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Radical Sylvester-Gallai Theorem for Cubics

Rafael Oliveira [*] Akash Kumar Sengupta [†]

Abstract

Let $\mathcal{F} = \{F_1, \ldots, F_m\}$ be a finite set of irreducible homogeneous multivariate polynomials of degree at most 3 such that F_i does not divide F_j for $i \neq j$. We say that $\mathcal{F}$ is a cubic radical Sylvester-Gallai configuration if for any two distinct F_i, F_j there exists a third polynomial F_k such that whenever F_i, F_j vanish, F_k also vanishes. In particular, for any two indices $i, j \in [m]$, there exists $k \in [m] \setminus \{i, j\}$ such that $F_k \in \text{rad}(F_i, F_j)$.

We prove that any cubic radical Sylvester-Gallai configuration is low-dimensional, that is

$$\dim \text{span}_{\mathbb{K}} \{\mathcal{F}\} = O(1).$$

This solves a conjecture of Gupta [Gup14] in degree 3 and generalizes the result in [Shp20], which proved that quadratic radical Sylvester-Gallai configurations are low-dimensional. Our result takes us one step closer towards solving the non-linear Sylvester-Gallai conjectures of Gupta [Gup14], which would yield the first deterministic polynomial time algorithm for the PIT problem for depth-4 circuits of bounded top and bottom fanins.

To prove our Sylvester-Gallai theorem, we develop several new tools combining techniques from algebraic geometry and elimination theory. Among our technical contributions, we prove a structure theorem characterizing non-radical ideals generated by two cubic forms, generalizing the structure theorems of [HP94, CTSSD87, Shp20]. Moreover, building upon the groundbreaking work [AH20a], we introduce the notion of wide Ananyan-Hochster algebras and show that these algebras allow us to transfer the local conditions of Sylvester-Gallai configurations into global conditions.

[*]Cheriton School of Computer Science, University of Waterloo, rafael@uwaterloo.ca.
[†]Columbia University, akashs@math.columbia.edu

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1 Introduction

In 1893, Sylvester asked a basic question in combinatorial geometry ([Syl93]): given a finite set of distinct points $v_1, \ldots, v_m \in \mathbb{R}^N$ such that the line defined by any pair of distinct points v_i, v_j contains a third point v_k in the set, must all points in the set be collinear? This question was independently answered in the affirmative by Melchior and Gallai [Mel40, Gal44], and is now known as the Sylvester-Gallai theorem.

Since then, generalizations of Sylvester's question have received considerable attention by mathematicians and computer scientists [EK66, Han66, Ser66, Kel86, BDWY11], and results such as the above are known as Sylvester-Gallai type theorems. In its most general form, a Sylvester-Gallai type configuration is a finite set of geometric objects which satisfy certain local conditions/dependencies. For instance, in the original question above the geometric objects are the points and the local dependencies are collinear dependencies amongst the points of the set. The general underlying theme of Sylvester-Gallai type problems is the following local-to-global phenomenon: must these local constraints on the geometric objects imply a global constraint on such configurations? The main question of concern is:

Are Sylvester-Gallai type configurations always low-dimensional?

For a thorough survey on earlier works on Sylvester-Gallai type problems, we refer the reader to [BM90], and for a survey on applications to computer science, we refer the reader to [SY10, Dvi12].

The perspective above highlights the *geometric* aspects of Sylvester-Gallai problems. If one looks at the problems through the *algebraic and computational* lens (via the algebra-geometry dictionary), one realizes that they are intrinsically related to the study of *cancellations/relations* in computational problems. In algebraic and computational terms, the Sylvester-Gallai questions become:

Must Sylvester-Gallai configurations depend only on "few variables"?

Since cancellations are remarkably powerful in computation, as we know from several results in boolean and algebraic complexity [Tar88, Raz92, RW92, GMOR15], it is no wonder that Sylvester-Gallai type theorems have appeared in the study of cancellations in algebraic complexity – the polynomial identity testing problem. Moreover, since Sylvester-Gallai problems are about cancellations, it is not surprising that the solutions of such problems has used sophisticated mathematics.

As an example, linear Sylvester-Gallai type problems have found applications in diverse subfields of theoretical computer science. In algebraic complexity theory, linear Sylvester-Gallai theorems were used in algorithms for polynomial identity testing (PIT) and reconstrution of depth-3 circuits of bounded top fanin [DS07, KS09, SS13, Sin16]. In coding theory, linear Sylvester-Gallai theorems were used to prove non-existence of 2-query LCCs over fields of characteristic zero [BDWY11].

In [Gup14], Gupta proposed far-reaching non-linear generalizations of the known Sylvester-Gallai type problems in order to give a deterministic polynomial time black-box PIT algorithm for the model of depth 4 algebraic circuits with constant top fanin. Gupta conjectured that the non-linear Sylvester-Gallai type problems that he proposed were also "low dimensional" and must depend on "few variables".

The main conjecture in Gupta's work that needs to be solved in order to prove that his black-box PIT algorithm runs in deterministic polynomial time is the following [Gup14, Conjecture 1].

Conjecture 1.1 ((k, d, c)-Sylvester-Gallai conjecture)**.** *Let $k, d, c \in \mathbb{N}$ be positive integers. There exists a function $B : \mathbb{N}^3 \to \mathbb{N}$ such that for any collection of k finite sets $\mathcal{F}_1, \dots, \mathcal{F}_k \subset \mathbb{K}[x_1, \dots, x_N]$ of irreducible polynomials of degree at most d satisfying*

- $\bigcap_i \mathcal{F}_i = \varnothing$,
- *for every $P_1, \dots, P_{k-1}$ each from a distinct set $\mathcal{F}_{i_j}$, there are forms $Q_1, \dots, Q_c$ in the remaining set such that $\prod_{i=1}^{c} Q_i \in \text{rad}(P_1, \dots, P_{k-1})$,*

the transcendence degree of the union $\bigcup_i \mathcal{F}_i$ is bounded above by $B(k, d, c)$. In particular, this bound is independent of the number of variables N or the size of the sets $\mathcal{F}_i$.

The first challenge in Gupta's series of conjectures is a non-linear version of the (dual) linear Sylvester-Gallai problem, which we now state. We begin with the definition of a radical Sylvester-Gallai configuration. For the remainder of the paper, we will adopt the usual notation and use the term *form* to refer to a homogeneous polynomial, and we will take $\mathbb{K}$ to be an algebraically closed field of characteristic zero.

Definition 1.2 (Radical Sylvester-Gallai configuration)**.** Let d be a positive integer. We say that a finite set $\mathcal{F} = \{F_1, \dots, F_m\} \subset \mathbb{K}[x_1, \dots, x_N]$ of irreducible forms of degree $\leqslant d$ is a d-radical-SG configuration if the following conditions hold:

1. $F_i \notin (F_j)$ for any $i \neq j \in [m]$, (forms are "pairwise independent")
2. for every pair F_i, F_j, there is $k \neq i, j$ such that $F_k \in \text{rad}(F_i, F_j)$. (radical SG condition)

Note that if $d = 1$, the definition above specializes to the dual of the classical Sylvester-Gallai condition for points in $\mathbb{K}^N$. A fundamental step towards resolving Conjecture 1.1 would be the following conjecture of Gupta, [Gup14, Conjecture 2]:

Conjecture 1.3 (Radical Sylvester-Gallai)**.** *There is a function $\lambda : \mathbb{N} \to \mathbb{N}$ such that the transcendence degree of any d-radical-SG configuration $\mathcal{F}$ is upper bounded by $\lambda(d)$.*

In [Shp20], Shpilka broke ground on the conjecture above, solving it for $d = 2$. In this work, we solve the above conjecture for $d = 3$. The geometry of cubic or higher degree forms is significantly more complex and richer than that of quadratic forms. For example, quadratic forms over an algebraically closed field are determined, up to isomorphism, by their rank or strength. However, the same statement does not hold for cubics. Therefore, there are many challenges that are unique to the cubic SG problem, which are not amenable to the existing approaches for quadratic or linear SG problems. We generalize the approach of [Shp20], introducing several new concepts and techniques from algebraic geometry and elimination theory to solve Conjecture 1.3 in degree 3, and we expect that these techniques will prove fundamental for the full resolution of the conjecture in general.

We develop an inductive framework for solving Conjecture 1.3 and prove that this inductive approach solves Conjecture 1.3 in degree 3. In inductive arguments, one often needs to start from a stronger hypothesis, which may be harder to prove but amenable to induction. In a similar spirit, one of our conceptual contributions towards attacking Conjecture 1.3 is the following. We generalize radical SG configurations to have weaker constraints (what we call SG configurations over an algebra – Definition 6.6) and prove our main result by reducing 3-radical-SG configurations to these generalized SG configurations for quadratic forms. Since many of our tools work for forms of arbitrary degree, our work provides strong evidence that these radical SG configurations over algebras have the potential to allow us to reduce d-radical-SG configurations to generalized $(d-1)$-radical-SG configurations over an algebra.

Since the solution of the quadratic radical SG problem by [Shp20] (that is, Conjecture 1.3 for $d = 2$), further progress on the conjectures of [Gup14] have been made, whenever the forms in consideration are *quadratic forms*. This progress was done in the sequence of works [PS20a, PS20b], where the first work solved a product-version of the radical SG problem and the second work managed to solve Conjecture 1.1 for the case when $(k, d, c) = (3, 2, 4)$.

Just as the solution of the radical SG problem for quadratic forms fueled further progress towards Conjecture 1.1, we expect that our techniques and inductive approach will lead to progress towards a full resolution to Conjecture 1.1, albeit much more work remains to be done.

In the next section, we formally state the main contributions of this paper.

1.1 Main results

The main result of this paper is a solution of the radical Sylvester-Gallai problem for cubics. More precisely, we prove the following theorem, which implies Conjecture 1.3 for $d = 3$.

Theorem 1.4 (Radical Sylvester-Gallai for cubics)**.** *If $\mathcal{F}$ is a* 3-radical-SG *configuration, then*

$$\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{F}\} = O(1).$$

First we note that in order to bound the dimension or transcendence degree of $\mathcal{F}$, it is enough to show that $\mathcal{F}$ is contained in a small subalgebra of the polynomial ring. Before we are able to construct such algebras, an important observation is that if the configuration $\mathcal{F}$ has indeed low transcendence degree, then it should depend on "few variables," and these variables are (in principle) the generators of the small subalgebra that we will construct. To prove the theorem above in an "inductive way," we would like to reduce the 3-radical-SG configuration $\mathcal{F}$ to a 2-radical-SG configuration. While this may not be possible in general, one natural approach to induct is to try to construct a small subalgebra of the polynomial ring which contains all the cubic forms in our configuration $\mathcal{F}$. Then we can reduce to a generalized 2-radical-SG configuration over an algebra. The main technically challenging and most interesting part of this approach is to construct a subalgebra containing the cubics. In order to achieve our goal, we develop several algebraic-geometric tools along the way and prove results which are of independent interest. We describe some of these results below.

Since the local constraint in our SG problem is given by radicals, we are lead to the fundamental question: when is the ideal generated by two cubic forms not radical? This brings us to one of our main technical contributions: a structure theorem for non-radical ideals generated by two irreducible cubics.

Theorem 1.5. *Let F_1, F_2 be two non-associate irreducible cubic forms in the polynomial ring $\mathbb{K}[x_1, \cdots, x_n]$ over an algebraically closed field $\mathbb{K}$. Then at least one of the following holds:*

1. *The ideal (F_1, F_2) is radical.*
2. *There exists a linear minimal prime of (F_1, F_2), i.e. there exist two linearly independent linear forms x, y such that $(F_1, F_2) \subset (x, y)$.*

3. *There exists a quadratic minimal prime of* (F_1, F_2), *i.e. there exists a prime ideal* (Q, ℓ) *where* Q *is a quadratic form,* ℓ *is a linear form and* $(F_1, F_2) \subset (Q, \ell)$.

4. *There exist linear forms* x, y *such that* $xy^2 \in \operatorname{span}_{\mathbb{K}}\{F_1, F_2\}$.

5. *There exists a minimal prime* $\mathfrak{p}$ *of* (F_1, F_2) *such that* $\mathfrak{p}$ *is the homogeneous prime ideal a variety of minimal degree. In particular,* $\mathfrak{p} = (Q_1, Q_2, Q_3)$ *where* Q_i *are the quadratic forms given by the maximal minors of a matrix* M *of the form*

$$\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} y_0 & y_2 & y_3 \\ y_1 & y_3 & y_4 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} y_0 & y_2 & y_4 \\ y_1 & y_3 & y_5 \end{pmatrix},$$

where $y_1, \ldots, y_5$ *are linearly independent linear forms.*

This theorem generalizes previous structural results for intersections of two quadrics, which were proved in [HP94, CTSSD87, Shp20]. In [CTSSD87, Section 1], the authors analyzed the cycle decomposition of a complete intersection of two quadrics. Inspired by their approach, we generalize their result to the case of cubic forms. We use the standard tools of primary decomposition and Hilbert-Samuel multiplicity to characterize minimal primes of non-radical ideals generated by two cubic forms. The main conceptual takeaway is: the ideal generated by two cubic forms is not radical only if the cubic forms "are close" to each other, in the sense that they "share many common variables." Therefore we may try to construct a small algebra containing $\mathcal{F}$ that depends on "few variables" globally.

Theorem 1.5 provides us insight into the structure of non-radical ideals. However, it does not provide us with a quantitative bound on how many such non-radical ideals can be there in a SG configuration, and such quantitiative bounds will also be crucial for us to obtain global constraints on the SG configuration. Thus we are lead to the following key question: given a subalgebra of a polynomial ring and an irreducible form P outside of the subalgebra, how many forms Q are there in the subalgebra such that the ideal (P, Q) is not radical? Bertini's theorem in algebraic geometry tells us that we should expect that there are not too many such forms Q. We provide a criterion for determining when such an ideal is radical and consequently we obtain quantitative bounds on the number of such forms Q in a fixed algebra (see Corollary 3.24). One of our technical contributions in Section 3 is a criterion for radical ideals in terms of disciminants of polynomials (see Lemma 3.22). We state a simplified version below.

Lemma 1.6. *Let* $P, Q \in \mathbb{K}[z_1, \cdots, z_r, x_1, \cdots, x_s]$ *be irreducible forms of arbitrary degree over an algebraically closed field. Suppose the following conditions hold:*

1. $Q \in \mathbb{K}[x_1, \cdots, x_s]$,

2. $P \notin (x_1, \cdots, x_s)$

3. *For all* $i \in [r]$ *such that* P *depends on the variable* z_i, *we have* $\mathrm{Disc}_{z_i}(P) \notin (Q)$.

Then the ideal (P, Q) *is radical.*

The result above is a generalization of the fact that the discriminant of a univariate polynomial P is zero iff P has multiple roots – in other words the ideal (P) is not radical. A key property in our proof of the above lemma is that the ideal (P, Q) is Cohen-Macaulay. In this case, the

Cohen-Macaulay property implies that it is enough to show that P has no multiple roots after generic evaluations in the variables x_i which belong to the zero set of Q. Cohen-Macaulay rings and ideals play a central role in our arguments throughout Section 3 since they enjoy several beautiful properties such as equidimensionality and uniqueness of primary decompositions.

The SG conditions on triples of forms $F_k \in \text{rad}(F_i, F_j)$ is a "*local condition*," involving only three forms. So we first construct a "nice algebra" which resembles a small polynomial subring, that contains these three polynomials. However, even if we construct a nice algebra for each such triple, we still have to cope with the *global* aspect of the SG condition, that is, that for any pair F_i, F_j, there is an F_k with the relation above. Therefore, if we fix a form F_i, the algebras that we construct for the triples better be "compatible" as we vary the indices j, k.

To achieve such *compatibility* among these small algebras that we are constructing, we need such algebras to be *robust* in a certain sense, which we make precise in Section 4. We introduce the notion of wide AH algebras, by building upon the strong algebras recently constructed by Ananyan and Hochster [AH20a, AH20b]. We prove that these wide algebras are robust with regards to our SG configuration. A key property of wide algebras is the *intersection flatness* property. This property allows us to apply algebraic-geometric and elimination theoretic tools to ideals within a wide algebra and extend properties of such ideals to the ambient polynomial ring via a *transfer principle*. For example, if the generators of an ideal I is contained in a wide algebra then, in order to verify properties of I such as primality, reducedness in the polynomial ring, it is enough to verify these properties inside the wide algebra (see Section 3).

Summary of contributions: Since this work introduces several new technical tools, the following points summarize the main technical takeaways of the paper.

1. We generalize the SG problem of [Gup14] to SG configurations over an algebra and develop an inductive framework for solving Conjecture 1.3. Using our inductive approach we solve Conjecture 1.3 in degree 3. Thus, we open up a concrete avenue to prove the general version of the problem without reducing it to robust versions or other variants.

   In the previous versions of the problem, it was unclear how one could prove the SG theorem inductively, and here we provide such a way.

2. We use tools from algebraic geometry to generalize the structure theorem for intersection of quadrics proved in [HP94, CTSSD87, Shp20]. This structure theorem (Theorem 1.5) is new and interesting on its own right. We use the standard tools of Hilbert-Samuel multiplicty and primary decomposition to characterize non-radical ideals generated by cubic forms. The structure theorem for cubics is relatively more involved as the geometry of cubic forms is significantly richer than that of quadratic forms.

3. We prove a criterion for determining when certain ideals generated by two forms of arbitrary degree is radical (Lemma 1.6). This result generalizes the classical fact that the discriminant of a univariate polynomial P is zero iff P has multiple roots.

4. We introduce the notion of wide Ananyan-Hochster algebras, by building upon the seminal work of Ananyan-Hochster in [AH20a, AH20b]. These wide algebras are "well-behaved" subalgebras that behave similar to subpolynomial rings and are particularly amenable to algebraic geometric and elimination theoretic tools. These wide algebras have robustness properties which are particularly useful for our inductive approach to the SG problem.

5. As a part of our inductive approach, we generalize the main result of [Shp20] to quadratic SG configurations over an algebra (see Appendix A). As a special case we obtain a new proof of the quadratic SG theorem in [Shp20].

Generality of our results: Many of our results (especially in Section 3) hold for polynomials of arbitrary degree and therefore we expect that these tools will pave a way for fully settling the main radical SG problem in Conjecture 1.3.

1.2 High-level ideas of the proof

Suppose we are given a 3-radical-SG configuration $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2 \sqcup \mathcal{F}_3 \subset \mathbb{K}[x_1, \ldots, x_n]$, where $\mathcal{F}_d$ is the set of irreducible forms of degree d in the configuration. As stated, the radical SG problem is not amenable to inductive arguments, as there seems to be no way to convert $\mathcal{F}$ into a 2-radical-SG configuration. However, if one could prove that there is a small polynomial subring $\mathbb{K}[y_1, \ldots, y_r] \subset \mathbb{K}[x_1, \ldots, x_n]$ where $r = O(1)$ and $\mathcal{F}_3 \subset \mathbb{K}[y_1, \ldots, y_r]$, then one could hope that the remaining configuration $\mathcal{F}_1 \sqcup \mathcal{F}_2$ would behave similarly to a 2-radical-SG configuration, which would make our problem more tractable.

This overall strategy is exactly the main idea behind our proof. However, several challenges need to be overcome, as the situation above may not happen at all. For instance, there could be a cubic form $C \in \mathcal{F}_3$ of really "high rank," that is, C cannot be decomposed as $C = \sum_{i=1}^{s} y_i Q_i$ for linear forms y_i and quadratic forms Q_i. Additionally, even if C were of "low rank," there could be a quadratic form in its decomposition which is of high rank, for instance $C = y_1(x_1 z_1 + \cdots + x_n z_n) + y_2^2$. In both cases, we have that the cubic form will depend on many linear variables, so at first glance it seems that the strategy described above is doomed to fail! Moreover, even if we managed to construct such a small polynomial ring, how do we solve the remaining configuration? In the set $\mathcal{F}_1 \sqcup \mathcal{F}_2$, we would have to account for SG pairs which have a dependency outside of the configuration, which will be inside of the small polynomial ring.

To address the first problem, instead of working with small polynomial subrings, we need to construct small and graded $\mathbb{K}$-subalgebras of $\mathbb{K}[x_1, \ldots, x_n]$ which *behave like polynomial rings*, in the sense that these subalgebras are generated by elements which are *as free as possible*, which is the main property of polynomial rings. But this is not the only property of such algebras that we need: we also need these algebras to *interact well* with the main polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ – and for us, the commutative algebraic property we need is called *intersection flatness*, which we formally define in Section 3. The main property from intersection flatness that we will need is that it *preserves prime ideals*, namely, if $\mathfrak{p}$ is a prime ideal in our subalgebra, then $\mathfrak{p}$ will also be a prime ideal over $\mathbb{K}[x_1, \ldots, x_n]$. This is important because if an algebra preserves primes, then it also preserves the structure of radical ideals, and thus such algebra should behave nicely with our SG configuration.

Suppose we have an algebra $\mathcal{A}$ with the properties from the previous paragraph. If this algebra contains our entire configuration $\mathcal{F}$ we are done, but this may not happen, as we will not be able to construct this algebra at once. Hence, another desirable property of the algebra $\mathcal{A}$ that we will need is that it can be *augmented* to a slightly larger algebra $\mathcal{B}$ **without losing** its structure – that is, we would like the generators of $\mathcal{A}$ (which one can think of as the "free variables" of $\mathcal{A}$) to be a subset of the generators of $\mathcal{B}$. We denote such property by *robustness*.

Robustness is desirable for several reasons – as it allows us to extend several results from elimination theory and algebraic geometry (which hold for polynomial rings) to such subalgebras.

But in a high level, the main advantage that robustness gives us is that it formally allows us to treat the subalgebra $\mathcal{A}$ as a polynomial ring with respect to our configuration $\mathcal{F}$. That is, with such robustness we are formally in the case where $\mathcal{A}$ is of the form $\mathbb{K}[g_1, \dots, g_s]$ (where the g_i's are not necessarily linear) where we can think of g_i's in the same way as if they were free linear forms – as far as they interact with $\mathcal{F}$.

To construct such algebras, we build on the groundbreaking result of Ananyan and Hochster [AH20a, AH20b] which in a high level say that if your algebra generators are of "sufficiently high rank" then your algebra will behave like a polynomial ring. In Section 4 we build on their work by constructing *wide Ananyan-Hochster algebras* (wide AH algebras), and proving that they have all the aforementioned properties, in particular showing that they are robust to small augmentations.

Once we construct such algebras, we are equipped to finally define our inductive strategy. In Section 6.2 we define our "inductive form of SG problem:" the radical SG problem over an algebra (which in practice we'll take it to be a wide AH algebra). In Section 7 we proceed to reducing the 3-radical-SG problem to a 2-radical-SG problem over a wide AH algebra, by proving that the cubics in $\mathcal{F}$ must be contained in a small wide AH algebra. Subsequently in Appendix A we prove that 2-radical-SG configurations over wide AH algebras have small vector-space dimension.

We now describe our approach to prove that forms in $\mathcal{F}_3$ are in a small wide AH algebra. Let $\{C_1, \dots, C_t\} = \mathcal{F}_3$ be the cubic forms in our SG configuration. If $\delta \in (0, 1)$ is a small constant, and each C_i is such that for at least $\delta(t-1)$ C_j's $\mathrm{span}_{\mathbb{K}}\{C_i, C_j\}$ contains another cubic from $\mathcal{F}_3$, then $\mathcal{F}_3$ is in fact a fractional linear SG configuration, in which case we know that $\dim \mathrm{span}_{\mathbb{K}}\{\mathcal{F}_3\} = O(1)$, and in this case we know how to construct a small wide AH algebra (see Section 4).

The main difficulty comes when $\mathcal{F}_3$ is not a fractional linear SG configuration, that is, we have many forms C_i for which *most* C_j's are such that $\mathrm{span}_{\mathbb{K}}\{C_i, C_j\}$ does not contain a third cubic in $\mathcal{F}_3$. In this case, the ideal (C_i, C_j) must not be radical, and we use our main structure theorem for ideals of the form (C_1, C_2) where C_1, C_2 are cubic forms (see Section 5) to show that there exists a simple prime ideal (of small height) containing most of the cubic forms. Once we have such a simple prime, we can invoke special quadratic SG configurations (see Section 6.3 and Section 6.5) to construct a small wide AH algebra that contains most of the cubic forms. As it turns out, we can also prove that if a small wide algebra contains most cubic forms, then it must contain **all** cubic forms. This concludes the reduction from a 3-radical-SG configuration to a 2-radical-SG configuration over a wide AH algebra, which we handle in Appendix A.

In summary, the high-level roadmap of our proof is as follows:

1. Given 3-radical-SG configuration $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2 \sqcup \mathcal{F}_3$, prove that $\mathcal{F}_3$ is contained in a wide AH algebra of constant dimension ("reducing max degree")
   (a) If each form in $\mathcal{F}_3$ spans a third element with many other forms in $\mathcal{F}_3$, then $\mathcal{F}_3$ is a linear SG configuration, and in this case we are done
   (b) If $\mathcal{F}_3$ is not a linear SG configuration, there must be many cubics $C_i \in \mathcal{F}_3$ such that (C_i, C_j) does not span a third element in $\mathcal{F}_3$, which implies (C_i, C_j) is not a radical ideal. In this case, use our structure theorem (see Theorem 1.5) and the disciminant criterion (see Lemma 1.6) to prove there is a small prime ideal containing most of $\mathcal{F}_3$.
   (c) Then we prove that SG subconfigurations within small prime ideals are constant dimensional. We use the structure of these primes to construct a small wide AH algebra containing most of $\mathcal{F}_3$ (Section 6).

(d) Once we have that most of our polynomials in $\mathcal{F}_3$ are contained in a small wide AH algebra, then we show that all of $\mathcal{F}_3$ must be contained in a small wide AH algebra (Section 6.4).

2. Once we have the small wide AH algebra containing $\mathcal{F}_3$, note that the set $\mathcal{F}_1 \sqcup \mathcal{F}_2$ is a 2-radical-SG over the wide AH algebra. We show that such configurations are constant dimensional (Appendix A). ("inductive step")

3. The two steps above prove that $\mathcal{F}$ must be a low-dimensional vector space.

It is worth noting that most of the technical work is needed in the steps 1. (b)-(d) above, in order to handle the loss linearity. As mentioned above, to handle this case, we need to develop and generalize several results from elimination theory and algebraic geometry to the setting of algebras generated by prime sequences, and in this step we also crucially use the robustness of the wide AH algebras.

1.3 Related Work

Radical Sylvester-Gallai problems: among the previous works, the one which mostly resemble ours are [Shp20] solution of the quadratic radical SG problem.

Our work generalizes [Shp20] by solving the cubic radical SG problem. Moreover, we also prove in Appendix A a more general version of the quadratic SG problem, allowing some of the SG dependencies to be in a fixed algebra. In Section 6.3 we also need to solve an important variant of quadratic SG configurations, which we term as "saturated SG configurations," where there is an extra linear form z such that the SG dependencies are of the form $zF_k \in \mathrm{rad}(F_i, F_j)$. Saturated configurations mostly resemble the product configurations from [PS20a], but they are not quite the same as the linear form z may not satisfy the SG condition with many of the forms in the configuration $\mathcal{F}$.

On a high level, our approach is similar to the one from [Shp20], and we were inspired by it, generalizing it in several ways. In his work, Shpilka proves his main theorem by reducing a quadratic SG configuration to a linear SG configuration, and therefore his approach is also inductive. To achieve this induction, Shpilka proves a structure theorem on the structure of dependencies of the form $Q_k \in \mathrm{rad}(Q_i, Q_j)$, [Shp20, Theorem 29]. This structure theorem falls slightly short of a structure theorem for intersections of quadrics, but it was sufficient for his application and motivated us to generalize his theorem for intersections of cubics. Once Shpilka has his structure theorem, his approach to construct a small algebra containing all forms in the configuration is by converting the radical dependencies (which are non-linear) into linear dependencies. To achieve this, Shpilka then uses general linear projections ([Shp20, Section 5.1, Claim 45]) to preserve the radical dependencies in a non-trivial way. Once Shpilka converts the radical dependencies into linear ones, he can finally apply the linear SG theorems from [BDWY11, DSW14]. As discussed in Section 1.2, we generalize all steps in this approach to make it work for cubic forms, providing several structural elements which generalize to the degree d case.

Progress on PIT: the past year has seen remarkable progress on the PIT problem for constant depth circuits. In [DDS21], the authors give a deterministic quasi-polynomial time algorithm for blackbox PIT for depth 4 circuits with bounded top and bottom fanins. Their approach is analytic in nature, allowing them to bypass the need of Sylvester-Gallai configurations. Another PIT result in this setting comes from the lower bound against low depth algebraic circuits proved by [LST22], which gives a deterministic weakly-exponential algorithm for PIT for these circuits via the hardness vs randomness paradigm for constant depth circuits [CKS19].

However, the SG-based approach of [Gup14] is the only one so far which could yield deterministic polynomial-time blackbox PIT algorithms for the subclass of depth-4 circuits with constant top and bottom fanins. Moreover, understanding these non-linear SG type configurations will improve the conceptual understanding of cancellation in algebraic computations, which is something inherent in the SG-based approach.

1.4 Organization

In Section 2 we establish the notation and basic facts that will be used throughout the paper. In Section 3, we establish necessary definitions and theorems needed from commutative algebra and algebraic geometry. Moreover, in this section we develop some transfer principles, showing that certain elimination-theoretic and algebro-geometric results also hold for subalgebras generated by prime sequences. In Section 4 we define and establish useful properties of the main object that we will use in our proof: wide Ananyan-Hochster algebras. In Section 5 we prove our main structure theorem and auxiliary lemmas about the structure of ideals generated by two cubic forms. In Section 6 we define the variants and generalizations of Sylvester-Gallai configurations that we need, as well as our "inductive-friendly" Sylvester-Gallai configurations over an algebra. In Section 7 we prove our main theorem, that 3-radical-SG configurations are low dimensional. In Section 8 we conclude and pose some open problems which would lead to a solution of Conjecture 1.1.

2 Preliminaries

In this section, we establish the notation which will be used throughout the paper and some important background which we shall need to prove our claims in the next sections.

2.1 General Facts and Notations

Throughout the paper, we will work over an algebraically closed field $\mathbb{K}$ with $\text{char}(\mathbb{K}) = 0$. From now on we will use boldface to denote a vector of variables or an element of the projective space. For instance, $\mathbf{x} = (x_0, x_1, \ldots, x_n)$ is the vector of variables $x_0, x_1, \ldots, x_n$ and $\mathbf{a} = [a_0 : a_1 : \cdots : a_n]$ is a point in $\mathbb{P}^n$.

We will denote by $S := \mathbb{K}[x_0, \cdots, x_n]$ our polynomial ring, with the standard grading by degree. That is, $S = \bigoplus_{d \geqslant 0} S_d$, where S_d is the $\mathbb{K}$-vector space of forms of degree d. For the rest of this paper, we will use the term *form* to refer to a homogeneous polynomial.

Given any set of polynomials $F_1, \ldots, F_r \in S$, we denote the ideal generated by the polynomials $F_1, \ldots, F_r$ by $(F_1, \ldots, F_r)$, and by $\text{rad}(F_1, \ldots, F_r)$ the radical of the ideal $(F_1, \ldots, F_r)$.

2.2 Quadratic forms

In this subsection we recall the following basic definitions and statements about quadratic forms borrowed from [Shp20, PS20a]. In Section 4, we generalize these facts to cubic forms.

Definition 2.1 (Rank of quadratic form)**.** If $Q \in S_2$ is a quadratic form, we denote by $\operatorname{rank}(Q)$ the minimal $r \in \mathbb{N}$ such that $Q = \sum_{i=1}^{r} a_i b_i$, where $a_i, b_i \in S_1$. We call any representation of Q of the form above a *minimal representation* of Q.

The next proposition appears in [PS20a, Claim 2.13]

Proposition 2.2. *If $Q \in S_2$ is such that $\operatorname{rank}(Q) = r$ and $Q = \sum_{i=1}^{r} a_i b_i = \sum_{i=1}^{r} f_i g_i$ are two minimal representations of Q, then $\operatorname{span}_{\mathbb{K}}\{f_1, g_1, \ldots, f_r, g_r\} = \operatorname{span}_{\mathbb{K}}\{a_1, b_1, \ldots, a_r, b_r\}$.*

The proposition above motivates the following definition, taken from [PS20a, Definition 2.14]

Definition 2.3. Let $Q \in S_2$ be a quadratic form, where $\operatorname{rank}(Q) = r$, and let $Q = \sum_{i=1}^{r} a_i b_i$ be a minimal representation of Q. Define $\operatorname{Lin} Q := \operatorname{span}_{\mathbb{K}}\{a_1, b_1, \ldots, a_r, b_r\}$.

The proposition below appears in [PS20a, Claim 2.17].

Proposition 2.4. *If $Q \in S_2$ is such that $\dim \operatorname{Lin} Q \geqslant 2$ and $x, y \in S_1$ such that $x, y \notin \operatorname{Lin} Q$, then for any $\alpha, \beta \in \mathbb{K}^*$, we have that $s(\alpha Q + \beta xy) = s(Q) + 1$.*

2.3 General Projections

In this subsection, we recall some properties of projection maps used in [Shp20, PS20a]. We begin by defining projections maps.

Definition 2.5 (Projection maps)**.** Let $S = \mathbb{K}[x_1, \cdots, x_n]$ be a polynomial ring. Let $W \subset S_1$ be a subspace of linear forms and $y_1, \cdots, y_t$ be a basis of W. Let $y_1, \cdots, y_n$ be a basis of S_1 that extends the basis $y_1, \cdots, y_t$ of W. Let z be a formal variable not in $\{y_1, \cdots, y_n\}$. For $\alpha = (\alpha_1, \cdots, \alpha_t) \in \mathbb{K}^t$, we define the projection map $\varphi_{\alpha,W}$ as the $\mathbb{K}$-algebra homomorphism $\varphi_{\alpha,W} : S \to \mathbb{K}[z, y_{t+1}, \cdots, y_n] = S[z]/(W)$ defined by

$$y_i \mapsto \begin{cases} \alpha_i z, \text{ if } 0 \leqslant i \leqslant t \\ y_i, \text{ otherwise} \end{cases}$$

For simplicity we will often drop the subscripts W or α, and write φ_α or φ for a projection map when there is no ambiguity about the vector space W or the vector α.

General projections. Fix a vector space $W \subset S_1$ as in Definition 2.5. We will say that a property holds for a *general projection* φ_α, if there exists a non-empty open subset $U \subset \mathbb{K}^t$ such that the property holds for all φ_α with $\alpha \in U$. Here $U \subset \mathbb{K}^t$ is open with respect to the Zariski topology, hence U is the complement of the zero set of finitely many polynomial functions on $\mathbb{K}^t$. The general choice of the element α defining a general projection φ_α allows us to say that such projection maps will avoid any finite set of polynomial constraints. As shown in [Shp20, PS20a], general projection maps preserve several important properties of polynomials.

Proposition 2.6. *Let $F \in S$ be a polynomial and $W \subset S_1$ be a vector space of linear forms.*

(a) If $F \notin \mathbb{K}[W]$, then $\varphi(F) \notin \mathbb{K}[z]$ for a general projection $\varphi : S \to S[z]/(W)$.

(b) If $F \neq 0$, then $\varphi(F) \neq 0$ for a general projection.

(c) Suppose F is a form which does not have any multiple factors and $F \in (W)$. If $\varphi(F) = z^k G$ where $G \notin (z)$, then G does not have any mulitple factors.

Proof. (a) Let $W = \mathrm{span}_{\mathbb{K}}\{y_1, \cdots, y_t\}$ and $y_1, \cdots, y_n$ be a basis of S_1. Let $R = \mathbb{K}[y_1, \cdots, y_t]$. We may write F as a polynomial in $y_{t+1}, \cdots, y_n$ with coefficients in R. Since $F \notin R$, we know that there is a non-zero polynomial $g(y_1 \cdots, y_t) \in R$ such that g is the coefficient of some monomial $\Pi_{j=t+1}^{n} y_j^{e_j}$ in F. If $\alpha \in \mathbb{K}^t \backslash V(g)$, then we must have that $\varphi_\alpha \notin \mathbb{K}[z]$. Hence $\varphi_\alpha \notin \mathbb{K}[z]$ for a general α.

(b) If $F \notin \mathbb{K}[W]$, then by part (a) we conclude that $F \neq 0$. If $F \in \mathbb{K}[W]$ then $F = F_0 + F_1 + \cdots + F_d$ where $F_i \in \mathbb{K}[W]$ is the degree i homogeneous part of F. Now $\varphi_\alpha(F) = F_0(\alpha) + F_1(\alpha)z + \cdots + F_d(\alpha)z^d$. If $F \neq 0$, then $F_i \neq 0$ for some i. If $\alpha \in \mathbb{K}^t \backslash V(F_i)$, then $\varphi_\alpha(F) \neq 0$.

(c) If $F \in \mathbb{K}[W]$ then $G \in \mathbb{K}$. So we may assume that $F \notin \mathbb{K}[W]$. Since F does not have any multiple factors, we have $\mathrm{Disc}_{y_j}(F) \neq 0$ for all $y_j \in \{y_{t+1}, \cdots, y_n\}$ such that F depends on y_j. Hence, for a general projection φ_α we have $\mathrm{Disc}_{y_j}(\varphi_\alpha(F)) = \varphi_\alpha(\mathrm{Disc}_{y_j}(F)) \neq 0$. Suppose that for some projection φ_α the polynomial G has a multiple factor. Let $y_j \in \{y_{t+1}, \cdots, y_n\}$ be a variable such that the multiple factor of G depends on y_j. Then $\mathrm{Disc}_{y_j}(G) = 0$ and hence $\mathrm{Disc}_{y_j}(\varphi_\alpha(F)) = 0$. Therefore G can not have any multiple factors for a general projection φ. □

The next proposition is from [PS20a, Claim 2.23].

Proposition 2.7. *Let $F, G \in S$ be two polynomials which have no common factor and $W \subset S_1$ a subspace of linear forms. For a general projection $\varphi : S \to S[z]/(W)$, we have $\gcd(\varphi(F), \varphi(G)) \in \mathbb{K}[z]$. In particular, if F, G are homogeneous then $\gcd(\varphi(F), \varphi(G)) = z^k$ for some $k \in \mathbb{N}$.*

The following corollary was proved in [PS20a, Corollary 2.24] for quadratic forms. The same argument applies to forms of arbitrary degree. We provide a proof for completeness.

Corollary 2.8. *Let $F, G \in S$ be linearly independent irreducible forms and $W \subset S_1$ be a vector space of linear forms. If $F, G \notin \mathbb{K}[W]$ then $\varphi(F), \varphi(G)$ are linearly independent, for a general projection $\varphi : S \to S[z]/(W)$.*

Proof. Since $F, G \notin \mathbb{K}[W]$, we have $\varphi(F), \varphi(G) \notin \mathbb{K}[z]$ for a general projection φ, by Proposition 2.6. Since F, G are irreducible, we know that any common factor of $\varphi(F), \varphi(G)$ must be of the form z^k. If $\varphi(F), \varphi(G)$ are linearly dependent, then we must have $\varphi(F), \varphi(G) \in \mathbb{K}[z]$, which is a contradicion. □

The next proposition follows from [PS20a, Claim 2.26].

Proposition 2.9. *Let $W \subset S_1$ be a vector space of linear forms. Let $\mathcal{F} \subset S_2$ be a finite set of quadratic forms. Suppose there is an integer $D > 0$ such that $\dim \mathrm{span}_{\mathbb{K}}\{\bigcup_{F \in \mathcal{F}} \mathrm{Lin}\varphi(F)\} \leqslant D$ for a general projection $\varphi : S \to S[z]/(W)$. Then $\dim \mathrm{span}_{\mathbb{K}}\{\bigcup_{F \in \mathcal{F}} \mathrm{Lin} F\} \leqslant (D+1) \cdot \dim W$.*

The proposition above can be sharpened if we have extra information about the linear forms in $\mathcal{F}$. We state this sharpening in the next proposition

Proposition 2.10. *Let $W \subset S_1$ be a vector space of linear forms and $\mathcal{F} \subset S_2$ be a finite set of quadratic forms such that $\mathcal{F} \cap (W)$ and $s(F) < s$ for each $F \in \mathcal{F}$. Suppose there is an integer $D > 0$ such that $\dim \mathrm{span}_{\mathbb{K}}\{\bigcup_{F \in \mathcal{F}} \mathrm{Lin}\varphi(F)\} \leqslant D$ for a general projection $\varphi : S \to S[z]/(W)$. Then $\dim \mathrm{span}_{\mathbb{K}}\{\bigcup_{F \in \mathcal{F}} \mathrm{Lin} F\} \leqslant (D+1) \cdot s$.*

The next result is an analogue of Proposition 2.9 for forms of higher degree.

Proposition 2.11. *Let $W \subset S_1$ be a vector space of linear forms. Let $\mathcal{F} \subset S_d$ be a finite set of forms of degree $d \geqslant 2$ such that $\mathcal{F} \subset (W)$. Suppose that there exists an integer $D > 0$ such that we have $\dim \mathrm{span}_{\mathbb{K}}\{\varphi(\mathcal{F})\} \leqslant D$ for a general projection φ of W. Then there exists a graded vector space*

$V = V_1 + \cdots + V_{d-1} \subset S$ *with* $V_k \subset S_k$, *such that* $\mathcal{F} \subset \mathbb{K}[V + W]$ *and* $\dim(V_k) \leqslant D \cdot \binom{\dim(W)+d-k-1}{d-k}$ *for all* $k \in [d-1]$.

Moreover, we have $\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{F}\} \leqslant D \cdot \left(\sum_{k=1}^{d-1} \binom{\dim(W)+k-1}{k}^2\right)$

Proof. Let $\mathcal{F} = \{F_1, \cdots, F_m\} \subset S_d$. Let $W = \operatorname{span}_{\mathbb{K}} \{y_1, \cdots, y_t\}$ and $y_1, \cdots, y_n$ be a basis of S_1. Let $R = \mathbb{K}[y_1, \cdots, y_t]$ and $A = \mathbb{K}[y_{t+1}, \cdots, y_n]$. For $k \in [d-1]$, let $r_k = \binom{t+k-1}{k}$ be the number of distinct monomials of degree k in R. Let $M_1, \cdots, M_{r_k}$ denote these monomials that span R_k. Consider F_i as a polynomial in $A[y_1, \cdots, y_t]$. Let $F_{ij}^k \in A$ denote the coefficient of the degree k monomial M_j in F_i. Note that if $F_{ij}^k \neq 0$, then it is a homogeneous polynomial of degree $d-k$. Let $V_{d-k} = \operatorname{span}_{\mathbb{K}} \left\{F_{ij}^k | i \in [m], j \in [r_k]\right\}$ and $V = V_1 + \cdots + V_{d-1}$. Then we have $\mathcal{F} \subset \mathbb{K}[V + W]$. We will show that $\dim(V_{d-k}) \leqslant r_k D$ for all $k \in [d-1]$.

Fix $k \in [d-1]$ and let F_{ij} denote the forms F_{ij}^k as defined above. For $\alpha \in \mathbb{K}^t$, we have $\varphi_\alpha(F_i) \in A[z]$ and the coefficient of z^k in $\varphi_\alpha(F_i)$ is given by $M_1(\alpha)F_{i1} + \cdots + M_{r_k}(\alpha)F_{ir_k}$. For a general $\alpha \in \mathbb{K}^t$, we know that $\dim \operatorname{span}_{\mathbb{K}} \{\varphi(\mathcal{F})\} \leqslant D$. Therefore the dimension of the span of the coefficients of z^k in all the $\varphi(F_i)$ is at most D, i.e. we have $\dim \operatorname{span}_{\mathbb{K}} \{M_1(\alpha)F_{i1} + \cdots + M_{r_k}(\alpha)F_{ir_k} | i \in [m]\} \leqslant D$. We may choose r_k number of general vectors $\alpha^{(1)}, \cdots, \alpha^{(r_k)} \in \mathbb{K}^{r_k}$, such that the matrix $\mathcal{M}$ defined by $\mathcal{M}_{ij} = M_i(\alpha^{(j)})$ is invertible. Then we have $\dim \operatorname{span}_{\mathbb{K}} \left\{F_{ij} | i \in [m], j \in [r_k]\right\} \leqslant r_k D$.

Let $F_i \in \mathcal{F}$ and $k \in [d-1]$. By the above argument, we know that the coefficients $\{F_{ij}\}$ of the degree k monomials $M_1, \cdots, M_{r_k}$ in F_i's span a vector space of dimension at most $r_k D$. Hence, the span of $\{M_\ell F_{ij} | i \in [m], j, \ell \in [r_k]\}$ is of dimension at most $r_k^2 D$. Therefore, $\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{F}\} \leqslant \sum_{k=1}^{d-1} r_k^2 D$. $\square$

Corollary 2.12. *Let* $W \subset S_1$ *be a vector space of linear forms. Let* $\mathcal{F} \subset S_{\leqslant 3}$ *be a finite set of forms of degree at most* 3 *such that* $\mathcal{F} \subset (W)$. *Suppose that there exists an integer* $D > 0$ *such that we have* $\dim \operatorname{span}_{\mathbb{K}} \{\varphi(\mathcal{F})\} \leqslant D$ *for a general projection* φ *of* W. *Then there exists a graded vector space* $V = V_1 + V_2$ *such that* $\mathcal{F} \subset \mathbb{K}[V + W]$ *and* $\dim(V_1) = O((\dim W)^2)$, $\dim(V_2) = O(\dim(W))$. *Moreover, we have* $\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{F}\} \leqslant O((\dim W)^4)$.

Proof. Let $\mathcal{F}_i = \mathcal{F} \cap S_i$ for $i \in [3]$. As $\mathcal{F} \subset (W)$, we have $\mathcal{F}_1 \subset W$. By Proposition 2.11, there exists a vector space $U_1 \subset S_1$ such that $\mathcal{F}_2 \subset \mathbb{K}[U_1 + W]$ and $\dim(U_1) \leqslant D \cdot \dim(W)$. Again, by applying Proposition 2.11 to $\mathcal{F}_3$, we obtain a graded vector space $U_1' + U_2'$ such that $\mathcal{F}_3 \subset \mathbb{K}[U_1' + U_2' + W]$ and $\dim(U_1') \leqslant D \cdot \binom{\dim(W)+1}{2}$, $\dim(U_2') \leqslant D \cdot \dim(W)$. Hence, we may take $V_1 = U_1 + U_1'$ and $V_2 = U_2'$.

Since $\mathcal{F}_1 \subset W$, we have $\dim(\mathcal{F}_1) \leqslant \dim(W)$. By Proposition 2.11, we know that $\dim(\mathcal{F}_2) \leqslant D \cdot (\dim W)^2$ and $\dim(\mathcal{F}_3) \leqslant D \cdot \left((\dim W)^2 + \binom{\dim(W)+1}{2}^2\right)$. Therefore, we have

$$\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{F}\} \leqslant \dim(W) + D \cdot \left(2(\dim W)^2 + \binom{\dim W + 1}{2}^2\right).$$

Hence $\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{F}\} \leqslant O((\dim W)^4)$. $\square$

3 Results from algebraic geometry

In this section we establish the necessary definitions and theorems needed from commutative algebra and algebraic geometry.

3.1 Regular sequences and Hilbert-Samuel multiplicity

Definition 3.1 (Regular sequence)**.** Let R be ring and M an R-module. A sequence of elements $f_1, f_2, \cdots f_n \in R$ is called an M-regular sequence if

(1) $(f_1, f_2, \cdots, f_n)M \neq M$, and

(2) for $i = 1, \cdots, n$, f_i is a non-zerodivisor on $M/(f_1, \cdots, f_{i-1})M$.

If $M = R$, then we simply call it a regular sequence.

If P and Q are two irreducible polynomials in the polynomial ring S such that P does not divide Q, then P, Q is an S-regular sequence. Indeed, since P is irreducible, we know that $S/(P)$ is a integral domain. Therefore Q is a non-zero element in $S/(P)$, and hence a non-zero divisor.

We also note that if $f_1, \cdots, f_m$ is a regular sequence of forms in S, then $f_1, \cdots, f_m$ are algebraically independent. Therefore the subalgebra generated by $f_1, \cdots, f_m$ is isomorphic to a polynomial ring. In particular, the ring homomorphism $k[y_1, \cdots, y_n] \to S$ defined by $y_i \mapsto f_i$ is an isomorphism onto its image.

Even though the $\mathbb{K}$-algebra $\mathbb{K}[f_1, \ldots, f_m] \subset S$ is isomorphic to a polynomial ring, its elements may not behave well when seen as elements of S. We next present a sufficient condition which will ensure to us that the subalgebra is well behaved with respect to S, in a way which we formalize later.

Definition 3.2 (R_η-property)**.** Let η be a non-negative integer. We say that a Noetherian ring R satisfies the R_η property if the local ring $R_\mathfrak{p}$ is a regular local ring for all prime ideals $\mathfrak{p} \subset R$ such that $\text{height}(\mathfrak{p}) \leqslant \eta$.

Definition 3.3. Let η be a non-negative integer and R a Noetherian ring. A sequence of elements $f_1, \ldots, f_n \in R$ is called a prime sequence (respectively an R_η-sequence) if

1. $f_1, \cdots, f_n$ is a regular sequence, and
2. $R/(f_1, \cdots, f_i)$ is an integral domain (respectively, satisfies the R_η property) for all $i \in [n]$.

Remark 3.4. *Since prime sequences and R_η-sequences are regular sequences, we know that if $f_1, \cdots, f_n$ is a prime sequence or R_η-sequence in the polynomial ring S, then $f_1, \cdots, f_n$ are algebraically independent.*

We note the following simple statement about radicals and regular sequences.

Lemma 3.5. *If F, P_1, P_2 is a regular sequence in S and $FG \in \text{rad}\,(FP_1, FP_2)$ then $G \in \text{rad}\,(F_1, F_2)$.*

Proof. Note that we have $(FG)^d = AFP_1 + BFP_2$ for some $A, B \in S$, and hence $F^{d-1}G^d = AP_1 + BP_2$. Since F is a non-zero divisor in $S/(P_1, P_2)$, we conclude that $G^d \in (P_1, P_2)$. □

Let $(R, \mathfrak{m})$ be a local ring and M be an R-module. Define the Hilbert-Samuel function of M to be

$$H_{\mathfrak{m},M}(n) = \text{length}\left(\frac{\mathfrak{m}^n M}{\mathfrak{m}^{n+1} M}\right).$$

By [Eis95, Proposition 12.2, Theorem 12.4], there exists a polynomial $P_{\mathfrak{m},M}$ of degree $\dim(M) - 1$ such that $P_{\mathfrak{m},M}(n) = H_{\mathfrak{m},M}(n)$ for $n \gg 0$. The polynomial $P_{\mathfrak{m},M}$ is called the Hilbert-Samuel polynomial of M. Let a_d be the leading coefficient of $P_{\mathfrak{m},M}$, where $d = \dim(M) - 1$. The Hilbert-Samuel multiplicity of M is defined as

$$e(\mathfrak{m}, M) = (d-1)! a_d.$$

Therefore, the leading coefficient of the Hilbert-Samuel polynomial $P_{\mathfrak{m},M}$ is $\frac{e(\mathfrak{m},M)}{(\dim M - 1)!}$.

Let $S_{\mathfrak{m}}$ be the localization of S at the irrelevant maximal ideal $\mathfrak{m} = (x_0, \cdots, x_n)$ and let I be a homogeneous ideal in S. Then the localization $(S/I)_{\mathfrak{m}}$ is an $S_{\mathfrak{m}}$-module. We will denote $e(S/I) := e(\mathfrak{m}, (S/I)_{\mathfrak{m}})$. Note that by [Eis95, Exercise 12.6], the number $e(S/I)$ is also equal to the degree of the projective variety defined by I in $\mathbb{P}^n$.

Let $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$ be an irredundant primary decomposition of I in S and $\mathfrak{p}_i = \mathrm{rad}(\mathfrak{q}_i)$ be a minimal prime of I for some $i \in [m]$. Then the localization $(S/I)_{\mathfrak{p}_i}$ is an $S_{\mathfrak{p}_i}$ module of finite length. We define the multiplicity of $\mathfrak{p}_i$ in the primary decomposition of I as

$$m(\mathfrak{p}_i) = \mathrm{length}((S/I)_{\mathfrak{p}_i}).$$

Remark 3.6. *Note that* $m(\mathfrak{p}_i) \geqslant 1$. *Moreover, we have* $m(\mathfrak{p}_i) = 1 \Leftrightarrow \mathfrak{q}_i = \mathfrak{p}_i$. *Indeed,* $\mathrm{length}((S/\mathfrak{q}_i)_{\mathfrak{p}_i}) \leqslant \mathrm{length}((S/I)_{\mathfrak{p}_i})$ *since we have a surjective homomorphism* $(S/I)_{\mathfrak{p}_i} \to (S/\mathfrak{q}_i)_{\mathfrak{p}_i}$. *Therefore we have* $\mathrm{length}((S/\mathfrak{q}_i)_{\mathfrak{p}_i}) = 1$. *If* $\mathfrak{q}_i \subsetneq \mathfrak{p}_i$, *then we have a strict chain of* $S_{\mathfrak{p}_i}$*-modules given by* $(0) \subsetneq \mathfrak{p}_i \subsetneq (S/\mathfrak{q}_i)_{\mathfrak{p}_i}$, *which is of length* 2. *That is a contradiction. Therefore we must have* $\mathfrak{q}_i = \mathfrak{p}_i$. *Hence if we have* $m(\mathfrak{p}_j) = 1$ *for all* j, *then* $I = \cap_j \mathfrak{p}_j$, *and* I *is a radical ideal.*

Conversely if I *is a radical ideal then* $m(\mathfrak{p}_i) = 1$ *for all minimal primes. Indeed for every* i *the ideal* $\mathfrak{p}_i(S/I)_{\mathfrak{p}_i}$ *is the nilradical of* $(S/I)_{\mathfrak{p}_i}$ *by the minimality of* $\mathfrak{p}_i$. *Since* S/I *has no nilpotents, we have* $\mathfrak{p}_i(S/I)_{\mathfrak{p}_i} = (0)_{\mathfrak{p}_i}$, *which implies* $(S/I)_{\mathfrak{p}_i}$ *is a field, and therefore has length* 1.

We recall the basic properties of the Hilbert-Samuel multiplicity below.

Proposition 3.7. *[Eis95, Exercises 12.7, 12.11] Let* I *be a homogeneous ideal in* S.

1. *Let* $J \subset S$ *be a homogeneous ideal such that* $I \subset J$. *Then* $e(S/J) \leqslant e(S/I)$.
2. *If* $I = (F)$ *for some form of degree* d, *then* $e(S/I) = d$.
3. *If* $I = (F_1, \cdots, F_m)$ *where* $F_1, \cdots, F_m$ *is a regular sequence of forms and* $\deg(F_i) = d_i$. *Then*
$$e(S/I) = d_1 \cdots d_m.$$
4. *If* $I = (F_1, \cdots, F_m)$ *where* $F_1, \cdots, F_m$ *is a regular sequence of forms, and* $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$ *are the minimal primes of* I *in* S. *Then*
$$e(S/I) = \sum_i m(\mathfrak{p}_i) e(S/\mathfrak{p}_i).$$

Lemma 3.8. *Let* P, Q *be irreducible forms and* $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ *be the minimal primes of* (P, Q) *in* S. *Suppose there exist linear forms* $\ell_j \in \mathfrak{p}_j$ *for all* $j \in [r]$. *Then we have*

$$\prod_j \ell_j^{m(\mathfrak{p}_j)} \in (P, Q).$$

Proof. Let I denote the ideal (P, Q) in S. We have a chain of $(S)_{\mathfrak{p}_j}$-submodules of $(S/I)_{\mathfrak{p}_j}$ given by

$$(S/I)_{\mathfrak{p}_j} \supset (\ell_j)_{\mathfrak{p}_j} \supset (\ell_j^2)_{\mathfrak{p}_j} \supset \cdots (\ell_j^{m_j})_{\mathfrak{p}_j} \supset 0$$

Since the local ring $(S/I)_{\mathfrak{p}_j}$ is an $S_{\mathfrak{p}_j}$-module of length m_j, we must have that $\ell_j^{m_j} = 0$ in $(S/I)_{\mathfrak{p}_j}$. Now consider that natural morphism to the localization

$$(S/I) \to (S/I)_{\mathfrak{p}_j}.$$

Since the image of $\ell_j^{m_j}$ is zero under the localization morphism, by [AM69, Corollary 10.21], we see that $\ell_j^{m_j}$ must be contained in all $\mathfrak{p}_j$-primary ideals of S/I. In particular $\ell_j^{m_j} \in \mathfrak{q}_j$ for all j. Therefore we have

$$\prod_j \ell_j^{m_j} \in \cap_j \mathfrak{q}_j = I.$$

$\square$

Lemma 3.9. *Let $P, Q, P_1, \cdots, P_m \in S$ be irreducible forms of degree at most d. Suppose $m \geqslant 2^{d^2}$. If $Q \in \mathrm{rad}(P, P_i)$ for all $i \in [m]$, then we must have $\mathrm{rad}\,(P, P_i) = \mathrm{rad}(P, P_k)$ for two distinct $i, k \in [m]$.*

Proof. Note that we have $\mathrm{rad}(P, Q) \subset \mathrm{rad}(P, P_i)$ for all $i \in [m]$, since $Q \in \mathrm{rad}(P, P_i)$. Let $\mathcal{S} = \{\mathfrak{p}_1, \cdots \mathfrak{p}_\ell\}$ and $\mathcal{S}_i = \{\mathfrak{p}_{i1}, \cdots, \mathfrak{p}_{i\ell_i}\}$ be the set of minimal primes of (P, Q) and (P, P_i) respectively. We have $\bigcap \mathfrak{p}_j = \mathrm{rad}(P, Q) \subset \mathrm{rad}(P, P_i) = \bigcap_j \mathfrak{p}_{ij}$. Therefore, by [AM69, Proposition 1.11], we must have that for all i, j, there exists k such that $\mathfrak{p}_k \subset \mathfrak{p}_{ij}$. Since we know that $\mathrm{ht}(\mathfrak{p}_j) = \mathrm{ht}(\mathfrak{p}_{ij}) = 2$ for all i, j, we must have that for all i, j, there exists k such that $\mathfrak{p}_k = \mathfrak{p}_{ij}$, i.e. $\mathcal{S}_i \subset \mathcal{S}$ for all $i \in [m]$.

Note that we have $\sum_j m(\mathfrak{p}_j)e(S/\mathfrak{p}_j) = e(S/(P, Q)) = \deg(P)\deg(Q) \leqslant d^2$. Since $m(\mathfrak{p}_j) \geqslant 1$ and $e(S/\mathfrak{p}_j) \geqslant 1$, we conclude that $|\mathcal{S}| \leqslant d^2$, i.e. there exist at most d^2 minimal primes of (P, Q). Therefore there are at most $2^{d^2} - 1$ number of distinct choices for the set of minimal primes $\mathcal{S}_i = \{\mathfrak{p}_{ij}\}$ of the ideals (P, P_i). By the pigeonhole principle, there exist distinct $i, k \in [m]$ such that $\mathcal{S}_i = \mathcal{S}_k$. Thus, $\mathrm{rad}(P, P_i) = \bigcap_{\mathfrak{q} \in \mathcal{S}_i} \mathfrak{q} = \bigcap_{\mathfrak{p} \in \mathcal{S}_k} \mathfrak{p} = \mathrm{rad}(P, P_k)$. $\square$

3.2 Intersection flatness and prime ideals

Recall that if $R \to R'$ is a flat ring homomorphism, then $IR' \bigcap JR' = (I \bigcap J)R'$ for any two ideals I, J in R. The notion of intersection flatness generalizes this fact to arbitrary intersections. A flat ring homomorphism $R \to R'$ is called intersection flat if for every family $\mathcal{I}$ of ideals in R, we have $\bigcap_{I \in \mathcal{I}}(IR') = (\bigcap_{I \in \mathcal{I}} I)R'$. The following result is from the discussion on extensions of prime ideals in in [AH20a, Section 2]. We rephrase it here in the form of the following proposition for convenience.

Proposition 3.10 (Intersection flatness)**.** *Let $g_1, \cdots, g_s$ be forms in $\mathbb{K}[y_1, \cdots, y_n]$ such that $g_1, \cdots, g_s$ form a regular sequence. Then the $\mathbb{K}$-algebra homomorphism $\mathbb{K}[x_1, \cdots, x_s] \to \mathbb{K}[y_1, \cdots, y_n]$, given by $x_i \mapsto g_i$, is intersection flat.*

Proof. Since $g_1, \cdots, g_s$ is a regular sequence, it can be extended to a homogeneous system of parameters $g_1, \cdots, g_n$. Since $g_1, \cdots, g_n$ is a homogeneous system of parameters, we know that $\mathbb{K}[y_1, \cdots, y_n]$ is a free module over $\mathbb{K}[g_1, \cdots, g_n]$. Now $\mathbb{K}[g_1, \cdots, g_n]$ is a free module over $\mathbb{K}[g_1, \cdots, g_s]$ via the inclusion homomorphism $\mathbb{K}[g_1, \cdots, g_s] \hookrightarrow \mathbb{K}[g_1, \cdots, g_n]$. Therefore $\mathbb{K}[y_1, \cdots, y_n]$ is free over $\mathbb{K}[g_1, \cdots, g_s]$ via the inclusion homomorphism. By [HH94, Page 41], free extensions are intersection flat. Therefore the ring homomorphism $\mathbb{K}[g_1, \cdots, g_s] \hookrightarrow \mathbb{K}[y_1, \cdots, y_n]$ is intersection flat. Since $g_1, \cdots, g_s$ is part of a homogeneous system of parameters, we know that $g_1, \cdots, g_s$ are algebraically independent over $\mathbb{K}$. Therefore the homomorphism $\mathbb{K}[x_1, \cdots, x_s] \to \mathbb{K}[y_1, \cdots, y_n]$, given by $x_i \mapsto g_i$ is an isomorphism onto its image $\mathbb{K}[g_1, \cdots, g_s]$ and thus it is intersection flat. $\square$

The following result from [AH20a] shows that if we further assume that $g_1, \cdots, g_s$ is a prime sequence, then prime ideals in $\mathbb{K}[g_1, \cdots, g_s]$ extend to prime ideals in $\mathbb{K}[y_1, \cdots, y_n]$.

Lemma 3.11. *[AH20a, Corollary 2.9] Let S be our polynomial ring and $g_1, \cdots, g_s$ be a prime sequence of forms in S. Then for any prime ideal $\mathfrak{p} \subset \mathbb{K}[g_1, \cdots, g_s]$, the extension ideal $\mathfrak{p}S$ is also prime.*

Proposition 3.12. *Let P, Q be non-associate irreducible forms in S. Let $g_1, \cdots, g_s$ be a prime sequence of forms such that $Q \in \mathcal{A} := \mathbb{K}[g_1, \cdots, g_s]$. Suppose $P \notin (g_1, \cdots, g_s)$. Let $\mathfrak{p} \subset S$ be a minimal prime over (P, Q). Then $\mathfrak{p} \cap \mathcal{A} = (Q)$ and $(P, Q) \cap \mathcal{A} = (Q)$.*

Proof. Let $\mathfrak{q} = \mathfrak{p} \cap \mathcal{A}$. Note that $\mathfrak{q} \subset (g_1, \cdots, g_s)$. Indeed, suppose we have $f \in \mathfrak{q} \backslash (g_1, \cdots, g_s)$. Then f is a polynomial in $\mathbb{K}[g_1, \cdots, g_s]$ with non-zero constant term. Since (P, Q) is a homogeneous ideal, the minimal prime $\mathfrak{p}$ over (P, Q) is also a homogeneous ideal. By homogeneity, $f \in \mathfrak{p}$ implies that $\mathfrak{p}$ contains a non-zero constant, which is a contradiction.

Since P is irreducible, we know that (P, Q) is a regular sequence. Hence we have $\mathrm{ht}(\mathfrak{p}) = 2$. By Lemma 3.11, the ideal $\mathfrak{q}S$ is prime. Therefore, $\mathrm{ht}(\mathfrak{q}S) \leqslant 2$ since $\mathfrak{q}S \subset \mathfrak{p}$. Note that for any prime ideal $\mathfrak{q}' \subset \mathcal{A}$ we have $\mathfrak{q}'S \cap \mathcal{A} = \mathfrak{q}'$. By Lemma 3.11, any strict chain of prime ideals $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_m \subsetneq \mathfrak{q}$ in $\mathcal{A}$ extends to a strict chain of prime ideals $\mathfrak{q}_0 S \subsetneq \cdots \subsetneq \mathfrak{q}_m S \subsetneq \mathfrak{q}S$ in S. Therefore $\mathrm{ht}(\mathfrak{q}) \leqslant \mathrm{ht}(\mathfrak{q}S) \leqslant 2$. Suppose $\mathrm{ht}(\mathfrak{q}) = 2$. Then we must have $\mathfrak{p} = \mathfrak{q} \cdot S$. This is a contradiction since $P \in \mathfrak{p}$ and $P \notin (g_1, \cdots, g_s)$. Therefore $\mathrm{ht}(\mathfrak{q}) = 1$, and hence $\mathfrak{q} = (Q)$, as (Q) is a prime ideal. Further, we have $(Q) \subset (P, Q) \cap \mathcal{A} \subset \mathfrak{p} \cap \mathcal{A} = (Q)$. Hence $(P, Q) \cap \mathcal{A} = (Q)$. $\square$

Corollary 3.13. *Let P, Q be non-associate irreducible forms in S. Let $g_1, \cdots, g_s$ be a prime sequence of forms such that $Q \in \mathcal{A} := \mathbb{K}[g_1, \cdots, g_s]$. Suppose $P \notin (g_1, \cdots, g_s)$. Let $Y \subset \mathbb{A}^n$ and $X \subset \mathbb{A}^s$ be the affine schemes defined by the ideals (P, Q) and (Q) respectively. Then every irreducible component of Y dominates X under the projection morphism $\pi : \mathbb{A}^n \to \mathbb{A}^s$ defined by $y \mapsto (g_1(y), \cdots, g_s(y))$.*

Proof. Let $Y = \bigcup_j Y_j$ be the irreducible decomposition of Y. We know that the irreducible components Y_j are in one-to-one correspondence with the minimal primes over (P, Q) in S. Let $\mathfrak{p}_j$ be the minimal prime corresponding to Y_j. Note that $\pi(Y) \subset X$. We have the projection morphism $\pi : Y_j \to X$, where $Y_j = \mathrm{Spec}(S/\mathfrak{p}_j)$ and $X = \mathrm{Spec}(\mathcal{A}/(Q))$. By Proposition 3.12, we have $\mathfrak{p}_j \cap \mathcal{A} = (Q)$. Therefore the generic point $\mathfrak{p}_j$ of Y_j maps to the generic point of X under the morphism π. Hence we conclude that the closure of $\pi(Y_j)$ is X, i.e. Y_j dominates X. $\square$

Lemma 3.14. *Let $\phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of finitely generated $\mathbb{K}$-algebras. Let $Y = \mathrm{Spec}(\mathcal{B})$, $X = \mathrm{Spec}(\mathcal{A})$ and $\pi : Y \to X$ be the corresponding morphism of affine schemes. Suppose that X is irreducible and reduced, i.e. $\mathcal{A}$ is an integral domain. Suppose that every irreducible component of Y dominates X.*

1. *If $\pi^{-1}(x)$ is irreducible for a general closed point $x \in X$, then Y is irreducible.*
2. *If Y is Cohen-Macaulay and $\pi^{-1}(x)$ is reduced for a general closed point $x \in X$, then Y is reduced.*

Proof. (1) Since $\mathcal{A} \to \mathcal{B}$ is a homomorphism of finitely generated $\mathbb{K}$-algebras, the morphsim π is of finite type. Therefore, by [Sta15, Tag 0554], the generic fiber of π is irreducible. Since there is bijection between the irreducible components of Y that dominate X and the irreducible components of the generic fiber, we conclude that Y is irreducible.

(2) Since $\mathcal{B}$ is Cohen-Macaulay, in order to show that $\mathcal{B}$ is reduced, it is enough to show that $\mathcal{B}$ is generically reduced, i.e. $\mathcal{B}_\mathfrak{p}$ is reduced for any minimal prime $\mathfrak{p}$ of $\mathcal{B}$.

Note that the generic point α of $\mathrm{Spec}(\mathcal{A})$ is the point corresponding to the prime ideal (0) in $\mathcal{A}$. Recall that the minimal primes $\mathfrak{p}$ of $\mathcal{B}$ correspond to the generic points of the irreducible components of Y. Since all irreducible components of Y dominate X, any such generic point $\mathfrak{p}$ maps to (0) under $\mathrm{Spec}(\mathcal{B}) \to \mathrm{Spec}(\mathcal{A})$. Therefore we have $\phi^{-1}(\mathfrak{p}) = (0)$. By [Sta15, Tag 0575] we know that the generic fiber Y_α is reduced, where α is the generic point of X. Hence the ring $\mathcal{B} \otimes_\mathcal{A} K(\mathcal{A})$ is reduced, as $Y_\alpha = \mathrm{Spec}(\mathcal{B} \otimes_\mathcal{A} K(\mathcal{A}))$ where $K(\mathcal{A})$ is the fraction field of $\mathcal{A}$.

Let $\frac{f}{t} \in \mathcal{B}_\mathfrak{p}$ be a nilpotent element. Therefore $sf^k = 0$ in $\mathcal{B}$ for some $s \notin \mathfrak{p}$. So $(sf)^k = 0$ in $\mathcal{B}$ and hence $(sf)^k \otimes 1 = 0$ in $\mathcal{B} \otimes_\mathcal{A} K(\mathcal{A})$. Therefore $sf \otimes 1 = 0$ in $\mathcal{B} \otimes_\mathcal{A} K(\mathcal{A})$ by reducedness. Let $T \subset \mathcal{A}$ be the multiplicatively closed set $\mathcal{A} \backslash \{0\}$. Consider the $\mathcal{A}$-bilinear map $\phi : \mathcal{B} \times K(\mathcal{A}) \to T^{-1}\mathcal{B}$ given by $(s, \frac{a}{b}) = \frac{a \cdot s}{b}$. If $sf \otimes 1 = 0$, then by the universal property of tensor products we must have $sf = \phi(sf, 1) = 0$ in $T^{-1}\mathcal{B}$. Therefore there exists an $a \in T$, such that $a \cdot sf = 0$ in $\mathcal{B}$. Note that $s \notin \mathfrak{p}$ and $\phi(a) \notin \mathfrak{p}$, as $\phi^{-1}(\mathfrak{p}) = (0)$. So we have $\phi(a)sf = a \cdot sf = 0 \in \mathfrak{p}$, where $\phi(a)s \notin \mathfrak{p}$.Therefore we conclude that $f = 0$ in $\mathcal{B}_\mathfrak{p}$. Thus $\mathcal{B}_\mathfrak{p}$ is reduced, since $\frac{f}{t}$ was an arbitrary nilpotent element. ☐

Lemma 3.15. *Let $\mathcal{A}$ be a subalgebra generated by a prime sequence of forms in S. Let $P_1, \dots, P_k \in S$ be irreducible polynomials and $d_i \in \mathbb{N}_+$ such that $\prod_{i=1}^{k} P_i^{d_i} \in \mathcal{A}$. Then $P_i \in \mathcal{A}$ for $i \in [k]$.*

Proof. Since a prime sequence is algebraically independent, we know that $\mathcal{A}$ is isomorphic to a polynomial ring, and hence an UFD. Let $F = \prod_{i=1}^{k} P_i^{d_i}$. Since $\mathcal{A}$ is a UFD and $F \in \mathcal{A}$, we have that $F = \prod_{i=1}^{t} Q_i^{e_i}$, where each $Q_i \in \mathcal{A}$ is irreducible. Therefore (Q_i) is a prime ideal in $\mathcal{A}$. By Lemma 3.11, we see that $(Q_i) \cdot S$ is a prime ideal in S. Therefore, Q_i is irreducible in S as well and hence, $F = \prod_{i=1}^{t} Q_i^{e_i}$ is an irreducible decomposition of F in S. Since S is a UFD, by uniqueness of irreducible factorization, we must have that $P_i = Q_i$ and $d_i = e_i$ (after potential permutation of indices). Hence $P_i \in \mathcal{A}$ for all $i \in [k]$. ☐

Note that, in Lemma 3.15, it is necessary to assume that $\mathcal{A}$ is generated by a prime sequence. For example, consider the subalgebra $\mathcal{A} = k[xy] \subset k[x, y]$. Now $\mathcal{A}$ is isomorphic to a polynomial ring in one variable and hence an UFD. However $xy \in \mathcal{A}$ but $x, y \notin \mathcal{A}$.

Another important consequence of intersection flatness and prime sequences is that radical ideals of complete intersections behave well across intersection flat ring homomorphisms. Let $\mathcal{A} \subset S$ be a sub-algebra. Let $I = (f_1, \cdots, f_r) \subset S$ be an ideal such that $f_1, \cdots, f_r \in \mathcal{A}$. Let $I_\mathcal{A}$ be the ideal generated by $f_1, \cdots, f_r$ in $\mathcal{A}$. Let $\mathrm{rad}_S(I)$ and $\mathrm{rad}_\mathcal{A}(I_\mathcal{A})$ denote the radical of I in S and $I_\mathcal{A}$ in $\mathcal{A}$ respectively. For example, if F, G are two non-associate cubic forms, and if the generators of $\mathcal{A}$ form a prime sequence, then the minimal primes of $I = (F, G)$ and $I_\mathcal{A}$ are in one-to-one correspondence, and the elimination ideal of $\mathrm{rad}_S(I)$ is exactly $\mathrm{rad}_\mathcal{A}(I_\mathcal{A})$.

Proposition 3.16. *Let $\mathcal{A} = \mathbb{K}[g_1, \dots, g_r] \subset S$ be a sub-algebra where $g_1, \dots, g_r$ form a homogeneous prime sequence in S. Let $F, G \in S$ be two irreducible forms such that F, G form a regular sequence in S. Suppose $F, G \in \mathcal{A}$. Let $I = (F, G)$ be the ideal generated by F, G in S. Then we have*

1. *If $\mathfrak{p} \subset \mathcal{A}$ is a minimal prime of $I_{\mathcal{A}}$ then $\mathfrak{p} \cdot S$ is a minimal prime of I in S. Conversely, any minimal prime $\mathfrak{q} \subset S$ of I is of the form $\mathfrak{q} = \mathfrak{p} \cdot S$ for some minimal prime $\mathfrak{p} \subset \mathcal{A}$ of $I_{\mathcal{A}}$.*

2. *Moreover, we have*

$$\mathrm{rad}_S(I) \cap \mathcal{A} = \mathrm{rad}_{\mathcal{A}}(I_{\mathcal{A}}) \text{ and } \mathrm{rad}_{\mathcal{A}}(I_{\mathcal{A}}) \cdot S = \mathrm{rad}_S(I).$$

Proof. Since $g_1 \cdots, g_r$ is a prime sequence, the subalgebra $\mathcal{A}$ is isomorphic to a polynomial ring in r variables. Note that F, G are irreducible as elements of $\mathcal{A}$, and hence F, G is a regular sequence in $\mathcal{A}$. Therefore every associated prime of the ideal (F, G) in $\mathcal{A}$ is minimal and of height 2. Thus we can write $\mathrm{rad}_{\mathcal{A}}(I_{\mathcal{A}}) = \bigcap_{i=1}^{a} \mathfrak{p}_i$ where $\mathfrak{p}_i \subset \mathcal{A}$ are the minimal primes of (F, G) in $\mathcal{A}$. Similarly, we have $\mathrm{rad}_S(I) = \bigcap_{j=1}^{b} \mathfrak{q}_j$ where $\mathfrak{q}_j \subset S$ are the minimal primes of (F, G) in S, and thus $\mathrm{ht}(\mathfrak{q}_j) = 2$.

Since $\mathrm{rad}_{\mathcal{A}}(I_{\mathcal{A}}) \subset \mathrm{rad}_S(I) \cap \mathcal{A}$, we have that $\mathrm{rad}_{\mathcal{A}}(I_{\mathcal{A}}) \subset (\mathfrak{q}_j \cap \mathcal{A})$ for all j. Moreover, since contractions of prime ideals are prime, we have that the ideal $\mathfrak{q}_j \cap \mathcal{A}$ is prime. Then $\bigcap_i \mathfrak{p}_i \subset (\mathfrak{q}_j \cap \mathcal{A})$ implies that $\mathfrak{p}_i \subset (\mathfrak{q}_j \cap \mathcal{A})$ for some i, by [AM69, Proposition 1.11]. Since $2 = \mathrm{ht}(\mathfrak{p}_i) \leqslant \mathrm{ht}(\mathfrak{q}_j \cap \mathcal{A}) \leqslant \mathrm{ht}(\mathfrak{q}_j) = 2$, all inequalities must be equalities and thus we must have $\mathfrak{p}_i = \mathfrak{q}_j \cap \mathcal{A}$. Additionally, by Lemma 3.11, we have that $\mathfrak{p}_i S$ is prime in S and since $\mathfrak{p}_i S \subset \mathfrak{q}_j$, the same height argument implies $\mathfrak{p}_i S = \mathfrak{q}_j$ in S.

Hence, the previous paragraph and pure dimensionality imply that $a = b$ and after relabeling we have $\mathfrak{q}_j = \mathfrak{p}_i S$, which implies that $\mathrm{rad}_S(I) \cap \mathcal{A} = \mathrm{rad}_{\mathcal{A}}(I_{\mathcal{A}})$ and $\mathrm{rad}_{\mathcal{A}}(I_{\mathcal{A}}) \cdot S = \mathrm{rad}_S(I)$. ☐

3.3 Elimination, Resultants and Radical ideals

We now prove some structural results on the elimination ideals of radical ideals of the form $\mathrm{rad}(P, Q)$. The following results show that certain eliminations will lead to principal ideals.

Lemma 3.17 (Principal Eliminations). *Let $P, Q \in S := \mathbb{K}[x_1, \ldots, x_r, y_1, \ldots, y_s]$ be irreducible polynomials. Let $S' = \mathbb{K}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r, y_1, \ldots, y_s]$. Suppose the following conditions hold:*

1. $Q \in R := \mathbb{K}[y_1, \ldots, y_s]$.

2. *the polynomial P depends on the variable x_i, i.e. $P = \sum_{k=0}^{e} p_k x_i^k$, where $0 < e$, $p_e \neq 0$ and $p_k \in S'$ for all k.*

3. $p_e \notin (Q)$.

Then the elimination ideal $\mathrm{rad}(P, Q)_{x_i} = \mathrm{rad}(P, Q) \cap R$ is generated by Q, i.e. $\mathrm{rad}(P, Q)_{x_i} = (Q)$.

Proof. By item (2), we have $P = \sum_{k=0}^{e} p_k x_i^k$, where $0 < e$ and $p_k \in S'$, with $p_e \neq 0$. Let $J = \mathrm{rad}(P, Q) \cap S'$. Since $Q \in J$, we have that $(Q) \subseteq J$.

To show the other inclusion, if $F \in J$, we have that

$$F^D = PA + QB$$

for some $A, B \in S$. We will now show that $A \in (Q)$. To prove this, it is enough to show that for any solution $A, B \in S$ to the equation above, we must have that Q divides the leading term of A, when considered as a polynomial in x_i with coefficients in S'.

Since $\deg_{x_i}(F^D) = \deg_{x_i}(Q) = 0$, we have that $\deg_{x_i}(PA) = \deg_{x_i}(B) =: \ell \geqslant e$, which implies that $A = \sum_{k=0}^{\ell-e} a_k x_i^k$, $B = \sum_{k=0}^{\ell} b_k x_i^k$, where $b_\ell \neq 0$. Therefore we have

$$p_e \cdot a_{\ell-e} + Qb_\ell = 0 \Rightarrow Q \mid a_{\ell-e}$$

since Q is irreducible and $p_e \notin (Q)$.

Thus, by induction, we must have that $A \in (Q)$, which proves that $F^D \in (Q)$. Since (Q) is prime, as Q is irreducible and S is a UFD, the latter implies $F \in (Q)$ and thus $J = (Q)$. $\square$

We can easily generalize the statement above to subalgebras generated by homogeneous prime sequences, and we do so in the next lemma. While the proof is essentially the same as in the above lemma, some additional technicalities are needed to make the proof go through, and therefore we add a proof for completeness.

Lemma 3.18 (Principal eliminations in prime sequences). *Let $\mathcal{A} = \mathbb{K}[x_1, \dots, x_r, y_1, \dots, y_s] \subset S$ be a subalgebra of our polynomial ring S such that $x_1, \dots, x_r, y_1, \dots, y_s$ are homogeneous polynomials which form a prime sequence. Let P, Q be two irreducible polynomials in S and let $\mathcal{A}' = \mathbb{K}[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r, y_1, \dots, y_s]$. Suppose the following holds:*

1. $P \in \mathcal{A}$ *and* $Q \in \mathcal{B} := \mathbb{K}[y_1, \dots, y_s]$.
2. P *depends on* x_i*, i.e.* $P = \sum_{k=0}^{e} p_k x_i^k$*, where* $0 < e$*,* $p_e \neq 0$ *and* $p_k \in \mathcal{A}'$ *for all* k.
3. $p_e \notin (Q)$.

Then the elimination ideal $\mathrm{rad}(P, Q)_{x_i} = \mathrm{rad}(P, Q) \cap \mathcal{A}'$ *is generated by* (Q)*, i.e.* $\mathrm{rad}(P, Q)_{x_i} = (Q)$.

Proof. Note that $\mathcal{B} \subset \mathcal{A}$ is an intersection flat subalgebra of $\mathcal{A}$ generated by a homogeneous prime sequence. Let $F \in \mathrm{rad}(P, Q) \cap \mathcal{A}'$. By Proposition 3.16, we know that if $F \in \mathcal{A}$ is such that $F \in \mathrm{rad}(P, Q)$, then there exist $A, B \in \mathcal{A}$ and $D \in \mathbb{N}_{>0}$ such that $F^D = PA + QB$.

Since P depends on x_i, we have $P = \sum_{k=0}^{e} p_k x_i^k$, where $0 < e < d$ and $p_k \in \mathcal{B}$ for each $0 \leqslant k \leqslant e$, with $p_e \notin (Q)$. Let $J = \mathrm{rad}(P, Q) \cap \mathcal{B}$. Since $Q \in J$, we have that $(Q) \subseteq J$.

To show the other inclusion, if $F \in \mathcal{B}$ is such that $F \in \mathrm{rad}(P, Q)$, by the first paragraph there exist $A, B \in \mathcal{A}$ and $D \in \mathbb{N}_{>0}$ such that $F^D = PA + QB$. Since $F, Q \in \mathcal{B}$, we have that $\deg_{x_i}(F) = \deg_{x_i}(Q) = 0$ over $\mathcal{A}$. Thus, we have that $\deg_{x_i}(PA) = \deg_{x_i}(B) =: \ell \geqslant e$, which implies that $A = \sum_{k=0}^{\ell-e} a_k x_i^k$, $B = \sum_{k=0}^{\ell} b_k x_i^k$, where $a_k, b_k \in \mathcal{B}$, $a_{\ell-e}, b_\ell \neq 0$ and thus

$$p_e \cdot a_{\ell-e} + Qb_\ell = 0 \Rightarrow Q \mid a_{\ell-e}$$

since $\mathcal{B}$ is a UFD, $Q \in \mathcal{B}$ is irreducible and $p_e \notin (Q)$.

Thus, by induction, we must have that $A \in (Q)$, which proves that $F^D \in (Q)$. Since (Q) is prime, as Q is irreducible and $\mathcal{B}$ is a UFD, the latter implies $F \in (Q)$ and thus $J = (Q)$. $\square$

Corollary 3.19. *Let $\mathcal{A} = \mathbb{K}[x_1, \dots, x_r, y_1, \dots, y_s] \subset S$ be a subalgebra of our polynomial ring S such that $x_1, \dots, x_r, y_1, \dots, y_s$ are homogeneous polynomials which form a prime sequence, and let $P, Q, F \in \mathcal{A}$ be three irreducible and pairwise coprime forms such that $Q \in \mathbb{K}[y_1, \dots, y_s]$ and $F \in \mathrm{rad}(P, Q)$. If P depends on x_i, then F also depends on x_i.*

Proof. Let $\mathcal{B} := \mathbb{K}[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r, y_1, \dots, y_s]$. By Lemma 3.18, we have that $\mathrm{rad}(P, Q)_{x_i} = (Q)$. Since F, Q are pairwise coprime, we have that $F \notin (Q)$, and therefore we must have $F \notin \mathcal{B}$. $\square$

Let S be our polynomial ring and P, Q be two polynomials in S. Let $g_1, \cdots, g_s$ be a prime sequence of forms. Let $\mathcal{A} = \mathbb{K}[g_1, \cdots, g_s]$ be the subalgebra generated by $g_1, \cdots, g_s$. Since $g_1, \cdots, g_s$ are algebraically independent, we have an isomorphism $\phi : \mathbb{K}[x_1, \cdots, x_s] \xrightarrow{\sim} \mathcal{A}$ given by $x_i \mapsto g_i$.

Definition 3.20. In the setting as above, suppose that $P, Q \in \mathcal{A}$. Let $\widetilde{P} = \phi^{-1}(P)$ and $\widetilde{Q} = \phi^{-1}(Q)$. We define the resultant of P, Q with respect to g_i in the subalgebra $\mathcal{A}$ as

$$\mathrm{Res}^{\mathcal{A}}_{g_i}(P, Q) = \phi(\mathrm{Res}_{x_i}(\widetilde{P}, \widetilde{Q})).$$

Similarly we define the discriminant of P with respect to g_i in the sublagebra $\mathcal{A}$ as

$$\mathrm{Disc}^{\mathcal{A}}_{g_i}(P) = \phi(\mathrm{Res}_{x_i}(\widetilde{P}, \frac{\partial \widetilde{P}}{\partial x_i})).$$

Using the isomorphism ϕ we can translate the usual properties of resultants in polynomial rings to similar properties in the subalgebra $\mathcal{A}$. Note that $\mathrm{Res}^{\mathcal{A}}_{g_i}(P, Q)$ is in the subalgebra generated by $\{g_1, \cdots, g_s\} \backslash \{g_i\}$ in S. Also, we have $\mathrm{Res}^{\mathcal{A}}_{g_i}(P, Q) \in (P, Q)$.

If we start with a form $P \in S$ such that $P \in \mathbb{K}[g_1, \cdots, g_s]$ and write P as a polynomial in $g_1 \cdots, g_s$, then P might not be homogeneous in $g_1, \cdots, g_s$ (if we consider each g_i having degree 1 in $\mathbb{K}[g_1, \ldots, g_s]$). Thus we need to consider possibly non-homogeneous polynomials in the following result.

Lemma 3.21. *Let $P, Q \in S = \mathbb{K}[z_1, \cdots, z_m, x_1, \cdots, x_s]$ be irreducible polynomials. If the following holds*

1. $Q \in R := \mathbb{K}[x_1, \cdots, x_s]$,
2. $\mathfrak{p} \cap R = (Q)$ *for any minimal prime* $\mathfrak{p}$ *of* (P, Q).
3. *For all* $i \in [m]$ *such that* P *depends on the variable* z_i, *we have* $\mathrm{Disc}_{z_i}(P) \notin (Q)$.

Then the ideal (P, Q) *is radical.*

Proof. Note that if $P \in R$, then condition 2 above implies that $P \in (Q)$. Therefore, in this case we have $(P, Q) = (Q)$, which is a prime ideal. Hence we may assume that $P \notin R$ and in particular, there exists at least one variable z_i such that P depends on z_i. Let $\pi : \mathbb{A}^{m+s} \to \mathbb{A}^s$ be the projection morphism onto the $(x_1, \cdots, x_s)$ coordinates. Let $\mathcal{B} = S/(P, Q)$ and $\mathcal{A} = R/(Q)$. Note that $\pi|_Y : Y \to X$ is the corresponding morphism of affine schemes, where $Y = \mathrm{Spec}(\mathcal{B})$ and $X = \mathrm{Spec}(\mathcal{A})$. Note that by assumption (2), every irreducible component of Y dominates X. Therefore, by Lemma 3.14, it is enough to show that $\pi|_Y^{-1}(x)$ is reduced for a general closed point $x \in X$.

Let $x = (c_1, \cdots, c_s) \in X$ be a closed point. Let $P_x \in \mathbb{K}[z_1 \cdots, z_m]$ denote the polynomial $P(z_1 \cdots, z_m, c_1, \cdots, c_s)$. Note that $\pi|_Y^{-1}(x) = \mathrm{Spec}(\mathbb{K}[z_1, \cdots, z_m]/(P_x))$. Now $\pi|_Y^{-1}(x)$ is not reduced iff the polynomial P_x has a multiple factor, i.e. $\mathrm{Disc}_{z_i}(P_x) \equiv 0$ for some variable z_i such that P_x depends on z_i. Let $Z_i \subset \mathbb{A}^{m+s}$ be the closed subscheme defined by $\mathrm{Disc}_{z_i}(P)$. Now, if P depends on a variable z_i, then $\mathrm{Disc}_{z_i}(P_x) \equiv 0$ iff $\pi^{-1}(x) \subset Z_i$. To summarize, for a closed point $x \in X$, we have that $\pi|_Y^{-1}(x)$ is reduced iff $\pi^{-1}(x) \not\subset Z_i$ for all i such that P depends on z_i.

Note that $\mathrm{Spec}(S/(Q)) = \pi^{-1}(X) = X \times \mathbb{A}^m$. By assumption (3), we have that $Z_i \cap \pi^{-1}(X)$ is of pure dimension $m + s - 2$. Indeed, since Q is irreducible and $\mathrm{Disc}_{z_i}(P) \notin (Q)$, we see that

$Q, \mathrm{Disc}_{z_i}(P)$ is a regular sequence. Recall that X is of dimension $s-1$. If $Z_i \cap \pi^{-1}(X)$ does not dominate X, then for a general $x \in X$ we have $\pi^{-1}(x) \not\subset Z_i$. If $Z_i \cap \pi^{-1}(X)$ dominates X, by [Har77, Exercise II.3.2], we conclude that there is a dense open subset $U \subset X$ such that $Z_i \cap \pi^{-1}(x)$ is of dimension $m-1$ for $x \in U$. Therefore for a general $x \in X$, we have $\pi^{-1}(x) \not\subset Z_i$ as $\dim(\pi^{-1}(x)) = m$. Hence, $\pi|_Y^{-1}(x)$ is reduced for a general closed point $x \in X$. □

Lemma 3.22. *Let $P, Q \in S$ be irreducible forms and $h_1 \cdots, h_m, g_1, \cdots, g_s$ be a prime sequence of forms in S such that $P, Q \in \mathcal{A} := \mathbb{K}[h_1, \cdots, h_m, g_1, \cdots, g_s]$. Suppose the following holds*

1. *$Q \in \mathbb{K}[g_1, \cdots, g_s]$,*
2. *$P \notin (g_1, \cdots, g_s)$ in S,*
3. *For all $i \in [m]$ such that P depends on h_i, then $\mathrm{Disc}^{\mathcal{A}}_{h_i}(P) \notin (Q)$.*

Then the ideal (P, Q) is a radical ideal in S.

Proof. Since $h_1 \cdots, h_m, g_1, \cdots, g_s$ is a prime sequence, we have an isomorphism

$$\phi : \mathbb{K}[z_1 \cdots, z_m, x_1, \cdots, x_s] \to \mathcal{A}$$

given by $z_i \mapsto h_i$ and $x_j \mapsto g_j$. Let $\widetilde{P} = \phi^{-1}(P)$ and $\widetilde{Q} = \phi^{-1}(Q)$. Note that the minimal primes of $(\widetilde{P}, \widetilde{Q})$ are of the form $\phi^{-1}(\mathfrak{p})$ for some minimal prime $\mathfrak{p}$ of (P, Q) in $\mathcal{A}$. Note that if $\mathfrak{p}$ is a minimal prime of (P, Q) in $\mathcal{A}$, then $\mathfrak{p}S$ is a minimal prime of (P, Q) in S by Lemma 3.11. Thus, by Proposition 3.12, we see that $\mathfrak{p} \cap \mathbb{K}[g_1, \cdots, g_s] = (Q)$. Therefore condition (2) of Lemma 3.21 is satisfied for $\widetilde{P}, \widetilde{Q}$. Note that conditions (1) and (3) of Lemma 3.21 are satisfied for $\widetilde{P}, \widetilde{Q}$ via the isomorphism ϕ. Therefore the ideal $(\widetilde{P}, \widetilde{Q})$ is radical in the polynomial ring $\mathbb{K}[z_1, \cdots, z_m, x_1, \cdots, x_s]$. Hence (P, Q) is radical in the subalgebra $\mathcal{A}$. By Lemma 3.11, prime ideals of $\mathcal{A}$ extend to prime ideals in S. Thus, by Proposition 3.10, the ideal (P, Q) is an intersection of prime ideals and hence radical. □

Example 3.23. *Note that in Lemma 3.22, we need that $\mathrm{Disc}^{\mathcal{A}}_{h_i}(P) \notin (Q)$ for all z_i such that P depends on z_i. It is not enough to assume that this property holds for just one such z_i. One can construct such an example as follows. Let $P = y^3 + vy^2 + (xu^2 - z^3)$ and $Q = xu^2 - z^3$ in $\mathbb{K}[x, y, z, u, v]$. Then $Q \in \mathbb{K}[x, u, z]$ and $\mathrm{Disc}_v(P) = y^2 \notin (Q)$. However, the ideal (P, Q) is not radical as $\sqrt{(P, Q)} = (y^2 + yv, xu^2 - z^3)$. This occurs because we have $\mathrm{Disc}_y(P) \in (Q)$.*

Corollary 3.24. *Let $P \in S$ be irreducible form of degree $\leqslant d$. Suppose $h_1 \cdots, h_m, g_1, \cdots, g_s$ is a prime sequence of forms in S such that $P \in \mathcal{A} := \mathbb{K}[h_1, \cdots, h_m, g_1, \cdots, g_s]$. If $P \notin (g_1, \cdots, g_s)$, then there exist at most $d^2(2d-1)$ irreducible forms $Q_i \in \mathbb{K}[g_1, \cdots, g_s]$ such that (P, Q_i) is not radical and $Q_i \notin (Q_j)$ for $i \neq j$.*

Proof. Let $\mathcal{A}' = \mathbb{K}[g_1, \cdots, g_s]$. Consider P as a polynomial in $\mathcal{A}'[h_1, \cdots, h_m]$. Since $P \notin (g_1, \cdots, g_s)$, there exists a monomial $\alpha \prod_j h_{i_j}^{e_j}$ in P, where α is a non-zero constant in $\mathbb{K}$ and $h_{i_j} \in \{h_1 \cdots, h_m\}$. Without loss of generality, let $h_1, \cdots, h_k$ be the elements appearing in this monomial. Note that we must have $k \leqslant d$ since $\deg(P) \leqslant d$. Note that $P \notin (h_{k+1}, \cdots, h_m, g_1 \cdots, g_s)$. Suppose for some $Q \in \mathbb{K}[g_1, \cdots, g_s]$ we have (P, Q) not radical. Then by Lemma 3.22, we know that $\mathrm{Disc}^{\mathcal{A}}_{h_i}(P) \in (Q)$ for some $i \in [k]$. Therefore, for any $Q \in \mathbb{K}[g_1, \cdots, g_s]$ such that (P, Q) is not radical, we must have that Q divides $\prod_{i=1}^k \mathrm{Disc}^{\mathcal{A}}_{h_i}(P)$. Now, $\deg(\mathrm{Disc}^{\mathcal{A}}_{h_i}(P)) \leqslant d(2d-1)$ for all i. Therefore, there can be at most $d^2(2d-1)$ such irreducible forms Q_i, where $Q_i \notin (Q_j)$ for $i \neq j$. □

Proposition 3.25. *Let* $P, Q \in S = \mathbb{K}[y, x_1, \cdots, x_n]$ *be irreducible polynomials. Suppose* $P = a_d y^d + \cdots + a_0$ *and* $Q = b_e y^e + \cdots + b_0$*, where* d, e *are positive integers,* $a_i, b_i \in \mathcal{A} := \mathbb{K}[x_1, \cdots, x_n]$*, and at least one of* a_d *or* b_e *is non-zero. Let* $\mathfrak{p} \subset S$ *be a minimal prime over the ideal* (P, Q) *and* $\mathfrak{q} = \mathfrak{p} \cap \mathcal{A}$*. Then the following holds:*

(1) If $\mathrm{ht}(\mathfrak{q}) = 1$*, then* $\mathfrak{q} = (f)$ *where* $f \in \mathcal{A}$ *is an irreducible factor of the resultant* $\mathrm{Res}_y(P, Q)$*.*

(2) If $\mathrm{ht}(\mathfrak{q}) = 2$*, then* $(a_0, \cdots, a_d, b_0, \cdots, b_e) \subset \mathfrak{p}$*.*

Proof. Note that $\mathrm{Res}_y(P, Q)$ is a non-zero polynomial in $(P, Q) \cap \mathcal{A}$. In particular, $\mathrm{Res}_y(P, Q) \in \mathfrak{q}$. Since $\mathfrak{q}$ is a prime ideal, we must have that $f \in \mathfrak{q}$ for some irreducible factor f of $\mathrm{Res}_y(P, Q)$. Now (f) is a prime ideal, since f is irreducible. Therefore if $\mathrm{ht}(\mathfrak{q}) = 1$, then we must have $\mathfrak{q} = (f)$.

Suppose $\mathrm{ht}(\mathfrak{q}) = 2$. Recall that $\mathfrak{q}S$ is a prime ideal. By Lemma 3.11, any strict chain of prime ideals in $\mathcal{A}$ extends to a strict chain of prime ideals in S. Thus, $\mathrm{ht}(\mathfrak{q}S) = 2$. Since (P, Q) is a regular sequence and $\mathfrak{p}$ is a minimal prime, we have $\mathrm{ht}(\mathfrak{p}) = 2$. Therefore $\mathfrak{p} = \mathfrak{q}S$.

Consider the projection morphism $\pi : \mathbb{A}^{n+1} \to \mathbb{A}^n$ corresponding to the inclusion $\mathcal{A} \subset S$. Let $Y \subset \mathbb{A}^{n+1}$ be the closed subscheme defined by (P, Q). Let $Z \subset \mathbb{A}^{n+1}$ be the irreducible component of Y given by $\mathrm{Spec}(S/\mathfrak{p})$, and $W \subset \mathbb{A}^n$ the closed subscheme corresponding to $\mathrm{Spec}(\mathcal{A}/\mathfrak{q})$. Note that the scheme-theoretic inverse image $\pi^{-1}(W)$ is given by $\mathrm{Spec}(S/\mathfrak{q}S)$. Since $\mathfrak{p} = \mathfrak{q}S$, we conclude that $Z = \pi^{-1}(W)$. Note that $\pi^{-1}(W) \simeq W \times \mathbb{A}^1$. Therefore $\dim(\pi|_Z^{-1}(w)) = 1$ for any $w \in W$.

Suppose $a_i \notin \mathfrak{q}$ for some i. Then for any closed point $w = (c_1, \cdots, c_n) \in W \backslash V(a_i)$, we have $P_w(y) = P(y, c_1, \cdots, c_n)$ is a non-zero polynomial in y. Therefore $P_w(y) = 0$ for at most finitely many $y \in \mathbb{A}^1$. Hence $\pi^{-1}(w) \cap Y$ is finite for any closed point $w \in W \backslash V(a_i)$. Since $\pi|_Z^{-1}(w) \subset \pi^{-1}(w) \cap Y$, we have a contradiction. Hence $a_i \in \mathfrak{q}$ for any i, and similarly $b_j \in \mathfrak{q}$ for any j. Therefore $(a_0, \cdots, a_d, b_0, \cdots, b_e) \subset \mathfrak{p}$. ☐

Corollary 3.26. *Let* $P, Q \in S = \mathbb{K}[y, x_1, \cdots, x_n]$ *be irreducible polynomials. Suppose* $P = a_d y^d + \cdots + a_0$ *and* $Q = b_e y^e + \cdots + b_0$*, where* d, e *are positive integers and* $a_i, b_i \in \mathcal{A} := \mathbb{K}[x_1, \cdots, x_n]$*, at least one of* a_d *or* b_e *being non-zero. Suppose* $(a_0, \cdots, a_d, b_0, \cdots, b_d) \not\subset \mathrm{rad}(P, Q)$*. Let* $C_3 \in \mathrm{rad}(P, Q)$ *be an irreducible polynomial. If* $C_3 \in \mathcal{A}$*, then* C_3 *divides the resultant* $\mathrm{Res}_y(P, Q)$*.*

Proof. Recall that $\mathrm{rad}(P, Q)$ is the intersection of all the minimal primes $\mathfrak{p}_i$ over (P, Q) in S. Since $C_3 \in \mathrm{rad}(P, Q)$, we know that $C_3 \in \cap_i \mathfrak{q}_i$, where $\mathfrak{q}_i = \mathfrak{p}_i \cap \mathcal{A}$. If $\mathrm{ht}(\mathfrak{q}_i) = 2$ for all i, then $(a_0 \cdots, a_d, b_0, \cdots, b_e) \subset \cap_i \mathfrak{p}_i = \mathrm{rad}(P, Q)$, by Proposition 3.25. Therefore there exists i such that $\mathrm{ht}(\mathfrak{q}_i) = 1$. By Proposition 3.25, we know that $\mathfrak{q} = (f)$ for some irreducible factor of $\mathrm{Res}_y(P, Q)$. Since C_3 is irreducible we conclude that $(C_3) = (f)$ and hence C_3 divides $\mathrm{Res}_y(P, Q)$. ☐

Proposition 3.27. *Let* $P, Q \in S = \mathbb{K}[y, x_1, \cdots, x_n]$ *be irreducible polynomials of positive degree in* y*. Suppose* $P = a_d y^d + \cdots + a_0$ *and* $Q = b_d y^d + \cdots + b_0$*, where* $a_i, b_i \in \mathcal{A} := \mathbb{K}[x_1, \cdots, x_n]$*. Let* f *be an irreducible factor of* $\mathrm{Res}_y(P, Q)$ *and let* $I_1 = (P, Q) \cap \mathcal{A}$*. Suppose there exist* $i, j \neq 0$*, such that* $f \nmid a_i$ *and* $f \nmid b_j$*. Then* $\mathrm{rad}(I_1) \subset (f)$*.*

Proof. Let $X_f \subset \mathbb{A}^n$ be the hypersurface defined by f. By our hypothesis, there exist a_i, b_j such that $X_f \backslash V(a_i b_j) \neq \varnothing$. Therefore $U = X_f \backslash V(a_i b_j)$ is a non-empty open subset of X_f. Since X_f is irreducible, we have $\overline{U} = X_f$. Now, for any $x \in U$, the polynomials $P_x(y) = a_d(x) y^d + \cdots + a_0(x)$ and $Q_x(y) = b_d(x) y^d + \cdots + b_0(x)$ are of positive degree in y. Since f is an irreducible factor of $\mathrm{Res}_y(P, Q)$, we see that $\mathrm{Res}_y(P, Q)(x) = 0$ for any $x \in X_f$. Therefore, $\mathrm{Res}_y(P_x(y), Q_x(y)) = 0$ for any $x \in U$. Hence, for any $x \in U$, the polynomials $P_x(y)$ and $Q_x(y)$ have a common root $y = c$ and we have $(c, x) \in V(I) \subset \mathbb{A}^{n+1}$, where $I = (P, Q)$. Therefore, for any $x \in U$, there exists a c such that $(c, x) \in V(I)$ and hence $U \subset \pi(V(I)) \subset V(I_1)$. Thus we have $X_f = \overline{U} \subset V(I_1)$ and $\mathrm{rad}(I_1) \subset (f)$. □

Proposition 3.28. *Let $P, Q \in S = \mathbb{K}[y, x_1, \cdots, x_n]$ be irreducible polynomials of positive degree in y. Suppose $P = a_d y^d + \cdots + a_0$ and $Q = b_d y^d + \cdots + b_0$, where $a_i, b_i \in A = \mathbb{K}[x_1, \cdots, x_n]$. Let I_1 be the elimination ideal $(P, Q) \cap A$. If $\mathrm{rad}(\mathrm{Res}_y(P, Q)) \neq \mathrm{rad}(I_1)$, then there exists an irreducible factor f of $\mathrm{Res}_y(P, Q)$, such that $(a_d, a_{d-1}, \cdots, a_1, b_d) \subset (f)$ or $(a_d, b_d, b_{d-1}, \cdots, b_1) \subset (f)$. In particular, if a_d and b_d do not have a common factor then $\mathrm{rad}(\mathrm{Res}_y(P, Q)) = \mathrm{rad}(I_1)$.*

Proof. Let $f_1, \cdots, f_k$ be the distinct irreducible factors of $\mathrm{Res}_y(P, Q)$. If $\mathrm{rad}(I_1) \subset (f_i)$ for all i, then $\mathrm{rad}(I_1) \subset (f_1 \cdots, f_k) = \mathrm{rad}(\mathrm{Res}_y(P, Q))$. Then $\mathrm{rad}(\mathrm{Res}_y(P, Q)) = \mathrm{rad}(I_1)$, as $\mathrm{Res}_y(P, Q) \in I_1$. Therefore we may assume that there is an irreducible factor f of $\mathrm{Res}_y(P, Q)$, such that $\mathrm{rad}(I_1) \not\subset (f)$. Then by Proposition 3.27, we have $f | a_i$ for all $i \neq 0$ or $f | b_j$ for all $j \neq 0$. Without loss of generality, let us assume that $f | a_i$ for all $i \neq 0$. We will show that $f | b_d$. Since $f | a_i$ for all $i \neq 0$, if we have that $\mathrm{Res}_y(P, Q) \equiv a_0^d b_d^d$ modulo (f). Since f is an irreducible factor of $\mathrm{Res}_y(P, Q)$, we have that $f | a_0^d b_d^d$. As P is irreducible, f can not divide a_0. Therefore $f | b_d$. □

The following result, from [GSS05, Proposition 23], states that birational projections are well-behaved with respect to primary decompositions.

Proposition 3.29. *Let $I \subset S = \mathbb{K}[y, x_1, \cdots, x_n]$ be an ideal. Suppose I contains a polynomial $f = gy + h$ such that $g, h \in R = \mathbb{K}[x_1, \cdots, x_n]$ and g is a non-zero divisor in S/I. Let $I_1 = I \cap R$ be the elimination ideal. Then*

(1) I is prime if and only if I_1 is prime.

(2) I is primary if and only if I_1 is primary.

(3) Any irredundant primary decomposition of I_1 lifts to an irredundant primary decomposition of I.

(4) I is radical if and only if I_1 is radical.

Proof. Parts $(1) - (3)$ is the content of [GSS05, Proposition 23]. We prove (4) here. Suppose I is a radical ideal, then I is an intersection of prime ideals. Therefore $I_1 = I \cap R$ is the intersection of the prime ideals containing I with R, and hence radical. Suppose I_1 is a radical ideal. Consider a minimal primary decomposition $I_1 = \cap_i q_i$. Since I_1 is a radical ideal, we must have that the primary ideals q_i are actually prime. By part (3), we can lift the primary decomposition $I_1 = \cap_i q_i$ to a primary decomposition of $I = \cap_j \tilde{q}_j$ where $\tilde{q}_j \cap R = q_i$ for some i. Now by part (1), we can conclude that the primary ideals $\tilde{q}_j$ are prime, and hence I is a radical ideal. □

3.4 Determinantal ideals

Let R be a Cohen-Macaulay ring and M a $p \times q$ matrix with entries in R. Let $I_k(M) \subset R$ be the ideal generated by the $k \times k$ minors of the matrix M. Then $\mathrm{ht}(I_k(M)) \leqslant (p - k + 1) \times (q - k + 1)$ [Eis95, Exercise 10.9]. It was proved by Eagon-Hochster [EH71, Theorem 1.1] that if $\mathrm{ht}(I_k(M)) = (p - k + 1) \times (q - k + 1)$ then $R/I_k(M)$ is Cohen-Macaulay (see [Eis95, Theorem 18.18]). We note the precise statement for determinanatal ideals in polynomial rings below.

Proposition 3.30. *Let* M *be a* $p \times q$ *matrix with entries in the polynomial ring* S*. Let* $I_k(M) \subset S$ *be the ideal generated by the* $k \times k$ *minors of* M*. If* $\mathrm{ht}(I_k(M)) = (p-k+1) \times (q-k+1)$*, then* $S/I_k(M)$ *is Cohen-Macaulay.*

Suppose M is a matrix whose entries are linear forms in the polynomial ring S. We say that M is 1-generic if every non-trivial linear combination of the rows of M consists of linearly independent linear forms [Eis05, Section 6B]. If M is a 1-generic matrix, then the ideal generated by the maximal minors of M is a prime ideal [Eis05, Theorem 6.4].

Proposition 3.31. *If* M *is a* 1*-generic matrix of linear forms in the polynomial ring* S*, of size* $p \times q$ *where* $p \leqslant q$*. Then the ideal* $I_p(M)$ *generated by the maximal minors of* M *is a prime ideal of height* $q-p+1$*. Furthermore,* $S/I_p(M)$ *is Cohen-Macaulay.*

Example 3.32. *Let* M *be one of the following matrices*

$$\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} y_0 & y_2 & y_3 \\ y_1 & y_3 & y_4 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} y_0 & y_2 & y_4 \\ y_1 & y_3 & y_5 \end{pmatrix},$$

where $y_1, \ldots, y_5 \in S_1$ *are linearly independent linear forms. Then* M *is a* 1*-generic matrix and by Proposition 3.31, the ideal* $I = I_2(M)$ *defined by the maximal minors of* M *is a prime ideal. Note that the ideal* I *does not contain any linear forms. Further, one can check the Hilbert-Samuel multiplicty of* I *is* 3*, i.e.* $e(S/I) = 3$*. In Corollary 3.38, we will show that these are the only such homogeneous prime ideals.*

One consequence of Proposition 3.30 is that certain determinantal varieties are prime, as the following lemma shows.

Lemma 3.33. *Let* $a, x_1, x_2, y_1, y_2 \in S_1$ *and* $P \in S_2$ *be such that* $\dim \mathrm{span}_{\mathbb{K}} \{a, x_1, x_2\} = 3$ *and* $P \bmod (a, x_1, x_2, y_1, y_2)$ *is irreducible. If* $x_1P - y_1a^2, x_2P - y_2a^2$ *and* $x_1y_2 - x_2y_1$ *are irreducible, then the ideal* $(x_1P - y_1a^2, x_2P - y_2a^2, x_1y_2 - x_2y_1)$ *is prime.*

Proof. Let $I = (x_1P - y_1a^2, x_2P - y_2a^2, x_1y_2 - x_2y_1)$ and $J = (x_1P - y_1a^2, x_2P - y_2a^2)$. Note that I is the determinantal ideal of

$$\begin{pmatrix} P & y_1 & y_2 \\ a^2 & x_1 & x_2 \end{pmatrix}.$$

Since $x_1P - y_1a^2, x_2P - y_2a^2$ are irreducible and $x_2P - y_2a^2 \notin (x_1P - y_1a^2)$, we have that $\mathrm{ht}(J) = 2$. Since $x_1P - y_1a^2, x_2P - y_2a^2, x_1y_2 - x_2y_1$ is not a regular sequence we have that $\mathrm{ht}(I) \leqslant 2$. Since the $\mathrm{ht}(I) \geqslant \mathrm{ht}(J)$, the above implies $\mathrm{ht}(I) = 2$. Hence, by Proposition 3.30, we have that the ideal I is Cohen-Macaulay and therefore equidimensional. In particular, all of the associated primes of I are minimal primes of height 2.

Let $W = \mathrm{span}_{\mathbb{K}} \{a, x_1, x_2, y_1, y_2\}$. Since $P \bmod (W)$ is irreducible, and $W \subset S_1$, we have that the generators of W and P form a prime sequence. By Lemma 3.11, in order to prove that I is prime in S, it is enough to prove that I is prime in $\mathcal{A} = \mathbb{K}[W, P]$. Since the generators of W and P form a prime sequence, we know that $\mathcal{A} \simeq \mathbb{K}[W, z]$ for a new variable z. Thus we may replace P by z and let $\mathcal{A} = \mathbb{K}[W, z]$. Then $I = (x_1z - y_1a^2, x_2z - y_2a^2, x_1y_2 - x_2y_1) \subset \mathcal{A}$. Let $\mathcal{B} := \mathbb{K}[W]$. Since the generators of I satisfy Buchberger's Criterion ([Eis95, Theorem 15.8]), they form a Gröbner basis. Hence, the elimination ideal $I_1 = I \cap \mathcal{B} = (x_1y_2 - x_2y_1)$ is prime, as $(x_1y_2 - x_2y_1)$ is irreducible.

Now, we will show that x_1 is a non-zero divisor in $\mathcal{A}/I$ and apply Proposition 3.29 to show that I is prime. Since I is generated in S by elements of degree $\geqslant 2$, we have that x_1 is not zero in $\mathcal{A}/I$. If x_1 is a zero divisor, there exists a minimal prime $I \subset \mathfrak{p}$ such that $x_1 \in \mathfrak{p}$. This implies $(x_1, x_1y_2 - x_2y_1) \subset \mathfrak{p}$. Therefore we have that either $\mathfrak{p} = (x_1, x_2)$ or $\mathfrak{p} = (x_1, y_1)$, since $\mathrm{ht}(\mathfrak{p}) = 2$.

However, note that $I \not\subset (x_1, y_1)$. Indeed, if $x_2P - y_2a^2 \notin (x_1, y_1)$ then we would either have $x_2, y_2 \in (x_1, y_1)$, which would contradict $x_1y_2 - x_2y_1$ being irreducible, or $x_2P \equiv y_2a^2 \not\equiv 0 \bmod (x_1, y_1)$, which would contradict P being irreducible over $S/(W)$.

Similarly, we have that $I \not\subset (x_1, x_2)$. Otherwise we would have $x_1P - a^2y_1 \in (x_1, x_2) \Rightarrow a^2y_1 \in (x_1, x_2) \Rightarrow y_1 \in (x_1, x_2)$ as $a \notin (x_1, x_2)$. Also, $x_2P - a^2y_2 \in (x_1, x_2) \Rightarrow y_2 \in (x_1, x_2)$, which would contradict the irreducibility of $x_1y_2 - x_2y_1$. This completes the proof that x_1 is a non-zero divisor.

Therefore, we can apply Proposition 3.29 with our ideal I, $f = x_1P - y_1a^2$, $y = P$ and $g = x_1$. Since I_1 is prime, we obtain that I is also prime. □

Corollary 3.34. *Let $a, x_1, x_2, y_1, y_2 \in S_1$ and $P \in S_2$ be such that $x_1 \notin \mathrm{span}_{\mathbb{K}}\{a, x_2, y_2\}$ and $P \bmod (a, x_1, x_2, y_1, y_2)$ is irreducible. Suppose the polynomials $x_1P - y_1a^2, x_2P - y_2a^2$ are non-associate irreducible forms. If $G \in S$ is an irreducible form such that $G \in \mathrm{rad}(x_1P - y_1a^2, x_2P - y_2a^2)$ then $\deg(G) \geqslant 3$.*

Proof. We have two cases to analyze, depending on whether $x_1y_2 - x_2y_1$ is irreducible or not.

Case 1: $x_1y_2 - x_2y_1$ is irreducible.

In this case, let $\mathfrak{p} := (x_1P - y_1a^2, x_2P - y_2a^2, x_1y_2 - x_2y_1)$. Since $x_1P - y_1a^2, x_2P - y_2a^2$ are not multiples of each other, and $x_1y_2 - x_2y_1$ is irreducible, we have $\dim \mathrm{span}_{\mathbb{K}}\{x_1, x_2\} = 2$. Therefore, $x_1 \notin \mathrm{span}_{\mathbb{K}}\{a, x_2, y_2\}$ implies that $\dim \mathrm{span}_{\mathbb{K}}\{a, x_2, x_1\} = 3$. Thus, Lemma 3.33 applies and we know that $\mathfrak{p}$ is prime with $\mathrm{ht}(\mathfrak{p}) = 2$. Note that $\mathrm{rad}(x_1P - y_1a^2, x_2P - y_2a^2) \subset (P, a) \cap \mathfrak{p}$. Hence, $G \in (P, a) \cap \mathfrak{p}$, which implies that $\deg(G) \neq 1$, as $\mathfrak{p}$ does not contain any linear form. Now $\deg(G) = 2$ and G irreducible would imply $x_1y_2 - x_2y_1 \in (G)$, but we know that $x_1y_2 - x_2y_1 \notin (P, a)$ as that would contradict the fact that P is irreducible over $S/(a, x_1, x_2)$. Thus, we must have $\deg(G) \geqslant 3$.

Case 2: $x_1y_2 - x_2y_1 = \ell_1\ell_2$ for some $\ell_1, \ell_2 \in S_1$.

In this case, we know that $\ell_1\ell_2 \equiv 0 \bmod (x_1, x_2)$ so we can assume w.l.o.g. that $\ell_1 = \alpha_1x_1 + \alpha_2x_2$. Moreover, we must have $\alpha_1, \alpha_2 \neq 0$. Otherwise, if $\alpha_2 = 0$ we would have $x_2y_1 \in (x_1)$, which implies $y_1 \in (x_1)$, as $\dim \mathrm{span}_{\mathbb{K}}\{x_1, x_2\} = 2$. This is a contradiction as $x_1P - y_1a^2$ is irreducible. The case where $\alpha_1 = 0$ is analogous.

Since $x_1 \notin \mathrm{span}_{\mathbb{K}}\{a, x_2, y_2\}$, we have that the ideal $\mathfrak{p} := (x_2P - y_2a^2, \ell_1)$ is prime. Indeed, note that ℓ_1, x_2, y_2, a, P form a prime sequence. Therefore, we may apply Proposition 3.29, to the ideal $I = (x_2P - y_2a^2, \ell_1)$ with $f = y = \ell_1$. Since the elimination ideal $I_1 = (x_2P - y_2a^2)$ is prime, we conclude that I is prime. Thus, we have $\mathrm{rad}(x_1P - y_1a^2, x_2P - y_2a^2) \subset (P, a) \cap \mathfrak{p}$. Hence, $G \in (P, a) \cap \mathfrak{p}$ which implies that $\deg(G) \neq 1$, otherwise $G \in (a) \cap (\ell_1) = 0$, which is a contradiction. Moreover, $\deg(G) = 2$ and $G \in \mathfrak{p}$ would imply $G \in (\ell_1)$ which contradicts irreducibility of G. Thus, we must have $\deg(G) \geqslant 3$. □

3.5 Varieties of minimal degree

In this section we discuss the classification of homogeneous prime ideals that define varieties of minimal degree in a projective space. The classification of these ideals will be a key ingredient for our structure theorem for ideals generated by two cubics.

Definition 3.35. A homogeneous prime ideal $\mathfrak{p} \subset S$ is degenerate if it contains a linear form.

The following proposition is a classical lower bound for the multiplicity of non-degenerate homogeneous primes (see [EH87, Proposition 0], [HMMS13, Proposition 2.12]).

Proposition 3.36. *Let $\mathfrak{p}$ be a homogeneous prime ideal in a polynomial ring over an algebraically closed field. If $\mathfrak{p}$ is non-degenerate, then $e(S/\mathfrak{p}) \geqslant \mathrm{ht}(\mathfrak{p}) + 1$.*

We will say that a homogeneous prime ideal is a prime ideal of minimal multiplicity if $e(S/\mathfrak{p}) = \mathrm{ht}(\mathfrak{p})+1$. If $\mathfrak{p} \subset S$ is a homogeneous prime ideal of minimal multiplicity, then the projective variety $X = V(\mathfrak{p})$ defined by $\mathfrak{p}$ in $\mathbb{P}^{n-1}$ is a variety of minimal degree, i.e. $\deg(X) = \mathrm{codim}(X) + 1$. Varieties of minimal degree have been classified by del Pezzo-Bertini and later by the works in [Har81, Xam81, EH87].

Theorem 3.37. *(see [EH87, Theorem 1]) Let $X \subset \mathbb{P}^r$ be a projective variety of minimal degree, i.e. $\deg(X) = \mathrm{codim}(X) + 1$. If X is smooth and $\mathrm{codim}(X) > 1$, then $X \subset \mathbb{P}^r$ is either a rational normal scroll or the veronese embedding $\mathbb{P}^2 \subset \mathbb{P}^5$. If X is not smooth, then X is a cone over a smooth variety of minimal degree.*

Suppose $F_1, F_2 \subset S_3$ are two non-associate irreducible cubic forms. In Section 5, we will be interested in the minimal primes of the ideal (F_1, F_2). Suppose $\mathfrak{p}$ is a minimal prime of (F_1, F_2). Then $\mathfrak{p}$ is a homogeneous prime ideal with $\mathrm{ht}(\mathfrak{p}) = 2$. If $\mathfrak{p}$ is non-degenerate and $e(S/\mathfrak{p}) = 3$, then $\mathfrak{p}$ is a homogeneous prime of minimal multiplicity. As a consequence of Theorem 3.37, we obtain a classification of such prime ideals as determinantal prime ideals. The following result is an immediate corollary of Theorem 3.37, we provide a proof for completeness.

Corollary 3.38. *Let $\mathfrak{p} \subset S$ be a non-degenerate homogeneous prime ideal with $\mathrm{ht}(\mathfrak{p}) = 2$ and $e(S/\mathfrak{p}) = 3$. Then $\mathfrak{p}$ is the ideal generated by maximal minors of a matrix M of the form*

$$\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} y_0 & y_2 & y_3 \\ y_1 & y_3 & y_4 \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} y_0 & y_2 & y_4 \\ y_1 & y_3 & y_5 \end{pmatrix},$$

where $y_1, \ldots, y_5 \in S_1$ are linearly independent linear forms.

Proof. Let $X = V(\mathfrak{p}) \subset \mathbb{P}^{n-1}$ be the projetcive variety defined by $\mathfrak{p}$. Since $\mathfrak{p}$ is a non-degenerate prime ideal, by the projective Nullstellensatz, we have $I(X) = \mathfrak{p}$, where $I(X)$ denotes the homogeneous ideal of X. Now X is a variety of minimal degree, as $\deg(X) = e(S/\mathfrak{p})$ and $\mathrm{codim}(X) = \mathrm{ht}(\mathfrak{p})$. Since $\mathrm{codim}(X) = 2$, we note that $X \subset \mathbb{P}^{n-1}$ can not be the veronese embedding $\mathbb{P}^2 \subset P^5$, as the image of the veronese embedding has codimension 3. Also, X can not be a cone over the veronese embedding, as the cone also has codimension 3. Therefore, by Theorem 3.37, we know that $X \subset \mathbb{P}^n$ is either a rational normal scroll, or a cone over a rational normal scroll. Since the cone is cut out by the same equations in a higher dimensional projective space, it is enough to prove the result when X is a smooth rational normal scroll. By [EH87, Page 6], the homogeneous ideal of a rational normal scroll X is given by the maximal minors of a matrix of the from

$$\begin{pmatrix} x_{0,0} & x_{0,1} & \cdots & x_{0,a_0-1} & | & x_{1,0} & \cdots & x_{1,a_1-1} & | & \cdots x_{d,a_d-1} \\ x_{0,1} & x_{0,2} & \cdots & x_{0,a_0} & | & x_{1,1} & \cdots & x_{1,a_1} & | & \cdots x_{d,a_d} \end{pmatrix}$$

where $\{x_{ij}\}$ are linearly independent linear forms and $0 \leqslant a_0 \leqslant a_1 \cdots \leqslant a_d$. Also we have $\sum_i a_i = \deg(X)$ and $\dim(X) = d + 1$. Since X is smooth, we have $a_i > 0$ for all i. Since $\deg(X) = 3$ and $\dim(X) = n - 4$, we see that $d \leqslant 3$ and $(a_0, \cdots, a_d)$ is $(1, 1, 1)$, $(1, 2)$ or (3). Therefore, $\mathfrak{p}$ is given by the maximal minors of a martix of the desired form. $\square$

4 Wide algebras

In this section we describe the main tool which we will use to prove our Sylvester-Gallai theorem – wide Ananyan-Hochster algebras. As mentioned in Section 1, we would like to construct small algebras which behave as polynomial rings and contain many of the polynomials in our configuration. As we saw in Section 3, for an algebra to "behave as a polynomial ring" it is enough to construct an algebra whose generators form a prime sequence. Another property that we would like from our algebras, is that they are robust with regards to certain augmentations, as we will increase such algebras to contain more and more forms from our configuration, and we would like to preserve the structure of the original algebra inside of the augmented one – a concept which we will make precise in Proposition 4.11.

In a recent breakthrough work by Ananyan-Hochster [AH20a], where they positively answer Stillman's conjecture, they show that given an ideal I in a polynomial ring S, one can construct a sub-algebra $\mathcal{A} \subset S$ such that $I \subset \mathcal{A}$ and the number of generators of $\mathcal{A}$ is uniformly bounded by a function of the degrees and number of the generators of the given ideal I.

Building on the work in [AH20a], we define the notion of a wide Ananyan-Hochster algebra. We show that these wide AH algebras have strong algebraic-geometric properties and are particularly suitable for applications to Sylvester-Gallai problems as described above. In order to construct such algebras, we need to define a notion of rank for a form in S. The notion of rank that we describe below is called strength and it was introduced in [AH20a, AH20b]. This notion can be seen as a symmetric version of the notion of slice rank of a tensor introduced by Tao.

Definition 4.1 (Collapse)**.** Given non-zero a form $F \in S_d$, we say that F has a k-*collapse* if there exist k forms $G_1, \dots, G_k$ such that $\deg(G_i) < d$ and $F \in (G_1, \dots, G_k)$.

Definition 4.2 (Strength)**.** Given a non-zero form $F \in S_d$, the *strength* of F, denoted by $s(F)$, is the least positive integer such that F has a $(s(F)+1)$-collapse but it has no $s(F)$-collapse. We say that $s(F) \geqslant t$ whenever F does not have a $t+1$-collapse.

Remark 4.3. *Note that by the definitions above, a linear form does not have a* k*-collapse for any* $k \in \mathbb{N}$*. Thus, we say that for any* $x \in S_1$*,* $s(x) = \infty$*.*

We will make the convention that $s(0) = -1$*.*

Definition 4.4 (Minimum collapse)**.** Given a non-zero form $F \in S_d$ and $s \in \mathbb{N}^*$ such that $s(F) = s-1$, a *minimum collapse* of F is any identity of the form $F = G_1H_1 + \cdots + G_sH_s$, where G_i, H_i are forms of degree $< d$.

Note that the definition of minimum collapse given above generalizes the definition of rank from [PS20a] (see Definition 2.1). We will see that cubics or quadratic forms of high enough strength can be used to construct subalgebras which are isomorphic to polynomial rings.

It is also useful to define the min and max strength of a pencil of forms of the same degree.

Definition 4.5 (Min and max strength)**.** Given a set of forms $F_1, \dots, F_m \in S_d$, we define $s_{\min}(F_1, \dots, F_m)$ as the *minimum* strength of a *non-zero* form in $\text{span}_{\mathbb{K}}\{F_1, \dots, F_m\}$ and $s_{\max}(F_1, \dots, F_m)$ as the *maximum* strength of a form in $\text{span}_{\mathbb{K}}\{F_1, \dots, F_m\}$.

In particular, given any vector space $V \subset S_d$, we can define $s_{\min}(V)$ ($s_{\max}(V)$) as the minimum (maximum) strength of any non-zero form in V.

For the rest of this section, we have that $V_d \subset S_d$ is a vector space of forms of degree d from S. The next theorem, proved in [AH20b, Theorem 1.10], states that a vector space of quadratics of high enough strength is always generated by an R_η-sequence.

Theorem 4.6 (Ananyan-Hochster quadratic algebras)**.** *Let V be a vector space of quadratic forms in S_2 of dimension n over $\mathbb{K}$. If $s_{\min}(V) \geqslant n - 1 + \lceil \eta/2 \rceil$, every sequence of linearly independent elements of V is an R_η-sequence.*

By Remark 3.4, we know that R_η-sequences in S are prime sequences whenever $\eta \geqslant 1$, so the above theorem is a way of obtaining prime sequences, and therefore algebras such that the tools developed in the previous section apply. As noted in [AH20a, Discussion 1.3], if $\eta \geqslant 3$ then we have the extra property that all quotients of $\mathbb{K}[V]$ by ideals generated by homogeneous linear combinations of the generators are UFDs.

An immediate corollary of the theorem above is the following result, which can be found in [Har77, Chapter II, Section 6, Exercise 6.5] or (a more basic version) [Eis95, Exercise 18.12].

Proposition 4.7 (High rank quadratic yields UFD)**.** *If $Q \in S_2$ such that $s(Q) \geqslant 2$ then $S/(Q)$ is a UFD.*

The theorem above motivates our definition of wide algebras. As we will see, wide Ananyan-Hochster algebras will also be robust to certain augmentations, and thus have all the algebro-geometric properties we need for the Sylvester-Gallai theorem.

Definition 4.8 (Wide Ananyan-Hochster Algebras)**.** Let $w, t \in \mathbb{N}_{>0}$. A graded vector space $V = V_1 + V_2 \subset S$ is called a (w, t)-wide AH vector space if $s_{\min}(V_2) \geqslant t(\dim(V) + w)$. The algebra generated by V will be denoted $\mathcal{A}_V$ or $\mathbb{K}[V]$. An algebra generated by a (w, t)-wide AH vector space is called a (w, t)-wide AH algebra.

We note the following simple result for convenience.

Proposition 4.9. *Let $V = V_1 + V_2$ be a (w, t)-wide AH-vector space in S.*

1. *Let $\ell \in S_1$. Then $\ell \in \mathbb{K}[V]$ if and only if $\ell \in V_1$.*
2. *Let $Q \in S_2$ such that $s(Q) < (t-1)\dim(V_1) + t(\dim(V_2) + w)$. Then $Q \in \mathbb{K}[V]$ if and only if $F \in \mathbb{K}[V_1]$. Also, $Q \in (V)$ if and only if $Q \in (V_1)$.*
3. *Let $C \in S_3$. If $C \in \mathbb{K}[V]$ then $C \in (V_1)$.*

Proof. Note that $\mathbb{K}[V] \cap S_1 = V_1$, hence the first statement holds. Let $Q \in \mathbb{K}[V] \cap S_2$ as in the second statement above. We have $Q = P_1 + P_2$ where $P_i \in S_2$ with $P_1 \in \mathbb{K}[V_1]$ and $P_2 \in V_2$. Since V is (w, t)-strong, we know that if $P_2 \neq 0$, then $s(P_2) \geqslant t(\dim(V) + w)$. Note that $s(P_1) \leqslant \dim(V_1)$. Therefore we have $s(P_2) \leqslant s(P_1) + s(Q) < t(\dim(V) + w)$, which is a contradiction. Therefore $P_2 = 0$ and hence $Q \in \mathbb{K}[V_1]$. Similarly, suppose $Q \in (V) \cap S_2$. Then we have $Q = P_1 + P_2$ where $P_i \in S_2$ with $P_1 \in (V_1)$ and $P_2 \in V_2$. If $P_2 \neq 0$, we have a contradiction as above. Let $C \in \mathbb{K}[V] \cap S_3$. Then we may write $C = x_1 Q_1 + \cdots + x_r Q_r$ where $x_i \in V_1$ and hence $C \in (V_1)$. □

As a corollary of Theorem 4.6, we have that the algebra generated by an (w, t)-wide AH vector space satisfies Serre's R_η property if $w \geqslant \lceil \eta/2 \rceil - 1$.

Proposition 4.10. *Let $V = V_1 + V_2$ be a (w, t)-wide AH-vector space where $w \geqslant \lceil \eta/2 \rceil - 1$. Let $y_1, \cdots, y_r$ and $Q_1, \cdots, Q_n$ be bases of the vector spaces V_1 and V_2 respectively. Then $y_1, \cdots, y_r, Q_1 \cdots, Q_n$ is an R_η-sequence in S.*

Proof. Note that any subset of $y_1, \cdots, y_r$ is an R_η-sequence as $y_i \in V_1$ for all $i \in [r]$. Consider $S' = S/(y_1, \cdots, y_r)$. Note that S' is isomorphic to a polynomial ring. For any quadratic form $Q \in S$, let $\overline{Q}$ denote the image of Q in S'. Since V is (w, t)-wide and $w \geq \lceil \eta/2 \rceil - 1$, we must have $s_{\min}(\overline{Q}_1, \cdots, \overline{Q}_n) \geq t(n - 1 + \lceil \eta/2 \rceil)$. Therefore, by Theorem 4.6, we have that $\overline{Q}_1, \cdots, \overline{Q}_n$ is an R_η-sequence in S'. Hence $y_1, \cdots, y_r, Q_1 \cdots, Q_n$ is an R_η-sequence in S. □

The next proposition tells us that, for any $w, t \in \mathbb{N}$, if we start with a graded vector space U which is not (w, t)-wide, we can construct a new vector space V (of larger total dimension) from it which is (w, t)-wide, and the algebra generated by the latter contains the former. Also, given a linear subspace $H \subset U_2$ such that all the quadratics in H are "sufficiently" strong, we show that we can construct the (w, t)-wide vector space V in such a way that $H \subset V_2$.

Proposition 4.11 (Constructing wide AH algebras)**.** *Given a graded vector space $U = U_1 + U_2$ in S, and parameters $w, t \in \mathbb{N}_{>0}$, there exists an (w, t)-wide AH vector space $V = V_1 + V_2$ such that*

1. $U_1 \subset V_1$ *and* $V_2 \subset U_2$
2. $\mathbb{K}[U] \subset \mathbb{K}[V]$
3. $\dim(V_1) \leq (2t + 1)^{(\dim(U_2)+1)} \cdot (\dim(U) + w)$

Furthermore, suppose $H \subset U_2$ is a sub-vector space of codimension r, such that

$$s_{\min}(H) > (2t + 1)^{(r+2)} \cdot (\dim(U) + w).$$

Then there exists a (w, t)-wide vector space V satisfying properties $(1) - (3)$ above such that $H \subset V_2$.

Proof. Let $U^{(0)} = U$. So long as $U^{(k)}$ is not (w, t)-wide, there exists a non-zero quadratic $Q \in U_2^{(k)}$ such that $s(Q) < t(\dim(U^{(k)}) + w)$. Let $\mathcal{B}^{(k)}$ be a basis of $U_2^{(k)}$ that contains Q. We can define $U^{(k+1)}$ as follows: $U_1^{(k+1)} = U_1^{(k)} + \mathrm{Lin}(Q)$, and $U_2^{(k+1)} = \mathrm{span}_{\mathbb{K}} \{\mathcal{B}^{(k)} \setminus \{Q\}\})$. At each step we have that

$$\dim(U_1^{(k+1)}) \leq \dim(U_1^{(k)}) + 2t(\dim(U^{(k)}) + w) \quad \text{and} \quad \dim(U_2^{(k+1)}) \leq \dim(U_2^{(k)}) - 1.$$

Thus, this process has to stop in $\leq \dim(U_2)$ iterations. Let V be the last vector space from this procedure. Then items 1 and 2 follow from the construction. The inequalities above imply that

$$\dim(U_1^{(k)}) \leq (2t + 1)^{(k+1)}(\dim(U) + w).$$

Since the iterative process stops after at most $\dim(U_2)$ steps, we have the desired inequality 3 above.

Now suppose $H \subset U_2$ is a linear subspace of dimension d. Note that $r = \dim(U_2) - d$, and let $\{P_1, \cdots, P_d\}$ be a basis of H. Suppose that that $s_{\min}(H) > (2t + 1)^{(r+2)} \cdot (\dim(U) + w)$. We will show that by induction, we can run the iterative process above in such a way that $H \subset U_2^{(k)}$ for all k. Note that $H \subset U_2^{(0)} = U_2$. Suppose at the k-th step of the iterative process above we have $H \subset U_2^{(k)}$. If $U^{(k)}$ is (w, t)-strong, then $V = U^{(k)}$ and we are done. So, let us assume that $U^{(k)}$ is not (w, t)-strong.

If $k \geqslant r$, then $\dim(U_2^{(k)}) \leqslant \dim(U_2) - k \leqslant d = \dim(H)$. In this case, we must have $k = r$ and $H = U_2^{(r)}$. Now, using the inequalities above we obtain,

$$s_{\min}(H) \geqslant (2t+1)^{(r+2)}(\dim(U) + w) > t(\dim(U^{(r)}) + w).$$

Hence $U^{(k)} = U^{(r)}$ is (w,t)-strong, which is a contradiction.

Thus, we may assume that $k < r$. Suppose $Q \in U_2^{(k)}$ is a non-zero quadratic form such that $s(Q) < t(\dim(U^{(k)}) + w)$. In particular, we have that $Q \notin H$ since $s_{\min}(H) > s(Q)$. Then, at this step of the iterative process, we may choose the basis $\mathcal{B}^{(k)}$ of $U_2^{(k)}$ that contains $Q, P_1, \cdots, P_d$. Therefore, we have $P_i \in U_2^{(k+1)}$ for all $i \in [d]$, and hence $H \subset U_2^{(k+1)}$. Thus we conclude by induction. □

Remark 4.12. *Note that the (w,t)-wide vector space constructed above is not unique, as it depends on the choice of basis at each step of the iterative process. Furthermore, given a linear subspace $H \subset U_2$ as above, the vector space V constructed in Proposition 4.11 depends on H.*

Corollary 4.13. *Let $w, t, s \in \mathbb{N}_{>0}$ and $G_1, \cdots, G_m \in S_3$ be cubic forms such that $s(G_i) \leqslant s-1$ for all $i \in [m]$. Suppose $\dim \operatorname{span}_{\mathbb{K}} \{G_1, \cdots, G_m\} = d$. Then there exists a (w,t)-wide vector space $V = V_1 + V_2$ such that*

1. $G_i \in \mathbb{K}[V]$ *for all* $i \in [m]$,
2. $\dim(V_1) \leqslant (2t+1)^{(ds+1)}(2ds + w)$, *and* $\dim(V_2) \leqslant ds$.

Proof. Without loss of generality let us assume that $G_1, \cdots, G_d$ is a basis of $\operatorname{span}_{\mathbb{K}} \{G_1, \cdots, G_m\}$. Suppose $s(G_i) = s_i$. Let $G_i = y_{i1}Q_{i1} + \cdots + y_{is_i}Q_{is_i}$ be a minimum collapse of G_i, where $y_{ij} \in S_1$ and $Q_{ij} \in S_2$. Let $U_1 = \operatorname{span}_{\mathbb{K}} \{y_{ij}\}$ and $U_2 = \operatorname{span}_{\mathbb{K}} \{Q_{ij}\}$. Consider the graded vector space $U = U_1 + U_2$. Note that $\dim(U_1) \leqslant ds$ and $\dim(U_2) \leqslant ds$. We have $G_i \in \mathbb{K}[U]$ for all $i \in [m]$. By Proposition 4.11, there is a (w,t)-wide vector space $V = V_1 + V_2$ with the desired properties. □

As a corollary, if three cubic forms of low strength depend on a sufficiently strong quadratic, then one can construct a wide algebra which contains this strong quadratic as a "variable." The following corollary formalizes this.

Corollary 4.14. *Let $w, t \in \mathbb{N}$, $Q \in S_2$ be such that $s(Q) > (2t+1)^8 \cdot (12 + w)$, $x, y \in S_1$ and $C_1, C_2, C_3 \in S_3 \cap (Q, x, y)$. There is a (w,t)-wide vector space $V = V_1 + V_2$ such that $C_i \in \mathbb{K}[V]$ and $Q \in V_2$.*

Proof. As $C_i \in (Q, x, y)$, we can write $C_i = z_iQ + xA_i + yB_i$, for some $z_i, \in S_1$ and $A_i, B_i \in S_2$. Let $U = U_1 + U_2$ where $U_1 = \operatorname{span}_{\mathbb{K}} \{x, y, z_1, z_2, z_3\}$ and $U_2 = \operatorname{span}_{\mathbb{K}} \{Q, A_1, A_2, A_3, B_1, B_2, B_3\}$. We apply Proposition 4.11 to the vector space U with $P = Q$ to obtain a (w,t)-wide vector space V with the desired properties. □

One desirable property of a strong subalgebra is that it is robust when we try to enlarge it by adding a couple of weak polynomials to it (and applying Corollary 4.13). The following proposition shows us that a wide algebra is robust under such changes, in the sense that it will remain wide (perhaps with a small deterioration in the width parameter) after we add a weak cubic to it.

Proposition 4.15 (Robustness of Wide Algebras). *Let $w, t, s \in \mathbb{N}_{>0}$ such that $w > 2s$ and $t > 3^{s+2}$. Let $U = U_1 + U_2$ be a (w, t)-wide AH-vector space. Let $C \in S_{\leqslant 3}$ be a form such that $s(C) = s - 1$ or $C \in S_1$. For any pair of integers $w_1, t_1 \in \mathbb{N}_{>0}$ such that $w_1 < (w - 2s)$ and $(2t_1 + 1)^{(s+2)} < t$, there exists a (w_1, t_1)-wide vector space $V = V_1 + V_2$ where $V_i \subset S_i$ and*

1. *$U_i \subset V_i$ for $i \in \{1, 2\}$, and hence $\mathbb{K}[U] \subset \mathbb{K}[V]$.*
2. *$C \in \mathbb{K}[V]$.*
3. *$\dim(V) \leqslant A(w, t, \dim(W))$ for some function $A : \mathbb{N}^3 \to \mathbb{N}$.*

Proof. If $C \in S_1$, then we define $V_1 = U_1 + \mathrm{span}_{\mathbb{K}}\{C\}$ and $V_2 = U_2$. Note that $V = V_1 + V_2$ is $(w - 1, t)$ wide. Hence V is (w_1, t_1)-wide and it satisfies the desired properties above.

If $C \in S_2$, let $C = z_1 y_1 + \cdots + z_s y_s$ be a minimum collapse of C. Let $V_1 = U_1 + \mathrm{Lin}(Q)$ and $V_2 = U_2$. Then $\dim \mathrm{Lin}(Q) \leqslant 2s$ and thus V is $(w - 2s, t)$-wide, which implies V is a (w_1, t_1)-wide vector space with the additional properties above.

Now suppose $C \in S_3$. Let $C = z_1 Q_1 + \cdots + z_s Q_s$ be a minimum collapse of C where $z_i \in S_1$ and $Q_i \in S_2$. Consider the vector spaces $U_1' = U_1 + \mathrm{span}_{\mathbb{K}}\{z_1, \cdots, z_s\}$ and $U_2' = U_2 + \mathrm{span}_{\mathbb{K}}\{Q_1, \cdots, Q_s\}$. Let $U' = U_1' + U_2'$. Note that $C \in \mathbb{K}[U']$ and $\dim(U') \leqslant \dim(U) + 2s$. Now $U_2 \subset U_2'$ is a subspace of codimension at most s. Since U is (w, t)-wide, we know that $s_{\min}(U_2) \geqslant t(\dim(U) + w)$. Thus,

$$s_{\min}(U_2) \geqslant t(\dim(U) + w) > (2t_1 + 1)^{(s+2)}(\dim(U') + w_1).$$

Therefore by applying Proposition 4.11 to the vector space $U' = U_1' + U_2'$ with $H = U_2$, we obtain a (w_1, t_1)-wide vector space $V = V_1 + V_2$ such that $U_2 \subset V_2$. Also, we have $U_1 \subset U_1' \subset V_1$ and $C \in \mathbb{K}[U'] \subset \mathbb{K}[V]$.

Note that in all the three cases above, we have $\dim(V)$ bounded above by some function of $w, t, \dim(W)$, which we call $A(w, t, \dim(W))$. ☐

The next lemma shows that if a cubic form has a minimum collapse in a wide-algebra, then this collapse can be taken to be from elements of the wide-algebra itself.

Lemma 4.16 (Minimum Collapse in Wide Algebra). *Let $\mathcal{A} = \mathbb{K}[V]$ be an algebra generated by a (w, t)-wide AH-vector space $V = V_1 + V_2$ in the polynomial ring S, where $w \geqslant 3, t \geqslant 1$. Let $1 \leqslant s < \frac{w}{2}$. Suppose $C \in S_3$ is a non-zero form such that $s(C) = s - 1$ and $C \in \mathcal{A}$. Then, there exist $z_1, \ldots, z_s \in \mathcal{A} \cap S_1$ and $P_1, \ldots, P_s \in \mathcal{A} \cap S_2$ such that*

$$C = z_1 P_1 + \cdots + z_s P_s.$$

Moreover, if $C = \ell_1 Q_1 + \cdots + \ell_s Q_s$ is any s-collapse of C then

1. *$\ell_i \in \mathcal{A}$ for all $i \in [s]$,*
2. *for any $i \in [s]$, we have $Q_i \in \mathcal{A}$ or there exists a quadratic form $R_i \in \mathcal{A}$ such that $s(Q_i - R_i) \leqslant s$.*

Proof. The proof is by induction on the strength of C. Note that, since $w \geqslant 3$, we know that the generators of the algebra $\mathcal{A}$ form a R_1-sequence (and hence a prime sequence) by Proposition 4.10. The base case, when $s = 1$, is the case where C is a reducible polynomial. If $C = \ell_1 Q_1$ then we have $\ell_1, Q_1 \in \mathcal{A}$ by Lemma 3.15. Note that in this case any 1-collapse is of this form. Assume that the lemma is true for forms of strength $\leqslant s - 2$ for some $1 < s < \frac{w}{2}$.

Suppose that $s(C) = s - 1$. Then C has an s-collapse, and we can write

$$C = \ell_1 Q_1 + \cdots + \ell_s Q_s$$

where $\ell_i \in S_1$ and $Q_i \in S_2$. If $\mathrm{span}_{\mathbb{K}}\{\ell_1, \ldots, \ell_s\} \cap V_1 \neq \{0\}$, we can assume w.l.o.g. that $\ell_1 \in V_1$. Then $\bar{C} \in \mathcal{A}/(\ell_1)$ is a non-zero cubic form in $S/(\ell_1)$ and $s(\overline{C}) = s - 2$. Note that $\mathcal{A}/(\ell_1)$ is a $(w-1, t)$-wide AH-algebra and $S/(\ell_1)$ is a polynomial ring. Therefore by induction we must have

$$\bar{C} = z_2 P_2 + \ldots + z_s P_s$$

where $z_i \in (\mathcal{A}/(\ell_1))_1$ and $P_i \in (\mathcal{A}/(\ell_1))_2$. This yields the following minimum collapse of C:

$$C = \ell_1 P + z_2 P_2 + \ldots + z_s P_s$$

for some $P \in S_2$. In particular, we have that $\ell_1 P = C - z_2 P_2 + \ldots + z_s P_s$ is a non-zero form in the algebra $\mathcal{A}$ and has strength 0. Thus, by Lemma 3.15 we have that $P \in \mathcal{A}_2$.

Thus, we are left with the case where $\mathrm{span}_{\mathbb{K}}\{\ell_1, \ldots, \ell_s\} \cap V_1 = \{0\}$. We will show that this leads to a contradiction. Let $r = \dim(V_1)$. Since $s(C) = s - 1$ it must be the case that $\dim(\mathrm{span}_{\mathbb{K}}\{\ell_1, \ldots, \ell_s\}) = s$. Thus we can extend $\{\ell_1, \ldots, \ell_s\}$ to a basis of S_1 given by $B = \{\ell_1, \ldots, \ell_s, x_1, \ldots, x_r, y_1, \ldots, y_t\}$ where $V_1 = \mathrm{span}_{\mathbb{K}}\{x_1, \ldots, x_r\}$. Note that B generates the polynomial ring S as a $\mathbb{K}$-algebra, i.e. $S = \mathbb{K}[B]$.

Let $G_1, \ldots, G_m$ be a basis for V_2. Since $C \in \mathcal{A}$, we have

$$C = \sum_{i=1}^{r} x_i F_i$$

where each

$$F_i = \sum_{j=i}^{r} x_j a_{ij} + H_i$$

where $a_{i,j} \in V_1/(x_1, \ldots, x_{j-1})$ and $H_i \in V_2$.

Since $s < \frac{w}{2}$ and V is (w, t)-wide, for any $G \in V_2$ we have $s(G) \geqslant t(r + s + \dim(V_2) + \frac{w}{2})$. Let $S' = S/(\ell_1, \cdots, \ell_s)$. Note that the image $\overline{V} = \overline{V}_1 + \overline{V}_2$ in the polynomial ring S' is an $(\frac{w}{2}, t)$-wide vector space, since for any element $G \in V_2$ we have $s(\overline{G}) \geqslant t(r + \dim(V_2) + \frac{w}{2})$. Now we have

$$\overline{C} = x_1 \overline{F}_1 + \cdots + x_r \overline{F}_r = 0$$

in S'. Since $x_1, \cdots, x_r$ is a regular sequence in S', we must have that $\overline{F}_i \in (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_r)$ for all i. Therefore we have $\overline{H}_i \in (x_1, \cdots, x_r)$ and hence $s(H_i) \leqslant r + s$ for all i. Since V is (w, t)-wide and $H_i \in V_2$, we must have $H_i = 0$ for all i. Thus we have $C \in \mathbb{K}[V_1]$. Note that $\mathbb{K}[V_1]$ is mapped isomorphically to its image in S' since $\mathrm{span}_{\mathbb{K}}\{\ell_1, \cdots, \ell_s\} \cap V_1 = (0)$. Therefore $\overline{C} = 0$ in S' implies that $C = 0$ in S, which is a contradiction. This proves the existence of the collapse in the algebra.

Now we prove that properties (1) and (2) hold for any minimum collapse $C = \ell_1 Q_1 + \cdots + \ell_s Q_s$. Recall that $\overline{C} \in \mathcal{A}/(\ell_1)$ is non-zero and $s(\overline{C}) = s - 2$. Hence we have a minimum collapse given by $\overline{C} = \overline{\ell_2 Q_2} + \cdots + \overline{\ell_s Q_s}$ in $S/(\ell_1)$. Moreover, by induction, we have $\overline{\ell_i} \in \mathcal{A}/(\ell_1)$ for $i = 2, \cdots, s$. Since $\ell_1 \in \mathcal{A}$, we conclude that $\ell_i \in \mathcal{A}$ for all i. Note that by induction, for all $i = 2, \cdots, s$, we have that $\overline{Q}_i \in \mathcal{A}/(\ell_1)$ or there exists $\overline{R}_i \in \mathcal{A}/(\ell_1)$ such that $s(\overline{Q}_i - \overline{R}_i) \leqslant s - 1$. Suppose for some i, we have $\overline{Q}_i \in \mathcal{A}/(\ell_1)$. Then there exists $R_i \in \mathcal{A}$ such that $Q_i - R_i = \ell_1 z_i$ for some linear form z_i. Hence $Q_i \in \mathcal{A}$ or $s(Q_i - R_i) = 0 \leqslant s$. Suppose for some i we have that $\overline{Q}_i \notin \mathcal{A}/(\ell_1)$ and there exists $\overline{R}_i \in \mathcal{A}/(\ell_1)$ such that $s(\overline{Q}_i - \overline{R}_i) \leqslant s - 1$. Let $R_i \in \mathcal{A} \cap S_2$ be a quadratic form such that the image of R_i under the quotient homomorphism $\mathcal{A} \to \mathcal{A}/(\ell_1)$ is $\overline{R}_i$. Then we have $s(Q_i - R_i) \leqslant s - 1 + 1 = s$. □

The next corollary shows that if a cubic form C has a strong enough quadratic in a minimum collapse, then C cannot be contained in a small wide algebra generated by linear forms.

Corollary 4.17. *Let $s \in \mathbb{N}_{>0}$ and $C \in S_3$ be such that $s(C) = s - 1$. Further, assume that C has a minimum collapse $C = x_1 Q_1 + \cdots + x_s Q_s$ where $s(Q_1) \geqslant r + s$. If $W \subset S_1$ is a vector space such that $C \in \mathbb{K}[W]$, then $\dim W > r$.*

Proof. Suppose $W \subset S_1$ is such that $C \in \mathbb{K}[W]$. Since W is (w, t)-wide for any $w, t \in \mathbb{N}_{>0}$, Lemma 4.16 applies and by item (2) of that lemma, there exists $R \in \mathbb{K}[W]$ such that $s(Q_1 - R) \leqslant s$. Since $s(R) < \dim W$, as $R \in \mathbb{K}[W]$, we have

$$r + s \leqslant s(Q_1) \leqslant s(Q_1 - R) + s(R) < s + \dim W,$$

which concludes our proof. □

5 Structure theorem and minimal primes

In this section we prove the main structural results of this paper. Our main result in this section is a structure theorem for non-radical ideals generated by two irreducible cubic forms. With this result at hand, we proceed to analyze certain important minimal primes which can appear in the cycle decomposition of such non-radical ideals. The extra structure of these minimal primes will give us crucial structure in the proof of the Sylvester-Gallai theorem.

5.1 Structure of non-radical ideals generated by two irreducible cubics

The following theorem is the main result about the structure of ideals generated by two cubic forms. The structure of ideals generated by two quadratic forms was considered in [HP94, CTSSD87], and later in the works of [Shp20, Theorem 4.1], [PS20a, Theorem 3.1], [GOS22, Proposition 1.4]. We generalize the structure theorem to ideals generated by two cubic forms and classify when such an ideal is not a radical ideal.

Theorem 1.5. *Let F_1, F_2 be two non-associate irreducible cubic forms in the polynomial ring $\mathbb{K}[x_1, \cdots, x_n]$ over an algebraically closed field $\mathbb{K}$. Then at least one of the following holds:*

1. *The ideal (F_1, F_2) is radical.*

2. *There exists a linear minimal prime of (F_1, F_2), i.e. there exist two linearly independent linear forms x, y such that $(F_1, F_2) \subset (x, y)$.*

3. *There exists a quadratic minimal prime of (F_1, F_2), i.e. there exists a prime ideal (Q, ℓ) where Q is a quadratic form, ℓ is a linear form and $(F_1, F_2) \subset (Q, \ell)$.*

4. *There exist linear forms x, y such that $xy^2 \in \mathrm{span}_{\mathbb{K}}\{F_1, F_2\}$.*

5. *There exists a minimal prime $\mathfrak{p}$ of (F_1, F_2) such that $\mathfrak{p}$ is the homogeneous prime ideal a variety of minimal degree. In particular, $\mathfrak{p} = (Q_1, Q_2, Q_3)$ where Q_i are the quadratic forms given by the maximal minors of a matrix M of the form*

$$\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} y_0 & y_2 & y_3 \\ y_1 & y_3 & y_4 \end{pmatrix} \quad \textit{or} \quad \begin{pmatrix} y_0 & y_2 & y_4 \\ y_1 & y_3 & y_5 \end{pmatrix},$$

where $y_1, \dots, y_5$ are linearly independent linear forms.

Proof. Let $I = (F_1, F_2)$ and $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$ be an irredundant primary decomposition of I in $S = \mathbb{K}[x_1 \cdots, x_n]$. Let $\mathfrak{p}_i = \mathrm{rad}(\mathfrak{q}_i)$. Note that F_1, F_2 is a regular sequence, hence all the primes $\mathfrak{p}_i$ are minimal primes of I and $\mathrm{ht}(\mathfrak{p}_i) = 2$ for all i. By Proposition 3.7, we have $\sum_i m(\mathfrak{p}_i)e(S/\mathfrak{p}_i) = e(S/I) = 9$.

Suppose there exists a minimal prime $\mathfrak{p}_i$ such that $e(S/\mathfrak{p}_i) \leqslant 2$. Then, by Proposition 3.36, the ideal $\mathfrak{p}_i$ must be degenerate. Let $x \in \mathfrak{p}_i$ be a linear form. Consider the image $\overline{\mathfrak{p}_i}$ in the quotient ring $S/(x)$. Then we have $\mathrm{ht}(\overline{\mathfrak{p}_i}) = 1$. Hence, by Krull's principal ideal theorem, there exists an irreducible homogeneous polynomial F such that $\overline{\mathfrak{p}_i} = (F)$. Hence $\mathfrak{p}_i = (x, F)$ for some irreducible form F. Now, $\deg(F) = e(S/\mathfrak{p}_i) \leqslant 2$. Therefore, I has a linear or quadratic minimal prime.

Therefore we may assume that $e(S/\mathfrak{p}_i) \geqslant 3$ for all i. Suppose I is not radical. Then, by Remark 3.6, we must have that $m(\mathfrak{p}_i) > 1$ for some i. Since $\sum_i m(\mathfrak{p}_i)e(S/\mathfrak{p}_i) = 9$, we see that we must have $e(S/\mathfrak{p}_i) = 3$ for all i. Furthermore, we see that either there is only one minimal prime $\mathfrak{p}_1$ with $m(\mathfrak{p}_1) = 3$ or there are exactly two minimal primes of I given by $\mathfrak{p}_1, \mathfrak{p}_2$ such that $m(\mathfrak{p}_1) = 1$ and $m(\mathfrak{p}_2) = 2$.

If there exists a minimal prime $\mathfrak{p}_i$ which is non-degenerate, then it is a homogeneous prime of minimal multiplicity and by Corollary 3.38, we have that $\mathfrak{p}_i$ is of the form as described in 5 above. Therefore we may assume that all the minimal primes are degenerate. If there is only one minimal prime $\mathfrak{p}_1$ with $m(\mathfrak{p}_1) = 3$. As $\mathfrak{p}_1$ is degenerate there exists a linear form $x \in \mathfrak{p}_1$ and by Lemma 3.8, we have $x^3 \in (F_1, F_2)$. In the other case we have two degenerate minimal primes $\mathfrak{p}_1, \mathfrak{p}_2$ and we have linear forms $x \in \mathfrak{p}_1$ and $y \in \mathfrak{p}_2$. Then by Lemma 3.8, we have $xy^2 \in (F_1, F_2)$. As F_1, F_2 are cubics, we conclude that $xy^2 \in \mathrm{span}_{\mathbb{K}}\{F_1, F_2\}$. □

Corollary 5.1. *Let F_1, F_2 be two non-associate irreducible cubic forms in the polynomial ring $\mathbb{K}[x_1, \cdots, x_n]$. Suppose the ideal (F_1, F_2) does not satisfy the conditions $1, 2, 4, 5$ in Theorem 1.5. Then there exists a quadratic minimal prime $\mathfrak{p} = (Q, \ell)$ of I such that $m(\mathfrak{p}) \geqslant 2$.*

Proof. Let $\mathfrak{p}_1, \cdots, \mathfrak{p}_m$ be the minimal primes of I. We have $\sum_i m(\mathfrak{p}_i)e(S/\mathfrak{p}_i) = e(S/I) = 9$. By Theorem 1.5, there exists at least one quadratic minimal prime, say $\mathfrak{p}_1 = (Q, \ell)$ with $e(S/\mathfrak{p}_1) = 2$. If $m(\mathfrak{p}_1) \geqslant 2$ then we are done. So we may assume that $m(\mathfrak{p}_1) = 1$. Therefore $\sum_{i \geqslant 2} m(\mathfrak{p}_i)e(S/\mathfrak{p}_i) = 7$. Since I does not have any linear minimal primes, we have $e(S/\mathfrak{p}_i) \geqslant 2$ for all i. Since I is not radical we must have $m(\mathfrak{p}_i) \geqslant 2$ for some $\mathfrak{p}_i$. Therefore, the above equation implies that there must exist a minimal prime $\mathfrak{p}_i$ such that $e(S/\mathfrak{p}_i) = 2$ and $m(\mathfrak{p}_i) \geqslant 2$. □

5.2 Minimal primes defining varieties of minimal degree

Let M be matrix of the form

$$\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} y_0 & y_2 & y_3 \\ y_1 & y_3 & y_4 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} y_0 & y_2 & y_4 \\ y_1 & y_3 & y_5 \end{pmatrix},$$

where $y_1, \ldots, y_5 \in S_1$ are linearly independent linear forms. Let $\mathfrak{p}$ be the prime ideal defined by the maximal minors of M. If F is a cubic form contained in $\mathfrak{p}$, then F has a very simple structure that we note below.

Corollary 5.2. *Suppose F is an irreducible cubic form contained in a prime ideal $\mathfrak{p}$ given by the maximal minors of a matrix M as above. Then there exists a vector space of linear forms $V \subset S_1$ such that $F \in \mathbb{K}[V]$ and $\dim(V) \leqslant 9$. Furthermore, if $s(F) = 2$ and W is a (w,t)-wide vector space with $w \geqslant 6$ and $F \in \mathbb{K}[W]$, then we have $y_i \in W$ for all the variables y_i that appear in M.*

Proof. Let Q_1, Q_2, Q_3 denote the maximal minors of M. Since $\deg(F) = 3$, there exist at most 3 linear forms ℓ_1, ℓ_2, ℓ_3 such that $F = \ell_1 Q_1 + \ell_2 Q_2 + \ell_3 Q_3$. Therefore we may take $V = \operatorname{span}_{\mathbb{K}} \{\ell_i, y_j | 1 \leqslant i \leqslant 3, 0 \leqslant j \leqslant 5\}$.

Now, if $s(F) = 2$ and W is a (w,t)-wide vector space with $w > 6$ such that $F \in \mathbb{K}[W]$, Lemma 4.16 implies that if $F \in (\ell_1, \ell_2, \ell_3)$ where $\ell_i \in S_1$, then we must have $\ell_i \in W$. Since for any row of the matrix M, which is of the form (y_i, y_j, y_k), we have $F \in (y_i, y_j, y_k)$, the above implies $y_i, y_j, y_k \in W$. $\square$

Corollary 5.3. *Let F be an irreducible cubic form such that $F = zQ + z_1 Q_1 + \cdots z_r Q_r$ where $s(Q) > 9 + r$ and $z \notin \operatorname{span}_{\mathbb{K}} \{z_1, \cdots, z_r\}$. Then F can not be contained in a prime ideal $\mathfrak{p}$ given by the maximal minors of a matrix M as above.*

Proof. If F is contained in a prime ideal $\mathfrak{p}$ given by the maximal minors of a matrix M as above, then by Corollary 5.2, there exists $V \subset S_1$ such that $F \in \mathbb{K}[V]$ and $\dim(V) \leqslant 9$. Then we have $\overline{z}\overline{Q} \in \mathbb{K}[\overline{V}]$ modulo $\operatorname{span}_{\mathbb{K}} \{z_1, \cdots, z_r\}$. Therefore, by Lemma 3.15, $\overline{Q} \in \mathbb{K}[\overline{V}]$, which is a contradiction since $s(Q) > 9 + r$. $\square$

5.3 Quadratic minimal primes

Proposition 5.4. *Let C be an irreducible cubic form and Q an irreducible quadratic form in S. If $\mathfrak{p} = (x, Q)$ is a prime ideal containing C then the local ring $(S/C)_{\mathfrak{p}}$ is a discrete valuation ring.*

Proof. Note that $\mathfrak{p}$ defines a prime ideal of height 1 in S/C. Therefore the local ring $(S/C)_{\mathfrak{p}}$ is one dimensional and it is enough to show that $(S/C)_{\mathfrak{p}}$ is a regular local ring. Note that $(S/C)_{\mathfrak{p}}$ is a regular local ring iff the Jacobian ideal $(\frac{\partial C}{\partial x_i} | i \in [n])$ is not contained in $\mathfrak{p}$. Suppose $(\frac{\partial C}{\partial x_i} | i \in [n]) \subset \mathfrak{p}$. Let $C = xP + yQ$. After a change of coordinates, we may assume that x is one of the variables in S and $\frac{\partial Q}{\partial x} = 0$ (after possibly changing P). Then $\frac{\partial C}{\partial x} \in \mathfrak{p}$ implies that $P \in \mathfrak{p}$ and hence $P = xz + \alpha Q$ for some $z \in S_1$, $\alpha \in \mathbb{K}$. We have $C = x^2 z + (\alpha x + y)Q$. Then $\frac{\partial C}{\partial x_i} \in \mathfrak{p}$ implies that $y \frac{\partial Q}{\partial x_i} \in \mathfrak{p}$. Since $y \notin \mathfrak{p}$, we must have that $\frac{\partial Q}{\partial x_i} \in \mathfrak{p}$. Since $\frac{\partial Q}{\partial x_i} \in S_1$, we must have $\frac{\partial Q}{\partial x_i} = \beta x$ for some $\beta \in \mathbb{K}$. Now $\frac{\partial C}{\partial x \partial x_i} = 0$ implies that $\beta = 0$. Hence $\frac{\partial Q}{\partial x_i} = 0$ for all the variables, which is a contradiction. $\square$

Lemma 5.5. *Let $V = V_1 + V_2$ be an (w,t)-wide AH-vector space in the polynomial ring S, with $w \geqslant 2$. Let C be an irreducible cubic form contained in $\mathcal{A} = \mathbb{K}[V]$. Suppose $C = xQ + \ell P$ where $P, Q \in S_2$ and $x, \ell \in S_1$. If $s(Q) < (t-1)\dim(V_1) + t(\dim(V_2) + w - 1)$ then $Q \in (V_1)$.*

Proof. Note that $C = xQ + \ell P$ is a minimal collapse, and by Lemma 4.16, we have $x, \ell \in V_1$. Consider the algebra $\mathcal{A}' = \mathbb{K}[V/(\ell)]$ in the polynomial ring $S/(\ell)$. Note that $\mathcal{A}'$ is $(w-1, t)$ wide. By Proposition 4.10, we know that any basis of V forms an R_3 sequence, and in particular a prime sequence. Now we have $\overline{x}\overline{Q} \in \mathcal{A}'$. By Lemma 3.15, we know that $\overline{Q} \in \mathcal{A}'$ and hence $Q = Q_1 + \ell u$, where $Q_1 \in \mathcal{A}$ and $u \in S_1$. Note that $s(Q_1) \leqslant s(Q) + 1 < (t-1)\dim(V_1) + t(\dim(V_2) + w - 1)$. Therefore, by Proposition 4.9, we have $Q_1 \in \mathbb{K}[V_1/(\ell)]$. Thus $Q \in (V_1)$ as $\ell \in V_1$. $\square$

Lemma 5.6 (Minimal prime with strong quadratic)**.** *Let* C, F *be irreducible cubic forms in* S*. Suppose there exists a minimal prime* $\mathfrak{p} = (Q, \ell)$ *of the ideal* (C, F) *such that* Q *is a quadratic form with* $s(Q) \geqslant 5$ *and* $\ell \in S_1$*. If* $m(\mathfrak{p}) \geqslant 2$*, then one of the following holds:*

1. *we have* $F = \alpha C + \beta \ell Q + \ell^2 z$ *for some* $\alpha, \beta \in \mathbb{K}$ *and* $z \in S_1$*. In particular* $F \in (C, \ell)$*, and* (C, F) *has a linear minimal prime.*
2. *there exists a quadratic form* G *such that* $(C, F) \subset (G, \ell^2)$ *and* $(G, \ell) = (Q, \ell)$*.*

Proof. Let $C = xQ + \ell P$ and $F = aQ + \ell B$ where $x, a \in S_1$ and $P, Q \in S_2$. If $B \in (Q, \ell)$ and $P \in (Q, \ell)$ then we are done. So we may assume that $B \notin (Q, \ell)$. By Proposition 5.4, we know that the local ring $(S/F)_{\mathfrak{p}}$ is a discrete valuation ring with a discrete valuation ν. Now $\overline{Q}\overline{a} \equiv -\overline{\ell B}$ in $(S/F)_{\mathfrak{p}}$. Since $B \notin \mathfrak{p}$, we know that $\overline{B}$ is an unit in $(S/F)_{\mathfrak{p}}$ and we have $\overline{\ell} = -\frac{\overline{Q}\overline{a}}{\overline{B}}$. If $\nu(\overline{Q}) > 1$, then $\nu(\overline{\ell}) > 1$ and we have $\mathfrak{p} \subset \mathfrak{p}^2$ in $(S/F)_{\mathfrak{p}}$, which is contradiction by Nakayama's lemma. Hence we must have $\nu(\overline{Q}) = 1$. Then we have $m(\mathfrak{p}) = \nu(C) = \nu(\overline{Q}\overline{x} + \overline{\ell P}) = \nu(\overline{Q}) + \nu(\overline{B}\overline{x} - \overline{a}\overline{P})$. Hence $m(\mathfrak{p}) > 1$ iff $\nu(\overline{B}\overline{x} - \overline{a}\overline{P}) > 0$, i.e. $Bx - aP \in \mathfrak{p} = (Q, \ell)$. Since $s(Q) \geqslant 5$, we know that $S/(Q, \ell)$ is a UFD. Now $\overline{B}\overline{x} \equiv \overline{a}\overline{P}$ in $S/\mathfrak{p}$. Therefore by unique factorization, we must have one the following two cases.

Case 1. We have $\overline{B} \equiv \alpha \overline{P}$ and $\overline{a} = \alpha \overline{x}$ in $S/(Q, \ell)$ where $\alpha \in \mathbb{K}$ is an unit. Then we have $B = \alpha P + \mu Q + \ell z$ and $a = \alpha x + \lambda \ell$ for some $\mu, \lambda \in \mathbb{K}$ and $z \in S_1$. Then we have $F = (\alpha x + \lambda \ell)Q + \ell(\alpha P + \mu Q + \ell z) = \alpha C + \beta \ell Q + \ell^2 z$ for some $\beta \in \mathbb{K}$.

Case 2. We have $\overline{a}|\overline{B}$ and $\overline{x}|\overline{P}$ in $S/(Q, \ell)$. Then we have $B = az + \ell y + \alpha Q$ and $P = xu + \ell v + \beta Q$ for some $\alpha, \beta \in \mathbb{K}$ and $z, y, u, v \in S_1$. Since $\overline{B}\overline{x} \equiv \overline{a}\overline{P}$ in $S/(Q, \ell)$, we have $ax(u - z) \equiv 0$ modulo (Q, ℓ). Therefore $u \equiv z$ modulo ℓ, and may write $u = z + \gamma \ell$ for some $\gamma \in \mathbb{K}$. Then $F = aQ + \ell(az + \ell y + \alpha Q) = a(Q + \ell z) + \ell(\ell y + \alpha Q) = a(Q + \ell z) + \ell(\ell y + \alpha(Q + \ell z) - \alpha \ell z)$. Hence $F = (a + \alpha \ell)G + \ell^2(y - \alpha z)$ where $G = Q + \ell z$. Similarly we have $C = xQ + \ell(xu + \ell v + \beta Q) = xQ + \ell(xz + \gamma x \ell + \ell v + \beta Q)$. Hence we have $C = (x + \beta \ell)G + \ell^2(\gamma x + v - \beta z)$. Therefore $(C, F) \subset (G, \ell^2)$ and we have $(G, \ell) = (Q, \ell)$. □

A useful corollary of the lemma above is the following:

Corollary 5.7. *Let* $V = V_1 + V_2$ *be an* (w, t)*-wide AH-vector space in the polynomial ring* S*, with* $w \geqslant 6$*. Let* C *be an irreducible cubic form contained in* $\mathcal{A} = \mathbb{K}[V]$*. Suppose* $F \in S_3$ *is an irreducible cubic form such that* (C, F) *has a quadratic minimal prime* $\mathfrak{p}$ *with* $m(\mathfrak{p}) \geqslant 2$ *and* $F \notin (V_1)$*. Then there exists* $Q \in S_2$ *and* $\ell \in S_1$ *such that* $\mathfrak{p} = (Q, \ell)$*, and we have* $s(Q) \geqslant (t-1)\dim(V_1) + t(\dim(V_2) + w - 1) - 1$ *and*

$$(C, F) \subset (Q, \ell^2).$$

Proof. Let $\mathfrak{p} = (Q_1, \ell)$. If $s(Q_1) < (t-1)\dim(V_1) + t(\dim(V_2) + w - 1)$, then by Lemma 5.5 we have $Q_1 \in (V_1)$. This is a contradiction since $F \notin (V_1)$. Thus we must have that $s(Q_1) \geqslant (t-1)\dim(V_1) + t(\dim(V_2) + w - 1)$. Then by Lemma 5.6, we know that $F \in (C, \ell)$ or there exists a quadratic form Q such that $(C, F) \subset (Q, \ell^2)$ and $(Q, \ell) = (Q_1, \ell)$. Let $C = xQ_1 + \ell P$. If $F \in (C, \ell)$, then $F \in (x, \ell)$. By Lemma 4.16, $x, \ell \in V_1$ and hence $F \in (V_1)$, which is a contradicion. Therefore, there must exist a quadratic form Q such that $(C, F) \subset (Q, \ell^2)$ and $(Q, \ell) = (Q_1, \ell)$. Since $Q \in (Q_1, \ell)$, we note that $s(Q) \geqslant s(Q_1) - 1 \geqslant (t-1)\dim(V_1) + t(\dim(V_2) + w - 1) - 1$. □

Lemma 5.8. *Let* $F = xQ + \ell^2 y \in S_3$ *be an irreducible cubic where* $Q \in S_2$ *is such that* $s(Q) \geqslant 3$*. If* $F \in (G, a^2)$ *where* $a \in S_1, G \in S_2$ *with* $s(G) \geqslant 3$*, then* $a \in (x, \ell)$*,* $G \in (Q, \ell, x)$ *and* $F = xG + a^2 b$ *for some* $b \in S_1$*. Moreover, if* $x \notin (a, \ell)$*, then* $a \in (\ell)$ *and* $(G, a^2) = (Q, \ell^2)$*.*

Proof. Since F is irreducible, we know that $\dim \operatorname{span}_{\mathbb{K}}\{x, \ell\} = 2$. Moreover, as $F \in (G, a^2)$, we can also write $F = zG + a^2 b$.

Let us begin by proving that $z = x$ (after a possible multiplication of G by a scalar). Since $zG \equiv -a^2 b \mod (x, \ell)$, $S/(x, \ell)$ is a UFD, and $s(G) \geqslant 3$, we must have that $z \equiv 0 \mod (x, \ell)$ and $a^2 b \equiv 0 \mod (x, \ell)$. Writing $z = \alpha x + \beta \ell$, we get $F = xQ + \ell^2 y = (\alpha x + \beta \ell)G + a^2 b$, which implies $x(Q - \alpha G) = \ell(\beta G - \ell y) + a^2 b$. Hence, $\ell(\beta G - \ell y) \equiv -a^2 b \mod (x)$, and together with $s(G) \geqslant 3$ implies that $\beta = 0$ and therefore we can write $z = \alpha x$, where $\alpha \neq 0$. Thus, after a possible multiplication of G by a scalar, we have $F = xG + a^2 b$.

Note that the above also implies that $a^2 b \equiv \ell^2 y \mod (x)$, which by factoriality of $S/(x)$ implies that $a \in (x, \ell)$. From $0 = x(Q - G) + \ell^2 y - a^2 b$, we have $x(Q - G) \equiv a^2 b \equiv \gamma x^2 b \mod (\ell)$, which implies that $Q - G \in (\ell, x)$ and thus $G \in (Q, \ell, x)$. This concludes the first part of the lemma.

For the moreover part, since we know that $a \in (x, \ell)$, if $x \notin (a, \ell)$ then we must have $a \in (\ell)$. In this case, $xQ \equiv xG \bmod (\ell)$, which implies $G = Q + \ell f$ for some $f \in S_1$. Hence, $F = xQ + \ell^2 y = x(Q + \ell f) + \ell^2 b$, which yields $xf = \ell(y - b)$, and therefore $f \in (\ell)$, otherwise $x \in \operatorname{span}_{\mathbb{K}}\{\ell\}$, which is a contradiction. However, $f \in (\ell) \Rightarrow G \in (Q, \ell^2)$ and therefore we can take $G = Q$. □

Remark 5.9. *An easy consequence of the above lemma is that given $F = xQ + \ell^2 y$, then there is only one primary ideal of the form (P, ℓ^2) such that $F \in (P, \ell^2)$. In particular, we must have that $(P, \ell^2) = (Q, \ell^2)$.*

6 Sylvester-Gallai configurations

In this section we formally define several variants of Sylvester-Gallai configurations and discuss some preliminary results – old and new – which we will need to prove our main theorem in Section 7. In particular, we will define the main variant that we will need – Sylvester-Gallai over algebras – in order to apply our reduction from the cubic case to the quadratic SG case. Throughout this section we will denote our Sylvester-Gallai configuration by $\mathcal{F}$, and $\mathcal{F}_d := \mathcal{F} \cap S_d$ is the subset of the forms in $\mathcal{F}$ which have degree d.

6.1 Linear Sylvester-Gallai configurations

Definition 6.1 (Robust linear SG configuration). Let $\mathcal{F} := \{\ell_1, \cdots, \ell_m\} \subset S_1$ be a finite set of pairwise linearly independent linear forms in S and let $\delta \in (0, 1]$. We say that $\mathcal{F}$ is a δ-linear-SG configuration if for every $i \in [m]$ there exist at least $\delta(m-1)$ values of j such that $|\mathcal{F} \cap \operatorname{span}_{\mathbb{K}}\{\ell_i, \ell_j\}| \geqslant 3$, i.e. there exists $k \neq i, j$ such that $\ell_k \in \operatorname{span}_{\mathbb{K}}\{\ell_i, \ell_j\}$. If $\delta = 1$, then we simply call it a linear SG configuration.

Given a Sylvester-Gallai configuration, we say that (ℓ_i, ℓ_j, ℓ_k) is a SG triple if i, j, k are distinct and $\ell_k \in \operatorname{span}_{\mathbb{K}}\{\ell_i, \ell_j\}$. Moreover, we say that (ℓ_i, ℓ_j) is a SG pair if there is $k \neq i, j$ such that (ℓ_i, ℓ_j, ℓ_k) is a SG triple.

It was proved in [BDWY11, DSW14] that the dimension of the span of a δ-linear-SG configuration is bounded by a function depending only on the robustness parameter δ. Below we state the sharpest known result, from [DSW14, Theorem 1.9].

Theorem 6.2. *If $\mathcal{F}$ is a* δ-linear-SG *configuration, then* $\dim(\operatorname{span}_{\mathbb{K}}\{\mathcal{F}\}) \leqslant \frac{12}{\delta}$.

We will also need a slight strengthening of the result above, which comes from considering a slightly more general type of linear SG configurations, where we also allow certain SG pairs (ℓ_i, ℓ_j) to intersect non-trivially a small dimensional vector space, instead of spanning a third element of the configuration.

Definition 6.3 (Robust linear Sylvester-Gallai configurations over a vector space)**.** Let $c \in \mathbb{N}$, $0 < \delta \leqslant 1$ and $\mathcal{F} := \{\ell_1, \ldots, \ell_m\} \subset S_1$ be a set of linear forms such that $\ell_i \notin (\ell_j)$ for any $i \neq j$. We say that $\mathcal{F}$ is a (c, δ)-linear-SG configuration if there exists a vector space $W \subset S_1$ of dimension at most c such that for any $\ell_i \in \mathcal{F} \backslash W$, there exist at least $\delta(m-1)$ indices $j \in [m] \backslash \{i\}$ such that $\ell_j \notin W$ and one of the following holds: $|\mathrm{span}_{\mathbb{K}} \{\ell_i, \ell_j\} \cap \mathcal{F}| \geqslant 3$ or $\mathrm{span}_{\mathbb{K}} \{\ell_i, \ell_j\} \cap W \neq 0$.

The following proposition is an easy corollary of Theorem 6.2, and it is a slightly more general version than [Shp20, Corollary 16]. For completeness, we provide a proof of this proposition in Appendix B.1.

Proposition 6.4 (Robust Linear SG Configurations)**.** *Let $\mathcal{F}$ be a (c, δ)-linear-SG configuration. Then* $\dim \mathrm{span}_{\mathbb{K}} \{\mathcal{F}\} \leqslant c + 25/\delta$.

6.2 Radical Sylvester-Gallai Configurations

Next we define a generalization of SG-configurations for radicals of ideals in a polynomial ring. The following generalization was defined in [Gup14, Section 6].

Definition 6.5 (Radical SG configuration)**.** Let $\mathcal{F} = \{F_1, \cdots, F_m\} \subset S$ be a set of irreducible forms such that $\deg(F_i) \leqslant d$ for all $i \in [m]$ and $F_i \notin (F_j)$ for $i \neq j$. We say that $\mathcal{F}$ is a (δ, d)-radical-SG configuration if for every $i \in [m]$ there exist at least $\delta(m-1)$ values of j such that $|\mathcal{F} \cap \mathrm{rad}(F_i, F_j)| \geqslant 3$. If $\delta = 1$ then we simply call it a d-radical-SG configuration.

We now define a slightly more flexible variant of a radical SG configuration, where we allow some dependencies to be inside of a predefined algebra.

Definition 6.6 (Radical SG over an algebra)**.** Let $d \in \mathbb{N}^*$ and $V \subset S_{\leqslant d}$ be a graded vector space. Let $\mathcal{F} = \{F_1, \cdots, F_m\} \subset S$ be a set of irreducible forms such that $\deg(F_i) \leqslant d$ and $F_i \notin (F_j)$ for $i \neq j$. We say that $\mathcal{F}$ is a (δ, d, V)-radical-SG configuration if for every $i \in [m]$, there exist at least $\delta(m-1)$ values of j such that $|\mathcal{F} \cap \mathrm{rad}(F_i, F_j)| \geqslant 3$ or $|\mathrm{rad}(F_i, F_j) \cap \mathbb{K}[V] \backslash (F_i) \cup (F_j)| \geqslant 1$. If $\delta = 1$ then we simply call it a (d, V)-radical-SG configuration.

The following proposition, whose proof can be found in Appendix A, says that any 2-radical-SG configuration over a small algebra must be in a slightly larger algebra. In case the initial algebra is of constant dimension, then the configuration must be in a constant dimensional vector space.

Proposition 6.7 (2-radical-SG configurations over small algebra)**.** *Let $\mathcal{F} \subset S_{\leqslant 2}$ be a finite set of irreducible forms such that for any $F, G \in \mathcal{F}$ we have $F \notin (G)$. Additionally, let $V = V_1 + V_2$ be a vector space of forms of degree at most 2. If $\mathcal{F}$ is a $(2, V)$-radical-SG configuration, then* $\dim \mathrm{span}_{\mathbb{K}} \{\mathcal{F}\} = O(1 + \dim(W)^2)$, *where W is any $(600, 8)$-wide vector space such that $\mathbb{K}[V] \subset \mathbb{K}[W]$.*

An immediate corollary of the above proposition, when $V = W = 0$, is the quadratic radical Sylvester-Gallai theorem [Shp20, Theorem 7].

6.3 Saturated radical Sylvester-Gallai theorem

In this section we consider a variant of the radical Sylvester-Gallai configuration, where now, in addition to our set of forms $\mathcal{F}$, there will be a special linear form $z \in S_1$ such that for any two polynomials $F_i, F_j \in \mathcal{F}$, there exists $k \neq i, j$ such that $zF_k \in \mathrm{rad}(F_i, F_j)$. In hindsight, as one would expect, such configurations can only appear inside small algebras, and in this section we prove this fact. We begin by formally defining such configurations.

Definition 6.8 (Saturated SG configurations)**.** Let z be a non-zero linear form in $S = \mathbb{K}[x_1, \cdots, x_n]$. Let $\mathcal{F} := \{z, F_1, \ldots, F_m\} \subset S_{\leqslant 2}$, where each F_i is either an irreducible form or a product of two distinct linear forms such that $\gcd(F_i, F_j) = \gcd(F_i, z) = 1$ for any $i \neq j \in [m]$. We say that $\mathcal{F}$ is a z-saturated radical Sylvester-Gallai configuration if for any two forms $F_i, F_j \in \mathcal{F}$, there is a third form $F_k \in \mathcal{F} \backslash \{F_i, F_j\}$ such that $zF_k \in \text{rad}(F_i, F_j)$.

In what follows, the next definitions shall be useful:

- $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2$ is our z-saturated configuration, with $|\mathcal{F}| = m$ and $|\mathcal{F}_i| = m_i$, for $i \in [2]$
- For $Q \in S_2$, define $\mathbb{L}(Q) = \begin{cases} \text{Lin}(Q), & \text{if } s(Q) \leqslant 2, \\ \text{span}_{\mathbb{K}}\{Q\}, & \text{otherwise} \end{cases}$
- given $Q \in \mathcal{F}_2$, denote by

$$\mathcal{F}_{span}(Q) := \{P \in \mathcal{F}_2 \mid |(P, Q) \cap \mathcal{F}_2| \geqslant 3\}$$
$$\mathcal{F}_{non-prime}(Q) := \mathcal{F}_2 \backslash \mathcal{F}_{span}(Q)$$

- Given a parameter $\delta \in (0, 1]$, define the set

$$\mathcal{F}_{span} := \{Q \in \mathcal{F}_2 \ \mid \ |\mathcal{F}_{span}(Q)| \geqslant \delta m_2\}.$$

  For the remainder of this subsection, we shall assume $\delta = 1/30$ and $\mathcal{F}_{span}$ is defined with respect to this choice of δ.

With the definitions above at hand, we will now prove some useful lemmas about such configurations. We begin by proving that if the quadratics are not a linear SG configuration, then we can find a small wide algebra which "approximates" all the quadratics.

Proposition 6.9. *Suppose $\mathcal{F}_2 \neq \mathcal{F}_{span}$. Let W be a graded vector space such that $\mathbb{K}[W] \cap \mathcal{F}_2 \backslash \mathcal{F}_{span} \neq \varnothing$. Then any $Q \in \mathcal{F}_2$ can be written as $Q = P + \alpha R$, where $\alpha \in \mathbb{K}$, $R \in \mathbb{K}[W] \cap \mathcal{F}_2$ and $s(P) \leqslant 3$.*

Proof. Let $F \in \mathbb{K}[W] \cap \mathcal{F}_2 \backslash \mathcal{F}_{span}$ and $\mathcal{F}_{non-prime}(F) := \{G_1, \ldots, G_t\}$. Since $F \notin \mathcal{F}_{span}$, we have that $t \geqslant (1 - \delta)m_2$. Moreover, since (F, G_i) does not span a third element, we must have that (F, G_i) is not prime. Thus, Proposition A.2 implies that either $G_i = F + a_i b_i$ for $a_i, b_i \in S_1$ or there are $x_i, y_i \in S_1$ such that $F, G_i \in (x_i, y_i)$.

Since $F \in \mathbb{K}[W]$, by the above paragraph, we have that each $G_i \in \mathcal{F}_{non-prime}(P)$ is such that $s(G_i) = 1$ or $G_i - F = a_i b_i \Rightarrow s(G_i - F) = 0$, i.e. $s_{\min}(F, G_i) \leqslant 1$. Thus G_i satisfies the conditions of the proposition with W.

Let $Q \in \mathcal{F}_{span}(F)$. If (Q, G_i) is not prime for some G_i, then $s_{\min}(Q, G_i) \leqslant 1$. Hence $s_{\min}(Q, F) \leqslant 3$. Therefore we may assume that (Q, G_i) is prime for all $i \in [t]$. However, since $(1 - \delta)m_2 > 2m_2/3$, by the pigeonhole principle this would imply that there are $G_i, G_j \in \mathcal{F}_{non-prime}(P)$ such that $Q \in (G_i, G_j)$. Hence we have $\min_{\alpha \in \mathbb{K}} s(Q + \alpha F) \leqslant \min_{\alpha \in \mathbb{K}} s(G_i + \alpha F) + \min_{\alpha \in \mathbb{K}} s(G_j + \alpha F) + 1 \leqslant 3$. ☐

Lemma 6.10. *Let $0 < \nu < 1$ be a constant and $w \in \mathbb{N}$ such that $w > 24/\nu + 10$. If $\mathcal{F}_2 \neq \mathcal{F}_{span}$ and W is a $(w, 3)$-wide vector space such that $|\mathcal{F} \cap \mathbb{K}[W]| \geqslant \nu|\mathcal{F}|$ and $z \in W$, then there is a $(w - 24/\nu, 1)$-wide vector space V such that $\mathbb{K}[W] \subset \mathbb{K}[V]$, $\dim(V) \leqslant 3(\dim(W) + 1) + 25(1 + 1/\nu)$ and $\mathcal{F} \subset \mathbb{K}[V]$.*

Proof. Let $m+1 := |\mathcal{F}|$ and let $\{F_1, \ldots, F_t\} = \mathcal{F} \cap \mathbb{K}[W]$. Hence, we know that $t \geqslant \nu m$. We begin by showing that there is a $(w - 24/\nu, 1)$-wide vector space X such that $\dim(X) \leqslant 3(\dim W + 1) + 24/\delta$, $\mathbb{K}[W] \subset \mathbb{K}[X]$ and $\mathcal{F} \subset (X)$.

If $\mathbb{K}[W]$ contains a form in $\mathcal{F}_2 \backslash \mathcal{F}_{span}$, then we know that for any $Q \in \mathcal{F}_2$, there is $R_Q \in \mathbb{K}[W]$ such that $s(Q - R_Q) \leqslant 3$. If this is not the case, let $Q \in \mathcal{F}_2 \backslash \mathcal{F}_{span}$. By Lemma A.3 there is $(w, 1)$-wide U such that $\dim U \leqslant 3(\dim W + 1)$, $W \subset \mathbb{K}[U]$ and $Q \in \mathbb{K}[U]$.

If $\mathcal{F} \subset (U)$, set $X = U$. Otherwise, let $P \in \mathcal{F} \backslash (U)$. If $P \in S_1$, then $Y := U + \mathrm{span}_{\mathbb{K}}\{P\}$ is $(w-1, 1)$-wide, and if $P \in \mathcal{F}_2$, we know that there is $R_P \in \mathbb{K}[U]$ such that $s(P - R_P) \leqslant 3$, which implies that $Y := U + \mathrm{Lin}(P - R_P)$ is $(w-8, 1)$-wide. In either case, we constructed Y $(w-8, 1)$-wide such that $U \subset Y$ and $P \in \mathbb{K}[Y]$. Since $w - 8 > 3$, we have that any homogeneous generators for Y form an R_3-sequence, and in particular Y is generated by a prime sequence (the same holds for U).

Lemma 3.22 applied to P, F_i, U and $\mathbb{K}[Y]$ (if $F_i = u_i v_i$, apply the lemma to $P, u_i, \mathbb{K}[Y]$ and $P, v_i, \mathbb{K}[Y]$) implies that for at most 8 values of $i \in [t]$ we have (P, F_i) not radical. Hence, we can say that (P, F_i) is radical for $i \leqslant t - 8$. Moreover, as $z \in U$ and $P \notin (U)$, we have that P, F_i, z is a regular sequence for each $F_i \in \mathbb{K}[U]$, and therefore we have that $zG_i \in \mathrm{rad}(P, F_i) \Rightarrow G_i \in \mathrm{rad}(P, F_i)$.

Thus, if $i \leqslant t-8$, the above implies that there is $G_i \in \mathcal{F} \backslash (U)$ such that $G_i \in (P, F_i) \subset (Y)$. Moreover, note that if $G_i \in (P, F_i) \cap (P, F_j)$, then it must be the case that $F_i, F_j \in U_1$ and $G_i = P + \alpha_i F_i F_j$, and thereby we have that each G_i can belong to at most 2 of the ideals (P, F_i). Therefore, we have that

$$|\mathcal{F} \cap (Y)| \geqslant |\mathcal{F} \cap (U)| + \frac{t-8}{2} \geqslant |\mathcal{F} \cap (U)| + \frac{\nu m}{3} \text{ and } |\mathcal{F} \cap \mathbb{K}[Y]| \geqslant \nu m.$$

Hence, setting $U = Y$ we can increase the number of forms in $\mathcal{F} \cap (U)$ by $\nu m/3$.

Given the above and the fact that $\dim(Y) \leqslant \dim(U) + 8$, applying the process above at most $3/\nu$ times we obtain X such that $\dim X \leqslant \dim U + 24/\nu \leqslant 3(\dim W + 1) + 24/\nu$ and $\mathcal{F} \subset (X)$. It is important to note that $\mathcal{F} \subset (X)$ implies that $\mathcal{F}_1 \subset X_1$. Hence, $\mathcal{F} \backslash \mathbb{K}[X]$ only has quadratic forms.

Now that we have $\mathcal{F} \subset (X)$ and X is $(w - 24/\nu, 1)$-wide, if we apply a general projection φ mapping $X_1 \mapsto z$, we have that the forms in $\mathcal{F} \backslash \mathbb{K}[X]$ become:

$$\varphi(F_i) = \begin{cases} z\ell_i, \text{ where } \ell_i \notin (z), \text{ if } F_i \in (X_1) \\ G_i + z\ell_i, \text{ where } G_i \in \mathrm{span}_{\mathbb{K}}\{\varphi(X_2)\} \text{ and } \ell_i \notin (z). \end{cases}$$

Since X is $(w - 24/\nu, 1)$-wide, we have that $s(G_i) \geqslant 10$.

Let $\mathcal{F} \backslash \mathbb{K}[X] =: \mathcal{H} = \{H_1, \ldots, H_r\}$, and for each $i \in [r]$, let ℓ_i be the linear form such that $\varphi(H_i) = z\ell_i$ or $\varphi(H_i) = G_i + z\ell_i$. Let $\mathcal{L} := \{\ell_1, \ldots, \ell_r\} \cup \{z\}$. To conclude the proof, by Proposition 2.9 it is enough to prove that $\mathrm{span}_{\mathbb{K}}\{\mathcal{L}\} = O(1)$. To do this, let $\mathcal{L}' = \{\ell_1, \ldots, \ell_s\} \cup \{z\}$ (after possibly relabeling) be the maximum subset of $\mathcal{L}$ such that $\ell_i \notin (\ell_j)$ for any $i \neq j$. Note that $\mathrm{span}_{\mathbb{K}}\{\mathcal{L}\} = \mathrm{span}_{\mathbb{K}}\{\mathcal{L}'\}$. We will show that $\mathcal{L}'$ is a $(1,1)$-linear-SG configuration, which proves that $\dim \mathrm{span}_{\mathbb{K}}\{\mathcal{L}'\} \leqslant 25$. Thus setting $V = X + \mathrm{span}_{\mathbb{K}}\{\mathcal{L}'\}$ we are done.

Let $i \neq j \in [s]$. If $z \in (\ell_i, \ell_j)$, then we know that ℓ_i, ℓ_j, z form a SG triple and we are done. So we can assume that $\dim \mathrm{span}_{\mathbb{K}}\{\ell_i, \ell_j, z\} = 3$. By the z-saturated SG condition on H_i, H_j, there is $F \in \mathcal{F} \backslash \{H_i, H_j\}$ such that $zF \in \mathrm{rad}(H_i, H_j)$. Hence, we have that $z\varphi(F) \in \mathrm{rad}(\varphi(H_i), \varphi(H_j))$. We have three cases to analyze:

Case 1: $\varphi(H_i) = z\ell_i, \varphi(H_j) = z\ell_j$.

In this case, we have $z\varphi(F) \in \mathrm{rad}(z\ell_i, z\ell_j) \subset (\ell_i, \ell_j) \Rightarrow \varphi(F) \in (\ell_i, \ell_j) \Rightarrow \varphi(F) \notin \mathbb{K}[X]$ and therefore $F = H_k$ for some $k \neq i, j$. Moreover, since $s(\varphi(H_k)) \leqslant 2$, as $\varphi(H_k) \subset (\ell_i, \ell_j)$, we must have that $\varphi(H_k) = z\ell_k$ and hence $\ell_k \in (\ell_i, \ell_j)$. By Corollary 2.8, we know that $\ell_k \notin (\ell_i) \cup (\ell_j)$, and hence ℓ_i, ℓ_j, ℓ_k form a SG triple.

Case 2: $\varphi(H_i) = z\ell_i, \varphi(H_j) = G_j + z\ell_j$.

In this case, we have $z\varphi(F) \in \text{rad}(z\ell_i, G_j + z\ell_j) \subset (\ell_i, G_j + z\ell_j) \Rightarrow \varphi(F) \in (\ell_i, G_j + z\ell_j) \Rightarrow \varphi(F) \notin \mathbb{K}[X]$ and therefore $F = H_k$ for some $k \neq i, j$. Since $s(G_j) \geqslant 10$, we have that $\varphi(H_k) = G_k + z\ell_k$, otherwise we would have $\varphi(H_k) = z\ell_k \Rightarrow \ell_k \in (\ell_i, G_j + z\ell_j) \Rightarrow \ell_k \in (\ell_i)$, contradicting Corollary 2.8.

From $G_k + z\ell_k \in (\ell_i, G_j + z\ell_j)$, we have $G_k = \alpha_k G_i$ for some $\alpha \in \mathbb{K}^*$ and hence we know that $\ell_k \neq \alpha_k \ell_j$, otherwise Corollary 2.8 would imply that $H_k \in (H_j)$, which is a contradiction. Therefore, $z(\ell_k - \alpha_k \ell_j) \in (\ell_i, G_j + z\ell_j)$, which implies $\ell_k - \alpha_k \ell_j \in (\ell_i) \Rightarrow \ell_i, \ell_j, \ell_k$ form a SG triple.

Case 3: $\varphi(H_i) = G_i + z\ell_i, \varphi(H_j) = G_j + z\ell_j$.

In this case, if $\dim \text{span}_{\mathbb{K}} \{G_i, G_j\} = 2$ then Proposition A.2 implies that $(G_i + z\ell_i, G_j + z\ell_j)$ is prime, as $s_{\min}(G_i + z\ell_i, G_j + z\ell_j) \geqslant 8$. Therefore, we would have $\varphi(F) \in (G_i + z\ell_i, G_j + z\ell_j)$ which implies that $F = H_k$ and $\varphi(H_k) = G_k + z\ell_k$. Hence, $G_k + z\ell_k \in (G_i + z\ell_i, G_j + z\ell_j) \Rightarrow G_k + z\ell_k = \alpha_i(G_i + z\ell_i) + \alpha_j(G_j + z\ell_j)$ where $\alpha_i, \alpha_j \neq 0$. This in turn implies $\ell_k = \alpha_i \ell_i + \alpha_j \ell_j$, which together with $\dim \text{span}_{\mathbb{K}} \{\ell_i, \ell_j\} = 2$ implies that ℓ_i, ℓ_j, ℓ_k is a SG triple.

On the other hand, if $\dim \text{span}_{\mathbb{K}} \{G_i, G_j\} = 1$ then there is $\beta \in \mathbb{K}^*$ such that $G_j = \beta G_i$. In this case, we have $(G_i + z\ell_i, G_j + z\ell_j) = (G_i + z\ell_i, z) \cap (G_i + z\ell_i, \ell_j - \beta\ell_i)$. This in turn implies that $\varphi(F) \in (G_i + z\ell_i, \ell_j - \beta\ell_i) \Rightarrow \varphi(F) \notin \mathbb{K}[X]$ and therefore $F = H_k$ for some $k \neq i, j$.

If $s(\varphi(H_k)) \leqslant 5$, then $\varphi(H_k) = z\ell_k$ and $z\ell_k = \varphi(H_k) \in (G_i + z\ell_i, \ell_j - \beta\ell_i)$ implies $\ell_k \in (\ell_j - \beta\ell_i)$ which implies that ℓ_i, ℓ_j, ℓ_k is a SG triple.
Otherwise, $\varphi(H_k) = G_k + z\ell_k$ which implies

$$G_k + z\ell_k = \varphi(H_k) \in (G_i + z\ell_i, \ell_j - \beta\ell_i) \cap (G_i + z\ell_i, z) = (G_i + z\ell_i, G_j + z\ell_j).$$

Hence, we have that $G_k + z\ell_k = \gamma_i(G_i + z\ell_i) + \gamma_j(G_j + z\ell_j)$, where $\gamma_i, \gamma_j \neq 0$. Since $s(G_i) \geqslant 10$ and $G_j, G_k \in (G_i)$, the above implies $\ell_k = \gamma_i \ell_i + \gamma_j \ell_j$, which implies that ℓ_i, ℓ_j, ℓ_k is a proper SG triple.

Conclusion: the above cases prove that for any pair ℓ_i, ℓ_j such that $\ell_j \notin (\ell_i)$, either $z \in (\ell_i, \ell_j)$ or there exists $k \in [s]$ such that ℓ_i, ℓ_j, ℓ_k is a SG triple. Thus, $\mathcal{L}'$ is a $(1,1)$-linear-SG configuration. □

Lemma 6.11. *If $\mathcal{F}$ is a z-saturated SG configuration such that $|\mathcal{F}_1| \geqslant 3 \cdot |\mathcal{F}|/4$ then* $\dim \text{span}_{\mathbb{K}} \{\mathcal{F}\} = O(1)$.

Proof. Let $\mathcal{F}_1 = \{z, x_1, \ldots, x_r\}$, $\mathcal{F}_2 = \{Q_1, \ldots, Q_s\}$ and $m := |\mathcal{F}| = 1 + r + s$. By the assumption of the lemma, we have $r \geqslant 3s$. If there exists a linear form x_i such that for $\geqslant \delta m$ x_j's we have that x_i, x_j, z is not a regular sequence, then we have that $|\text{span}_{\mathbb{K}} \{x_i, z\} \cap \mathcal{F}_1| \geqslant \delta m$, which by Lemma 6.10 implies that $\text{span}_{\mathbb{K}} \{\mathcal{F}\} = O(1)$. Hence, we will assume that for each x_i at most δm x_j's do not form a regular sequence with x_i, z.

Given $x_i \in \mathcal{F}_1$, let $\mathcal{F}_{bad}(x_i)$ be the set of linear forms x_j such that x_i, x_j, z is regular but no linear form x_k is such that $x_k \in (x_i, x_j)$. We will now prove that $|\mathcal{F}_{bad}(x_i)| \leqslant r - 2\delta m$ for each $i \in [r]$.

If the above is not the case, we can assume w.l.o.g. that x_1 is such that $|\mathcal{F}_{bad}(x_1)| > r - 2\delta m > 2s$. Letting $\mathcal{F}_{bad}(x_1) := \{x_2, \ldots, x_t\}$ where $t > 2s$, by the SG condition and the fact that x_1, x_i, z is regular, we must have that there is $Q \in \mathcal{F}_2$ such that $Q \in (x_1, x_i)$. We will now show that for any distinct $i, j, k \in \mathcal{F}_{bad}(x_1)$, we must have $\mathcal{F}_2 \cap (x_1, x_i) \cap (x_1, x_j) \cap (x_1, x_k) = \varnothing$.

Let $\overline{x_i} = x_i \bmod (x_1)$. Note that $\overline{x_i}$'s are all independent since $x_i \in \mathcal{F}_{bad}(x_1)$, so we don't have linear dependencies. If $\mathcal{F}_2 \cap (x_1, x_i) \cap (x_1, x_j) \cap (x_1, x_k) \neq \varnothing$, we would have $Q \in \mathcal{F}_2 \cap$

$(x_1, x_i) \cap (x_1, x_j) \cap (x_1, x_k)$, which implies that $\overline{Q} \in (\overline{x_i} \cdot \overline{x_j} \cdot \overline{x_k})$ over $S/(x_1)$, which implies $\overline{Q} \equiv 0$, contradicting the fact that $\gcd(Q, x_1) = 1$. Hence, we have that each $Q \in \mathcal{F}_2$ is in at most two ideals of the form (x_1, x_i), which implies that $t \leqslant 2s$, which is a contradiction.

Since $|\mathcal{F}_{bad}(x_i)| \leqslant r - 2\delta m$ for each x_i, we have that $\mathcal{F}_1$ is a $(1, \delta)$-linear-SG, which implies that $\mathrm{span}_{\mathbb{K}} \{\mathcal{F}_1\} = O(1)$, which by Lemma 6.10 implies that we are done. □

Lemma 6.12. *If there exists $Q \in \mathcal{F}_2$ such that $s(Q) \geqslant 4$ and $Q \notin \mathcal{F}_{span}$, then there exists $V \subset S_1$ with $\dim(V) = O(1)$ such that any algebra $\mathcal{A}$ containing V and Q is such that $|\mathcal{A} \cap \mathcal{F}| \geqslant \delta|\mathcal{F}|$.*

Proof. Let $|\mathcal{F}_2| = s$ and $m = |\mathcal{F}|$. We can assume that $s \geqslant m/4$, otherwise Lemma 6.11 implies that we are done. If $Q \notin \mathcal{F}_{span}$, then let $\mathcal{F}_{non-prime}(Q) = \{Q_1, \ldots, Q_t\}$ be the set of all forms in $\mathcal{F}_2$ such that (Q, Q_i) do not span a third element in $\mathcal{F}_2$. In this case, $t \geqslant s - \delta m$. Note that $Q_i \in \mathcal{F}_{non-prime}(Q) \Rightarrow s(Q_i) \geqslant 3$ and $s_{\min}(Q, Q_i) = 0$, as either Q, Q_i, z is not a regular sequence, or (Q, Q_i) is not radical.

We will show that $P \in \mathcal{F}_2 \Rightarrow s_{\min}(P, Q) \leqslant 1$. Suppose it is not the case, that is, we have $P \in \mathcal{F}_2$ such that $s_{\min}(P, Q) \geqslant 2$. Note that P must be irreducible, as $s(P) \geqslant s_{\min}(P, Q) \geqslant 2$. By Proposition A.2, we have that (P, Q_i) is prime for any $Q_i \in \mathcal{F}_{non-prime}(Q)$, as $s_{\min}(P, Q_i) \geqslant s_{\min}(P, Q) - 1 \geqslant 1$ and $s_{\max}(P, Q_i) \geqslant s(Q_i) \geqslant s(Q) - 1 \geqslant 3$. Hence, we have that (P, Q_i) must span an element $G_i \in \mathcal{F}_2 \setminus \mathcal{F}_{non-prime}(Q)$. Since $\mathrm{span}_{\mathbb{K}} \{P, Q_i\} \cap \mathrm{span}_{\mathbb{K}} \{P, Q_j\} = \mathrm{span}_{\mathbb{K}} \{P\}$, as $s_{\min}(P, Q) \geqslant 2$, we have that $G_i \notin (G_j)$ for each $i \neq j \in [t]$. However, this would imply that $\delta m \geqslant |\mathcal{F}_2 \setminus \mathcal{F}_{non-prime}(Q)| \geqslant t - \delta m$, which is a contradiction.

Now that we know that $P \in \mathcal{F}_2 \Rightarrow s_{\min}(P, Q) \leqslant 1$, we have two cases to analyze:

Case 1: there is $P \in \mathcal{F}_2$ such that $s_{\min}(P, Q) = 1$.

In this case, we have that $P = \alpha Q + R$, where $\alpha \in \mathbb{K}$ and $s(R) = 1$. Let $\mathcal{A}$ be an algebra containing $Q, \mathrm{Lin}(R), z$. We prove that $|\mathcal{A} \cap \mathcal{F}| \geqslant \delta m$ by proving that $|\mathcal{F}_{non-prime}(Q) \setminus \mathcal{A}| \leqslant \delta m$. This would conclude this case by taking $V = \mathrm{span}_{\mathbb{K}} \{z, \mathrm{Lin}(R)\}$. After relabeling, we can assume that $\{Q_1, \ldots, Q_r\} := \mathcal{F}_{non-prime}(Q) \setminus \mathcal{A}$ for $r \leqslant t$.

Let $i \in [r]$. Since $s_{\min}(Q, Q_i) = 0$ and $s(Q_i) \geqslant 3$, we have $Q_i = Q + u_i v_i$. In particular, this implies that (Q_i, ℓ) is prime for any linear form $\ell \in S_1$.

We first prove that P, Q_i, z form a regular sequence. If this is not the case then $P \in (Q_i, z)$, which implies that $P = \beta Q_i + z\ell_i \Rightarrow (\alpha - \beta)Q = z\ell_i - R$. Since $s(Q) \geqslant 4$ and $s(s\ell_i - R) \leqslant 2$ we must have $\beta = \alpha$, which implies $s(R) = 0$, which is a contradiction to $s(R) = 1$.

Moreover, (P, Q_i) is radical, otherwise by Proposition A.2 there is $\ell_i \in S_1$ such that $\beta P + \gamma Q_i = \ell_i^2$ which implies $(\alpha\beta + \gamma)Q = \ell_i^2 - R$ and analogously this contradicts $s(R) = 1$.

Hence, it must be the case that (P, Q_i) must span a third element of $\mathcal{F}_2$. If there is $Q_j \in (P, Q_i) \cap \mathcal{F}_2 \setminus \{P, Q_i\}$ such that $s_{\min}(Q, Q_j) = 0$, writing $Q_j = Q + u_j v_j$, there is $\beta, \gamma \in \mathbb{K}^*$ such that

$$Q_j = \beta P + \gamma Q_i \Rightarrow (1 - \alpha\beta - \gamma)Q = \beta R + \gamma u_i v_i - u_j v_j.$$

Since $s(Q) \geqslant 4$ and $s(\beta R + \gamma u_i v_i - u_j v_j) \leqslant 3$ we must have $1 - \alpha\beta - \gamma = 0$ and $\beta R = u_j v_j - \gamma u_i v_i$. Since $\beta \neq 0$ and $s(R) = 1$ Proposition 2.2 implies that $u_i, v_i \in \mathrm{Lin}(R) \subset \mathcal{A}$ which contradicts $Q_i \notin \mathcal{A}$.

Thus, the only possibility is that (P, Q_i) must span an element $F \in \mathcal{F}_2$ such that $s_{\min}(F, Q) = 1$. In particular, we know that $F \in \mathcal{F}_2 \setminus \mathcal{F}_{non-prime}(Q) =: \{F_1, \ldots, F_k\}$. Since $|\mathcal{F}_{non-prime}(Q)| \geqslant s - \delta m$, we have $k \leqslant \delta m$. Since $Q_i \notin \mathcal{A}$ for $i \in [r]$, we must have that $\mathrm{span}_{\mathbb{K}} \{P, Q_i\} \cap \mathrm{span}_{\mathbb{K}} \{P, Q_j\} = \mathrm{span}_{\mathbb{K}} \{P\}$, otherwise $P \in \mathrm{span}_{\mathbb{K}} \{Q_i, Q_j\}$ which contradicts $Q_i, Q_j \notin \mathcal{A}$. Hence, for each $i \in [r]$, there exists $F_i \in \mathrm{span}_{\mathbb{K}} \{P, Q_i\}$ and for $i \neq j \in [r]$, we have that $F_i \notin (F_j)$. This implies that $r \leqslant k \leqslant \delta m$. Since $r = |\mathcal{F}_{non-prime}(Q) \setminus \mathcal{A}|$ we are done with this case.

Case 2: all forms in $F_i \in \mathcal{F}_2$ satisfy $s_{\min}(Q, F_i) = 0$, that is, are of the form $F_i = \alpha_i Q + u_i v_i$, where $\alpha_i \in \mathbb{K}$ and $u_i, v_i \in S_1$.

In this case, partition $\mathcal{F}_2 = \Lambda \sqcup \mathcal{H}$, where $F_i \in \Lambda$ if $s(F_i) = 0$ (that is, $\alpha_i = 0$) and $F_i \in \mathcal{H}$ if $s(F_i) \geqslant 3$ (that is $\alpha_i \neq 0$). Let $\widehat{\Lambda} = \{u_i, v_i \mid F_i \in \Lambda\}$, and let $\Gamma := \mathcal{F}_1 \cup \widehat{\Lambda}$.

The SG dependencies from $\mathcal{F}$ and the fact that $s(F) \geqslant 3$ for $F \in \mathcal{H}$ imply that Γ is a $(1,1)$-linear-SG configuration, and thus $\dim \operatorname{span}_{\mathbb{K}} \{\Gamma\} = O(1)$. Let $W = \operatorname{span}_{\mathbb{K}} \{\Gamma\}$. Since we are assuming that $z \in \mathcal{F}_1$, we also have that $z \in W$. Moreover, if $|\mathcal{F}_1 \cup \Lambda| \geqslant \delta m$ we have that $|\mathcal{F} \cap \mathbb{K}[W]| \geqslant \delta m$ and we are done by setting $V = W$. Thus, we can assume that $|\mathcal{H}| \geqslant (1-\delta)m$. Since $s = |\mathcal{F}_2| \geqslant |\mathcal{H}| \geqslant (1-\delta)m$, this implies $t \geqslant (1-2\delta)m$. We now need to handle the forms in $\mathcal{H}$.

Any $Q_i \in \mathcal{F}_{non-prime}(Q)$ is of the form $Q_i = Q + a_i^2$ (in case (Q, Q_i) not radical) or $Q_i = Q + zb_i$ where $b_i \notin (z)$ (in case (Q, Q_i) is radical but Q, Q_i, z not regular). Moreover, any $F \in \mathcal{H} \setminus \mathcal{F}_{non-prime}(Q) \cup \{Q\}$ is of the form $F = Q + uv$ where $\dim \operatorname{span}_{\mathbb{K}} \{u, v\} = 2$ and $uv \notin (z)$.

We can assume that for any $U \subset S_1$ for which $z \in U$ and $\dim U \leqslant 4$, we have $|\mathcal{H} \cap \mathbb{K}[Q, U]| \leqslant \delta m$, otherwise we are done by simply taking $\mathcal{A}$ to be any algebra containing Q, U.

We now show that $\mathcal{H} = \mathcal{F}_{non-prime}(Q) \cup \{Q\}$. If $\mathcal{H} \neq \mathcal{F}_{non-prime}(Q) \cup \{Q\}$, let $F \in \mathcal{H} \setminus \mathcal{F}_{non-prime}(Q) \cup \{Q\}$. By the argument above, $F = Q + uv$, where $\dim \operatorname{span}_{\mathbb{K}} \{u, v\} = 2$ and $uv \notin (z)$. Let $U = \operatorname{span}_{\mathbb{K}} \{u, v, z\}$.

If $Q_i \in \mathcal{F}_{non-prime}(Q)$ is such that $Q_i = Q + a_i^2$, then we claim that $a_i \in \mathbb{K}[U]$. Suppose, for the sake of contradiction, that it is not the case. Then Proposition 2.4 implies that $s(\alpha uv + \beta a_i^2) = 1$ for all $\alpha, \beta \neq 0$, which by Proposition A.2 implies (F, Q_i) prime. Thus, there is a polynomial $G \in \mathcal{F}_2 \setminus \{F, Q_i\}$ such that $G \in (F, Q_i)$. But in this case we have $G = (\beta + \gamma)Q + \beta uv + \gamma a_i^2$, for some $\beta, \gamma \neq 0$, which contradicts the fact that $s_{\min}(Q, G) = 0$.

Now, we will prove that if $Q_i \in \mathcal{F}_{non-prime}(Q)$ is such that $Q_i = Q + zb_i$, then $b_i \in \mathbb{K}[U]$. Suppose this is not the case. Then, Proposition B.2 implies that $\alpha uv + \beta zb_i$ is irreducible for all $\alpha, \beta \in \mathbb{K}^*$. Hence, Proposition A.2 implies (F, Q_i) is prime, and analogously to the previous paragraph any $G \in \mathcal{F}_2 \cap (F, Q_i) \setminus \{F, Q_i\}$ is such that $s_{\min}(G, Q) = 1$, which is a contradiction.

The two paragraphs above show that if $\mathcal{H} \neq \mathcal{F}_{non-prime}(Q) \cup \{Q\}$, then there is a quadratic that has no SG dependence, which contradicts the assumption that $\mathcal{F}$ is a SG configuration. Thus, in this case $\mathcal{H} = \mathcal{F}_{non-prime}(Q) \cup \{Q\}$. Now, we can partition $\mathcal{F}_{non-prime}(Q) = \mathcal{N} \sqcup \mathcal{R}$, where $\mathcal{N} := \{Q_i \mid Q + a_i^2 \text{ and } a_i \notin (z)\}$ and $\mathcal{R} := \{Q_i \mid Q_i = Q + zb_i\}$.

If there is $P \in \mathcal{R}$ such that $\dim \operatorname{span}_{\mathbb{K}} \{z, b\} = 2$, then $\mathcal{F}_{non-prime}(Q) = \mathcal{R}$ or $\mathcal{N} \in \mathbb{K}[Q, z, b]$. To see this, note that if $Q_1 \in \mathcal{N} \setminus \mathbb{K}[Q, z, b]$, we have that $Q_1 = Q + a_1^2$ where $a_1 \notin (z, b)$. Thus, Proposition B.2 implies $s(\alpha zb + \beta a_1^2) = 1$ for all $\alpha, \beta \in \mathbb{K}^*$, which implies that (P, Q_1) is prime. By the SG condition, there must be $R \in (P, Q_1) \cap \mathcal{F} \setminus \{P, Q_1\}$, which implies $R = \alpha P + \beta Q_1$ for $\alpha, \beta \in \mathbb{K}^*$, which contradicts the fact that $s_{\min}(R, Q) = 0$.

Since $\mathcal{F}_2 = \Lambda \cup \mathcal{H}$, $\Lambda \subset \mathbb{K}[W]$ and $\mathcal{H} = \mathcal{F}_{non-prime}(Q) \cup \{Q\}$, by the above paragraph we have two cases to analyze: $\mathcal{F}_2 \setminus \mathbb{K}[Q, W] \subset \mathcal{N}$ or $\mathcal{F}_2 \setminus \mathbb{K}[Q, W] \subset \mathcal{R}$.

Case 2.1: If $\mathcal{F}_2 \setminus \mathbb{K}[Q, W] \subset \mathcal{N}$, let $\{Q_1, \ldots, Q_s\} = \mathcal{F}_{non-prime}(Q) \setminus \mathbb{K}[Q, W]$. Hence, we have $Q_i = Q + a_i^2$, where $a_i \notin (W)$. If $s > 0$, we claim that $Q_i \in \mathbb{K}[Q, W, a_1]$, which concludes this case. To see this, suppose $a_2 \notin (W, a_1)$. This implies Q_1, Q_2, z is a regular sequence, and (Q_1, Q_2) is radical. Thus, there is $j \neq 1, 2$ such that $Q_j \in (Q_1, Q_2)$, which implies $a_j^2 = \alpha a_1^2 + \beta a_2^2$ with $\alpha, \beta \in \mathbb{K}^*$, which is a contradiction.

Case 2.2: If $\mathcal{F}_2 \backslash \mathbb{K}[Q,W] \subset \mathcal{R}$, let $\{Q_1, \dots, Q_s\} = \mathcal{F}_{non-prime}(Q) \backslash \mathbb{K}[Q,W]$. Hence, we have $Q_i = Q + zb_i$, where $b_i \notin (W)$. W.l.o.g., we can assume that $\mathcal{L} := \{b_1, \dots, b_r\}$ is a maximal subset of $\{b_1, \dots, b_s\}$ such that $b_j \notin (b_i)$ for all $i \neq j$. We now show that $\mathcal{L}$ forms a $(\dim(W), 1)$-linear-SG configuration, which will end this case. Let $i \neq j \in [r]$. Since $(Q_i, Q_j) = (Q, z) \cap (Q_i, b_i - b_j)$, we have that (Q_i, Q_j) is radical, and by the saturated SG configuration condition, there is $F_k \in \mathcal{F}$ such that $zF_k \in (Q_i, Q_j)$, which implies $F_k \in (Q_i, b_i - b_j)$. If $F_k \in \mathbb{K}[W]$, we have that $\mathrm{span}_{\mathbb{K}} \{b_i, b_j\} \cap W \neq 0$, which means that b_i, b_j forms a SG pair over W. Else, we must have $F_k = Q + zb_k$ for some $b_k \notin W$, where $k \in [s]$. This implies that $F_k \in (Q_i, Q_j)$. Hence, there is $\alpha \in \mathbb{K} \backslash \{0, 1\}$ such that $F_k = \alpha Q_i + (1-\alpha) Q_j \Rightarrow b_k = \alpha b_i + (1-\alpha) b_j$, which implies that $b_k \in \mathcal{L} \backslash \{b_i, b_j\}$, which concludes the proof that $\mathcal{L}$ is a $(\dim(W), 1)$-linear-SG configuration. ☐

Lemma 6.13. *If there exists $Q \in \mathcal{F}_2$ such that $s(Q) \leqslant 3$ and $Q \notin \mathcal{F}_{span}$, then there exists $V \subset S_1$ with $\dim(V) = O(1)$ such that $\mathcal{F} \subset \mathbb{K}[V]$.*

Proof. Let $|\mathcal{F}_1| = r$, $|\mathcal{F}_2| = s$ and $m = |\mathcal{F}| = r + s$. By Lemma 6.11, we can assume that $s \geqslant m/4$, otherwise we are done. Let $W = \mathrm{Lin}(Q) + \mathrm{span}_{\mathbb{K}} \{z\}$. Since $s(Q) \leqslant 3$, we have that $\dim W \leqslant 9$. If $|\mathcal{F} \cap \mathbb{K}[W]| \geqslant \delta m$ we are done by Lemma 6.10, hence we will assume this is not the case.

Since $Q \notin \mathcal{F}_{span}$, denoting by $\mathcal{B} := \mathcal{F}_2 \backslash (\mathcal{F}_{span}(Q) \cup \mathbb{K}[W]) = \{Q_1, \dots, Q_t\}$, we have that $t \geqslant s - 2\delta m$. Now, if $Q_i \in \mathcal{B}$, then one of the following holds:

- Q, Q_i, z is not a regular sequence, which implies $Q_i \in (W)$ and $s(Q_i) \leqslant s(Q) + 1$, as Q_i must be contained in a minimal prime of (Q, z) and all minimal primes of (Q, z) are in (W)
- $Q_i \notin (W)$ and Q, Q_i is not radical: in this case by Proposition A.2 we must have $Q_i = Q - y_i^2$, for some $y_i \notin W$.

In both cases above, after a general projection φ mapping $W \overset{\varphi}{\mapsto} z$, we have

$$Q_i \mapsto \begin{cases} z \cdot u_i, \ u_i \notin (W) \\ (y_i - \alpha_i z)(y_i - \beta_i z), \ y_i \in S_1/(W) \text{ and } \alpha_i, \beta_i \in \mathbb{K}^* \end{cases}$$

After relabeling the Q_i's, we can assume that $\varphi(Q_i) = zu_i$ for $i \leqslant a$ and $\varphi(Q_i) = (y_i - \alpha_i z)(y_i - \beta_i z)$ for $a < i \leqslant t$. Note that the above implies that $s(Q_i) \leqslant s(Q) + 1 \leqslant 4$ for any form in $\mathcal{B}$, and Proposition 6.9 implies that $s(F) \leqslant 6$ for any form $F \in \mathcal{F}_2$.

We now show that every form in $\mathcal{F}_2$ will factor after projection. Suppose that is not the case, that is, there is $F \in \mathcal{F}_2$ such that $\varphi(F)$ is irreducible. Hence, we must have that $F \notin (W) \cup \mathcal{B}$. Let $U = \mathrm{Lin}(F) + W$. As $s(F) \leqslant 6$, we have that $\dim(U) \leqslant 23$. By Lemma 6.10, we can assume that $|\mathcal{F} \cap \mathbb{K}[U]| \leqslant \delta m$, otherwise we are done. For each $Q_i \in \mathcal{B}$ such that $Q_i \notin \mathbb{K}[U]$, let $G_i \in \mathcal{F}$ such that $zG_i \in \mathrm{rad}(F, Q_i)$. After projection, we have $z\varphi(G_i) \in \mathrm{rad}(\varphi(F), zu_i)$ or $z\varphi(G_i) \in \mathrm{rad}(\varphi(F), (y_i - \alpha_i z)(y_i - \beta_i z))$. In either case, there is a linear form $\ell_i \notin U$ such that $\varphi(G_i) \in (\varphi(F), \ell_i)$ and the latter ideal is prime, since $\varphi(F)$ is irreducible after quotienting by ℓ_i. As Proposition 2.7 implies that $G_i \notin (\ell_i)$, we must also have that $\varphi(G_i)$ is irreducible after projection. Hence, by the pigeonhole principle we reach a contradiction, as we have $t \geqslant s - 2\delta m$ ideals of the form (F, Q_i) and $\leqslant 2\delta m$ G_i's.

Thus, after projection we have that for any $F \in \mathcal{F}_2$:

- if $F \in (W) \backslash \mathbb{K}[W]$ then $\varphi(F) = zu$, where $u \notin (W)$
- if $F \notin (W)$, then $\varphi(F) = uv$, where $u, v \notin (W)$

Thus, we can write $\mathcal{F}_2 \backslash \mathbb{K}[W] = \{F_1, \dots, F_a\} \cup \{H_1, \dots, H_b\}$, where $F_i \in (W)$ and $H_i \notin (W)$. In particular, we have that $a + b \geqslant s - \delta m$, and after projection, we have that $\varphi(F_i) = z\ell_i$ and $\varphi(H_j) = u_j v_j$ where $\ell_i, u_j, v_j \notin (z)$ for all i, j.

Let $\mathcal{F}_1 = \{z, x_1, \dots, x_{r-1}\}$. We can also assume that after projection, we still have $\varphi(x_i) \notin \varphi(x_j) \cup (z)$ for any $i \neq j \in [r]$, otherwise we can just pick one representative for each $\varphi(x_i)$. Hence, for simplicity we will denote $\varphi(x_i)$ by x_i.

Let $\mathcal{G}_1 := \{z, x_1, \dots, x_{r-1}\} \cup \{\ell_1, \dots, \ell_a\} \cup \{u_1, v_1, \dots, u_b, v_b\}$. That is, $\mathcal{G}_1$ is the set of all linear forms appearing in $\varphi(\mathcal{F})$. By Proposition 2.7 and the fact that the forms in $\mathcal{F}$ are all pairwise coprime, we have that the set of linear forms in $\mathcal{G}_1$ are pairwise linearly independent.

We will now prove that the linear forms in the "projected configuration" $\mathcal{G}_1$ have small vector space dimension. To do that, we will prove that $\mathcal{G}_1$ is a $(1, 1/2)$-linear-SG configuration, which follows from the following observation: let $F_i, F_j \in \mathcal{F} \backslash \mathbb{K}[W]$, and let $\ell_i \mid \varphi(F_i)$, $\ell_j \mid \varphi(F_j)$ such that $\ell_i \ell_j \notin (z)$. If $z \in (\ell_i, \ell_j)$ then we are done, since ℓ_i, ℓ_j would be a valid SG pair. If $\dim \mathrm{span}_{\mathbb{K}} \{\ell_i, \ell_j, z\} = 3$, then

$$zF_k \in \mathrm{rad}(F_i, F_j) \Rightarrow z\varphi(F_k) \in \mathrm{rad}(\varphi(F_i), \varphi(F_j)) \subset (\ell_i, \ell_j) \Rightarrow \varphi(F_k) \in (\ell_i, \ell_j)$$

as $z \notin (\ell_i, \ell_j)$ we have $F_k \notin \mathbb{K}[W]$, otherwise $\varphi(F_k) = z^2$, so there is $\ell_k \mid \varphi(F_k)$ and $\ell_k \in \mathcal{G}_1$. $\square$

We are now ready to prove the main result of this section: saturated SG configurations can only happen in small subalgebras.

Theorem 6.14. *Let $z \in S_1$ be a non-zero linear form. If $\mathcal{F}$ is a z-saturated radical SG configuration, then*

$$\dim(\mathrm{span}_{\mathbb{K}} \{\mathcal{F}\}) = O(1).$$

Proof. Let $m := |\mathcal{F}|$, $\delta = 1/30$, and define $\mathcal{F}_{span}$ with respect to δ. By Lemma 6.11 we can assume that $|\mathcal{F}_2| \geqslant m/4$, otherwise we are done. We can also assume that there is no $(24/\delta + 10, 3)$-wide vector space W such that $\dim(W) = O(1)$ and $|\mathcal{F} \cap \mathbb{K}[W]| \geqslant \delta m$, otherwise Lemma 6.10 implies that we are done.

By Lemma 6.12 and Lemma 6.13, we have that $\mathcal{F}_2 = \mathcal{F}_{span}$, otherwise we are done. Hence, $\mathcal{F}_2$ is a $(0, \delta)$-linear-SG configuration, and Theorem 6.2 implies that $\dim \mathrm{span}_{\mathbb{K}} \{\mathcal{F}_2\} = O(1/\delta) = O(1)$.

Now, by Proposition 4.11 applied to $\mathrm{span}_{\mathbb{K}} \{z, \mathcal{F}_2\}$, we can construct a $(10, 1)$-wide vector space W such that $\{z\} \cup \mathcal{F}_2 \subset \mathbb{K}[W]$ and $\dim(W) = O(1)$. With W at hand, we will prove that $\mathcal{F}_1$ is a $(\dim(W), 1)$-linear-SG configuration, which will finish the proof.

For any $x_i, x_j \in \mathcal{F}_1$, there exists $F_k \in \mathcal{F}$ such that $zF_k \in (x_i, x_j)$. If x_i, x_j, z are not a regular sequence, then $z \in (x_i, x_j)$ and thus x_i, x_j is a valid $(\dim(W), 1)$-SG pair. On the other hand, if x_i, x_j, z is a regular sequence, then we must have $F_k \in (x_i, x_j)$. If $F_k \in \mathcal{F}_1$ we have that x_i, x_j, F_k are a SG triple. Otherwise, $F_k \in \mathcal{F}_2$ and $s(F_k) \leqslant 1$, which implies that $\mathrm{span}_{\mathbb{K}} \{x_i, x_j\} \cap W_1 \neq 0$. Thus, either $\mathrm{span}_{\mathbb{K}} \{x_i, x_j\} \cap W_1 \neq 0$, in which case (x_i, x_j) is a valid $(\dim(W), 1)$-SG pair, or $\mathrm{span}_{\mathbb{K}} \{x_i, x_j\} \cap W_1 = 0$ in which case there exists $x_k \in \mathcal{F}_1$ such that $x_k \in x_i, x_j$. $\square$

As a corollary of the results above, we have that 3-radical-SG configurations which are contained in small linear ideals must be low dimensional. This is essentially the content of the next corollary. However, we slightly generalize it to also work for $(2, W)$-radical-SG configurations, and hence the statement looks a bit more technical.

Corollary 6.15. *Let $w, t \in \mathbb{N}$ with $t \geqslant 2$ and $W = W_1 + W_2$ be a (w, t)-wide graded vector space. Let $\mathcal{F} = \{F_1, \cdots, F_m\} \subset S_{\leqslant 3}$ be a finite set of irreducible forms of degree at most 3 such that $F_i \notin (F_j)$ for* *any* $i \neq j$. *Suppose that for any two distinct* $F_i, F_j \in \mathcal{F}$, *either* $|\operatorname{rad}(F_i, F_j) \cap \mathcal{F}| \geqslant 3$ *or there exists a form* $G \in \mathbb{K}[W]$, *of degree at most* 2 *such that* $G \in \operatorname{rad}(F_i, F_j)$. *If* $\mathcal{F} \subset (W_1)$, *then*

$$\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{F}\} = O((\dim W)^4).$$

Moreover, there exist constants C_1, C_2 *such that for any* $w', t' \in \mathbb{N}$, *there exists a* (w', t')*-wide vector space* V *such that* $\mathcal{F} \cup \mathbb{K}[W] \subset \mathbb{K}[V]$ *with*

$$\dim(V_1) \leqslant (2t'+1)^{(C_1 \dim(W)+1)}(C_2(\dim W)^2 + w')$$

and $\dim(V_2) = O(\dim(W))$. *In particular,* $\dim(V) \leqslant B(w', t', \dim(W))$ *for some function* $B : \mathbb{N}^3 \to \mathbb{N}$.

Proof. Let $\varphi : S \to S[z]/(W_1)$ be a general projection, where z is free from $x_1, \ldots, x_n$. Denote by $R := S[z]/(W_1)$. Since $\mathcal{F} := \{F_1, \ldots, F_m\} \subset (W_1)$, we have that $\varphi(F_i) = z^{d_i} G_i$, for some $d_i \geqslant 1$. Note that $G_i \in R \backslash (z)$ is square-free by Proposition 2.6. Let $\mathcal{G} = \{z, G_1, \cdots, G_m\}$.

We will show that the set $\mathcal{G}$ is a z-saturated radical SG configuration in R. By Proposition 2.7, we know that for all i, j, the polynomials G_i, G_j do not have a common factor and $z \nmid G_i$. Hence any two distinct elements of $\mathcal{G}$ do not have a common factor. For $G_i, G_j \in \mathcal{G}$, suppose that there is $k \neq i, j$ such $F_k \in \operatorname{rad}(F_i, F_j)$. Then we have $z^{d_k} G_k \in \operatorname{rad}(z^{d_i} G_i, z^{d_j} G_j) \subset \operatorname{rad}(G_i, G_j)$. Thus, we have $zG_k \in \operatorname{rad}(G_i, G_j)$.

For some $G_i, G_j \in \mathcal{G}$, suppose there exists $G \in \operatorname{rad}(F_i, F_j) \cap \mathbb{K}[W]$ with $\deg(G) \leqslant 2$. Then $G \in (W_1) \cap \mathbb{K}[W]$. Since W is (w, t)-wide and $\deg G \leqslant 2$, by Proposition 4.9 $G \in \mathbb{K}[W_1]$ and $\varphi(G) = z^d$ for some $d \geqslant 1$. Therefore $z^d \in \operatorname{rad}(z^{d_i} G_i, z^{d_j} G_j) \subset \operatorname{rad}(G_i, G_j)$. Hence $z \in \operatorname{rad}(G_i, G_j)$.

Therefore, $\mathcal{G}$ is a z-saturated SG configuration and by Theorem 6.14 $\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{G}\} = O(1)$. Hence, there exists a constant $D > 0$ such that for a general projection φ, $\dim \operatorname{span}_{\mathbb{K}} \{\varphi(\mathcal{F})\} \leqslant D$. By Corollary 2.12, we conclude that $\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{F}\} = O((\dim W)^4)$.

Also, by Corollary 2.12, there exists a graded vector space $U = U_1 + U_2$, such that $\mathcal{F} \subset \mathbb{K}[U + W]$ where $\dim(U_1) = O(\dim(W)^2)$ and $\dim(U_2) = O(\dim(W))$. For any $w', t' \in \mathbb{N}$, we apply Proposition 4.11 to the graded vector space $(U_1 + W) + U_2$ to obtain a (w', t')-wide vector space $V = V_1 + V_2$ such that $\mathcal{F} \cup \mathbb{K}[W] \subset \mathbb{K}[V]$. Furthermore, by Proposition 4.11, there exist constants C_1, C_2 such that

$$\dim(V_1) \leqslant (2t'+1)^{(C_1 \dim(W)+1)}(C_2(\dim W)^2 + w')$$

and $\dim(V_2) = O(\dim(W))$. □

6.4 Cubic Sylvester-Gallai over a small algebra

We now prove that if a 3-radical-SG configuration has a constant fraction of its elements in a small wide algebra, then there is a slightly larger wide algebra which contains the entire configuration.

Proposition 6.16. *Let* $0 < \nu < 1$ *be a constant. There exists a function* $B_\nu : \mathbb{N}^3 \to \mathbb{N}$ *such that the following holds:*

Let $\mathcal{F} = \{F_1, \cdots, F_m\} \subset S_{\leqslant 3}$ *be a set of irreducible forms of degree at most* 3 *such that* $|\mathcal{F}| \geqslant 2^{11}/\nu$ *and* $W = W_1 + W_2$ *be a* (w, t)*-wide AH-vector space. Suppose* $\mathcal{F}$ *is a* 3-radical-SG *configuration and we have*

1. *(Low strength). For all* $F \in \mathcal{F}_{\geqslant 2}$ *we have* $3s(F) < w$ *and* $3^{s(F)+3} < t$.
2. *(constant fraction in the algebra). we have* $|\mathcal{F} \cap \mathbb{K}[W]| \geqslant \nu|\mathcal{F}|$.

Then there exists a (w, t)*-wide vector space* V *such that* $\mathcal{F} \cup \mathbb{K}[W] \subset \mathbb{K}[V]$ *and* $\dim(V) \leqslant B_\nu(w, t, \dim W)$.

Proof. We will construct the vector space V iteratively. At each step of the iterative process we will preserve the property that $\mathbb{K}[W] \subset \mathbb{K}[V]$ and we will increase the cardinality of $\mathcal{F} \cap \mathbb{K}[V]$.

1. Set $V = W$.
2. While $\mathcal{F} \not\subset \mathbb{K}[V]$:
   - If $\mathcal{F} \subset (V_1)$, then apply Corollary 6.15 to the set $\mathcal{F} \cap (V_1)$ and the algebra $\mathbb{K}[V]$, to obtain a (w, t)-wide vector space U such that $\mathcal{F} \cup \mathbb{K}[V] \subset \mathbb{K}[U]$. Set $V = U$ and stop.
   - Else:
     - Pick $P \in \mathcal{F} \backslash (V_1)$. Apply Proposition 4.15 to P and V to obtain a (w_1, t_1)-wide vector space U such that $P \in \mathbb{K}[U]$ and $V \subset U$, where $w_1, t_1 \in \mathbb{N}$ satisfy the inequalities in Proposition 4.15.
     - Apply Corollary 6.15 to $\mathcal{F} \cap (U_1)$ and $\mathbb{K}[U]$ to obtain a (w, t)-wide vector space U' such that $(\mathcal{F} \cap (U_1)) \cup \mathbb{K}[U] \subset \mathbb{K}[U']$. Set $V = U'$.

Termination. We will show that this iterative process terminates after at most $2^{10}/\nu$ iterations of the While loop. First, we note that each step of the iterative process preserves the inclusion $\mathbb{K}[W] \subset \mathbb{K}[V]$. Therefore we always have $|\mathcal{F} \cap \mathbb{K}[V]| \geqslant \nu m$.

Suppose we have $\mathcal{F} \not\subset (V_1)$. Let $P \in \mathcal{F} \backslash (V_1)$ and U be the (w_1, t_1)-wide vector space as constructed above. Since $P \notin \mathbb{K}[V]$, there is a generator $x \in U \backslash V$ such that P depends on x. Since $\deg(P) \leqslant 3$, the leading coefficient of x in $P \in \mathbb{K}[U]$ has degree at most 2. Thus there can be at most two forms $F_i \in \mathcal{F} \cap \mathbb{K}[V]$, such that the coefficient of x in $P \in \mathbb{K}[U]$ is divisible by F_i. Therefore, by Lemma 3.18, we know that $\mathrm{rad}(P, F_i) \cap \mathbb{K}[V] = (F_i)$ for at least $\nu m - 2$ forms $F_i \in \mathcal{F} \cap \mathbb{K}[V]$. Hence, for each such F_i, there exists $G_i \in \mathrm{rad}(P, F_i) \cap \mathcal{F}$, such that $G_i \notin \mathbb{K}[V]$. As $P, F_i \in \mathbb{K}[U]$, by Proposition 4.9 we know that $P, F_i \in (U_1)$ and hence $\mathrm{rad}(P, F_i) \subset (U_1)$. Therefore, $G_i \in (U_1) \cap \mathcal{F}$ for at least $\nu m - 2$ forms $F_i \in \mathbb{K}[V]$.

Let U' be the vector space constructed in the iterative process above. Then we have $G_i \in \mathbb{K}[U'] \backslash \mathbb{K}[V]$ for at least $\nu m - 2$ forms F_i. Since $m \geqslant 2^{11}/\nu$, we know that $(\nu m - 2)/2^9 \geqslant 2$. If there exist less than $(\nu m - 2)/2^9$ number of distinct G_i's, then by the pigeonhole principle, there exist at least 2^9 forms F_j such that some $G_k \in \mathrm{rad}(P, F_j)$ for fixed k. By Lemma 3.9, we must have $\mathrm{rad}(P, F_i) = \mathrm{rad}(P, F_j)$ for some $i \neq j$, which is a contradiction since $\mathrm{rad}(P, F_i) \cap \mathbb{K}[V] = (F_i)$ for all such F_i. Hence we must have at least $(\nu m - 2)/2^9$ number of distinct $G_i \in \mathbb{K}[U'] \backslash \mathbb{K}[V]$. Therefore, at this step of the iterative process, when we update $V = U'$, the cardinality $|\mathcal{F} \cap \mathbb{K}[V]|$ increases by at least $(\nu m - 2)/2^9$. Hence, after each iteration of the While loop, the cardinality $|\mathcal{F} \cap \mathbb{K}[V]|$ increases by at least $(\nu m - 2)/2^9$. Since $|\mathcal{F}| = m$, the iterative process stops after $\leqslant 2^{10}/\nu$ steps.

Dimension bound. If we apply Proposition 4.15 to a polynomial P and the vector space V to obtain the vector space U, then $\dim(U) \leqslant A(w, t, \dim(V))$ for the function $A : \mathbb{N}^3 \to \mathbb{N}$ in Proposition 4.15. Further, if we apply Corollary 6.15 to $\mathcal{F} \cap (U_1)$ and $\mathbb{K}[U]$ to obtain the vector space U', then $\dim(U')$ is bounded above by $B(w, t, \dim(U))$, where $B : \mathbb{N}^3 \to \mathbb{N}$ is the function in Corollary 6.15. Therefore after the i-th step of the iterative process $\dim(V)$ is bounded above by $B_i(w, t, \dim(W))$ for some function $B_i : \mathbb{N}^3 \to \mathbb{N}$. Indeed, we may take $B_i = B(w, t, A(w, t, B_{i-1}))$. Since the iterative process terminates after at most $2^{10}/\nu$ steps, we have $\dim(V) \leqslant B_k(w, t, \dim(W))$ where $k = \lceil \frac{2^{10}}{\nu} \rceil$. $\square$

Corollary 6.17. *Let $0 < \nu < 1$ be a constant. There exists a function $D_\nu : \mathbb{N}^3 \to \mathbb{N}$ such that the following holds:*

*Given a 3-*radical-SG *configuration $\mathcal{F}$, and a (w, t)-wide AH-vector space $W = W_1 + W_2$ such that*

1. *(Low strength). For all $F \in \mathcal{F}_{\geqslant 2}$ we have $3s(F) < w$ and $3^{s(F)+3} < t$.*
2. *(constant fraction in the algebra). we have $|\mathcal{F} \cap \mathbb{K}[W]| \geqslant \nu|\mathcal{F}|$.*

There exists a (w, t)-wide vector space V such that $\mathcal{F} \cup \mathbb{K}[W] \subset \mathbb{K}[V]$ and $\dim(V) \leqslant D_\nu(w, t, \dim W))$.

Proof. Let $\mathcal{F} \cap S_i = \mathcal{F}_i$. If $|\mathcal{F}| \leqslant 2^{10}/\nu$. Then by Corollary 4.13, we have a vector space $U = U_1 + U_2$ such that $\mathcal{F}_3 \subset \mathbb{K}[U]$ and $\dim(U) \leqslant C_\nu(w, t)$ for some function C_ν. Consider the graded vector space $U' = (\text{span}_{\mathbb{K}} \{\mathcal{F}_1\} + U_1 + W_1) + (\text{span}_{\mathbb{K}} \{\mathcal{F}_2\} + U_2 + W_2)$. Note that the $\dim(U')$ is bounded above by a function of $\nu, w, t, \dim(W)$. Therefore, by Proposition 4.11, we have a (w, t)-wide vector space V such that $\mathcal{F} \cup \mathbb{K}[W] \subset \mathbb{K}[V]$ and $\dim(V) \leqslant A_\nu(w, t, \dim W)$, for some function A_ν.

Otherwise, we may assume that $|\mathcal{F}| > 2^{10}/\nu$. Then by Proposition 6.16, we have a (w, t)-wide vector space V such that $\mathcal{F} \cup \mathbb{K}[W] \subset \mathbb{K}[V]$ and $\dim(V) \leqslant B_\nu(w, t, \dim W)$. We may take $D_\nu = \max(A_\nu, B_\nu)$. □

Proposition 6.18. *Let $0 < \delta < 1$ and $w, t \in \mathbb{N}$. Let $\mathcal{F} = \{\mathbb{F}_1, \cdots, F_m\} \subset S_{\leqslant 3}$ be a set of irreducible forms such that $F_i \notin (F_j)$ for $i \neq j$. Let $U = U_1 + U_2$ be a (w, t)-wide vector space such that $|\mathcal{F} \cap \mathbb{K}[U]| \geqslant (1-\delta)|\mathcal{F}|$. Let $F \in \mathcal{F} \backslash \mathbb{K}[U]$ be a cubic form such that $3s(F) < w$ and $3^{s(F)+3} < t$. Then there are at most $2^9 \cdot \delta \cdot |\mathcal{F}| + 2$ forms $G \in \mathcal{F} \cap \mathbb{K}[U]$ such that $|\operatorname{rad}(F, G) \cap \mathcal{F}| \geqslant 3$.*

Proof. By Proposition 4.15, we have a (w_1, t_1)-wide vector space $V = V_1 + V_2$ such that $F \in \mathbb{K}[V]$ and $U_i \subset V_i$. Since $F \notin \mathbb{K}[U]$, there exists a generator $x \in \mathbb{K}[V]$ such that F depends on x. Suppose there exist r forms $G_1, \cdots, G_r \in \mathcal{F} \cap \mathbb{K}[U]$ such that $|\operatorname{rad}(F, G_i) \cap \mathcal{F}| \geqslant 3$. Since $\deg(F) = 3$, note that there exist at most two forms $G_j \in \mathcal{F} \cap \mathbb{K}[U]$ such that G_j divides the leading coefficient of x in $F \in \mathbb{K}[V]$. Then by Lemma 3.18, we have $\operatorname{rad}(F, G_i) \cap \mathbb{K}[U] = (G_i)$ for at least $r - 2$ forms G_i. Therefore, if $H_i \in \operatorname{rad}(F, G_i) \cap \mathcal{F}$, then we must have $H_i \notin \mathbb{K}[U]$ for at least $r-2$ forms $G_i \in \mathcal{F} \cap \mathbb{K}[U]$. Note that by assumption $|\mathcal{F} \backslash \mathbb{K}[U]| < \delta m$. If $r - 2 > 2^9 \delta m$, then by the pigeon-hole principle, there exists $H_k \in \mathcal{F}$ such that $H_k \in \operatorname{rad}(F, G_i)$ for at least $2^9 + 1$ such G_i's. Now, by Lemma 3.9, we must have $\operatorname{rad}(F, G_i) = \operatorname{rad}(F, G_j)$ for some $i \neq j$, which is a contradiction. Therefore $r \leqslant 2^9 \delta m + 2$. □

6.5 Radical Sylvester-Gallai configurations within wide quadratic ideals

In this section, we consider a special kind of 3-radical-SG configuration $\mathcal{F}$, where the entire SG configuration is contained in a prime ideal (Q, x, y) generated by a strong quadratic form Q and linear forms x, y. We will show that the span of these special SG-configurations have constant dimension.

Let $\mathcal{F}$ be a 3-radical-SG configuration. Let (Q, x, y) be a prime ideal such that $\mathcal{F} \subset (Q, x, y)$. We partition $\mathcal{F}$ as

$$\mathcal{F} := \{F_1, \ldots, F_r\} \sqcup \{G_1, \ldots, G_s\} \sqcup \{H_1, \ldots, H_t\}$$

where $G_i \in (x, y)$, $F_i \in S_3 \cap (Q, x, y) \backslash (x, y)$ and $H_i \in S_2 \cap (Q, x, y) \backslash (x, y)$. Henceforth, we shall always work with such partitions. We begin with a remark on the structure of such configurations, which allows us to drop the quadratics H_i from our configuration.

Proposition 6.19. *Let $x, y \in S_1$ and $Q \in S_2$ be such that $s(Q) \geqslant 5$. Let $\mathcal{F} = \{F_1, \dots, F_r\} \sqcup \{G_1, \dots, G_s\} \sqcup \{H_1, \dots, H_t\}$ be a 3-radical-SG configuration as above where $\mathcal{F} \subset (Q, x, y)$. Then, $\{F_1, \dots, F_r\} \sqcup \{G_1, \dots, G_s\}$ is also a 3-radical-SG configuration.*

Proof. Let $\mathcal{G} := \{F_1, \dots, F_r\} \sqcup \{G_1, \dots, G_s\}$. We write $H_k = Q + R_k$, where $R_k \in (x, y) \cap S_2$ and let $z_i \in S/(x, y)$ be such that $F_i \equiv z_i Q \bmod (x, y)$ and $z_i \neq 0$. It is enough to show that given two forms $P_1, P_2 \in \mathcal{G}$, we must have $\mathrm{rad}(P_1, P_2) \cap \mathcal{F} = \mathrm{rad}(P_1, P_2) \cap \mathcal{G}$. And to show this, it is enough to show that $H_k \notin \mathrm{rad}(P_1, P_2)$ for any $k \in [t]$. We have three cases to consider:

1. $P_1 = F_i$, $P_2 = F_j$. In this case, $H_k \in \mathrm{rad}(F_i, F_j) \Rightarrow Q \in \mathrm{rad}(z_i Q, z_j Q)$ in $S/(x, y)$. Hence $Q \in (z_i, z_j)$ in $S/(x, y)$, which is a contradiction since $s(Q) \geqslant 5$.

2. $P_1 = F_i, P_2 = G_j$. In this case, $H_k \in \mathrm{rad}(F_i, G_j) \Rightarrow Q \in \mathrm{rad}(z_i Q)$ in $S/(x, y)$ which is a contradiction.

3. $P_1 = G_i, P_2 = G_j$. In this case, $H_k \in \mathrm{rad}(G_i, G_j) \Rightarrow H_k \in (x, y)$ which is a contradiction.

□

We also need the following facts:

Proposition 6.20. *Let $F_i = x_i Q - a^2 y_i$ be non-associate irreducible forms for $i \in [3]$, where $a, x_i, y_i \in S_1$ and $Q \in S_2$ such that $s(Q) \geqslant 10$. Suppose $x_2 \notin (a, x_1, y_1)$. Then we have*

$$\mathrm{rad}(F_1, F_2) = \mathrm{rad}(F_1, F_3) \Leftrightarrow F_3 \in (F_1, F_2).$$

Moreover, if $x_1 y_2 - x_2 y_1 \notin (a)$ and is square-free, then $F_3 \in \mathrm{rad}(F_1, F_2) \Leftrightarrow F_3 \in (F_1, F_2)$.

Proof. Since F_i, F_j are non-associate forms, we have $F_3 \in (F_1, F_2) \Rightarrow (F_1, F_2) = (F_1, F_3) \Rightarrow \mathrm{rad}(F_1, F_3) = \mathrm{rad}(F_1, F_2)$.

For the other direction, let $\mathcal{B} := \mathbb{K}[a, x_1, x_2, x_3, y_1, y_2, y_3]$ and $\mathcal{A} := \mathbb{K}[a, x_1, x_2, x_3, y_1, y_2, y_3, Q]$. Let $W = \mathrm{span}_{\mathbb{K}} \{a, x_1, x_2, x_3, y_1, y_2, y_3\}$. Let $w_1, \cdots, w_k$ be a basis of W for some $k \leqslant 7$. Note that $w_1, \cdots, w_k, Q$ is a prime sequence, as $s(Q) \geqslant 10$. Therefore the polynomial ring S is intersection flat over the subalgebras $\mathcal{A}$ and $\mathcal{B}$. By applying Proposition 3.28 in the algebra $\mathcal{A}$ and eliminating the variable Q, we see that the radical of the elimination ideal is given by $\mathrm{rad}((F_1, F_2) \cap \mathcal{B}) = \mathrm{rad}(\mathrm{Res}_Q(F_1, F_2)) = \mathrm{rad}(a(x_1 y_2 - x_2 y_1))$. We have two cases to analyze:

Case 1: If $x_1 y_2 - x_2 y_1$ is irreducible, then Lemma 3.33 implies $(F_1, F_2, x_1 y_2 - x_2 y_1)$ is prime. Thus, $F_3 \in \mathrm{rad}(F_1, F_2) \Rightarrow F_3 \in (F_1, F_2, x_1 y_2 - x_2 y_1)$. Let $F_3 = \alpha F_1 + \beta F_2 + \ell(x_1 y_2 - x_2 y_1)$. Since $F_3 \in (Q, a^2)$ and $S/(Q)$ is a UFD, we must have $a^2 | \ell(x_1 y_2 - x_2 y_1)$ in $S/(Q)$. If $\ell \neq 0$, then $x_1 y_2 - x_2 y_1 \in (Q, a)$, which is a contradiction since $s(Q) \geqslant 10$. Therefore, we must have $\ell = 0$ and $F_3 \in (F_1, F_2)$.

Case 2: $x_1 y_2 - x_2 y_1 = gh$ for some $g, h \in S_1$. Here, without loss of generality, we may assume that $g \in (x_1, x_2)$, and writing $g = \alpha_1 x_1 - \alpha_2 x_2$. We have $\alpha_1 \alpha_2 \neq 0$, as $\alpha_1 = 0 \Rightarrow F_2 \in (x_2)$ and $\alpha_2 = 0 \Rightarrow F_1 \in (x_1)$, which contradicts irreducibility of F_1 or F_2. Rearranging $x_1 y_2 - x_2 y_1 = (\alpha_1 x_1 - \alpha_2 x_2) h$, we get that $x_1(y_2 - \alpha_1 h) = x_2(y_1 - \alpha_2 h)$. Hence $y_1 = \beta x_1 + \alpha_2 h$ and $y_2 = \beta x_2 + \alpha_1 h$, for some $\beta \in \mathbb{K}$. Hence, $F_1 = x_1(Q - \beta a^2) - \alpha_2 a^2 h$ and $F_2 = x_2(Q - \beta a^2) - \alpha_1 a^2 h$.

Note that (F_1, g) is a prime ideal. Indeed, we first note that $(F_1, g) \cap \mathcal{B} = (g)$ by Lemma 3.18. Hence the elimination ideal is a prime ideal. Next note that x_1 is a non-zero divisor modulo (F_1, g). Otherwise, there is a minimal prime $\mathfrak{p}$ of (F_1, g) that contains x_1, hence $\mathfrak{p} = (x_1, x_2)$. However $F_1 \notin (x_1, x_2)$, as F_1 is irreducible and $x_2 \notin (a, x_1, y_1)$. Therefore we may apply Proposition 3.29 to conclude that (F_1, g) is prime. Also, we have $\alpha_2 F_2 = \alpha_1 F_1 - g(Q - \beta a^2) \in (F_1, g)$ and hence $\operatorname{rad}(F_1, F_2) \subset (F_1, g)$. Therefore we have $F_3 \in \operatorname{rad}(F_1, F_2) \Rightarrow F_3 \in (F_1, g)$. Thus $F_3 \in (Q, a^2) \cap (F_1, g) \Rightarrow F_3 = \alpha F_1 + gP$, where $P \in (Q, a^2)$, since $g \notin (Q, a)$. In particular, if $P = \mu Q + \nu a^2$ we have $F_3 = (\alpha + \mu\alpha_1)F_1 - \mu\alpha_2 F_2 + (\mu\beta + \nu)ga^2$. We have two subcases to analyze, if $h \in (a)$ or not.

Case 2.1: $h \notin (a)$. Note that $(Q - \beta a^2, h)$ is a prime ideal, as $s(Q) \geqslant 10$. Therefore we have $F_3 \in \operatorname{rad}(F_1, F_2) \Rightarrow F_3 \in (Q - \beta a^2, h)$. Hence, we must have $gP \in (Q - \beta a^2, h)$. If $g = \alpha_1 x_1 - \alpha_2 x_2 \in (Q - \beta a^2, h)$, then we have $h \in (x_1, x_2)$. Hence $y_1 = \beta x_1 + \alpha_2 h \in (x_1, x_2)$ which is a contradiction. Therefore we must have $P = \mu Q + \nu a^2 \in (Q - \beta a^2, h)$ and thus $(\mu\beta + \nu)a^2 \in (Q - \beta a^2, h)$. If $\mu\beta + \nu \neq 0$, then $a \in (Q - \beta a^2, h)$ which is a contradiction as $a \notin (h)$. Therefore $\mu\beta + \nu = 0$ and $F_3 \in (F_1, F_2)$.

Case 2.2: $h \in (a)$. In this case, we have $F_1 = x_1(Q - \beta a^2) - \alpha_2 a^3$ and $F_2 = x_2(Q - \beta a^2) - \alpha_1 a^3$. Then $(F_1, F_2) \subset (Q - \beta a^2, a^3) \cap (F_1, g)$. Therefore the quadratic minimal prime (Q, a) has multiplicity at least 3 in the primary decomposition of (F_1, F_2). Since the cubic minimal prime (F_1, g) must have multiplicity at least 1, we see that (Q, a) and (F_1, g) must be the only minimal primes of (F_1, F_2). Hence, we have $\operatorname{rad}(F_1, F_3) = \operatorname{rad}(F_1, F_2) = (Q, a) \cap (F_1, g)$. Now $x_1 \notin (Q, a) \cup (F_1, g)$ and hence x_1 is a non-zero divisor modulo (F_1, F_3). By Proposition 3.29 a minimal primary decomposition of $(F_1, F_3) \cap \mathcal{B}$ lifts to a minimal primary decomposition of (F_1, F_3).

Note that $\operatorname{Res}_Q(F_1, F_3) = a^2 g((\mu\beta + \nu)x_1 + \mu\alpha_2 a)$. If $\mu\beta + \nu \neq 0$, then $\operatorname{rad}((F_1, F_3) \cap \mathcal{B}) = (a) \cap (g) \cap ((\mu\beta + \nu)x_1 + \mu\alpha_2 a)$ and $((\mu\beta + \nu)x_1 + \mu\alpha_2 a)$ is a minimal prime of $(F_1, F_3) \cap \mathcal{B}$. Hence, we must have that $(\mu\beta + \nu)x_1 + \mu\alpha_2 a \in (Q, a) \cup (F_1, g)$, which is a contradiction. Thus $\mu\beta + \nu = 0$ and $F_3 \in (F_1, F_2)$.

For the moreover part, note that $x_1 y_2 - x_2 y_1 \notin (a)$ corresponds to cases 1 and 2.1 above, where it is sufficient to have $F_3 \in \operatorname{rad}(F_1, F_2)$. $\square$

Proposition 6.21. *Let $Q_1, Q_2 \in S_2$ such that $s(Q_i) \geqslant 20$. Let F_1, F_2 be irreducible cubics of the form $F_i = x_i Q_i - y_i^2 z_i$, where $x_2 \notin \operatorname{span}_{\mathbb{K}}\{x_1, y_1, z_1\}$. If $F \in S_3$ is irreducible such that $F \notin (x_1, y_1, z_1, x_2, y_2, z_2)$ and (F, F_i) is not radical for $i \in \{1, 2\}$ then there is $P \in S_2$ and $\ell \in S_1$ such that $s(P) \geqslant 17$ and $F_1, F_2, F \in (P, \ell^2)$.*

Proof. Note that the algebra $\mathbb{K}[Q_i, x_i, y_i, z_i]$ is $(6, 2)$-wide. If (F, F_i) is contained in a linear minimal prime (x, y), then by Lemma 4.16, we have $x, y \in \operatorname{span}_{\mathbb{K}}\{x_i, y_i, z_i\}$. This is a contradiction as $F \notin (x_1, y_1, z_1, x_2, y_2, z_2)$. Hence (F, F_i) does not have a linear minimal prime. By Corollary 5.3, we know that (F, F_i) does not have minimal prime defining a variety of minimal degree. By Theorem 1.5 and Corollary 5.7, the conditions of the proposition imply that $(F, F_i) \subset (P_i, \ell_i^2)$, where $s(P_i) \geqslant 5$. Thus, for $i \in \{1, 2\}$, we can write $F_i = u_i P_i - \ell_i^2 v_i$, for $u_i, v_i \in S_1$. Note that $u_1 \in (x_1)$, otherwise $x_1 Q_1 \equiv y_1^2 z_1 - \ell_1^2 v_1 \bmod (u_1) \Rightarrow Q_1 \bmod (u_1) \in \mathbb{K}[y_1, z_1, \ell_1, v_1]$ by by Lemma 3.15. This contradicts $s(Q_1) \geqslant 20$. Similarly $u_2 \in (x_2)$. If $u_i = \alpha_i x_i$, then we have $x_i(Q_i - \alpha_i P_i) \in \mathbb{K}[y_i, z_i, \ell_i, v_i]$. Therefore by Lemma 3.15, we have $Q_i - \alpha_i P_i \in \mathbb{K}[y_i, z_i, \ell_i, v_i]$ and hence, $s(P_i) \geqslant s(Q_i) - 3 \geqslant 17$ for $i \in \{1, 2\}$. Moreover, Lemma 5.8 applied to $F_i, (P_i, \ell_i^2)$ implies that $\ell_i \in (x_i, y_i)$ for $i \in \{1, 2\}$.

Since $F \in (P_1, \ell_1^2) \cap (P_2, \ell_2^2)$, we can write $F = a_1 P_1 - \ell_1^2 b_1 = a_2 P_2 - \ell_2^2 b_2$. Note that $a_2 \in (a_1)$, otherwise we would have $a_2P_2 \equiv \ell_2^2 b_2 - \ell_1^2 b_1 \bmod (a_1) \Rightarrow P_2 \bmod (a_1) \in \mathbb{K}[b_1, b_2, \ell_1, \ell_2]$ by contradicting $s(P_2) \geqslant 17$. Thus, we have $\ell_1^2 b_1 \equiv \ell_2^2 b_2 \bmod (a_1)$, which by factoriality of $S/(a)$ implies $\ell_2 \in (a_1, \ell_1)$.

Therefore we have $\ell_2 \in (a_1, \ell_1) \cap (x_2, y_2)$ and $\ell_1 \in (x_1, y_1)$. Hence we must have that $(\ell_2) = (\ell_1)$, otherwise we would have $a_1 \in (\ell_1, \ell_2) \subset (x_1, x_2, y_1, y_2)$, which contradicts $F \notin (x_1, y_1, z_1, x_2, y_2, z_2) \Rightarrow a_1 \notin (x_1, y_1, x_2, y_2)$. Thus, we can take $(\ell) := (\ell_1) = (\ell_2)$. As $a_1 \notin (\ell)$, we may apply Lemma 5.8 to F, to conclude that $(P_1, \ell^2) = (P_2, \ell^2)$ and hence we can take $P = P_1$. $\square$

In the next lemma, we show that such configurations are in fact contained in an ideal of small codimension generated by linear forms.

Lemma 6.22. *Let $\mathcal{F}$ be a 3-radical-SG configuration, $x, y \in S_1$ linearly independent, and let $Q \in S_2$ such that $s(Q) \geqslant 20$ and $\mathcal{F} \subset (Q, x, y)$. If $\{F_1, \dots, F_r\} = \mathcal{F}_3 \cap (Q, x, y) \backslash (x, y)$, that is, such that $F_i \equiv z_i Q \not\equiv 0 \bmod (x, y)$, then there is a $(20, 1)$-wide vector space $V = V_1 + V_2$ such that $\dim V = O(1)$, $\mathcal{F} \subset \mathbb{K}[V]$ and $\operatorname{span}_{\mathbb{K}} \{x, y, z_1, \dots, z_r\} \subset V_1$. In particular, $\dim \operatorname{span}_{\mathbb{K}} \{x, y, z_1, \dots, z_r\} = O(1)$*

Proof. Since we are only interested in the F_i's, by Proposition 6.19 we can assume that $\mathcal{F} = \{F_1, \dots, F_r\} \cup \{G_1, \dots, G_s\}$, where $G_i \in \mathcal{F} \cap (x, y)$. Let $m := |\mathcal{F}|$, where $r + s = m$. Note that $\{G_1, \dots, G_s\}$ is a 3-radical-SG configuration, since (x, y) is a prime ideal. Corollary 6.15 applied with vector space $\operatorname{span}_{\mathbb{K}} \{x, y\}$ implies that $\dim \operatorname{span}_{\mathbb{K}} \{G_1, \dots, G_s\} = O(1)$. Furthermore, there is a $(20, 1)$-wide vector space U with $\dim U = O(1)$ such that $x, y, G_i \in \mathbb{K}[U]$.

We would like to have a wide algebra containing G_i's such that Q is one of the generators of the algebra. Let $U_1' = U_1$ and $U_2' = U_2 + \operatorname{span}_{\mathbb{K}} \{Q\}$. Consider the vector space $U' = U_1' + U_2'$. If Q is sufficiently strong, then we apply Proposition 4.11 to U' and with $H = \operatorname{span}_{\mathbb{K}} \{Q\}$, to obtain a $(40, 3^{30})$-wide algebra W such that $\dim W = O(1)$ such that $x, y, Q, G_i \in \mathbb{K}[W]$. In this case $Q \in W_2$. If Q is not strong enough to apply Proposition 4.11 with $H = \operatorname{span}_{\mathbb{K}} \{Q\}$, then we have $\mathcal{F} \subset (x, y, \operatorname{Lin}(Q))$ and $\dim(\operatorname{span}_{\mathbb{K}} \{x, y\} + \operatorname{Lin}(Q)) = O(1)$. In this case we conclude by applying Corollary 6.15. Therefore we may assume that we have a $(40, 3^{30})$-wide algebra W such that $\dim W = O(1)$ such that $x, y, Q, G_i \in \mathbb{K}[W]$ and $Q \in W_2$.

Let $\delta = 2^{-20}$. We can assume that $s \leqslant \delta m$. Otherwise Proposition 6.16 yields a $(20, 3^6)$-wide V with $\dim(V) = O(1)$ such that $\mathcal{F} \cup \mathbb{K}[W] \subset \mathbb{K}[V]$. Then $z_i Q \bmod (x, y)$ is in the algebra $\mathbb{K}[V/(x, y)]$. Lemma 3.15 implies $z_i \in V_1 + \operatorname{span}_{\mathbb{K}} \{x, y\}$ for all i, and thus we would be done.

Given any form F_i, let $W^{(i)} = W_1^{(i)} + W_2^{(i)}$ be a $(20, 1)$-wide vector space that we obtain when applying Proposition 4.15 to W and F_i. Thus, we have $W \subset W^{(i)}$ and $\dim W^{(i)} = O(1)$. Also, $F_i \in \mathbb{K}[W^{(i)}]$, $z_i, x, y \in W_1^{(i)}$ and $Q \in W_2^{(i)}$. In particular, this implies that $F_j \in (W_1^{(i)}) \Leftrightarrow z_j \in (W_1^{(i)})$. Analogously to the previous paragraph, we can also assume that $\mathcal{F} \cap (W_1^{(i)}) \leqslant \delta m$. Otherwise we could apply Corollary 6.15 to $(W_1^{(i)})$ and Proposition 6.16 to the resulting algebra and be done.

Let $\mathcal{F}_{good}(F_i) := \mathcal{F} \backslash (W_1^{(i)})$. Then $|\mathcal{F}_{good}(F_i)| \geqslant (1 - 2\delta)m$ and $G_j \notin \mathcal{F}_{good}(F_i)$ for any $i \in [r], j \in [s]$. Note that for any $F_j \in \mathcal{F}_{good}(F_i)$, $\dim(x, y, z_i, z_j) = 4$ and (F_i, F_j) is not contained in a linear minimal prime, otherwise we would have $F_j \in (W_1^{(i)})$ by Lemma 4.16. Additionally, let $\mathcal{F}_{span}(F_i) := \{F_j \in \mathcal{F}_{good}(F_i) \mid |\mathcal{F} \cap (F_i, F_j)| \geqslant 3\}$ and define $\mathcal{F}_{span} := \{F_i \in \mathcal{F} \backslash (W) \mid |\mathcal{F}_{span}(F_i)| \geqslant \delta m\}$.

Let $\mathcal{T} := \{F_1, \dots, F_r\} \backslash \mathcal{F}_{span}$. If $|\mathcal{T}| \leqslant 6$ then $\{F_1, \dots, F_r\}$ form a $(\dim W^3 + 6, \delta)$-linear-SG configuration with respect to the vector space $\operatorname{span}_{\mathbb{K}} \{\mathcal{G}, \mathcal{T}\}$ and we are done by Proposition 6.4. Thus, after relabeling, we can assume $\mathcal{T} = \{F_1, \dots, F_t\}$ where $t > 6$, and $\mathcal{F}_{span} = \{F_{t+1}, \dots, F_r\}$.

Note that if $\mathcal{T} \subset (W_1^{(1)})$, we are also done, as in this case we have $\mathcal{F} \cap (W_1^{(1)})$ has $O(1)$ dimension by Corollary 6.15 and $\mathcal{F}_{span}$ forms a $(O(1), \delta)$-linear-SG configuration. Thus, after relabeling we can assume that $F_2 \notin (W_1^{(1)})$. Let $X = X_1 + X_2$ be a $(20, 1)$-wide vector space such that $W^{(1)} + W^{(2)} \subset \mathbb{K}[X]$ and $Q \in X_2$. We now prove that $|\mathcal{F} \cap (X_1)| \geqslant \delta m$, and thus we are done.

Suppose $|\mathcal{F} \cap (X_1)| \leqslant \delta m$. Let $\mathcal{G} := \{F_k \in \mathcal{F} \setminus (X_1) \mid |(F_b, F_k) \cap \mathcal{F}| = 2 \text{ for } 1 \leqslant b \leqslant 2\}$. Since $F_1, F_2 \in \mathcal{T}$ and $|\mathcal{F} \cap (X_1)| \leqslant \delta m$, we have $|\mathcal{G}| \geqslant (1 - 5\delta)m$. Moreover, by definition of X we have that $\mathcal{G} \subset \mathcal{F}_{good}(F_1) \cap \mathcal{F}_{good}(F_2)$. We now show that there is $\ell \in S_1, P \in S_2$ such that $\mathcal{G} \subset (P, \ell^2)$ and $F_1, F_2 \in (P, \ell^2)$.

In what follows, $b \in \{1, 2\}$. For $F_k \in \mathcal{G}$, we have that (F_k, F_b) is not radical, since it does not span a third element of $\mathcal{F}$. By Theorem 1.5 and $F_k \in \mathcal{F}_{good}(F_b)$ we must have that (F_k, F_b) has a quadratic minimal prime. Since $F_k, F_b \in (Q, x, y) \setminus (x, y)$ the quadratic in the minimal prime must have strength $\geqslant 17$. Thus, Lemma 5.6 implies $(F_k, F_b) \subset (P_{kb}, \ell_{kb}^2)$ for $P_{kb} \in S_2$ and $\ell_{kb} \in S_1$ with $s(P_{kb}) \geqslant s(Q) - 3 \geqslant 20$.

In particular, the above implies $F_1 = u_1 P_1 - a_1^2 v_1$ and $F_2 = u_2 P_2 - a_2^2 v_2$, where $a_b, u_b, v_b \in W^{(b)}$ and $P_b \in S_2$ such that $s(P_b) \geqslant 20$. Moreover $F_2 \notin (W^{(1)}) \Rightarrow u_2 \notin (u_1, a_1, v_1)$. Hence, Proposition 6.21 applied to F_1, F_2, F_k, where $F_k \in \mathcal{G}$ implies that there is $P_k \in S_2$ and $\ell_k \in S_2$ such that $F_1, F_2, F_k \in (P_k, \ell_k^2)$. Thus, $\ell_k \in W_1^{(1)}$ for each such k, and in particular we have that $u_2 \notin (\ell_k, \ell_j)$ for any two distinct ℓ_k, ℓ_j. Hence, Remark 5.9 applied to F_2 and the ideals (P_k, ℓ_k^2) imply that $(P_k, \ell_k^2) = (P_j, \ell_j^2)$ for any two distinct $F_j, F_k \in \mathcal{G}$. This proves that there is (P, ℓ^2) such that $\mathcal{G} \subset (P, \ell^2)$ and $s(P) \geqslant 20$.

By the above paragraph, we can write $F_k \in \mathcal{G}$ as $F_k = u_k P - \ell^2 v_k$, as well as $F_1 = u_1 P - \ell^2 v_1$ and $F_2 = u_2 P - \ell^2 v_2$. By Proposition 6.20 and Lemma 3.9, there are at most $2^9 \cdot 5 \cdot \delta m$ $F_k \in \mathcal{G}$ such that $|\operatorname{rad}(F_1, F_k) \cap \mathcal{G}| = 2$. Since $|\mathcal{G}| \geqslant (1 - 5\delta)m \gg 2^9 \cdot 5 \cdot \delta m$, let us take $F_k \in \mathcal{G}$ such that $|\operatorname{rad}(F_1, F_k) \cap \mathcal{G}| \geqslant 3$ and $\operatorname{rad}(F_1, F_k) \cap \mathcal{F} \subset \mathcal{G}$.

By the moreover part of Proposition 6.20, and the fact that $u_1 v_k - u_k v_1$ is square-free, we must have $u_1 v_k - u_k v_1 \in (\ell)$ and thus $u_1 v_k - u_k v_1 = g\ell$ where $g = \alpha u_1 - \beta u_k$ for $\alpha\beta \neq 0$. Let $F_i \in \operatorname{rad}(F_1, F_k)$ be their SG dependency. Since $\ell^2(u_1 v_i - u_i v_1) = \mathrm{Res}_P(F_1, F_i) \in (\ell g)$ we have that $u_1 v_i - u_i v_1 \in (g)$. However, since $\operatorname{rad}(F_1, F_k) \cap \mathcal{F} \subset \mathcal{G}$ we must have $\operatorname{rad}(F_1, F_i) \cap \mathcal{F} \subset \mathcal{G}$ and thus we must also have $u_1 v_i - u_i v_1 \in (\ell g)$, which implies $\operatorname{rad}(F_1, F_k) = \operatorname{rad}(F_1, F_i)$ which by Proposition 6.20 implies $F_i \in (F_1, F_k)$, which is a contradiction. $\square$

7 Radical Sylvester-Gallai for cubics

In this section, we prove our second main theorem: the radical Sylvester-Gallai theorem for irreducible cubics. Throughout this section, our SG configuration will be denoted by $\mathcal{F} := \mathcal{F}_1 \sqcup \mathcal{F}_2 \sqcup \mathcal{F}_3$, where $\mathcal{F}_d := \mathcal{F} \cap S_d$, and we will denote $m := |\mathcal{F}|$ and $m_d := |\mathcal{F}_d|$ for $d \in [3]$. Moreover, we will write $\mathcal{F}_3 := \{C_1, \ldots, C_{m_3}\}$.

Before stating our main theorem, we need a couple of definitions which will be useful when handling the cases which appear in our analysis. These definitions are motivated by our structure theorem Theorem 1.5.

Definition 7.1. Given a radical Sylvester-Gallai configuration $\mathcal{F}$, and a cubic $C \in \mathcal{F}_3$, divide the cubic polynomials as follows:

$$\begin{aligned}
\mathcal{F}_{\mathtt{span}}(C) &:= \{P \in \mathcal{F}_3 \mid |\mathrm{span}_{\mathbb{C}}\{P, C\} \cap \mathcal{F}| \geqslant 3\} \\
\mathcal{F}_{\mathtt{linear}}(C) &:= \{P \in \mathcal{F}_3 \mid (P, C) \subset (x, y),\ x, y \in S_1\} \\
\mathcal{F}_{\mathtt{quad}}(C) &:= \{P \in \mathcal{F}_3 \mid (P, C) \subset (x, Q),\ x \in S_1, Q \in S_2\} \\
\mathcal{F}_{\mathtt{md}}(C) &:= \{P \in \mathcal{F}_3 \mid (P, C) \subset I_V,\ \text{where } I_V \text{ ideal of variety of minimal degree}\} \\
\mathcal{F}_{\mathtt{factor}}(C) &:= \{P \in \mathcal{F}_3 \mid xy^2 \in \mathrm{span}_{\mathbb{C}}\{P, C\},\ \text{for } x, y \in S_1\}
\end{aligned}$$

And for a parameter $\nu \in (0, 1)$ define the following subset of $\mathcal{F}$:

$$\mathcal{F}_{\mathtt{span}} := \{C \in \mathcal{F}_3 \mid |\mathcal{F}_{\mathtt{span}}(C)| \geqslant \nu \cdot |\mathcal{F}_3|\}$$

Another good definition to have is the set of pairs of cubics $P, Q \in S_3$ which not only do not span a third cubic (and therefore (P, Q) is not radical) but the radical of (P, Q) has its SG dependency in lower degrees. We term such pairs "bad pairs" and give the formal definition below.

Definition 7.2 (Bad Pairs)**.** Given a radical SG configuration $\mathcal{F}$, we say that two cubics $C_1, C_2 \in \mathcal{F}_3$ form a *bad pair* if $|\mathrm{rad}(C_1, C_2) \cap \mathcal{F}_3| = 2$. That is, we have the their SG dependency is in $\mathcal{F}_1 \cup \mathcal{F}_2$. Given a polynomial $C \in \mathcal{F}_3$, let $\mathcal{F}_{\mathtt{bad}}(C)$ be the set of cubics $F \in \mathcal{F}_3$ such that (C, F) is a bad pair.

The following proposition also allows us to assume that $m_3 = \omega(1)$, otherwise we essentially have that $\mathcal{F}$ is a 2-radical-SG configuration over a small algebra.

Proposition 7.3. *If $\mathcal{F}$ is a* 3-radical-SG *configuration and* $m_3 = O(1)$*, then* $\dim \mathrm{span}_{\mathbb{K}}\{\mathcal{F}\} = O(1)$.

Proof. Let $C_1, \ldots, C_t \in \mathcal{F}_3$ be the cubics from $\mathcal{F}_3$ for which $s(C_i) = 1$. Since $t \leqslant m = O(1)$, Corollary 4.13 implies that there is a $(20, 1)$-wide W with $\dim W = O(1)$ such that $C_1, \ldots, C_t \in \mathbb{K}[W]$.

Now, if $\mathcal{G} := \mathcal{F}_1 \sqcup \mathcal{F}_2$, the SG dependencies for $\mathcal{F}$ imply that $\mathcal{G}$ is a $(2, W)$-radical-SG configuration, and hence Proposition 6.7 implies that $\dim \mathrm{span}_{\mathbb{K}}\{G\} = O(1)$ and we are done. ☐

Given the above proposition, we shall henceforth assume that $m_3 = \omega(1)$.

7.1 Controlling the Cubic Forms

In this section, we will prove that the strong forms in $\mathcal{F}_3$ are in $\mathcal{F}_{\mathtt{span}}$.

Lemma 7.4. *Let $\delta \leqslant 1/30$ and define $\mathcal{F}_{\mathtt{span}}$ with respect to δ as in Definition 7.1. If $C \in \mathcal{F}_3$ is such that $s(C) \geqslant 3$, then $C \in \mathcal{F}_{\mathtt{span}}$.*

Proof. We will prove this lemma by first showing that for all but $2\sqrt{m_3}$ cubics $F \in \mathcal{F}_3 \setminus \mathcal{F}_{\mathtt{span}}(C)$, we can find a third cubic $G \in \mathcal{F}_3 \cap \mathrm{rad}(C, F)$ such that (C, G) is a radical ideal. Thus, for such F, we can associate to the pair $\{C, F\}$ a third cubic $G \in \mathcal{F}_{\mathtt{span}}(C) \cap \mathrm{rad}(C, F)$. Then, we will show that for at most 8 such polynomials $F_1, \cdots, F_8 \in \mathcal{F}_3 \setminus \mathcal{F}_{\mathtt{span}}(C)$, the associated cubics G_i that we obtain can be the same. This will imply that $|\mathcal{F}_{\mathtt{span}}(C)| \geqslant m_3/20$ and thus $C \in \mathcal{F}_{\mathtt{span}}$. Indeed, let $|\mathcal{F}_{\mathtt{span}}(C)| = m'$. Then $|\mathcal{F}_3 \setminus \mathcal{F}_{\mathtt{span}}(C)| = m_3 - m'$, and we must have $\frac{m_3 - m' - 2\sqrt{m_3}}{8} \leqslant m'$. We may assume that $\sqrt{m_3} \leqslant \frac{m_3}{4}$. Then the previous inequality implies that we must have $\frac{m_3}{20} \leqslant m'$.

If $C_i \notin \mathcal{F}_{\mathtt{span}}(C)$, we must have that (C, C_i) is not radical. Since $s(C) \geqslant 3$, Theorem 1.5 implies $C_i = C + x_i y_i^2$ or $C_i = C + x_i^3$ for linearly independent forms $x_i, y_i \in S_1$. We have two cases:

Case 1: $C_i = C + x_i y_i^2$, with $\dim \operatorname{span}_{\mathbb{K}} \{x_i, y_i\} = 2$.

Note that we have $\operatorname{rad}(C, C_i) = (C, x_i y_i)$ by Proposition B.3. Thus we cannot have a linear or irreducible quadratic form in $\operatorname{rad}(C, C_i)$. Therefore the SG condition implies the existence of $C_j \in \operatorname{rad}(C, C_i) = (C, x_i y_i)$, which implies that $C_j = C + x_i y_i z_j$. If $z_j \notin \operatorname{span}_{\mathbb{K}} \{x_i\} \cup \operatorname{span}_{\mathbb{K}} \{y_i\}$, we have that (C, C_j) is radical, by Theorem 1.5 and therefore $C_j \in \mathcal{F}_{span}(C)$. Note that if $z_j \in \operatorname{span}_{\mathbb{K}} \{y_i\}$, then we would have $C_j \in \operatorname{span}_{\mathbb{K}} \{C, C_i\}$, and hence $C_i \in \mathcal{F}_{span}(C)$ which is a contradiction. Thus we may assume that $C_j = C + \alpha x_i^2 y_i$ for some $\alpha \in \mathbb{K}^*$. If $C_j \in \mathcal{F}_{span}(C)$ we are done, so assume this is also not the case.

Note that $s(C) \geqslant 3 \Rightarrow (C_i, C_j) = (C_i, x_i y_i (y_i - \alpha x_i))$ is a radical ideal. Since $C \notin (C_i, C_j)$, there is $k \neq i, j$ such that $C_k \in (C_i, C_j)$ and $C_k \notin (C)$. Hence, we can write $C_k = C + x_i y_i z_k$, where $z_k \in (x_i, y_i)$. Note that $k \neq i, j$ implies $z_k \notin \operatorname{span}_{\mathbb{K}} \{x_i\} \cup \operatorname{span}_{\mathbb{K}} \{y_i\}$, otherwise either C_i or C_j would be in $\mathcal{F}_{span}(C)$. In this case, we have that (C, C_k) is radical by Theorem 1.5 and thus $C_k \in \mathcal{F}_{span}(C) \cap \operatorname{rad}(C, C_i)$.

Case 2: $C_i = C + x_i^3$.

Note that $\operatorname{rad}(C, C_i) = (C, x_i)$ by Proposition B.3. Thus we can not have an irreducible quadratic form in $\operatorname{rad}(C, C_i)$. Therefore the SG condition implies that either there is a cubic $C_j \in \operatorname{rad}(C, C_i) = (C, x_i)$, or there exists a linear form $x \in \mathcal{F}_1 \cap \operatorname{rad}(C, C_i)$. Note that in the latter case we must have that $x \in (x_i)$, and $x = \alpha_i x_i$for some $\alpha \in \mathbb{K}^*$.

Case 2.1: there is a cubic $C_j \in \operatorname{rad}(C, C_i) = (C, x_i)$.

In this case, we have $C_j = C + x_i y_j z_j$ or $C_j = C + x_i Q_j$ where Q_j is an irreducible quadratic. Note that if the latter happens, by Theorem 1.5 we have that (C, C_j) is radical, and therefore $C_j \in \mathcal{F}_{span}(C)$. Thus, we are left with the case where $C_j = C + x_i y_j z_j$ for $y_j, z_j \in S_1$.

If $y_j \notin \operatorname{span}_{\mathbb{K}} \{x_i\} \cup \operatorname{span}_{\mathbb{K}} \{z_j\}$ then by Theorem 1.5 we have that (C, C_j) is radical and we are done. Otherwise, we have that $C_j = C + x_i z_j^2$ or $C_j = C + x_i^2 z_j$, which we already handled in the previous case.

Case 2.2: we have $\alpha_i x_i \in \operatorname{rad}(C, C_i)$ is the only SG dependence of C, C_i in $\mathcal{F}$, where $\alpha_i \in \mathbb{K}^*$.

Let $\Gamma := \{F_1, \ldots, F_r\} \subset \mathcal{F}_3$ be the set of cubic forms which satisfy this case, where $F_i = C + x_i^3$, with $x_i \in S_1$. For each $i \neq j \in [r]$, by Theorem 1.5 we have that (F_i, F_j) is radical. Indeed, since $s(C) \geqslant 3$ the ideal (F_i, F_j) can not have a linear or quadratic minimal prime. If $F_i = F_j + uv^2$, then we have $x_i^3 - x_j^3 = uv^2$. By unique factorization, we must have $x_i \in (x_j)$. Then we would have $F_j \in \operatorname{rad}(C, F_i)$ which is a contradiction. Suppose there is a minimal prime $\mathfrak{p}$ of (F_i, F_j) which defines a variety of minimal degree. Then we must have $x_i^3 - x_j^3 \in \mathfrak{p}$, which is a contradiction since the prime ideal $\mathfrak{p}$ cannot contain a linear form. Therefore F_i, F_j must span a third element in $\mathcal{F}_3$, of the form $G_{i,j} := C + \lambda x_i^3 + (1 - \lambda) x_j^3$ (after proper scalar multiplication). By Proposition B.4 we have that for each pair $\{i, j\} \subset [r]$, we must have a distinct form $G_{i,j}$ of the form above. Thus, we have that $\binom{r}{2} \leqslant m_3$, which implies that $r \leqslant 2\sqrt{m_3}$. Therefore, we can disregard this case.

Showing that spanned polynomials are mostly distinct: Now, let $C_1, \ldots, C_t$ be the polynomials in $\mathcal{F}_3 \backslash \mathcal{F}_{span}(C)$ which satisfy cases 1 and 2.1, i.e. $C_i = C + x_i y_i^2$ for some linear forms x_i, y_i. Let $G_1, \ldots, G_t \in \mathcal{F}_{span}(C)$ be the polynomials that we obtained from the process above such that $(C, G_i) \subset \operatorname{rad}(C, C_i)$. Note that C_i's are distinct polynomials, whereas the G_j's may not be.

Note that by our construction (C, G_j) is radical and we have $G_j = C + u_j v_j w_j$ where u_j, v_j, w_j are pairwise linearly independent. Suppose $G_j \in \mathrm{rad}(C, C_i)$ for some i. If $C_i = C + x_i y_i^2$ then we must have $x_i, y_i \in \mathrm{span}_{\mathbb{K}}\{u_j\} \cup \mathrm{span}_{\mathbb{K}}\{v_j\} \cup \mathrm{span}_{\mathbb{K}}\{w_j\}$. Therefore we must have that C_i is of the form $C + \beta x y^2$ where $x, y \in \{u_j, v_j, w_j\}$. If we have more than 8 such C_i's, then there must be two distinct C_ℓ, C_k which are of the same form. Then we will have $C_k \in \mathrm{span}_{\mathbb{K}}\{C, C_\ell\}$, which is a contradiction because $C_\ell \notin \mathcal{F}_{span}(C)$. Therefore at most 8 of the C_i's can give us the same G_j. ☐

Lastly, we also need the following auxiliary lemma:

Proposition 7.5. *Let $\mathcal{A}$ be a $(10, 1)$-wide AH algebra and $C \in \mathcal{A} \cap S_3$ be an irreducible cubic form. Suppose $F = C - xy^2$ is an irreducible cubic form where $x, y \notin \mathcal{A}$. Then, up to multiplication by an unit, there exists at most one irreducible cubic $P \in \mathcal{A} \cap S_3$ such that $C \notin (P)$ and (F, P) is not a radical ideal.*

Proof. Let $V = V_1 + V_2$ be the $(10, 1)$-wide AH vector space that generates $\mathcal{A}$. Let $z_1, \cdots, z_r$ and $Q_1, \cdots, Q_m$ be bases for V_1 and V_2 respectively. Let $V_1' = V_1 + \mathrm{span}_{\mathbb{K}}\{x, y\}$. Then $V' = V_1' + V_2$ is an $(8, 1)$-wide AH vector space. Let $\mathcal{A}' = \mathbb{K}[V']$. Note that if x, y are linearly independent modulo V_1, then we have that $x, y, z_1, \cdots, z_r, Q_1, \cdots, Q_m$ is an R_3-sequence that generates $\mathcal{A}'$. Otherwise we may assume that $y, z_1, \cdots, z_r, Q_1, \cdots, Q_m$ is an R_3-sequence that generates $\mathcal{A}'$.

Let $P_1, P_2 \in \mathcal{A} \cap S_3$ be irreducible cubic forms such that $C \notin (P_i)$ and $P_1 \notin (P_2)$. Suppose that (F, P_i) is not radical for $i = 1, 2$. Note that $F = C - xy^2 \notin (V)$ as $x, y \notin \mathcal{A}$. Therefore, by Lemma 3.22, we must have $\mathrm{Disc}_x^{\mathcal{A}'}(F) \cdot \mathrm{Disc}_y^{\mathcal{A}'}(F) \in (P_i)$ for $i = 1, 2$. If x, y are linearly independent modulo V_1, then we have that $\mathrm{Disc}_x^{\mathcal{A}'}(F) = -y^2$ and $\mathrm{Disc}_y^{\mathcal{A}'}(F) = 4x^2 C$. Hence we have a contradiction since $C \notin (P_i)$. If x, y are not linearly independent modulo V_1, suppose $x = z + \alpha y$, where $z \in V_1$ and $\alpha \in \mathbb{K}^*$. In this case, we must have that $\mathrm{Disc}_y^{\mathcal{A}'}(F) \in (P_i)$ for $i = 1, 2$. Now we have $\mathrm{Disc}_y^{\mathcal{A}'}(F) = C(4\alpha z^3 - 27\alpha^3 C)$. Since $C \notin (P_i)$, we must have that $(4\alpha z^3 - 27\alpha^3 C) = \beta_i P_i$ for $i = 1, 2$ where $\beta_i \in \mathbb{K}^*$, which is a contradiction. ☐

7.2 Proof of Theorem 1.4

We begin by restating our main theorem on radical Sylvester-Gallai configurations. Our high level strategy is as follows: we first construct a small algebra that contains $\mathcal{F}_3$, and once we construct this algebra, we apply our "inductive step" by using Proposition 6.7 that a quadratic radical SG configuration over a strong algebra has small dimension.

We begin by proving that in a 3-radical-SG configuration, if $\mathcal{F}_3$ is not a $(0, \delta)$-linear-SG configuration, for a constant $\delta \in (0, 1/30]$, then all polynomials in $\mathcal{F}_3$ must be of low strength.

Proposition 7.6. *Let $\delta \leqslant 1/30$ and define $\mathcal{F}_{span}$ with respect to δ as in Definition 7.1. Suppose that $\mathcal{F}_3 \neq \mathcal{F}_{span}$. Then we have $s(F) \leqslant 5$ for all $F \in \mathcal{F}_3$.*

Proof. Suppose, for the sake of a contradiction, that there is $C \in \mathcal{F}_3 \setminus \mathcal{F}_{span}$ and a cubic $F \in \mathcal{F}_3$ such that $s(F) \geqslant 6$. By Lemma 7.4, we know that $F \neq C$, as $s(C) \leqslant 2$. Let $\mathcal{L} := \mathcal{F}_3 \setminus \mathcal{F}_{span}(C) = \{G_1, \ldots, G_t\}$. Now, $C \notin \mathcal{F}_{span} \Rightarrow t \geqslant (1 - \delta)m_3$. By definition of $\mathcal{L}$, we have that, for $i \in [t]$, (G_i, C) is not radical and therefore, by Theorem 1.5 we have that $s_{\max}(G_i, C) \leqslant 3$, where if equality happens then $G_i = C + x_i y_i^2$, for some $x_i, y_i \in S_1$.

In particular, the above implies that $s_{\max}(G_i, G_j) \leqslant 5$ for any $i \neq j \in [t]$. Since $s(G_i) \leqslant 3$, Theorem 1.5 implies that (F, G_i) is radical and thus $G_i \in \mathcal{F}_{span}(F)$. However, since $t \geqslant (1 - \delta)m_3 > m_3/2$, there exist $i \neq j \in [t]$ such that $G_j \in (F, G_i)$, which is a contradiction, as the latter inclusion implies that either $G_j \in (G_i)$ (contradicting the hypothesis of our SG configuration) or that $F \in (G_i, G_j) \Rightarrow 6 \leqslant s(F) \leqslant s_{\max}(G_i, G_j) \leqslant 5$. $\square$

With the above observation at hand, we will prove that there exists a w-AH vector space $V = V_1 + V_2$ and a space of cubics $U \subset S_3$ where any nonzero $C \in U$ satisfies $s(C) \geqslant 3$ such that $\dim(U + V) = O(1)$ and $\mathcal{F}_3 \subset \mathbb{C}[U, V]$. Moreover, if $C \in \mathcal{F}_3$ is such that $s(C) \leqslant 2$, then $C \in \mathbb{K}[V]$. Once we prove this structure, we will show how to reduce the cubic radical SG problem to the quadratic radical SG problem with the strong subalgebra $\mathbb{K}[V]$.

Theorem 1.4. If $\mathcal{F}$ is a 3-radical-SG configuration, then

$$\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{F}\} = O(1).$$

Proof. Let $\delta = 2^{-10}$ and $\mathcal{F}_{span}$ be defined with respect to δ as in Definition 7.1.

Constructing small subalgebra containing $\mathcal{F}_3$. We now will construct a $(10, 1)$-wide vector space $V = V_1 + V_2$ with $\dim V = O(1)$, and a vector space $U \subset S_3$ with $\dim(U) = O(1)$ such that $\mathcal{F}_3 \subset \mathbb{K}[U, V]$ where any $C \in \mathcal{F}_3$ such that $s(C) \leqslant 2$ is contained in $\mathbb{K}[V]$.

We divide this part of the proof into two cases:

Case 1: $|\mathcal{F}_3 \backslash \mathcal{F}_{span}| \leqslant 3$

In this case, we have that $\mathcal{F}_3$ is a $(3, \delta)$-linear-SG configuration, and Proposition 6.4 gives us $\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{F}_3\} = O(1)$. Take a basis $C_1, \ldots, C_t \in \mathcal{F}_3$ such that $W = \operatorname{span}_{\mathbb{K}} \{C_1, \ldots, C_t\} = \operatorname{span}_{\mathbb{K}} \{\mathcal{F}_3\}$. By iteratively picking basis elements of lowest strength, we can construct a basis $F_1, \ldots, F_r, G_1, \ldots, G_{t-r}$ such that $s(F_i) \leqslant 2$, and any element of W that depends non-trivially on $G_1, \ldots, G_{t-r}$ has strength $\geqslant 3$. Let V be the $(10, 1)$-wide vector space from Corollary 4.13 applied to $\operatorname{span}_{\mathbb{K}} \{F_1, \ldots, F_r\}$. Hence $\dim V = O(1)$. Moreover, by our choice of basis, if $C \in \mathcal{F}_3$ is such that $s(C) \leqslant 2$, then $C \in \mathbb{K}[V]$. Taking $U = \operatorname{span}_{\mathbb{K}} \{G_1, \ldots, G_{t-r}\}$ concludes this case.

Case 2: $\mathcal{F}_3 \backslash \mathcal{F}_{span} = \{C_1, \ldots, C_k\}$, where $k \geqslant 3$.

By Proposition 7.6, we have that $s(C) \leqslant 5$ for all $C \in \mathcal{F}_3$. Moreover, by Lemma 7.4, we know that $s(C_i) \leqslant 2$ for all $i \in [k]$. We can assume w.l.o.g. that $2 \geqslant s(C_1) \geqslant s(C_2) \geqslant \cdots \geqslant s(C_k) \geqslant 1$. Since each $C_i \notin \mathcal{F}_{span}$, we have that $|\mathcal{F}_{span}(C_i)| \leqslant \delta m_3$ and therefore at least $(1 - \delta)m_3$ of the polynomials in $\mathcal{F}_3$ do not form a radical ideal with C_i.

In this case, we will first prove that there exists a subspace $W_1 \subset S_1$ such that:

1. $\dim(W_1) = O(1)$
2. there exists a subset $\mathcal{G} \subset \mathcal{F}_3 \cap (W_1)$ such that $|\mathcal{G}| \geqslant (1 - 3\delta)m_3$

Case 2.1: $s(C_1) = 2$.

Let $W = W_1 + W_2$ be the $(3, 10)$-wide algebra that one obtains from a minimal collapse of C_1, C_2, C_3. Let $\mathcal{G} := \mathcal{F} \backslash \mathcal{F}_{span}(C_1) \cup \mathcal{F}_{span}(C_2) \cup \mathcal{F}_{span}(C_3)$. Since $C_1, C_2, C_3 \notin \mathcal{F}_{span}$, we have that $|\mathcal{G}| > (1 - 3\delta)m_3$. In this case, we will show that $\mathcal{G} \subset (W_1)$.

Since $s(C_1) = 2$, Theorem 1.5 implies that $\mathcal{F}_3 \backslash \mathcal{F}_{span}(C_1) = \mathcal{F}_{md}(C_1) \cup \mathcal{F}_{factor}(C_1)$. Since $s(C_1) = 2$, if $\mathcal{F}_{md}(C_1) \neq \varnothing$ then $C_1 \in \mathbb{K}[W_1]$ and therefore by Lemma 4.16 any variety of minimal degree containing C_1 will have all its defining linear forms in W_1. Hence $\mathcal{F}_{md}(C_1) \subset (W_1)$.

Thus, if a polynomial $F \in \mathcal{G}$ is not in (W_1), by the above it must be the case that $F \in \mathcal{F}_{factor}(C_1)$, since (F, C_1) is not radical and $F \notin (W_1)$. In this case we must have $F = C_1 - xy^2$, where $x, y \notin \mathbb{K}[W]$. However, we also know that (F, C_2) and (F, C_3) are not radical, and this contradicts Proposition 7.5. Hence, we have that $\mathcal{G} \subset (W_1)$, as we wanted.

Case 2.2: $s(C_1) = 1$.

Similarly to the previous case, let $U = U_1 + U_2$ be the $(3, 10)$-wide algebra that one obtains from a minimal collapse of C_1, C_2, C_3. Let $\mathcal{G} = \mathcal{F} \backslash \mathcal{F}_{span}(C_1) \cup \mathcal{F}_{span}(C_2) \cup \mathcal{F}_{span}(C_3)$. Since $C_1, C_2, C_3 \notin \mathcal{F}_{span}$, we have that $|\mathcal{G}| > (1 - 3\delta)m_3$. In this case, we will now show that there is a vector space $X \subset S_1$ with $\dim X = O(1)$ such that $\mathcal{G} \subset (U_1 + X)$, and thus we can take $W_1 = U_1 + X$.

If $\mathcal{G} \subset (U_1)$, we can take $X = 0$ and we are done. Therefore let's assume that $\mathcal{G} \not\subset (U_1)$. Consider $F \in \mathcal{G} \backslash (U_1)$. By Theorem 1.5 and the fact that $F \notin (U_1)$ the only possible cases are the following:

1. $F \in \mathcal{F}_{factor}(C_b)$ for some $b \in [3]$.

   In this case, we can assume $F \in \mathcal{F}_{factor}(C_1)$, which implies that $F = C_1 + xy^2$ where $x, y \notin \mathbb{K}[U]$. However, by Proposition 7.5, there is at most one irreducible cubic $P \in \mathbb{K}[U]$ such that $P \notin (C_1)$ and (F, P) is not radical. This contradicts the fact that $F \in \mathcal{G}$, as that implies (F, C_2) and (F, C_3) are both not radical. Thus, this subcase is ruled out.

2. $F \in \mathcal{F}_{quad}(C_b)$ for some $b \in [3]$ and $F \notin \mathcal{F}_{factor}(C_b)$ for any $b \in [3]$.

   In this case, we can assume that $F \in \mathcal{F}_{quad}(C_1)$. Hence, we must have $F, C_1 \in (Q, \ell)$ for some $\ell \in S_1$ and $Q \in S_2$. If there is a linear minimal prime (x, y) then, by Lemma 4.16, we have $x, y \in U_1$. This is a contradiction as $F \notin (U_1)$. Suppose the minimal prime (Q, ℓ) has multiplicity 1. If we have a minimal prime $\mathfrak{p}$ with $e(\mathfrak{p}) = 3$ or $e(\mathfrak{p}) = 4$, and $m(\mathfrak{p}) \geqslant 2$, then we must have a linear minimal prime, which is again a contradiction. Therefore, by Theorem 1.5 we may assume that (Q, ℓ) is a minimal prime of (C_1, F) with multiplicity at least 2. Thus, Corollary 5.7 applies, and we have that $F, C_1 \in (Q, \ell^2)$, so we can write $F = yQ + \ell^2 z$ for some $y \in S_1 \backslash W_1, z \in S_1$. As $C_1 \in (Q, \ell^2)$, we must have $s(Q) \geqslant 10$, otherwise we would have $C_1 \in \mathbb{K}[U_1]$, which implies $Q \in (U_1)$, contradicting $F \notin (U_1)$.

   Hence, Theorem 1.5 and the facts that $F \notin (U_1)$ and $s(Q) > 10$ imply that $F \in \mathcal{F}_{quad}(C_1) \cap \mathcal{F}_{quad}(C_2) \cap \mathcal{F}_{quad}(C_3)$. By Lemma 5.8, we have that $C_2, C_3 \in (Q, \ell^2)$. Thus, we can write $C_b = x_b Q + \ell^2 u_b$, where $x_b, u_b, \ell \in U_1$.

   The above shows that any polynomial $F \notin (U_1)$ which satisfies this case must be in $\mathcal{F}_{quad}(C_1)$. By Lemma 5.8, we have that $F \in \mathcal{F}_{quad}(Q_1) \Rightarrow F \in (Q, x_1, \ell)$. Hence, Lemma 6.22 implies that there exists $X \subset S_1$ with $\dim(X) = O(1)$ such that $\mathcal{F}_{quad}(C_1) \subset (U + X)$. Therefore, in this case we can take $W_1 = U_1 + X$.

3. $F \in \mathcal{F}_{md}(C_b)$ for all $b \in [3]$.

   In this case, Corollary 5.2 implies that $U = U_1$. Also by Corollary 5.2, there is $Z \subset S_1$ with $d := \dim(Z) \leqslant 9$ such that $F \in \mathbb{K}[Z]$. As we are assuming that $F \notin (U_1)$, we can take a basis $z_1, \ldots, z_t$ for $U_1 + Z$ such that F is monic in z_1 and $U_1 \subset \mathrm{span}_{\mathbb{K}} \{z_2, \ldots, z_t\}$. Hence, we have $C_b \in \mathbb{K}[z_2, \ldots, z_t]$ and $F \in \mathbb{K}[z_1, \ldots, z_t]$ such that $F \notin (z_2, \ldots, z_t)$. By Lemma 3.21 we have that (F, C_b) not radical iff $\mathrm{Disc}_{z_1}(F) \in (C_b)$, and since each C_b is irreducible and pairwise independent, we must have $\mathrm{Disc}_{z_1}(F) \in (C_1 C_2 C_3)$. However, we have that $\deg \mathrm{Disc}_{z_1}(F) = 6$, so the foregoing would imply that $\mathrm{Disc}_{z_1}(F) = 0$, which contradicts the fact that F is irreducible. Hence, this case cannot happen.

Hence, in both cases there is W_1 satisfying $\dim(W_1) = O(1)$ and $|\mathcal{F}_3 \cap (W_1)| \geqslant (1-3\delta)m_3$. By Corollary 6.15, if we set $\mathcal{H} := \mathcal{F} \cap (W_1)$ we know that $\dim \mathrm{span}_{\mathbb{K}} \{\mathcal{H}\} = O(1)$ and that there is a $(20, 3^{10})$-wide vector space $Y = Y_1 + Y_2$ such that $\mathcal{H} \subset \mathbb{K}[Y]$ and $\dim Y = O(1)$. Thus $|\mathcal{F}_3 \cap \mathbb{K}[Y]| \geqslant (1-3\delta)m_3$.

If $\mathcal{F}_3 \subset (Y_1)$ we are done with this part of the proof, as Corollary 6.15 would imply that $\dim \mathrm{span}_{\mathbb{K}} \{\mathcal{F}_3\} = O(1)$. So assume this is not the case and let $\mathcal{B} := \mathcal{F}_3 \backslash (Y_1)$. As $\mathcal{F}_3 \cap \mathbb{K}[Y] \subset (Y_1)$, we have $|\mathcal{B}| \leqslant 3\delta m_3$. Also, let $\Gamma := \mathcal{F}_3 \cap \mathbb{K}[Y] = \{G_1, \ldots, G_t\}$, where $t \geqslant (1-3\delta)m_3$.

Let $F \in \mathcal{B}$. Note that $|\mathcal{F}_{span}(F) \cap \Gamma| \leqslant 3\delta m_3$ as for each $G_i \in \mathcal{F}_{span}(F) \cap \Gamma$, if $H_i \in \mathrm{span}_{\mathbb{K}} \{F, G_i\}$, we must have that $H_i \in \mathcal{B}$, and for any two $G_i, G_j \in \Gamma$ we have that $\mathrm{span}_{\mathbb{K}} \{F, G_i\} \cap \mathrm{span}_{\mathbb{K}} \{F, G_j\} = \mathrm{span}_{\mathbb{K}} \{F\}$. Thus, we have that (F, G_i) does not span for $\geqslant (1-6\delta)m_3$ C_i's from Γ, and in particular, we must have (F, G_i) not radical. As $(1-6\delta)m_3 > 50$, Corollary 3.24 implies that $F \in (Y)$.

We can assume that (F, G_i) is not radical for $i \in [r]$, where $(1-6\delta)m_3 \leqslant r \leqslant t$. Since $F \in (Y) \backslash (Y_1)$ and $s(F) \leqslant 5$, Corollary 4.17 implies that F is not in any fully linear algebra with dimension $\leqslant 9$. Moreover, Proposition 7.5 implies $F \notin \mathcal{F}_{factor}(G_i)$ for any $G_i \in \Gamma$. Thus, by Corollary 5.1 and Corollary 5.7, we have that $(F, G_i) \subset (Q, \ell^2)$, where $s(Q) > 10$ and $Q, \ell \in \mathbb{K}[Y]$. In particular, $G_i \in \mathcal{F}_{quad}(F)$ for $i \in [r]$.

Writing $F = xQ - \ell^2 z$, where $x \notin Y_1$, by Lemma 5.8 and the fact that $G_i \in (Y_1)$, we have that $G_i \in (Q, \ell^2)$ for all $i \in [r]$. Thus, we can write $G_i = x_i Q - \ell^2 z_i$, and by Corollary 3.34 we have that the SG condition implies $|\mathrm{rad}(F, G_i) \cap \mathcal{F}_3| \geqslant 3$. However, Proposition 6.18 applied to $\mathcal{F}_3$ implies that $|\mathrm{rad}(F, G_i) \cap \mathcal{F}_3| \geqslant 3$ for at most $2^9 \cdot \delta \cdot m_3 \leqslant m/3/2 < (1-6\delta)m_3 \leqslant r$, which is a contradiction. Hence, we must have $\mathcal{F}_3 \subset (Y_1)$ and we are done with this part.

Concluding the proof: Now that we have shown that there is a $(600, 8)$ wide vector space V such that $\dim V = O(1)$ and $\mathcal{F}_3 \subset \mathbb{K}[V]$, we note that $\mathcal{F}_1 \cup \mathcal{F}_2$ is a quadratic radical SG configuration over $\mathbb{K}[V]$, which by Proposition 6.7 implies that $\dim \mathrm{span}_{\mathbb{K}} \{\mathcal{F}_1 \cup \mathcal{F}_2\} = O(1)$. Therefore, we have that $\dim \mathrm{span}_{\mathbb{K}} \{\mathcal{F}\} \leqslant \dim \mathrm{span}_{\mathbb{K}} \{\mathcal{F}_1 \cup \mathcal{F}_2\} + \dim V = O(1)$ as we wanted. □

8 Conclusion and Open Problems

In this paper, we proved that cubic radical Sylvester-Gallai configurations must be low dimensional, therefore solving Conjecture 1.3 for the case when $d = 3$. This generalizes the approach and results of [Shp20], who first broke ground on this question by solving it for $d = 2$.

To solve the cubic radical Sylvester-Gallai problem, we devised a structure theorem for ideals generated by two cubics, classifying when they are not radical. While such structure theorems may be hard to generalize, given the richness in structure of prime ideals of codimension 2, the main message of such structure theorems should be that two low-degree forms generate a non-radical ideal only if they are "close to each other."

Once we have the structure theorem, we studied variants of the quadratic radical SG problem, which allow us to "induct" onto these lower degree variants. However, one difficulty in the cubic case, is that we have to consider a couple of distinct variants since now we may not guarantee at first that all cubic forms are contained in a small prime ideal generated by *forms of the same degree*. Thus, we have to work harder in order to be able to use the general projections to reduce the degree and proceed by induction. We believe this is the main technical issue that needs to be overcome for the general case, in order to fully solve Conjecture 1.3, as our approach should generalize if one proves the existence of a small prime sequence of forms of *same degree* which contains the part of the configuration of highest degree.

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A Quadratic radical Sylvester-Gallai theorem over an algebra

In this section we prove that a quadratic radical SG configuration over an algebra of constant dimension must be contained in a constant dimensional vector space. For convenience, we recall the definition of a radical SG configuration over an algebra.

Definition 6.6 (Radical SG over an algebra)**.** Let $d \in \mathbb{N}^*$ and $V \subset S_{\leqslant d}$ be a graded vector space. Let $\mathcal{F} = \{F_1, \cdots, F_m\} \subset S$ be a set of irreducible forms such that $\deg(F_i) \leqslant d$ and $F_i \notin (F_j)$ for $i \neq j$. We say that $\mathcal{F}$ is a (δ, d, V)-radical-SG configuration if for every $i \in [m]$, there exist at least $\delta(m-1)$ values of j such that $|\mathcal{F} \cap \mathrm{rad}(F_i, F_j)| \geqslant 3$ or $|\mathrm{rad}(F_i, F_j) \cap \mathbb{K}[V] \backslash (F_i) \cup (F_j)| \geqslant 1$. If $\delta = 1$ then we simply call it a (d, V)-radical-SG configuration.

Our main result of this section, formally stated, is:

Proposition 6.7 (2-radical-SG configurations over small algebra)**.** *Let $\mathcal{F} \subset S_{\leqslant 2}$ be a finite set of irreducible forms such that for any $F, G \in \mathcal{F}$ we have $F \notin (G)$. Additionally, let $V = V_1 + V_2$ be a vector space of forms of degree at most 2. If $\mathcal{F}$ is a $(2, V)$-radical-SG configuration, then $\dim \mathrm{span}_{\mathbb{K}} \{\mathcal{F}\} = O(1+\dim(W)^2)$, where W is any $(600, 8)$-wide vector space such that $\mathbb{K}[V] \subset \mathbb{K}[W]$.*

Remark A.1. *Note that by Proposition 4.11, there exists a $(600, 8)$ wide vector space W such that $\mathbb{K}[V] \subset \mathbb{K}[W]$. Also, for any such W as above, we have that $\mathcal{F}$ is a $(2, W)$-radical-SG-configuration. Since Proposition 6.7 bounds the dimension of the configuration by a function of the dimension of a wide algebra containing $\mathbb{K}[V]$, we shall henceforth assume that V itself is a $(600, 8)$-wide vector space.*

The main idea in the proof of the above proposition is to show that in a $(2, V)$-radical-SG configuration, there exists a small wide subalgebra $\mathcal{A}$ which contains a constant fraction of the forms in the configuration. The approach to show this is based on the following: if a constant fraction forms a linear SG sub-configuration, then we can apply the linear SG theorems to construct the subalgebra $\mathcal{A}$. In case the configuration is not linear, then we can use the structure theorem for ideals generated by two quadrics to construct the subalgebra $\mathcal{A}$. Once we construct $\mathcal{A}$, we can use an iterative process to construct a slightly larger algebra that contains the entire configuration.

We will use the following definitions and notation throughout this section:

- $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2$ is our $(2, V)$-radical-SG configuration, with $|\mathcal{F}| = m$ and $|\mathcal{F}_i| = m_i$, for $i \in [2]$
- For $x \in S_1$, define $\mathbb{L}(x) = \mathrm{span}_{\mathbb{K}} \{x\}$ and for $Q \in S_2$, define $\mathbb{L}(Q) = \begin{cases} \mathrm{Lin}(Q), & \text{if } s(Q) \leqslant 3, \\ \mathrm{span}_{\mathbb{K}} \{Q\}, & \text{otherwise} \end{cases}$
- in analogy with the cubics, given $Q \in \mathcal{F}_2$, denote by

$$\mathcal{F}_{span}(Q) := \{P \in \mathcal{F}_2 \mid |(P, Q) \cap \mathcal{F}_2| \geqslant 3 \text{ or } |\mathrm{span}_{\mathbb{K}} \{P, Q\} \cap \mathbb{K}[V] \backslash (P) \cup (Q)| \geqslant 1\}$$
$$\mathcal{F}_{not-span}(Q) := \mathcal{F}_2 \backslash \mathcal{F}_{span}$$

- Whenever $|\mathrm{rad}(P, Q) \cap \mathbb{K}[V] \backslash (P) \cup (Q)| \geqslant 1$, we will say that $\mathrm{rad}(P, Q)$ meets $\mathbb{K}[V]$ non-trivially.
- Given a parameter $\nu \in (0, 1]$, define the set

$$\mathcal{F}_{span} := \{Q \in \mathcal{F}_2 \;\mid\; |\mathcal{F}_{span}(Q)| \geqslant \nu m_2\}.$$

We now state the structure theorem for ideals generated by two quadratics, whose proof can be found in [GOS22, Proposition 1.4].

Proposition A.2 (Radical Structure Theorem for Quadratics)**.** *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $Q_1, Q_2 \in S = \mathbb{K}[x_1, \cdots, x_n]$ be two forms of degree 2. Then one of the following holds:*

1. *The ideal (Q_1, Q_2) is prime.*
2. *The ideal (Q_1, Q_2) is radical, but not prime. Furthermore, one of the following cases occur:*
   (a) *There exist two linearly independent linear forms $x, y \in S_1$ such that $xy \in \mathrm{span}(Q_1, Q_2)$.*
   (b) *There exists a minimal prime $\mathfrak{p}$ of (Q_1, Q_2), such that $\mathfrak{p} = (x, y)$ for some linearly independent forms $x, y \in S_1$*
3. *The ideal (Q_1, Q_2) is not radical and one of the following cases occur:*
   (a) *Q_1, Q_2 have a common factor and $Q_1 = xy$, $Q_2 = x(\alpha x + \beta y)$ for some linear forms x, y and $\alpha, \beta \in k$. In this case, we have $x^2 \in \mathrm{span}(Q_1, Q_2)$.*
   (b) *Q_1, Q_2 do not have a common factor. There exists a minimal prime $\mathfrak{p}$ of (Q_1, Q_2) such that $\mathfrak{p} = (x, Q)$, where $x \in S_1$, $Q \in S_2$ and Q is irreducible modulo x, and we also have $x^2 \in \mathrm{span}(Q_1, Q_2)$.*
   (c) *Q_1, Q_2 do not have a common factor and there exists a minimal prime $\mathfrak{p}$ of (Q_1, Q_2), such that $\mathfrak{p} = (x, y)$ for some linearly independent forms $x, y \in S_1$, and the (x, y)-primary ideal $\mathfrak{q}$ has multiplicity $e(S/\mathfrak{q}) \geqslant 2$.*

We now show how one can slightly augment a wide algebra to contain an extra quadratic form.

Lemma A.3. *Let V be a (w, t)-wide vector space, where $w > 6$ and $t \geqslant 4$. Given any $Q \in S_2$, there exists a $(w, t^{1/2}/2)$-wide vector space W such that $\dim W \leqslant 3(\dim V + w)$, $V \subset W$ and $Q \in \mathbb{K}[W]$.*

Proof. We have two cases to analyze:

Case 1: $s(Q) \leqslant 2$.

In this case, just note that $W = V + \mathbb{L}(Q)$ is a $(w - 6, t)$-wide vector space, and since $w > 6$, W is also $(w, t - 2)$-wide. Moreover, $\dim W \leqslant \dim V + \dim \mathbb{L}(Q) = \dim V + 6$.

Case 2: $s(Q) \geqslant 3$.

If $s(P) \geqslant \sqrt{t}/2 \cdot (\dim(V) + 1 + w)$ for every $P \in \mathrm{span}_{\mathbb{K}}\{Q, V_2\}$, then $W := V_1 + \mathrm{span}_{\mathbb{K}}\{Q, V\}$ is $(w, \sqrt{t}/2)$-wide. Else, given that V is (w, t)-wide, then $P = Q + R$ for some $R \in V_2$. Since $s(P) < \sqrt{t}/2 \cdot (\dim(V) + 1 + w)$, we have $\dim \mathrm{Lin}(P) \leqslant \sqrt{t} \cdot (\dim(V) + 1 + w)$ and therefore, the space $W := V + \mathrm{Lin}(P)$ is $(w, \sqrt{t}/2)$-wide, as any element $F \in W_2 = V_2$ is such that $s(F) \geqslant t(\dim(V) + w) \geqslant \sqrt{t}/2 \dim(W) + w$.

In both cases, we constructed a $(w, t^{1/2}/2)$-wide vector space W such that $V \subset W$, $Q \in \mathbb{K}[W]$ and $\dim(W) \leqslant 3(\dim(V) + w)$. ∎

With the above at hand, we now address the quadratics in any $(2, V)$-radical-SG configuration. From here onwards, we set $\nu = 1/20$ and define $\mathcal{F}_{span}$ with respect to ν.

We begin by first showing that if $\mathcal{F}_2 \neq \mathcal{F}_{span}$, then we can construct a slightly larger algebra such that each quadratic is "close" to it.

Proposition A.4. *If $\mathcal{F}_2 \neq \mathcal{F}_{span}$ and $Q \in \mathcal{F}_2 \backslash \mathcal{F}_{span}$, then there exists a $(600, 8)$-wide algebra W such that $\dim W = O(\dim V + 1)$, $\mathbb{K}[Q, V] \subset \mathbb{K}[W]$ and for every $P \in \mathcal{F}_2 \backslash \mathbb{K}[W]$, there is $R_P \in \mathbb{K}[W]$ such that $s(P - R_P) \leqslant 3$.*

Proof. Let $Q \in \mathcal{F}_2 \backslash \mathcal{F}_{span}$ and let $\mathcal{F}_{not-span}(Q) = \{Q_1, \dots, Q_r\}$, where $r \geqslant (1 - \nu)m_2$. By Lemma A.3, there is $(600, 8)$-wide W such that $\dim W = O(\dim V + 1)$, $V \subset \mathbb{K}[W]$ and $Q \in \mathbb{K}[W]$.

We will now show that for each $i \in [r]$ such that $Q_i \notin \mathbb{K}[W]$, there is $\alpha_i \in \mathbb{K}$ such that $s(Q_i - \alpha_i Q) \leqslant 1$. If this is not the case, w.l.o.g. we can assume Q_1 is such that $Q_1 \notin \mathbb{K}[W]$ and $s(Q_1 - \alpha Q) \geqslant 2$ for any $\alpha \in \mathbb{K}$, which by Proposition A.2 implies (Q_1, Q) is prime. As $Q_1 \in \mathcal{F}_{not-span}(Q)$, by the SG condition on (Q, Q_1) the only possible case is that (Q_1, Q) intersects $\mathbb{K}[V]$ nontrivially. However, by Lemma 3.18 we have that $(Q_1, Q) \cap \mathbb{K}[W] = (Q)$, which implies $(Q_1, Q) \cap \mathbb{K}[V] \subset (Q)$, which is a contradiction.

Now that we know that each $Q_i \in \mathcal{F}_{not-span}(Q)$ is either in $\mathbb{K}[W]$ or satisfy $s(Q_i - \alpha_i Q) \leqslant 1$, we are left with proving that any $P \in \mathcal{F}_{span}(Q) \backslash \mathbb{K}[W]$ is such that $s(P - R) \leqslant 3$ for some $R \in \mathbb{K}[W]$. Suppose there is $P \in \mathcal{F}_{span}(Q)$ such that $s(P - R) \geqslant 4$ for all $R \in \mathbb{K}[W]$. In particular, Proposition A.2 implies that (P, Q_i) is a prime ideal for each $i \in [r]$, and hence $P \in \mathcal{F}_{span}(Q_i)$ or (P, Q_i) intersects $\mathbb{K}[V]$ non-trivially. Since $s(Q_i - \alpha_i Q) \leqslant 1$ or $Q_i \in \mathbb{K}[W]$, the vector space $U := W + \mathbb{L}(Q_i - \alpha_i Q)$ is $(25, 8)$-wide, and hence Lemma 3.18 implies $(P, Q_i) \cap \mathbb{K}[U] = (Q_i)$, which implies $(P, Q_i) \cap \mathbb{K}[V] \subset (Q_i)$ and therefore, we have $P \in \mathcal{F}_{span}(Q_i)$ for every $i \in [t]$.

Note that $\text{span}_{\mathbb{K}}\{P, Q_i\} \cap \text{span}_{\mathbb{K}}\{P, Q_j\} = \text{span}_{\mathbb{K}}\{P\}$, otherwise $P \in \text{span}_{\mathbb{K}}\{Q_i, Q_j\}$, which contradicts $s(P - R) \geqslant 4$ for all $R \in \mathbb{K}[W]$. Hence, there must exist $G_i \in \mathcal{F}_{span}(Q) \backslash \{P\}$ such that $G_i \in \text{span}_{\mathbb{K}}\{P, Q_i\}$. By the pigeonhole principle, as $|\mathcal{F}_{span}(Q)| < \nu m_2 < (1 - \nu)m_2$, there exist Q_i, Q_j such that $\text{span}_{\mathbb{K}}\{P, Q_i\} \cap \text{span}_{\mathbb{K}}\{P, Q_j\} \neq \text{span}_{\mathbb{K}}\{P\}$, which is a contradiction. $\square$

Proposition A.5. *Let $0 < \varepsilon < 1$ be a constant and $\mathcal{F}$ a $(2, V)$-radical-SG configuration such that $m \geqslant 4/\varepsilon$. If $w \geqslant 24/\varepsilon + 310$ and W is a $(w, 1)$-wide AH-vector space such that*

1. *$V \subset \mathbb{K}[W]$*
2. *(Close to algebra). For each $F \in \mathcal{F}_2$, there is $G_F \in \mathbb{K}[W]$ such that $s(F - G_F) \leqslant 3$.*
3. *(constant fraction in the algebra). we have $|\mathcal{F} \cap \mathbb{K}[W]| \geqslant \varepsilon m$.*

Then there exists a $(10, 1)$-wide vector space X such that $\mathcal{F} \cup \mathbb{K}[W] \subset \mathbb{K}[X]$ and $\dim(X) = \dim W + O(1)$.

Proof. We will construct the vector space X iteratively. At each step of the iterative process we will preserve the property that $\mathbb{K}[W] \subset \mathbb{K}[X]$ and we will increase the cardinality of $\mathcal{F} \cap \mathbb{K}[X]$.

1. Set $X = W$.
2. While $\mathcal{F} \not\subset \mathbb{K}[X]$:
   - If $\mathcal{F} \subset (X)$, set $X \leftarrow X + \text{span}_{\mathbb{K}}\left\{\bigcup_{P \in \mathcal{F} \backslash \mathbb{K}[X]} \mathbb{L}(P - G_P)\right\}$.
   - Else, pick $P \in \mathcal{F} \backslash (X)$ and set $X \leftarrow X + \mathbb{L}(P - G_P)$.

Termination. We show this iterative process terminates after $\leqslant 3/\varepsilon$ iterations of the While loop. First, we note that each step of the iterative process preserves the inclusion $\mathbb{K}[W] \subset \mathbb{K}[X]$. Therefore we always have $|\mathcal{F} \cap \mathbb{K}[X]| \geqslant \varepsilon m$. Let $\{F_1, \ldots, F_t\} \subset \mathcal{F} \cap \mathbb{K}[W]$, where by assumption $t \geqslant \varepsilon m$.

Suppose we have $\mathcal{F} \not\subset (X)$ and let $P \in \mathcal{F} \backslash (X)$. By assumption 2, there is $G_P \in \mathbb{K}[X]$ such that $s(P - G_P) \leqslant 3$. Let $U = X + \operatorname{span}_{\mathbb{K}} \{\mathbb{L}(P - G_P)\}$. Note that $P \in \mathbb{K}[U]$ and U is $(w-8, 1)$-wide, since $s(P - G_P) \leqslant 3$. By Corollary 3.24, we have that (P, F_i) is radical for $\geqslant t - 20$ forms $i \in [t]$.

Since $P \notin (X)$ and $s(P - G_P) \leqslant 3$, there is a linear form $x \in U_1/X_1$ such that P is monic on x. Therefore, by Lemma 3.18, we know that $\operatorname{rad}(P, F_i) \cap \mathbb{K}[X] = (F_i)$ for all $i \in [t]$. Thus, we have just showed that for $\geqslant t - 20$ indices $i \in [t]$, we have that (P, F_i) is radical and $(P, F_i) \cap \mathbb{K}[X] = (F_i)$, and hence the SG condition implies that there is $G_i \in (P, F_i) \cap \mathcal{F} \backslash \{P, F_i\}$ and such that $G_i \notin \mathbb{K}[X]$.

Without loss of generality, let us assume that the above holds for all $1 \leqslant i \leqslant t - 20$. Note that any $G \in \mathcal{F}$ can be in at most 2 ideals (P, F_i), since if $\deg P = 2$ and $G \in (P, F_i) \cap (P, F_j) \cap \mathcal{F} \Rightarrow F_i, F_j \in \mathcal{F}_1$ and in this case there is $\alpha \in \mathbb{K}$ such that $G - \alpha P \in (F_i F_j)$ so F_i, F_j are uniquely defined. And in case $\deg P = 1$, we have $G \in (P, F_i) \cap (P, F_j) \cap \mathcal{F} \Rightarrow G \in (P, F_i F_j)$ and since $P \notin \mathbb{K}[W]$ once again F_i, F_j are uniquely defined. Since $G_i \in (P, F_i) \Rightarrow G_i \in (U)$, we have shown

$$|\mathcal{F} \cap (U)| \geqslant |\mathcal{F} \cap (X)| + (t - 20)/2 \geqslant |\mathcal{F} \cap (X)| + \varepsilon m/2 - 10.$$

Thus, in at most $3/\varepsilon$ iterations we will have constructed X such that $\mathcal{F} \subset (X)$.

Dimension bound. After a general projection φ of X_1, we have that each $P \in \mathcal{F} \backslash \mathbb{K}[W]$ is such that $\varphi(P) = F_P + z\ell_P$, where $F_P \in W_2$ (and therefore $s(F_P) \geqslant w - 24/\varepsilon$) and $\ell_P \notin (z)$. Let $\mathcal{L} := \{z\} \cup \{\ell_P \mid P \in \mathcal{F} \backslash \mathbb{K}[w]\}$, where we do not include forms with repetition (that is, if they are a scalar multiple of a previously included linear form). The SG condition on $\mathcal{F}$ implies that $\mathcal{L}$ is a $(1, 1/2)$-linear-SG configuration, thus Proposition B.6 implies $\dim \operatorname{span}_{\mathbb{K}} \left\{\bigcup_{P \in \mathcal{F}_2} \mathbb{L}(\varphi(P - G_P))\right\} = \dim \mathcal{L} \leqslant 50$ and hence by Proposition 2.10 we have that $\dim \operatorname{span}_{\mathbb{K}} \left\{\bigcup_{P \in \mathcal{F}_2} \mathbb{L}(P - G_P)\right\} \leqslant 300$. Thus, we get the following bound:

$$\dim(X) \leqslant \dim(W) + 8 \cdot 3/\varepsilon + 300 = \dim W + O(1)$$

and X is $(w - 24/\varepsilon - 300, 1)$-wide, as we have added at most $24/\varepsilon + 300$ linear forms to W. $\square$

Remark A.6. *Note that Proposition A.5 implies that* $\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{F}\} = O((\dim W + 1)^2)$, *since* $\mathcal{F} \subset \mathbb{K}[X]$.

Lemma A.7. *Let* V *be a* $(600, 3)$*-wide vector space and* $\mathcal{F}$ *be a* $(2, V)$*-radical-SG configuration. If* $1 \leqslant m_2 < 3m/10$, *then for any* $Q \in \mathcal{F}_2$ *there exists a* $(600, 1)$*-wide vector space* W *with* $\dim W \leqslant 3(\dim V + 600)$ *and* $\mathbb{K}[Q, V] \subset \mathbb{K}[W]$ *such that* $|\mathbb{K}[W] \cap \mathcal{F}| \geqslant m/10$.

Proof. If $Q \in \mathbb{K}[V]$ then we are done simply by setting $W = V$. So we can assume $Q \notin \mathbb{K}[V]$. By Lemma A.3 we can construct W which is $(600, 1)$-wide, $\dim W \leqslant 3(\dim V + 600)$, $\mathbb{K}[Q, V] \subset \mathbb{K}[W]$.

For each $\ell \in \mathcal{F}_1 \backslash \mathbb{K}[W]$, Lemma 3.21 implies $\operatorname{rad}(Q, \ell) = (Q, \ell)$. Moreover, Lemma 3.18 implies $(Q, \ell) \cap \mathbb{K}[W] = (Q)$, which implies $(Q, \ell) \cap \mathbb{K}[V] = 0$, as $Q \notin \mathbb{K}[V]$. Hence, we have that $|(Q, \ell) \cap \mathcal{F}_2| \geqslant 2$ and there is $R \in (Q, \ell) \cap \mathcal{F}_2 \backslash (Q)$. In particular, we have that $R = Q + \ell a$, for some $a \in S_1$, which implies that R can be in at most two ideals of the form (Q, ℓ), for $\ell \in \mathcal{F}_1 \backslash \mathbb{K}[W]$. Hence, we have that $|\mathcal{F}_1 \backslash \mathbb{K}[W]| \leqslant 2 \cdot m_2$, which implies

$$m = |\mathcal{F}| \leqslant |\mathcal{F}_1 \cap \mathbb{K}[W]| + 3m_2$$

and therefore either $|\mathcal{F}_1 \cap \mathbb{K}[W]| \geqslant m/10$ or $m_2 \geqslant 3m/10$. $\square$

Lemma A.8. *Let* V *be a* $(600, 8)$*-wide vector space and* $\mathcal{F}$ *be a* $(2, V)$-radical-SG *configuration. Let* $U \subset S_1$ *with* $\dim U \leqslant 6$*. There is a* $(600, 1)$*-wide vector space* W *such that* $\dim(W) \leqslant 8(\dim(V) + 600)$*,* $U + V \subset \mathbb{K}[W]$ *and* $\mathcal{F} \cap (U) \subset \mathcal{F} \cap \mathbb{K}[W]$*.*

Proof. Let $\mathcal{G} := \mathcal{F} \cap (U) = \{G_1, \ldots, G_t\}$. Since (U) is a prime ideal, as it is generated by linear forms, we have that $\mathcal{G}$ is also a $(2, V)$-radical-SG configuration. As $\mathcal{G} \subset (U)$, after a generic projection φ, we have that $\varphi(G_i) = z\ell_i$ if $G_i \notin \mathbb{K}[U]$ or $\varphi(G_i) = z^{\deg(G_i)}$. Moreover, by Corollary 2.8 we have that $\ell_i \notin (\ell_j)$ for any $i \neq j$.

Let $\mathcal{L} := \{\ell_1, \ldots, \ell_r\}$ be the set of the linear forms that we get from $\varphi(G_i)$ of all $G_i \notin \mathbb{K}[U]$. Since the non-linear SG dependencies from $\mathcal{F}$ imply linear dependencies for $\mathcal{L}$ over the vector space V_1, we have that $\mathcal{L}$ is a $(1 + \dim V_1, 1)$-linear-SG configuration. By Proposition 6.4, we have that $\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{L}\} \leqslant 26 + \dim V_1$. Hence, Proposition 2.9 implies

$$\dim \operatorname{span}_{\mathbb{K}} \left\{ \bigcup_{i \in [t]} \operatorname{Lin}(G_i) \right\} \leqslant 7 \cdot (26 + \dim V_1).$$

Let $Z := \operatorname{span}_{\mathbb{K}} \left\{ \bigcup_{i \in [t]} \operatorname{Lin}(G_i) \right\}$. Setting $W = V + Z$, and noting that $Z \subset S_1$, we have that W is $(600, 1)$-wide and $\dim W \leqslant \dim V + \dim Z \leqslant 8(\dim V + 600)$. ☐

For the remainder of the section, we will let $\nu = 1/20$ and define $\mathcal{F}_{span}$ with respect to ν.

Lemma A.9. *Let* V *be a* $(600, 8)$*-wide vector space and* $\mathcal{F}$ *be a* $(2, V)$-radical-SG *configuration. If* $Q \in \mathcal{F}_2 \backslash \mathbb{K}[V]$ *then either* $Q \in \mathcal{F}_{span}$ *or there is a* $(600, 1)$*-wide vector space* W *such that* $\dim(W) \leqslant 8(\dim(V) + 600)$*,* $\mathbb{K}[Q, V] \subset \mathbb{K}[W]$ *and* $|\mathcal{F} \cap \mathbb{K}[W]| \geqslant m/10$*.*

Proof. We begin by constructing a $(600, 1)$-wide vector space W such that $\mathcal{F} \cap (\mathbb{L}(Q)) \subset \mathbb{K}[W]$ and $V \subset W$. If $s(Q) \geqslant 4$ then $\mathbb{L}(Q) = \operatorname{span}_{\mathbb{K}} \{Q\}$ and by Lemma A.3 there is W $(600, 1)$-wide, $\dim W \leqslant 3(\dim V + 600)$ and $\mathbb{K}[Q, V] \subset \mathbb{K}[W]$. If $s(Q) \leqslant 3$, then $\mathbb{L}(Q) = \operatorname{Lin}(Q)$ and $\dim \mathbb{L}(Q) \leqslant 8$, so by Lemma A.8 there is W $(600, 1)$-wide such that $\dim W \leqslant 8(\dim V + 600)$ such that $\mathbb{L}(Q) + V \subset W$ and $\mathcal{F} \cap (\mathbb{L}(Q)) \subset \mathbb{K}[W]$.

By Lemma A.7 we can assume $m_2 \geqslant 3m/10$, otherwise we are done. Moreover, we can also assume that $|\mathcal{F} \cap \mathbb{K}[W]| \leqslant m/10$.

Note that $\mathcal{F}_2 = \{Q\} \sqcup \mathcal{F}_{span}(Q) \sqcup \mathcal{F}_{not-span}(Q)$. Partition $\mathcal{F}_{not-span}(Q) \backslash \mathbb{K}[W] \cup (\mathbb{L}(Q))$ as

$$\mathcal{F}_{bad}(Q) := \mathcal{F}_{not-span}(Q) \backslash \mathbb{K}[W] \cup (\mathbb{L}(Q)) = \{F_1, \ldots, F_r\} \sqcup \{G_1, \ldots, G_s\} \sqcup \{H_1, \ldots, H_t\},$$

where $|\operatorname{rad}(Q, F_i) \cap \mathcal{F}_2| \geqslant 3$; $|\operatorname{rad}(Q, G_i) \cap \mathcal{F}_2| = 2$ but $\operatorname{rad}(Q, G_i)$ meets $\mathbb{K}[V]$ non-trivially; and lastly $|\operatorname{rad}(Q, H_i) \cap \mathcal{F}_2| = 2$, $\operatorname{rad}(Q, H_i)$ intersects $\mathbb{K}[V]$ trivially and $|\operatorname{rad}(Q, H_i) \cap \mathcal{F}_1| \geqslant 1$.

Since (Q, F_i) is not radical, otherwise $F_i \in \mathcal{F}_{span}(Q)$, and $F_i \notin (\mathbb{L}(Q))$, Proposition A.2 implies $\alpha_i F_i = Q + \ell_i^2$ for some $\ell_i \in S_1 \backslash W$ and non-zero scalar α_i. Let $R_i \in \operatorname{rad}(F_i, Q) \cap \mathcal{F}_2 \backslash \{Q, F_i\}$. Since $\operatorname{rad}(F_i, Q) = (Q, \ell_i)$, we have that $R_i = \beta_i Q + \ell_i a_i$ for some $a_i \in S_1$ and non-zero scalar β_i. Since $F_i \notin \mathcal{F}_{span}(Q)$, we must have that $a_i \notin (\ell_i)$. Moreover, note that for any $i \neq j$, we must have that $\ell_i \notin (\ell_j)$, and thus $R_i \in (R_j) \Rightarrow R_i = R_j$ and hence $(\beta_i - \beta_j)Q = \ell_j a_j - \ell_i a_i$. If $\beta_i - \beta_j \neq 0$, then $\ell_i, \ell_j, a_i, a_j \in \operatorname{Lin}(Q) = \mathbb{L}(Q) \subset W$, which is a contradiction. Hence $\beta_i = \beta_j$, and we have $a_i \in (\ell_j)$ and $\ell_j \in (a_i)$. In particular, we must have that each R_i can be in at most two of the radicals $\operatorname{rad}(Q, F_i)$, which implies $r \leqslant 2|\mathcal{F}_{span}(Q)|$.

Since $\mathrm{rad}(Q, G_i)$ meets $\mathbb{K}[V]$ non-trivially, we must have $G_i \in \mathbb{K}[W]$, otherwise Lemma 3.18 implies $\mathrm{rad}(Q, G_i) \cap \mathbb{K}[W] = (Q)$, which implies that $\mathrm{rad}(Q, G_i) \cap \mathbb{K}[V] = 0$. Thus, we have $s = 0$.

Since (Q, H_i) is not radical, as $|\mathrm{rad}(Q, H_i) \cap \mathcal{F}_1| \geqslant 1$, and since $H_i \notin (\mathbb{L}(Q))$, there is $\ell_i \in \mathcal{F}_1$ such that $\mathrm{rad}(Q, H_i) = (Q, \ell_i)$ and $H_i = \gamma_i Q + \ell_i^2$ for some non-zero scalar α_i. Since $H_i \notin \mathbb{K}[W]$, we must have $\ell_i \notin \mathbb{K}[W]$, and therefore $s(H_i) \geqslant 2$. Indeed, if $s(Q) \geqslant 4$ and $\ell_i \in \mathrm{Lin}(Q)$, then $s(H_i) = s(Q) - 1 \geqslant 2$. Otherwise if $s(Q) \leqslant 3$ then $\mathrm{Lin}(Q) = \mathbb{L}(Q) \subset W$ and we must have $\ell_i \notin \mathrm{Lin}(Q)$. Hence $s(H_i) = s(Q) + 1 \geqslant 2$. Moreover, we must have that $\ell_i \notin (\ell_j)$ for any $i \neq j$ otherwise H_i would be in $\mathcal{F}_{span}(Q)$. In particular, we have that $(H_1, H_i) = (H_1, \ell_1 - \ell_i) \cap (H_1, \ell_1 + \ell_i)$ and since $s(H_1) \geqslant 2$ we have that (H_1, H_i) is radical.

We will now show that (H_1, H_i) intersects $\mathbb{K}[V]$ trivially. If (H_1, H_i) intersect $\mathbb{K}[V]$ non-trivially, then $(H_1, \ell_1 - \ell_i)$ and $(H_1, \ell_1 + \ell_i)$ also intersect $\mathbb{K}[V]$ non-trivially. Thus there exist irreducible forms F, G of degree at most 3 such that $F \in (H_1, \ell_1 - \ell_i) \cap \mathbb{K}[V]$ and $G \in (H_1, \ell_1 + \ell_i) \cap \mathbb{K}[V]$. If $\deg(F) = 3$, then by Lemma 4.16, we must have $\ell_i \in \mathrm{span}_{\mathbb{K}} \{\ell_1, W_1\}$. If $\deg F = 1$, then $F \in (\ell_1 - \ell_i)$ and hence $\ell_i \in \mathrm{span}_{\mathbb{K}} \{\ell_1, W_1\}$. If $\deg(F) = 2$, then $\ell_1^2 \in \mathbb{K}[W]$ modulo $(\ell_1 - \ell_i)$, and hence $\ell_i \in \mathrm{span}_{\mathbb{K}} \{\ell_1, W_1\}$. Thus, $\ell_i = \alpha_i \ell_1 + u_i$ for $\alpha_i \in \mathbb{K}^*$ and $u_i \in W_1$. In $\mathbb{K}[\ell_1, W]$, we have $\mathrm{Res}_{\ell_1}(H_i, \ell_i - \ell_1) = (\alpha_i - 1)^2 Q + u_i^2$ and $\mathrm{Res}_{\ell_1}(H_i, \ell_i + \ell_1) = (\alpha_i + 1)^2 Q + u_i^2$. By Proposition 3.28, we have $\mathrm{rad}(\mathrm{Res}_{\ell_1}(H_i, \ell_i - \ell_1)) = \mathrm{rad}((H_1, \ell_i - \ell_1)_{\ell_1})$ and $\mathrm{rad}(\mathrm{Res}_{\ell_1}(H_i, \ell_i + \ell_1)) = \mathrm{rad}((H_1, \ell_i + \ell_1)_{\ell_1})$. Since Q is irreducible, we have $(\alpha_i - 1)^2 Q + u_i^2$ is reduced if $\alpha_i - 1 \neq 0$, and similarly for $\alpha_i + 1$. Thus we have, $\mathrm{rad}((H_1, \ell_i - \ell_1)_{\ell_1}) = ((\alpha_i - 1)^2 Q + u_i^2)$ if $\alpha_i - 1 \neq 0$, otherwise $\mathrm{rad}((H_1, \ell_i + \ell_1)_{\ell_1}) = (u_i)$, and similarly for $\alpha_i + 1$. Now $F \in \mathrm{rad}((H_1, \ell_i - \ell_1)_{\ell_1}) \cap \mathbb{K}[V]$ and $G \in \mathrm{rad}((H_1, \ell_i + \ell_1)_{\ell_1}) \cap \mathbb{K}[V]$. Since F, G are irreducible, we have F is a scalar multiple of $(\alpha_i - 1)^2 Q + u_i^2$ if $\alpha_i - 1 \neq 0$, otherwise a scalar multiple of u_i, and similarly for $\alpha_i + 1$ and G. Therefore $(\alpha_i - 1)^2 Q + u_i^2, (\alpha_i + 1)^2 Q + u_i^2 \in \mathbb{K}[V]$, which would imply that $Q \in \mathbb{K}[V]$, a contradiction.

Hence (H_1, H_i) do not intersect $\mathbb{K}[V]$ non-trivially, and since (H_1, H_i) is radical we must have $R_i \in (H_1, H_i) \cap \mathcal{F}_2$. This implies, after proper scalar multiplication, $R_i = \mu_i Q + \alpha \ell_1^2 + (1 - \alpha) \ell_i^2$, where $\alpha \neq 0, 1$ and $\mu_i \neq 0$. In particular, this implies $R_i \notin \{H_1, \dots, H_t\} \cup \mathbb{K}[W] \cup (\mathbb{L}(Q))$, and hence $R_i \in \{F_1, \dots, F_r\} \cup \mathcal{F}_{span}(Q)$. Note that, $R_i \neq R_j$, otherwise $H_j \in (H_1, H_i)$ which is a contradiction as above. This implies $t - 1 \leqslant |\mathcal{F}_{span}(Q)| + r$.

Thus, we have

$$\begin{aligned} m_2 &\leqslant |\mathcal{F}_{span}(Q)| + |\mathcal{F}_2 \cap \mathbb{K}[W] \cup (\mathbb{L}(Q))| + |\mathcal{F}_{bad}(Q)| \\ &\leqslant |\mathcal{F}_{span}(Q)| + m/10 + r + t \leqslant 2 \cdot |\mathcal{F}_{span}(Q)| + m/10 + 2r \leqslant 6 \cdot |\mathcal{F}_{span}(Q)| + m_2/3 \end{aligned}$$

which implies that $Q \in \mathcal{F}_{span}$. □

We are now ready to prove the main result of this section: that $(2, V)$-radical-SG configurations over an algebra are low-dimensional. As a reminder to the reader, we are assuming that V is $(600, 8)$-wide, $\nu = 1/20$ and define $\mathcal{F}_{span}$ with respect to ν.

Proof of Proposition 6.7. Suppose $\mathcal{F}_2 = \varnothing$. If $(\ell_i, \ell_j) \cap \mathcal{F}_1 = \varnothing$ for some $\ell_i, \ell_j \in \mathcal{F}_1 \backslash V_1$, then there exists $F \in \mathbb{K}[V]$ of degree at most 3 such that $F \in (\ell_i, \ell_j)$. If $\deg(F) > 1$, then we have a minimal collapse $F = \ell_i f + \ell_j g$. Therefore, by Lemma 4.16 and Proposition 2.2, we have $\ell_i, \ell_j \in V_1$ which is a contradiction. Thus, $\deg(F) = 1$ and then we have that $\mathcal{F}$ is a $(1, \dim(V))$-linear-SG configuration, which by Proposition 6.4 implies that $\dim \mathrm{span}_{\mathbb{K}} \{\mathcal{F}\} = O(\dim V)$ and we are done. Hence we can assume that $\mathcal{F}_2 \neq \varnothing$.

If $\mathcal{F}_2 = \mathcal{F}_{span}$, then $\mathcal{F}_2$ is a $(O(\dim V)^2), \nu)$-linear-SG configuration, and by Proposition 6.4 we have $\dim \mathrm{span}_{\mathbb{K}} \{\mathcal{F}_2\} \leqslant O(\dim(V)^2) + 25/\nu$. In particular, if $\mathcal{G} = \{Q \in \mathcal{F}_2 \mid s(Q) = 1\}$, we have that $U = \sum_{Q \in \mathcal{G}} \mathbb{L}(Q)$ is such that $\dim U \leqslant 4 \dim \operatorname{span}_{\mathbb{K}}\{\mathcal{F}_2\} \leqslant 4\dim(V) + 100/\nu$. Since two linear forms $x, y \in \mathcal{F}_1$ can only have dependencies $Q \in \mathcal{F}_2$ where $s(Q) = 1$, we have that $\mathcal{F}_1$ is a $(\dim V + \dim U, 1)$-linear-SG configuration, and thus $\dim \operatorname{span}_{\mathbb{K}}\{\mathcal{F}\} = O(\dim(V) + 1)$.

Now suppose $\mathcal{F}_2 \neq \mathcal{F}_{span}$. If $\mathcal{F}_2 \subset \mathbb{K}[V]$, then $\dim(\operatorname{span}_{\mathbb{K}}\{\mathcal{F}_2\}) = O(\dim(V)^2)$. Hence, by the argument above, we have $\mathcal{F}_1$ is a $(O(\dim(V)^2), 1)$-linear-SG configuration and hence $\dim \operatorname{span}_{\mathbb{K}}\{\mathcal{F}\} = O(\dim(V)^2 + 1)$. If $\mathcal{F}_2 \notin \mathbb{K}[V]$, then by Lemma A.9, there is a $(600, 1)$-wide vector space W with $\dim(W) \leqslant 8(\dim V + 600)$ such that $|\mathcal{F} \cap \mathbb{K}[W]| \geqslant m/10$. In this case, Proposition A.5 and Remark A.6 imply that $\dim \operatorname{span}_{\mathbb{K}}\{\mathcal{F}\} = O((\dim W + 1)^2) = O((\dim V + 1)^2)$. $\square$

B Auxiliary Claims

Proposition B.1. *If $x, y, z, w, \ell \in S_1$ are such that $\dim \operatorname{span}_{\mathbb{K}}\{x, y\} = 2$ and $xy - zw = \ell^2$, then we have that $z, w, \ell \in \operatorname{span}_{\mathbb{K}}\{x, y\}$.*

Proof. If $xy \in (\ell)$, then w.l.o.g. $x = \alpha\ell$ for some $\alpha \neq 0$ which implies that $zw = \ell(\alpha y - \ell)$ and unique factorization implies that $z, w \in \operatorname{span}_{\mathbb{K}}\{\ell, \alpha y\} = \operatorname{span}_{\mathbb{K}}\{x, y\}$ and we are done. If $xy \notin (\ell)$, then $zw \equiv xy \not\equiv 0 \bmod (\ell)$, which by factoriality of $S/(\ell)$ implies $z = \alpha x + \beta\ell$ and $\alpha w = y + \gamma\ell$ for some $\alpha \in \mathbb{K}^*$ and $\beta, \gamma \in \mathbb{K}$. Hence, we have $\alpha\ell^2 = \alpha xy - (\alpha x + \beta\ell)(y + \gamma\ell)$, which implies $(\alpha + \beta\gamma)\ell + \alpha\gamma x + \beta y = 0$. Since $\dim \operatorname{span}_{\mathbb{K}}\{x, y\} = 2$ and $\alpha \neq 0$, the last equation implies that $\ell \in \operatorname{span}_{\mathbb{K}}\{x, y\}$ and we are done. $\square$

Proposition B.2. *If $x, y, z, w \in S_1$ are such that $\dim \operatorname{span}_{\mathbb{K}}\{x, y\} = 2$, $xy \notin z$ and $w \notin \mathbb{K}[x, y, z]$, then $\alpha xy + \beta zw$ is irreducible for any $\alpha, \beta \in \mathbb{K}^*$.*

Proof. Let $\alpha, \beta \in \mathbb{K}^*$. Since x, y are independent, we have that $\operatorname{span}_{\mathbb{K}}\{x, z\} \cap \operatorname{span}_{\mathbb{K}}\{y, z\} = \operatorname{span}_{\mathbb{K}}\{z\}$. If $\alpha xy + \beta zw = ab$ for $a, b \in S_1$, note that $a, b \notin \operatorname{span}_{\mathbb{K}}\{z\}$, otherwise the equation $\alpha xy + \beta zw = ab$ would imply that $xy \in (z)$. Hence we must have, w.l.o.g., that $a \in \operatorname{span}_{\mathbb{K}}\{x, z\}$ and $b \in \operatorname{span}_{\mathbb{K}}\{y, z\}$. Thus, $\beta zw = ab - \alpha xy \in \mathbb{K}[x, y, z]$, which implies that $w \in \mathbb{K}[x, y, z]$. $\square$

Proposition B.3. *Let $C \in S_3$ be a cubic form such that $s(C) \geqslant 3$. Let $x, y \in S_1$ be linearly independent linear forms.*

1. *The minimal primes of the ideal (C, xy^2) are (C, x), (C, y), and we have $\operatorname{rad}(C, xy^2) = (C, xy)$*
2. *The only minimal prime of the ideal (C, x^3) is (C, x) and hence we have $\operatorname{rad}(C, x^3) = (C, x)$.*

Proposition B.4. *If $x, y, z, w \in S$ are linear forms such that $x^3 - y^3 = z^3 - w^3$, then we must have, $z, w \in (x) \cup (y)$.*

Proof. By factoriality of S, we have that $z, w \in (x, y)$. Suppose, for the sake of a contradiction, that the conclusion does not hold. We can assume, w.l.o.g., that $z \notin (x) \cup (y)$. In this case, we must have $z = \alpha x + \beta y$, where $\alpha, \beta \in \mathbb{K}^*$. Letting $w = \gamma x + \delta y$, we have that $x^3 - y^3 = z^3 - w^3$ iff the following equations hold: $\alpha^3 - \gamma^3 = 1 = \beta^3 - \delta^3$ and $\alpha^2\beta - \gamma^2\delta = 0 = \alpha\beta^2 - \gamma\delta^2$. Note that the latter equations imply that $\gamma, \delta \in \mathbb{K}^*$.

Now, multiplying the latter equations we obtain $\alpha^3\beta^3 = \gamma^3\delta^3$, which using the former equations imply $1 + \gamma^3 + \delta^3 = 0$. In particular, we have $\alpha^3 = -\delta^3$ and $\beta^3 = -\gamma^3$. If $\omega = e^{i\pi/3}$, we have that $\alpha = \omega^a\delta$ and $\beta = \omega^b\gamma$, where $a, b \in \{-1, 1, 3\}$.

From $\alpha^2\beta = \gamma^2\delta$ we obtain that $\omega^{2a+b}\gamma\delta^2 = \gamma^2\delta \Rightarrow \omega^{2a+b}\delta = \gamma$. However, this latter equation implies that $\gamma^3 = -\delta^3$, which contradicts the earlier equation $1 + \gamma^3 + \delta^3 = 0$. This concludes the proof of our claim. $\square$

Proposition B.5. *Let $C = xQ + yP \in S_3$ be irreducible, where $x, y \in S_1$ and $Q, P \in S_2$ such that $s(Q) \geqslant 8$ and $s(P) = \min_{\alpha \in \mathbb{K}} s(P + \alpha Q)$. If $C \in (G, a^2)$, for some $G \in S_2$ and $a \in S_1$, then we must have $\alpha \in \mathbb{K}^*$ such that $C = \alpha x G + a^2 b$ for some $b \in S_1$, and $\alpha G - Q \in (a, y)$.*

Proof. From $C \in (G, a^2)$ we have that $xQ + yP = C = zG + a^2 b$ for some $a, z \in S_1$. Hence, we have that $xQ \equiv a^2 b \bmod (z, y)$, which by $s(Q) \geqslant 8$ implies that $x \in (z, y)$ and $a^2 b \in (z, y)$. As C is irreducible, we know that $\dim \operatorname{span}_{\mathbb{K}} \{x, y\} = 2$ and thus we can write $z = \alpha x + \beta y$, for $(\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0, 0)\}$. In particular, the above implies

$$x(Q - \alpha G) \equiv a^2 b \bmod (y)$$
$$y(P - \beta G) \equiv a^2 b \bmod (x).$$

Since $\dim \operatorname{span}_{\mathbb{K}} \{x, y\} = 2$, the above congruences imply that $\operatorname{rank}(Q - \alpha G) \leqslant 2$ and $\operatorname{rank}(P - \beta G) \leqslant 2$, which in turn imply $\operatorname{rank}(\beta Q - \alpha P) \leqslant 4$. As $s(Q) \geqslant 8$, we must have $\alpha \neq 0$, which by definition of P implies $\operatorname{rank}(P) \leqslant \operatorname{rank}(\alpha P - \beta Q) \leqslant 4$. From $\operatorname{rank}(P) \leqslant 4$, we deduce that $\beta = 0$, otherwise $Q = \frac{1}{\beta}(\beta Q - \alpha P) + \frac{\beta}{\alpha} P \Rightarrow \operatorname{rank}(Q) \leqslant \operatorname{rank}(\beta Q - \alpha P) + \operatorname{rank}(P) \leqslant 8$, contradicting $s(Q) \geqslant 8$.

Therefore, we have that $a^2 b \in (x, y)$ and since $yP \notin (x)$ as C is irreducible, we must have $0 \not\equiv yP \equiv a^2 b \bmod (x)$. This gives us two cases:

Case 1: if $a \notin (x, y)$ the last congruence implies $P = x\ell + \gamma a^2$ and $b = \gamma y + \delta x$ for some $\gamma \in \mathbb{K}^*$ and $\delta \in \mathbb{K}$. Hence, as $x(Q - \alpha G) = a^2 b - yP$, we get $x(Q - \alpha G) = x(\delta a^2 - y\ell)$, which implies $\alpha G = Q + y\ell - \delta a^2$.

Case 2: if $a \in (x, y)$, then we have $a = \delta x + \gamma y$ and the last congruence implies $P = x\ell + \gamma^2 y b$ for some $\gamma \in \mathbb{K}^*$, $\delta \in \mathbb{K}$. Using $x(Q - \alpha G) = a^2 b - yP$, we get $x(Q - \alpha G) = x(\delta^2 x + y(2\delta\gamma b - \ell))$, which implies $\alpha G = Q - \delta^2 x - y(2\delta\gamma b - \ell)$. $\square$

B.1 Proof of Proposition 6.4

We give here a proof of Proposition 6.4. For the convenience of the reader, we restate the definition of a (c, δ)-linear-SG configuration.

Definition 6.3 (Robust linear Sylvester-Gallai configurations over a vector space)**.** Let $c \in \mathbb{N}$, $0 < \delta \leqslant 1$ and $\mathcal{F} := \{\ell_1, \ldots, \ell_m\} \subset S_1$ be a set of linear forms such that $\ell_i \notin (\ell_j)$ for any $i \neq j$. We say that $\mathcal{F}$ is a (c, δ)-linear-SG configuration if there exists a vector space $W \subset S_1$ of dimension at most c such that for any $\ell_i \in \mathcal{F} \setminus W$, there exist at least $\delta(m-1)$ indices $j \in [m] \setminus \{i\}$ such that $\ell_j \notin W$ and one of the following holds: $|\operatorname{span}_{\mathbb{K}} \{\ell_i, \ell_j\} \cap \mathcal{F}| \geqslant 3$ or $\operatorname{span}_{\mathbb{K}} \{\ell_i, \ell_j\} \cap W \neq 0$.

We will need the corollary from [Shp20, Corollary 16].

Proposition B.6. *A $(1, \delta)$-linear-SG configuration has dimension at most $1 + 24/\delta$.*

Proposition 6.4 (Robust Linear SG Configurations)**.** *Let $\mathcal{F}$ be a (c, δ)-linear-SG configuration. Then $\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{F}\} \leqslant c + 25/\delta$.*

Proof. Let $\mathcal{F} = \{\ell_1, \ldots, \ell_m\}$, and $W \subset S_1$ be the vector space given by Definition 6.3. Let $\varphi : S \to S[z]/(W)$ be a generic projection, and let $y_i = \varphi(\ell_i)$. We know that $\ell_i \notin W \Rightarrow y_i \notin (z)$. Moreover, by Corollary 2.8, for any pair $\ell_i, \ell_j \in \mathcal{F} \backslash W$, we have $y_i \notin (y_j)$. In particular, this implies that $\varphi(\mathcal{F})$ is a $(1, \delta)$-linear-SG configuration, which by Proposition B.6 implies that $\dim \operatorname{span}_{\mathbb{K}} \{\varphi(\mathcal{F})\} \leqslant 1 + 24/\delta$. Since $\mathcal{F} \subset \operatorname{span}_{\mathbb{K}} \{\varphi(\mathcal{F}), z, W\}$, we have $\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{F}\} \leqslant c + 1 + 24/\delta \leqslant c + 25/\delta$. $\square$

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