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Linear Relaxed Locally Decodable and Correctable Codes Do Not Need Adaptivity and Two-Sided Error

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Abstract

Relaxed locally decodable codes (RLDCs) are error-correcting codes in which individual bits of the message can be recovered by querying only a few bits from a noisy codeword. Unlike standard (non-relaxed) decoders, a relaxed one is allowed to output a "rejection" symbol, indicating that the decoding failed. To prevent the decoder from always rejecting, we demand that if its input is a valid codeword, then for every bit, the decoder is correct with high probability.

We study the power of adaptivity and two-sided error for RLDCs. Our main result is that if the underlying code is linear, *adaptivity and two-sided error do not give any power to relaxed local decoding.* We construct a reduction from adaptive, two-sided error relaxed decoders to non-adaptive, one-sided error ones. That is, the reduction produces a relaxed decoder that never errs or rejects if its input is a valid codeword and makes queries based on its internal randomness (and the requested index to decode), independently of the input. The reduction does not change the query complexity (nor the underlying code), and for any input, the decoder's error probability increases at most two-fold.

The idea behind the reduction is our new notion of *additive* promise problem. A promise problem is additive if the sum of any two YES-instances is a YES-instance (i.e., the YES instances are a subspace) and the sum of any NO-instance and a YES-instance is a NO-instance (i.e., the NO-instances are a collection of cosets).

We prove that relaxed decoding, interpreted as a promise problem, satisfies this definition. We construct a reduction that applies to *any* additive promise problem, allowing us to obtain the result for RLDCs. Our result also holds for relaxed locally *correctable* codes (RLCCs), where a *codeword* bit should be recovered.

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1 Introduction

Suppose you receive a binary string (a "message"), and would like to know the value of the message at some index i. How many queries do you need to make? The answer is obviously 1, as you can query index i and get the value. But what if some of the message bits are corrupted, because they were, say, transmitted over a noisy channel? The bit at the needed index i might have been corrupted.

Error-correcting codes might help. Such codes allow encoding the message with extra redundancy as a codeword, and the original message can be recovered even if some bits of the codeword were corrupted. However, one needs to read the entire codeword to recover the original message. As our goal was to read only one bit from the message, this solution seems inefficient.

Locally decodable codes (LDCs), introduced by Katz and Trevisan [KT00], are aimed at solving this problem. These codes are equipped with a local decoding algorithm ("decoder") that recovers each message bit by querying a few bits from a codeword, instead of reading all of it. Two main measures of efficiency for LDCs are the query complexity of the decoder (which we want to be as small as possible) and the rate of the code (which we want to be high). A similar notion, originated in works on program checking by Blum and Kannan [BK95] and Lipton [Lip90], is of locally *correctable* codes (LCCs). These are error-correcting codes that admit a local algorithm (now called "corrector") that not only recovers each message bit, but is also required to correct any bit from the *codeword*.

LDCs and LCCs have profoundly impacted theoretical computer science and found numerous applications. Despite the extensive research, current constructions require adding a large amount of redundancy. Motivated by this, Ben-Sasson et. al. [BGH+06] defined the notion of Relaxed Locally Decodable Codes (RLDCs). Such codes could achieve a dramatically better tradeoff between query complexity and rate. The relaxation is in the following sense: A relaxed decoder is allowed to output a "rejection" symbol, indicating that the decoding failed. To prevent the decoder from always rejecting, they demand that if its input is a valid codeword, then for every bit its output is correct with high probability.

More formally, a relaxed decoder of radius $\rho > 0$ for a code C with soundness error $\epsilon_{\text{soundness}}$ and completeness error $\epsilon_{\text{completeness}}$ is a procedure that gets oracle access to $w \in \{0,1\}^n$, that is ρ-close[1] to some codeword $c = C(x)$ and an index i, and satisfies the following two requirements:

1. (completeness) If w is a valid codeword (that is, $w = c$) then for every i the relaxed decoder outputs x_i with probability at least $1 - \epsilon_{\text{completeness}}$.

2. (relaxed local correction) Otherwise, the relaxed decoder outputs x_i or a "reject" symbol $\perp$, indicating the decoding failed with probability at least $1 - \epsilon_{\text{soundness}}$.

Gur, Ramnarayan, and Rothblum [GRR17] considered an analogous relaxation for local *correction*, where the corrector either recovers the desired codeword bit, or rejects in case it detects a corruption. They named such codes Relaxed Locally *Correctable* Codes, or RLCCs (see Definition 2.2).

A relaxed decoder / corrector is allowed to err, with a small probability, even if its input is a valid codeword. If it never errs for such inputs, we say it has *one-sided error* (and that it has *two-sided error* otherwise). Understanding the power of two-sided errors algorithms vs. one-sided ones is fundamental in the theory of randomized algorithms. For RLDCs / RLCCs, we ask the following: are one-sided error relaxed decoders / correctors weaker than two-sided error ones?

Adaptivity is a crucial consideration in the field of local algorithms. We say that an algorithm is *non-adaptive* if it determines its queries based on its internal coin tosses, independently of the queries' answers. We say it is *adaptive* otherwise. Adaptive algorithms are syntactically stronger than non-adaptive ones. The question arises whether adaptive relaxed decoders / correctors are truly stronger than non-adaptive ones. Does adaptivity provide them any meaningful strength?

[1] Two strings are ρ-close to each other if the normalized Hamming distance between them is at most ρ.

1.1 Our results

Our main result is that for linear codes, two-sided error and adaptivity do not give any strength to RLDCs and RLCCs.

We show a reduction that starts with a relaxed decoder (resp., corrector) that might query adaptively and err on valid codewords, and ends with a relaxed decoder (resp., corrector) for the same code, that is non-adaptive and never errs or rejects on valid codewords. The reduction does not change the query complexity. The new soundness error is the sum of the completeness error and soundness error of the original algorithm. Hence the gap between completeness and soundness stays the same. The error probability, which is the probability to err on *any* specific input (i.e, $\max(\epsilon_{\text{completeness}}, \epsilon_{\text{soundness}})$) is at most doubled.

Theorem 1. *Let C be a* linear *code.*

If C has a decoder of radius ρ with completeness error $\epsilon_{completeness}$, soundness error $\epsilon_{soundness}$ and query complexity q, then it has a one-sided error, non-adaptive *decoder of radius ρ, with soundness error $\epsilon_{completeness} + \epsilon_{soundness}$ and query complexity q.*

Theorem 2. *Let C be a* linear *code.*

If C has a corrector of radius ρ with completeness error $\epsilon_{completeness}$, soundness error $\epsilon_{soundness}$ and query complexity q, then it has a one-sided error, non-adaptive *corrector of radius ρ, with soundness error $\epsilon_{completeness} + \epsilon_{soundness}$ and query complexity q.*

Linearity: Our reduction only works for linear codes. Nevertheless, linear codes are a significant type of error-correcting and are widely studied in the literature. Virtually all known RLDCs and RLCCs (and their non-relaxed counterparts) constructions are of linear codes.[2]

Known reductions: We note two well-known immediate reductions from adaptive to non-adaptive local algorithms. These reductions date back to [KT00], which stated them for non-relaxed LDCs, but they also apply to relaxed ones.

The first reduction is to replace each of the q adaptive queries with multiple non-adaptive ones. We replace the i-th query with 2^{i-1} queries, one for each possible result of the previous queries. This reduction yields an exponential blowup in the query complexity. The second reduction is to "guess" the result of the first $q-1$ queries, and query a set of indices based on that guess. This reduction exponentially decreases the algorithm's soundness.

In light of the above, even a reduction with a polynomial blowup in the query complexity would have been an exciting result. Our reduction maintains the query complexity of the algorithm, while only doubling its error probability.

Our contribution: Our reductions are based on the results of Ben-Sasson, Harsh and Raskhodnikov [BHR03] (namely, Theorem 2 and Theorem 3 there). They showed a reduction from adaptive, two-sided error *testers* for linear properties, to non-adaptive, one-sided error testers. Our contribution is in defining additive promise problems (see Definition 1.1), generalizing their result to the new framework, and applying the result to relaxed correction and decoding.

1.2 Motivation

The main motivation for this work is to gain better understanding of RLDCs and RLCCs. Beyond this, we next list some concrete motivations.

[2] A remarkable exception is multiplicity codes, which are not linear. Fortunately, their codewords constitute a subgroup (the sum of two codewords is a codeword), so our framework still covers them. See Definition 1.1.

Lower bounds: The main application for our result is for lower bounds. Our results show that, for linear codes, any lower bound for non-adaptive, one-sided error RLDCs (resp., RLCCs) can be transformed to a lower bound for adaptive, two-sided error RLDCs (resp., RLCCs).

Gur and Lachish [GL20] showed a lower bound for RLDCs, proving that such codes cannot simultaneously achieve constant positive rate and constant query complexity. Namely, they show that any *non-adaptive, one-sided error* RLDC with constant query complexity q must have rate roughly $n = \Omega(k^{1+1/q^2})$. For adaptive RLDCs, however, they get a lower bound with exponentially worse dependency in the query complexity. Namely, they use the trivial reduction from above to show that any (adaptive, one-sided error) RLDC that makes at most q queries must have block length roughly $n = \Omega(k^{1+\frac{1}{2q^2}})$.

A direct corollary of our result, applied to the lower bound of [GL20], is that any *adaptive, two-sided error* (and linear) RLDC with query complexity q must have block length roughly $n = \Omega(k^{1+1/q^2})$. However, a followup work of Dall'Agnol, Gur and Lachish [dSDGL21] extends the result of [GL20] to adaptive, two-sided error RLDCs, for all codes, including not linear ones.

Our results are still relevant for lower bounds for a few reasons. First, the extension of the lower bound from non-adaptive to adaptive of [dSDGL21] is highly involved. Our reduction is arguably much simpler, which gives an alternative, simpler proof for the same lower bound for linear codes. Second, the lower bound of [dSDGL21] is not known to be tight. Our result is a general reduction, hence can be applied to improved lower bounds that might be found in the future.

Third, for linear codes any lower bound for RLDCs is also a lower bound for RLCCs, but not vice versa. This is because every linear code can be represented systematically, where the first bits of the codeword are the original message. Hence, for such codes, every corrector is also a decoder, so RLCCs are stronger objects than RLDCs. Our reduction applies to each type separately. That means it can be applied for lower bounds for RLCCs, which might be tighter than for RLDCs.

Constructions: Virtually all known constructions of RLDCs and RLCCs are non-adaptive, linear, and have one-sided error. Our results show that this should not be a surprise. It is impossible to use adaptivity or two-sided error to improve constructions of linear RLDCs / RLCCs.

Definitions: In some works (e.g, [CGS20, GRR20, AS21, CY22]), the definition of RLDC requires it to have one-sided error (i.e., it is not an additional property that a decoder might have). Other works (e.g., [BGH+06, dSDGL21]) use the same definition we gave above. Our result settles this nuance in the definitions - both are equivalent (for linear codes).

1.3 Technical overview

We next give a high-level sketch of the reduction.

Promise problems: We prove Theorem 1 and Theorem 2 by proving a general result on a family of promise problems. First, a promise problem is a couple of two disjoint sets, Y (the YES-instances) and N (the NO-instances). A randomized algorithm for a promise problem gets as input $x \in Y \cup N$ (we sometimes call $Y \cup N$ "the promise") and outputs YES or NO. If $x \in Y$ then the algorithm must output YES with high probability. Similarly, if $x \in N$ it must output NO with high probability.

The main new idea we introduce in this work is of *additive* promise problems. The definition is as follows:

Definition 1.1. *A promise problem $(Y, N) \subseteq \{0,1\}^n$ is* additive *if it satisfies the following conditions:*

1. *(YES-instances are a linear subspace[3]) For every $x, y \in Y$, $x + y \in Y$*
2. *(NO-instances are a collection of cosets) For every $x \in N, y \in Y$, $x + y \in N$*

This definition can be generalized to any abelian group instead of $\{0,1\}^n$. For simplicity, in this work we focus on $\{0,1\}^n$.

Testing linear properties: As a demonstration for the new definition, we next show that property testing, when the tested property $\Pi \subseteq \{0,1\}^n$ is a linear subspace, is an additive promise problem. The YES-instances in this case are the elements of the tested property. That is, $Y = \Pi$. The NO-instances are the elements ϵ-far from every YES-instances. Namely,

$$N = \{x \in \{0,1\}^n \mid \forall y \in Y,\ \mathrm{dist}(x, y) > \epsilon\}$$

The first item of Definition 1.1 follows from the assumptions that the property Π is linear. For the second item, let $x \in N, y \in Y$. We need to show that $x + y \in N$. I.e, that $x + y$ is ϵ-far from every $y' \in Y$. Indeed, for every $y' \in Y$ we have

$$\mathrm{dist}(x + y, y') = \mathrm{dist}(x, y' - y) > \epsilon$$

The equality holds because in general, $\mathrm{dist}(a, b) = \mathrm{dist}(a + c, b + c)$ for every a, b, c. The inequality holds because, since y and y' are in the linear space Y then $y' - y \in Y$, and $x \in N$ and hence ϵ-far from every element in Y.

In Section 3, we show how to interpret relaxed decoding and correction of linear codes as promise problems. We use similar arguments as in the proof above to show that the resulting promise problems are additive.

The reduction: Next, we construct a reduction from adaptive, two-sided error local algorithms to non-adaptive, one-sided error ones that works for *any* additive promise problem.

Theorem 3. *Let $(Y, N) \subseteq \{0,1\}^n$ be an additive promise problem. If (Y, N) has an adaptive algorithm A with completeness error ϵ_Y, soundness error ϵ_N and query complexity q, it has a non-adaptive, one-sided error algorithm A' with soundness error $\epsilon_Y + \epsilon_N$ and query complexity q.*

By applying Theorem 3 to relaxed decoding we prove Theorem 1, and by applying it to relaxed correction we prove Theorem 2.

The reduction works in two steps. The first step ensures that the algorithm never errs on YES-instances. We start with an adaptive, two-sided error arbitrary algorithm, and transform it to have one-sided error (and it remains adaptive). This step does not increase the query complexity. If the original algorithm errs on YES-instances with probability at most ϵ_Y and on NO-instances with probability at most ϵ_N, then the transformed algorithm errs on NO-instances with probability at most $\epsilon_Y + \epsilon_N$.

The second step handles adaptivity. We start with an adaptive, one-sided error algorithm, and transform it to a non-adaptive algorithm (that still has one-sided error). This step maintains the query complexity and the soundness error.

We next describe the two reductions.

[3]Strictly speaking, the requirement is that the YES-instances are a *subgroup*. For $\{0,1\}^n$ these requirements are equivalent.

Two-sided to one-sided error: Every randomized algorithm A can be described as a distribution over a set of deterministic decision trees. Each leaf of each decision tree is labeled with YES or NO, which is the output of the algorithm when that tree is chosen. The first step of the reduction is to *relabel* the leaves of all trees, in the following way: If there is an input $x \in Y$ that "leads" to this leaf, then it is relabeled to YES. This step is necessary to get a one-sided error algorithm. However, this transformation may not maintain the algorithm's soundness. In Section 4.1, we explain the issue in detail.

The solution is to modify the algorithm. Instead of using the (relabeled) decision trees of A with the given input x, choose a random YES-instance y, and use the tree as if the input was $x + y$. Since (Y, N) is an additive promise problem, if $x \in Y$ then $x + y \in Y$ for any (randomly chosen) y, and the original algorithm's completeness can be used. Similarly, if $x \in N$ then $x + y \in N$ for any y, and the soundness of the original algorithm can be used. In Lemma 4.2, we prove that with this modification the transformation maintains the sum of soundness and completeness error.

Adaptivity: Consider an adaptive, one-sided error algorithm A. Without loss of generality, the only freedom A has is in choosing its queries. Once it queried an input x, it must output YES if there exists a YES-instance consistence with the queries. Otherwise, when no YES-instance is consistence, then x cannot be a YES-instance and w.l.o.g A outputs NO.

The new non-adaptive algorithm A' works as follows: On input x, choose a random YES-instance y. Query x on all indices A would have queried y, and output YES if the partial view of x is consistent with some YES-instance (which might be different than y).

The new algorithm is non-adaptive since now it determines its queries independently of its input. Its query complexity is maintained, and it has one-sided error (as it always outputs YES for YES-instances). In Lemma 4.5, we show that its soundness error is also maintained. This is done by relating the probability A' outputs YES on some specific x to the *average* probability A outputs YES for a random element of the set $x + Y$.

1.4 Related work

Error correcting codes date back to the seminal works of Shannon [Sha49] and Hamming [Ham50]. LDCs, LCCs and their relaxed counterparts have attracted significant attention in recent years. See the works of Yekhanin [Yek12] and Kopparty and Saraf [KS17] and references within for comprehensive surveys of LDCs, LCCs and their applications.

RLDCs and RLCCs constructions: The constructions of RLDCs and RLCCs can be separated into two regimes of parameters: constant query complexity, and constant rate. In the constant rate regime, the state-of-the-art code is the construction of Cohen and Yankovitz [CY22]. This construction is of a linear RLCC with rate arbitrarily close to 1, and query complexity $q = (\log n)^{O(\log\log\log n)}$. This construction builds upon the result of [GRR20], which shows a similar code but with query complexity $q = (\log n)^{O(\log\log n)}$.

In the constant query regime, the original work of [BGH+06] achieves RLDC with constant query complexity $O(q)$ and block length $n = O(k^{1+1/\sqrt{q}})$. The work of [GRR20] introduced the notion of RLCCs, and constructed such a code with constant query complexity, but with worse block length. Chiesa, Gur, and Shinkar [CGS20] constructed an improved RLCC, matching the block length of [BGH+06].

The current state-of-the-art construction is of Asadi and Shinkar [AS21], which builds upon [GRR20] and [CGS20]. Their construction is of RLCC and RLDC with constant query complexity $O(q)$ and block length $n = O(k^{1+1/q})$. This is the first construction of RLDC, in the constant query regime, with better block length than of [BGH+06].

We remark that all the above constructions are linear, non-adaptive, and have one-sided error.

Lower bounds: In recent decades, there has been extensive research concerned with lower bounds for (non-relaxed) LDCs in various regimes [KT00, KdW03, Woo07, Woo12, AGKM22]. Gur and Lachish [GL20] showed the first lower bound for relaxed LDCs. Such lower bounds are arguably harder to obtain, as RLDCs are weaker objects than LDCs. The result of [GL20] was extended to more settings, such as proofs of proximity, property testing and more by Dall'Agnol, Gur and Lachish [dSDGL21]. In the context of this work, [dSDGL21] also handles adaptive algorithms. See Section 1.2 for more details.

Locally testable codes and RLDCs: Our work follows the theme of extending ideas from the area of *LTCs* (locally testable codes [GS06]) to RLDCs. Loosely speaking, a code is locally testable if it has a *tester*, a probabilistic algorithm that accepts valid codewords, and rejects inputs that are "far" from any codeword, while making a small number of queries. LTC can be viewed as a special case of property testing (see the book [Gol17] and references therein).

Kaufman and Viderman [KV10] showed that LDCs do *not* imply locally testable codes and vice versa. In contrast, *relaxed* LDCs (and LCCs) seem to be closely related to LTC.

One RLCC construction from [GRR20] is based on an LTC construction of Kopparty, Meir, Ron-Zewi and Saraf [KMRS17]. Indeed, this is the same construction, with [GRR20] showing that the LTC is also an RLDC by describing a corrector for the code.

Furthermore, the lower bound of [GL20] uses techniques from the work of Fischer, Lachish and Vasudev [FLV15], developed in the context of property testing.

Our work generalizes a result of [BHR03] for testing linear properties. We do that by finding a framework capturing both linear property testing and linear RLDCs/RLCCs, giving concrete evidence to the connection between LTCs and RLDCs/RLCCs.

We note one remarkable difference between LTCs and RLDCs. Recently, a work of Dinur et. al. [DEL+22] showed the existence of LTC with constant rate and constant query complexity (and constant distance). In contrast, the lower bound of [GL20] implies that an RLDC with such parameters cannot exist.

1.5 Open problems

We conclude this section with a few questions we leave for future research.

Non-linear codes: Our result holds only for linear codes. A natural open problem is to extend the result to non-linear codes, or to show that for non-linear codes, adaptivity and / or two-sided error *do* give more power. Our reduction heavily relies on linearity, so we believe other techniques than the ones used in this paper will be needed.

Adaptivity of LDCs and LCCs: The power of adaptivity for (non-relaxed) LDCs and LCCs is an interesting open problem. As we discuss in Section 3, our framework of additive problems does not seem to cover them. This leaves the question open, even for linear codes. To the best of our knowledge, there are currently no known reductions from adaptive to non-adaptive LDCs, besides the ones described above. Even a reduction that polynomially increases the query complexity, or slightly increasing the code block length, would be an interesting result.

A main open problem is to separate the power of LDCs and that of relaxed LDCs. Constructing an adaptive (linear) LDC might aid in this effort, showing that LDCs and relaxed LDC differ regarding adaptivity.

Additive promise problems: The notion we introduce in this work allows tackling RLDCs, RLCCs and linear property testing simultaneously. We wonder if there are additional natural problems that fit into this new framework. Our main reduction will hold for any such problems.

2 Definitions and preliminaries

We begin with some basic notations:

- Denote by $[n]$ the set $[n] = \{1, 2, \ldots, n\}$.
- For a distribution D, denote by $x \sim D$ a random variable x that is chosen according to the distribution D.
- For a distribution D and an element x in its support, denote by $D(x)$ the probability that x is drawn from D.
- For a string $x \in \{0,1\}^n$ and a set $S \subseteq [n]$, denote by $x|_S$ the restriction of x to the indices S.
- For $x, y \in \{0,1\}^n$, denote by $\mathrm{dist}(x, y)$ the relative Hamming distance between x and y. Namely, $\mathrm{dist}(x, y) = \frac{|\{x_i \neq y_i | i \in [n]\}|}{n}$. For $\epsilon > 0$, if $\mathrm{dist}(x, y) \leq \epsilon$, we say that x is ϵ-close to y, and otherwise we say that x is ϵ-far from y.

2.1 Error correcting codes

Throughout, an error correcting code C with *message length* k and *block length* n is a function $C : \{0,1\}^k \to \{0,1\}^n$. For simplicity, we consider only binary alphabet in this work. We identify a code with its image, i.e, $C \subseteq \{0,1\}^n$.

A code C is *linear* if it is a linear function (or, equivalently, if C as a set is closed to addition).

Definition 2.1. *Let $C : \{0,1\}^k \to \{0,1\}^n$ be an error correcting code. A relaxed* decoder *of radius $\rho > 0$ for C is a randomized procedure A, that gets as inputs an oracle access to $x \in \{0,1\}^n$, and explicit input $i \in [k]$, outputs an element of $\{0, 1, \perp\}$, and needs to satisfy the following two requirement:*

1. *(completeness) If $x = C(y)$ for some $y \in \{0,1\}^k$ then $A^x(i) = y_i$ with probability at least $1 - \epsilon_{completeness}$.*
2. *(relaxed local decoding) If there exists $y \in \{0,1\}^k$ such that $dist(x, C(y)) < \rho$, then $A^x(i) \in \{y_i, \perp\}$ with probability at least $1 - \epsilon_{soundness}$.*

Where the probability is over the internal randomness of A.

Definition 2.2. *Let $C \subseteq \{0,1\}^n$ be an error correcting code. A relaxed* corrector *of radius $\rho > 0$ for C is a randomized procedure A, that gets as inputs an oracle access to $x \in \{0,1\}^n$, and explicit input $i \in [n]$, outputs an element of $\{0, 1, \perp\}$, and needs to satisfy the following two requirement:*

1. *(completeness) If $x \in C$ then $A^x(i) = x_i$ with probability at least $1 - \epsilon_{completeness}$.*
2. *(relaxed local correction) If there exists $c \in C$ such that $dist(x, c) < \rho$, then $A^x(i) \in \{c_i, \perp\}$ with probability at least $1 - \epsilon_{soundness}$.*

Where the probability is over the internal randomness of A.

In both definitions we call $\epsilon_{\text{completeness}}$ the *completeness error*, and $\epsilon_{\text{soundness}}$ the *soundness error*.

In what follows we use the term *local algorithm* to refer to an algorithm which is a relaxed decoder or a relaxed corrector. A local algorithm has *one-sided error* if its completeness error is 0 (i.e., if it never errs on valid codewords). We say that a local algorithm has *query complexity* $q = q(n)$ if, on

input i, and oracle access to any $x \in \{0,1\}^n$, the corrector makes at most $q(n)$ queries. We say that a local algorithm is *non-adaptive* if it determines all its queries based on its explicit input (namely, the index to decode / correct) and internal coin tosses, independently of the specific x to which it is given oracle access. Otherwise, we say that it is *adaptive*.

For an RLDC, we view its decoder A of a code as a set of k decoders $A_1, \ldots, A_k$, where $A_i(x) = A^x(i)$. We call A_i the *decoders* of C. Similarly, we view the corrector of a RLCCs as a set of n correctors, $A_1, \ldots, A_n$, and call them the *correctors* of C. The benefit of this view is that each A_i is now an algorithm that gets a single (implicit) input $x \in \{0,1\}^n$.

2.2 Promise problems

Definition 2.3. *A* Promise Problem *is couple* $(Y, N) \subseteq \{0,1\}^n$ *such that* $Y \cap N = \emptyset$. *We call* Y *the* YES-instances *of the problem, and* N *the* NO-instances *of the problem.*

Definition 2.4. *An* algorithm for a promise problem $(Y, N) \subseteq \{0,1\}^n$ *with* completeness error $\epsilon_Y > 0$ *and* soundness error $\epsilon_N > 0$ *is a randomized procedure that gets as input an oracle access to* $x \in \{0,1\}^n$, *outputs YES or NO, and satisfies the following conditions:*

1. *(completeness) If* $x \in Y$ *then it outputs YES with probability at least* $1 - \epsilon_Y$.
2. *(soundness) If* $x \in N$ *then its outputs NO with probability at least* $1 - \epsilon_N$.

We define query complexity and adaptivity of promise problem algorithms as they are defined in Definition 2.2. We say that a promise problem algorithm has *one-sided* error if its completeness error is 0 (i.e., if it never errs on YES-instances).

The main new definition of this work is of additive promise problems. See Definition 1.1.

3 Relaxed decoding and correction as additive promise problems

In this section, we show how to interpret relaxed decoding and relaxed correction of linear codes as promise problems. We do that in Section 3.1. We then show, in Section 3.2, that the resulting promise problems are additive.

3.1 Interpretation as promise problems

We start by showing how to interpret relaxed correction as a promise problem. We then explain how relaxed decoding can be considered as a restrictive case of relaxed correction.

There are three possible values for the output of a relaxed corrector: $0, 1$ or $\perp$.[4] In contrast, an algorithm for a promise problem has only two possible outputs: YES and NO. To interpret relaxed correction as a promise problem, we need to specify how to translate a correction problem (with multiple possible output values) to a yes/no question. The following observation allows us to that.

Claim 3.1. *If a code has a corrector* A, *then it has a corrector* A' *(with the same parameters) such that for every* $x \in \{0,1\}^n$, *the output of* A' *for index* i *is* x_i *or* $\perp$.

Proof. The new corrector A'_i works according to the following rule:

$$A'_i(x) = \begin{cases} A_i(x), & \text{if } A_i(x) \in \{x_i, \perp\} \\ \perp, & \text{otherwise} \end{cases}$$

[4] For a larger alphabet, the number of possible outputs of a corrector is the size of the alphabet $+1$.

From the construction, the output of $A'_i(x)$ is x_i or $\perp$ for every input x (and never $1 - x_i$). We next show that A'_i satisfies the required completeness and soundness (Definition 2.2). For completeness, if $x \in C$ then $A_i(x) = x_i$ with probability $1 - \epsilon_{\text{completeness}}$, and hence $A'_i(x) = x_i$ with the same probability.

For soundness, assume there exists $c \in C$ such that $\text{dist}(x, c) < \rho$. We need to show $A'_i(x) \in \{c_i, \perp\}$ with high probability. Consider the case $c_i = x_i$. From the soundness of A_i, $A_i(x) \in \{x_i, \perp\}$ with probability at least $1 - \epsilon_{\text{soundness}}$, and hence $A'_i(x) \in \{x_i, \perp\} = \{c_i, \perp\}$ with the same probability.

Otherwise, $c_i \neq x_i$. With probability at least $1 - \epsilon_{\text{soundness}}$, the output of A_i is c_i or $\perp$. From the construction, whenever the output of A_i is c_i or $\perp$, the output of A'_i is $\perp$. Hence, with the same probability, $A'_i(x) = \perp \in \{c_i, \perp\}$. □

By using Claim 3.1, we can replace item 2 of Definition 2.2 with the following:

Definition 3.2. *(alternative definition of RLCCs)*

2. *(soundness) If there exists $c \in C$ such that $dist(x, c) < \rho$ and $x_i \neq c_i$, then $A_i(x) = \perp$ with probability at least $1 - \epsilon_{soundness}$.*

Definition 3.2 allows us to treat a corrector as having a "binary" output; We say that A_i *accepts* x if $A_i(x) = x_i$, and that it *rejects* x otherwise (namely, if $A_i(x) = \perp$)

We can now phrase relaxed correction as a promise problem, as follows[5]:

Definition 3.3. *Let $C \subset \{0,1\}^n$ be an error correcting code, let $\rho > 0$ and let $i \in [n]$. The* promise problem of relaxed correction *of C at index i with correction radius ρ is defined by:*

1. *(YES-instances are the codewords)*
   $Y = C$

2. *(NO-instances are the inputs a corrector rejects)*
   $N = \{x \in \{0,1\}^n \mid \exists c \in C \text{ with } dist(x, c) < \rho \text{ and } x_i \neq c_i\}$.

The promise problem of relaxed correction is equivalent to relaxed correction of codes (of Definition 3.2) in the following sense:

Claim 3.4. *An algorithm A_i is a corrector for index i of C with correction radius ρ, completeness error $\epsilon_{completeness}$ and soundness error $\epsilon_{soundness}$ according to Definition 3.2, if and only if it is an algorithm for the promise problem of relaxed correction of C at index i with correction radius ρ, completeness error $\epsilon_{completeness}$ and soundness error $\epsilon_{soundness}$ according to Definition 2.4*

□

Relaxed decoding: We can assume w.l.o.g that the a linear code is systematic. Namely, we assume that the first k bits of each codeword are the message encoded in it. Hence, a decoder is simply a corrector that needs to "correct" only the first k bits of the input. The observation above (Claim 3.1) holds for relaxed decoders as well. Hence, we can assume w.l.o.g that the output of the decoder, for input x and index $i \in [k]$, is either x_i or $\perp$.

[5] Another possible formulation is $Y = \{(x, b) \mid x \in C \text{ and } b = x_i\}$ and $N = \{(x, b) \mid \exists c \in C \text{ with } \text{dist}(x, c) < \rho \text{ and } c_i \neq b\}$. This formulation preserves better the decoding "flavor" of the problem. We use the formulation of Definition 3.2, as it emphasizes the similarity to testing, with $\perp$ corresponding to "reject" and any other output corresponding to "accept".

This allows us to replace the two requirements of Definition 2.1 (for linear codes) with:

Definition 3.5. *(alternative definition of RLDCs)*

1. *(completeness) If $x \in C$ then $A^x(i) = x_i$ with probability at least $1 - \epsilon_{completeness}$.*
2. *(soundness) If there exists $c \in C$ such that $dist(x, c) < \rho$ and $x_i \neq c_i$, then $A_i(x) = \perp$ with probability at least $1 - \epsilon_{soundness}$.*

Accordingly, we phrase relaxed decoding as a promise problem in the same way relaxed correction, except that the input index is in $[k]$ (instead of in $[n]$):

Definition 3.6. *Let $C \subset \{0,1\}^n$ be an error correcting code, let $\rho > 0$ and let $i \in [k]$. The* promise problem of relaxed decoding *of C at index i with decoding radius ρ is defined by:*

1. *(YES-instances are the codewords)*
   $Y = C$
2. *(NO-instances are the inputs a decoder rejects)*
   $N = \{x \in \{0,1\}^n \mid \exists c \in C \text{ with } dist(x, c) < \rho \text{ and } x_i \neq c_i\}$.

3.2 Additive promise problems

In this section, we show that the promise problems formulated above for relaxed decoding and relaxed correction, have the special property of being *additive* (Definition 1.1).

Claim 3.7. *The relaxed correction and relaxed decoding promise problems for linear codes is additive.*

We prove the claim for relaxed correction. The proof for relaxed decoding is the same, with the restriction that $i \in [k]$ (instead of in $[n]$).

Proof. Let C be a *linear* error correcting code, let $\rho > 0$ and let $i \in [n]$. Let Y, N be as in Definition 3.3.

From the linearity assumption $Y = C$ is a linear subspace of $\{0,1\}^n$, and the first item of Definition 1.1 holds.

To show the second item, let $x \in N, y \in Y$. We need to show that $x+y \in N$. $x \in N$, so there exists a codeword $c \in C$ such that $\text{dist}(x, c) < \rho$ and $x_i \neq c_i$. Define $c' = c+y$. c' is a codeword in C, since it is a sum of two codewords, and C is a linear code. We get $\text{dist}(x+y, c') = \text{dist}(x+y, c+y) = \text{dist}(x, c) < \rho$, and $(x + y)_i = x_i + y_i \neq c_i + y_i = c'_i$. Hence $x + y \in N$. □

Non-relaxed LDCs: We remark that *non-relaxed* decoding / correction (for a specific index) does not seem to fit into our new framework. First, it needs to be clarified how to formulate decoding / correction as a promise problem. Say we define the YES-instances (resp. NO-instances) as the inputs on which the corrector must output 0 (resp., 1) with high probability. Then the set of YES-instances is *not* a linear subspace, as it contains strings outside C.

Another possible formulation is to define the YES-instances as the strings on which the corrector must output x_i. This formulation has the same problem, as again the set of YES-instances contains strings outside C, so it is not necessarily a subspace.

4 The reduction

In this section, we prove our main results, Theorem 1 and Theorem 2.

We prove these theorems by constructing the following, more general reduction:

Theorem 4.1. *(Restatement of Theorem 3) Let $(Y, N) \subseteq \{0,1\}^n$ be an additive promise problem. If (Y, N) has an adaptive algorithm A with completeness error ϵ_Y, soundness error ϵ_N and query complexity q, it has a* one-sided error, non-adaptive algorithm *A' with soundness error $\epsilon_Y + \epsilon_N$ and query complexity q.*

Theorem 1 is a direct corollary of Theorem 4.1 applied to the relaxed decoding promise problem (Definition 3.6). Theorem 2 is a direct corollary of Theorem 4.1 applied to the relaxed correction promise problem (Definition 3.3).

The proof of Theorem 4.1 has two steps. The first step is a reduction from two-sided error to one-sided error algorithms. We show this reduction in Section 4.1. The second step is a reduction from one-sided, adaptive algorithms to one-sided, non-adaptive algorithms. We show this reduction in Section 4.2.

4.1 From two-sided to one-sided error

In this section, we show a reduction from two-sided error algorithms to one-sided error ones for additive promise problems. The reduction does not change the query complexity of the algorithm, and maintains the sum of the completeness and soundness errors.

Lemma 4.2. *Let $(Y, N) \subseteq \{0,1\}^n$ be an additive promise problem. If (Y, N) has an (adaptive) algorithm A with completeness error ϵ_Y, soundness error ϵ_N and query complexity q, it has an (adaptive) one-sided error algorithm A' with soundness error $\epsilon_Y + \epsilon_N$ and query complexity q.*

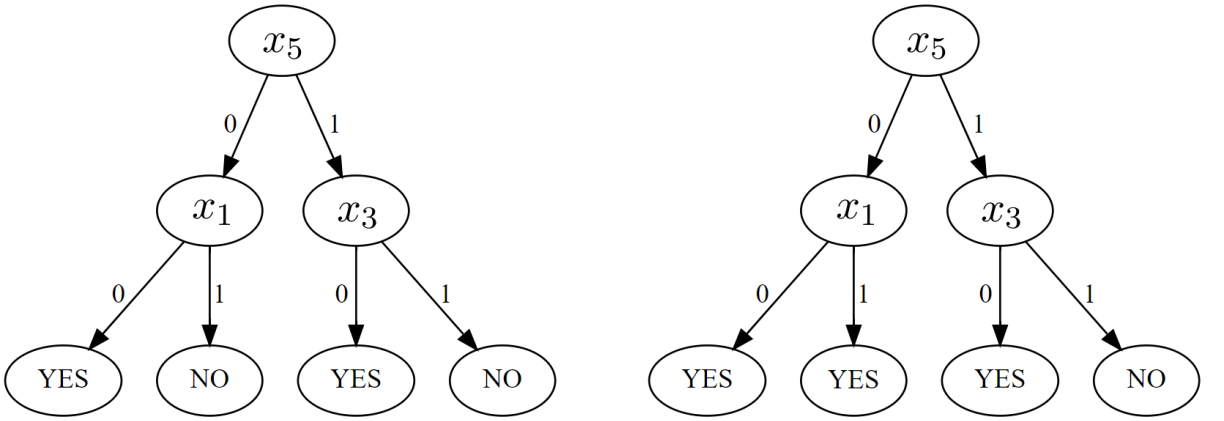
Randomized algorithms as distributions over decision trees: Consider some randomized algorithm A for a promise problem. A can be described as a distribution D_A over a set of deterministic decision trees $\Upsilon_A = \{\Gamma_1, \Gamma_2, \ldots\}$. We denote by $\Gamma \sim D_A$ a tree chosen randomly from Υ_A according to the distribution D_A. Each leaf ℓ of each tree corresponds to a set of indices $I = (i_1, \ldots, i_t) \in [n]^t$ that are queried along the path leading to ℓ, and the corresponding values $\sigma = (\sigma_1, \ldots, \sigma_t) \in \{0,1\}^t$ at these indices. We identify each leaf with the corresponding indices and values and write $\ell = (I, \sigma)$.

For an input x, we denote by $\Gamma(x)$ the *leaf* of Γ at the end of the path corresponding to querying x. That is, $\Gamma(x) = \ell$ if $x|_I = \sigma$, where $\ell = (I, \sigma)$, and $x|_I$ is the restriction of x to indices I. Each leaf is labeled with YES or NO.

We can now describe the operation of A on input x as follows: It chooses a $\Gamma \sim D_A$ and outputs the label of $\Gamma(x)$.

Relabeling decision trees: To convert an algorithm A to have one-sided error, we first go over all of $\Gamma \in \Upsilon_A$ and relabel them, so each Γ will have one-sided error. The relabeling works as follows: For every leaf ℓ of Γ, if there exists a YES-instance y such that $\Gamma(y) = \ell$, relabel ℓ with YES. We denote the relabeled tree by Γ_c and call it the *one-sided error relabeling* of Γ. See Figure 1 on page 12 for an illustration of a relabeling that happens if, for example, the string "11100" $\in \{0,1\}^5$ is in Y.

This step is necessary to get a one-sided error algorithm; as long as there exists $y \in Y$ and Γ such that the label of $\Gamma(y)$ is NO, there is some probability that A outputs NO for a YES-instance. The issue, however, is that this transformation may not maintain the algorithm's soundness. There might be NO-instances that the new algorithm (wrongly) accepts with a high probability.



(a) A decision tree

(b) The same tree after relabeling the leaf corresponding to $I = (5, 1)$ and $\sigma = (0, 1)$

Figure 1: An example for relabeling a decision tree

The soundness issue with relabeling: Pretend the transformed algorithm A' worked as follows. For an input x, choose a random decision tree $\Gamma \sim D_A$, relabel the tree to get Γ_c, and output the label of $\Gamma_c(x)$ (instead of using $\Gamma(x)$ as the original algorithm did). This new algorithm has one-sided error. In fact, we can transform *any* algorithm this way to have one-sided error, so we should not expect it to maintain soundness.

To see that the soundness is not necessarily maintained, consider some leaf ℓ of a tree Γ that was relabeled from NO to YES. Let $x \in \{0,1\}^n$ such that $\Gamma(x) = \ell$. If $x \in Y$, the relabeling was beneficial. Before the relabeling, the algorithm returned a wrong output for x (whenever it used the tree Γ), and now it returns the correct output. However, what if x is a NO-instance? In this case, after the relabeling we return the *wrong* output for x each time Γ is used. The tree Γ is sampled according to the distribution D_A, which is arbitrary. If D_A gives much weight to Γ (say, it is chosen with probability $\frac{1}{2}$), then the new algorithm returns the wrong output for x with high probability.[6] This implies that A' does not have the required soundness property, as it should hold for *every* NO-instance.

The actual reduction: The solution is to modify the algorithm in the following way. Instead of deterministically returning the label of $\Gamma_c(x)$ (after Γ was chosen at random), the transformed algorithm outputs the label of $\Gamma_c(x + y)$ for a random $y \in Y$. We give a formal description of the transformation in Algorithm 1. Now, even if $\Gamma_c(x)$ was relabeled to YES, we output its label with a small probability. In the proof of Lemma 4.2, we show that the soundness error of the transformed algorithm is at most $\epsilon_Y + \epsilon_N$.[7]

Query complexity: The relabeling does not change the query complexity (i.e., depth) of the decision trees. Hence, the query complexity of A' is the same as that of A.

One-sided error: Let $x \in Y$. We argue that A' always outputs YES for x. From the definition of A', its output for x is the label of $\Gamma_c(x + y)$ for some $y \in Y$. Since (Y, N) is an additive problem and

[6] Even if there is no one heavy tree, relabels of many leaves in different trees might have the same affect if their total weight is high.

[7] Recall that ϵ_N is error probability of the original algorithm for NO-instances, and ϵ_Y is its error probability for YES-instances.

Algorithm 1 one-sided error local algorithm, A'
Input: Oracle access to $x \in \{0,1\}^n$.
Output: YES or NO.

1. Choose $\Gamma \sim D_A$.
2. Choose $y \in Y$ uniformly at random.
3. Output the label of $\Gamma_c(x+y)$.

$x \in Y$, we get that $x+y \in Y$. From the relabeling scheme for Γ_c, the label of $\Gamma_c(y')$ is YES for every $y' \in Y$, and in particular for $y' = x+y$. Hence, the output of A' for x is YES.

Before arguing about the soundness of the transformed algorithm, we need the following preparations.

Probability of hitting a specific leaf: We stated above that for an input x, the modified algorithm uses the label of a specific leaf ℓ with a small probability. We next calculate this probability (conditioning on first choosing the tree Γ_c of ℓ). There are $|Y|$ possible options for y. The leaf ℓ is used if the chosen y satisfies $\Gamma_c(x+y) = \ell$. Hence, we choose the label of ℓ with probability $\frac{|(x+Y)\cap\Gamma^{-1}(\ell)|}{|Y|}$, where $x+Y = \{x+y \mid y \in Y\} \subseteq \{0,1\}^n$, and $\Gamma^{-1}(\ell) = \{z \mid \Gamma(z) = \ell\} \subseteq \{0,1\}^n$. We argue that this probability is either 0 (when there is no $y \in Y$ such that $\Gamma(x+y) = \ell$ and $(x+Y)\cap\Gamma^{-1}(\ell)$ is empty), or equals to a quantity not depending on x.

Lemma 4.3. *Let $Y \subseteq \{0,1\}^n$ be a subspace of $\{0,1\}^n$, and fix a decision tree Γ and a leaf $\ell = (I,\sigma)$. Define $U = \{u \in Y \mid u|_I = 0\}$. Then for every $x \in \{0,1\}^n$, if there exists $y \in Y$ such that $\Gamma(x+y) = \ell$, then $|(x+Y)\cap\Gamma^{-1}(\ell)| = |U|$.*

Proof. First, notice that U is not empty since the all-zeros string is in Y (as Y is a linear subspace). Furthermore, U is a subspace of Y, since if $u, u' \in U$ then $(u+u')|_I = u|_I + u'|_I = 0$ and $u+u' \in U$.

We argue that $(x+Y)\cap\Gamma^{-1}(\ell)$ is a coset of U, hence having the same size as U. Namely, we claim that:

$$(x+Y)\cap\Gamma^{-1}(\ell) = x+y+U$$

where y is an element of Y such that $\Gamma(x+y) = \ell$.

We begin by proving the inclusion $(x+Y)\cap\Gamma^{-1}(\ell) \subseteq x+y+U$. Let $x+y' \in (x+Y)\cap\Gamma^{-1}(\ell)$. That is, $(x+y')|_I = \sigma$. Define $u = y'-y$. Since $(x+y)|_I = (x+y')|_I = \sigma$, we have $u|_I = ((x+y')-(x+y))|_I = 0$ and $u \in U$. Therefore $x+y' = x+y+u \in x+y+U$.

To prove that $(x+Y)\cap\Gamma^{-1}(\ell) \supseteq x+y+U$, let $u \in U$. Since $y \in Y$ and $u \in U \subseteq Y$, and Y is closed to addition, we have $y+u \in Y$ and $x+y+u \in x+Y$. Next, $(x+y+u)|_I = (x+y)_I + u|_I = (x+y)_I = \sigma$ and hence $x+y+u \in \Gamma^{-1}(\ell)$. □

We are now ready to prove Lemma 4.2.

Proof. (of Lemma 4.2) We proved above that the reduction maintains the query complexity of the algorithm, and that the transformed algorithm has one-sided error. We are left with arguing about its soundness.

Let $x \in N$. We need to show that A' outputs NO for x with probability at least $1-\epsilon_Y-\epsilon_N$. That is, we need to prove $\Pr[A'(x) = \text{ YES }] < \epsilon_Y + \epsilon_N$.

The probability for (wrongly) outputting YES for x may increase due to the transformation. Nevertheless, it does not increase too much. We argue the transformation does not decrease the gap

between the expected probability of returning YES for a random element of Y and the expected probability of outputting YES for a random element of $x+Y$. That is, we claim:

Claim 4.4. *For every $x \in N$:*

$$\mathop{\mathbb{E}}_{y\in Y}[\Pr[A(y) = \mathit{YES}]] - \mathop{\mathbb{E}}_{y\in Y}[\Pr[A(x+y) = \mathit{YES}]] \leq \mathop{\mathbb{E}}_{y\in Y}[\Pr[A'(y) = \mathit{YES}]] - \mathop{\mathbb{E}}_{y\in Y}[\Pr[A'(x+y) = \mathit{YES}]]$$

where the probabilities are over the internal randomness of A and A'.

Proof. (of Claim 4.4) To prove this claim, it is enough to show that relabeling one leaf ℓ of one decision tree does not decrease the gap. Then we obtain the claim by relabeling one leaf at a time, to get A' from A.

Assume ℓ was relabeled from NO to YES. Let $G := Y \cap \Gamma^{-1}(\ell)$ be the strings $y \in Y$ such that $\Gamma(y) = \ell$ (these are the "Good" strings, which are now labeled YES and are in Y). The set G is not empty since we relabel ℓ only if there exists $y \in Y$ such that $\Gamma(y) = \ell$, i.e. $y \in G$. Let $B := (x+Y) \cap \Gamma^{-1}(\ell)$ be the strings $x+y \in x+Y$ such that $\Gamma(x+y) = \ell$ (these are the "Bad" strings, which are now labeled YES but in $x+Y \subseteq N$).

Every string in $G \cup B$ was rejected before the relabeling but is now accepted. The algorithm's behavior on the other elements in Y and $x+Y$ is unaltered. Hence, $\mathop{\mathbb{E}}_{y\in Y}[\Pr[A(y) = \text{YES}]]$ increases by $D(\Gamma) \cdot \frac{|G|}{|Y|}$ when ℓ is relabeled. Similarly, $\mathop{\mathbb{E}}_{y\in Y}[\Pr[A(x+y) = \text{YES}]]$ increases by $D(\Gamma) \cdot \frac{|B|}{|Y|}$. It suffices to show that $|G| \geq |B|$. Intuitively, this means any "harm" done to x by the relabeling (an element of B that increases the probability to wrongly output YES) is "compensated" by the relabeling (by an element of G improving the algorithm's completeness).

If B is empty, we are done. Otherwise, due to Lemma 4.3, $|B| = |U|$ and $|G| = |U|$, and we conclude that $|G| = |B|$. □

From the claim, the soundness error for x gets worse by an amount bounded by the completeness improvement. Since the completeness error reduces from ϵ_{Y} to 0, the soundness error for x increases by at most ϵ_{Y}.

More formally, from Claim 4.4, the prefect completeness of A' and the completeness error of A:

$$\begin{aligned}\mathop{\mathbb{E}}_{y\in Y}[\Pr[A'(x+y) = \text{YES}]] &\leq \mathop{\mathbb{E}}_{y\in Y}[\Pr[A'(y) = \text{YES}]] - \mathop{\mathbb{E}}_{y\in Y}[\Pr[A(y) = \text{YES}]] + \mathop{\mathbb{E}}_{y\in Y}[\Pr[A(x+y) = \text{YES}]] \\ &\leq 1 - (1-\epsilon_{\mathrm{Y}}) + \mathop{\mathbb{E}}_{y\in Y}[\Pr[A(x+y) = \text{YES}]] = \mathop{\mathbb{E}}_{y\in Y}[\Pr[A(x+y) = \text{YES}]] + \epsilon_{\mathrm{Y}}\end{aligned} \tag{1}$$

Now, $x+y \in N$ for every $y \in Y$, since $x \in N$ and (Y, N) is additive. Hence, applying the soundness of A to *every* $x+y$, we have $\mathop{\mathbb{E}}_{y\in Y}[\Pr[A(x+y) = \text{ YES }]] < \epsilon_{\mathrm{N}}$.

In addition, from the definition of A', $\Pr[A'(x+y) = \text{YES}] = \Pr[A'(x) = \text{YES}]$ for every $y \in Y$. Hence, from equation 1 we get that:

$$\Pr[A'(x) = \text{YES}] < \epsilon_{\mathrm{Y}} + \epsilon_{\mathrm{N}}$$

□

4.2 From adaptive to non-adaptive algorithms

In this section, we show a reduction from one-sided error adaptive to (one-sided error) non-adaptive algorithms for additive promise problems. The reduction does not change the algorithm's query complexity or it soundness error.

Lemma 4.5. *Let $(Y, N) \subseteq \{0,1\}^n$ be an additive promise problem. If (Y, N) has an* one-sided error, adaptive algorithm A *with soundness error ϵ_N and query complexity q, it has an one-sided error* non-adaptive algorithm A' *with soundness error ϵ_N and query complexity q.*

Proof. Let $D = D_A$ be a distribution over decision trees $\Upsilon = \Upsilon_A$ corresponding to A.

We first observe that we can assume w.l.o.g that the label of each leaf $\ell = (I, \sigma)$ is YES if and only $\exists y \in Y$ such that $y|_I = \sigma$. Since A never errs on YES-instances, if there exists $y \in Y$ such that $y|_I = \sigma$, then ℓ must be labeled YES. On the other hand, we can assume that if no such $y \in Y$ exists, then ℓ is labeled NO. Otherwise, ℓ can be relabeled from YES to NO while only improving the algorithm's soundness and maintaining its one-sided error.

Description of A': At a high level, the new non-adaptive algorithm works as follows. On input x, choose a random $y \in Y$. Query x on all indices A would have queried y, and output YES if the partial view of x is consistent with some $y' \in Y$. We give a formal description of the non-adaptive algorithm in Algorithm 2.

Algorithm 2 Non-adaptive local algorithm, A'
Input: Oracle access to $x \in \{0,1\}^n$.
Output: YES or NO.

---

1. Choose $\Gamma \sim D$.
2. Choose $y \in Y$ uniformly at random.
3. Let $\ell = (I, \sigma) = \Gamma(y)$.
4. Query x on the indices I.
5. Output "YES" if $\exists y' \in Y$ such that $x|_I = y'|_I$, and "NO" otherwise.

Analysis: The algorithm A' is non-adaptive, as its queries depend only on its internal randomness (the choice of Γ and y). It has the same query complexity as A, since it uses the same decision trees. The algorithm has one-sided error since if $x \in Y$, we can take $y' = x$ at the last step of the algorithm, and $x|_I = y'|_I$ for every I.

We are left with proving that the transformation does not decrease the soundness error. Towards this end, we relate the acceptance probability of A' to the *average* acceptance probability of A.

Claim 4.6. *For every $x \in \{0,1\}^n$:*

$$\Pr[A'(x) = \mathit{YES}] = \mathop{\mathbb{E}}_{y \in Y}[\Pr[A(x+y) = \mathit{YES}]]$$

where the probabilities are over the internal randomness of A and A'.

This claim shows that soundness is maintained. If $x \in N$, then since (Y, N) is additive, $x + y \in N$ for every $y \in Y$. Hence, by using the soundness of A for *every* $x + y$, we get that

$$\Pr[A'(x) = \text{YES}] = \mathop{\mathbb{E}}_{y \in Y}[\Pr[A(x+y) = \text{YES}]] < \epsilon_{\text{N}}$$

□

Proof. (of Claim 4.6) We begin by calculating $\mathop{\mathbb{E}}_{y\in Y}[\Pr[A(x+y)=\text{YES}]]$. Identifying the output YES with 1, we can take expectation instead of probability. We denote by $\Gamma(Y)$ the set of leaves in Γ labeled YES[8] , and get:

$$\mathop{\mathbb{E}}_{y\in Y}[\Pr[A(x+y)=\text{YES}]] = \mathop{\mathbb{E}}_{y\in Y}\mathop{\mathbb{E}}_{\Gamma\sim D}[A(x+y)]] = \mathop{\mathbb{E}}_{\Gamma\sim D}\mathop{\mathbb{E}}_{y\in Y}[A(x+y)]] = \mathop{\mathbb{E}}_{\Gamma\sim D}[\frac{1}{|Y|}\cdot\sum_{\ell\in\Gamma(Y)}|(x+Y)\cap\Gamma^{-1}(\ell)|] \tag{2}$$

where in the last equality, we take into account all $y\in Y$ by iterating over each leaf $\ell\in\Gamma(Y)$ (for other leaves the algorithm's output is 0) and counting the number of y values for which $\Gamma(x+y)=\ell$ (i.e., that lead the algorithm to output the label of ℓ).

On the other hand, for any $I\subseteq\{0,1\}^n$ and $x\in\{0,1\}^n$ define:

$$H_I(x)=\begin{cases}1, & \text{if } \exists y\in Y \text{ such that } x|_I=y|_I\\ 0, & \text{otherwise}\end{cases}$$

With this notation, the last step of A' can be described as "output $H_I(x)$".

We get:

$$\Pr[A'(x)=\text{YES}]=\mathbb{E}[A'(x)]=\mathop{\mathbb{E}}_{\Gamma\sim D}\mathop{\mathbb{E}}_{\substack{y\in Y\\(I,\sigma)\leftarrow\Gamma(y)}}[H_I(x)]=\mathop{\mathbb{E}}_{\Gamma\sim D}[\frac{1}{|Y|}\cdot\sum_{\ell=(I,\sigma)\in\Gamma(Y)}|Y\cap\Gamma^{-1}(\ell)|\cdot H_I(x)] \tag{3}$$

Here it is sufficient to iterate over the leaves in $\Gamma(Y)$: the algorithm A never errs on YES instances, so if ℓ is labeled NO, there cannot be $y\in Y$ such that $\Gamma(y)=\ell$.

From equations 2 and 3 it is enough to show that for every $\ell=(I,\sigma)\in\Gamma(Y)$:

$$|(x+Y)\cap\Gamma^{-1}(\ell)|=|Y\cap\Gamma^{-1}(\ell)|\cdot H_I(x)$$

Since (I,σ) is labeled YES, and as discussed above, there exists $y'\in Y$ such that $y'|_I=\sigma$ and $Y\cap\Gamma^{-1}(\ell)$ is not empty.

Consider the case $H_I(x)=0$. We claim $(x+Y)\cap\Gamma^{-1}(\ell)$ is empty and hence the equality holds. Assume towards contradiction that this set is not empty. Then there exists $(x+y)\in(x+Y)$ such that $(x+y)|_I=\sigma$. Hence $(x+y)|_I=y'|_I$ and $x|_I=(y'-y)|_I$, which implies $H_I(x)=1$ (since $y'-y\in Y$).

Next, consider the case $H_I(x)=1$. We argue that $(x+Y)\cap\Gamma^{-1}(\ell)$ is not empty. $H_I(x)=1$ implies there exists $y\in Y$ such that $x|_I=y|_I$, and $(x-y)|_I=0$. Now $(x-y+y')|_I=(x-y)|_I+y'|_I=\sigma$, and hence $x-y+y'\in(x+Y)\cap\Gamma^{-1}(\ell)$ (as $-y+y\in Y$). Since $(x+Y)\cap\Gamma^{-1}(\ell)$ is not empty is not empty, and from Lemma 4.3, we get that $|(x+Y)\cap\Gamma^{-1}(\ell)|=|U|$. The set $Y\cap\Gamma^{-1}(\ell)$ is also not empty, and again from Lemma 4.3 $|Y\cap\Gamma^{-1}(\ell)|=|U|$. We conclude that $|(x+Y)\cap\Gamma^{-1}(\ell)|=|Y\cap\Gamma^{-1}(\ell)|$. □

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[8] This definition is equivalent to $\Gamma(Y)=\{\Gamma(y)\mid y\in Y\}$ since, as discussed above, a leaf ℓ is labeled YES if and only if there exists $y\in Y$ such that $\Gamma(y)=\ell$.

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