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A Proof of the Generalized Union-Closed Set Conjecture assuming the Union-Closed Set Conjecture

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Abstract

The Union Closed Set Conjecture states that if a set system $X \subseteq \mathcal{P}([n])$ is closed under pairwise unions, then there exists $a \in [n]$ in at least half of the sets of X. We show that there is a very natural generalization of the union closed set conjecture which gives a lower bound for k-set subsets of $[n]$. This a stronger version of a Conjecture of (Nagel, 2022). We then prove the Conjecture conditional on the Union Closed Set Conjecture using invariants of Union-Closed sets. Additionally, we prove that there exists a k-set in $.38^k|F|$ sets of a union closed set X for every $n \geq k > 0$. We explain why our result suggests a lack of sharpness of the original conjecture.

1 Definitions

In our discussion, we require the following basic definitions.

Definition 1. A set system X is a family of subsets of a universe of elements $\mathcal{U}(X)$. In our paper X always occurs as subsets of the $[n]$.

Definition 2. $\mathcal{U}(X)$. If X is a set system, then $\mathcal{U}(X) := \cup_{X' \in X} X'$.

In our paper, we will usually assume the above for convenience, without loss of generality.

Definition 3. $X(a)$. If X is a set system, for all $a \in \mathcal{U}(X)$, $X(a) := \{X' \in X | a \in X'\}$. We also sometimes write this as $X(\{a\})$.

Definition 4. $X(A)$. If X is a set system, for all $A \subseteq \mathcal{U}(X)$, $X(A) := \{X' \in X | A \subseteq X'\}$.

2 Introduction

A seemingly simple if not trivial problem postulated by Frankl in 1979, the Union-Closed Sets Conjecture has escaped proof for decades:

Conjecture 1. (Union-Closed Set Conjecture) (Frankl, 1979). If X is a union-closed set system of a universe $[n]$, then there exists $a \in [n]$ such that $|X(a)| \geq |X|/2$.

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At times we refer to the above simply as UCSC. Here we reformulate Conjecture 1 and show its equivalence to the following Conjecture 2 highlighting a manner of its relevance to computational complexity theory and in particular, boolean circuit complexity.

Roughly speaking, the following formulation of the conjecture states that any set of length n strings of 0,1 entries which is closed under bitwise OR, has some entry of the strings which is 1 for at least half of all of the strings. This formulation first appears in (Karpas, 2017).

First, we will require the following definition:

Definition 4. If a $x \in \{0,1\}^n$, then x^s denotes the sth entry of x.

We can now state the following:

Conjecture 2. If $\{x_i\}_{0 \leq i \leq |X|} = X \subseteq \{0,1\}^n$ and for all $x_i, x_j \in X$ we have that $\vee(x_i, x_j) \in X$ where $\vee(.,.)$ is bitwise OR, then there exists s such that $1 \leq s \leq n$ and $X' \subset X$ of cardinality at least $|X|/2$ such that for all $x \in X'$, we have that $x^s = 1$.

This formulation is obtained from Conjecture 1 by corresponding each element of the universe $[n]$ to a position on an n-string string setting each position to be 1 or 0 on the string for a set $A \subseteq [1, 2, .., n]$ iff the set A contains or does not contain the element corresponding to that entry respectively.

In this paper, we will treat the conjecture in set-theoretic form and we will state its generalization in such a manner as well. First we present the recent generalization of (Nagel, 2022):

Conjecture 3. (Generalized-Union Closed Set Conjecture I) (Nagel, 2022)

If X is a union-closed set system of a universe $[n]$, then for every $n \geq k > 0$, there exists distinct elements $a \in [n]$ such that $|X(a)| \geq 2^{-k}|X|$.

We note that for non-integer fractions we take the floor function of $2^{-k}|X|$.

In this paper we prove the following:

Theorem 1. Conjecture 3 is equivalent to Conjecture 1.

We further generalize this to stating essentially that not only are there additional elements which occur in smaller fractions, but whole subsets of size k for every $k \in [n]$ occur. In other words, the elements occur together.

Conjecture 4. (Generalized-Union Closed Set Conjecture II) If X is a union-closed set system of a universe $[n]$, then for every $n \geq k > 0$, there exists a k-set $A \subseteq [n]$ such that $|X(A)| \geq 2^{-k}|X|$.

In regards to Conjecture 4, we prove the following:

Theorem 2. Conjecture 4 is equivalent to Conjecture 1.

While clearly Theorem 2 implies Theorem 1, it is possible to prove Theorem 1 without proving Theorem 2. The proofs emerge upon taking a deep look at the structural properties of union-closed sets, both in general (eg. Lemma 1.1, Lemma 2.1) as well as assuming the Union-Closed Sets Conjecture.

In this paper, we also show the following unconditional result:

Theorem 3. If $X \subseteq \mathcal{P}([n])$ is union-closed, for every $n \geq k > 0$ there exists a k-element subset of $[n]$ A, such that $|X(A)| \geq .38^k|X|$.

This relies on the following recent progress on Conjecture 1 due to (Gilmer, 2022), (Sawin, 2022), (Alweiss et al, 2022), (Lovett et al, 2022):

Theorem 4. If $X \subseteq \mathcal{P}([n])$ is union-closed, there exists $i \in [n]$ such that $|X(i)| \geq .38|X|$.

We will show that Theorem 3 follows from a modification of the argument of Theorem 2 incorporating Theorem 4 instead of the UCSC.

3 Equivalence Theorems

We now show that Conjecture 3 is in fact equivalent to Conjecture 1.

Theorem 1. If a set $X \subseteq \mathcal{P}([n])$ is union-closed, there necessarily exists an element $a \in [n]$ such that $|X(a)| \geq |X|/2$ if and only if there exists distinct elements $(a_i)_{n \geq i > 0}$, $a_i \in [n]$ such that, such that $|X(a_i)| \geq 2^{-i}$ for every $n \geq i > 0$. In particular, Conjecture 1 and Conjecture 3 are equivalent.

Proof.

Conjecture 1 implies Conjecture 2 if $k = 1$.

For clarity, we denote X as X_n to indicate that it is of the universe of n elements.

Let X_n be an arbitrary union-closed set of universe $[n]$.

Then from the UCSC, there exists $a \in [n]$ such that $|X_n(a)| \geq |X|/2$.

Let $X_n(a)$ be labelled such that $X_n(a) = \{X_n^1(a), ..., X_n^{|X_n(a)|}(a)\}$. Then we define a set X_{n-1} on $n-1$ elements as follows $X_{n-1} := \{X_n^1(a) - \{a\}, X_n^2(a) - \{a\}, ..., X_n^{|X_n|}(a) - \{a\}\} \cup X_n|X_n(a)$.

Intuitively, X_{n-1} is the set obtained by removing $\{a\}$ from every set. And if two sets become identical by removal of $\{a\}$, then we save only one of these.

We now how the following two claims about X_{n-1}.

Claim 1.1. X_{n-1} is union closed.

Proof.

Choose two arbitrary sets X, X' within X_{n-1}. In the original set X_n, we any one of either: $\Upsilon_1 = \{X, X'\}$, $\Upsilon_2 = \{X \cup \{a\}, X'\}$, $\Upsilon_2 = \{X, X' \cup \{a\}\}, \Upsilon_4 = \{X \cup \{a\}, X' \cup \{a\}\}$ were contained within the original set systemX_n. And since X_n is union-closed by assumption, the union of $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4$ which is either $X' \cup X$ or $(X' \cup X) \cup \{a\}$ was contained in X_n. In the case of Υ_1, $(X' \cup X) \in X|X(a)$ and therefore is in X_{n-1}. In the cases $\Upsilon_2, \Upsilon_3,$ and Υ_4, $(X' \cup X) \cup \{a\} \in \{X_1(a), X_2(a), ..., X_{|X|}(a)\}$

and so $(X' \cup X) \in \{X_1(a) - \{a\}, X_2(a) - \{a\}, ..., X_{|X|}(a) - \{a\}\}$ which from the definition of X_{n-1}, contained in X_{n-1}. $(X' \cup X) \in X_n$. Therefore, for any two X, X' contained in X_{n-1}, it follows that $X \cup X' \in X_{n-1}$ and so X_{n-1} is union-closed. □

Claim 1.2. $|X_{n-1}| \geq |X_n|/2$.

Proof.

X_n can be reconstructed at most by at most taking (1) a copy of X_{n-1} and (2) a copy of X_{n-1} with $\{a\}$ appended to every set. To see this, consider that in to reconstruct X_n, (1) any two identical sets after the removal of $\{a\}$ absorbed into each other are made distinct once again, and otherwise, (2) for every set which had no identical set after the removal of $\{a\}$ set is added back to X_n with $\{a\}$ appended if the set contained $\{a\}$ originally in X_n. (In the latter case, whether $\{a\}$ is added or not, does not make a cardinal difference to the reconstructed set.) Hence, formally we have the following:

$X_n \subseteq X_{n-1} \cup \{X' \cup \{a\} | X' \in X_{n-1}\}$ and therefore $|X_n| \leq |X_{n-1}| + |\{X' \cup \{a\} | X' \in X_{n-1}\}| = 2|X_{n-1}|$. □

Since X_{n-1} is a union-closed family of the universe $[n]|\{a\}$ which is of size at least $|X|/2$, from the UCSC, there exists an element within $|X|/4$ sets. Now Conjecture 3 will follow by applying Claim 1.1 and Claim 1.2 recursively. This looks like doing induction, but backwards.

In particular, assume that we've constructed a set of size $2^{-k}|X| = X_{n-(k-1)}$ such that at least half of it contains an element (ie. within $2^{-(k+1)}|X|$ sets) and that such an element existed for each $0 \leq k' \leq k$. If $k = n$ then we are done. Otherwise, assume $k < n$. Remove this element from all of the sets in which it appears and consider the resulting set. From Claim 1.1, it is union-closed and from Claim 1.2 it contains at least $2^{-(k+1)}|X|$ sets. Then the UCSC implies that this set contains an element within $2^{-(k+2)}|X|$ sets. □

We now prove our stronger result.

Theorem 2. If a set $X \subseteq \mathcal{P}([n])$ is union-closed, there necessarily exists an element $a \in [n]$ such that $|X(a)| \geq |X|/2$ if and only if there exists a k-set $A \subseteq [1, 2, .., n]$, such that $|X(A)| \geq 2^{-k}$ for every $n \geq k > 0$. In particular, Conjecture 1 and Conjecture 4 are equivalent.

Proof.

To prove this theorem, we prove the following more general lemma that will make the next steps easy The proof will critically rely on a special case of the following Invariance Lemma.

Lemma 2.1 (Union Closure Invariance Lemma). If $X \subseteq \mathcal{P}([n])$ is union-closed, then for any $A \subseteq \mathcal{U}(X)$, $X(A)$ is union-closed.

Proof.

Let $X \subseteq P([n])$ be an arbitrary union-closed set and choose an arbitrary $A \subseteq \cup_{X' \in X} X' = \mathcal{U}(X)$. Choose X, $X' \in X$ arbitrarily. By definition of $X(A)$, $A \subseteq X$ and $A \subseteq X'$ and therefore, $A \subseteq X' \cup X$. Since $X' \cup X \in X$ by the union-closedness of X, and since $X(A)$ is the set of all subsets of X which contain A, clearly $X' \cup X \in X(A)$. □

We will use the above lemma recursively in order to construct such a k-set subset of $[n]$ common to $2^{-k}|X|$ sets in X. To begin this process, we now claim the following:

Claim 2.2.

There exists $a_1, a_2 \in \mathcal{U}(X)$ such that $|X(\{a_1, a_2\})| \geq |X|/2^{-2}$.

Proof.

We know that there exists $a_1 \in \mathcal{U}(X)$ such that $|X(\{a_1\})| \geq |X|/2$ from the UCSC. Since $X(\{a_1\})$ is union closed by Lemma 2.1, therefore, removing $\{a_1\}$ from each of the sets of $X(\{a_1\})$, we have the set $X(\{a_1\}) - \{a_1\} := \{X' - \{a\} : X' \in X(\{a_1\})\}$ is clearly union-closed. Then the UCSC implies that there exists an element $a_2 \in [1, 2, ..., n]|\{a_1\}$ such that a_2 occurs in at least half of $X(\{a_1\})$ and therefore, returning the elements $\{a_1\}$ to $X(\{a_1\}) - \{a_1\}$, we have that $|X(\{a_1, a_2\})| \geq |X(\{a_1\})|/2 \geq |X|/4$ □

We can now finish the proof as follows:

Suppose we have constructed a $2^{-k}|X|$ size subset of X with a common k-set $A \subseteq [1, 2, ..., n]$. If $k = n$, then we are done. Otherwise, assume $k < n$. Then $X(A)$ is union-closed by Lemma 2.1 and therefore, following the pattern of Claim 2.1, there exists $a_{k+1} \in [1, 2, ..., n]|A$ such that $|X(A \cup \{a_{k+1}\})| \geq |X(A)|/2 \geq 2^{-(k+1)}|X|$. □

4 An Unconditional Result

Relying on recent progress on the conjecture due to (Alweiss et al, 2022), we can say the following:

Theorem 3. There exists a k-set subset of $[n]$ which occurs in $.38^k|X|$ sets in X.

Proof Sketch.

We simply repeat the argument of Theorem 2 with .38 instead of 1/2 and replace the assumption of the UCSC with the Theorem 4 (see Section 2) and Theorem 3 follows. □

5 Conclusion

In this paper, we show that if the Union-Closed Set Conjecture is true, then not only is there a 1-element subset common to half of the sets, but in fact, we can amplify that result to show that there is a k-set which is common of $|X|/2^{-k}$ sets for each $k \in [n]$.

We run through a basic example of Theorem 1:

Example 1. The Power Set. The power set is a union-closed set where there is a k-set contained within a $|X|2^{-k}$ fraction of the sets.

This example also gives us a hint towards the lack of sharpness of the conjecture since of course only the power set can contain all of the elements which are in $|X|/2^{-n}$.

However for ANY union-closed family which contains all $[n]$ elements, its easy to see the following:

Lemma 3 If $X \subseteq \mathcal{P}([n])$ is union-closed and contains all n elements, then $\cup_{X' \in X} X' \in X$.

Proof.

This follows from the more general lemma which is the following:

Lemma 3.1 If $X \subseteq \mathcal{P}([n])$ is union-closed, then X is closed under arbitrary unions.

Proof.

Assume it is closed under unions of size k. For each of those unions of size k is another set in X, and therefore its union with any other set in X is contained within X by union closure. $\square$

We can then take a union which is of size $|X|$ (ie. of all the sets in X) and it is contained within X. $\square$

It is clear that there could be many union-closed sets containing the universe. Consider the power set excluding any specific number of sets in order of the least to greatest size. This shows that there are at least $2^n - 1$ union-closed sets containing the universe. Clearly such a set is still union closed and the union is $\{1, 2, ..., n\}$ as long as all the sets are not removed.

However, our theorem only implies the containment of the universe set when the set is equivalent to the power set. And as we showed there are in fact multiple sets which satisfy the property which contain the universe. This indicates that the conjecture is actually quite weak and researchers should keep this in mind when attempting to prove the conjecture. Perhaps a strengthening of it will determine the problem better making it easier to prove.

References (Coming in next revision)

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