $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\sigma_Y = i\sigma_X\sigma_Z$.

3.4 Tensor codes and dual tensor codes: robustness

Definitions and results for this section are taken from [LZ22], to which we refer for proofs and comments.

Let A and B be two sets of size Δ . We define codes on the ambient space $\mathbb{F}_2^{A \times B}$ that we may think of as the space of matrices whose rows (columns) are indexed by A (by B). If $C_A \subset \mathbb{F}_2^A$ and $C_B \subset \mathbb{F}_2^B$ are two linear codes, we define the *tensor* (or product) code $C_A \otimes C_B$ as the space of matrices x such that for every $b \in B$ the column vector $(x_{ab})_{a \in A}$ belongs to C_A and for every $a \in A$ the row vector $(x_{ab})_{b \in B}$ belongs to C_B . It is well known that $\dim(C_A \otimes C_B) = \dim(C_A) \dim(C_B)$ and that the minimum distance of the tensor code is $d(C_A \otimes C_B) = d(C_A)d(C_B)$.

Consider the codes $C_A \otimes \mathbb{F}_2^B$ and $\mathbb{F}_2^A \otimes C_B$ consisting respectively of the space of matrices whose columns are codewords of C_A and whose rows are codewords of C_B . We may consider their sum $C_A \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B$ which is called a *dual tensor* code, since it is the dual code of the tensor code $C_A^\perp \otimes C_B^\perp = (C_A^\perp \otimes \mathbb{F}_2^B) \cap (\mathbb{F}_2^A \otimes C_B^\perp)$. It is relatively straightforward to check that $d(C_A \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B) = \min(d(C_A), d(C_B))$.

Definition 4. Let $0 \leq w \leq \Delta^2$. Let C_A and C_B be codes of length Δ with minimum distances d_A and d_B . We shall say that the dual tensor code $C = C_A \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B$ is w -robust, if for any codeword $x \in C$ of Hamming weight $|x| \leq w$, there exist $A' \subset A, B' \subset B$, $|A'| \leq |x|/d_B$, $|B'| \leq |x|/d_A$, such that $x_{ab} = 0$ whenever $a \notin A', b \notin B'$.

Proposition 5. Let C_A and C_B be codes of length Δ with minimum distances d_A and d_B , and suppose $C = C_A \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B$ is w -robust with $0 < w < d_A d_B$. Then for any codeword $x \in C$ such that $|x| \leq w$, there exist $A' \subset A, B' \subset B$, $|A'| \leq |x|/d_B$, $|B'| \leq |x|/d_A$ and a decomposition $x = c + r$, with $c \in C_A \otimes \mathbb{F}_2^{B'}$ and $r \in \mathbb{F}_2^{A'} \otimes C_B$.

Proof. To see this, apply the definition and write $x = r' + c'$, with $r'_{ab} = c'_{ab}$ for any $(a, b) \in (A \setminus A') \times (B \setminus B')$. The restrictions of r' and c' to $(A \setminus A') \times (B \setminus B')$ both belong to the code obtained by tensoring C'_A and C'_B , the punctured codes deduced from C_A and C_B by throwing away coordinates of A' and B' . This code is the same as the punctured code obtained from $C_A \otimes C_B$ by throwing away the coordinates $A' \times B \cup A \times B'$. Therefore, there exists a tensor codeword of $C_A \otimes C_B = C_A \otimes \mathbb{F}_2^B \cap \mathbb{F}_2^A \otimes C_B$ that coincides with $c' = r'$ on $(A \setminus A') \times (B \setminus B')$: adding this tensor codeword to both c' and r' yields the required pair r, c such that $x = r + c$. \square

Proposition 6. Let C_A and C_B be codes of length Δ and minimum distances d_A, d_B such that the dual tensor code $C_A \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B$ is w -robust with $w \leq d_A d_B / 2$. Then, any word x close to both the column and row code is also close to the tensor code: precisely, if $d(x, C_A \otimes \mathbb{F}_2^B) + d(x, \mathbb{F}_2^A \otimes C_B) \leq w$ then

$$d(x, C_A \otimes C_B) \leq \frac{3}{2} (d(x, C_A \otimes \mathbb{F}_2^B) + d(x, \mathbb{F}_2^A \otimes C_B)). \quad (4)$$

Definition 7. Let $C_A \subset \mathbb{F}_2^A$ and $C_B \subset \mathbb{F}_2^B$. For integers w, p , let us say that the dual tensor code $C_A \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B$ is w -robust with p -resistance to puncturing, if for any

$A' \subset A$ and $B' \subset B$ such that $|A'| = |B'| = \Delta - w'$, with $w' \leq p$, the dual tensor code

$$C_{A'} \otimes \mathbb{F}_2^{B'} + \mathbb{F}_2^{A'} \otimes C_{B'}$$

is w -robust.

We shall need the following result on the robustness of random dual tensor codes.

Theorem 8. *Let $0 < \rho_A < 1$ and $0 < \rho_B < 1$. Let $0 < \varepsilon < 1/2$ and $1/2 + \varepsilon < \gamma < 1$. Let C_A be a random code obtained from a random uniform $\rho_A \Delta \times \Delta$ generator matrix, and let C_B be a random code obtained from a random uniform $(1 - \rho_B) \Delta \times \Delta$ parity-check matrix. With probability tending to 1 when Δ goes to infinity, the dual tensor code*

$$C_A \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B$$

is $\Delta^{3/2-\varepsilon}$ -robust with Δ^γ -resistance to puncturing.

4 Detailed description of quantum Tanner codes

We recall the ingredients that enable us to define the family of asymptotically good quantum LDPC codes introduced in [LZ22]. The only difference with [LZ22] is that we prefer to work here with a quadripartite version of the Cayley complex since it gives a simpler description of the decoding algorithm. This quadripartite version was mentioned in passing in [LZ22], but quadripartite square complexes are particular instances of bipartite square complexes, and any bipartite square complex with the required expansion properties was acceptable for the the lower bound of Theorem 1 on the minimum distance to hold. Below we discuss the quadripartite structure in some more detail.

4.1 Left-right Cayley complexes (quadripartite version)

A *left-right Cayley complex* X is introduced in [DEL⁺21] from a group G and two sets of generators $A = A^{-1}$ and $B = B^{-1}$. As in [DEL⁺21] we will restrict ourselves, for the sake of simplicity, to the case $|A| = |B| = \Delta$. The complex is made up of vertices, A -edges, B -edges, and squares. The vertex set consists of four copies of the group G in the quadripartite version, $V = V_{00} \cup V_{10} \cup V_{01} \cup V_{11}$ with $V_{ij} = G \times \{ij\}$. We will also use the notation $V_0 := V_{00} \cup V_{11}$ and $V_1 := V_{01} \cup V_{10}$. The A -edges are pairs of vertices of the form $\{(g, i0), (ag, i1)\}$ and B -edges are of the form $\{(g, 0j), (gb, 1j)\}$ for $g \in G, a \in A, b \in B, i, j = 0, 1$. We denote by E_A and E_B these two edge sets. The associated graphs are denoted by $\mathcal{G}_A = (V, E_A)$ and $\mathcal{G}_B = (V, E_B)$. A *square* is a set of four vertices of the form $\{(g, 00), (ag, 01), (gb, 10), (agb, 11)\}$. The set of squares (or quadrangles) of the complex is denoted by Q . Every vertex is incident to exactly Δ^2 squares. For a vertex v , the set of incident squares is called the *link* of v , or Q -neighbourhood, and denoted by $Q(v)$.

The sets of generators A and B will be chosen so that the Cayley graphs $\text{Cay}(G, A)$ and $\text{Cay}(G, B)$ are non-bipartite Ramanujan graphs. It should be understood that when writing $\text{Cay}(G, A)$ we implicitly mean the Cayley graph defined by left multiplication

by elements of A , while $\text{Cay}(G, B)$ stands for the Cayley graph defined by right multiplication by elements of B . The sets A and B could in principle be chosen to be identical, but we keep a distinct notation for both sets, in particular in order to allow the above abuse of notation to be non-confusing.

We see that the subset of edges of E_A that connect vertices of V_{00} to vertices of V_{01} make up a double cover of the Cayley graph $\text{Cay}(G, A)$, the edges E_A that connect V_{10} to V_{11} make up a second copy of the same double cover. Therefore, the graph \mathcal{G}_A is a disjoint union of two copies of the double cover of $\text{Cay}(G, A)$. Similarly, \mathcal{G}_B is a disjoint union of two copies of the double cover of $\text{Cay}(G, B)$. We will regularly talk about expansion in \mathcal{G}_A (or \mathcal{G}_B) to mean expansion in either of the connected components of \mathcal{G}_A (\mathcal{G}_B).

Let us introduce one additional graph that exists on the complex X , and that we denote by \mathcal{G}^\square . This graph puts an edge between all pairs of vertices of the form $\{(g, i), (agb, i)\}$, $g \in G, a \in A, b \in B, i = 0, 1$. The graph \mathcal{G}^\square is therefore made up of two connected components, on V_0 and V_1 , that we denote by \mathcal{G}_0^\square and \mathcal{G}_1^\square . We note that \mathcal{G}^\square is regular of degree Δ^2 , and may have multiple edges.

If $\text{Cay}(G, A)$ and $\text{Cay}(G, B)$ are Ramanujan, then \mathcal{G}^\square inherits some of their expansion properties. Specifically:

Lemma 9. *Assume that $\text{Cay}(G, A)$ and $\text{Cay}(G, B)$ are Ramanujan graphs, then*

$$\lambda(\mathcal{G}_0^\square) \leq 4\Delta, \quad \lambda(\mathcal{G}_1^\square) \leq 4\Delta.$$

The proof follows from the fact that the adjacency matrix of \mathcal{G}^\square is the product of the adjacency matrices of \mathcal{G}_A and \mathcal{G}_B , and that these two adjacency matrices commute, by definition of the square complex. See [LZ22] for a little more detail.

4.2 Labelling Q -neighbourhoods

We will define Tanner codes on \mathcal{G}_0^\square and \mathcal{G}_1^\square , which implies a labelling of the coordinates in every Q -neighbourhood $Q(v)$. There is a natural labeling of $Q(v)$ by the set $A \times B$, namely a one-to-one map $\phi_v : A \times B \rightarrow Q(v)$, which we now state explicitly.

We set

$$\begin{aligned} \text{for } v = (g, 00) \in V_{00}, \quad \phi_v(a, b) &= \{(g, 00), (ag, 01), (gb, 10), (agb, 11)\}, \\ \text{for } v = (g, 01) \in V_{01}, \quad \phi_v(a, b) &= \{(g, 01), (a^{-1}g, 00), (gb, 11), (a^{-1}gb, 10)\}, \\ \text{for } v = (g, 10) \in V_{10}, \quad \phi_v(a, b) &= \{(g, 10), (ag, 11), (gb^{-1}, 00), (agb^{-1}, 01)\}, \\ \text{for } v = (g, 11) \in V_{11}, \quad \phi_v(a, b) &= \{(g, 11), (a^{-1}g, 10), (gb^{-1}, 01), (a^{-1}gb^{-1}, 00)\}. \end{aligned}$$

The map ϕ_v thus defined is obviously one-to-one, and one easily checks that:

Any two vertices $v = (g, i0)$ and $w = (gb, i1)$, $i = 0, 1$, that are connected through a B -edge (labelled b), have a common ‘‘column’’, i.e. their Q -neighbourhoods share exactly Δ squares that are labelled (a, b) , $a \in A$, in both $Q(v)$ and $Q(w)$.

Similarly,

Any two vertices $v = (g, 0i)$ and $w = (ag, 1i)$, $i = 0, 1$, that are connected through an A -edge (labelled a), have a common row, i.e. their Q -neighbourhoods share exactly Δ squares that are labelled $(a, b), b \in B$, in both $Q(v)$ and $Q(w)$.

The situation is illustrated on Figure 2. Summarising, any two vertices connected by B -edge (an A -edge) have a common column (row) in their Q -neighbourhoods, that is labelled by the same $b \in B$ ($a \in A$). This is the reason for the possibly intriguing inversions in the definition of ϕ_v : without these inversions the Q -neighbourhoods of two neighbouring vertices would still share a common row or a common column, but their indexes in their respective local views would be inverse of each other. This would still be manageable but slightly less convenient.

4.3 Local codes

The constraints of a Tanner code consist of local constraints from small codes enforced on the local view of each vertex. For quantum Tanner codes, now that all local Q -neighbourhoods are isomorphic to $A \times B$, we may put local constraints that are codewords of the tensor codes $C_A \otimes C_B$ and $C_A^\perp \otimes C_B^\perp$.

Recall that the generators of the quantum Tanner code correspond to a basis of $C_A \otimes C_B$ in each local view of $V_{00} \cup V_{11}$ (for the σ_Z -type generators) and to a basis of $C_A^\perp \otimes C_B^\perp$ in each local view of $V_{01} \cup V_{10}$ (for the σ_X -type generators). The classical code $\mathcal{C}_0 \subset \mathbb{F}_2^Q$ correcting σ_X -type errors is the Tanner code on the graph \mathcal{G}_0^\square with local constraints corresponding to the dual tensor code $(C_A \otimes C_B)^\perp = C_A^\perp \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B^\perp$. With the notation of Section 3.2, $\mathcal{C}_0 = \text{Tan}(\mathcal{G}_0^\square, C_A^\perp \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B^\perp)$. Similarly, the classical code $\mathcal{C}_1 \subset \mathbb{F}_2^Q$ correcting σ_Z -type errors is the Tanner code on the graph \mathcal{G}_1^\square with local constraints corresponding to the dual tensor code $(C_A^\perp \otimes C_B^\perp)^\perp = C_A \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B$, i.e. $\mathcal{C}_1 = \text{Tan}(\mathcal{G}_1^\square, C_A \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B)$.

Summary. A large enough Δ is chosen, together with an infinite family of groups G with generating sets A, B , $|A| = |B| = \Delta$, such that the left Cayley graph $\text{Cay}(G, A)$ and the right Cayley graph $\text{Cay}(G, B)$ are Ramanujan. The quadripartite left-right square complex X is defined by G, A, B .

For the conditions for the component codes that together with the above square complexes X will yield asymptotically good quantum codes, we recall Theorem 16 from [LZ22].

Theorem 10. Fix $\rho \in (0, 1/2)$, $\varepsilon \in (0, 1/2)$, $\gamma \in (1/2 + \varepsilon, 1)$ and $\delta > 0$. If Δ is large enough and C_A and C_B are codes of length Δ such that

1. $0 < \dim C_A \leq \rho \Delta$ and $\dim C_B = \Delta - \dim C_A$,
2. the minimum distances of $C_A, C_B, C_A^\perp, C_B^\perp$ are all $\geq \delta \Delta$,
3. both dual tensor codes $C_0^\perp = (C_A \otimes C_B)^\perp$ and $C_1^\perp = (C_A^\perp \otimes C_B^\perp)^\perp$ are $\Delta^{3/2-\varepsilon}$ -robust with Δ^γ -resistance to puncturing (see Definition 7),

then the quantum code $\mathcal{Q} = (\mathcal{C}_0, \mathcal{C}_1)$ has length $|Q|$, dimension at least $(1 - 2\rho)^2|Q|$ and minimum distance at least $|Q|\delta/4\Delta^{3/2+\varepsilon}$.

It was proved in [LZ22] (see also Theorem 8), that for any fixed ρ and $\delta > 0$ such that $-\delta \log_2 \delta - (1 - \delta) \log_2(1 - \delta) < \rho$, then for any fixed ε, γ and Δ large enough, randomly choosing C_A and C_B with the required rates will yield codes that satisfy conditions 2 and 3 in Theorem 10 with high probability. In the sequel we will set, somewhat arbitrarily, $\gamma = 1 - \varepsilon$, so as to minimise the number of constants, and we will naturally assume that the codes C_A and C_B satisfy the conditions of Theorem 10.

5 The decoding strategy

We recall that we consider without loss of generality a bit-flip error $\mathbf{e} \in \mathbb{F}_2^Q$ and wish to correct it with the help of the classical Tanner code \mathcal{C}_1 . The goal of the decoder is to output some guess $\hat{\mathbf{e}} \in \mathbb{F}_2^Q$ and it is successful if $\mathbf{e} + \hat{\mathbf{e}} \in \mathcal{C}_0^\perp$.

For a vertex $v \in V$, denote by e_v the local view of \mathbf{e} on $Q(v)$, i.e. its restriction to $Q(v)$, which we also extend back to \mathbb{F}_2^Q by padding it with 0s. The error \mathbf{e} can be identified with the collection of local views $\mathbf{e} = \{e_v\}_{v \in V_1}$ and we note that they satisfy $\sum_{v \in V_{01}} e_v = \sum_{v \in V_{10}} e_v$, since $(Q(v))_{v \in V_{01}}$ and $(Q(v))_{v \in V_{10}}$ are both partitions of Q . The Hamming weight of \mathbf{e} is

$$|\mathbf{e}| = \sum_{v \in V_{01}} |e_v| = \sum_{v \in V_{10}} |e_v|.$$

As observed in [LZ22], a possible approach to decoding a Tanner code is to consider the *mismatch* of the error \mathbf{e} . It is defined as follows. For each vertex $v \in V_1$, one can compute an error ε_v of minimal Hamming weight yielding the corresponding local syndrome. This gives a decomposition of the local views of the error $e_v = \varepsilon_v + c_v + r_v$ with $c_v \in C_A \otimes \mathbb{F}_2^B$, $r_v = \mathbb{F}_2^A \otimes C_B$, and ε_v of minimal Hamming weight. In the case where $e_v = \varepsilon_v$ for all $v \in V_1$, then the decoder has succeeded in recovering the true error. In general, however, we have $e_v \neq \varepsilon_v$ for some $v \in V_1$. The mismatch vector in \mathbb{F}_2^Q defined as $Z := \sum_{v \in V_1} \varepsilon_v$ then characterises the inconsistency between the local views in V_{10} and V_{01} . From $\sum_{v \in V_1} e_v = 0$, we obtain that

$$Z = \sum_{v \in V_1} r_v + c_v.$$

We observe that the minimality of ε_v implies that $|e_v + c_v + r_v| \leq |e_v|$ and therefore $|c_v + r_v| \leq 2|e_v|$. This immediately shows that

$$|Z| \leq \sum_{v \in V_1} |r_v + c_v| \leq 2 \sum_{v \in V_1} |e_v| = 4|\mathbf{e}|.$$

We note that the quantum Tanner codes are *a priori* not locally testable, and therefore the Hamming weight of the mismatch can be much smaller than that of the error (otherwise a simple test for detecting an error would be to sample bits of the mismatch).

The idea behind the decoder will be to find a decomposition $\{\hat{r}_v, \hat{c}_v\}_{v \in V_1}$ such that $Z = \sum_{v \in V_1} \hat{r}_v + \hat{c}_v$. In that case, the decoder will return the following guess for the decomposition of the error: $\{\hat{e}_v\}_{v \in V_1}$ with

$$\hat{e}_v := \varepsilon_v + \hat{r}_v + \hat{c}_v.$$

In particular, the vectors \mathbf{e} and $\hat{\mathbf{e}}$ differ by an element of \mathcal{C}_1 since they both give the same syndrome, and a sufficient condition to guarantee the success of the decoder is that $|\mathbf{e} + \hat{\mathbf{e}}| < d_{\min}(\mathcal{Q})$.

Since each column $A \times \{b\}$ of a Q -neighbourhood $Q(v)$ appears also in the Q -neighbourhood of a neighbouring vertex, we have that for either $v \in V_{00}$ or $v \in V_{10}$, each column subvector of c_v is a codeword of C_A that appears in two local views, one indexed by a vertex of V_{10} and one by a vertex of V_{00} : for $v \in V_{01} \cup V_{11}$, we have that any column subvector of c_v lies both in the local view around a vertex of V_{01} and around a vertex of V_{11} .

A similar observation also holds for row codewords: This implies that one can also define couples (c_v, r_v) for all vertices of V_0 .

It is convenient to define four vectors associated with the collection $\{r_v, c_v\}_{v \in V_1}$,

$$C_0 = \sum_{v \in V_{10}} c_v, \quad R_0 = \sum_{v \in V_{01}} r_v, \quad C_1 = \sum_{v \in V_{01}} c_v, \quad R_1 = \sum_{v \in V_{10}} r_v,$$

that provide a decomposition of the mismatch:

$$Z = C_0 + R_0 + C_1 + R_1.$$

Note that (C_0, R_0, C_1, R_1) and $\{r_v, c_v\}_{v \in V}$ are in one-to-one correspondence.

We will denote this decomposition as

$$\mathcal{Z} = (C_0, R_0, C_1, R_1) = \{r_v, c_v\}_{v \in V}.$$

The decoder then simply aims at finding a valid decomposition $\hat{\mathcal{Z}} = (\hat{C}_0, \hat{R}_0, \hat{C}_1, \hat{R}_1)$ of Z and this decomposition will yield the correct guess provided that $|\mathbf{e} + \hat{\mathbf{e}}| = |\sum_{v \in V_{10}} c_v + r_v + \hat{c}_v + \hat{r}_v| = |C_0 + R_1 + \hat{C}_0 + \hat{R}_1|$ is less than the minimal distance of the quantum Tanner code.

A natural idea for the decoder is to progressively decrease the weight of Z , so as to obtain a consistent assignment that will correspond to the output of the decoder. To do this, one can look for some vertex v , as well as some local codeword $c_v + r_v$ of the dual tensor code such that $|Z + c_v + r_v| < |Z|$. If it was always possible to guarantee the existence of such a vertex and local codeword, then one would simply repeat the operation until reaching a complete decomposition of the mismatch. At each step, the weight of the mismatch decreases by at least 1, so at most $|Z|$ steps are required. In addition, we observe that $|\hat{C}_0 + \hat{R}_1| \leq \Delta^2 |Z| \leq 4\Delta^2 |\mathbf{e}|$ and we recall that $|C_0 + R_1| = \sum_{v \in V_{10}} |c_v + r_v| \leq 2|\mathbf{e}|$. We conclude that if we could always find a vertex and a local codeword that could be added to the mismatch to decrease its weight, then the decoder would return a valid correction provided the initial error is not too large:

$$|\mathbf{e}| \leq \frac{d_{\min}(\mathcal{Q})}{4\Delta^2 + 2}.$$

Unfortunately, we only know how to prove the existence of local codewords decreasing the weight of Z if the local dual tensor code is w -robust with $w > \Delta^{3/2}$, and we recall that Theorem 8 only asserts that randomly chosen codes $C_A \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B$ and $C_A^\perp \otimes \mathbb{F}_2^B + \mathbb{F}_2^A \otimes C_B^\perp$ are w -robust for $w = O(\Delta^{3/2-\varepsilon})$ for random codes C_A, C_B .

For this reason, we need to tweak this natural decoder in order to deal with error configurations (described by their mismatch Z) whose weight cannot be decreased locally by the addition of some $c_v + r_v$. For this purpose, we will add a *parallel* procedure that will be described in detail in Section 6. The actual decoder will apply the sequential local procedure described above, though we will only need to apply it to vertices of V_0 , and resort to the parallel procedure whenever the sequential decoder is stalled. The parallel procedure will consist of two phases, the first will be applied to vertices of V_1 , and the second to vertices of V_0 . While it may not be possible to decrease locally the Hamming weight of the mismatch when the sequential decoder is stalled, we will show that together, the two steps of the parallel procedure will significantly decrease the Hamming weight of Z , by a factor which becomes close to $\sqrt{\Delta}$ when the exponent ε that defines the robustness parameter $w = \Delta^{3/2-\varepsilon}$ tends to 0. Since the sequential procedure decreases $|Z|$ at every step, the number of sequential steps is at most linear in $|Z|$, and therefore in the Hamming weight $|e|$ of the initial error, provided it is not too large. Since any parallel step divides $|Z|$ by a constant, the total number of parallel steps is logarithmic in $|e|$.

In both cases, sequential and parallel, the decoder needs only look at Q -neighbourhoods on which the mismatch $|Z|$ is nonzero: therefore, by carefully keeping track of the set of vertices for which the mismatch is nonzero, the decoder can run in linear-time in the standard uniform cost computational model.

Norm and minimal decomposition of the mismatch. The analysis of the decoder will rest on a careful study of a situation where the sequential decoder is stalled, that is, where there does not exist any vertex (in V_0) where one can add a codeword of the dual tensor code that would decrease the Hamming weight of Z . (If the sequential decoder is never stalled, then there is essentially nothing to prove.) The parallel procedure that will unlock the situation will consist of two steps. While the combination of both steps will reduce the Hamming weight $|Z|$ of the mismatch, it is not the case of the first part alone. In order to keep track of the progress of the decoding algorithm, we will therefore require another metric beside the Hamming weight: this role will be played by the norm of the decomposition, that will be a proxy for the number of vertices of V_0 for which $c_v + r_v \neq 0$.

First, note that there are many valid decompositions of the mismatch since adding any codeword of the tensor code $C_A \otimes C_B$ to both r_v and c_v does not change their sum. For an element $r \in \mathbb{F}_2^A \otimes C_B$ or $c \in C_A \otimes \mathbb{F}_2^B$, we denote by $\|r\|$ or $\|c\|$ the number of nonzero rows (for r) or columns (for c) in its support:

$$\begin{aligned} \|r\| &:= \min\{|A'| \text{ s.t. } A' \subseteq A, \text{supp}(r) \subseteq \mathbb{F}_2^{A'} \otimes C_B\}, \\ \|c\| &:= \min\{|B'| \text{ s.t. } B' \subseteq B, \text{supp}(c) \subseteq C_A \otimes \mathbb{F}_2^{B'}\}. \end{aligned}$$

The norm $\|\mathcal{Z}\|$ of the decomposition is then simply:

$$\|\mathcal{Z}\| := \|C_0\| + \|R_0\| + \|C_1\| + \|R_1\|,$$

with

$$\|C_0\| := \sum_{v \in V_{10}} \|c_v\|, \quad \|R_0\| := \sum_{v \in V_{01}} \|r_v\|, \quad \|C_1\| := \sum_{v \in V_{01}} \|c_v\|, \quad \|R_1\| := \sum_{v \in V_{10}} \|r_v\|.$$

It will be useful for us to consider a *minimal* decomposition of the mismatch, $\mathcal{Z} = (C_0, R_0, C_1, R_1)$, which simply corresponds to a decomposition of minimal norm. We define the norm $\|Z\|$ of the mismatch to be the norm of a such minimal decomposition:

$$\|Z\| := \min_{\mathcal{Z} \text{ is a decomposition of } Z} \|\mathcal{Z}\|.$$

A minimal decomposition has the property that for each vertex, it is not possible to add a codeword of $C_A \otimes C_B$ that decreases the norm of the local view: for each $v \in V$, and each $u \in C_A \otimes C_B$, it holds that

$$\|r_v + c_v + u\| \geq \|r_v + c_v\|.$$

6 Analysis of the decoder

6.1 The decoding algorithm

The goal of the algorithm is to find a small decomposition of Z . We will keep track of 5 vectors initialized as follows

$$\hat{Z} := Z, \quad \hat{C}_0 := 0, \quad \hat{R}_0 := 0, \quad \hat{C}_1 := 0, \quad \hat{R}_1 := 0.$$

It will alternate between two procedures:

Sequential procedure. This procedure is relatively natural, and consists of modifying \hat{Z} locally, *i.e.* in some $Q(v)$, so as to decrease the Hamming weight of \hat{Z} . Specifically, while there exists some $v \in V_{00}$ or $v \in V_{11}$, and $c_v \in C_A \otimes \mathbb{F}_2^B, r_v \in \mathbb{F}_2^A \otimes C_B$ such that $|\hat{Z} + c_v + r_v| < |\hat{Z}|$, perform the update:

$$\hat{Z} \leftarrow \hat{Z} + c_v + r_v, \quad \hat{C}_j \leftarrow \hat{C}_j + c_v, \quad \hat{R}_i \leftarrow \hat{R}_i + r_v.$$

Parallel procedure. Ideally, we would like the sequential procedure to be applicable until we have $\hat{Z} = 0$, at which point decoding would be complete. However, we cannot always guarantee the existence of a vertex for which we can decrease the Hamming weight of \hat{Z} . The sequential decoder may be stalled. At this point we switch to an alternative procedure which will unlock the situation. We remark that we could try to extend the sequential procedure to vertices of V_1 and not just apply it to vertices of V_0 as specified above. However, we cannot guarantee that this extended sequential decoder will not stall

as well, and we will deal with the stalled decoder through a procedure whose analysis only needs to know that sequential decoder is stalled on V_0 . This alternative procedure consists of two *parallel* decoding steps.

The *first parallel decoding step* is the most involved. It consists of looking for all vertices v of V_1 for which the weight of \hat{Z} can be decreased, not on *all* their Q -neighbourhood $Q(v)$, but on a *sufficiently large subset* of $Q(v)$, *i.e.* indexed by some $A_0 \times B_0$ for $A_0 \subset A$, $B_0 \subset B$. We give below the precise criteria for vertices v for which the decoder updates \hat{Z} on $Q(v)$. We cannot guarantee that this first parallel decoding step decreases the Hamming weight of \hat{Z} . However, we will show that this first parallel decoding step significantly decreases *the number of active vertices of V_0* : the active vertices v of \hat{Z} , for $v \in V_{ij}$, are the vertices on whose Q -neighbourhoods we have $R_i + C_j \neq 0$, for a relevant decomposition $\mathcal{Z} = (C_0, R_0, C_1, R_1)$ of \hat{Z} . We shall be more precise with the definition of this set when we start the analysis. Specifically it divides this number by a quantity close to Δ . Proving this (as stated in Theorem 23) will be the most technical part of the analysis of the decoder.

The *second parallel decoding step* is conceptually simpler. It consists simply of looking for all vertices of V_0 on which the value of $|\hat{Z}|$ can be decreased and *simultaneously* updating \hat{Z} on all such vertices. Precisely:

Second parallel decoding step. Simultaneously, for each $v \in V_0$, search for codewords $c_v \in C_A \otimes \mathbb{F}_2^B, r_v \in \mathbb{F}_2^A \otimes C_B$ such that $|\hat{Z} + c_v + r_v| < |\hat{Z}|$. If such c_v, r_v exist, choose c_v, r_v that maximize the difference $|\hat{Z}| - |\hat{Z} + c_v + r_v|$ and perform the update

$$\hat{Z} \leftarrow \hat{Z} + c_v + r_v, \quad \hat{C}_j \leftarrow \hat{C}_j + c_v, \quad \hat{R}_i \leftarrow \hat{R}_i + r_v$$

for the appropriate $i, j \in \{0, 1\}$.

Just by relying on the fact that the number of active vertices of V_0 has been greatly reduced after the first parallel decoding step, it will be relatively straightforward to show that after the second parallel decoding step, the Hamming weight of \hat{Z} must incur a significant reduction from its value from *before the first parallel decoding step*. Together, both parallel steps, first on V_1 and then on V_0 , enable us to unlock the situation and divide the Hamming weight of \hat{Z} by a quantity that we will show to be close to $\sqrt{\Delta}$.

We now finish this description of the decoder by giving the precise description of the first parallel decoding step. We need to recall that the dual tensor code $C_A \otimes \mathbb{F}^B + \mathbb{F}^A \otimes C_B$ is w -robust with resistance to puncturing p with $w = \Delta^{3/2-\varepsilon}$ and $p \geq \Delta^\gamma$, where $\varepsilon > 0$ can be taken arbitrarily close to 0 and $\gamma < 1$ can be taken arbitrarily close to 1. For simplicity we will choose $\gamma = 1 - \varepsilon$.

First parallel decoding step. The *tweaked* decoder does the following: on V_1 it looks, *in parallel*, for vertices v for which it can identify a partition $A = A_0 \cup A''$ and a partition $B = B_0 \cup B''$ such that $|A''| \leq \Delta^\gamma/2$ and $|B''| \leq \Delta^\gamma/2$ and which satisfy the following properties:

- the value of $|\hat{Z}|$ on the restricted local view $A_0 \times B_0$ of $Q(v)$ is $> w/2$,

- there exists $r_v + c_v \in C_{A_0} \otimes \mathbb{F}^{B_0} + \mathbb{F}^{A_0} \otimes C_{B_0}$ such that $|r_v + c_v| > w$, and replacing \hat{Z} by $\hat{Z} + r_v + c_v$ decreases the value of $|\hat{Z}|$ on $A_0 \times B_0$ so that it becomes $< w/2$.

The behaviour of the decoder on the Q -neighbourhood of such a vertex v is then to

- first update \hat{Z} to $\hat{Z} + r_v + c_v$, update \hat{R}_i to $\hat{R}_i + r_v$, and \hat{C}_j to $\hat{R}_j + c_v$,
- look for individual rows and columns on $Q(v)$, on which it is possible to decrease the weight of \hat{Z} by adding a single (column) codeword of C_A or a single (row) codeword of C_B . If such rows or columns exist it updates $\hat{Z}, \hat{R}_i, \hat{C}_j$ correspondingly, and stops its treatment of $Q(v)$ when there are no more such rows or columns.

We stress that this whole procedure is applied *in parallel* to all Q -neighbourhoods from the original stalled \hat{Z} .

6.2 Analysis of the parallel procedure

We suppose the sequential decoder is stalled. We let $\mathcal{Z} = (R_0, C_0, R_1, C_1)$ be a minimum representation of the current stalled mismatch \hat{Z} , meaning that $\|\mathcal{Z}\| = \|\hat{Z}\|$. In particular we will keep in mind that \mathcal{Z} is *locally minimal*, meaning that its norm cannot be decreased by adding a codeword of the tensor code in a single local view. The decoder does not know of such a representation, but analysing it will be key to understanding what the decoder is able to do from the sole observation of \hat{Z} , which it does see. We will denote by $\mathcal{Z}' = (C'_0, R'_0, C'_1, R'_1)$ the representation of the updated value of \hat{Z}' of \hat{Z} after the first parallel decoding step. This representation is deduced from \mathcal{Z} through local updates as specified by the above description of the algorithm during the first parallel decoding step. We note that \mathcal{Z}' does not have to be a minimal representation. Similarly, we define $\mathcal{Z}'' = (C''_0, R''_0, C''_1, R''_1)$ to be the representation of the updated mismatch \hat{Z}'' after the second parallel decoding step, again deduced from \mathcal{Z}'' through the specified updates of \hat{R}_i, \hat{C}_j .

Let us denote by $S_0 = S_{00} \cup S_{11}$ the set of vertices $v \in V_{ii}$ of the stalled sequential decoder such that $R_i + C_i$ is non-zero on $Q(v)$. Note that S_0 depends on the choice \mathcal{Z} of the representation of \hat{Z} . We call these vertices the *active vertices* of \mathcal{Z} (or simply of \hat{Z} when its representation is implicit).

It will be useful to keep in mind that for any vertex v of $V_{ii} \subset V_0$, the fact that the sequential decoder cannot decrease $|\hat{Z}|$ locally means that on $Q(v)$, the Hamming weight of $(R_0 + C_0) \cap (R_1 + C_1)$ is at least half the Hamming weight of $R_i + C_i$: otherwise we could decrease $|\hat{Z}|$ by updating \hat{Z} to $\hat{Z} + r_v + c_v$ where $r_v + c_v$ is the value of $R_i + C_i$ on $Q(v)$.

We first start by drawing consequences from this fact that will tell us a lot about the structure of S_0 and the local views of $R_0 + C_0$ and $R_1 + C_1$ on the Q -neighbourhoods of V_0 .

Let $0 < \beta < 1/2$. Let us define Col_0^β to be the set of columns in all Q -neighbourhoods of V_{00} (we could also view them in Q -neighbourhoods of V_{10}), that share at least $\beta\Delta$ coordinates with C_1 , *i.e.* that have at least $\beta\Delta$ coordinates on which C_1 is equal to 1.

Similarly, we define Col_1^β to be the set of columns in all Q -neighbourhoods of V_{11} that share at least $\beta\Delta$ coordinates with C_0 . We also define Row_0^β to be the set of rows in all Q -neighbourhoods of V_{00} that share at least $\beta\Delta$ coordinates with R_1 and likewise Row_1^β denotes the set rows that share at least $\beta\Delta$ coordinates with R_0 .

Our main technical lemma reads:

Lemma 11. *Suppose $|S_0| \leq \frac{\delta}{8\Delta}|V_0|$. For Δ large enough, There is a constant $\kappa(\beta)$, depending only on β , such that, whenever the decoder is stalled: for $i = 0, 1$,*

$$\begin{aligned} |\text{Col}_i^\beta| &\leq \frac{\kappa(\beta)}{\Delta^{1-2\varepsilon}}|S_0|. \\ |\text{Row}_i^\beta| &\leq \frac{\kappa(\beta)}{\Delta^{1-2\varepsilon}}|S_0|. \end{aligned}$$

A word of comment may be helpful at this point. Consider a column (say) in some local view, of a vertex of V_{00} (without loss of generality). Suppose this column supports a non-zero C_A -codeword that contributes to the makeup of the global C_0 vector in the current \mathcal{Z} -decomposition of \hat{Z} . What prevents us from adding this codeword to \hat{Z} and decrease its weight? It is naturally the presence of contributions of either R_0 , or R_1 , or C_1 (or all three) to the support of this column vector. What we want to point out here, is that the contribution of R_0 is completely described by a unique Q -neighbourhood of a single vertex of V_{00} , and that the contribution of R_1 is similarly described by the Q -neighbourhood of just one vertex of V_{10} . However, the contribution of C_1 stems from up to Δ different Q -neighbourhoods, and its nature is very much different. What Lemma 11 essentially says is that this contribution is going to be negligible and that it will suffice for us (and the decoder) to focus on the contributions of R_0 and R_1 .

The proof of Lemma 11 will rely upon the subgraphs, both of \mathcal{G}_0^\square and of \mathcal{G}_A , induced by the active vertices, and rely upon expansion in both these graphs. We first need to define S_{ii}^ℓ to be the subset of S_{ii} that consists of either

- (a) the set of vertices v for which the local view on $Q(v)$ of $R_i + C_i$ is made up of at least ℓ non-zero columns vectors of C_i ,
- (b) or the set of vertices v for which the local view on $Q(v)$ of $R_i + C_i$ is made up of at least ℓ non-zero row vectors of R_i .

Lemma 12. *For $i = 0, 1$, if $|S_0| \leq \frac{\delta}{8\Delta}|V_0|$, we have*

$$\begin{aligned} |S_{ii}^\ell| &\leq \frac{256}{\delta^2 \ell^2} |S_0| && \text{if } \delta \ell \leq \Delta^{1/2-\varepsilon} \\ |S_{ii}^\ell| &\leq \frac{256}{\Delta^{1-2\varepsilon}} |S_0| && \text{if } \delta \ell \geq \Delta^{1/2-\varepsilon}. \end{aligned}$$

Proof. We prove the result for $i = 1$ and definition (a) of S_{11}^ℓ , the other cases being identical. We consider $E(S_{11}^\ell, S_{00})$ in the subgraph of \mathcal{G}^\square where the edges belong to $(R_0 + C_0) \cap (R_1 + C_1)$. We argue that the degree of every vertex of S_{11} is at least half

the weight of its local view in $R_1 + C_1$, so at least half of $\ell\delta\Delta$ when this quantity is not more than $\Delta^{3/2-\varepsilon}$, by robustness of the local dual tensor code. This gives

$$\frac{1}{2}\ell\delta\Delta|S_{11}^\ell| \leq |E(S_{11}^\ell, S_{00})|.$$

We now apply the Expander mixing Lemma 3 in \mathcal{G}_0^\square . This gives

$$\begin{aligned} \frac{1}{2}\ell\delta\Delta|S_{11}^\ell| &\leq \Delta^2 \frac{|S_{11}^\ell||S_{00}|}{|V_{00}|} + 4\Delta\sqrt{|S_{11}^\ell||S_{00}|} \\ &\leq \frac{1}{4}\delta\Delta|S_{11}^\ell| + 4\Delta\sqrt{|S_{11}^\ell||S_{00}|} \\ &\quad \text{since } |S_{00}|/|V_{00}| \leq 2|S_0|/|V_0| \text{ and } |S_0| \leq \delta|V_0|/8\Delta \\ \ell\delta\sqrt{|S_{11}^\ell|} &\leq 16\sqrt{|S_{00}|} \\ |S_{11}^\ell| &\leq \frac{256}{\delta^2\ell^2}|S_{00}| \end{aligned}$$

which gives the first statement of the lemma by writing $|S_{00}| \leq |S_0|$. The second statement follows analogously, by arguing that when $\delta\ell \geq \Delta^{1/2-\varepsilon}$ the weight of the local view is not less than $\Delta^{3/2-\varepsilon}$. \square

Lemma 13. *Let $T_{10}^{\beta,\ell}$ be the set of vertices of V_{10} whose Q -neighbourhoods contain at least ℓ columns of Col_0^β . Under the hypothesis $|S_0| \leq \frac{\delta}{8\Delta}|V_0|$, for Δ large enough, we have*

$$|T_{10}^{\beta,\ell}| \leq \frac{64}{\beta^2\Delta}|S_{11}^{\ell\beta/2}|.$$

Proof. Consider a the subarray of the Q -neighbourhood of a vertex of $T_{10}^{\beta,\ell}$ consisting of the ℓ columns of Col_0^β . One easily checks, by counting the number coordinates in the subarray on which \hat{C}_1 must be equal to 1, that there are at least $\beta\Delta/2$ rows on which \hat{C}_1 has weight at least $\beta\ell/2$. This in turn implies that, in the graph \mathcal{G}_A , this vertex has at least $\beta\Delta/2$ outgoing edges that fall into $|S_{11}^{\ell\beta/2}|$.

Let us estimate $|E(T_{10}^{\beta,\ell}, S_{11}^{\ell\beta/2})|$ in \mathcal{G}_A . We have just proved

$$|T_{10}^{\beta,\ell}| \frac{1}{2}\beta\Delta \leq |E(T_{10}^{\beta,\ell}, S_{11}^{\ell\beta/2})|.$$

We now apply the Expander mixing Lemma 3 in \mathcal{G}_A . We get

$$|T_{10}^{\beta,\ell}| \frac{1}{2}\beta\Delta \leq \Delta \frac{|T_{10}^{\beta,\ell}||S_{11}^{\ell\beta/2}|}{|V_{11}|} + 2\sqrt{\Delta}\sqrt{|T_{10}^{\beta,\ell}||S_{11}^{\ell\beta/2}|}.$$

Writing $|S_{11}^{\ell\beta/2}|/|V_{11}| \leq |S_{11}|/|V_{11}| \leq 2|S_0|/|V_0|$, we get, for fixed β and Δ large enough,

$$\begin{aligned} |T_{10}^{\beta,\ell}| \frac{1}{4}\beta\Delta &\leq 2\sqrt{\Delta}\sqrt{|T_{10}^{\beta,\ell}||S_{11}^{\ell\beta/2}|} \\ |T_{10}^{\beta,\ell}|^{1/2} &\leq \frac{8}{\beta\Delta^{1/2}}|S_{11}^{\ell\beta/2}|^{1/2} \end{aligned}$$

whence the claimed result after squaring. \square

Proof of Lemma 11. We prove the Lemma for $|\text{Col}_0^\beta|$, the other cases being essentially identical. Write

$$|\text{Col}_0^\beta| \leq \sum_{\ell < \ell_{\max}} \ell |T_{10}^{\beta, \ell}| + \Delta |T_{10}^{\beta, \ell_{\max}}| \quad (5)$$

where ℓ_{\max} is chosen so as to have $\delta \ell_{\max} \beta / 2 = \Delta^{1/2-\varepsilon}$. Applying Lemmas 13 and 12 we obtain

$$\begin{aligned} |\text{Col}_0^\beta| &\leq \sum_{\ell < \ell_{\max}} \ell \frac{64}{\beta^2 \Delta} |S_{11}^{\ell \beta / 2}| + \Delta \frac{64}{\beta^2 \Delta} |S_{11}^{\ell_{\max} \beta / 2}| \\ &\leq \sum_{\ell} \frac{2^{16}}{\delta^2 \ell \beta^4} \frac{1}{\Delta} |S_0| + \frac{2^{14}}{\delta^2 \beta^2 \Delta^{1-2\varepsilon}} |S_0| \end{aligned}$$

from which the result follows, after writing that $\sum_{\ell} 1/\ell \leq \ln \Delta \leq \Delta^{2\varepsilon}$. \square

Definition 14 (normal vertices of S_{00} and S_{11}). *Let us say that a vertex v of S_{00} is normal if $\|r_v\| + \|c_v\| < \Delta^{1/2-\varepsilon}/4$, where r_v, c_v are the local components of R_0, C_0 . Normal vertices of S_{11} are defined similarly. A vertex which is not normal will be said to be exceptional.*

Below we will talk about normal columns of C_0 and normal rows of R_1 . This will mean that in their respective local views in V_{00} and V_{11} they belong to the local view of a normal vertex.

The terminology “normal” and “exceptional” is justified by the following lemma.

Lemma 15. *Let S_0^e be the set of exceptional vertices of S_0 . Under the hypothesis $|S_0| \leq \frac{\delta}{8\Delta} |V_0|$, we have*

$$|S_0^e| \leq \frac{\kappa}{\delta^2 \Delta^{1-2\varepsilon}} |S_0|$$

for some absolute constant κ .

Proof. Let S_{00}^e (S_{11}^e) be the set of exceptional vertices of S_{00} (S_{11}). The definition of an exceptional (not normal) vertex v of S_{00} implies that $R_0 + C_0$ has weight at least $\delta \Delta^{3/2-\varepsilon}/8$ on $Q(v)$, and therefore that $(R_0 + C_0) \cap (R_1 + C_1)$ has weight at least $\delta \Delta^{3/2-\varepsilon}/16$ on $Q(v)$. The weight of $(R_0 + C_0) \cap (R_1 + C_1)$ on $\bigcup_{v \in S_{00}^e} Q(v)$ is therefore at least

$$|S_{00}^e| \delta \Delta^{3/2-\varepsilon} / 16$$

and it is also not more than the number of edges $|E(S_{00}^e, S_{11})|$ in the graph \mathcal{G}_0^\square . By the Expander mixing Lemma in \mathcal{G}_0^\square , we have

$$\frac{1}{16} \delta \Delta^{3/2-\varepsilon} |S_{00}^e| \leq |E(S_{00}^e, S_{11})| \leq \Delta^2 \frac{|S_{00}^e| |S_{11}|}{|V_{00}|} + 4\Delta \sqrt{|S_{00}^e| |S_{11}|}.$$

Writing $|S_{11}|/|V_{00}| \leq 2|S_0|/|V_0|$, for Δ large enough, we straightforwardly obtain that

$$\begin{aligned} \frac{1}{32} \delta \Delta^{3/2-\varepsilon} |S_{00}^e|^{1/2} &\leq 4\Delta |S_{11}|^{1/2} \\ |S_{00}^e| &\leq \frac{2^{14}}{\delta^2} \frac{1}{\Delta^{1-2\varepsilon}} |S_{11}|. \end{aligned}$$

We obtain similarly

$$|S_{11}^e| \leq \frac{2^{14}}{\delta^2} \frac{1}{\Delta^{1-2\varepsilon}} |S_{00}|.$$

Hence the result, by summing the two inequalities. \square

Similarly to $S_0 = S_{00} \cup S_{11}$, let us set $S_1 = S_{10} \cup S_{01}$ where S_{10} and S_{01} are the set of active vertices of V_{10} and V_{01} relative to \hat{Z} . In other words, S_{10} (S_{01}) is the set of vertices of V_{10} (V_{01}) on which the local view of $R_1 + C_0$ ($R_0 + C_1$) is non-zero.

Definition 16 (agglutinating vertices of V_{ij} , $i \neq j$). *Let us say that a vertex v of S_{ij} is agglutinating if $\|r_v\| \geq \delta\Delta/3$, where r_v is the local component of R_i , or if $\|c_v\| \geq \delta\Delta/3$, where c_v is the local component of C_j .*

Definition 17 (decodable agglutinating vertex). *Let us say that an agglutinating vertex v of S_{10} (resp. S_{01}) is decodable, if at most $\Delta^\gamma/2$ columns in its local view are in common with exceptional vertices of S_{00} (resp. S_{11}), and at most $\Delta^\gamma/2$ rows in its local view are in common with exceptional vertices of S_{11} (resp. S_{00}).*

We now show that every decodable agglutinating vertex really is decodable in the sense of the first parallel decoding step:

Proposition 18. *The tweaked decoder that is applied during the first parallel decoding step, described in Section 6.1, identifies correctly every decodable agglutinating vertex v of S_{10} or of S_{01} , i.e. it finds the required non-zero $r_v + c_v$.*

Proof. Let v be a decodable agglutinating vertex of S_{10} , the case $v \in S_{01}$ being essentially identical. We set A_0 to consist of all the indexes of all normal rows in $Q(v)$ and B_0 to consist of all the indexes of normal columns. We now show that the local component $r_v + c_v$ of $R_1 + C_0$, restricted to $A_0 \times B_0$ satisfies all the requirements of the tweaked decoder. Indeed, we must have $|(r_v + c_v)_{A_0 \times B_0}| > w$, otherwise robustness (with resistance to puncturing) would imply that $r_v + c_v$ would be expressible as sum of fewer than $O(\Delta^{1/2})$ non-zero rows and columns of $C_A \otimes \mathbb{F}_2^B$ and $\mathbb{F}_2^A \otimes C_B$, which would contradict local minimality of $\|r_v\| + \|c_v\|$, since we have supposed either $\|r_v\| \geq \delta\Delta/3$, or $\|c_v\| \geq \delta\Delta/3$, by definition of an agglutinating vertex.

Furthermore, the contribution of R_0 on any normal column is less than $\Delta^{1/2-\varepsilon}/4$ by definition of a normal column. Therefore, the weight of R_0 on $A \times B_0$ is less than $\Delta \cdot \Delta^{1/2-\varepsilon}/4 = w/4$. Similarly, the weight of C_1 on $A_0 \times B$ is less than $w/4$, and therefore the weight of $R_0 + C_1$ on $A_0 \times B_0$ is $< w/2$. After removing $r_v + c_v$, the updated value $R'_1 + C'_0$ on $A_0 \times B_0$ is zero, so the value of $|\hat{Z}|$ on $A_0 \times B_0$ is that of $|R_0 + C_1|$ and is $< w/2$.

Finally, since the weight of $R_0 + C_1$ on $A_0 \times B_0$ is $< w/2$ and the weight of $R_1 + C_0$ on $A_0 \times B_0$ is $> w$, we have that the weight of $\hat{Z} = R_0 + C_1 + R_1 + C_0$ on $A_0 \times B_0$ is $> w/2$. \square

We remark that the above proof tells us in passing that the Hamming weight of \hat{Z} on the local view of a decodable agglutinating vertex is at least $w/2$. We note this for future reference.

Proposition 19. *Whenever the sequential decoder is stalled, the Hamming weight of \hat{Z} restricted to $Q(v)$, for v a decodable agglutinating vertex of S_{10} or of S_{01} , is at least equal to $\Delta^{3/2-\varepsilon}/2$.*

Let us denote by K_{10}^γ (K_{01}^γ) the set of vertices of V_{10} (V_{01}) whose Q -neighbourhoods have at least $\Delta^\gamma/2$ columns (rows) in common with exceptional vertices of S_{00} , or that have at least $\Delta^\gamma/2$ rows (columns) in common with exceptional vertices of S_{11} . We then set $K^\gamma = K_{10}^\gamma = K_{01}^\gamma$. These can be thought of as the vertices of V_1 whose Q -neighbourhoods share an exceptionally large number of rows or of columns with exceptional vertices of S_0 .

Lemma 20. *Suppose $|S_0| \leq \frac{\delta}{8\Delta}|V_0|$. Then there is an absolute constant κ such that*

$$|K^\gamma| \leq \frac{\kappa}{\delta^2} \frac{1}{\Delta^{2-4\varepsilon}} |S_0|.$$

Proof. We give an upper bound for K_{10}^γ , an identical one follows for K_{01}^γ by symmetry. Consider the case when K_{10}^γ has $\Delta^\gamma/2$ columns in common with the Q -neighbourhoods of exceptional vertices of S_{00}^e , the other case being similar. This last fact means that all vertices of K_{10}^γ have degree at least $\Delta^\gamma/2$ in the subgraph of \mathcal{G}_B induced by K_{10}^γ and S_{00}^e . By the Expander mixing Lemma in \mathcal{G}_B we therefore have

$$|K_{10}^\gamma| \Delta^\gamma/2 \leq |E(K^\gamma, S_0^e)| \leq \Delta \frac{|K_{10}^\gamma||S_{00}^e|}{|V_{00}|} + 2\sqrt{\Delta}\sqrt{|K_{10}^\gamma||S_0^e|}.$$

Bounding $|S_{00}^e|/|V_{00}|$ from above by $2|S_0|/|V_0|$ we easily get

$$|K_{10}^\gamma| \Delta^\gamma/4 \leq 2\sqrt{\Delta}\sqrt{|K_{10}^\gamma||S_0^e|}$$

whence

$$\begin{aligned} |K_{10}^\gamma|^{1/2} &\leq 8\Delta^{1/2-\gamma}|S_0^e|^{1/2} \\ |K_{10}^\gamma| &\leq 64\Delta^{1-2\gamma}|S_0^e| \\ &\leq \frac{\kappa}{\delta^2}\Delta^{2\varepsilon-2\gamma}|S_0| \end{aligned}$$

by applying Lemma 15. Hence the result, since we have supposed $\gamma = 1 - \varepsilon$. \square

Proposition 21. *Let v be a vertex of S_{10} (resp. S_{01}) that is not in K^γ and that is decoded by the tweaked decoder. Let A_n and B_n be the index sets of its normal rows and its normal columns respectively. After (tweaked) decoding, the contribution of $R'_1 + C'_0$ (resp. $R'_0 + C'_1$) to the coordinate set indexed by $A_n \times B_n$ in $Q(v)$ is supported by columns of Col_0^β and Row_1^β (resp. Col_1^β and Row_0^β) for some $\beta > 1/3$.*

Proof. We treat the case $v \in S_{10}$, the case $v \in S_{01}$ being essentially identical. First consider the dual-tensor codeword $r'_v + c'_v$ that the decoder has identified on $A_0 \times B_0$ (which may not be equal to $A_n \times B_n$). We compare it to $r_v + c_v$, the reduced local view of $R_1 + C_0$. We recall, that by definition of the tweaked decoder, we must have that $|\hat{Z} + r'_v + c'_v|$ is less than $w/2$ on $A_0 \times B_0$. So in particular it must be less than $w/2$ on $(A_0 \cap A_n) \times (B_0 \cap B_n)$. Since the value of $|\hat{Z} + r_v + c_v|$ is less than $w/2$ on $A_n \times B_n$, by definition of normal rows and columns, we must have that $|r_v + c_v + r'_v + c'_v| \leq w$ on $(A_0 \cap A_n) \times (B_0 \cap B_n)$. Now $A_0 \cap A_n$ and $B_0 \cap B_n$ both have cardinality at least $\Delta - \Delta^\gamma/2 - \Delta^\gamma/2 = \Delta - \Delta^\gamma$, so that we may apply robustness with resistance to puncturing to $r_v + c_v + r'_v + c'_v$ and deduce that $\|r_v\| + \|c_v\| + \|r'_v\| + \|c'_v\|$ is $O(\Delta^{1/2} + \Delta^\gamma)$. This in turn implies that, after tweaked decoding, every normal column (*i.e.* indexed by $b \in B_n$) has at most $O(\Delta^{1/2} + \Delta^\gamma)$ coordinates on which R_0 or R'_1 equals 1: therefore, if a column, indexed by $b \in B_n$ say, is the support of a non-zero column codeword $c \in C_A$ in C'_0 , the only way this column has not been decoded is for the support of c on $A \times \{b\}$ to coincide with the support of C_1 on almost half its coordinates (minus $O(\Delta^{1/2} + \Delta^\gamma)$), otherwise more than half the support of c contributes to \hat{Z} , and adding c would decrease $|\hat{Z}|$. Summarising, if the column under discussion does not belong to Col_0^β , it will be decoded by the tweaked decoder. A similar argument holds for normal rows. \square

Note that the choice $\beta > 1/3$ in this last proposition is somewhat arbitrary and could in principle be chosen arbitrarily close to $1/2$.

Proposition 22. *Let v be a vertex of S_{10} (resp. S_{01}) that is not in K^γ and that is not identified as decodable (hence not decoded) by the tweaked decoder. Let A_n and B_n be the index sets of its normal rows and its normal columns respectively. Then the contribution of $R_1 + C_0$ (resp. $R_0 + C_1$) to the coordinate set indexed by $A_n \times B_n$ in $Q(v)$ is supported by columns of Col_0^β and Row_1^β (resp. Col_1^β and Row_0^β) for $\beta = 1/6 + o_\Delta(1)$.*

Proof. Suppose $v \in S_{10}$ is such a vertex. The vertex v cannot be agglutinating by Proposition 18. Indeed, if it were agglutinating, then since we suppose it not in K^γ , it would be a decodable agglutinating vertex and would be decoded by the tweaked decoder. Therefore, we have $\|r_v\| < \delta\Delta/3$ and $\|c_v\| < \delta\Delta/3$, where r_v, c_v are the components of R_1 and C_0 on $Q(v)$. For every column of $Q(v)$, the contribution of R_1 to its support is therefore $< \delta\Delta/3$. For every normal column, the contribution of R_0 to its support is $< \Delta^{1/2-\varepsilon}/4$ by definition of normality. Therefore, on every normal column on which C_0 is non-zero we must have a contribution of C_1 that is at least $\delta\Delta/2 - \delta\Delta/3 - \Delta^{1/2-\varepsilon}/4$. We argue similarly for normal rows in $Q(v)$. \square

Situation after the first parallel decoding step. We now argue that this tweaked decoding step has drastically reduced the number of active vertices of \hat{Z} in V_0 .

A remaining active vertex of V_0 may be one of the exceptional vertices of $S_0^\varepsilon \subset S_0$. By Lemma 15 there are at most $\kappa|S_0|/\Delta^{1-2\varepsilon}$ of them, for some constant κ .

To bound from above the additional number of active vertices, we simply bound from above the number of C_A -column vectors of C'_0 and C'_1 (and of C_B -row vectors of R'_0 and

R'_1) that are supported by the Q -neighbourhood of a non-exceptional vertex $v \notin S_e$. Such a row or column codeword may

- belong to the Q -neighbourhood of a vertex of V_1 that has many columns or rows in common with exceptional vertices of V_0 : the set of such vertices was defined as K^γ , and by Lemma 20, there are not more than $\kappa|S_0|/\Delta^{2-4\epsilon}$ of them. These vertices therefore contribute at most $2\Delta|K^\gamma|$ row and column vectors, so not more than $\kappa|S_0|/\Delta^{1-4\epsilon}$ for some constant κ .
- Remaining row or column codewords (that have not been removed by the decoder or have wrongly been introduced by the decoder) may also be present in the Q -neighbourhoods of vertices of V_1 that have been decoded by the tweaked decoder. By Proposition 21, all these rows and columns must belong to Row_i^β and Col_i^β , for some fixed value β and $i = 0, 1$. By Lemma 11 there are at most $\kappa|S_0|/\Delta^{1-2\epsilon}$ such column and row vectors, for some constant κ .
- The remaining row or column codewords must be present in the the Q -neighbourhoods of vertices of V_1 that are not in K^γ and have not been decoded by the tweaked decoder. By Proposition 22, all these rows and columns must belong to Row_i^β and Col_i^β , for some fixed value β and $i = 0, 1$. Again, by Lemma 11 there are at most $\kappa|S_0|/\Delta^{1-2\epsilon}$ such column and row vectors, for some constant κ .

Summing everything, we have shown the following:

Theorem 23. *We suppose $|S_0| \leq \frac{\delta}{8\Delta}|V_0|$. Let S'_0 be the set of active vertices of V_0 after the first parallel decoding step, i.e. the set of vertices of V_{00} on whose Q -neighbourhood $R'_0 + C'_0$ is non-zero, together with the set of vertices of V_{11} on whose Q -neighbourhood $R'_1 + C'_1$ is non-zero. We have $|S'_0| \leq \kappa|S_0|/\Delta^{1-4\epsilon}$ for some constant κ .*

Before analysing the second parallel decoding step, we need the following consequence of the above analysis.

Proposition 24. *When the sequential decoder is stalled, we have $|\hat{Z}| \geq \kappa|S_0|\Delta^{1/2-\epsilon}$ for some constant κ .*

Proof. By Lemma 15, the number of exceptional vertices in S_0^e is a small fraction of the set of active vertices S_0 . There are therefore at least at least $|S_0|/2$ (say) normal rows and columns. By Lemma 20 only a small fraction of the set of normal rows and columns spanned by S_0 can end up in the Q -neighbourhood of a vertex of K^γ in V_1 . Furthermore, by arguing as in the proof of Proposition 22, if a normal row or a normal column is not in the Q -neighbourhood of a decodable agglutinating vertex, it must belong to one of the sets Row_i^β or Col_i^β for $\beta = 1/6 + o_\Delta(1)$, otherwise it would have already been decoded by the sequential decoder. Therefore, most normal rows and columns of S_0 must belong to the Q -neighbourhood of a decodable agglutinating vertex of S_1 (hence the terminology 'agglutinating'). The number of decodable agglutinating vertices of S_1 is therefore at least equal to $\kappa|S_0|/\Delta$ for some constant κ , and the result follows from Proposition 19. \square

Analysis of the second parallel decoding step. This step is simpler than the first. Recall that the second parallel decoding step is applied on vertices of V_0 and uses regular decoding (*i.e.* decreases $|\hat{Z}|$ as much as possible on complete, unpunctured, local views).

To avoid confusion between the different stages of the decoder, let us denote by $\hat{Z}, \hat{Z}', \hat{Z}''$ respectively the values of the mismatch \hat{Z} of the stalled serial decoder before the first parallel decoding step, just after the first parallel decoding step on V_1 , and finally after the second parallel decoding step on V_0 . Accordingly we write $\hat{Z} = C_0 + R_0 + C_1 + R_1$, $\hat{Z}' = C'_0 + R'_0 + C'_1 + R'_1$, $\hat{Z}'' = C''_0 + R''_0 + C''_1 + R''_1$, and use the notation $S_0 = S_{00} \cup S_{11}$, $S'_0 = S'_{00} \cup S'_{11}$, $S''_0 = S''_{00} \cup S''_{11}$ for the set of active vertices of V_0 at each stage of decoding.

We recall also that $\mathcal{Z} = (C_0, R_0, C_1, R_1)$ has been chosen to be a minimal representation of \hat{Z} . The representations $\mathcal{Z}' = (C'_0, R'_0, C'_1, R'_1)$ and $\mathcal{Z}'' = (C''_0, R''_0, C''_1, R''_1)$ are however simply deduced from \mathcal{Z} by the decoder's increments.

Lemma 25. *Let $x_0 \geq 1$ (resp. $x_1 \geq 1$) be such that $x_0\delta\Delta$ (resp. $x_1\delta\Delta$) is the average weight of $R''_0 + C''_0$ (resp. $R''_1 + C''_1$) on the union of the Q -neighbourhoods of S''_{00} (resp. S''_{11}). Under the hypothesis $|S_0| \leq \frac{\delta}{8\Delta}|V_0|$, we have*

$$\begin{aligned} |S''_{00}| &\leq \frac{256}{x_0^2\delta^2}|S'_0| \\ |S''_{11}| &\leq \frac{256}{x_1^2\delta^2}|S'_0|. \end{aligned}$$

Proof. Observe that $x_i \geq 1$, $i = 0, 1$, because $\delta\Delta$ is the minimum distance of the dual tensor code. Now for any vertex v of V_{00} , we must have

$$|(R''_0 + C''_0)|_{Q(v)}| \leq 2|(R'_1 + C'_1)|_{Q(v)}| \quad (6)$$

(otherwise the zero vector is a better choice than $R''_0 + C''_0$ on $Q(v)$, since it gives a lower value of $|\hat{Z}''|$).

From (6) we have that the average degree of vertices of S''_{00} in the subgraph of \mathcal{G}_0^\square induced by S''_{00} and S'_{11} is at least $x\delta\Delta/2$. Applying the Expander mixing Lemma in \mathcal{G}_0^\square we have

$$\frac{1}{2}|S''_{00}|x\delta\Delta \leq |E(S''_{00}, S'_{11})| \leq \Delta^2 \frac{|S''_{00}||S'_{11}|}{|V_{00}|} + 4\Delta\sqrt{|S''_{00}||S'_{11}|}.$$

Writing $|S'_{11}|/|V_{00}| \leq 2|S'_0|/|V_0| \leq 2|S_0|/|V_0|$ by Theorem 23 and applying the hypothesis $|S_0| \leq \frac{\delta}{8\Delta}|V_0|$, we get

$$\begin{aligned} \frac{1}{4}|S''_{00}|x_0\delta\Delta &\leq 4\Delta\sqrt{|S''_{00}||S'_0|} \\ |S''_{00}|^{1/2} &\leq \frac{16}{x_0\delta}|S'_0|^{1/2} \\ |S''_{00}| &\leq \frac{256}{x_0^2\delta^2}|S'_0|. \end{aligned}$$

The same reasoning is valid for S''_{11} . □

The following theorem is the core result of the analysis of the whole parallel decoding procedure, it states that together the two steps of the parallel decoding procedure decrease the Hamming weight of \hat{Z} .

Theorem 26. *Under the hypothesis $|S_0| \leq \frac{\delta}{8\Delta}|V_0|$, after the second parallel decoding step we have $|\hat{Z}''| \leq \kappa|S_0|\Delta^{4\epsilon}$ for some constant κ .*

Proof. From Lemma 25 we obtain that

$$|R_0'' + C_0''| = \sum_{v \in S_{00}''} |(R_0'' + C_0'')_{Q(v)}| = |S_{00}''|x_0\delta\Delta \leq \frac{256}{x_0\delta}|S_0''|\Delta$$

and applying Theorem 23 we obtain

$$|R_0'' + C_0''| \leq \kappa|S_0|\Delta^{4\epsilon}$$

for some constant κ (remembering that $x_0 \geq 1$). We have naturally the same upper bound for $|R_1'' + C_1''|$ and since

$$|\hat{Z}''| \leq |R_0'' + C_0''| + |R_1'' + C_1''|$$

we have the claimed result. \square

For the proof of Theorem 2 we will also need the following Lemma. Recall that $\|\hat{Z}\|$ denotes the minimum weight $\|\mathcal{Z}\|$ of a representation of \hat{Z} .

Lemma 27. *Let \hat{Z} and \hat{Z}'' be the states of \hat{Z} just before a parallel decoding procedure and just after the second step of the parallel decoding procedure. We have $\|\hat{Z}''\| \leq \|\hat{Z}\|/2$.*

Proof. Recall that we have taken $\mathcal{Z} = (C_0, R_0, C_1, R_1)$ such that $\|\mathcal{Z}\| = \|\hat{Z}\|$. As mentioned just before Lemma 25, the two representations $\mathcal{Z}' = (C_0', R_0', C_1', R_1')$ and $\mathcal{Z}'' = (C_0'', R_0'', C_1'', R_1'')$ that are deduced from \mathcal{Z} through the parallel decoder's updates are not necessarily minimal, but nevertheless it suffices to show that $\|\mathcal{Z}''\| \leq \|\mathcal{Z}\|/2$.

Let x_0, x_1 be defined as in Lemma 25. We consider two cases

- (i) $x_0 > \Delta^{1/4}$. In this case Lemma 25 and Theorem 23 imply $|S_{00}''| \leq \frac{\kappa\Delta^{4\epsilon}}{\delta^2\Delta^{3/2}}|S_0|$ for some constant κ . We deduce $\|R_0''\| + \|C_0''\| \leq |S_{00}''|2\Delta \leq |S_0|/4$ for Δ large enough.
- (ii) $x_0 \leq \Delta^{1/4}$. All vertices of S_{00}'' on which $R_{00}'' + C_{00}''$ has Hamming weight less than $\Delta^{3/2-\epsilon}$ contribute, by applying robustness with $w = \Delta^{3/2-\epsilon}$, at most $\frac{2}{\delta}\Delta^{1/2-\epsilon}|S_{00}''|$ to $\|R_0''\| + \|C_0''\|$. The vertices of S_{00}'' on which $R_{00}'' + C_{00}''$ has Hamming weight $\geq \Delta^{3/2-\epsilon}$ number not more than $\frac{\delta}{\Delta^{1/4-\epsilon}}|S_{00}''|$, and we again easily deduce from Lemma 25 and Theorem 23 that $\|R_0''\| + \|C_0''\| \leq |S_0|/4$ for Δ large enough.

We therefore always have $\|R_0''\| + \|C_0''\| \leq |S_0|/4$. We also have the symmetrical inequality $\|R_1''\| + \|C_1''\| \leq |S_0|/4$, and summing both we get

$$\|R_0''\| + \|C_0''\| + \|R_1''\| + \|C_1''\| \leq |S_0|/2 \leq \|\hat{Z}\|/2.$$

\square

Proof of Theorem 2. Let \mathbf{e} be the original error vector, with local view e_v for vertices $v \in V$. We recall that the decoder initially computes a local estimation ε_v of the error on all Q -neighbourhoods of vertices $v \in V_1$. We have $\varepsilon_v = e_v + r_v + c_v$ where $r_v + c_v$ is a local dual tensor codeword. Over the partition $(Q(v))_{v \in V_1}$ of the set Q of coordinates, the global vector ε_{01} defined by the local views ε_v differs from the error vector \mathbf{e} by a vector $R_0 + C_1$, similarly the global vector ε_{10} defined by the local views ε_v for $v \in V_0$ differs from \mathbf{e} by a vector $R_1 + C_0$. The two vectors $R_0 + C_1$ and $R_1 + C_0$ do not a priori coincide, and their sum defines the original mismatch Z . The goal of the decoder is to find a decomposition $Z = \hat{C}_0 + \hat{R}_0 + \hat{C}_1 + \hat{R}_1$ and output $\hat{\mathbf{e}} = \varepsilon_{01} + \hat{R}_0 + \hat{C}_1$.

We will have proved that our decoder works after ensuring two things:

- that it indeed finds a decomposition $Z = \hat{C}_0 + \hat{R}_0 + \hat{C}_1 + \hat{R}_1$,
- that the corresponding $\hat{\mathbf{e}}$ is such that $|\mathbf{e} + \hat{\mathbf{e}}| = |R_0 + C_1 + \hat{R}_0 + \hat{C}_1| < d_{\min}$ where d_{\min} is the minimum distance of the quantum code.

By Proposition 24 and Theorem 26 we have that, for a sufficiently small value of ε (namely $\varepsilon < 1/10$), the Hamming weight $|\hat{Z}|$ decreases during parallel decoding steps whenever we have $|S_0| \leq \frac{\delta}{8\Delta}|V_0|$. Furthermore, the Hamming weight $|\hat{Z}|$ decreases at every sequential decoding step by definition. Therefore to ensure convergence to some valid output $\hat{\mathbf{e}}$ we only need to ensure that the condition $|S_0| \leq \frac{\delta}{8\Delta}|V_0|$ is satisfied throughout the decoding process.

To estimate the number $|S_0|$ of active vertices in V_0 , we can simply upper bound it by $\|\hat{Z}\|$ and track the evolution of $\|\hat{Z}\|$ during the course of the decoding algorithm.

At the initial stage, we have $\hat{Z} = Z$ which is the original mismatch vector. We note that we will have $r_v + c_v \neq 0$ only if $|e_v| \geq \delta\Delta/2$. Since Q is partitioned into Q -neighbourhoods of vertices of V_{01} , we have that the number of vertices of V_{01} for which $r_v + c_v \neq 0$ is at most $2|e|/\delta\Delta$, with the same conclusion holding for V_{10} . Crudely upper bounding the contribution of every such vertex to $\|Z\|$ by 2Δ , we have $\|Z\| \leq 8|e|/\delta$. Strictly speaking, we have upper bounded the number of terms of one possible expression of Z as a sum of local row and column vectors, but the conclusion holds anyway since $\|Z\|$ refers to the minimum number of terms of such a decomposition of Z .

Similarly, we remark that $|r_v + c_v| \leq 2|e_v|$, from which we have the upper bound for the Hamming weight of Z , $|Z| \leq 4|e|$, and also

$$|R_0 + C_1| \leq 2|e|, \tag{7}$$

for (R_0, C_0, R_1, C_1) the original decomposition of Z .

For any sequential decoding step, we clearly have that the weight $\|\hat{Z}\|$ is increased by at most 2Δ . Since the weight $\|\hat{Z}\|$ only decreases during a parallel decoding step, we have that $\|\hat{Z}\|$ stays upper bounded by $8|e|/\delta + |Z|2\Delta \leq 8|e|(\Delta + 1/\delta)$ so that the requirement $|S_0| \leq \frac{\delta}{8\Delta}|V_0|$ is always satisfied whenever

$$8|e|(\Delta + \frac{1}{\delta}) \leq \frac{\delta}{8\Delta}|V_0|$$

i.e. whenever

$$|e| \leq \frac{\delta^2}{64\Delta(\delta\Delta + 1)} |V_0| = \frac{\delta^2}{32\Delta^3(\delta\Delta + 1)} |Q|$$

since $|Q| = |V_0|\Delta^2/2$. We have therefore proved that under the hypothesis of Theorem 2, the decoder always finds a decomposition of Z .

Next we evaluate an upper bound on the Hamming weight $|\hat{R}_0 + \hat{C}_1|$ of the output of the decoder. To this end we upper bound the increment of $|\hat{R}_0 + \hat{C}_1|$ during sequential decoding, during all the first parallel decoding steps, and finally during all second parallel decoding steps.

We note that during a sequential decoding step, the Q -neighbourhood of a single vertex is modified, this vertex being in V_{00} or in V_{11} . This translates either into a modification of \hat{R}_0 or into a modification of \hat{C}_1 , which affects at most Δ rows or Δ columns. Therefore a sequential decoding step translates into an augmentation of $|\hat{R}_0 + \hat{C}_1|$ by at most Δ^2 . Since the value of $|\hat{Z}|$ decreases at every step, the total contribution of sequential decoding does not exceed $|Z|\Delta^2 \leq 4|e|\Delta^2$.

During a first parallel decoding step, the only modification to $|\hat{R}_0 + \hat{C}_1|$ is induced by the parallel decoder's action on vertices v of V_{01} . We note that the decoder modifies the associated Q -neighbourhood only if it decreases $|\hat{Z}|$ on a subset of $Q(v)$ by at least $w/2 = \Delta^{3/2-\varepsilon}/2$. Since all these sub- Q -neighbourhoods are disjoint, their number is at most $2|\hat{Z}|/\Delta^{3/2-\varepsilon}$ and the total increment of $|\hat{R}_0 + \hat{C}_1|$ during this step is at most Δ^2 times this number of vertices, namely $2|\hat{Z}|\Delta^{1/2+\varepsilon}$. Now by Proposition 24 and Theorem 26, the next time a first parallel decoding step occurs the value of $|\hat{Z}|$ will have been divided by at least $\kappa\Delta^{1/2-5\varepsilon}$ for some constant κ , therefore the sum of the contributions to $|\hat{R}_0 + \hat{C}_1|$ of all parallel decoding steps is at most $4|Z|\Delta^{1/2+\varepsilon} \leq 16|e|\Delta^{1/2+\varepsilon}$ by summing the converging geometric series.

Finally, for the contribution of the second parallel decoding steps, we argue as before that the increment of $|\hat{R}_0 + \hat{C}_1|$ is at most Δ^2 per vertex on which the decoder takes action. The number of such vertices is not more than $|S'_0| + |S''_0|$, $|S'_0|$ being an upper bound on the number of active vertices on which the decoder may act, and $|S''_0|$ being an upper bound on the number of inactive vertices which will become active because the decoder decides to update them. Applying Theorem 23 and Lemma 25 (which also holds for S''_{11}), we have that $|S'_0| \leq \kappa|S_0|/\Delta^{1-4\varepsilon}$ and $|S''_0| \leq \kappa|S_0|/\Delta^{1-4\varepsilon}$ for some constant κ . The increment to $|\hat{R}_0 + \hat{C}_1|$ is therefore at most

$$\kappa|S_0|\Delta^{1+4\varepsilon} \leq \kappa\|\hat{Z}\|\Delta^{1+4\varepsilon}$$

for some constant κ , where \hat{Z} refers to the value just before the parallel decoding procedure is applied. As stated above, the maximum value of $\|\hat{Z}\|$ is bounded from above by $8|e|(\Delta+1/\delta)$. Since every parallel decoding procedure divides the current value of $\|\hat{Z}\|$ by at least 2 (by Lemma 27), it is an easy exercise in combinatorics to see that the sum of all the values of $\|\hat{Z}\|$ before each parallel decoding procedure cannot exceed $16|e|(\Delta+1/\delta)$, irrespective of the distribution of the parallel decoding steps during the whole decoding procedure. We obtain therefore that total contribution of the second parallel decoding

steps is at most

$$\kappa|e|\Delta^{2+4\epsilon}$$

for some constant κ and Δ large enough.

Summarising, the upper bound on the contribution of the second parallel decoding steps dominates the other terms and we have that the final output of the decoder is $\hat{R}_0 + \hat{C}_1$ of Hamming weight

$$|\hat{R}_0 + \hat{C}_1| \leq \kappa|e|\Delta^{2+4\epsilon}$$

for some constant κ . From (7) we therefore obtain

$$|e + \hat{e}| = |R_0 + C_1 + \hat{R}_0 + \hat{C}_1| \leq |R_0 + C_1| + |\hat{R}_0 + \hat{C}_1| \leq c|e|\Delta^{2+4\epsilon}$$

for some constant κ . Applying the lower bound on d_{\min} from Theorem 1 we obtain that $|e + \hat{e}| < d_{\min}$ which concludes the proof. \square

7 Link with the Lifted Product codes

In this section, we briefly recall the construction of the lifted product codes introduced by Pantelev and Kalachev [PK21a] and explain how quantum Tanner codes can be obtained from lifted product codes, or more generally from hypergraph product codes. Finally, we show that the decoder we studied in the previous sections immediately gives an efficient decoder for lifted product codes.

7.1 Construction of the Lifted Product codes

Take a group G and two symmetric subsets $A = \{a_i\} \subset G$, $B = \{b_i\} \subset G$ of size Δ . Let h_A and h_B be two parity-check matrices on $\mathbb{F}_2 \subset \mathbb{F}_2[G]$ of size $m \times \Delta$. We further define two diagonal matrices of size Δ over G :

$$D_A := \text{diag}(a_1, \dots, a_\Delta), \quad D_B := \text{diag}(b_1, \dots, b_\Delta)$$

and set $H_A := h_A D_A$ and $H_B := h_B D_B$.

Consider the graph with two vertices and Δ parallel edges, and define two Tanner codes T_A and T_B on this graph, where the local constraints on both vertices are given by the codes $\ker h_A$ and $\ker h_B$, respectively. The corresponding Tanner graphs have vertex sets $A \cup (C_0 \cup C_1)$ for T_A and $B \cup (D_0 \cup D_1)$ for T_B . In particular, we have $|C_0| = |C_1| = |D_0| = |D_1| = m$.

The lifted product codes of [PK21a] are defined on qubits indexed by ($|G|$ copies of) $A \times B \cup (C_0 \cup C_1) \times (D_0 \cup D_1)$, and have σ_X -generators indexed by ($|G|$ copies of) $A \times (D_0 \cup D_1)$, and σ_Z -generators by ($|G|$ copies of) $(C_0 \cup C_1) \times B$. We will use the following identification between \mathbb{F}_2^G and the group algebra $\mathbb{F}_2[G]$:

$$(x_g)_{g \in G} \leftrightarrow \sum_{g \in G} x_g g$$

where $x_g \in \mathbb{F}_2$. This allows us to describe a subset of $A \times B \times G$, *e.g.* the support of a stabilizer, as a matrix in $\mathbb{F}_2[G]^{A \times B}$.

A relatively simple way of describing the lifted product codes is *via* their stabilizer elements. For instance, a general σ_Z -stabilizer is associated with two matrices (U_0, U_1) with entries indexed by $C_0 \times B$ and $C_1 \times B$, and values in $\mathbb{F}_2[G]$. The corresponding operator is a product of σ_Z -Pauli matrices with a support given by

$$\underbrace{h_A^T U_0 + H_A^T U_1}_{A \times B} + \underbrace{U_0 h_B^T}_{C_0 \times D_0} + \underbrace{U_0 H_B^T}_{C_0 \times D_1} + \underbrace{U_1 h_B^T}_{C_1 \times D_0} + \underbrace{U_1 H_B^T}_{C_1 \times D_1}.$$

We leave the factor G implicit for ease of notation, and represent the support of a set by its indicator vector. Similarly, a general σ_X -stabilizer element (V_0, V_1) has support

$$\underbrace{V_0 h_B + V_1 H_B}_{A \times B} + \underbrace{h_A V_0}_{C_0 \times D_0} + \underbrace{H_A V_0}_{C_1 \times D_0} + \underbrace{h_A V_1}_{C_0 \times D_1} + \underbrace{H_A V_1}_{C_1 \times D_1}.$$

A generator of the code is obtained by taking (U_0, U_1) , or (V_0, V_1) , of Hamming weight equal to 1. The generators of the lifted product codes have weight at most 2Δ .

7.2 Quantum Tanner codes from Lifted Product codes

Let us define matrices g_A indexed by $C' \times A$ and g_B by $D' \times B$ to be the parity-check matrices of $(\ker h_A)^\perp$ and $(\ker h_B)^\perp$. Note that $|C'| = |D'| = \Delta - m$ and that

$$g_A h_A^T = 0, \quad g_B h_B^T = 0.$$

We further define $G_A := g_A D_A^{-1}$ and $G_B := g_B D_B^{-1}$ that satisfy $G_A H_A^T = 0$ and $G_B H_B^T = 0$.

The quantum Tanner code is nothing but the code with σ_Z -stabilizers $(U'_0, U'_1) \in \mathbb{F}_2[G]^{(C_0 \cup C_1) \times D'}$ and σ_X -stabilizers $(V'_0, V'_1) \in \mathbb{F}_2[G]^{(C' \times (D_0 \cup D_1))}$ corresponding to the stabilizers $(U_0, U_1) := (U'_0 G_B, U'_1 g_b)$ and $(V_0, V_1) := (g_A^T V'_0, G_A^T V'_1)$ of the Lifted Product code. They take the following form:

$$\begin{aligned} \sigma_Z - \text{stabilizer } (U'_0, U'_1) : & \quad \underbrace{h_A^T U'_0 G_B + H_A^T U'_1 g_b}_{A \times B} + \underbrace{U'_0 G_B h_B^T}_{C_0 \times D_0} + \underbrace{U'_0 G_B H_B^T}_{C_0 \times D_1} + \underbrace{U'_1 g_b h_B^T}_{C_1 \times D_0} + \underbrace{U'_1 g_b H_B^T}_{C_1 \times D_1} \\ & = \underbrace{h_A^T U'_0 G_B + H_A^T U'_1 g_b}_{A \times B} + \underbrace{U'_0 G_B h_B^T}_{C_0 \times D_0} + \underbrace{U'_1 g_b H_B^T}_{C_1 \times D_1}, \\ \sigma_X - \text{stabilizer } (V'_0, V'_1) : & \quad \underbrace{g_A^T V'_0 h_B + G_A^T V'_1 H_B}_{A \times B} + \underbrace{h_A g_A^T V'_0}_{C_0 \times D_0} + \underbrace{H_A g_A^T V'_0}_{C_1 \times D_0} + \underbrace{h_A G_A^T V'_1}_{C_0 \times D_1} + \underbrace{H_A G_A^T V'_1}_{C_1 \times D_1} \\ & = \underbrace{g_A^T V'_0 h_B + G_A^T V'_1 H_B}_{A \times B} + \underbrace{H_A g_A^T V'_0}_{C_1 \times D_0} + \underbrace{h_A G_A^T V'_1}_{C_0 \times D_1}. \end{aligned}$$

We immediately notice that the supports of the σ_X -stabilizers and the σ_Z -stabilizers only overlap on $A \times B$. In particular, we can simply discard all the qubits in $(C_0 \cup C_1) \times (D_0 \cup D_1)$

and define a shorter stabilizer code on $A \times B$ with the stabilizers corresponding to (U'_0, U'_1) and (V'_0, V'_1) . This is nothing but the quantum Tanner code $Q = (\mathcal{C}_0, \mathcal{C}_1)$ with local tensor codes $C_A \otimes C_B$ and $C_A^\perp \times C_B^\perp$ where we define

$$C_A = \ker g_A, \quad C_B = \ker h_B, \quad C_A^\perp = \ker h_A, \quad C_B^\perp = \ker g_B.$$

7.3 Decoding Lifted Product codes

An error vector $\mathbf{e} = (\mathbf{e}^X, \mathbf{e}^Z)$ for the lifted product code is described a σ_X -type error \mathbf{e}^X and a σ_Z -type error \mathbf{e}^Z :

$$\mathbf{e}^{X/Z} = \mathbf{e}_{AB}^{X/Z} + \mathbf{e}_{00}^{X/Z} + \mathbf{e}_{01}^{X/Z} + \mathbf{e}_{10}^{X/Z} + \mathbf{e}_{11}^{X/Z}$$

with $\mathbf{e}_{AB}^{X/Z} \in \mathbb{F}_2[G]^{A \times B}$ and $\mathbf{e}_{ij}^{X/Z} \in \mathbb{F}_2[G]^{C_j \times D_i}$. This error gives rise to a syndrome (S_0, S_1, T_0, T_1) with (S_0, S_1) detected by σ_Z -generators and (T_0, T_1) detected by σ_X -generators, as follows:

$$\begin{aligned} S_0 &= h_A \mathbf{e}_{AB}^Z + \mathbf{e}_{00}^Z h_B + \mathbf{e}_{10}^Z H_B && \in \mathbb{F}_2[G]^{C_0 \times B}, \\ S_1 &= H_A \mathbf{e}_{AB}^Z + \mathbf{e}_{01}^Z h_B + \mathbf{e}_{11}^Z H_B && \in \mathbb{F}_2[G]^{C_1 \times B}, \\ T_0 &= \mathbf{e}_{AB}^X h_B^T + h_A^T \mathbf{e}_{00}^X + H_A^T \mathbf{e}_{01}^X && \in \mathbb{F}_2[G]^{A \times D_0}, \\ T_1 &= \mathbf{e}_{AB}^X H_B^T + h_A^T \mathbf{e}_{10}^X + H_A^T \mathbf{e}_{11}^X && \in \mathbb{F}_2[G]^{A \times D_1}. \end{aligned}$$

One of the features of this quantum code (and also of the hypergraph product code of two classical Tanner codes) is that one can easily remove the error from $C_0 \times D_1 \cup C_1 \times D_0$ by adding a stabilizer of weight $O(|e_{10} \cup e_{01}|)$. Since the code is LDPC, it means that this can be done without increasing the error weight too much.

Lemma 28. *For any error \mathbf{e} , there exists an equivalent error \mathbf{e}' such that $\mathbf{e}'_{01} = 0$, $\mathbf{e}'_{10} = 0$ and*

$$|\mathbf{e}'| \leq |\mathbf{e}| + 4\Delta^2(|e_{01}| + |e_{10}|).$$

Proof. Consider a general error $(\mathbf{e}^X, \mathbf{e}^Z)$. We focus on the σ_Z -part $(\mathbf{e}_{AB}^Z, \mathbf{e}_{00}^Z, \mathbf{e}_{10}^Z, \mathbf{e}_{01}^Z, \mathbf{e}_{11}^Z)$. The σ_X -part is treated identically. Let U_0, U_1 be matrices such that

$$U_0 H_B = \mathbf{e}_{10}^Z, \quad U_1 h_B = \mathbf{e}_{01}^Z.$$

We have that $|U_0| \leq \Delta |\mathbf{e}_{10}^Z|$ and $|U_1| \leq \Delta |\mathbf{e}_{01}^Z|$. These stabilizers induce new errors $h_A^T U_0 + H_A^T U_1 + U_0 h_B^T + U_1 H_B^T$, which have a weight at most $2\Delta(|U_0| + |U_1|) \leq 2\Delta^2(|e_{10}| + |e_{01}|)$. \square

Thanks to Lemma 28, we now focus on a σ_Z -error \mathbf{e} such that $\mathbf{e}_{01} = 0$ and $\mathbf{e}_{10} = 0$. Its syndrome reads:

$$T_0 = \mathbf{e}_{AB} h_B^T + h_A^T \mathbf{e}_{00}, \quad T_1 = \mathbf{e}_{AB} H_B^T + H_A^T \mathbf{e}_{11}.$$

We can multiply these syndromes by g_A and G_A respectively and obtain

$$g_A T_0 = g_A \mathbf{e}_{AB} h_B^T, \quad G_A T_1 = G_A \mathbf{e}_{AB} H_B^T,$$

which is exactly the decoding problem for quantum Tanner codes. We note that a similar strategy was already exploited in [QC21]. The decoder presented in the previous sections can correct this error provided its weight is small enough (Theorem 2). More precisely, if $|e_{AB}| \leq \kappa n / \Delta^4$, then the decoder outputs an equivalent error \hat{e}_{AB} for the quantum Tanner code. In other words, there exists a stabilizer element associated with (U_0, U_1) such that

$$\hat{e}_{AB} = e_{AB} + h_A^T U'_0 G_B + H_A^T U'_1 g_B,$$

as described in the previous subsection.

Let us introduce a matrix n_A indexed by $C_0 \times A$ such that $n_A h_A^T = \mathbb{1}_{C_0}$. We assume here that h_A is a full-rank matrix. We also denote $N_A := n_A D_A^{-1}$, which implies $N_A H_A^T = \mathbb{1}_{C_1}$.

The decoder for the lifted product code will simply return the triple $(\hat{e}_{AB}, \hat{e}_{00}, \hat{e}_{11})$ with

$$\hat{e}_{00} := n_A(T_0 + \hat{e}_{AB} h_B^T), \quad \hat{e}_{11} := N_A(T_1 + \hat{e}_{AB} H_B^T).$$

Straightforward manipulations show that

$$\begin{aligned} \hat{e}_{00} &= n_A((e_{AB} h_B^T + h_A^T e_{00}) + (e_{AB} + h_A^T U'_0 G_B + H_A^T U'_1 g_B) h_B^T) \\ &= e_{00} + U'_0 G_B h_B^T \end{aligned}$$

and

$$\begin{aligned} \hat{e}_{11} &= N_A((e_{AB} H_B^T + H_A^T e_{11}) + (e_{AB} + h_A^T U'_0 G_B + H_A^T U'_1 g_B) H_B^T) \\ &= e_{11} + U'_1 g_B H_B^T \end{aligned}$$

which means that the true error and the error returned by the decoder differ by

$$\underbrace{h_A^T (U'_0 G_B) + H_A^T (U'_1 g_B)}_{A \times B} + \underbrace{(U'_0 G_B) h_B^T}_{C_0 \times D_0} + \underbrace{(U'_1 g_B) H_B^T}_{C_1 \times D_1}.$$

In other words, they differ by the stabilizer element $(U_0, U_1) = (U'_0 G_B, U'_1 g_B)$ of the lifted product code, meaning that they are equivalent and that the decoder succeeded.

We have therefore proved the following:

Theorem 29. *There exists a constant κ , depending only on δ , a lower bound for the minimum distances of both component codes C_A and C_B , such that for large enough fixed Δ , the above decoding algorithm corrects all error patterns of weight less than $\kappa n / \Delta^6$ for the lifted product code of length $n = |Q|$.*

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