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A Proof of the Generalized Union-Closed Set Conjecture assuming the Union-Closed Set Conjecture

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Abstract

The Union Closed Set Conjecture states that if a set system $X \subseteq \mathcal{P}([n])$ is closed under pairwise unions, then there exists $a \in [n]$ in at least half of the sets of X. We show that there is a very natural generalization of the union closed set conjecture which gives a lower bound for k-set subsets of $[n]$. This a stronger version of a Conjecture of (Nagel, 2022). We then prove the Conjecture conditional on the Union Closed Set Conjecture using invariants of Union-Closed sets. Additionally, we prove that there exists a k-set in $.38^k|F|$ sets of a union closed set X for every $n \geq k > 0$. We explain why our result suggests a lack of sharpness of the original conjecture.

1 Definitions

In our discussion, we require the following basic definitions.

Definition 1. A set system X is a family of subsets of a universe of elements R.
In our paper X always occurs as a universe of subsets of $[n]$.

Definition 2. If X is a set system, then $X(a) := \{X' \in X | a \in X'\}$.

Definition 3. If X is a set system, then $X(A) := \{X' \in X | A \subseteq X'\}$.

2 Introduction

A seemingly simple if not trivial problem, the following conjecture has escaped proof for decades:

Conjecture 1. (Union Closed Set Conjecture) (Frankl, 1979). A union-closed set system X of a universe $[n]$, then there exists $a \in [n]$: $|X(a)|/2$.

We show that Conjecture 1 is equivalent to Conjecture 2 which shows its relevance to computational complexity theory and in particular, boolean circuit complexity.

Roughly speaking, the following formulation of the conjecture states that any set of length n strings of 0,1 entries which is closed under bitwise OR, has some entry of the strings which is 1 for at least half of all of the strings.

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We will require the following definition:

Definition 4. If a $x \in \{0,1\}^n$, then let x^s denote the sth entry of x.

We can now state the following:

Conjecture 2. If $\{x_i\}_{0 \leq i \leq |X|} = X \subseteq \{0,1\}^n$ and for all $x_i, x_j \in X$ $\vee(x_i, x_j) \in X$ where $\vee(.,.)$ is bitwise OR, then there exists $1 \leq s \leq n$ and $X' \subset X$ of cardinality at least $|X|/2$ such that for all $x \in X'$, we have that $x^s = 1$.

This formulation is obtained by corresponding each element of the universe $[n]$ to position on an n-string string setting each position to be 1 or 0 on an entry iff the set contains the element corresponding to that entry.

We will treat the conjecture in set-theoretic form and we will state its generalization in such a manner as well. First we present a recent generalization of (Nagel, 2022):

Conjecture 3. (Generalized Union Closed Set Conjecture I) (Nagel, 2022)

If X is a union-closed set system of a universe $[n]$, then for every $n \geq k > 0$, there exists an element $a \in [n]$ such that $|X(a)| \geq 2^{-k}|X|$.

We generalize this to stating essentially that not only are there additional elements occur, but whole subsets of size k for every $k \in [n]$ occur:

Conjecture 4. (Generalized Union Closed Set Conjecture II) If X is a union-closed set system of a universe $[n]$, then for every $n \geq k > 0$, there exists a k-set $A \subseteq [n]$ such that $|X(A)| \geq 2^{-k}|X|$.

We note that for non-integer fractions we take the floor function of the bound.

3 Main Theorem

We now show that Conjecture 3 is in fact equivalent to Conjecture 1.

Theorem 1. Conjecture 1 and Conjecture 2 are equivalent.

Proof.

Conjecture 1 implies Conjecture 2 if $k = 1$.

Lemma 1. If X is union-closed there's a 2-set element contained within $|X|/4$.

Proof.

Let X_n be an arbitrary union-closed set of universe $[n]$.

Then there exists $a \in [n]$ such that $|X_n(a)| \geq |X|/2$.

We define a set X_{n-1} on $n-1$ elements as follows $X_{n-1} := \{X_1(a) - \{a\}, X_2(a) - \{a\}, ..., X_{|X|}(a) - \{a\}\} \cup X|X(a)$.

Intuitively, X_{n-1} is the set obtained by removing $[n]$ from every set. And if two sets become identical by removal of $[n]$, then we save only one of these.

Claim 1.1. X_{n-1} is union closed.

Proof.

Choose two arbitrary sets X, X' within X_{n-1}. In the original set X_n, we any one of either: $\Upsilon_1 = \{X, X'\}$, $\Upsilon_2 = \{X \cup \{a\}, X'\}$, $\Upsilon_2 = \{X, X' \cup \{a\}\}, \Upsilon_4 = \{X \cup \{a\}, X' \cup \{a\}\}$ were contained within the original set systemX_n. And since X_n is union-closed by assumption, the union of $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4$ which is either $X' \cup X$ or $(X' \cup X) \cup \{a\}$ was contained in X_n. In the case of Υ_1, $(X' \cup X) \in X|X(a)$ and therefore is in X_{n-1}. In the cases $\Upsilon_2, \Upsilon_3,$ and Υ_4, $(X' \cup X) \cup \{a\} \in \{X_1(a), X_2(a), ..., X_{|X|}(a)\}$ and so $(X' \cup X) \in \{X_1(a) - \{a\}, X_2(a) - \{a\}, ..., X_{|X|}(a) - \{a\}\}$ which from the definition of X_{n-1}, contained in X_{n-1}. $(X' \cup X) \in X_n$. Therefore, for any two X, X' contained in X_{n-1}, it follows that $X \cup X' \in X_{n-1}$ and so X_{n-1} is union-closed. $\square$

Claim 1.2. $|X_{n-1}| \geq |X|/2$.

Proof.

X_n can be reconstructed at most by at most taking (1) a copy of X_{n-1} taking the element $\{n\}$ and (2) a copy of X_{n-1} with $\{n\}$ appended to every set. Formally we have the following:

$X_n \subseteq X_{n-1} \cup \{X' \cup \{a\} | X' \in X_{n-1}\}$. Recall the definition of X_{n-1} as:

$X_{n-1} := \{X_1(a) - \{a\}, X_2(a) - \{a\}, ..., X_s(a) - \{a\}\} \cup X|X(a)$.

The sets absorbed into each other are made distinct once again.

This clearly follows from the definition of X_{n-1}

It follows that $|X_n| \leq 2|X_{n-1}|$. $\square$

Since X_{n-1} is a union-closed family of the universe $[n]|\{a\}$ which is of size at least $|X|/2$, there exists an element within $|X|/4$ sets. Let us now denote a as a_1 and the common element in X_{n-1} as a_2. We will show the stronger claim that there are such elements which occur in the same sets in at least $|X|/4$ sets in X.

Before moving onto the next claim, we prove a more general lemma that will make the next claim easy. The proof will critically rely on a special case of the following Invariance Lemma.

Lemma 3 (Union Closure Invariance Lemma). If X is a union-closed subset, then for any $A \subseteq P([n])$, $X(A)$ is union-closed.

Proof.

Let $X \subseteq P([n])$ be an arbitrary union-closed set and choose an arbitrary $A \subseteq [n]$. Every set which is within $X(A)$ contains A and and so if X contains the union of every pair of sets within X, then for all $X, X' \in X$, $X' \cup X \in X$ and since $A \subseteq X' \cup X$, hence $X' \cup X \in X(A)$. $\square$

The theorem applies this lemma recursively by setting A to be the 1-element set contained within at least half of the sets of X, taking the set system after removing the element and repeating the process.

Claim 3.

There exists $a_1, a_2 \in [1, 2, .., n]$ such that $|X(\{a_1, a_2\})| \geq |X|/4$.

Proof.

We know that $|X(\{a_1\})| \geq |X|/2$ and is union closed by Lemma 3. Therefore, removing $\{a_1\}$ from each of the sets, the set is still union-closed. Then the Union-Closed Conjecture implies that there exists an element $a_2 \in [1, 2, ..., n]|\{a_1\}$ such that a_2 occurs in at least half of $X(\{a_1\})$ and therefore $|X(\{a_1, a_2\})| \geq 1/4$ $\square$

The remainder of the theorem follows by induction. □

Relying on recent progress on the conjecture due to (Alweiss et al, 2022), we can say the following:

Theorem 2. There exists a k-set subset of $[n]$ which occurs in $.38^k|X|$ sets in X.
Proof Sketch.
We simply repeat the argument of Theorem 1 with .38 instead of 1/2 and Theorem 2 follows. □

4 Conclusion

In this paper, we show that if the Union-Closed Set Conjecture is true, then not only is there a 1-element subset common to half of the sets, but in fact, we can amplify that result to show that there is a k-set which is common of $|X|/2^{-k}$ sets for each $k \in [n]$.
We run through a basic example of Theorem 1:

Example 1. The Power Set. The power set is a union-closed set where there is a k-set contained within a $|X|2^{-k}$ fraction of the sets.

This example also gives us a hint towards the lack of sharpness of the conjecture since of course only the power set can contain all of the elements which are in $|X|/2^{-n}$.
However for ANY union-closed family which contains all $[n]$ elements, its easy to see the following:

Lemma 2 If $X \subseteq P([n])$ is union-closed and contains all n elements, then $\cup_{X' \in X} X' \in X$.
Proof.
This follows from the more general lemma which is the following:

Lemma 2.1 If $X \subseteq P([n])$ is union-closed, then X is closed under arbitrary unions.
Proof.
Assume it is closed under unions of size k. For each of those unions of size k is another set in X, and therefore its union with any other set in X is contained within X by union closure. □

We can then take a union which is of size $|X|$ (ie. of all the sets in X) and it is contained within the X. □

It is clear that there are many union-closed sets containing the universe. Consider the power set excluding any specific set from within (along with the removal of all other sets which are necessary to maintain union closure). Clearly the set is still union closed and the union is $\{1, 2, ..., n\}$.

However, our theorem only implies the containment of the universe set when the set is equivalent to the power set. And as we showed there are in fact many sets which satisfy the property which contain the universe. This indicates that the conjecture is actually quite weak and researchers should keep this in mind when attempting to prove the conjecture. Perhaps a strengthening of it will determine the problem better making it easier to prove.

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