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Coboundary and cosystolic expansion without dependence on dimension or degree

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Abstract

We give new bounds on the cosystolic expansion constants of several families of high dimensional expanders, and the known coboundary expansion constants of order complexes of homogeneous geometric lattices, including the spherical building of $SL_n(\mathbb{F}_q)$. The improvement applies to the high dimensional expanders constructed by Lubotzky, Samuels and Vishne, and by Kaufman and Oppenheim.

Our new expansion constants do not depend on the degree of the complex nor on its dimension, nor on the group of coefficients. This implies improved bounds on Gromov's topological overlap constant, and on Dinur and Meshulam's cover stability, which may have applications for agreement testing.

In comparison, existing bounds decay exponentially with the ambient dimension (for spherical buildings) and in addition decay linearly with the degree (for all known bounded-degree high dimensional expanders). Our results are based on several new techniques:

- We develop a new "color-restriction" technique which enables proving dimension-free expansion by restricting a multi-partite complex to small random subsets of its color classes.
- We give a new "spectral" proof for Evra and Kaufman's local-to-global theorem, deriving better bounds and getting rid of the dependence on the degree. This theorem bounds the cosystolic expansion of a complex using coboundary expansion and spectral expansion of the links.
- We derive absolute bounds on the coboundary expansion of the spherical building (and any order complex of a homogeneous geometric lattice) by constructing a novel family of very short cones.

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1 Introduction

High dimensional expansion, which is a generalization of graph expansion to higher dimensional objects, is an active topic in recent years. The importance of graph expansion across many areas of computer science and mathematics, suggests that high dimensional expansion may also come to have significant impact. So far we have seen several exciting applications including analysis of convergence of Markov chains [Ana+19], and constructions of locally testable codes and quantum LDPC codes [Din+22; PK22].

Several notions of expansion that are equivalent in graphs, such as convergence of random walks, spectral expansion, and combinatorial expansion, turn out to diverge into two main notions in higher dimensions.

The first is the notion of local link expansion which has to do with the expansion of the graph underlying each of the links of the complex; where a link is a sub-complex obtained by taking all faces that contain a fixed lower-dimensional face. This notion is qualitatively equivalent to convergence of random walks, it implies agreement testing, and it captures a spectral similarity between a (possibly sparse) high dimensional expander and the dense complete complex. It allows a spectral decomposition of functions on the faces of the complex in the style of Fourier analysis on the Boolean hypercube, see [Dik+18; KO20; GLL22; Baf+22; Gai+22].

The second notion is coboundary and cosystolic expansion. Here we look at the complex not only as a combinatorial object but also as a sequence of linear maps, called coboundary maps, defined by the incidence relations of the complex. The i-th coboundary map δ_i maps a function on the i-faces to a function on the $i+1$-faces,

$$C^0 \xrightarrow{\delta_0} C^1 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{d-1}} C^d$$

where $C^i = C^i(X, \mathbb{F}_2) = \{f : X(i) \to \mathbb{F}_2\}$ is the space of functions on i faces with coefficients in $\mathbb{F}_2$ (we will consider general groups of coefficients, beyond $\mathbb{F}_2$). The coboundary map δ_i is defined in a very natural way: the value of $\delta f(s)$ for any $s \in X(i+1)$ is the sum of $f(t)$ for all $s \supset t \in X(i)$ (the precise definition is in Section 2).

Coboundary (or cosystolic[1]) expansion captures how well the coboundary map tests its own kernel, in the sense of property testing. Given $f \in C^i$ such that $\delta f \approx 0$, coboundary expansion guarantees existence of some $g \in \ker \delta_i$ such that $f \approx g$. More precisely, a complex is a β coboundary (or cosystolic) expander if

$$wt(\delta f) \geqslant \beta \cdot \min_{g \in Ker \delta} \mathrm{dist}(f, g)$$

where $wt(\delta f)$ is the hamming weight of δf. We denote by $h^i(X)$ the largest value of β that satisfies the above inequality for all f.

Whereas for $i = 0$ coboundary expansion coincides with the combinatorial definition of edge expansion, for larger i, it may appear at first glance to be quite mysterious. However, this definition is far from being a merely syntactical generalization of the $i = 0$ case and turns out to provide a rich connection between topological and cohomological concepts and between several important concepts in TCS, which we describe briefly below.

The study of coboundary and cosystolic expansion was initiated independently by Linial, Meshulam and Wallach [LM06], [MW09] in their study of connectivity of random complexes, and by Gromov [Gro03] in his work on the topological overlapping property. Kaufman and Lubotzky [KL14] were the first to realize

[1]The difference between coboundary and cosystolic expansion is just whether the cohomology is 0 or not (i.e. whether $Ker\delta_{i+1} = Im\delta_i$). This distinction is not important for this exposition and the expansion inequality is the same in both cases.

the connection between this definition and property testing. This point of view is important in the recent breakthroughs constructing locally testable codes and quantum LDPC codes [Din+22; PK22] (see also earlier works [EKZ20]).

Moreover, the coboundary maps come from a natural way to associate a (simplicial) complex to a constraint satisfaction problem. Attach a Boolean variable to each i-face, and view the $(i+1)$-faces as parity constraints. The value that an assignment $f : X(i) \to \mathbb{F}_2$ on gives on $s \in X(i+1)$ is $\delta f(s)$. This connection to CSPs has been harnessed towards showing that the CSPs derived from certain cosystolic expanders are hard to refute for resolution and for the sum of squares hierarchy, [Din+; HL22].

In addition, cosystolic expansion of 1-chains (with non-abelian coefficients) of a complex has been connected to the stability of its topological covers [DM22]. Informally, a complex is cover-stable if slightly faulty simplicial covers are always "fixable" to valid simplicial covers. Surprisingly, this is related to agreement testing questions, particularly in the small 1% regime, which is a basic PCP primitive and part of the initial motivation for this work. Along this vein, very recently Gotlib and Kaufman [GK22] use coboundary expansion of 1-chains to construct a variation of agreement tests that they call list-agreement testing.

In light of all of the above, we believe that cosystolic expansion is a fundamental notion that merits a deeper systematic study. Along with the aim of exploring its various implications, a more concrete research goal would be to give strong bounds, and ultimately nail down exactly, the correct expansion values for the most important and well-studied high dimensional expanders. We mention that to the best of our knowledge even for the simplest cases, such as expansion of k-chains in the n-simplex, exact expansion values are not yet completely determined.

In this work we provide new bounds for the coboundary expansion of the spherical building, and the cosystolic expansion of known bounded-degree high dimensional expanders including the complexes of [LSV05b; LSV05a; KO21].

Two of the most celebrated results in this area are the works of [KKL14] and [EK16] showing that the bounded-degree families of Ramanujan complexes of [LSV05a] are cosystolic expanders. These works introduce an elegant local-to-global criterion, showing that if the links are coboundary expanders, and further assuming spectral expansion, then the entire complex is a cosystolic expander.

The estimates proven by [KKL14; EK16] for the coboundary expansion parameters are roughly

$$h^k(X) \geqslant \min(\frac{1}{Q}, (d!)^{-O(2^k)}).$$

(where X is a d dimensional LSV complex and Q is the maximal degree of a vertex which is roughly equal to $1/\lambda^{O(d^2)}$ in these complexes, where λ is the spectral bound on the expansion of the links). We completely get rid of the dependence on the ambient dimension d and on the maximal degree Q, and prove

Theorem 1.1. *For every integer $d > 1$ and every small enough $\lambda > 0$ let X be a d-dimensional LSV complex whose links are λ-one-sided expanders. For every group* [2] *Γ, every small enough $\lambda > 0$ and every integer $k < d - 1$, $h^k(X, \Gamma) \geqslant \exp(-O(k^6 \log k))$.*

Our bounds for h^k only depend on the dimension k of the chains, so for $k = 1$ they are absolute constants. For larger k we still suffer an exponential decay, although not doubly exponential. We do not know what the correct bound should be and whether dependence on k is at all necessary.

[2] The theorem holds for every group Γ for which cohomology is defined, namely, abelian groups for $k > 1$ and any group for $k = 1$.

The case of $k = 1$ is interesting even in complexes whose dimension is $d \gg 1$, because h^1 controls the cover stability of the complex, as shown in [DM22]. Our bounds also immediately give an improvement for the topological overlap constants, when plugged into the Gromov machinery [Gro10; DKW18; EK16]. We elaborate on both of these applications later below.

The result is proven by enhancing the local-to-global criterion of [EK16], and introducing a variant of the local correction algorithm that makes local fixes only if they are sufficiently cost-effective. This is inspired by and resembles the algorithms in [EK16; Din+22; PK22].

Our analysis is novel and departs from previous proofs: instead of relying on the so-called "fat machinery" of [EK16], our proof is 100% fat free and relies on the up/down averaging operators on *real-valued functions*. Our main argument is to show that, for a function h that is the indicator of the support of a (locally minimal) k-chain,

$$\|D \cdots Dh\|^2 \gtrsim \cdots \gtrsim \|DDh\|^2 \gtrsim \|Dh\|^2 \gtrsim \|h\|^2,$$

where D is the down averaging operator, and we write $a \gtrsim b$ whenever $a \geqslant \Omega(b)$. From here we easily derive a lower bound on $\|h\|^2$ showing that either the correction algorithm has found a nearby cocycle, or else the coboundary of our function was initially very large to begin with.

This method gives universal bounds on the cosystolic expansion of any complex whose links have both sufficient coboundary-expansion and sufficient local spectral expansion,

Theorem 1.2. *Let $\beta, \lambda > 0$ and let $k > 0$ be an integer. Let X be a d-dimensional simplicial complex for $d \geqslant k + 2$ and assume that X is a λ-one-sided local spectral expander. Let Γ be any group. Assume that for every non-empty $r \in X$, X_r is a coboundary expander and that $h^{k+1-|r|}(X_r, \Gamma) \geqslant \beta$. Then*

$$h^k(X, \Gamma) \geqslant \frac{\beta^{k+1}}{(k+2)! \cdot 4} - e\lambda.$$

Here $e \approx 2.71$ is Euler's number.

Armed with an improved local-to-global connection, we derive Theorem 1.1 from Theorem 1.2 by further strengthening the coboundary expansion of the links of the LSV complexes, namely spherical buildings. The best previously known bound on coboundary expansion of k-cochains in spherical buildings is due to [Gro10] and [LMM16]. They proved a lower bound of $\left(\binom{d+1}{k+1}(d+2)!\right)^{-1}$. This decays exponentially with the ambient dimension d, and with the cochain level k. We remove the dependence on d by developing a new technique which we call "color-restriction". The d-dimensional spherical buildings are colored, namely, they are $d+1$-partite. For a set of ℓ colors $F \subset [d+1]$, the color restriction X^F is the complex induced on vertices whose color is contained in F. The restriction to the the colors of F reduces the dimension of X from d to $\ell - 1$. We say that a color restriction X^F is a β-local coboundary expander, if X^F is a β-coboundary expander, and the same holds for the intersection of X^F with links (neighbourhoods) of faces whose color is disjoint from F. We show that if a typical color-restriction is a local coboundary expander, then the entire complex is a coboundary expander, and the expansion is independent of the dimension. Namely,

Theorem 1.3. *Let k, ℓ, d be integers so that $k + 2 \leqslant \ell \leqslant d$ and let $\beta, p \in (0, 1]$. Let X be a $(d+1)$-partite d-dimensional simplicial complex so that*

$$\mathop{\mathbb{P}}_{F \in \binom{[d+1]}{\ell}} \left[X^F \textit{ is a } \beta\textit{-locally coboundary expander} \right] \geqslant p.$$

Then $h^k(X) \geqslant \frac{p\beta^{k+1}}{e(k+2)!}$.

Finally, to prove that the spherical building satisfies the conditions of this theorem, we need to show that a typical random color-restriction is a good coboundary expander. For this we rely on the "cone machinery" developed by Gromov [Gro10], Kozlov and Meshulam [KM19], and Kaufman and Oppenheim [KO21]. We construct in Section 5, a novel family of short cones, thus proving the following.

Theorem 1.4. *Let* $k \geqslant 0$. *There is an absolute constant* $\beta_k = \exp(-O(k^5 \log k)) \geqslant 0$ *so that the following holds. Let* X *be the* $SL_n(\mathbb{F}_q)$*-spherical building for any integer* $n \geqslant k+1$ *and prime power* q. *Let* Γ *be any group. Then* X *is a coboundary expander with constant* $h^k(X,\Gamma) \geqslant \beta_k$.

In fact, we prove a more general version of this theorem, that holds for the order complex of any homogeneous geometric lattice, see Theorem 5.1.

Most earlier works on cosystolic expansion focus on $\mathbb{F}_2$ coefficients (see [KM18] and [DM22] for two exceptions). This is an important case especially in light of Gromov's result connecting $\mathbb{F}_2$-expansion and topological overlap. However, expansion (of 1-chains) with respect to more general coefficients is necessary for results on topological covers and in turn for agreement testing. The theorems stated above show expansion of k-chains with respect to coefficients not only in $\mathbb{F}_2$ but in general abelian groups Γ, and when $k=1$ also for non abelian groups Γ. In other words, the theorems hold for all groups of coefficients where the cohomology is defined.

Finally, we end with an upper bound. While most of our work is focused on lower bounds for coboundary and cosystolic expansion, we show in Appendix B that families of dense simplicial complexes cannot have cosystolic expansion greater than $1+o(1)$. This implies that high degree, in some weak sense, limits cosystolic expansion. It is interesting to compare this to a result of Kozlov and Meshulam that shows upper bounds on coboundary expansion of complexes with bounded degree [KM19].

1.1 Applications of cosystolic expansion

We describe two applications of cosystolic expansion for deriving topological properties of simplicial complexes.

Topological overlap. Cosystolic expansion was studied by [Gro10] to give a combinatorial criterion for the topological overlapping property. Let $f : X \to \mathbb{R}^k$ be continuous mapping (with respect to the natural topology on X), i.e. f realizes X in $\mathbb{R}^k$. A point $p \in \mathbb{R}^k$ is called c-heavily covered if

$$\mathop{\mathbb{P}}_{s \in X(k)}[p \in f(s)] \geqslant c.$$

A well known result by [FK81] showed that for every affine map from the complete 2-dimensional complex to the plane, there exists a $\frac{1}{27}$-heavily covered point. Gromov's greatly generalized this theorem to all *continuous* functions (instead of only affine functions), all dimensions k (instead of $k=2$) and complexes that are cosystolic expanders (instead of the complete complex), with c that depends on the dimension of the map k, as well as the cosystolic expansion constant. For a precise statement, see Section 6.

The motivation for [EK16] was to show that there exists families of bounded degree simplicial complexes which have this property. They use [LSV05a] complexes and achieve a lower bound of $c \geqslant \min(\frac{1}{Q}, (d!)^{-O(2^k)})$, which comes from their bound on cosystolic expansion. Here again, d is the dimension of X, which may be much larger than k, and Q is the maximal degree of a vertex in X.

Plugging in our bounds into Gromov's theorem gives an improved bound $c \geqslant \exp(-O(k^7 \log k))$ for the topological overlapping property. This bound is also free of the ambient dimension and of the degree.

Cover stability. [DM22] studied a topological locally testable property called *cover stability*. This property is equivalent to cosystolic expansion of 1-chains. A covering map between two simplicial complexes X, Y is a surjective t-to-1 simplicial map[3] $\rho : Y(0) \to X(0)$ such that for every $\tilde{u} \in Y(0)$ and $\rho(\tilde{u}) = u \in X(0)$, it holds that the links of $\tilde{u}, u$ are isomorphic $Y_{\tilde{u}} \cong X_u$.

Graph covers (also known as lifts) have been quite useful in construction of expander graphs. Bilu and Linial showed that random covers of Ramanujan graphs are almost Ramanujan [BL06]. A celebrated result by [MSS15] used these techniques to construct bipartite Ramanujan graphs of every degree. Recently, [Dik22] showed that random covers could also be applied for constructing new simplciail complexes that are local spectral expanders.

Dinur and Meshulam [DM22] show that there exists a test that for any simplicial complex X and an alleged cover given by a simplicial map $\rho : Y \to X$ samples q points $(u_i, \rho(u_i))$ and measures how close ρ is to an actual covering map. The query complexity of the test is $q = 3t$ points. Its soundness is affected by the cosystolic expansion of 1-chains. Using our new bounds on cosystolic expansion, we show that the complexes constructed in [LSV05a] or in [KO21] are cover-stable, i.e. that there exists some universal constant $c > 0$, such that for every $\rho : Y(0) \to X(0)$

$$\mathop{\mathbb{P}}_{(u_i, \rho(u_i))_{i=1}^{q}} [\text{test fails}] \geqslant c \cdot \min \left\{ \text{dist}(\rho, \psi) \mid \ \psi : Y(0) \to X(0) \text{ is a cover} \right\},$$

where the distance is Hamming distance.

Kaufman and Gotlib recently used cover stability to analyze new agreement tests on high dimensional expanders [GK22].

1.2 Related work

Coboundary and Cosystolic expansion was defined indpendently by Gromov [Gro10], and by Linial, Meshulam and Wallach [LM06], [MW09]. Gromov studied cosystolic expansion as a proxy for showing the topological overlapping property. Linial, Meshulam and Wallach were interested in analyzing high dimensional connectivity of random complexes.

Kaufman, Kazhdan and Lubotzky [KKL14] introduced an elegant local to global argument for proving cosystolic expansion of 1-chains in the *bounded-degree* Ramanujan complexes of [LSV05b; LSV05a]. This was significantly extended by Evra and Kaufman [EK16] to cosystolic expansion in all dimensions, thereby resolving Gromov's conjecture about existence of bounded degree simplicial complexes with the topological overlapping property in all dimensions. Kaufman and Mass [KM18] generalized the work of Evra and Kaufman from $\mathbb{F}_2$ to other groups as well, and used this to construct lattices with good distance.

Following ideas that appeared implicitly in Gromov's work, Lubotzky Mozes and Meshulam analyzed the expansion of many "building like" complexes [LMM16]. Kozlov and Meshulam [KM19] abstracted the main lower bound in [LMM16] to the definition of cones (which they call chain homotopies), in order to analyze the coboundary expansion of geometric lattices and other complexes. Their work also connects coboundary expansion to other homological notions, and gives an upper bound to the coboundary expansion of bounded degree simplicial complexes. In [KO21], Kaufman and Oppenheim defined the notion of cones in order to

[3] simplicial means that every i-face in Y is sent to an i-face in X.

analyze the cosystolic expansion of their high dimensional expanders (see [KO18]). In addition, they also come up with a criterion for showing that complexes admit short cones. They prove lower bounds on the cosystolic expansion of their complexes for 0- and 1-chains. The case of k-chains with $k \geqslant 2$ is still open.

Several works tried to define quantum LDPC codes as cohomologies of simplicial complexes. Cosystolic expansion is used for analyzing the distance of the quantum code. Works by Evra, Kaufman and Zémor [EKZ20] and by Kaufman and Tessler [KT21] used cosystolic expansion in Ramanujan complexes to construct quantum codes that beat the $\sqrt{n}$-distance barrier. This sequence of works culminated in the breakthrough work of [PK22] that construct quantum LDPC codes with constant rate and distance. This later code is a cohomology of a certain chain complex, albeit not a simplicial complex; and it is analyzed essentially through the cosystolic expansion. Developing new techniques for cosystolic expansion can be potentially useful in this domain as well.

1.3 Open questions

The works by [LMM16], [KM19] and [KO21] analyze a variety of symmetric complexes (that support a transitive group action). Could one combine our "color restriction" technique with the cone machinery to get lower bounds independent of degree and dimension on these complexes as well? There are a number of concrete constructions of local spectral high dimensional expanders that have excellent local spectral properties [CLY20; LMY20; Gol21; OP22; Dik22]. Are any of them cosystolic expanders?

Another intriguing direction of research is to develop more techniques for analyzing coboundary or cosystolic expansion. The current techniques are limited to complexes that either have a lot of symmetry, or have excellent local expansion properties. Are there other complexes with these properties?

Our expansion bounds still have a dependence on the level (k) of the chains. In the complete complex, for instance, this is not necessary. The complete complex is a $\beta = 1 + o(1)$ coboundary expander for all k-chains [LMM16]. It is not clear whether a dependence on k is necessary even in the spherical building. Which complexes have coboundary expansion that does not decay with the size of the chains?

Finally, the notion of coboundary and cosystolic expansion is closely related to locally testable codes and quantum LDPC codes. They also have connections to agreement expanders. It is interesting to find more applications for these expanders.

1.4 Overview of the proof of Theorem 1.1

We start with a complex X that is a finite quotient of the affine building, as constructed by [LSV05a]. Our goal is to lower bound the cosystolic expansion of X. The proof has three components:

- (Theorem 1.2) A new local-to-global argument that derives cosystolic expansion of the complex from coboundary and spectral expansion of its links.
- (Theorem 1.3) A general color restriction technique that reduces the task of analyzing the coboundary expansion of a partite complex, to that of analyzing the local coboundary expansion of random color restrictions of it.
- (Theorem 1.4) Bounds on random color restrictions of (links of) the spherical building. Towards this end we construct a novel family of short cones for the spherical building (not based on apartments as in previous works [LMM16]).

Below we give a short overview of each of these steps. For simplicity we assume in this subsection that $\Gamma = \mathbb{F}_2$, which captures the main ideas.

The local to global argument, Theorem 1.2. Let X be our simplicial complex. We describe a correction algorithm, that takes as input a k-chain $f : X(k) \to \mathbb{F}_2$, with small coboundary $\mathbb{P}[\delta f \neq 0] = \varepsilon$ and outputs a k-chain $\tilde{f} : X(k) \to \mathbb{F}_2$ close to f that has no coboundary, i.e. $\delta \tilde{f} = 0$. For this overview, we focus on $k = 1$, i.e. f is a function on edges, which already exhibits the main ideas.

Let $\eta > 0$ be some predetermined parameter. Our algorithm locally fixes "stars" of lower dimensional faces, that is, sets $A_r = \{s \in X(k) \mid s \supseteq r\}$ for $r \in X(j)$ (when $j \leqslant k$). The fix takes place only if it is sufficiently useful: whenever it decreases the weight of δf by at least $\eta \mathbb{P}[A_r]$. In the case at hand, $k = 1$, so r is either a vertex or an edge, so

1. If $r \in X(1)$, $A_r = \{r\}$ and a fix just means changing the value of $f(r)$.
2. If $r \in X(0)$, $A_r = \{ru\}_{u \sim r}$ are all edges adjacent to r. Here a fix means changing the values of all $\{f(ru) \mid u \sim r\}$ simultaneously.

Algorithm 1.5.

1. Set $f_0 := f$. Set $i = 0$.
2. While there exists a vertex or edge $r \in X(0) \cup X(1)$ so that A_r has an assignment that satisfies a $\eta \mathbb{P}[A_r]$-fraction of faces more than the current assignment.
   - Let $fix_r : A_r \to \Gamma$ be an optimal assignment to A_r.
   - Set $f_{i+1}(s) = \begin{cases} f_i(t) & r \nsubseteq s \\ fix_r(s) & r \subseteq s \end{cases}$.
   - Set i:=i+1.
3. Output the final function $\tilde{f} := f_i$.

The fact that we correct f locally only if the fix satisfies η fraction more triangles will promise that $\text{dist}(f, \tilde{f}) \leqslant \frac{1}{\eta} wt(\delta f)$. The output of the algorithm, $\tilde{f}$, is *not* necessarily locally minimal in the sense of [KKL14; EK16], but it is *"η-locally-minimal"*.

Notation: For functions $g, h : X(\ell) \to \mathbb{R}$ we denote by $\langle g, h \rangle = \mathbb{E}_{r \in X(\ell)}[g(r)h(r)]$ the usual inner product. For $\ell = 1, 2$, denote by D^ℓ the *down operator* that takes $h : X(2) \to \mathbb{R}$ and outputs $D^\ell h : X(2 - \ell) \to \mathbb{R}$ via averaging. Namely $D^\ell h(r)$ is the average of $h(s)$ over $s \supseteq r$, $\mathbb{E}_{s \supseteq r}[h(s)]$.

Let $h : X(2) \to \mathbb{R}$ indicate the support of a $\delta \tilde{f}$, so $h(t) = 1$ iff $\delta \tilde{f} \neq 0$. Our main argument is to show

$$\|D^3 h\|^2 \gtrsim \|D^2 h\|^2 \gtrsim \|Dh\|^2 \gtrsim \|h\|^2.$$

Eventually $D^3 h = \mathbb{E}[h]^2$ is just a constant function. This shows that $(\mathbb{E}[h])^2 = const \cdot \mathbb{E}[h]$ which implies that either the algorithm corrected f to a cosystol, i.e. $h = 0$, or that h has large weight, which implies that δf had large weight to begin with.

Let us show for example that $\|D^3 h\|^2 \gtrsim \|D^2 h\|^2$ given that $\|D^2 h\|^2 \gtrsim \|Dh\|^2 \gtrsim \|h\|^2$. To do so, we define an auxiliary averaging operator N based on a random walk from vertices to triangles, and use the fact that in local spectral expanders,

$$\|D^3 h\|^2 \approx \langle Nh, D^2 h \rangle. \tag{1.1}$$

The operator $N : \ell_2(X(2)) \to \ell_2(X(0))$ is defined by $Nh(v) = \mathbb{E}_s[h(s)]$, where s is sampled according to the following walk: Given $v \in X(0)$, sample some $t \in X(3)$ such that $v \in t$, and then go to the triangle $s = t \setminus \{v\}$. The proof of (1.1) follows by localizing the expectation to the links and relying on the link expansion as in [Opp18], [Dik+18, Claim 8.8] and in [KO20].

The key lemma in the proof shows that if there are many faces $s' \supseteq v_0$ such that $h(s') = 1$, then there are many s such that $v \notin s$, $\{v\} \cup s = t \in X(3)$, where $h(s) = 1$. More precisely, we will show that for every $v \in X(0)$ it holds that

$$Nh(v) \gtrsim \beta(D^2h(v) - \eta). \tag{1.2}$$

This immediately implies that

$$\begin{aligned}
\langle Nh, D^2h\rangle &= \mathop{\mathbb{E}}_{v}\left[D^2h(v)Nh(v)\right] \\
&\overset{(1.2)}{\gtrsim} \beta(\mathop{\mathbb{E}}_{v}\left[(D^2h(v))^2\right] - \eta \mathop{\mathbb{E}}_{v}\left[D^2h(v)\right]) \\
&\gtrsim \beta\|D^2h\|^2 - \beta\eta\|h\|^2 \\
&\gtrsim \beta\|D^2h\|^2.
\end{aligned}$$

The second inequality follows from $\mathbb{E}_v\left[D^2h(v)\right] = \mathbb{E}_s[h(s)] = \|h\|^2$. The last inequality follows from the assumption that $\|h\|^2 = O(\|D^2h\|^2)$. Combining this with (1.1) gives us the desired inequality.

Let us understand what is written in (1.2). On the right-hand side, $D^2h(v) = \mathbb{P}_{xy\in X_v(1)}[h(vxy) = 1]$ is the fraction of triangles vxy containing v, such that $\delta\tilde{f}(vxy) \neq 0$. On the left-hand side, $Nh(v)$ is the fraction of s that complete v to some $t = v \cup s \in X(3)$, so that $\delta\tilde{f}(s) \neq 0$. For such an $s = uxy$,

$$0 = \delta\delta\tilde{f}(vuxy) = \delta\tilde{f}(uxy) + (\delta\tilde{f}(vux) + \delta\tilde{f}(vuy) + \delta\tilde{f}(vxy)). \tag{1.3}$$

Set $g : X_v(1) \to \mathbb{F}_2$ to be $g(xy) = \delta\tilde{f}(vxy)$, and note that g has the following properties:

1. By (1.3), $\delta\tilde{f}(uxy) = 1 \iff \delta g(uxy) = 1$.
2. $\mathbb{P}[g \neq 0] = \mathbb{P}_{s\ni v}\left[\delta\tilde{f}(s) \neq 0\right] = D^2h(v)$.
3. η-local-minimality: $\mathrm{dist}(g, B^1(X_v)) \geqslant \mathbb{P}[g \neq 0] - \eta$, where $B^1(X_v) = \{\delta\psi \mid \psi : X_v(0) \to \mathbb{F}_2\}$ is the set of coboundaries.

We explain the third item. Assume towards contradiction that $\mathrm{dist}(g, B^1(X_v)) < \mathbb{P}[g \neq 0] - \eta$ and let $\delta\psi$ be a coboundary closest to g. Then by changing the values of $\tilde{f}$ on A_v to be $\tilde{f}'(vu) := \tilde{f}(vu) + \psi(u)$, we have that whenever $g(xy) = \delta\psi(xy)$, then the fixed function satisfies $\delta\tilde{f}'(vxy) = 0$. I.e.

$$\mathrm{dist}(g, \delta\psi) = \mathop{\mathbb{P}}_{vxy}\left[\delta\tilde{f}'(vxy) = 0\right] < \mathop{\mathbb{P}}_{vxy}\left[\delta\tilde{f}(vxy) = 0\right] - \eta.$$

This is a contradiction to the η-local minimality of $\tilde{f}$ which is guaranteed by the algorithm.

Here is where the coboundary expansion of X_v comes into play. By coboundary expansion, we have that $\mathbb{P}[\delta g(uxy) = 1] \geqslant \beta\,\mathrm{dist}(g, B^1(X_v))$. By combining the above we will get that

$$Nh(v) = \mathop{\mathbb{P}}_{uxy\in X_v(2)}\left[\delta\tilde{f}(uxy) \neq 0\right] \geqslant \beta(\mathop{\mathbb{P}}_{xy\in X_v(1)}[g(xy) \neq 0] - \eta) = \beta(D^2h(v) - \eta).$$

The "color restriction" technique, Theorem 1.3. For this overview, assume that $k = 2$ The full details are in Section 4. Let Y be a d-dimensional $(d+1)$-partite complex so that a p-fraction of its color restrictions Y^F are β-local-coboundary expanders. We begin with a 2-chain $f : Y(2) \to \mathbb{F}_2$ with small coboundary, namely $\mathbb{P}_{s \in Y(3)}[\delta f(s) \neq 0] = \varepsilon$. We need to find a 1-chain $g : Y(1) \to \mathbb{F}_2$ so that $\mathrm{dist}(f, \delta g) \leqslant O(\frac{\varepsilon}{\beta^3 p})$.

We first select a random color restriction, i.e. a set of colors so that Y^F is a local coboundary expander, that the weight of δf when restricted to triangles whose colors are in F is close to weight of δf on all Y. Averaging arguments guarantee that such F exists. Using this F, we construct g in three steps. In the first step we define g on edges with both endpoints colored in F, $uv \in Y^F$. In the second step we define g on edges with one endpoint colored in F, i.e. $uv \in Y(1)$ where $u \in Y^F$ and $v \notin Y^F$. In the third step we define g on edges $uv \in X(1)$ with neither endpoints colored in F, i.e. where $u, v \notin Y^F$. Every step uses the values of g that were constructed in the step before. For $k > 2$ the $(k-1)$-chain is constructed following a similar sequence of $k+1$ steps.

1. We start with the values of g on edges $vu \in Y^F(1)$. By the choice of F, the weight of δf inside Y^F is roughly ε. Local coboundary expansion implies that there exists a 1-chain g_0 whose coboundary is close to f on Y^F. We set $g(uv) = g_0(uv)$ for all $uv \in Y^F(1)$.

2. Next we define g on edges vu so that $v \notin Y^F$ and $u \in Y^F$. Fix some $v \notin Y^F$. Let $Y_v^F = \left\{ s \in Y^F \;\middle|\; s \cup v \in Y \right\}$. This is the color restriction of the link of v. We wish to set values for $g(vu)$ for all edges vu such that $u \in Y_v^F(0)$. We describe a system of equations that we use to set the values of g on the edges vu so as to satisfy a maximal number of equations. For every $u_1u_2 \in Y_v^F(1)$, the triangle vu_1u_2 defines an equation:

$$f(vu_1u_2) + g(u_1u_2) = g(u_1v) + g(u_2v). \tag{1.4}$$

   Note that the left-hand side of the equation is known since we have the values of f on all triangles, and we already constructed g for edges $u_1u_2 \in Y^F(1)$. So the above is an equation with two unknowns. We set $g(vu)$ simultaneously for all $u \in Y_v^F(1)$ to be an assignment that satisfies the largest fraction of equations (ties broken arbitrarily).

   The idea behind this step is the following. Obviously, we'd like that $f(vu_1u_2) = g(u_1u_2) + g(u_1v) + g(u_2v)$ for as many triangles as possible, so it makes sense to define g to satisfy the largest amount of equations (1.4). Let $h_v : Y_v^F(1) \to \mathbb{F}_2$ be the left-hand side of (1.4), i.e. $h_v(u_1u_2) = f(vu_1u_2) + g(u_1u_2)$. We want to find an assignment $g_v : Y_v^F(0) \to \mathbb{F}_2$ so that $h_v(u_1u_2) = g_v(u_1) + g_v(u_2)$ for as many equations (1.4) as possible (and set $g(vu) = g_v(u)$). Finding a solution $g_v : Y_v^F(0) \to \mathbb{F}_2$ that satisfies (1.4) is equivalent finding g_v so that $h_v(u_1u_2) = \delta g_v(u_1u_2)$. Hence, to find an assignment that satisfies most of the equations is the same showing that h_v is close to a coboundary. In the analysis we show that $\delta h_v \approx 0$. This together with the local coboundary expansion of Y^F (which says that $h^1(Y_v^F, \mathbb{F}_2) \geqslant \beta$) will show that indeed we can find satisfying $\{g_v\}_{v \notin Y^F}$ so that $f \approx \delta g$ where the distance is over edges uv where $v \notin Y^F, u \in Y^F$.

3. Finally we need to define the values of g on edges vu so that $v, u \notin Y^F$. Let vu be such an edge. Every triangle uvw where $w \in Y_{vu}^F(0)$ defines a constraint on $g(vu)$:

$$f(uvw) + g(uw) + g(vw) = g(uv). \tag{1.5}$$

   As in the previous case, $f(uvw)$ is known, and $g(uw), g(vw)$ were determined in step 2. We set

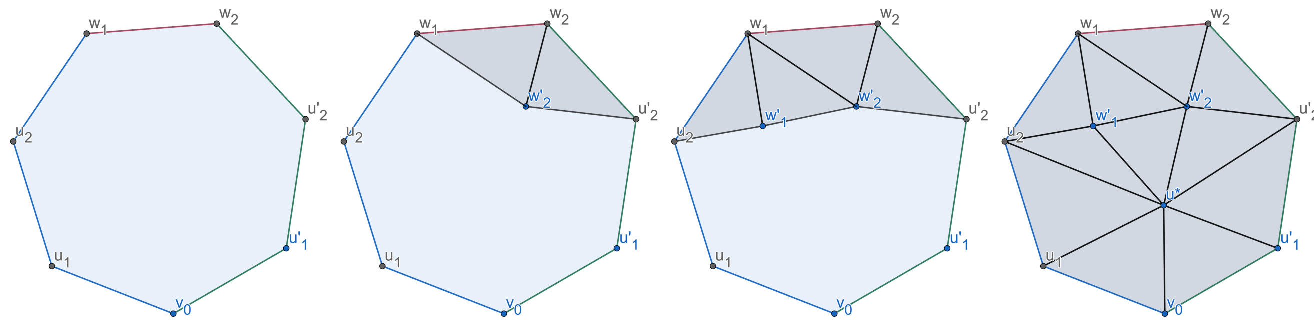

Figure 1: Tiling a cycle

$g(vu) = maj\left\{ f(uvw) + g(uw) + g(vw) \;\middle|\; w \in Y^F_{uv}(0) \right\}$. Ties are broken arbitrarily. Here we use the local coboundary expansion of Y^F in a way similar to the previous step, to show that indeed $f \approx \partial g$.

New bounds on color-restrictions of the spherical building via cones, Theorem 1.4. In order to apply the color restriction technique we need to show that for a d-dimensional spherical building, many color restrictions are coboundary expanders[4]. For this overview we assume that $k = 1$ and $|F| = 5$. Let us see how to bound coboundary expansion by constructing short cones.

It turns out easier to do so when the set of colors is a set of colors that are geometrically increasing (e.g. for $k = 1$ we need colors $F = \{i_1, i_2, ..., i_5\}$ so that $i_j \geqslant 10 i_{j-1}$). The fraction of such sets of colors F is a constant that doesn't depend on d (it may depend on k). For example, there is a constant probability that we select colors F so that for $j = 1, 2, .., 5$, $\frac{d}{10^{16-3j}} \leqslant i_j < \frac{2d}{10^{16-3j}}$, since each of these intervals are a constant fraction of the interval $[1, 2, ..., d]$. When these inequalities hold then $i_j \geqslant 10 i_{j-1}$.

Denote by Y the $SL_d(\mathbb{F}_q)$-spherical building. Let Y^F be a complex induced by the subspaces of dimensions (i.e., colors) $F = \{i_1, i_2, ..., i_5\}$ so that $i_j \geqslant 10 i_{j-1}$). Using the cone technology described in Section 5, showing the Y^F is a coboundary expander reduces to showing that there is a short 1-cone on Y^F. A 1-cone consists of three things:

1. A vertex $v \in X(0)$ (sometimes called the apex).
2. For every u, a path p_u from the apex v to u in $Y^F(1)$.
3. For every edge $uw \in$, a tiling by triangles $t_{uw} \subset Y^F(2)$ of the cycle that consists of the path p_u from v to u, the edge uw and the path p_w from w back to v. Denote this cycle by $p_u \circ uw \circ p_w$. Here a tiling is a set of triangles whose boundary is the edges of the cycle.

We give a formal and general definition of cones in Section 5. The radius of a cone is $rad((v, \{p_u\}_{u \in Y^F(0)}, \{t_{uw}\}_{uw \in Y^F(1)})) = \max_{uw \in X(1)} |t_{uw}|$.

We start by choosing an apex $v = v_0$ of dimension i_1 arbitrarily. Next we choose our paths to be as short as possible, and to consist of subspaces of dimension as low as possible. Explicitly we do the following.

1. For u adjacent to v_0, set $p_u = (v_0, u)$.

[4] In fact, we need to show that the links of the color restrictions are also coboundary expanders, but we ignore this point in the overview for brevity.

2. For u of the same dimension as v_0 we find some w of dimension i_2 so that w is a neighbour of v_0 and u, and set $p_u = (v_0, w, u)$. This is always possible since the dimension of $u + v_0$ is at most $2i_1$, so we can take any w of dimension $i_2 \geqslant 2i_1$ that contains the sum of spaces. (Notice how the fact that dimensions are geometrically increasing is important here).

3. For other $u \in Y^F(0)$, we first take some $w_2 \subseteq u$ of dimension i_1. Then we find some w_1 who is a neighbour of v_0 and of w_2 and we set $p_u = (v_0, w_1, w_2, u)$.

Constructing $t_{w_1 w_2}$ requires more care. Let us first consider the easier case. If $dim(w_1), dim(w_2) \leqslant i_4$ then the cycle $p_{w_1} \circ w_1 w_2 \circ p_{w_2}$ contains at most 7 vertices, all of dimension $\leqslant i_4$. In particular, the sum of all the vertices/subspaces is of dimension at most $7i_4 \leqslant i_5$, so there is a vertex u^* of dimension i_5 that contains all the vertices in the cycle. The set of triangles $u^* xy$ for all edges xy in the cycles is indeed a tiling of the cycle.

In the general case, it could be that the dimension of (say) w_1 is i_5. For example, assume that $dim(w_1) = i_5, dim(w_2) = i_4$ (in particular $w_2 \subseteq w_1$. It is useful to read this description while looking at Figure 1. In this case, we first find a tiling that "shifts" the cycle to a cycle of low dimension vertices. More explicitly, we find some $w_2' \subseteq w_2$ of dimension i_3, that is also connected to w's neighbours in the cycle. These neighbours are w_1 (and any subspace of w_2 is connected to it), and some u_2' of dimension $\leqslant i_2$, so we can indeed find some w_2' that is connected to u and u_2' of dimension i_3. We tile the cycle with $w_2 w_2' u_2', w_2 w_2' w_1$. This exchanges w_2 with w_2' in the untiled cycle. We perform a similar vertex-switch, for w_1 as well, finding some w_1' of dimension i_4 that is connected to w_1 neighbours in the untiled cycle. After these two steps, we can find a u^* that is connected to all the (now low-dimensional) cycle as in the previous case.

1.5 Organization of this paper

Section 2 contains preliminaries. We prove Theorem 1.2 that connects coboundary expansion in links to cosystolic expansion in Section 3 via the local correction algorithm. We develop the "color restriction" technique and prove Theorem 1.3 in Section 4. We analyze the expansion of the spherical building and other homogeneous geometric lattices in Section 5. We tie everything up and prove Theorem 1.1 in Section 6. In this section we present applications of our new bounds for better cover stability and topological overlap. In Appendix B we show an upper bound on the cosystolic expansion of dense complexes.

2 Preliminaries and notation

Simplicial complexes. A pure d-dimensional simplicial complex X is a set system (or hypergraph) consisting of an arbitrary collection of sets of size $d+1$ together with all their subsets. The sets of size $i+1$ in X are denoted by $X(i)$, and in particular, the vertices of X are denoted by $X(0)$. We will sometimes omit set brackets and write for example $uvw \in X(2)$ instead of $\{u, v, w\} \in X(2)$. As convention $X(-1) = \{\emptyset\}$. Unless it is otherwise stated, we always assume that X is finite. Let X be a d-dimensional simplicial complex. Let $k \leqslant d$. We denote the set of oriented k-faces in X by $\vec{X}(k) = \{(v_0, v_1, ..., v_k) \mid \{v_0, v_1, ..., v_k\} \in X(k)\}$. For $s = (v_0, v_1, ..., v_k) \in \vec{X}(k)$ we denote $set(s) = \{v_i\}_{i=0}^k$, but when its clear from context we abuse notation and write s for its underlying set instead of $set(s)$. For an oriented face $s \in \vec{X}(k)$ and an index $i \in \{0, 1, ..., k\}$, we denote by s_i the face obtained by removing the i-th vertex of s.

Probability over simplicial complexes. Let X be a simplicial complex and let $\mathbb{P}_d : X(d) \to (0,1]$ be a density function on $X(d)$ (i.e. $\sum_{s \in X(d)} \mathbb{P}_d(s) = 1$). This density function induces densities on lower level faces $\mathbb{P}_k : X(k) \to (0,1]$ by $\mathbb{P}_k(t) = \frac{1}{\binom{d+1}{k+1}} \sum_{s \in X(d), s \supset t} \mathbb{P}_d(s)$. We can also define a probability over directed faces, where we choose an ordering uniformly at random. Namely, for $s \in \vec{X}(k)$, $\mathbb{P}_k(s) = \frac{1}{(k+1)!} \mathbb{P}_k(set(s))$. When it's clear from the context, we omit the level of the faces, and just write $\mathbb{P}[T]$ or $\mathbb{P}_{t \in X(k)}[T]$ for a set $T \subseteq X(k)$.

2.1 Coboundary and cosystolic expansion

Asymmetric functions. Let X be a d-dimensional simplicial complex. Let $-1 \leqslant k \leqslant d$ be an integer. Let Γ be a group. A function $f : \vec{X}(k) \to \Gamma$ is *asymmetric* if for every $(v_0, v_1, ..., v_k) \in \vec{X}(k)$, and every permutation $\pi : [k] \to [k]$ it holds that

$$f(v_0, v_1, ..., v_k) = f(v_{\pi(0)}, v_{\pi(1)}, ..., v_{\pi(k)})^{sign(\pi)}.$$

We denote the set of these functions by $C^k(X, \Gamma)$. We note that by fixing some order to the vertices $X(0) = \{v_0, v_1, ..., v_n\}$, there is a bijection between functions $f : X(k) \to \Gamma$ and asymmetric functions $\vec{f} : \vec{X}(k) \to \Gamma$. Given $f : X(k) \to \Gamma$ and a set $s = \{v_{i_0}, v_{i_1}, ..., v_{i_k}\}$ so that $i_0 < i_1 < ... < i_k$, we set $\vec{f}(v_{\pi(i_0)}, v_{\pi(i_1)}, ..., v_{\pi(i_k)}) = f(s)^{\text{sign}(\pi)}$.

We record the following useful relation.

Claim 2.1. Let $s \in X(j)$. For every $x \in X_s$ and every asymmetric function $g : X(k) \to \Gamma$ it holds that $\sum_{i_1=0}^{j} \sum_{i_2=0}^{j-1} (-1)^{i_1+i_2} g((s_{i_1})_{i_2} \circ x) = 0$.

Let $f : \vec{X}(k) \to \Gamma$. The weight of f is $wt(f) = \mathbb{P}_{t \in X(k)}[f(t) \neq 0]$. For two functions $f, g : \vec{X}(k) \to \Gamma$ the distance between f and g is $\text{dist}(f,g) = wt(f-g) = \mathbb{P}_{t \in X(k)}[f(t) \neq g(t)]$.

Cohomology. Let Γ be an abelian group. The coboundary operator $\delta_k : C^k(X,\Gamma) \to C^{k+1}(X,\Gamma)$ is defined by

$$\delta_k f(s) = \sum_{i=0}^{k} (-1)^i f(s_i).$$

It is a direct calculation to verify that $\delta_k f$ is indeed an asymmetric function, and that $\delta_{k+1} \circ \delta_k = 0$.

Let $B^k(X,\Gamma) = \text{I}m(\delta_{k-1})$ be the space of coboundaries. Let $Z^k(X,\Gamma) = \text{K}er(\delta_k)$ be the space of cosystols. As $\delta_{k+1} \circ \delta_k = 0$, it holds that $B^k(X,\Gamma) \subseteq Z^k(X,\Gamma)$. The k-cohomology is $H^k(X,\Gamma) = Z^k(X,\Gamma)/B^k(X,\Gamma)$.

Coboundary expansion. For a function $f : \vec{X}(k) \to \Gamma$ let $\text{dist}(f, B^k) = \min_{g \in C^{k-1}} \text{dist}(f, \delta g)$, be the minimal distance between f and a coboundary. The k-th coboundary constant of a complex X (with respect to an abelian group Γ) is

$$h^k(X,\Gamma) = \min_{f \in C^k \setminus B^k} \frac{wt(\delta f)}{\text{dist}(f, B^k)}.$$

where $B^k = B^k(X,\Gamma)$. Note that $h^k(X,\Gamma) > 0$ if and only if $H^k = 0$.

Cosystolic expansion. A very related high dimensional notion of expansion is cosystolic expansion. The k-th cosystolic expansion constant of X (with respect to an abelian group Γ) is

$$h^k(X, \Gamma) = \min_{f \in C^k \setminus Z^k} \frac{wt(\delta f)}{\operatorname{dist}(f, Z^k)},$$

where $Z^k = Z^k(X, \Gamma)$. Notice that when $B^k(X, \Gamma) = Z^k(X, \Gamma)$, namely, when $H^k = 0$, this coincides with the definition of coboundary expansion, and this justifies using the same notation h^k, where the term coboundary expansion (as opposed to cosystolic expansion) is taken to indicate $H^k = 0$.

Another useful way to understand the constant is the following. $h^k(X, \Gamma) \geqslant \beta$ if and only if for every $f : \overrightarrow{X}(k) \to \Gamma$ there is some $h \in Z^k(X, \Gamma)$ so that $\beta \operatorname{dist}(f, h) \leqslant wt(\delta f)$. We note that in the work of [EK16] cosystolic expanders were also required to have no small weight $f \in Z^k(X, \Gamma) \setminus B^k(X, \Gamma)$. We don't focus on this notion in our work.

Non abelian coboundary and cosystolic expansion. For $k = 0, 1$ we can define the cohomology with respect to non abelian groups as well. Let Γ be a non abelian group. As before, for every k we can define $C^k(X, \Gamma)$. We define the coboundary operators as follows:

1. $\delta_{-1} : C^{-1}(X, \Gamma) \to C^0(X, \Gamma)$ is $\delta_{-1} h(v) = h(\emptyset)$.
2. $\delta_0 : C^0(X, \Gamma) \to C^1(X, \Gamma)$ is $\delta_0 h(vu) = h(v)h(u)^{-1}$.
3. $\delta_1 : C^1(X, \Gamma) \to C^2(X, \Gamma)$ is $\delta_1 h(vuw) = h(vu)h(uw)h(wv)$.

It is easy to check that $\delta_{k+1} \circ \delta_k f = e$ where $e \in \Gamma$ is the unit. The definitions for $h^k(X, \Gamma)$ and coboundary expansion are the same as in the abelian case for $k = 0, 1$.

Edge expander graphs.

Definition 2.2. Let $G = (V, E)$ be a graph and let $\lambda > 0$. We say that G is a λ-edge expander if for every $S \subseteq V$ so that $0 < \mathbb{P}[S] \leqslant \frac{1}{2}$, it holds that $\mathbb{P}[E(S, V \setminus S)] \geqslant \lambda \mathbb{P}[S]$.

It is well known that any λ-one-sided spectral expander is a $\frac{1-\lambda}{2}$-edge expander. The following claim shows that every λ-edge expander graph G also has $h^0(G, \Gamma) \geqslant \frac{\lambda}{2}$ for any group Γ.

Claim 2.3. Let $G = (V, E)$ be a λ-edge expander. Let $S_1, S_2, ..., S_m \subseteq V$ be mutually disjoint sets so that $V = \bigcup_{j=1}^m S_j$ and so that $< \varepsilon$ edges cross between the sets for $\varepsilon \leqslant \frac{\lambda}{2}$. Then there exists j so that $\mathbb{P}[S_j] \geqslant 1 - \varepsilon/\lambda$.

For any group Γ and $h : X(0) \to \Gamma$, we take as sets $S_g = h^{-1}(g)$. Edges so that $\delta h \neq 0$ are edges that cross between $S_g, S_{g'}$ for some $g \neq g'$. By Claim 2.3, when there are ε-edges crossing the cut, then there exists some $g : X(-1) \to \Gamma$ so that $\mathbb{P}[S_g] \geqslant 1 - \frac{\varepsilon}{\lambda}$. Then $\mathbb{P}[h \neq \delta g] \geqslant \frac{\varepsilon}{\lambda}$.

Proof of Claim 2.3. Denote by $E' \subseteq E$ the edges that cross between sets. We first show that there must exist some j so that $\mathbb{P}[S_j] > \frac{1}{2}$.

Assume that every S_i has measure less than $\frac{1}{2}$. Then

$$\varepsilon > \mathbb{P}[E] = \frac{1}{2} \sum_{j=1}^{m} \mathbb{P}[E(S_j, V \setminus S_j)] \geqslant \frac{\lambda}{2} \sum_{j=1}^{m} \mathbb{P}[S_j] \geqslant \frac{\lambda}{2}.$$

This contradicts the assumption that $\varepsilon \leqslant \frac{\lambda}{2}$.

Hence $\mathbb{P}[S_j] > \frac{1}{2}$. Thus

$$\varepsilon \geqslant \mathbb{P}[E(S_j, V \setminus S_j)] \geqslant \lambda \mathbb{P}[V \setminus S_j] = \lambda(1 - \mathbb{P}[S_j])$$

and $\mathbb{P}[S_j] \geqslant 1 - \frac{\varepsilon}{\lambda}$. □

2.2 Local properties of simplicial complexes

Links of faces. Let X be a d-dimensional simplicial complex. Let $k < d$ and $s \in X(k)$. the link of s is a $d-k-1$-dimensional simplicial complex defined by $X_s = \{t \setminus s \mid t \in X, t \supseteq s\}$. We point out that the link of the empty set is $X_\emptyset = X$.

Let $s \in X(k)$ for some $k \leqslant d$. The density function $\mathbb{P}_d$ on X induces on the link is $\mathbb{P}^s_{d-k-1} : X(d-k-1) \to (0, 1]$ where $\mathbb{P}^s_{d-k-1}[t] = \frac{\mathbb{P}[t \cup s]}{\mathbb{P}[s]\binom{d+1}{k+1}}$. We usually omit s in the probability, and for $T \subseteq X_s(k)$ we write $\mathbb{P}_{t \in X_s(k)}[T]$ instead.

High dimensional local spectral expanders. Let X be a d-dimensional simplicial complex. Let $k \leqslant d$. The k-skeleton of X is $X^{\leqslant k} = \bigcup_{j=-1}^{k} X(j)$. In particular, the 1-skeleton of X is a graph.

Definition 2.4 (high dimensional local spectral expander)**.** Let X be a d-dimensional simplicial complex. Let $\lambda \geqslant 0$. We say that X is a λ-one sided (two sided) local spectral expander if for every $s \in X^{\leqslant d-2}$, the 1-skeleton of X_s is a λ-one sided (two sided) spectral expansion.

Partite complexes. A $(d+1)$-partite simplicial complex is a d-dimensional complex that has a partition $X(0) = V_0 \cupdot V_1 \cupdot \ldots \cupdot V_d$ so that for every $s \in X(d)$ and every $i = 0, 1, \ldots, d$ it holds that $|s \cap V_i| = 1$. Let X a $(d+1)$-partite simplicial complex. A color of a face $t \in X(k)$ is $col(t) = \{i \in [d] \mid t \cap V_i \neq \emptyset\}$. Let $F \subseteq [d]$. We denote by $X[F] = \{s \in X \mid col(s) = F\}$, and by $X^F = \{s \in X \mid col(s) \subseteq F\}$. A probability density on X induces a probability density on X^F, $\mathbb{P}^F : X^F(|F|-1) \to (0, 1]$ by $\mathbb{P}^F(s) = \sum_{t \in X(d)} \mathbb{P}[t]$.

2.3 Complexes of interest

The $SL_d(\mathbb{F}_q)$-spherical building. We do not define here spherical buildings in full generality; for a general definition see e.g. [Bjö84]. Let q be a prime power, and let $\mathbb{F}_q$ be the field with q elements. Let $d > 1$ be integers. The $SL_d(\mathbb{F}_q)$-spherical building is the following $d-1$-partite simplicial complex

$$X(0) = \left\{ W \subseteq \mathbb{F}_q^d \;\middle|\; W \text{ is a vector subspace and } W \neq \{0\}, \mathbb{F}_q^d \right\}.$$

$$X(d-2) = \{\{W_1, W_2, \ldots, W_{d-1}\} \mid W_1 \subsetneq W_2 \subsetneq \ldots \subsetneq W_{d-1}\}.$$

The probability over $X(d-2)$ is uniform. The color of every vertex is its dimension.

Geometric lattices. Let $(P, \preceq)$ be a finite poset. The order complex of P, denoted X_P, is the simplicial complex on the vertex set P whose faces are all $\{v_0, \ldots, v_d\}$ so that $v_0 \prec v_1 \prec \ldots \prec v_d$, see [Koz08]. A poset $(P, \preceq)$ is a lattice if any two elements $x, y \in P$ have a unique minimal upper bound $x \vee y$ and a unique maximal lower bound $x \wedge y$. Let P be a lattice with minimal element $\hat{0}$ and maximal element $\hat{1}$. Then it has a rank function $rk : P \to \mathbb{N}$, with $rk(\hat{0}) = 0$ and $rk(y) = rk(x) + 1$ whenever y is a minimal element of $\{z : z \succ x\}$. P is a *geometric lattice* if $rk(x) + rk(y) \geqslant rk(x \vee y) + rk(x \wedge y)$ for any $x, y \in P$, and any element in P is a join of atoms (i.e., rank 1 elements). An example for a geometric lattice is the lattice whose elements are all subspaces of $\mathbb{F}_q^d$, with the containment partial order. It's order complex is the $SL_d(\mathbb{F}_Q)$-spherical building.

Let P be a geometric lattice, we denote by $\bar{P} = P \setminus \{\hat{0}, \hat{1}\}$. It is known (see [Koz08]) that if P is a geometric lattice, then $X_{\bar{P}}$ is pure and d-partite, where $d = rk(\hat{1}) - 1$. The color of every vertex is its rank. A homogeneous lattice P is a lattice so that $Aut(P)$ act transitively on $X_{\bar{P}}(d)$ (by the action $\pi(s) = \{\pi(v) \mid v \in s\}$). We show in Section 5 that homogeneous geometric lattices have constant coboundary expansion.

Geometric lattices have spectral expansion properties.

Claim 2.5. Let P be a homogeneous geometric lattice of of rank $\geqslant 2$. Then for every i and $j \geqslant 2i$ it holds that the bipartite graph between $X_P[i], X_P[j]$ is a $\frac{1}{\sqrt{2}}$-one sided spectral expander.

Proof of Claim 2.5. Note that every $u, v \in X_P[i]$ there is a path of length 2 of the form (u, wv) where $w \geqslant u \vee v$. This is because $rk(u \vee v) \leqslant 2i \leqslant j$, and because in a rank d geometric lattice we can always embed any chain in a chain of length d (i.e. we can always find $w \succeq u \vee v$ of rank j). Moreover, by homogeneity, the degree of every w is some constant $D \geqslant 2$. Thus the operator of taking two steps in this graph (starting at $X_P[i]$) is equal to $\frac{1}{D} Id + (1 - \frac{1}{D})C$ where C is the operator of the complete graph. The second largest eigenvalue of this operator is $\frac{1}{D} \leqslant \frac{1}{\sqrt{2}}$. As this is a 2-step walk according to the bipartite graph's operator, it follows that the second largest eigenvalue of the original graph is at most $\frac{1}{\sqrt{2}}$. □

[LSV05a] complexes. Lubotzky, Samuels and Vishne constructed the first bounded degree high dimensional expanders. They construct them by taking quotients of Bruhat-Titz buildings.

Theorem 2.6 ([LSV05a])**.** *For any prime power q and integer $d > 1$, there is a family $\mathcal{X}_{q,d} = \{X_n\}_{n=1}^{\infty}$ of connected complexes whose links are (isomorphic copies of) the $SL_d(\mathbb{F}_q)$-spherical building. In particular, For every $\lambda > 0$ there is some q_0 so that every X_n is a λ-one sided high dimensional expander when $q \geqslant q_0$.*

[KO21] complexes. Kaufman and Oppenheim created give another construction of bounded degree partite high dimensional expanders, by a technique called group development [KO21]. We state the properties in their construction that are necessary for our needs.

Theorem 2.7 ([KO21])**.** *For every $\lambda > 0$ there exists a family of 4-partite complexes $\mathcal{Y}_\lambda = \{Y_n\}_{n=1}^{\infty}$ so that*

1. *Y_n is a λ-one sided high dimensional expander.*
2. *There exists a constant $\beta > 0$ (independent of λ) so that for every abelian group Γ and every $s \in Y_n(0)$, the link of s has $h^1(Y_s, \Gamma) \geqslant \beta$.*

3 Cosystolic expansion

In this section we prove that local spectral expanders whose links are coboundary expanders are cosystolic expanders.

Theorem (Restatement of Theorem 1.2)**.** *Let $\beta, \lambda > 0$ and let $k > 0$ be an integer. Let X be a d-dimensional simplicial complex for $d \geqslant k + 2$ and assume that X is a λ-one-sided local spectral expander. Let Γ be any* *group. Assume that for every non-empty $r \in X$, X_r is a coboundary expander and that $h^{k+1-|r|}(X_r, \Gamma) \geqslant \beta$. Then*

$$h^k(X, \Gamma) \geqslant \frac{\beta^{k+1}}{(k+2)! \cdot 4} - e\lambda.$$

Here $e \approx 2.71$ is Euler's number.

In fact, we prove a slightly more general statement, allowing for different coboundary expansion in every level.

Theorem 3.1. *Let $k > 0$ be an integer and let $\beta_0, \beta_1, \beta_2, ..., \beta_k \in (0, 1]$ and $\lambda > 0$. Let X be a d-dimensional simplicial complex for $d \geqslant k + 2$ and assume that X is a λ-one-sided local spectral expander. Let Γ be any group. Assume that for every $0 \leqslant \ell \leqslant k$ and $r \in X(\ell)$, X_r is a coboundary expander and that $h^{k-\ell}(X_r, \Gamma) \geqslant \beta_{k-\ell}$. Then*

$$h^k(X, \Gamma) \geqslant \frac{\prod_{\ell=0}^{k} \beta_\ell}{(k+2)! \cdot 4} - e\lambda.$$

Here $e \approx 2.71$ is Euler's number.

Obviously, Theorem 1.2 follows from Theorem 3.1 by setting $\beta_\ell = \beta$ for every $\ell = 0, 1, 2, ..., k$.

The following proposition, that is important for the topological overlapping property will also be proven via similar arguments.

Proposition 3.2. *Let $k > 0$ be an integer and let $\beta_0, \beta_1, \beta_2, ..., \beta_{k-1} \in (0, 1]$ and $\lambda > 0$. Let X be a d-dimensional simplicial complex for $d \geqslant k + 1$ and assume that X is a λ-one-sided local spectral expander. Let Γ be any group. Assume that for every $0 \leqslant \ell \leqslant k - 1$ and $r \in X(\ell)$, X_r is a coboundary expander and that $h^{k-\ell}(X_r, \Gamma) \geqslant \beta_{k-\ell-1}$. Then every $g \in Z^k(X, \Gamma) \setminus B^k(X, \Gamma)$, has $wt(g) \geqslant \frac{\prod_{\ell=0}^{k-1} \beta_\ell}{(k+1)!} - e\lambda$.*

We remark that the when Γ is non abelian, these statements make sense only when $k = 1$.

Turning back to Theorem 3.1, we present a correction algorithm. We will show that when $f \in C^k(X, \Gamma)$ has a small coboundary, then the algorithm below returns some $\tilde{f} \in Z^k(X, \Gamma)$ that is close to f.

Algorithm 3.3. Input: A function $f : \vec{X}(k) \to \Gamma$, a parameter $\eta \leqslant 1$. Output: A function $\tilde{f} : \vec{X}(k) \to \Gamma$.

1. Set $f_0 := f$. Set $i = 0$.
2. While there exists $\ell \leqslant k$, and a face $s \in \vec{X}(\ell)$ so that $A_r = \{s \in X(k) \mid \ r \subseteq s\}$ has an assignment that satisfies a $\eta \mathbb{P}[A_r]$-fraction of faces more than the current assignment, do:
   - Let $fix_r : A_r \to \Gamma$ be an optimal assignment to A_r, satisfying the maximal number of $k+1$-faces containing s.
   - Set $f_{i+1}(s) = \begin{cases} f_i(s) & r \nsubseteq s \\ fix_r(s) & r \subseteq s \end{cases}$.
   - Set i:=i+1.
3. Output $\tilde{f} := f_i$.

3.1 Properties of Algorithm 3.3

Before proving Theorem 3.1 we record some properties of Algorithm 3.3.

Claim 3.4. Algorithm 3.3 halts on every input.

Claim 3.5. Let $f : \overrightarrow{X}(k) \to \Gamma$ and let $\eta \leqslant 1$. Let $\tilde{f} : \overrightarrow{X}(k) \to \Gamma$ be the output of Algorithm 3.3 on (f, η). Then $\eta \operatorname{dist}(f, \tilde{f}) \leqslant wt(\delta f)$.

Proof of Claim 3.4. Denote by $\varepsilon_j = \mathbb{P}_{t \in X(k+1)} [\delta f_j(t) \neq 0]$. If we show that $\varepsilon_{j+1} < \varepsilon_j$, then as X is finite the algorithm must halt. Indeed, fix j and let r be the face that was fixed in the j-th step. If $t \in X(k+1)$ doesn't contain r then it holds that $\delta f_j(t) = \delta f_{j+1}(t)$ so

$$\mathbb{P}_{t \in X(k+1)} \left[\delta f_j(t) \neq 0 \middle| \ t \not\supseteq r\right] = \mathbb{P}_{t \in X(k+1)} \left[\delta f_{j+1}(t) \neq 0 \middle| \ t \not\supseteq r\right].$$

For faces containing t, Algorithm 3.3 changed the values of f_j to satisfy more of the $k+1$-faces containing t, so all in all $\varepsilon_{j+1} < \varepsilon_j$. □

Proof of Claim 3.5. Let i be so that Algorithm 3.3 returned $\tilde{f} = f_i$. By the triangle inequality $\operatorname{dist}(f, f_i) \leqslant \sum_{j=0}^{i-1} \operatorname{dist}(f_j, f_{j+1})$. Following the notation of Claim 3.4, let $\varepsilon_j = wt(\delta f_j)$. If we show that

$$\eta \operatorname{dist}(f_j, f_{j+1}) \leqslant \varepsilon_j - \varepsilon_{j+1}$$

then

$$\eta \operatorname{dist}(f, f_i) \leqslant \sum_{j=0}^{i-1} \varepsilon_j - \varepsilon_{j+1} = \varepsilon_0 - \varepsilon_i = wt(\delta f) - \varepsilon_i \leqslant wt(\delta f).$$

Indeed, fix j and let r be the ℓ-face that was selected in the j-th step of the algorithm. Recall that A_r is the set of k-faces that contain r. Then $\operatorname{dist}(f_j, f_{j+1}) \leqslant \mathbb{P}[A_r]$, since the change between f_j, f_{j+1} was only on faces in A_r. However, the difference between $\varepsilon_j - \varepsilon_{j+1} \geqslant \eta \mathbb{P}[A_r]$, since otherwise the algorithm wouldn't change anything. Combine the two inequalities:

$$\eta \operatorname{dist}(f_j, f_{j+1}) \leqslant \eta \mathbb{P}[A_r] \leqslant \varepsilon_j - \varepsilon_{j+1}.$$

□

3.2 Local minimality

Definition 3.6 (Restriction)**.** Let $g \in C^k(X, \Gamma)$ and let $r \in X$ of size $0 < |r| \leqslant k$. The restriction of g to r is the function $g_r \in C^{k-|r|}(X_r, \Gamma)$ is defined by $g_r(p) = g(r \circ p)$.

Definition 3.7 (Local minimality)**.** Let $\eta \geqslant 0$ and let $g \in C^k(X, \Gamma)$. We say that g is η-locally minimal, if for every $r \in X, r \neq \emptyset$, and every $h \in C^{k-|r|-1}(X_r, \Gamma)$ it holds that

$$wt(g_r) \leqslant wt(g_r + \delta h) + \eta.$$

The non-abelian case. If Γ is non-abelian we need the correct analogy to adding coboundaries. The definition of η-minimality is as follows. If $k = 1$, we say that g is η-locally minimal if for every $v \in X(0)$, and every $\gamma \in \Gamma$, it holds that

$$wt(g_v) \leqslant wt(\gamma \cdot g_r) + \eta.$$

If $k = 2$, we say that g is locally minimal if:

1. For every edge uv and every $\gamma \in \Gamma$, it holds that $wt(g_{uv}) \leqslant wt(\gamma \cdot g_{uv}) + \eta$.

2. For every vertex v and every function $h : X_v(0) \to \Gamma$, it holds that $wt(g_v) \leqslant wt(g_v^h) + \eta$, where $g_v^h(uw) = h^{-1}(u) g_v(uw) h(w)$.

Claim 3.8. Let $f : \vec{X}(k) \to \Gamma$ and let $\eta \leqslant 1$. Let $\tilde{f} : \vec{X}(k) \to \Gamma$ be the output of Algorithm 3.3 on (f, η). Then $\delta \tilde{f}$ is η-locally minimal.

Proof of Claim 3.8. Assume towards contradiction that there is some $r \in X(j)$ and a $h \in C^{k-|r|-1}(X_r, \Gamma)$ so that $wt((\delta \tilde{f})_r) > wt((\delta \tilde{f})_r + \delta h) - \eta$. We define $fix_r : X(k) \to \Gamma$ to be $fix_r(t) = h(p)$ if $t = r \circ p$, and zero if $r \not\subseteq t$.

By definition

$$wt((\delta \tilde{f})_r) = \mathop{\mathbb{P}}_{t \in X(k+1), t \supseteq r} \left[\delta(\tilde{t}) \neq 0 \right] \tag{3.1}$$

and

$$wt((\delta \tilde{f})_r + \delta h) = \mathop{\mathbb{P}}_{t \in X(k+1), t \supseteq r} \left[\delta(\tilde{f} + fix_r)(t) \right].$$

But then Algorithm 3.3 would have added fix_r to $\tilde{f}$ (or another even better function), and wouldn't return $\tilde{f}$, a contradiction.

We remark that the same idea holds in the non-abelian case where $\delta \tilde{f} \in C^2(X, \Gamma)$, even though the case analysis is cumbersome. Equation (3.1) is still true. Thus,

1. Let $\gamma \in \Gamma$ and $r = uv \in X(1)$. For every triangle $uvw \in X(2)$, the value of $\gamma (\delta \tilde{f})_r(w) = (\gamma \cdot \tilde{f}(uv)) \tilde{f}(vw) \tilde{f}(wu)$. By Algorithm 3.3, changing the value of $\tilde{f}(uv)$ to $\gamma \cdot \tilde{f}(uv)$ cannot decrease the weight of $\delta \tilde{f}$ by more than η.

2. If $r \in X(0)$ and $h : X_r(0) \to \Gamma$. Then for every triangle $rvw \in X(2)$, it holds that

$$(\delta \tilde{f})_r^h(vw) = h(v) \tilde{f}(rv) \tilde{f}(vw) \tilde{f}(wr) h(w)^{-1} = (h(v) \tilde{f}(rv)) \tilde{f}(vw) (h(w) \tilde{f}(rw))^{-1}.$$

   By Algorithm 3.3, changing the values of $\{\tilde{f}(rx)\}$ for the edges rx adjacent to r, to the values $h(x)\tilde{f}(rx)$ cannot decrease the weight of $\delta \tilde{f}$ by more than η.

□

3.3 Locally minimal cosystols are heavy

The following lemma states that non-zero functions that are locally minimal must have large weight.

Lemma 3.9. *Let $\beta_0, ..., \beta_{k-1}$ and λ be as in Theorem 3.1. Let X be such that for every $0 \leqslant \ell \leqslant k-1$ and every $s \in X(\ell)$ it holds that X_s is a coboundary expander and $h^{k-\ell-1}(X_s, \Gamma) \geqslant \beta_{k-\ell-1}$. Assume further that X is a λ-local spectral expander. Let $g \in Z^k(X, \Gamma)$ be non-zero and η-locally minimal. Then*

$$wt(g) \geqslant \frac{\prod_{\ell=0}^{k-1} \beta_\ell}{(k+1)!} - e(\eta + \lambda).$$

This lemma implies Theorem 1.2 and Proposition 3.2 directly.

Proof of Theorem 3.1, given Lemma 3.9. Fix $\eta = \frac{\prod_{\ell=0}^{k} \beta_\ell}{4((k+2)!)}$. Let $\tilde{f}$ be the output of Algorithm 3.3 for some function f and η. If $wt(\delta f) \geqslant \frac{\prod_{\ell=0}^{k} \beta_\ell}{4(k+2)!} - e\lambda$ there is nothing to prove, so we assume that $wt(\delta f) < \frac{\prod_{\ell=0}^{k} \beta_\ell}{4(k+2)!} - e\lambda$. Then $\delta \tilde{f} \in Z^{k+1}(X, \Gamma)$ is an η-locally minimal function so that $wt(\delta \tilde{f}) \leqslant wt(\delta f)$. Hence by Lemma 3.9 (applied with $k+1$ instead of k), $\delta \tilde{f} = 0$ and $\tilde{f}$ is a cosystol. By Claim 3.5, $\eta \operatorname{dist}(f, \tilde{f}) \leqslant wt(\delta f)$, and we are done. $\square$

Proof of Proposition 3.2, given Lemma 3.9. For every $r \in X(j)$ and $h \in C^{k-j-1}(X_r, \Gamma)$, we define $h^\uparrow : X(k) \to \Gamma$ by

$$h^\uparrow(t) = \begin{cases} h(p) & s = r \circ p \\ 0 & r \nsubseteq s. \end{cases}.$$

It is easy to see that $g_r + \delta h = (g + \delta h^\uparrow)_r$.

Now let $0 \neq g \in Z^k(X, \Gamma) \setminus B^k(X, \Gamma)$ be a cosystol that has the minimal among all $Z^k(X, \Gamma) \setminus B^k(X, \Gamma)$. By the above, g is also 0-locally minimal (since otherwise we could have found some non-zero coboundary $\delta h^\uparrow$ to add to g and decrease its weight). Thus $wt(g) \geqslant \frac{\prod_{\ell=0}^{k-1} \beta_\ell}{(k+1)!} - e\lambda$ as required.

We remark that the case where Γ is non-abelian and $k = 1$ is similar. Given $g \in Z^1(X, \Gamma) \setminus B^1(X, \Gamma)$ that is non-zero and has minimal weight over all such functions. First we establish that it is locally minimal. Indeed, assume towards contradiction that there is some vertex $v \in X(0)$ and $\gamma \in \Gamma$ so that $wt(g_v) < wt(\gamma g_v)$. Then the function

$$g'(xy) = \begin{cases} \gamma g(xy) & x = v \\ g(xy)\gamma^{-1} & y = v \\ g(xy) & otherwise \end{cases}.$$

is also a cosystol. Taking some triangle $vuw \in X(2)$ that contains v, the value of

$$\delta g'(vuw) = \gamma \delta g(vuw) \gamma^{-1} = e$$

(the identity in Γ). For any triangle uwx that doesn't contain v we have that $\delta g'(uwx) = \delta g(uwx) = e$. On the other hand, $wt(g') < wt(g)$ so g' is trivial, which implies that $g = \delta h$ where $h(v) = \gamma$ and $h(u) = e$. A contradiction to the fact that $g \notin B^1(X, \Gamma)$. $\square$

The remainder of this section is devoted to proving Lemma 3.9. For this we need to define averaging operators that play a crucial role in the theory behind local-spectral expanders. We will only define what we need so for a more thorough exposition see e.g. [Dik+18]. Let $\ell_2(X(j))$ be the Hilbert space of all functions $f : X(j) \to \mathbb{R}$ where the inner product is $\langle f, g \rangle = \mathbb{E}_{r \in X(j)} [f(r)g(r)]$. Let $D_k : \ell_2(X(k)) \to \ell_2(X(k-1))$ be the following operator

$$D_k f(s) = \mathop{\mathbb{E}}_{t \supseteq s} [f(t)].$$

This operator's adjoint is $U_{k-1} : \ell_2(X(k-1)) \to \ell_2(X(k))$ that is defined by

$$U_{k-1} f(t) = \mathop{\mathbb{E}}_{s \subseteq t} [f(s)].$$

As a shorthand we write $D_k^\ell = D_{k-\ell+1} D_{k-\ell+2} ... D_k$ for $\ell \geqslant 1$ (and the same for U). For $\ell = 0$ $D_k^0 = U_k^0 = Id$. We record that $D_k^\ell f$ is a function whose domain is $X(k-\ell)$, and that $U_k^\ell f$ is a function whose domain is

$X(k+\ell)$.

Let $j \leqslant k < d$. The operator $N_{k\to j} : \ell_2(X(k)) \to \ell_2(X(j))$ is defined by

$$N_{k\to j} f(r) = \mathop{\mathbb{E}}_{t\in X(k+1), t\supseteq r} \left[\mathop{\mathbb{E}}_{s\subseteq t, r\not\subseteq s} [f(s)] \right].$$

Let us spell out this expression. We average over $f(s)$ where s is chosen according to the following rule. We first sample some $t \supseteq r$ in $X(k+1)$, and then we sample $s \subseteq t$ given that it does not contain r.

When j, k is clear from the context we simply write D, U, N.

The following is an operator norm inequality that is similar to [Dik+18], but for the one-sided case. We prove it in the end of this section.

Claim 3.10. Let X be a λ-one-sided local spectral expander. Then $U_j^{k-j} N_{k\to j} \preceq U_{j-1}^{k-j+1} D_k^{k-j+1} + \lambda Id$ for every $j \leqslant k$.

Proof of Lemma 3.9. Let $h = \mathbf{1}_{g\neq 0}$. We will prove that $wt(g) = \mathbb{E}[h] \geqslant \frac{\prod_{\ell=0}^{k-1}\beta_\ell}{(k+1)!} - e(\eta+\lambda)$. We do this by showing that

1. $\|D_k h\|^2 \geqslant \frac{1}{k+1}\|h\|^2 - \lambda\|h\|^2$.
2. For $0 \leqslant j < k$, $\|D_k^{k-j+1} h\|^2 \geqslant \frac{\beta_{k-j-1}}{j+1} \cdot \|D_k^{k-j} h\|^2 - \left(\frac{\beta_{k-j-1}\eta}{j+1} + \lambda\right)\|h\|^2$.

We note that $D^{k+1}h$ is a constant - the average of h on all faces. Hence $\|D^{k+1}h\|^2 = \mathbb{E}[h]^2$. By iteratively applying these inequalities we get that

$$\begin{aligned}
\mathbb{E}[h]^2 &= \|D^{k+1}h\|^2 \\
&\geqslant \beta_{k-1}\|D^k h\|^2 - (\beta_{k-1}\eta + \lambda)\|h\|^2 \\
&\geqslant \frac{\beta_{k-1}\beta_{k-2}}{2}\|D^{k-1}h\|^2 - \beta_{k-1}\left(\frac{\beta_{k-2}\eta}{2} + \lambda\right)\|h\|^2 - (\beta_{k-1}\eta+\lambda)\|h\|^2 \\
&\dots \\
&\geqslant \|h\|^2 \cdot \left(\frac{\prod_{\ell=0}^{k-1}\beta_\ell}{(k+1)!} - \eta\sum_{j=0}^{k-1}\frac{\beta_j}{(k-j+1)!} - \lambda\left(1 + \sum_{j=0}^{k-1}\frac{\beta_j}{(k-j+1)!}\right)\right).
\end{aligned}$$

By assuming $\beta_j \leqslant 1$, we upper bound $\sum_{j=0}^{k-1}\frac{\beta_j}{(k-j+1)!} \leqslant \sum_{j=0}^{\infty}\frac{1}{j!} = e$, and get $\mathbb{E}[h]^2 \geqslant \|h\|^2 \cdot \frac{\prod_{\ell=0}^{k-1}\beta_\ell}{k!} - e(\eta + \lambda)$. As $\|h\|^2 = \mathbb{E}[h]$ the lemma follows.

Let us begin with the first item. we call $s \in X(k)$ *active* if $h(s) = 1$. By assumption, $g \in Z^k(X,\Gamma)$, i.e.

$$\delta g(t) = \sum_{i=0}^{k+1}(-1)^i g(t_i) = 0.$$

Thus if $t \in X(k+1)$ contains an active $s = t_{i_1}$, then it must also contain a second active $s' = t_{i_2}$. This implies that $N_{k\to k}h(s) \geqslant \frac{1}{k+1}h(s)$, and so

$$\langle h, N_{k\to k}h\rangle = \mathop{\mathbb{E}}_t[h(t)N_{k\to k}h(t)] \geqslant \frac{1}{k+1}\|h\|^2.$$

By Claim 3.10 $N^{k\to k} \preceq UD + \lambda Id$, so

$$\frac{1}{k+1}\|h\|^2 \leqslant \langle N_{k\to k}h, h\rangle \leqslant \langle UDh, h\rangle + \lambda\|h\|^2 = \|Dh\|^2 + \lambda\|h\|^2$$

so the first item is proven.

Next, we will prove the second item. As before, we will show that

$$\langle U^{k-j}N_{k\to j}h, h\rangle \geqslant \frac{\beta_{k-j-1}}{j+1}\cdot(\|D^{k-j}h\|^2 - \eta\|h\|^2). \tag{3.2}$$

Then we rely on Claim 3.10

$$\|D^{k-j+1}h\|^2 \geqslant \langle U^{k-j+1}N_{k\to j}h, h\rangle - \lambda\|h\|^2. \tag{3.3}$$

Combining these inequalities completes the proof.

We now state the following claim, which is proven using the coboundary expansion of X_r where r is a j-face.

Lemma 3.11 (Key lemma)**.** *Let $r \in X(j)$. Then*

$$N_{k\to j}h(r) \geqslant \frac{\beta_{k-j-1}}{j+1}(D^{j-1}h(r) - \eta).$$

From this pointwise inequality, (3.2) follows easily:

$$\begin{aligned}\langle U^{k-j}N_{k\to j}h, h\rangle = \langle N_{k\to j}h, D^{k-j}h\rangle \geqslant \mathop{\mathbb{E}}_{r}\left[D^{k-1}h(r)\cdot\frac{\beta_{k-j-1}}{j+1}\cdot(D^{k-1}h(r)-\eta)\right] \\ = \frac{\beta_{k-j-1}}{j+1}\cdot(\|D^{k-1}h\|^2 - \eta\|h\|^2)\end{aligned} \tag{3.4}$$

□

We will prove Lemma 3.11 under the assumption that Γ is abelian since additive notation is more convenient. For non-abelian groups, see Remark 3.12.

Proof of Lemma 3.11. First, let us understand the meaning of the inequality in Lemma 3.11. Recall that $N_{k\to j}h(r)$ is an average of $h(s)$ over faces $s \in X(k)$ so that $r, s \subseteq t$ for some $t \in X(k+1)$ and $r \not\subseteq s$. As h is an indicator function this is the same as writing

$$N_{k\to j}h(r) = \mathop{\mathbb{P}}_{t,s}[h(s)=1],$$

where t, s are as above. On the other side of the inequality there is $D^{k-j}h(r) = \mathbb{P}_{s\supseteq r}[h(s)=1]$. Hence, we need to show that if there are many active faces that contain r, there must also be many active faces that "complete" r to a $(k+1)$-face.

We first note that

$$N_{k\to j}h(r) = \mathop{\mathbb{P}}_{t,s}[h(s)=1] \geqslant \frac{1}{j+1}\mathop{\mathbb{P}}_{t}[\exists s\subseteq t\ h(s)=1 \text{ and } r\not\subseteq s], \tag{3.5}$$

so we shall actually lower bound $\mathbb{P}_t[\exists s\subseteq t\ h(s)=1 \text{ and } r\not\subseteq s]$.

As $g \in Z^k(X,\Gamma)$, for every $t = r \circ p \in X(k+1)$

$$0 = \delta g(r \circ p) = \sum_{i=0}^{j} (-1)^i g(r_i \circ p) + (-1)^j \sum_{i=0}^{k-1-j} (-1)^i g(r \circ p_i). \tag{3.6}$$

And in particular

$$\sum_{i=0}^{k-1-j} (-1)^i g(r \circ p_i) \neq 0 \iff \sum_{i=0}^{j} (-1)^i g(r_i \circ p) \neq 0. \tag{3.7}$$

Recall that the restriction of g is $g_r : X_r(k-j-1) \to \Gamma$, defined by $g_r(p) = g(r \circ p)$. As we can see, $\delta g_r(p)$ is the left-hand side of (3.7). Thus

$$\mathop{\mathbb{P}}_{t}\left[\exists s \subseteq t \; h(s) = 1 \text{ and } r \nsubseteq s\right] \geqslant \mathop{\mathbb{P}}_{t=r\circ p}\left[\sum_{i=0}^{k-1-j} (-1)^i g(r \circ p_i) \neq 0\right] = \mathop{\mathbb{P}}_{p \in X_r(k-j+1)}\left[\delta g_r \neq 0\right]. \tag{3.8}$$

By assumption X_r is a β_{k-j-1}-coboundary expander, this is at most $\beta_{k-j-1} \cdot \mathrm{dist}(g_r, B^{k-j-1}(X_r,\Gamma))$.

To conclude we need to show that

$$\mathrm{dist}(g_r, B^{k-j-1}(X_r,\Gamma)) \geqslant \mathop{\mathbb{P}}_{s \supseteq r}\left[g(s) \neq 0\right] - \eta. \tag{3.9}$$

But

$$\mathrm{dist}(g_r, B^{k-j-1}(X_r,\Gamma)) = \min_{f \in C^{k-j-2}(X_r,\Gamma)} \{wt(g_r + \delta f)\} \geqslant wt(g_r) - \eta. \tag{3.10}$$

where the inequality follows from η-minimality of g. As $wt(g_r) = \mathbb{P}_{s \supseteq r}\left[h(s) = 1\right]$ we have proven

$$N_{k \to j} h(r) \geqslant \frac{\beta_{k-j-1}}{j+1} \mathrm{dist}(g_r, B^{k-j-1}(X_r,\Gamma)) \geqslant \frac{\beta_{k-j-1}}{j+1}\left(\mathop{\mathbb{P}}_{s \supseteq r}\left[h(s) = 1\right] - \eta\right).$$

□

Remark 3.12 (The non-abelian case). The first place where we need to accomodate for the non-abelianity is in (3.5) which implies (3.6). We note that the same relations hold in the non-abelian case. For example, if $r \in X(0)$ and $g \in Z^1(X,\Gamma)$, and $ruw \in X(2)$ we can write

$$e = \delta g(ruw) = g(ru)g(uw)g(wr)$$

instead of (3.6). This implies that

$$g(uw) = g(ur)g(rw) = \delta g_r(uw),$$

and in particular $g(uw) \neq 0 \iff \delta g_r(uw) \neq 0$. This is the same conclusion as we get in (3.6). The case where $r \in X(1)$ is similar.

The second equality we need to modify is (3.10). For example, assume that $r \in X(0)$, and $\delta h \in B^1(X_r,\Gamma)$ is a closest coboundary to g_r. Then

$$\mathrm{dist}(g_r, \delta h) = \mathbb{P}\left[g_r(vu) \neq h(v)h(u)^{-1}\right] = wt(g_r^h) \geqslant wt(g_r) - \eta.$$

The case where $r \in X(1)$ is similar.

Proof of Claim 3.10. We begin by showing the following inequality for operators on $\ell_2(X(k-1))$ on a d-dimensional simplicial complex for $d \geqslant k$:

$$U^{k-1}S_{k-1} \preceq U^k D^k_{k-1} + \lambda Id. \tag{3.11}$$

where $S_{k-1} : \ell_2(X(k-1)) \to \ell_2(X(0))$ is defined by $S_{k-1}f(v) = \mathbb{E}_{s \in X(k-1), s \cup \{v\} \in X(k)} [f(s)]$.

Note that $U^k D^k_{k-1}$ is the operator that averages a function on all $k-1$ faces (independently of the starting face), so it is sends the constant function to itself, and it sends everything perpendicular to the constant function to 0. The operator $U^{k-1}S_{k-1}$ also sends the constant to itself, and the functions perpendicular to the constant functions are an $U^{k-1}S_{k-1}$-invariant subspace.

Hence, to show (3.11) need to show that the second largest eigenvalue of $U^{k-1}S_{k-1}$ is less or equal to λ. Recall that for every two linear operators $Y : A \to B$ and $Z : B \to A$ the eigenvalues of $YZ : B \to B$ are equal to the eigenvalues of $ZY : A \to A$ (up to the multiplicity of the 0 eigenvalue). Therefore, applying this to $S_{k-1} : \ell_2(X(k-1)) \to \ell_2(X(0))$ and $U_0^{k-1} : \ell_2(X(0)) \to \ell_2(X(k-1))$,

$$\lambda(U_0^{k-1}S_{k-1}) = \lambda(S_{k-1}U_0^{k-1})$$

and we will easily bound the right-hand side. The operator $S_{k-1}U_0^{k-1}$ is nothing but the random walk on the 1-skeleton of the graph: indeed start with a vertex $v \in X(0)$, go up to $t \supset v, t \in X(k-1)$ and then traverse to u so that $u \cup s \in X(k)$. This is the same as choosing $u \sim v$ a random neighbor of v (and $s \cup \{u\}$ a random k face containing this edge). By spectral expansion of the 1-skeleton of X, $\lambda(S_{k-1,0}U^{k-1}) \leqslant \lambda$ and (3.11) is proven.

Now let us prove the claim. Fix some $f \in \ell_2(X(k))$. We need to show that

$$\langle f, U^{k-j}N_{k\to j}f\rangle \leqslant \langle f, U^{k-j+1}D^{k-j+1}f\rangle + \lambda\langle f, f\rangle$$

The lefthand side is equal to $\mathbb{E}_s\left[f(s) \cdot (U^{k-j}N_{k\to j}f)(s)\right] = \mathbb{E}_{s,s'}\left[f(s)f(s')\right]$ where s' is chosen according to the distribution of $U^{k-j}N_{k\to j}$.

We localize this expectation using $p = r \cap s$. Namely, the above random process of choosing an pair s, s' is equivalent to the following:

1. Choose $p \in X(j-1)$.
2. Choose $v \in X_p(0)$ and set $r = \{v\} \cup p$.
3. Choose $s' \supseteq r$.
4. In X_p walk from v to $q \in X_p(k-j)$ so that $\{v\} \cup q \in X_p(k-j+1)$ (Namely, apply the swap walk $S_{k-j,0}$ in the link of p, which takes a $(k-j)$-face to a vertex). Set $s = p \cup q \in X(k)$.

Thus we can write

$$\begin{aligned}\langle f, U^{k-j}N_{k\to j}f\rangle &= \mathop{\mathbb{E}}_{(s,t,r,p,s')}\left[f(s)f(s')\right] \\ &= \mathop{\mathbb{E}}_{p}\left[\mathop{\mathbb{E}}_{(v,q,s')}\left[f(p\cup q)f(p\cup(s'\setminus p))\right]\right]\end{aligned}$$

$$= \mathop{\mathbb{E}}_{p}\left[\mathop{\mathbb{E}}_{(v,q,s')}\left[f_p(q)f_p(s' \setminus p)\right]\right],$$

where $f_p : X_p(k-j) \to \mathbb{R}$ is defined by $f_p(x) = f(p \cup x)$. For a fixed p, the choice of $q, s' \setminus p$ is just using the random walk $U^\ell S_{\ell,0}$ in the link of p, with $\ell = k - j$. This is equal to

$$\mathop{\mathbb{E}}_{p}\left[\langle f_p, (U^\ell S_{\ell,0})_p f_p\rangle\right] \overset{(3.11)}{\leqslant} \mathop{\mathbb{E}}_{p}\left[\langle f_p, (U^{k-j+1}D^{k-j+1})_p f_p\rangle\right] + \mathop{\mathbb{E}}_{p}\left[\lambda\langle f_p, f_p\rangle\right] \tag{3.12}$$

$$= \langle f, U^{k-j+1}D^{k-j+1}f\rangle + \lambda\langle f, f\rangle. \tag{3.13}$$

The last equality holds since

$$\langle f, U^{k-j+1}D^{k-j+1}f\rangle = \mathop{\mathbb{E}}_{p}\left[\mathop{\mathbb{E}}_{s \supseteq p}[f(s)]^2\right] = \mathop{\mathbb{E}}_{p}\left[\mathop{\mathbb{E}}_{s \setminus p \in X_p(j-k)}[f_p(s)]^2\right] = \mathop{\mathbb{E}}_{p}\left[\langle f_p, (U^{k-j+1}D^{k-j+1})_p f_p\rangle\right].$$

Hence the claim is proven. □

4 Coboundary expansion via color restriction

In this section we develop a technique for bounding coboundary expansion in partite complexes with large dimension.

Recall that for a d-partite complex X, and some $F \subseteq [d]$ the complex $X^F = \{s \in X \mid col(s) \subseteq F\}$. We call these complexes color restrictions of X. Fix a group Γ and integer k. We say that a color restriction X^F is (k,β)-*locally coboundary expanding* (with Γ coefficients) if $H^k(X^F, \Gamma) = 0$, $h^k(X^F, \Gamma) \geqslant \beta$ and for every face $s \in X(j)$ so that $col(s) \cap F = \emptyset$, it holds that $H^k(X^F_s, \Gamma) = 0$ and $h^{k-|s|}(X^F_s, \Gamma) \geqslant \beta$ (for $j \leqslant k$). The next theorem states that when a color restriction X^F is β-locally coboundary expanding for typical F, then X itself is an $\Omega(\beta^{k+1})$-coboundary expander.

Theorem 4.1. *Let k, ℓ, d be integers so that $k + 2 \leqslant \ell \leqslant d$ and let $\beta, p \in (0,1]$. Let Γ be some group (that is abelian if $k > 1$). Let X be a d-partite simplicial complex so that*

$$\mathop{\mathbb{P}}_{F \in \binom{[d]}{\ell}}\left[X^F \text{ is a } (k,\beta)\text{-locally coboundary expander}\right] \geqslant p.$$

Then X is a coboundary expander with $h^k(X, \Gamma) \geqslant \frac{p\beta^{k+1}}{e(k+2)!}$. Here $e \approx 2.71$ is Euler's number.

We use this theorem in Section 5 to show the coboundary expansion of homogeneous geometric lattices. The reason we need this theorem is that lower bounds obtained in other techniques usually depend on the dimension of the complex (e.g. the cone technique in Section 5). Color restricting X^F reduces the dimension to $|F|$, which allows us to use the dimension dependent techniques on X^F, as long as $|F|$ is a function of k and not of d. Hence, this theorem allows us to overcome the dependence on dimension.

Remark 4.2. We shall write this section in additive notation, i.e. we assume that Γ is an abelian group. However, the same proof holds for non-abelian groups when $k = 1$ also. See Remark 4.5 after the proof of Theorem 4.1.

4.1 Additional notation

Averaging out real valued functions. Let X be a $(d+1)$-partite complex and let $I \subseteq J \subseteq [d]$. Let $A \subseteq X(k)$ be a set with relative size ε. For a face $s \in X(k)$ we denote by $\varepsilon_s = \mathbb{P}_{t \in X(k)}[A \mid t \supseteq s]$. We denote by $\varepsilon_{J,I} = \mathbb{P}_{t \in X(k)}[A \mid col(t) \cap J = I]$, the probability of A given that we sampled $t \in X(k)$ so that $col(t) \cap J = I$. We denote by $\varepsilon_{J,j} = \mathbb{P}_{t \in X(k)}[A \mid |col(t) \cap J| = j]$.

Conditional distances. Let $f_1, f_2 : \vec{X}(k) \to \Gamma$ be two asymmetric functions. The distance between f_1, f_2 is

$$\operatorname{dist}(f_1, f_2) = \mathop{\mathbb{P}}_{t \in \vec{X}(k)}[f_1(t) \neq f_2(t)].$$

For a fixed set F of colors in a partite complex X, we denote by

$$\operatorname{dist}_{\bar{F},i}(f_1, f_2) = \mathop{\mathbb{P}}_{t \in \vec{X}(k)}[f_1(t) \neq f_2(t) \mid |col(t) \cap \bar{F}| = i].$$

That is, the relative hamming distance between f_1 and f_2 on the set of k-faces so that exactly i-of the vertices have colors outside of F. When the set is clear from the context we omit $\bar{F}$ and denote this by $\operatorname{dist}_i(f_1, f_2)$.

4.2 Proof of Theorem 4.1

Proof. Fix Γ and let $f : \vec{X}(k) \to \Gamma$ be an asymmetric function. Assume that $wt(\delta f) = \mathbb{P}[\delta f \neq 0] = \varepsilon$. We need to find some $g : \vec{X}(k-1) \to \Gamma$ so that $\frac{p\beta^{k+1}}{e(k+2)!}\operatorname{dist}(f, \delta g) \leqslant \varepsilon$. We start by finding a set of colors $F \in \mathcal{F}$ so that most $(k+1)$-faces with F-colored vertices are satisfied. We recall that $\varepsilon_{F,j} = \mathbb{P}_{t \in X(k+1)}[\delta f(t) \neq 0 \mid |col(t) \cap F| = j]$.

Claim 4.3. There is some $F \in \mathcal{F}$ of size $k+2$ so that for any $j = 1, ..., k+2$, $\varepsilon_{F,j} \leqslant (k+2)p^{-1}\varepsilon$.

The claim follows by standard averaging and will be proven later below. Now construct $g : \vec{X}(k-1) \to \Gamma$ in $(k+1)$ steps as follows. Fix some global order on the vertices $X(0) = \{v_0, v_1, ..., v_n\}$.

1. In the first step we define g for $(k-1)$-faces whose colors are contained in F. Note that X^F has no cohomology, $h^k(X^F, \Gamma) \geqslant \beta$ and $\varepsilon_{F,k+2} = \varepsilon_{F,F} \leqslant (k+2)p^{-1}\varepsilon$ by Claim 4.3. Coboundary expansion of X^F implies that there exists a function $g_0 : \vec{X}^F(k-1) \to \Gamma$ so that $\beta \operatorname{dist}_{k+1}(f, \delta g_0) \leqslant \varepsilon_{F,F}$ which implies

$$\frac{p\beta}{(k+2)}\operatorname{dist}_0(f, \delta g_0) \leqslant \varepsilon \text{ [5]} \tag{4.1}$$

   We set $g = g_0$ on faces $s \in \vec{X}(k-1)$ so that $col(s) \subseteq F$.

2. Let $i > 1$. In the i-th step, we define g on faces t so that $|col(t) \cap F| = k - (i-1)$ as follows. Assume that g was defined for all faces t so that $|col(t) \cap F| = k - i$. Every face t with $i-1$ colors outside F can be viewed as $t = s \circ r$ for $s \in \vec{X}^{[d] \setminus F}(i-2)$ and $r \in \vec{X}_s^F(k-i)$.

   Fix s and let us define $g(r \circ s)$ for all r. We assume $s = (v_{j_0}, v_{j_1}, ..., v_{j_i})$ for $j_0 < j_1 < ... < j_i$, and

[5] Here we recall that $\operatorname{dist}_0(f, \delta g_0)$ is the distance between f and δg_0 over $k-1$-faces in F. See Section 4.1.

defining g for other s follows directly by asymmetry. Let us define $h_s : \overset{\to F}{X_s}(k-i+1) \to \Gamma$ by

$$h_s(a) = f(a \circ s) - (-1)^{|a|} \sum_{\ell=0}^{i-2} (-1)^\ell g(a \circ s_\ell).$$

Here we use the fact that g has been defined for all the $a \circ s_\ell$. Next we find (an arbitrary) $g_0^s : \overset{\to F}{X_s}(k-i) \to \Gamma$ that minimizes $\operatorname{dist}(h, \delta g_0^s)$. We set $g(r \circ s) = g_0^s(r)$.

As alluded to in the overview, the motivation behind defining h_s for every $s \in X(i-2)$ this way is because for every element in $\left\{ a \circ s \,\middle|\, a \in \overset{\to F}{X_s}(k-i) \right\}$, we have an equation

$$\delta g(a \circ s) = f(a \circ s)$$

which translates to

$$f(a \circ s) - (-1)^{|a|} \sum_{\ell=0}^{i-2} (-1)^\ell g(a \circ s_\ell) = \sum_{\ell=0}^{k-i} (-1)^\ell g(a_\ell \circ s). \tag{4.2}$$

We consider the set of equations (4.2) one per a. The left hand side (which is $h_s(a)$) has the values we have defined in the $(i-1)$-th step, and the right hand side are the "unknowns", that is, the values of g we wish to define in the i-th step. Translating this to the language of coboundaries:

1. $h_s(a)$ is the "free coefficient" in every equation.
2. Assignments to the "unknowns", i.e. $g(a_\ell \circ s)$ are functions $g_0^s : \overset{\to F}{X_s}(k-i-1) \to \Gamma$, so that $g(r \circ s) = g_0^s(r)$.
3. The equation that a defines (4.2) is satisfied by a solution g_0 if and only if $h_s(a) = \delta g_0^s(a)$. That is, we want to find some g_0^s that minimizes $\operatorname{dist}(h, \delta g_0^s)$ in the link of s (and as discussed before, we will do this by showing that $\delta h_s \approx 0$ and using coboundary expansion).

We need to prove we indeed constructed a function g so that δg is close to f.

Lemma 4.4.

$$\frac{p\beta^{k+1}}{e(k+2)!} \operatorname{dist}(f, \delta g) \leqslant \varepsilon.$$

□

Remark 4.5. When Γ is non-abelian and $k = 1$, we construct $g : \vec{X}(0) \to \Gamma$ in a similar manner to the abelian case. First, we find the values of g on $\overset{\to F}{X}(0)$ using coboundary expansion as in the first step of the abelian case. Next, for every v whose color isn't in f, we define $h_v : \overset{\to F}{X_v}(0) \to \Gamma$ by $h_v(u) = f(vu)g(u)$ (where $g(u)$ has been previously defined since $col(u) \in F$). Then we find $\gamma \in \Gamma = C^{-1}(X, \Gamma)$ so that $\operatorname{dist}(h_v, \delta\gamma)$ and set $g(v) = \gamma$.

The proof in this case is identical to the abelian case, so we won't repeat it.

Proof of Lemma 4.4. Recall that dist_i is the hamming distance on k-faces where i out of $k+1$ vertices are not in F. We show that for $i \leqslant k$

$$\operatorname{dist}_i(f, \delta g) = \mathop{\mathbb{P}}_{t \in X(k)} \left[f(t) \neq \delta g(t) \,\middle|\, |col(t) \cap F| = i \right] \leqslant (k+2)(i!)\beta^{-(i+1)} p^{-1} \varepsilon \sum_{j=0}^{i} \frac{1}{j!}, \tag{4.3}$$

and that for $i = k+1$

$$\mathrm{dist}_{k+1}(f,\delta g) \leqslant (k+2)!\beta^{-(k+1)}p^{-1}\varepsilon \sum_{j=1}^{k+1}\frac{1}{j!} \leqslant \frac{e(k+2)!}{p\beta^{k+1}}\varepsilon. \tag{4.4}$$

Since

$$\mathrm{dist}(f,\delta g) = \mathop{\mathbb{P}}_{t\in X(k)}[f(t)\neq \delta g(t)] = \sum_{i=0}^{k+1}\mathop{\mathbb{P}}_{t\in X(k)}\left[|col(t)\cap \bar{F}| = i\right]\mathrm{dist}_i(f,\delta g),$$

it holds that $\mathrm{dist}(f,\delta g) \leqslant \max_i \mathrm{dist}_i(f,\delta g)$ and the lemma follows. We show (4.3) by induction over i. When $i=0$, by the definition of g in the first step, see (4.1), it follows that $\mathrm{dist}_0(f,\delta g) \leqslant (k+2)\beta^{-1}p^{-1}\varepsilon$. Let us assume that (4.3) holds for i, and show it for $i+1$.

We want to bound the fraction of $t = r\circ s \in \overrightarrow{X}(k)$, where $|s| = i+1$, $col(r)\subseteq F$ and $col(s)\cap F = \emptyset$, so that

$$f(r\circ s) \neq \delta g(r\circ s) = \sum_{j=0}^{k-i-1}(-1)^j g(r_j\circ s) + \sum_{j=k-i}^{k}(-1)^{|r|+i} g(r\circ s_j).$$

Claim 4.6. For $i \leqslant k$,

$$f(r\circ s) \neq \delta g(r\circ s) \Leftrightarrow \delta g_0^s(r) \neq h_s(r),$$

where $g_0^s(r) = g(r\circ s)$.

Thus we wish to bound $\mathbb{P}_{s,r}\left[\delta g_0^s(r)\neq h_s(r)\right]$. As $h^{k-|s|}(X_s^F,\Gamma)\geqslant \beta$, it is enough to bound the probability that $\delta h \neq 0$ (up to multiplying by a factor of β^{-1}). That is, we need to show that

$$wt(\delta h_s) \leqslant (k+2)(i+1)!\beta^{-(i+1)}p^{-1}\varepsilon\sum_{j=0}^{i+1}\frac{1}{j!}. \tag{4.5}$$

Combining (4.5) with β-coboundary expansion (in the link of every s) we have that

$$\mathop{\mathbb{P}}_{s,r}\left[g_0^s(r)\neq h_s(r)\right] = \mathop{\mathbb{E}}_{s}\left[\mathop{\mathbb{P}}_{r}\left[g_0^s(r)\neq h_s(r)\right]\right] \leqslant \beta^{-1}\mathop{\mathbb{E}}_{s}\left[wt(\delta h_s)\right] \leqslant (k+2)(i+1)!\beta^{-(i+2)}p^{-1}\varepsilon\sum_{j=0}^{i+1}\frac{1}{j!},$$

since when $i+1\leqslant k$, g_0^s was chosen to minimize the distance between h_s and any coboundary. Towards this end, we claim the following:

Claim 4.7. Let s,r so that $\delta h_s(r)\neq 0$. Then

1. Either $r\circ s$ has some sub-face $r\circ s_j$ so that $f(r\circ s_j)\neq \delta g(r\circ s_j)$.
2. Or $\delta f(r\circ s)\neq 0$.

When s,r are chosen at random so that $|s| = i+1$, $col(r)\subseteq F$ and $col(s)\cap F=\emptyset$, the probability that the first item in Claim 4.7 occurs is bounded by

$$(i+1)\cdot \mathop{\mathbb{P}}_{t\in X(k)}\left[f(t)\neq\delta g(t)\mid\ |col(t)\cap\bar{F}| = i\right] = (i+1)\,\mathrm{dist}_i(f,\delta g),$$

since sampling $r\circ s$ as above, and then sampling a random $t = r\circ s_j$ in it, has the same marginal distribution as just sampling $t\in X(k-1)$ so that $|col(t)\setminus F| = i$. By the induction hypothesis this is bounded by

$$(i+1)\cdot(i!)(k+2)\beta^{-(i+1)}p^{-1}\varepsilon\sum_{j=0}^{i}\frac{1}{j!}. \tag{4.6}$$

The probability that the second item in Claim 4.7 occurs is $\varepsilon_{F,k-(i-1)}$ which is less or equal to $(k+2)p^{-1}\varepsilon$ by Claim 4.3. In conclusion

$$\begin{aligned}\mathop{\mathbb{P}}_{s,r}[\delta h_s(r)\neq 0] &\overset{\text{Claim 4.7}}{\leqslant} (i+1)\mathop{\mathbb{P}}_{s_i,r}[g(s_i\circ r)\neq f(s_i\circ r)]+(k+2)p^{-1}\varepsilon\\ &\overset{(4.6)}{\leqslant} (k+2)(i+1)!\beta^{-(i+1)}p^{-1}\varepsilon\sum_{j=0}^{i}\frac{1}{j!}+(i+1)!\frac{1}{(i+1)!}(k+2)p^{-1}\varepsilon\\ &\leqslant (k+2)(i+1)!\beta^{-(i+1)}p^{-1}\varepsilon\sum_{j=0}^{i+1}\frac{1}{j!}.\end{aligned}$$

Here the last inequality is just simplification.

Proving (4.4) is similar to the above (assuming for (4.3) holds for $i=k$). We need to bound the probability that $f(s)\neq\delta g(s)$. Similar to Claim 4.7 we note that $f(s)-\delta g(s)\neq 0$ implies that for every $w\in X_s^F(0)$:

1. Either there is some i so that $f(w\circ s_i)\neq g(w\circ s_i)$.
2. Or $\delta f(w\circ s)\neq 0$.

Otherwise take some $w\in X_s^F(0)$ so that $\delta f(w\circ s)=0$ and so that $f(w\circ s_i)=g(w\circ s_i)$ for all i. We have that

$$\begin{aligned}&0=\delta f(w\circ s)\\ &f(s)+\sum_{i=1}^{k+1}(-1)^i f((w\circ s)_i)=\\ &f(s)+\sum_{i=1}^{k+1}(-1)^i\delta g((w\circ s)_i)=\\ &f(s)-\delta g(s)+\sum_{i=0}^{k+1}\delta g((w\circ s)_i)=\\ &f(s)-\delta g(s)+\delta\delta g(s)=\\ &f(s)-\delta g(s).\end{aligned} \tag{4.7}$$

Hence

$$\operatorname{dist}_{k+1}(f,\delta g)\leqslant \mathop{\mathbb{E}}_{s\in\vec{X}(k)}\left[\mathop{\mathbb{P}}_{w\in X_s^F(0)}[\delta f(w\circ s)\neq 0]+\sum_{i=0}^{k}\mathop{\mathbb{P}}_{w\in X_s^F(0)}[f(w\circ s_i)\neq\delta g(w\circ s_i)]\right].$$

As

$$\mathop{\mathbb{E}}_{s\in\vec{X}(k)}\left[\mathop{\mathbb{P}}_{w\in X_s^F(0)}[\delta f(w\circ s)\neq 0]\right]=\varepsilon_{F,1}\leqslant(k+2)p^{-1}\varepsilon$$

and

$$\mathop{\mathbb{E}}_{s\in\vec{X}(k)}\left[\mathop{\mathbb{P}}_{w\in X_s^F(0)}[f(w\circ s_i)\neq\delta g(w\circ s_i)]\right]=\operatorname{dist}_k(f,\delta g).$$

This is less or equal to

$$\leqslant (k+2)p^{-1}\varepsilon + (k+1)\,\mathrm{dist}_k(f,\delta g)$$

$$\leqslant (k+2)\frac{(k+1)!}{(k+1)!}p^{-1}\beta^{-(k+1)}\varepsilon + (k+2)(k+1)(k!)p^{-1}\beta^{-(k+1)}\varepsilon\sum_{j=0}^{k}\frac{1}{j!}$$
$$=(k+2)!\beta^{-(k+1)}p^{-1}\varepsilon\sum_{j=0}^{k+1}\frac{1}{j!}.$$

and (4.4) follows. □

4.3 Proof of remaining claims

Proof of Claim 4.3. Consider the distribution where we sample some $(i,F,I) \in \{1,2,...,k+2\} \times \binom{[d]}{k+2} \times \binom{[d]}{\ell}$, so that F is uniform, i is uniform, and $I \subseteq F$ is uniform given that $|I| = i$. Let $\psi : X(k+1) \to \mathbb{R}$ be the indicator of the set $\{\delta f \neq 0\}$. Denote by $\varepsilon = \mathbb{E}[\psi]$. Then $\mathbb{E}_{(i,F,I)}\left[\varepsilon_{F,I}\right] = \varepsilon$ (where $\varepsilon_{F,I}$ is the expectation of ψ over faces t so that $col(t) \cap F = I$). Let $\mathcal{F}$ be the set of colors F so that X^F is locally coboundary expanding as defined in Theorem 4.1. By assumption $\mathbb{P}\left[\mathcal{F}\right] \geqslant p$. Thus we have that

$$\mathop{\mathbb{E}}_{(i,F,I):F\in\mathcal{F}}\left[\varepsilon_{F,I}\right] \leqslant p^{-1}\varepsilon.$$

In particular,

$$\mathop{\mathbb{E}}_{F\in\mathcal{F}}\left[\sum_{i=1}^{k+2}\varepsilon_{F,i}\right] \leqslant (k+2)p^{-1}\varepsilon,$$

where $\varepsilon_{F,i} = \mathbb{E}_{I\subseteq F,|I|=i}\left[\varepsilon_{F,I}\right]$. We conclude by taking some $F \in \mathcal{F}$ so that the sum of $\varepsilon_{F,i}$ is less than the expectation. This is the F we need. □

Proof of Claim 4.6. Recall that by definition $g(r' \circ s) = g_0^s(r')$ for some g_0^s so that $\delta g_0^s(a) = \sum_{\ell=0}^{k-i}(-1)^\ell g_0^s(a_\ell)$ is the closest coboundary to the function h_s. The function h_s was defined by

$$h_s(a) = f(a \circ s) - (-1)^{|a|}\sum_{\ell=0}^{i-2}(-1)^\ell g(a \circ s_\ell).$$

$\delta g_0^s(a) = h(a)$ if and only if

$$\delta g(a \circ s) = \sum_{j=0}^{k}(-1)^j g((a\circ s)_j) =$$

$$\delta g_0^s(a) + \sum_{j=k-i+1}^{k}(-1)^j g((a\circ s)_j) \overset{\delta g_0^s(a)=h(a)}{=}$$

$$h_s(a) + \sum_{j=k-i}^{k}(-1)^j g((a\circ s)_j) =$$

$$f(s\circ r) - \sum_{\ell=0}^{i-1}(-1)^{\ell+(k-i+1)}g(a\circ s_\ell) + \sum_{j=k-i+1}^{k}(-1)^j g((r\circ s)_j) = f(r\circ s).$$

The last equality is due to a change of variables in $\sum_{\ell=0}^{i-2}(-1)^{\ell+(k-i+1)}g(a\circ s_\ell)$ from ℓ to $j := \ell + (k-i+1)$. □

Proof of Claim 4.7. Let s, r be so that $|s| = i+1$, both $\delta f(r\circ s) = 0$ and for every $s_j \subseteq s$, it holds that $\delta g(r\circ s_j) = f(r\circ s_j)$.

Observe that,

$$\delta h_s(r) = \sum_{j=0}^{k-i}(-1)^j h_s(r_j) = \sum_{j=0}^{k-i}(-1)^j f(r_j\circ s) - (-1)^{k-i}\sum_{j=0}^{k-i}\sum_{\ell=0}^{i}(-1)^{j+\ell}g(r_j\circ s_\ell)$$

where the second equality is by the definition of h_s. By Claim 2.1 this is equal to

$$\sum_{j=0}^{k-i}(-1)^j f(r_j\circ s) - (-1)^{k-i}\sum_{j=0}^{k-i}\sum_{\ell=0}^{i}(-1)^{j+\ell}g(r_j\circ s_\ell) - (-1)^{k-i}\sum_{j=0}^{i}\sum_{\ell=0}^{i-1}(-1)^{(k-i)+1+j+\ell}g(r\circ(s_\ell)_j).$$

Note that in change of variables above we are using the fact that $r\circ(s_\ell)$ is equal to $(r\circ s)_{\ell+|r|}$ We change variables in the rightmost sum $\ell := \ell + (k-i+1)$, so this is equal to

$$\sum_{j=0}^{k-i}(-1)^j f(r_j\circ s) - (-1)^{k-i}\sum_{j=0}^{k-i}\sum_{\ell=0}^{i}(-1)^{j+\ell}g(r_j\circ s_\ell) - (-1)^{k-i}\sum_{j=0}^{i}\sum_{\ell=k-i+1}^{k}(-1)^{j+\ell}g((r\circ s_j)_\ell).$$

Rearranging, we get that this is equal to

$$\sum_{j=0}^{k-i}(-1)^j f((r\circ s)_j) - \sum_{j=0}^{i}(-1)^{j+(k-i)}\delta g(r\circ s_j) =$$

$$\sum_{j=0}^{k-i}(-1)^j f((r\circ s)_j) + \sum_{j=k-i+1}^{k}(-1)^j\delta g((r\circ s)_j).$$

If for all $r\circ s_j$ we have that $\delta g(r\circ s_j) = f(r\circ s_j)$, then this is equal to

$$\sum_{j=0}^{k}(-1)^j f((r\circ s)_j) = \delta f(r\circ s).$$

Finally, if $\delta f(r\circ s) = 0$ then indeed $\delta h_s(r) = 0$. □

5 k-coboundary expansion of order complexes of geometric lattices

In this section we analyze the coboundary expansion of k-chains in a d-dimensional spherical building, and prove lower bounds that depend on k but are independent of the ambient dimension d. Our analysis holds for any d-dimensional order complex of a homogeneous geometric lattice (see Section 2.3), a setting that generalizes the spherical building.

Theorem 5.1. *Let $k+2<d$. There exists constants $\beta_k = \exp(-O(k^5\log k)) > 0$ so that for every group Γ*

and every order complex of a homogeneous geometric lattice X, X is a coboundary expander with constant

$$h^k(X, \Gamma) \geqslant \beta_k.$$

We did not try to optimize the constants β_k.

Recall that $h^k(X, \Gamma)$ is defined for abelian groups Γ for all $k \geqslant 1$, and for $k = 0, 1$ for all groups Γ. In this section we assume that Γ is abelian. Section A treats the case of non-abelian Γ and $k = 1$. The case for $k = 0$ is straightforward. Claim 2.5 establishes that geometric lattices are edge expanders, and Claim 2.3 asserts that edge expansion implies constant coboundary expansion.

Note that an order complex of a $d-1$-graded lattice is naturally d-partite. For any $F \subseteq [d]$, we defined the restriction of X to the colors of F to be the complex $X^F = \{s \in X \mid \mathit{col}(s) \subseteq F\}$. A color restriction X^F is said to be (k, β)-*locally coboundary expanding* (with respect to Γ) if $h^k(X^F, \Gamma) \geqslant \beta$ and for every face $s \in X(j)$ so that $\mathit{col}(s) \cap F = \emptyset$, $h^{k-|s|}(X^F_s, \Gamma) \geqslant \beta$ (for $j \leqslant k$).

Our main effort will be to show that X^F is a local coboundary expander for many possible sets of colors F. Theorem 4.1 will then imply a lower bound on the coboundary expansion of X itself.

Lemma 5.2. *Let $c_k = \frac{k^2+5k+4}{2}$. Let $d \geqslant c_k$ and let X be a d-dimensional homogeneous geometric lattice. There are constants $p_k = \exp(-O(k^5 \log k))$ and $\beta'_k = \exp(-O(k^2 \log k))$ depending only on k so that for every abelian group Γ,*

$$\mathop{\mathbb{P}}_{F \in \binom{[d+1]}{c_k}} \left[X^F \textit{ is a } (k, \beta'_k)\textit{-locally coboundary expander with respect to } \Gamma \right] \geqslant p_k.$$

The proof of Theorem 5.1 is direct given this lemma and Theorem 4.1. We have it explicitly at the end of this section.

To prove this lemma, we use the cone machinery developed by [Gro10; LMM16; KM19; KO21]. We take a detour to define and explain their machinery in the next subsection.

Throughout the proof, we shall assume that $d \gg k$ (in particular, $d \geqslant \frac{k^2+5k+4}{2}$). The work of [KM19] gives constant coboundary expansion when d is smaller (see [KM19, Section 3.3]).

5.1 Boundaries and cones

We fix X to be a d-dimensional simplicial complex for $d \geqslant k$. We consider $C_k(X, \mathbb{Z})$. It will be convenient to write members of $C_k(X, \mathbb{Z})$ as a formal sum, so we prepare some conventions first. We identify $\vec{X}(k) \subseteq C_k(X, \mathbb{Z})$ where $t \in \vec{X}(k)$ is identified with the function

$$f_t(s) = \begin{cases} 1 & s = \pi(t) \text{ for some permutation } \pi \text{ with an even sign} \\ -1 & s = \pi(t) \text{ for some permutation } \pi \text{ with an odd sign} \\ 0 & \text{otherwise} \end{cases}.$$

These functions span $C_k(X, \mathbb{Z})$ [6]. Thus we can write every $f \in C_k(X, \mathbb{Z})$ as $f = \sum_{t \in \vec{X}(k)} \alpha_t t$.

The support of a function $\mathit{supp}(f) \subseteq X(k)$ is the set of all $t \in X(k)$ so that $f(\vec{t}) \neq 0$ (for any ordering $\vec{t}$ of t). The vertex support $vs(f) \subseteq X(0)$ is the set of all vertices that are contained in some $t \in \mathit{supp}(f)$.

[6] by choosing an ordering $\vec{t}$ for every $t \in X(k)$, the functions $\left\{ \vec{t} \;\middle|\; t \in X(k) \right\}$ are a basis for $C_k(X, \mathbb{Z})$.

The boundary operator. Let $\partial_k : C_k(X, \mathbb{Z}) \to C_{k-1}(X, \mathbb{Z})$ be the operator defined by

$$\partial f = \sum_{t \in \vec{X}(k)} \alpha_t \sum_{i=0}^{k} (-1)^i t_i.$$

This is the k-th boundary operator. It is a direct calculation to verify that this is well defined, i.e. that it does not depend on the choice of orientations of the faces in the sum of f.

Restriction to a vertex. Let $w \in X(0)$ and let $f \in C_k(X, \mathbb{Z})$ be $f = \sum_{t \in X(k)} \alpha_t t$. The restriction to w is the following function $f_w := \sum_{t \in X(k), t \ni w} \alpha_t t$.

Appending a vertex. Let $w \in X(0)$ and let $f \in C_k(X, \mathbb{Z})$ be a function that is supported in the link of w. Namely, $f = \sum_{t \in X_w(k)} \alpha_t t$. We denote by $f^w \in C_{k+1}(X, \mathbb{Z})$ the function $f^w = \sum_{t \in X_w(k)} \alpha_t (w \circ t)$. We note that this too does not depend on the representation of f. We record the following equality that will be useful later.

$$\partial_{k+1}(f^w) = f - (\partial_k f)^w \tag{5.1}$$

For example, if $f = uv$, and w is some vertex, then $f^w = wuv$ and

$$\begin{aligned} \partial f^w &= uv - wu + wv \\ &= f - ((u - v)^w) = f - (\partial f)^w. \end{aligned}$$

Cones. Let $\ell < d$. An *ℓ-cone* $\boldsymbol{\psi} = (\psi_i : C_i(X, \mathbb{Z}) \to C_{i+1}(X, \mathbb{Z}))_{i=-1}^{\ell}$ is a sequence of functions so that:

1. Every ψ_i is $\mathbb{Z}$-linear, that is $\psi(a \cdot s + b \cdot s') = a\psi(s) + b\psi(s')$.
2. The main property: for every $j \leqslant \ell$ and every $s \in X(j)$, it holds that

$$\partial \psi_j(s) = s - \sum_{i=0}^{j} (-1)^i \psi_{j-1}(s_i).$$

We illustrate the first levels of cones.

1. A (-1)-cone is just given by a single vertex $\psi_{-1} : C_{-1}(X, \mathbb{Z}) \to C_0(X, \mathbb{Z})$; $\psi_{-1}(\emptyset) = v \in X(0)$ (up to multiplying by an integer).
2. Let $v_0 = \psi_{-1}(\emptyset)$. To extend ψ to a 0-cone we choose walks $(v_0, v_1, ..., v_m = u)$ from v_0 to u for every $u \in X(0)$ and set $\psi(u) = \sum_{i=0}^{m-1} v_i v_{i+1}$. It is direct that $\partial \psi(u) = u - v_0 = u - \psi_{-1}(\emptyset)$. We note that it is always possible to add a cycle to $\psi(u)$ (i.e. any $R \in C_2(X, \mathbb{Z})$ so that $\partial R = 0$).
3. To extend ψ to a 1-cone, we need to find some $\psi(uw) \in C_2(X, \mathbb{Z})$ such that $\partial \psi(uw) = uw - \psi(w) + \psi(u)$. By the above, this is a cycle containing v_0 and the edge uw, so we need to "fill" this cycle with triangles.

When it is clear from context, we omit the index and just write $\psi(s)$. We note (and later use) the following.

Claim 5.3. Let $\boldsymbol{\psi}$ be any cone and let $s \in X(j)$. Then

$$\partial(s - \sum_{i=0}^{j}(-1)^i \psi(s_i)) = 0. \tag{5.2}$$

□

The reader may prove this by computing directly.

For $\ell' < \ell$ and an ℓ-cone $\boldsymbol{\psi} = (\psi_i : C_i(X,\mathbb{Z}) \to C_i(X,\mathbb{Z}))_{i=-1}^{\ell}$, the partial sequence $\boldsymbol{\psi}_{\ell'} = (\psi_i : C_i(X,\mathbb{Z}) \to C_i(X,\mathbb{Z}))_{i=-1}^{\ell'}$ is an ℓ'-cone.

The radius of a cone is

$$rad(\boldsymbol{\psi}) := \max_{s \in X} |supp\{\psi(s)\}|.$$

The following theorem connects cones with small radius to coboundary expansion.

Theorem 5.4 ([KO21] Theorem 3.8, see also [KM19], Theorem 2.5)**.** *Let X be a k dimensional simplicial complex so that there exists a group G that acts transitively on its k-dimensional faces. Assume that the $\ell+1$-skeleton has an ℓ cone with radius B. Then X is a coboundary expander and $h^\ell(X,\Gamma) \geqslant \frac{1}{B\binom{k+1}{\ell+1}}$ for any abelian group Γ.*

Remark 5.5. In previous works this theorem was proven for $\Gamma = \mathbb{F}_2$. Following the same steps for arbitrary Γ gives exactly the same statement. We omit it from the paper.

Let $X = X_P$ be the order complex of a homogeneous geometric lattice P, and let F any set of ℓ colors. Then $Aut(P)$ acts transitively on the ℓ-faces of X^F. The same is true for top-level faces of color restrictions of links X_s^F. Hence Lemma 5.2 will follow from Theorem 5.4 if we show there are enough colors F so that X^F and its local views X_s^F have small radius k-cones.

5.2 Proof of Lemma 5.2

Let P be a homogeneous geometric lattice, and let X be its d-dimensional order complex.

Recall that the colors in an order complex of a graded lattice correspond to the ranks of the elements. We will construct cones for certain sets of colors that correspond to ranks that are roughly exponentially increasing. For $i = 0, \ldots, k$ we will keep track of the parameters c_i, n_i, D_i which we now define inductively.

- c_i is the number of colors we need for constructing an i-dimensional cone,

$$c_0 = 2 \text{ and } c_i = c_{i-1} + (i+2) = \frac{i^2+5i+4}{2}.$$

- D_i upper bounds the radius of the i-cones,

$$D_0 = 3 \text{ and } D_i = (i+2)(i+1)(D_{i-1}+1).$$

  We record that $D_i = \exp(O(i^2 \log i))$.

- n_i upper bounds the size of the vertex support of the i-cones,

$$n_0 = 4 \text{ and } n_i = 2(i+2) - (i+1)^2 + (i+1)n_{i-1}\text{[7]}.$$

[7] One can show by induction that $n_i \geqslant i+1$ and in particular that this sequence is positive.

We record that $n_i = \exp(O(i \log i))$.

Let $\ell \geqslant c_k$. A set of colors $F = \{i_1 < i_2 < ... < i_\ell\}$ is called k*-suitable* if the following holds $i_2 \geqslant 2i_1$, and for every $j = 0, 1, ..., k-1$ and $m = 1, 2, ..., j+2$ it holds that $i_{c_j+m} \geqslant n_j \cdot i_{c_j} + \sum_{m'=1}^{m-1} i_{c_j+m'}$. While the exact inequalities may seem opaque, intuitively F is k-suitable if every i_{j+1} is sufficiently larger than its previous color i_j. For example, if for every j, $i_{j+1} > (k+3)n_k i_j$ then F is k-suitable.

Proposition 5.6 (Key Proposition)**.** *Let $F = \{i_1 < i_2 < ... < i_\ell\}$ be a set of k-suitable colors. Let X be either an order complex of a geometric lattice, or a link of said complex. Then X^F has a k-cone of radius $\leqslant D_k$.*

Proof of Lemma 5.2 (given Proposition 5.6). By Proposition 5.6, for every k-suitable set of colors F that has size c_k, and every $s \in X(j-1)$ so that $col(s) \cap F = \emptyset$, it holds that X_s^F has a $(k-j)$-cone of radius $D_{k-j} \leqslant D_k$. It follows from Theorem 5.4 that that $h^{k-j}(X_s^F, \Gamma) \geqslant \frac{1}{\binom{c_k+1}{k+1} D_k} := \beta'_k$. Note that $D_k = \exp(O(k^2 \log k))$ and $\binom{c_k}{k+1} \leqslant \binom{k^2}{k} = \exp(O(k \log k))$ so indeed $\beta'_k = \exp(-O(k^2 \log k))$.

We need to show that the set of suitable colors $\mathcal{F} \subseteq \binom{d}{c_k}$ is a constant fraction of all colors (that depends on k only). The intuitive idea is that a set is suitable if each color is in a strip $[\alpha d, \beta d)$ that guarantees they satisfy a sequence of inequalities of the form $i_{j+1} \geqslant B \cdot i_j$. The probability that this happens for a random set of colors is sufficient for our purpose. Indeed, let $F = \{i_1 < i_2 < ... < i_{c_k}\}$ be a set of colors. Let $B = (k+3)n_k$. If for every $j = 1, 2, ..., c_k$ it holds that

$$i_j \in \left[\frac{d}{(2B)^{c_k+1-j}}, \frac{2d}{(2B)^{c_k+1-j}}\right),$$

then in particular $i_{j+1} \geqslant B \cdot i_j$. One can easily verify that this implies that F is k-suitable, since $i_2 \geqslant Bi_1 \geqslant 2i_1$ and $i_{c_j+m} \geqslant (k+3)n_k \cdot i_{c_j+m-1} \geqslant n_j i_{c_j} + \sum_{m'=1}^{m-1} i_{c_j+m'}$. On the other hand, there are at least

$$\prod_{j=1}^{c_k} \left(\frac{2d}{(2B)^{c_k+1-j}} - \frac{d}{(2B)^{c_k+1-j}} \right) = d^{c_k} \cdot \frac{1}{(2B)^{c_k^2 + c_k - \sum_{j=1}^{c_k} j}}.$$

such sets F. As $\binom{d}{c_k} \leqslant d^{c_k}$, it holds that $\mathbb{P}_{F \in \binom{[d]}{c_k}} [\mathcal{F}] \geqslant \frac{1}{(2B)^{c_k^2 + c_k - \sum_{j=1}^{c_k} j}} := p_k$ as required.

Finaly, we draw our readers attention to the fact that $B = \exp(O(k \log k))$ and $c_k = O(k^2)$ so $p_k = \exp(-O(k^5 \log k))$. ◻

It remains to prove Proposition 5.6. We will prove a stronger statement that is amenable to an inductive proof:

Proposition 5.7. *Let $F = \{i_1 < i_2 < ... < i_\ell\}$ be a set of k-suitable colors. Let X be either an order complex of a geometric lattice, or a link of said complex. Then X^F has a k-cone of radius $\leqslant D_k$. Denote this cone ψ. It has the following properties:*

1. *For every $j \leqslant k$ and $s \in X(j)$, it holds that $|vs(\psi_j(s))| \leqslant n_j$.*
2. *For every $j \leqslant k$ and $s \in X(j)$ and every $u \in vs(\psi_j(s)) \setminus s$, it holds that $col(u) \leqslant i_{c_j}$.*

The proof of Proposition 5.7 will use the following properties of our complex.

Claim 5.8. Let X be either an order complex of a geometric lattice, or a link in said order complex. Then X has the following properties.

1. The bipartite graph $(X[i_1], X[i_2])$ has diameter 2.

2. If $col(v) > col(u)$ and $v \sim u$ then $X_v[\{i \leqslant col(u)\}] \supseteq X_u[\{i \leqslant col(u)\}]$.

3. (Bound on join rank) For every $j = 0, 1, ..., k-1$ and $m = 1, 2, ..., j+2$ the following holds. Let $D \subseteq X(0)$ be a set of at most n_j vertices of colors $\leqslant i_{c_j}$. Let $M = \{v_1, v_2, ..., v_r\}$ be so that $i_{c_j+1} \leqslant col(v_1) < col(v_2) < ... < col(v_r) \leqslant i_{c_j+m-1}$. Then there exists a vertex $w \in X[i_{c_j+m}]$ so that the complex induced by $D \cup M$ is contained in X_w.

4. The property in item 3 holds in X_s for every $s \in X$ so that all colors in s are all greater than i_{c_j+m}.

We prove this claim after proving the proposition.

Proof of Proposition 5.7. We construct a cone $\boldsymbol{\psi}$ inductively. We note that as all the ψ_j are $\mathbb{Z}$-linear and $C_k(X, \mathbb{Z})$ is generated by $X(k) \subseteq C(X, \mathbb{Z})$, it is enough to define ψ_j on $s \in \vec{X}(k)$ so that it respects anti-symmetry (i.e. $\psi_j(\pi(s)) = \text{sign}(\pi)\psi_j(s)$).

The first two steps of our construction, corresponding to $\ell = -1, 0$, are as follows:

1. $\psi_{-1}(\emptyset) = v_0$, where $v_0 \in X[i_1]$ is chosen arbitrarily.

2. For $u \in X(0)$ we construct $\psi_0(u)$.

   - If $u = v_0$ then $\psi_0(v_0) = 0$.
   - If $v_0 u \in X(1)$ then $\psi_0(u) = v_0 u$.
   - If $u \in X[i_1]$ then by the assumption that $(X[i_1], X[i_2])$ has diameter 2, they have a common neighbor $w \in X[i_2]$. We assign $\psi_0(u) = v_0 w + wu$.
   - Finally, for other $u \in X[i_j]$ first find a neighbor of u $w_1 \in X[i_1]$. Select some $w_2 \in X[i_2]$ that is a common neighbour of v_0 and w_1. Finally we set $\psi_0(u) = v_0 w_2 + w_2 w_1 + w_1 u$.

As we can see from the description above, there are many choices for ψ, so we choose arbitrarily. Notice that the first two parts of the cone $\boldsymbol{\psi}_0 = \{\psi_{-1}, \psi_0\}$ have radius $D_0 = 3$. The number of vertices in $\psi(u)$ is at most $n_0 = 4$. Finally, the vertices participating in $\psi(u)$ are either u or vertices whose colors are i_1, i_2.

For $\ell \geqslant 1$ given $\boldsymbol{\psi}_\ell$ we construct $\psi_{\ell+1}$, as described in the algorithm appearing in Figure 2.

Algorithm 5.9. Input: $s \in X(\ell+1)$

1. Set
$$R_0 = s - \sum_{i=0}^{\ell+1}(-1)^i \psi(s_i),$$
$$s_0 = \left\{ v \in s \middle| \ col(v) > i_{c_j} \right\},$$

2. Order the vertices in $s_0 = (v_0, v_2, ..., v_t)$ so that $col(v_1) < col(v_2) < ... < col(v_t)$.

3. (Shifting step) For $j = 0$ to t:

   (a) If $col(v_j) \neq i_{c_\ell + j+1}$:

      i. Find a vertex v'_j so that $supp((R_j)_{v_j}) \subseteq X_{v'_j}$ and so that $col(v'_j) = i_{c_\ell+j+1}$.

      ii. Set $T_j = ((R_j)_{v_j})^{v'_j}$, and set $R_{j+1} = R_j - \partial T_j$.

   (b) Else: set $T_j = 0$ and set $R_{j+1} = R_j$.

4. (Star step) Find some $u \in X[i_{c_{\ell+1}}]$ so that R_{t+1} is in the link of u.

5. Output $\psi(s) = R^u_{t+1} + \sum_{r=0}^t \partial T_r$.

Figure 2: Constructing $\psi(s)$

Fix $s \in \overrightarrow{X}(\ell+1)$ and let $R_0 = s - \sum_{i=0}^{\ell}(-1)^i \psi(s_i)$. Before constructing ψ formally, we describe the main idea.

We will first find a sequence of "shifted chains" $R_0, R_1, ..., R_{t+1}$ so that all vertices in the support of R_{t+1} are of colors $< i_{c_{\ell+1}}$ and each R_j is R_{j-1} "shifted" by T_{j-1}. Namely, there is a sequence of $T_0, T_1, ..., T_t \in C_{\ell+1}(X, \mathbb{Z})$ where $R_{j+1} = R_0 - \sum_{r=0}^{j} \partial T_r$. We will explain below how to construct such chains.

Next we will use item three in Claim 5.8 to argue that due to the fact that R_{t+1} is supported only on vertices of colors $< i_{c_{\ell+1}}$, there exists some $u \in X[i_{c_{\ell+1}}]$ so that the complex induced by the vertices of R_{t+1} is contained in X_u. Finally we will set

$$\psi(s) := \sum_{r=0}^{t} T_r + R^u_{t+1}. \tag{5.3}$$

A direct calculation shows that $\partial\psi(s) = R_0$:

$$\begin{aligned}
\partial\psi(s) &= \partial(\sum_{r=0}^{t} T_r + R^u_{t+1}) \\
&\overset{(5.1)}{=} \sum_{r=0}^{t} \partial T_r + R_{t+1} + (\partial R_{t+1})^u \\
&= \sum_{r=0}^{t} \partial T_r + R_0 - \sum_{r=0}^{t} \partial T_r + (\partial R_0 + \sum_{r=0}^{t} \partial\partial T_r)^u
\end{aligned}$$

$$= R_0 + (\partial R_0 + \sum_{r=0}^{t} \partial\partial T_r)^u$$
$$\overset{\partial\partial=0}{=} R_0 + (\partial R_0)^u = R_0.$$

The last equality is due to Claim 5.3 which states $\partial R_0 = 0$.

Let us understand how to perform the shifting step. Note that by the assumption on $\boldsymbol{\psi}_\ell$ any vertex in $vs(R_0)$ of color $> i_{c_\ell}$ must come from s itself. So let $v_0, v_1, ..., v_t \subseteq s$ be the vertices in $vs(R_0)$ of colors $> i_{c_\ell}$ (the vertices of color $\geqslant i_{c_{\ell+1}}$ are a subset of these vertices). The sets T_j we want will have the property that $vs(R_{j+1}) = vs(R_j - \partial T_j) = (vs(R_j) \setminus \{v_j\}) \cup \{v'_j\}$, where the replacing vertices $v'_0, v'_1, ..., v'_t$ are of low-colors $(col(v'_j) = i_{c_\ell+j+1})$.

We construct T_j as follows. Assume without loss of generality that $v_0 \prec v_1 \prec ... \prec v_t$ according to the order of the lattice. This implies that $col(v_j) \geqslant i_{c_\ell+j+1}$. If $col(v_j) = i_{c_\ell+j+1}$ then by setting $T_j = 0$ we are done. Otherwise, $col(v_j) > i_{c_\ell+j+1}$. We will find a vertex v'_j of color $i_{c_\ell+j+1}$ so that for every face $s' \in supp(R_j)$ that contains v_j, $s' \cup \{v'_j\} \in X$. Then we take $T_j = ((R_j)_{v_j})^{v'_j}$ (i.e. T_j takes all faces that contain v_j in R_j and adds v'_j to them). We will show below that v_j is no longer in the support of $R_{j+1} = R_j - \partial T_j$.

The reason we can find such a v'_j is the fourth item of Claim 5.8; this item promises us existence of some $v'_j \prec v_j$ so that all the vertices in $vs((R_j)_{v_j})$ of color $< col(v_j)$ are also neighbors of v'_j. Moreover, as $v'_j \prec v_j$ it also holds that $v'_j \prec v_{j+1} \prec ... \prec v_t$, this promises us that *all* $vs((R_{j-1})_{v_j})$ are also neighbors of v'_j. Thus $(R_{j-1})_{v_j}$ is contained in the link of v'_j and T_j is well defined as a chain in X.

We summarize this process in the following claim. Its proof, which technically formalizes this discussion, is given below.

Claim 5.10.

1. The shifting step is always possible. That is, there exists v'_j so that $supp((R_j)_{v_j}) \subseteq X_{v'_j}$ and so that $col(v'_j) = i_{c_\ell+j+1}$. Moreover,
2. If $R_j \neq R_{j-1}$ then $vs(R_j) \subseteq (vs(R_{j-1}) \cup \{v'_j\}) \setminus \{v_j\}$ and
3. $|supp(R_j)| \leqslant |supp(R_{j-1})| \leqslant ... \leqslant |supp(R_0)|$.

Note that the last item will be necessary when we will bound $|\psi(s)|$.

As alluded to in the discussion, after obtaining R_{t+1} we need to show that there exists some $u \in X[i_{c_{\ell+1}}]$ so that $R_{t+1} \subseteq X_u$.

Let $M \subseteq vs(R_{t+1})$ be the vertices of colors $> i_{c_\ell}$. By the shifting step there is at most one vertex of each color in M. Let $B \subseteq vs(R_{t+1})$ be the rest of the vertices, all of which are of color $\leqslant i_{c_\ell}$. By the last item of Claim 5.10 there are at most $|vs(R_{t+1})| \leqslant |vs(R_0)|$ vertices in B. This is at most $n_{\ell+1}$ by induction hypothesis on the cone already constructed. Hence by item 3 in Claim 5.8, there exists some $u \in X[i_{c_{\ell+1}}]$ so that the complex induced by $M \cup B = vs(R_{t+1})$ is in the link of u. This shows that $\psi(s)$ is well defined.

Now that we established that $\boldsymbol{\psi}$ is a cone, we bound its radius and verify its other properties.

The radius. Obviously $|supp(R_{t+1})^u| = |supp(R_{t+1})|$, and by last item in Claim 5.10, this is $\leqslant |\operatorname{supp} R_0|$. By the inductive assumption on $\boldsymbol{\psi}_\ell$,

$$|supp(R_0)| \leqslant 1 + \sum_{i=0}^{\ell+1} |supp(\psi(s_i))| \leqslant (\ell+2) rad(\boldsymbol{\psi_\ell}) + 1 \leqslant (\ell+2)(1 + D_\ell).$$

Moreover, in every iteration of the shifting step we added T_j to $\psi(s)$. The faces in $T_j = ((R_j)_{v_j})^{v'_j}$, the support of this function is again of size at most $(\ell+2)(1+D_\ell)$ (since by the last item of the size of Claim 5.10, the support of R_j is less or equal to the size of R_0's support). There are at most $\ell+2$ iterations in the shifting step. So the support of $\sum_{j=0}^{t} T_j$ is at most $(\ell+2)^2(1+D_\ell)$. In total, for every $s \in X(\ell+1)$,

$$|supp(\psi(s))| = |supp\left((R_{t+1})^u + \sum_{j=0}^{t} \partial T_j\right)| \leqslant (\ell+3)(\ell+2)(1+D_\ell) = D_{\ell+1}.$$

Colors of $vs(\psi(s)) \setminus s$. The vertices in $(vs(\psi(s)) \setminus s$ that come from $R_0 = s - \sum_{i=0}^{\ell+1} \psi(s_i)$ have colors $\leqslant i_{c_\ell}$ by the induction hypothesis on $\boldsymbol{\psi_\ell}$. The other vertices in $vs(\psi(s)) \setminus s$ either come from the shifting step, in which case these are the v'_j that have colors $\leqslant i_{c_{\ell+1}+j+1}$, or the vertex of color $i_{c_{\ell+1}}$ from the star step. To conclude, all vertices of $vs(\psi(s)) \setminus s$ have colors $\leqslant i_{c_{\ell+1}}$.

The size of $vs(\psi(s))$. The vertex support of every T_j consists only of the vertex support of R_0 and the new v'_j that we added. There are at most $\ell+2$ vertices v'_j introduced by the sets T_j. The vertex support of $(R_{t+1})^u$ is u together with the vertex support of (R_{t+1}). Thus $vs(\psi(s)) \subseteq vs(R) \cup \{v'_1, ..., v'_m\} \cup \{u\}$. The vertex support of R_0 satisfies

$$|vs(R_0)| = |vs(s - \sum_{i=0}^{j} \psi(s_i))| \leqslant |s| + 1 + \sum_{i=0}^{\ell+1} (|vs(\psi(s_i))| - |s_i| - 1)\,.$$

Here we subtract $|s_i|+1$ from the sums since we need to count the vertices of s and u_0 only once. Hence $|vs(R_0)| \leqslant \ell+3-(\ell+2)^2+(\ell+2)n_\ell$. Plugging this back in we get that

$$|vs(\psi(s))| \leqslant \ell+3-(\ell+2)^2+(\ell+2)n_\ell+(\ell+3) = n_{\ell+1}.$$

□

Proof of Claim 5.8. Here when we write $u \leqslant v$ we mean by the order of the lattice, and when we write $col(u) > col(v)$ we mean the usual order of integers.

1. Let $u_1, u_2 \in X[i_1]$ the join $u_1 \vee u_2$ has rank $\leqslant 2i_1$. By properties of the lattice, there exists some $v \in X[i_2]$ so that $v \succeq u_1 \vee u_2 \succeq u_1, u_2$. In particular, there is a path u_1, v, u_2 and the diameter is 2.

2. If $col(v) > col(u)$ and $v \sim u$ this implies that $v \succeq u$. In particular, $w \in X_u[\{i \leqslant col(u)\}]$ if and only if $w \preceq u \preceq v$ and thus $w \in X_v[\{i \leqslant col(u)\}]$.

3. Let u be the join of all the elements in $M \cup D$. The rank of u is at most the sum of the ranks of the elements in $M \cup D$. That is, $col(u) \leqslant n_j \cdot i_{c_j} + \sum_{m'=1}^{m-1} i_{c_j+m'}$. By assumption on the colors $i_{c_j+m} \geqslant col(u)$. Therefore, there is some $w \in X[i_{c_j+m}]$ that contains u (if $col(u) < i_{c_j+m}$ we can always add atoms to u one by one, each increasing the rank by 1, until we reach some $w \succ u$ whose color is i_{c_j+m} as required). In particular the complex induced by $M \cup D$ is in the link of w.

4. Let $v \in s$ be the vertex with the smallest rank. By the second property proven above, it holds that $X_s[\{i_1, ..., i_{c_j+m}\}] = X_v[\{i_1, ..., i_{c_j+m}\}]$. Note that $X_v[\{i_1, ..., i_{c_j+m}\}]$ is a partially ordered set whose elements are strictly less than v in the order of the lattice. This is also a geometric lattice (or a link of said lattice), thus the proof for item 3 holds there as well.

□

This following claim shall be used in the proof of Claim 5.10.

Claim 5.11. Let $R \in C_k(X, \mathbb{Z})$ so that $\partial R = 0$. Then $w \notin vs(\partial R_w)$ or equivalently $(\partial R_w)_w = 0$.

Proof of Claim 5.11. We write $R = R_w + R' = \sum_{w \in t} \alpha_t t + R'$ and get that $0 = \partial R = \partial R_w + \partial R'$. In particular $(\partial R_w)_w = (-\partial R')_w$. As $w \notin vs(R')$ it is also not in $vs(\partial R')$, so $(\partial R')_{w_2} = 0$. In conclusion $(\partial R_w)_w = 0$. □

Proof of Claim 5.10. We prove this by strong induction on j. Assume that the claim holds for all $j' < j$.

Let us begin with the first item. Our goal is to show that the conditions in the last item of Claim 5.8 hold on $(R_{j-1})_{v_j}$. By the induction hypothesis, we note that the vertex support of $(R_{j-1})_{v_j}$ is contained in the union of:

1. The vertices of $s' = \left\{v \in s \,\middle|\, col(v) \geqslant col(v_j)\right\} \subseteq s$.
2. A subset $M \subseteq \{v'_1, ..., v'_{j-1}\}$ which were added to the support in one of the previous steps.
3. Other vertices D that came from the support of R_0. There are at most n_ℓ of those, and they all have colors $\leqslant i_{c_j}$ by our assumption on the cone in the previous steps.

Let $Q = \left\{t \setminus s' \,\middle|\, t \in supp((R_{j-1})_{v_j})\right\}$. Q, is a subcomplex of the complex induced by $M \cup D$, and is contained in $X_{v_j}[\{i \leqslant col(v_j)\}]$. Also, we note that by item 2 in Claim 5.8, this is equal to $X_{s'}[\{i \leqslant col(v_p)\}]$ (since v_j is the vertex that has the smallest color out of s'). By the last item in Claim 5.8, there indeed exists some $v'_j \in X_{s'}$ so that the complex induced by $M \cup D$ is contained in $X_{v'_j \cup s'}$. In particular, this implies that the support of Q is contained in the $X_{v'_j}$. As all vertices $v \in s'$ that are not v_j are greater than v_j (in the lattice order), it also holds that the support of $(R_j)_{v_j}$ is in $X_{v'_j}$.

We turn towards the second item. Assume that $R_j \neq R_{j-1}$. By construction, $vs(R_p) \subseteq vs(R_{j-1}) \cup \{v'_j\}$. So to show that $vs(R_p) \subseteq (vs(R_{j-1}) \cup \{v'_j\}) \setminus \{v_j\}$ we need to show that $v_j \notin vs(R_j)$.

$$\begin{aligned} R_j &= R_{j-1} - \partial T_j \\ &= R_{j-1} - \partial(((R_{j-1})_{v_j})^{v'_j}) \\ &\overset{(5.1)}{=} R_{j-1} - (R_{j-1})_{v_j} + (\partial((R_{j-1})_{v_j})^{v'_j}. \end{aligned}$$

In particular, restricting both sides to the support of v_j we have that

$$(R_j)_{v_j} = (R_{j-1})_{v_j} - (R_{j-1})_{v_j} + (\partial(R_{j-1})_{v_j})_{v_j}^{v'_j} = (\partial(R_{j-1})_{v_j})_{v_j}^{v'_j}.$$

Note that the right-hand side is equal to $((\partial(R_{j-1})_{v_j})_{v_j}))^{v'_j}$, i.e. we can first restrict to v_j and then append v'_j. However, this is 0 by Claim 5.11 (and the fact that $\partial R_{j-1} = \partial R_0 - \sum_{r=0}^{j-1} \partial\partial T_r = 0$). So in conclusion we have that $(R_j)_{v_j} = 0$ and $v_j \notin vs(R_j)$.

Finally we explain why the last item holds. Indeed, we notice that $supp(R_{j-1}) \setminus supp(R_j)$ contains all faces in $supp((R_{j-1})_{v_j})$. On the other hand, the faces in $supp(R_j) \setminus supp(R_{j-1})$ are faces of the form $(s \setminus v_j) \cup \{v_{j'}\}$ for faces $s \in supp((R_{j-1})_{v_j})$. This is because faces in $supp(R_j) \setminus supp(R_{j-1})$ can come from ∂T_j. Faces in ∂T_j that are not of the form $(s \setminus v_j) \cup \{v_{j'}\}$ for faces $s \in supp((R_{j-1})_{v_j})$, must contain v_j (since $T_j = ((R_{j-1})_{v_j})^{v'_j}$ and thus all its faces contain v_j) but as we saw $v_j \notin vs(R_j)$ so all these faces are not in the support of R_j. Hence $|supp(R_j)| \leqslant |supp(R_{j-1})|$. □

5.3 Proof of Theorem 5.1

Proof. Lemma 5.2 says that the homogeneous geometric lattice satisfies the assumptions of Theorem 4.1, with $\beta = \beta'_k = \exp(-O(k^2 \log k))$ and $p = p_k = \exp(-O(k^5 \log k))$. Thus concluding that X is a coboundary expander and that

$$h^k(X, \Gamma) \geqslant \frac{p_k \beta'^{k+1}_k}{e(k+2)!} = \exp(-O(k^5 \log k)).$$

□

6 Applications to known bounded degree complexes

6.1 Cosystolic expansion of known complexes

6.1.1 [LSV05a] complexes

We now put everything together and show that complexes constructed by Lubotzky, Samuels and Vishne have degree and dimension independent cosystolic expansion. Recall the following properties of their construction.

Theorem (Restatement of Theorem 2.6)**.** *For any prime power q and integer $d > 1$, there is a family $\mathcal{X}_{q,d} = \{X_n\}_{n=1}^{\infty}$ of connected complexes whose links are (isomorphic copies of) the $SL_d(\mathbb{F}_q)$-spherical building. In particular, For every $\lambda > 0$ there is some q_0 so that every X_n is a λ-one sided high dimensional expander when $q \geqslant q_0$.*

Theorem 6.1. *For every $k > 0$ there is some constant $\beta_k = \exp(-O(k^6 \log k)) > 0$ and integer q_0 so that for every prime power $q > q_0$, integer $d > k + 2$, group Γ, and $X \in \mathcal{X}_{q,d}$ it holds that*

$$h^k(X, \Gamma) \geqslant \beta_k.$$

Proof of Theorem 6.1. We wish to apply Theorem 1.2 to the complexes of Theorem 2.6. By Theorem 5.1 there is a sequence of constants $\{\beta_\ell = \exp(-O(\ell^5 \log k))\}_{\ell=0}^{k}$ so that for every q, d the spherical building $SL_d(\mathbb{F}_q)$ satisfies $h^\ell(SL_d(\mathbb{F}_q), \Gamma) \geqslant \beta_\ell$ for every group Γ. There is some q_0 so that for every $q > q_0$, X_n are also λ-expanders for $\lambda = \exp(-O(k^6 \log k))$. Applying Theorem 1.2, we get that $h^\ell(X_n, \Gamma) \geqslant \exp(-O(\ell^6 \log \ell))$ for every $\ell \leqslant k$, and in particular, this holds for h^k as claimed. □

Observe that our theorem (as well as previous bounds of [KKL14; EK16]) holds only for LSV complexes $\mathcal{X}_{q,d}$ with sufficiently large $q > q_0$. It seems reasonable that even for $q = 2$ the theorem should hold, but we leave this as an open question.

6.1.2 [KO21] complexes

Kaufman and Oppenheim showed that links of their complexes are spectral expanders, and also coboundary expanders for 1-chains.

Theorem (Restatement of Theorem 2.7)**.** *For every $\lambda > 0$ there exists a family of 4-partite complexes $\mathcal{Y}_\lambda = \{Y_n\}_{n=1}^{\infty}$ so that*

1. *Y_n is a λ-one sided high dimensional expander.*

2. *There exists a constant $\beta > 0$ (independent of λ) so that for every abelian group Γ and every $s \in Y_n(0)$, the link of s has $h^1(Y_s, \Gamma) \geqslant \beta$.*

Applying our Theorem 1.2 implies stronger bounds on the cosystolic expansion of Y_n:

Theorem 6.2. *For every d there exists some $\lambda > 0$ and some $\beta' = \frac{\beta^2}{2}(1 - O(\lambda))$ so that for every abelian group Γ and every $Y_n \in \mathcal{Y}_{\lambda,d}$*

$$h^1(Y_n, \Gamma) \geqslant \beta'.$$

The proof of Theorem 6.2 is just applying Theorem 1.2 on every $Y_n \in \mathcal{Y}_{\lambda,d}$, and is therefore omitted.

6.2 Topological overlap of LSV complexes

Let X be d-dimensional simplicial complex. X is also a topological space constructed by taking a d-simplex for every $t \in X(d)$ and gluing two simplexes t_1, t_2 over their intersection $t_1 \cap t_2$.

Definition 6.3. Let X be a d-dimensional simplicial complex and let $c > 0$. We say that X has (c, k)-topological overlap if for every continuous map $f : X \to \mathbb{R}^k$ there exists a point $p \in \mathbb{R}^k$ so that

$$\mathop{\mathbb{P}}_{t \in X(d)} [p \in f(t)] \geqslant c.$$

We call such a point p a c-heavily covered point (with respect to f), since if the measure on $X(d)$ is uniform, this is proportional to the number of d-faces covering p.

Theorem 6.4 ([Gro03])**.** *Let X be a simplicial complex so that*

1. *For every $\ell \leqslant k$, $h^\ell(X, \mathbb{F}_2) \geqslant \beta$.*
2. *For every $\ell \leqslant k$ and every $g \in Z^k(X, \mathbb{F}_2) \setminus B^k(X, \mathbb{F}_2)$, $wt(g) \geqslant \nu$.*
3. *$\max_{v \in X(0)} \mathbb{P}[v] \leqslant \varepsilon$.*

Then the k-skeleton of X has (c, k)-topological overlap (i.e. for continuous functions to R^k) for $c = \frac{\nu \beta^{k+1}}{2(k+1)!} - \varepsilon k^2 \beta^{-(2k+1)}$.

The constant c was estimated by [DKW18]. In addition, it turns out that one can replace R^k with any k-dimensional manifold that admits a piecewise-linear triangulation, and this theorem still holds.

The work by [EK16] used this theorem to show that [LSV05a] complexes have the topological overlap properties. Namely, they showed that these complexes have

$$h^k(X, \beta) = \Omega(\min\{\frac{1}{Q}, (d!)^{-O(2^k)}\}),$$

where d is the dimension of the complex, and Q is the maximal number of edges adjacent to a vertex. They also show that $g \in Z^k(X, \mathbb{F}_2) \setminus B^k(X, \mathbb{F}_2)$ has weight at least $\nu = (d!)^{-O(2^k)}$. Plugging this in Theorem 6.4 gives a topological overlap constant of $c = Q^{-k} \exp(-O(2^k d \log d))$.

A direct application of our cosystolic expansion bounds, together with Proposition 3.2, gives us bounds that are independent of the degree of the vertices, and the ambient dimension of the complex. In addition, it gives a better dependence in k (exponential instead of doubly exponential).

Corollary 6.5. *Let $\{X_n\}$ be the simplicial complexes in Theorem 6.1. Then X_n have the (c,k)-topological overlap with $c = \exp(-O(k^7 \log k)) - \varepsilon \cdot \exp(O(k^7 \log k))$, where $\varepsilon = \frac{1}{|X_n(0)|}$ (and goes to 0 independent of k).*

We note that a similar corollary holds for the spherical building and other homogenuous lattices.

Proof. The corollary follows from plugging in the parameters of the complexes in Theorem 6.1, to Theorem 6.4. Theorem 6.1 gives us $h^k(X, \mathbb{F}_2) = \exp(-O(k^6 \log k))$ cosystolic expansion. Proposition 3.2 give us a bound of $\nu = \exp(-O(k^6 \log k))$ on the weight of all $g \in Z^k(X, \mathbb{F}_2) \setminus B^k(X, \mathbb{F}_2)$. □

Our bounds also directly imply that the 2-skeletons of all complexes constructed in [KO21] have $\Omega(1)$-topological overlap (where previously the bound depended on the maximal degree of a vertex). We omit the direct proof.

Corollary 6.6. *Let $\{Y_n'\}$ be the two skeletons of complexes in Theorem 6.2 with a sufficiently large number of vertices. Then every Y_n is a $(c,1)$-topological overlap for some universal constant $c > 0$.* □

6.3 Cover stability

Dinur and Meshulam studied local testability of covers [DM22], and showed that covering maps of a simplicial complex X are locally testable if and only if X is a cosystolic expander on 1-chains. We briefly describe their result below, and show that our new bounds on cosystolic expansion of 1-chains of [LSV05a] and [KO21] complexes, show that covering maps to these complexes are locally testable.

Definition 6.7 (covering map)**.** Let X, Y be pure simplicial complexes. A covering map is a surjective simplicial map[8] $\rho : Y(0) \to X(0)$ such that for every $\tilde{u} \in Y(0)$ that maps to $\rho(\tilde{u}) = u \in X(0)$, it holds that $\rho|_{Y_{\tilde{u}}(0)} : Y_{\tilde{u}}(0) \to X_u(0)$ is an isomorphism. If there exists such a map ρ we say that Y is a cover of X.

While covering maps are described combinatorially in simplicial complexes, they are a well-known topological notion in general topological spaces. They are classified by the fundamental group of the complex X [Sur84]. This is an interesting example for a non-trivial topological property that is locally testable.

The one-dimensional case, i.e. graph covers, have been useful in construction of expander graphs. Bilu and Linial showed that random covers of Ramanujan graphs are almost Ramanujan [BL06]. A celebrated result by [MSS15] used these techniques to construct bipartite Ramanujan graphs of every degree. Recently, [Dik22] showed that random covers could also be applied for constructing new local spectral expanders.

Local testability of covering maps. Given a map $\rho : Y(0) \to X(0)$, we wish to test whether (Y, ρ) is close to a covering map (in Hamming distance), while querying only a few of the values $(u, \rho(u))$.

To describe such a test restrict ourselves to the following family of maps. Fix X to be some pure d-dimensional simplicial complex. Let S be a set, and $\Gamma \leqslant Sym(S)$ be a group acting on S. The family of functions we consider are $M(\Gamma, S)$ (suppressing X in the notation). These are all (ρ, Y) such that

1. $Y(0) = X(0) \times S$,
2. $\rho(u, s) = u$ and
3. For every $uv \in X(1)$ the complex induced by the vertices $\{(u,s),(v,s) \mid s \in S\} \subseteq Y(0)$ is a perfect matching where $(u,s) \sim (v, \gamma_{uv}.s)$ for some $\gamma_{uv} \in \Gamma$.

[8] that is, for every $i \leqslant d$ and every $s \in Y(i)$, $\rho(s) \in X(i)$.

The distance between two maps is

$$\mathrm{dist}((Y_1,\rho_1),(Y_2,\rho_2)) = \underset{uv\in X(1)}{\mathbb{P}}\left[\gamma_{uv}^{Y_1} \neq \gamma_{uv}^{Y_2}\right],$$

where $\gamma_{uv}^{Y_i}$ is the member in Γ that describes the bipartite graph induced by $\{(u,s),(v,s) \mid\ s \in S\} \subseteq Y_i$. We note that when S is finite this is proportional to the Hamming distance between $Y(1), Y(2)$.

While this restriction seems arbitrary, it turns out that for every pair $(Y, \rho : Y(0) \to X(0))$ where ρ is a covering map, there exists some Γ, S and an identification $Y(0) \cong X(0) \times S$ such that $\rho(u,s) = u$ [Sur84]. However, being in $M(\Gamma,S)$ is not sufficient to being a covering map. A map in $M(\Gamma,S)$ above is a covering map if and only if for every $uvw \in \vec{X}(2)$,

$$\gamma_{uv}\gamma_{vw}\gamma_{wu} = e. \tag{6.1}$$

We denote by $M_0(\Gamma,S) \subseteq M(\Gamma,S)$ the family of covering maps (Y,ρ) satisfying (6.1).

The local condition (6.1) gives rise to a local test.

Test 6.8. Input: $(Y,\rho) \in M(\Gamma,S)$.

1. Sample $uvw \in \vec{X}(2)$.
2. Accept if (6.1) holds for uvw.

We note that this test could be realized by sampling $3|S|$ points in $Y(0)$ and their values. We denote by $c(Y,\rho) = \mathbb{P}_{uvw\in X(2)}[\text{test fails}]$, and for a complex X we define its (G,S)-*cover-stability* to be

$$C(X,\Gamma,S) = \min_{(Y,\rho)\in M(\Gamma,S)\setminus M_0(\Gamma,S)} \frac{C(Y,\rho)}{\mathrm{dist}((Y,\rho), M_0(\Gamma,S))}.$$

We say that X is c-cover stable, if for all Γ, S it holds that $C(X,\Gamma,S) \geqslant c$.

Theorem 6.9 ([DM22])**.** *Let X be a d-dimensional simplicial complex for $d \geqslant 2$. Then $C(X,\Gamma,S) = h^1(X,\Gamma)$.*

This test has been used as a component in the agreement test by [GK22, Lemma 3.26][9]. Our results show directly that the complexes of [LSV05a] and [KO21] are cover-stable.

Corollary 6.10. *There exists some absolute constant $c > 0$, such that*

- *Order complexes of homogenuous lattices,*
- *the complexes in Theorem 6.1,*
- *and the complexes in Theorem 6.2.*

are all c-cover stable. □

We point out in particular the meaning of this theorem for the case of the spherical building. The fundamental group of the spherical building is trivial, so it has no non-trivial covers. In other words, any cover is just a bunch of disjoint copies of the original complex. The theorem says that any approximate cover of the spherical building must approximately split into a bunch of disjoint copies.

[9]While they don't use this language to define their test, one may verify that their test is equivalent to this one.

References

[FK81] Z. Furedi and J. Komlos. "The eigenvalues of random symmetric matrices." In: *Combinatorica* 1 (1981), pp. 233–241. DOI: 10.1007/BF02579329.

[Bjö84] Anders Björner. "Some combinatorial and algebraic properties of Coxeter complexes and Tits buildings." In: *Advances in Mathematics* 52 (1984), pp. 173–212.

[Sur84] David Surowski. "Covers of simplicial complexes and applications to geometry." In: *Geometriae Dedicata* 16 (Apr. 1984), pp. 35–62. DOI: 10.1007/BF00147420.

[Gro03] M. Gromov. "Random walk in random groups." In: *Geom. Funct. Anal.* 13 (2003), pp. 73–146. DOI: 10.1007/s000390300002.

[LSV05a] Alexander Lubotzky, Beth Samuels, and Uzi Vishne. "Explicit constructions of Ramanujan complexes of type $\tilde{A}_d$." In: *European J. Combin.* 26.6 (2005), pp. 965–993. DOI: 10.1016/j.ejc.2004.06.007. eprint: math/0406217.

[LSV05b] Alexander Lubotzky, Beth Samuels, and Uzi Vishne. "Ramanujan complexes of type $\tilde{A}_d$." In: *Israel J. Math.* 149.1 (2005), pp. 267–299. DOI: 10.1007/BF02772543. eprint: math/0406208.

[BL06] Yehonatan Bilu and Nathan Linial. "Lifts, Discrepancy and Nearly Optimal Spectral Gap." In: *Combinatorica* 26.1439-6912 (2006), pp. 495–519. DOI: https://doi.org/10.1007/s00493-006-0029-7.

[LM06] Nathan Linial and Roy Meshulam. "Homological Connectivity Of Random 2-Complexes." In: *Combinatorica* 26 (2006), pp. 475–487. DOI: 10.1007/s00493-006-0027-9.

[Koz08] Dmitry N. Kozlov. *Combinatorial Algebraic Topology.* Vol. 21. Algorithms and computation in mathematics. Springer, 2008. ISBN: 978-3-540-73051-4. DOI: 10.1007/978-3-540-71962-5. URL: https://doi.org/10.1007/978-3-540-71962-5.

[MW09] Roy Meshulam and N. Wallach. "Homological connectivity of random k-dimensional complexes." In: *Random Struct. Algorithms* 34.3 (2009), pp. 408–417. DOI: 10.1002/rsa.20238. URL: https://doi.org/10.1002/rsa.20238.

[Gro10] M. Gromov. "Singularities, Expanders and Topology of Maps. Part 2: from Combinatorics to Topology Via Algebraic Isoperimetry." In: *Geom. Funct. Anal.* 20 (2010), pp. 416–526. DOI: 10.1007/s00039-010-0073-8.

[KKL14] Tali Kaufman, David Kazhdan, and Alexander Lubotzky. "Ramanujan Complexes and Bounded Degree Topological Expanders." In: *Proc. 55th IEEE Symp. on Foundations of Comp. Science (FOCS)*. 2014, pp. 484–493. DOI: 10.1109/FOCS.2014.58. eprint: 1408.6351.

[KL14] Tali Kaufman and Alexander Lubotzky. "High dimensional expanders and property testing." In: *Innovations in Theoretical Computer Science, ITCS'14, Princeton, NJ, USA, January 12-14, 2014*. 2014, pp. 501–506.

[MSS15] Adam W. Marcus, Daniel A. Spielman, and Nikhil Srivastava. "Interlacing families I: Bipartite Ramanujan graphs of all degrees." In: *Annals of Mathematics* 182 (2015), pp. 307–325. DOI: 10.4007/annals.2015.182.1.7.

[EK16] Shai Evra and Tali Kaufman. "Bounded degree cosystolic expanders of every dimension." In: *Proc. 48th ACM Symp. on Theory of Computing (STOC)*. 2016, pp. 36–48. DOI: 10.1145/2897518.2897543. eprint: 1510.00839.

[LMM16] Alexander Lubotzky, Roy Meshulam, and Shahar Mozes. "Expansion of building-like complexes." In: *Groups, Geometry, and Dynamics* 10.1 (2016), pp. 155–175.

[Dik+18] Yotam Dikstein, Irit Dinur, Yuval Filmus, and Prahladh Harsha. "Boolean function analysis on high-dimensional expanders." In: *Proc. 20th International Workshop on Randomization and Computation (RANDOM)*. Vol. 116. Princeton, NJ: RANDOM/APPROX, 2018, 37:1–37:21. DOI: 10.4230/LIPIcs.APPROX/RANDOM.2018.37. eprint: 1804.08155.

[DKW18] Dominic Dotterrer, Tali Kaufman, and Uli Wagner. "On expansion and topological overlap." In: *Geometriae Dedicata* 195 (2018), pp. 307–317.

[KM18] Tali Kaufman and David Mass. "Good Distance Lattices from High Dimensional Expanders." (manuscript). 2018. eprint: 1803.02849.

[KO18] Tali Kaufman and Izhar Oppenheim. "Construction of new local spectral high dimensional expanders." In: *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC 2018, Los Angeles, CA, USA, June 25-29, 2018*. 2018, pp. 773–786.

[Opp18] Izhar Oppenheim. "Local Spectral Expansion Approach to High Dimensional Expanders Part I: Descent of Spectral Gaps." In: *Discrete Comput. Geom.* 59.2 (2018), pp. 293–330. DOI: 10.1007/s00454-017-9948-x. eprint: 1709.04431.

[Ana+19] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. "Log-concave polynomials II: high-dimensional walks and an FPRAS for counting bases of a matroid." In: *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, STOC 2019, Phoenix, AZ, USA, June 23-26, 2019*. Ed. by Moses Charikar and Edith Cohen. ACM, 2019, pp. 1–12. DOI: 10.1145/3313276.3316385. URL: https://doi.org/10.1145/3313276.3316385.

[KM19] Dmitry N Kozlov and Roy Meshulam. "Quantitative aspects of acyclicity." In: *Research in the Mathematical Sciences* 6.4 (2019), pp. 1–32.

[CLY20] Michael Chapman, Nathan Linial, and Peled Yuval. "Expander Graphs — Both Local and Global." In: *Combinatorica* 40 (2020), pp. 473–509. DOI: 10.1007/s00493-019-4127-8.

[EKZ20] Shai Evra, Tali Kaufman, and Gilles Zémor. "Decodable quantum LDPC codes beyond the square root distance barrier using high dimensional expanders." In: *61st IEEE Annual Symposium on Foundations of Computer Science, FOCS 2020, Durham, NC, USA, November 16-19, 2020*. Ed. by Sandy Irani. IEEE, 2020, pp. 218–227. DOI: 10.1109/FOCS46700.2020.00029. URL: https://doi.org/10.1109/FOCS46700.2020.00029.

[KO20] Tali Kaufman and Izhar Oppenheim. "High Order Random Walks: Beyond Spectral Gap." In: *Comb.* 40.2 (2020), pp. 245–281. DOI: 10.1007/s00493-019-3847-0. URL: https://doi.org/10.1007/s00493-019-3847-0.

[LMY20] Siqi Liu, Sidhanth Mohanty, and Elizabeth Yang. "High-Dimensional Expanders from Expanders." In: (2020).

[Gol21] Louis Golowich. "Improved Product-Based High-Dimensional Expanders." In: *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2021)*. Vol. 207. 2021. DOI: 10.4230/LIPIcs.APPROX/RANDOM.2021.38.

[KO21] Tali Kaufman and Izhar Oppenheim. "Coboundary and Cosystolic Expansion from Strong Symmetry." In: *48th International Colloquium on Automata, Languages, and Programming, ICALP 2021, July 12-16, 2021, Glasgow, Scotland (Virtual Conference)*. Vol. 198. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021, 84:1–84:16. DOI: 10.4230/LIPIcs.ICALP.2021.84. URL: https://doi.org/10.4230/LIPIcs.ICALP.2021.84.

[KT21] Tali Kaufman and Ran J. Tessler. "New cosystolic expanders from tensors imply explicit Quantum LDPC codes with $\Omega(\sqrt{n}\log^k n)$ distance." In: *STOC '21: 53rd Annual ACM SIGACT Symposium on Theory of Computing, Virtual Event, Italy, June 21-25, 2021*. ACM, 2021, pp. 1317–1329. DOI: 10.1145/3406325.3451029. URL: https://doi.org/10.1145/3406325.3451029.

[Baf+22] Mitali Bafna, Max Hopkins, Tali Kaufman, and Shachar Lovett. "High Dimensional Expanders: Eigenstripping, Pseudorandomness, and Unique Games." In: *Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms, SODA 2022, Virtual Conference / Alexandria, VA, USA, January 9 - 12, 2022*. Ed. by Joseph (Seffi) Naor and Niv Buchbinder. SIAM, 2022, pp. 1069–1128. DOI: 10.1137/1.9781611977073.47. URL: https://doi.org/10.1137/1.9781611977073.47.

[Dik22] Yotam Dikstein. "New High Dimensional Expanders from Covers." 2022.

[Din+22] Irit Dinur, Shai Evra, Ron Livne, Alexander Lubotzky, and Shahar Mozes. "Locally testable codes with constant rate, distance, and locality." In: *STOC '22: 54th Annual ACM SIGACT Symposium on Theory of Computing, Rome, Italy, June 20 - 24, 2022*. Ed. by Stefano Leonardi and Anupam Gupta. ACM, 2022, pp. 357–374. DOI: 10.1145/3519935.3520024. URL: https://doi.org/10.1145/3519935.3520024.

[DM22] Irit Dinur and Roy Meshulam. "Near coverings and cosystolic expansion." In: *Archiv der Mathematik* 118.5 (May 2022), pp. 549–561. ISSN: 1420-8938. DOI: 10.1007/s00013-022-01720-6. URL: https://doi.org/10.1007/s00013-022-01720-6.

[Gai+22] Jason Gaitonde, Max Hopkins, Tali Kaufman, Shachar Lovett, and Ruizhe Zhang. "Eigenstripping, Spectral Decay, and Edge-Expansion on Posets." In: *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX/RANDOM 2022, September 19-21, 2022, University of Illinois, Urbana-Champaign, USA (Virtual Conference)*. Ed. by Amit Chakrabarti and Chaitanya Swamy. Vol. 245. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022, 16:1–16:24. DOI: 10.4230/LIPIcs.APPROX/RANDOM.2022.16. URL: https://doi.org/10.4230/LIPIcs.APPROX/RANDOM.2022.16.

[GK22] Roy Gotlib and Tali Kaufman. *List Agreement Expansion from Coboundary Expansion*. 2022. eprint: arXiv:2210.15714.

[GLL22] Tom Gur, Noam Lifshitz, and Siqi Liu. "Hypercontractivity on high dimensional expanders." In: *STOC '22: 54th Annual ACM SIGACT Symposium on Theory of Computing, Rome, Italy, June 20 - 24, 2022*. Ed. by Stefano Leonardi and Anupam Gupta. ACM, 2022, pp. 176–184. DOI: 10.1145/3519935.3520004. URL: https://doi.org/10.1145/3519935.3520004.

[HL22] Max Hopkins and Ting-Chun Lin. "Explicit Lower Bounds Against \ Omega (n)-Rounds of Sum-of-Squares." In: *arXiv preprint arXiv:2204.11469* (2022).

[OP22] Ryan O'Donnell and Kevin Pratt. "High-Dimensional Expanders from Chevalley Groups." In: *37th Computational Complexity Conference, CCC 2022, July 20-23, 2022, Philadelphia, PA, USA*. Ed. by Shachar Lovett. Vol. 234. LIPIcs. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2022, 18:1–18:26. DOI: 10.4230/LIPIcs.CCC.2022.18. URL: https://doi.org/10.4230/LIPIcs.CCC.2022.18.

[PK22] Pavel Panteleev and Gleb Kalachev. "Asymptotically good Quantum and locally testable classical LDPC codes." In: *STOC '22: 54th Annual ACM SIGACT Symposium on Theory of Computing, Rome, Italy, June 20 - 24, 2022*. Ed. by Stefano Leonardi and Anupam Gupta. ACM, 2022, pp. 375–388. DOI: 10.1145/3519935.3520017. URL: https://doi.org/10.1145/3519935.3520017.

[Din+] Irit Dinur, Yuval Filmus, Prahladh Harsha, and Madhur Tulsiani. "Explicit SoS lower bounds from high-dimensional expanders." In: *12th Innovations in Theoretical Computer Science (ITCS 2021)*. URL: https://arxiv.org/abs/2009.05218.

A Non-abelian 1-coboundary expansion in geometric lattices

In this appendix we give a proof that geometric lattices have constant 1-coboundary expansion, that applies to coefficients coming from non-abelian groups.

Theorem A.1. *Let $d \geqslant 3$ be and integer. Let X be the poset complex for some non-abelian homogeneous lattice P of rank d. Let Γ be any group. Then $H^1(X,\Gamma) = 0$ and*

$$h^1(X,\Gamma) \geqslant \frac{1}{324\left(2+\frac{28}{6-3\sqrt{2}}\right)}.$$

For this theorem we need the following lemma, whose proof is elementary (although it uses ideas that also appear in Theorem 4.1).

Lemma A.2. *Let $i,j,k \in [d]$ so that $j > 3i, k > 3j$, or $i = 1, j = 2, k = 3$. Then $H^1(X^{\{i,j,k\}},\Gamma) = 0$ and $h^1(X^{\{i,j,k\}},\Gamma) \geqslant \left(2+\frac{28}{6-3\sqrt{2}}\right)^{-1}$ for any group Γ.*

Proof of Theorem A.1. Let $\mathcal{F} \subseteq \binom{[d]}{3}$ be the set of all $\{i,j,k\}$ so that $j > 2i, k > 3j$ or so that $i = 1, j = 2, k = 3$. It is easy to see that $\mathbb{P}[\mathcal{F}] \geqslant \frac{1}{54}$. By Lemma A.2, For every $F \in \mathcal{F}$ it holds that $h^1(X^F,\Gamma) \geqslant \left(2+\frac{28}{6-3\sqrt{2}}\right)^{-1}$. It is also true that for every F and v so that $col(v) \notin F$ it holds that $h^0(X_v^F,\Gamma) \geqslant \left(2+\frac{28}{6-3\sqrt{2}}\right)^{-1}$ (recall that Claim 2.5 implies that every link is an edge expander, which by Claim 2.3 says h^0 of the link is bounded). By Theorem 4.1, we get that

$$h^1(X) \geqslant \frac{1}{324\left(2+\frac{28}{6-3\sqrt{2}}\right)}.$$

□

Proof of Lemma A.2. Fix i,j,k as in the lemma statement and set $Y = X^{\{i,j,k\}}$. Let $f : Y(1) \to \Gamma$ and denote by $T = \{t \in Y(2) \mid \delta f(t) \neq 0\}$. Let $\varepsilon = \mathbb{P}[T]$. We need to find some $g : X(0) \to \Gamma$ so that $\frac{1}{\left(2+\frac{28}{6-3\sqrt{2}}\right)}\mathrm{dist}(f,\delta g) \leqslant \varepsilon$ or equivalently $\mathrm{dist}(f,\delta g) \leqslant \left(2+\frac{28}{6-3\sqrt{2}}\right)\varepsilon$.

To do so, we exploit the short cycle structure of the graph between $Y[i], Y[j]$.

Claim A.3. There is a distribution over (w, c) where $w \in Y[k]$ and $c = (u_0, v_0, u_1, v_1, u_2, v_2)$ is a 6-cycle, that has the following properties:

1. For every $\ell \in \{0, 1, 2\}$, $u_\ell \in Y[\ell]$, $v_\ell \in Y[j]$.
2. The vertices and edges of the cycle c is contained in Y_w.
3. Moreover, for every $\ell = 0, 1, 2$, the marginals (w, u_ℓ, v_ℓ) and $(w, v_\ell, u_{\ell+1})$ has the distribution of triangles in Y. That is, the uniform distribution. Here the index ℓ is taken modulo 3.
4. The choice of the vertex u_0 and the edge $v_1 u_2$ are uniformly random given that $u_0 \neq u_2$.
5. Given u_0, v_1, u_2 in c, that is $c = (u_0, *, *, v_1, u_2, *)$, the path between u_0 and v_1 and the path between u_0 and u_2 are chosen independently from one another.

Next we define a randomized labeling $\boldsymbol{g}$ to the vertices of $Y^{\{i,j\}}(0)$ based on the cycle distribution above. We'll show that the expected distance between f and $\delta \boldsymbol{g}$ over $Y^{\{i,j\}}(0)$ is $\leqslant 6\varepsilon$. We'll take some g whose distance is less or equal to the expectation. Then we'll show, using a similar argument to that of Theorem 4.1 that we can extend g to $Y[k]$ so that its distance to f is still $O(\varepsilon)$.

We define a randomized labeling $\boldsymbol{g}$ to $Y[i] \cup Y[j]$ as follows.

1. We choose some $\hat{u} \in Y[i]$. Set $\boldsymbol{g}(\hat{u}) = 0$.
2. For $v \in Y[j]$ We choose some cycle $c = (u_0, ..., v_2)$ so that $u_0 = \hat{u}$ and $v_1 = v$. We set $g(v) = f(u_0 v_0) \cdot f(v_0 u_1) \cdot f(u_1 v_1)$.
3. For $u \in Y[i] \setminus \{\hat{u}\}$ We choose some cycle $c = (u_0, ..., v_2)$ so that $u_0 = \hat{u}$ and $u_2 = u$. We set $g(u) = f(u_0 v_2) \cdot f(v_2 u_2)$.

Claim A.4. $\mathbb{E}_{\boldsymbol{g}} \left[\mathbb{P}_{vu} \left[f(vu) \neq \boldsymbol{g}(u)^{-1} \cdot \boldsymbol{g}(v) \right] \right] \leqslant 6\varepsilon$.

Let $g : Y^{\{i,j\}}(0) \to \Gamma$ be some labeling so that

$$\mathop{\mathbb{P}}_{vu} \left[f(vu) \neq g(u)^{-1} \cdot g(v) \right] \leqslant \mathop{\mathbb{E}}_{\boldsymbol{g}} \left[\mathop{\mathbb{P}}_{vu} \left[f(vu) \neq \boldsymbol{g}(u)^{-1} \cdot \boldsymbol{g}(v) \right] \right] \leqslant 6\varepsilon.$$

We extend g to $Y[k]$ in a similar way as in Theorem 4.1. For every $r \in Y[k]$ we set

$$g(w) = \text{maj} \left\{ g(x) f(wx)^{-1} \;\middle|\; w \in Y_r(0) \right\}.$$

To complete the proof we bound the distance between f and δg over edges wx where $w \in Y[k]$.

Note that $f(wx) \neq g(w)^{-1} g(x)$ if and only if x doesn't agree with the majority vote on $g(w)$. Fix $w \in Y[k]$ and partition $Y_w(0)$ to sets $S_r = \left\{ x \in Y_w(0) \;\middle|\; g(x) f(wx)^{-1} = r \right\}$. By Claim 2.5 Y_w is a $\frac{1 - \frac{1}{\sqrt{2}}}{2}$-edge-expander, ergo by Claim 2.3, it holds for $r' = g(w)$ that

$$\mathop{\mathbb{P}}_{x \in Y_w(0)} \left[g(w) \neq g(x) f(wx)^{-1} \right] = \mathbb{P} \left[V \setminus S_{r'} \right] \leqslant \frac{\sqrt{2}}{\sqrt{2} - 1} \mathop{\mathbb{P}}_{xy \in Y_r(1)} \left[g(x) f(wx)^{-1} \neq g(y) f(wy)^{-1} \right],$$

that is, $\frac{\sqrt{2}}{\sqrt{2}-1}$ times the probability that an edge xy crosses between two sets S_{r_1}, S_{r_2}.

Taking expectation over w, we get that

$$\begin{aligned}\underset{w\in Y[k],wx\in Y(1)}{\mathbb{P}}\left[f(wx)\neq g(x)^{-1}g(w)\right] &\leqslant \underset{w}{\mathbb{E}}\left[\underset{x\in Y_w(0)}{\mathbb{P}}\left[f(wx)\neq g(x)^{-1}g(w)\right]\right] \\ &\leqslant \frac{\sqrt{2}}{\sqrt{2}-1}\underset{wxy\in Y(2)}{\mathbb{P}}\left[g(x)f(wx)^{-1}\neq g(y)f(wy)^{-1}\right].\end{aligned} \tag{A.1}$$

When $\delta f(wxy)=e$ and $f(xy)=g(x)^{-1}g(y)$ then it holds that $g(x)f(wx)^{-1}=g(y)f(wy)^{-1}$ since

$$\begin{gathered}[g(y)^{-1}\cdot g(x)]f(wx)^{-1}\cdot f(wy)= \\ f(yx)f(wx)^{-1}f(wy)=f(yx)f(xw)f(yw)=\delta f(yxw)=e.\end{gathered}$$

Hence

$$\begin{gathered}\underset{wxy\in Y(2)}{\mathbb{P}}\left[g(x)f(wx)^{-1}\neq g(y)f(wy)^{-1}\right]\leqslant \\ \underset{xy\in Y^{ij}(0)}{\mathbb{P}}\left[f(xy)\neq g(x)^{-1}g(y)\right]+\underset{wxy\in Y(2)}{\mathbb{P}}\left[\delta f(wxy)\neq e\right]\leqslant 7\varepsilon.\end{gathered}$$

Plugging this back in (A.1), we get that

$$\underset{wx\in Y(1),w\in Y[k]}{\mathbb{P}}\left[f(wx)\neq g(x)^{-1}g(w)\right]\leqslant\frac{7\sqrt{3}}{\sqrt{3}-1}\varepsilon.$$

Combining this with the distance of $f,\delta g$ on Y^{ij} we have that

$$\mathrm{dist}(f,\delta g)=\frac{1}{3}\cdot 6\varepsilon+\frac{2}{3}\cdot\frac{7\sqrt{2}}{\sqrt{2}-1}\varepsilon=\left(2+\frac{28}{6-3\sqrt{2}}\right)\varepsilon.$$

□

Proof of Claim A.3. First let us ignore w, and describe the choice of $c=(u_0,v_0,u_1,v_1,u_2,v_2)$:

1. We choose $u_0\in Y[i]$.
2. We choose an edge $v_1u_2\in Y^{\{i,j\}}(1)$ (where $v_1\in Y[j],u_2\in Y[i]$), given that $u_2\neq u_0$.
3. We choose a random neighbour $u_1\sim v_1$.
4. We choose $v_0,v_2\in Y[j]$ independently and uniformly at random given that $u_0\vee u_1\subseteq v_0$ and $u_0\vee u_2\subseteq v_2$.

Recall that every u_ℓ is has color i, and every v_ℓ has color $j\geqslant 2i$ (since $rk(a\vee b)\leqslant rk(a)+rk(b)=2i$).

It is easy to see that For every $\ell\in\{0,1,2\}$, $u_\ell\in Y[\ell]$, $v_\ell\in Y[j]$. Also, by definition the choice of u_0 and the edge v_1u_2 is independent given that $u_0\neq u_2$, and done according to the distribution over $Y[i]$ and $Y^{\{i,j\}}(1)$ respectively.

Moreover, note that every edge in the cycle is indeed distributed uniformly over all edges (i.e. a random j-colored element and i-colored element less or equal to it). For example, let us consider the edge u_0v_0. The vertex u_0 is a random i-element. The vertex u_1 is also random (since we chose it by choosing a random v_1 and then choosing a random element less or equal to it). Thus the choice of v_1 is also uniformly at random over elements that contain u_0, i.e. the edge u_0v_0 is chosen uniformly at random.

It is also apparent from the construction that given $c = (u_0, *, *, v_1, u_2, *)$ (which is determined in the first two steps), the path between u_0 and v_1 and the path between u_0 and u_2 are chosen independently from one another.

Finally, the distribution of (w, c) is done choosing c and then choosing uniformly some $w \in Y[k]$ that is greater or equal to vertices in c.

1. If $k \geqslant 3j$ then this is possible since $rk(v_0 \vee v_1 \vee v_2) \leqslant 3j$ and since this join is randomly chosen, then we can just choose some c that contains this join.

2. If $i = 1, j = 2, k = 3$ then we explain why $rk(v_0 \vee v_1 \vee v_2) \leqslant 3$. If all u_0, u_1, u_2 are distinct, then $v_\ell = u_\ell \vee u_{\ell+1}$ and in particular $v_0 \vee v_1 \vee v_2 = u_0 \vee u_1 \vee u_2$ (and this has rank at most 3 since every element has rank 1). Otherwise, either $u_0 = u_1 \neq u_2$ or $u_0 \neq u_1 = u_2$. Let's consider the case where $u_0 = u_1 \neq u_2$ (the second case is similar). Then $v_2 = v_1 = u_0 \vee u_2$. The element $v_0 = u_0 \vee u^*$ for some u^* of rank one. It follows that $v_0 \vee v_1 \vee v_2 = u_0 \vee u_2 \vee u^*$ and the join has rank 3.

It needs to be verified for every $\ell = 0, 1, 2$, the marginals (w, u_ℓ, v_ℓ) and $(w, v_\ell, u_{\ell+1})$ has the distribution of triangles in Y. But this holds since by the above, the distribution of (u_ℓ, v_ℓ) and $(v_\ell, u_{\ell+1})$ is the distribution of edges in $Y_w^{\{i,j\}}(1)$, and w is chosen uniformly given that $w \geqslant v_\ell$. ◻

Proof of Claim A.4.

$$\mathop{\mathbb{E}}_{\boldsymbol{g}} \left[\mathop{\mathbb{P}}_{vu} \left[f(vu) \neq \boldsymbol{g}(u)^{-1} \cdot \boldsymbol{g}(v) \right] \right] = \mathop{\mathbb{E}}_{g,vu} \left[\mathbf{1}_{f(vu) \neq \boldsymbol{g}(u)^{-1} \cdot \boldsymbol{g}(v)} \right] \tag{A.2}$$

Fix and edge vu so that $v \in Y[j], u \in Y[i]$. The marginal of the values of $\boldsymbol{g}(u), \boldsymbol{g}(v)$ could be described as follows:

1. Choose u_0.

2. Choose (w, c) as in Claim A.3 given that $c = (u_0, *, *, v_1 = v, u_2 = u, *)$ (or if $u_0 = u_2$ given that $c = (0_0, *, *, v_1 = v, *, *)$).

3. Set $\boldsymbol{g}(v) = f(u_0 v_0) \cdot f(v_0 u_1) \cdot f(u_1 v_1)$ and set $\boldsymbol{g}(u) = f(u_0 v_2) \cdot f(v_2 u_2)$ (or if $u_0 = u_2$ then $\boldsymbol{g}(u_2) = e$).

The reason we can choose *one* cycle and not two in the second step in the common case where $u_0 \neq u_2$, is because given that $c = (u_0, *, *, v_1 = v, u_2 = u, *)$, the paths from u_0 to v_1 and from u_2 to u_2 are independent.

Let us analyze the more common case where $u_2 \neq u_0$ first. In this case (A.2) is equal to

$$\begin{aligned} &\mathop{\mathbb{P}}_{c} \left[f(v_1 u_2) \neq \boldsymbol{g}(u_2)^{-1} \cdot \boldsymbol{g}(v_1)^{-1} \mid u_2 \neq u_0 \right] = \\ &\qquad \mathop{\mathbb{P}}_{c} \left[f(v_1 u_2) \neq [f(u_2 v_2) \cdot f(v_2 u_0)] \cdot [f(u_0 v_0) \cdot f(v_0 u_1) \cdot f(u_1 v_1)] \right] = \\ &\qquad \mathop{\mathbb{P}}_{c} \left[f(u_0 v_0) \cdot f(v_0 u_1) \cdot f(u_1 v_1) \cdot f(v_1 u_2) \cdot f(u_2 v_2) \cdot f(v_2 u_0) \neq e \right] \end{aligned} \tag{A.3}$$

Suppose there is some $w \in Y[k]$ that contains c so that for every (u_ℓ, v_ℓ) and every $(v_\ell, u_{\ell+1})$ it holds that $\delta f(w u_\ell v_\ell) = \delta f(w v_\ell u_{\ell+1}) = e$. By multiplying by $e = f(u_\ell w) f(w u_\ell) = f(v_\ell w) f(w v_\ell)$ between the original edges we get that

$$
\begin{aligned}
f(u_0v_0)\cdot f(v_0u_1)\cdot f(u_1v_1)\cdot f(v_1u_2)\cdot f(u_2v_2)\cdot f(v_2u_2) = &\\
& f(u_0w)[f(wu_0)f(u_0v_0)f(v_0w)]\cdot\\
& \cdot[f(wv_0)f(v_0u_1)f(u_1w)]\\
& \cdot \ldots \cdot [f(wv_2)f(v_2u_0)f(u_0w)]f(wu_0) =\\
& f(u_0w)\cdot \delta f(wu_0v_0)\cdot \ldots \cdot \delta f(wv_2u_0)\cdot f(wu_0) =\\
& e
\end{aligned} \tag{A.4}
$$

Thus the probability in (A.3) is upper bounded by $\mathbb{P}_{(w,c)}\left[\exists uv \in c\ \delta f(wuv) \neq 0\right]$. As the marginal of every wuv for $uv \in c$ is the probability of sampling a uniform triangle, and there are 6 edges in c, by union bound

$$\mathop{\mathbb{E}}_{\boldsymbol{g}}\left[\mathop{\mathbb{P}}_{vu}\left[f(vu) \neq \boldsymbol{g}(u)^{-1}\cdot \boldsymbol{g}(v)\right]\right] \leqslant \mathop{\mathbb{P}}_{(w,c)}\left[\exists uv \in c\ \delta f(wuv) \neq 0\right] \leqslant 6\varepsilon.$$

The case where $u_0 = u_2$ is even simpler. In this case, $f(u_0v_1) \neq \boldsymbol{g}(u_0)^{-1}\boldsymbol{g}(v_2) = f(u_0v_0)f(v_0u_1)f(u_1v_1)$. Here again, if there is a $w \in Y[k]$ that contains c so that the coboundaries

$$\delta f(wu_0v_0) = \delta f(wv_0u_1) = \delta f(wu_1v_1) = \delta f(wv_1u_0) = 0$$

then the same analysis above (where we multiply by elements of the form $f(wx)f(xw)$) shows that $f(u_0v_1) = \boldsymbol{g}(u_0)^{-1}\boldsymbol{g}(v_2)$. This happens with probability $\leqslant 4\varepsilon$. □

B Cosystolic expansion of dense complexes is at most $1 + o(1)$

In this section we give a simple upper bound on cosystolic expansion of dense complexes.

Proposition B.1. *Let $\varepsilon > 0$. Let X be a d-dimensional simplicial complex for $d \geqslant k+1$. Assume that for every j, the probability of $s \in X(j)$ is $\frac{1}{|X(j)|}$. Let $\varepsilon = \max(\sqrt{\frac{8|X(k-1)|}{|X(k)|}}, \sqrt{\frac{9}{|X(k+1)|}})$ and assume that $\varepsilon \leqslant \frac{1}{2}$. Then $h^k(X, \mathbb{F}_2) \leqslant 1 + 8\varepsilon$.*

For example, this proposition implies the following. If a $\{X_n\}$ are a family of simplicial complexes whose vertex set is growing to infinity, and so that $\lim_{n\to\infty} \frac{|X(k-1)|}{|X(k)|} = 0$ then for every $\varepsilon > 0$, all but finitely many X_n have that $h^k(X_n, \mathbb{F}_2) \leqslant 1 + \varepsilon$.

There are many complexes so that these conditions hold. For example, the complete complex, the complete bipartite complex, and the spherical building (when $d > \frac{k}{2}$).

Proof. We sample $f \in C_k(X, \mathbb{F}_2)$ uniformly at random (i.e. we fix some global order of the vertices. For every $s \in \overrightarrow{X}(k)$ that is ordered according to this global order we sample $f(s) \in \mathbb{F}_2$, and extend this asymmetrically). We note that every s was chosen independently.

Fixed some $g \in C_{k-1}(X, \mathbb{F}_2)$. Let $X_g(f) = \mathbb{P}_{s\in X(k)}\left[f(s) = \delta g(s)\right]$. Obviously, $X_g(f) = \sum_{s\in X(k)} \mathbb{P}\left[s\right] \mathbf{1}_{f(s)=\delta g(s)}$. As all the random variables $\mathbf{1}_{f(s)=g(s)}$ are independent, and have expectation $\frac{1}{2}$ it holds by Chernoff's bound that

$$\mathop{\mathbb{P}}_{f}\left[X_g(f) \geqslant \frac{1}{2} + \varepsilon\right] \leqslant e^{-\frac{\varepsilon^2}{1+\varepsilon}|X(k)|}.$$

There are $|B^k(X,\mathbb{F}_2)| \leqslant |C_{k-1}(X,\mathbb{F}_2)| \leqslant 2^{|X(k-1)|}$ possible coboundaries. Hence, the probability that there exists some g so that $X_g(f) \geqslant \frac{1}{2}+\varepsilon$ is at most

$$e^{|X(k-1)|\ln 2} e^{-\frac{\varepsilon^2}{1+\varepsilon}|X(k)|},$$

which is less than $\frac{1}{2}$ by the assumption on ε. Note that when $X_g(f) \leqslant \frac{1}{2}+\varepsilon$ for all g this implies that $\mathrm{dist}(f, Z_k(X,\mathbb{F}_2)) \geqslant \frac{1}{2}-\varepsilon$.

On the other hand, let $Y(f) = \mathbb{P}_{t\in X(k+1)}[f(t)=0]$. As above it holds that $\mathbb{E}[Y] = \frac{1}{2}$. Furthermore, $Y = \sum_{t\in X(k+1)} \mathbb{P}[t]\, \mathbf{1}_{\delta f(t)=0}$ where the set $\{\mathbf{1}_{\delta f(t)=0}\}_{t\in X(k+1)}$ are pairwise independent. This is because any two distinct $t, t' \in X(k+1)$ share only one k-face. By pairwise independence, $Var(Y(f)) = \sum_{t\in X(k+1)} Var(\mathbb{P}[t]\, \mathbf{1}_{\delta f(t)=0}) = \frac{1}{4|X(k+1)|}$. In particular,

$$\mathop{\mathbb{P}}_{f}\left[Y(f) \leqslant \frac{1}{2}-\varepsilon\right] \leqslant \frac{1}{4|X(k+1)|\varepsilon^2}.$$

Which is also strictly less than $\frac{1}{2}$ by assumption.

In particular, as the events $\{\forall g; X_g(f) \geqslant \frac{1}{2}+\varepsilon\}$ and $\{Y(f) \leqslant \frac{1}{2}-\varepsilon\}$ happen with probability less than $\frac{1}{2}$, there exists some f so that for every $g \in C_{k-1}(X,\Gamma)$, $\mathrm{dist}(f,\delta g) \geqslant \frac{1}{2}-\varepsilon$ but $wt(\delta f) \geqslant \frac{1}{2}+\varepsilon$. Hence $h^k(f,\Gamma) \leqslant \frac{\frac{1}{2}+\varepsilon}{\frac{1}{2}-\varepsilon} \leqslant 1+8\varepsilon$. □

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