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Radical Sylvester-Gallai Theorem for Tuples of Quadratics

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Abstract

We prove a higher codimensional radical Sylvester-Gallai type theorem for quadratic polynomials, simultaneously generalizing [Han65, Shp20]. Hansen's theorem is a high-dimensional version of the classical Sylvester-Gallai theorem in which the incidence condition is given by high-dimensional flats instead of lines. We generalize Hansen's theorem to the setting of quadratic forms in a polynomial ring, where the incidence condition is given by radical membership in a high-codimensional ideal. Our main theorem is also a generalization of the quadratic Sylvester–Gallai Theorem of [Shp20].

Our work is the first to prove a radical Sylvester–Gallai type theorem for arbitrary codimension $k \geqslant 2$, whereas previous works [Shp20, PS20, PS21, OS22] considered the case of codimension 2 ideals. Our techniques combine algebraic geometric and combinatorial arguments. A key ingredient is a structural result for ideals generated by a constant number of quadratics, showing that such ideals must be radical whenever the quadratic forms are far apart. Using the wide algebras defined in [OS22], combined with results about integral ring extensions and dimension theory, we develop new techniques for studying such ideals generated by quadratic forms. One advantage of our approach is that it does not need the finer classification theorems for codimension 2 complete intersection of quadratics proved in [Shp20, GOS22].

1 Introduction

Let $v_1, \dots, v_m$ be a set of points in $\mathbb{R}^n$ with the property that the line joining any two points passes through a third point. The Sylvester–Gallai theorem states that $v_1, \dots, v_m$ must all be collinear. This result was conjectured by Sylvester [Syl93], and proved independently by Melchior [Mel40] and Gallai [Gal44].

The inflection points of a cubic curve are a set of nine points in $\mathbb{C}^2$ such that the line joining any two of them passes through a third ([Dic14]). However, these points are not collinear. In fact, Kelly [Kel86] suggested that this was the motivation behind Sylvester's conjecture, to check if all inflection points can have real coordinates. In the same paper, Kelly observed that Hirzebruch's work on line arrangements [Hir83] directly implies that every configuration of points in $\mathbb{C}^n$ satisfying the Sylvester–Gallai condition must be coplanar, and thereby answered a question of Serre [Ser66]. This shows that the Sylvester–Gallai theorem crucially depends on the underlying field. If the underlying field is finite, then such configurations no longer have finite dimension. In light of these results, we fix our underlying field to $\mathbb{C}$ in this work, though our results hold for algebraically closed fields of characteristic zero.

A number of variations and generalisations of the Sylvester–Gallai theorem have been studied in combinatorial geometry such as a robust version [BDYW11], colored version [EK66], higher

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dimensional flats [Han65, BDYW11] and many more. The main underlying theme in all such results is that the *local* linear relations between the points in a Sylvester–Gallai configuration must imply that such configurations can only happen in *low dimension*, which is a *global condition* on the configuration. Once one translates such geometric relations into algebraic terms, one sees that the study of Sylvester-Gallai configurations is a study about *cancellations* in algebraic geometry. In summary, Sylvester-Gallai type questions ask the following: given a set of algebraic geometric objects (e.g. vectors, linear forms or polynomials), whether "many" *local cancellations or syzygies* (such as the SG incidence conditions) imply *global constraints* on the configuration (such as being low-dimensional or dependence on a low number of variables).

Many results in algebraic and boolean complexity, as well as in cryptography, show that cancellations are very powerful in computation [RW92, Raz92, Tar88, GMOR15, Val79, HY09, MM82, BS88]. Therefore, it is no surprise that proofs of Sylvester-Gallai theorems, which deal with limitations on the power of cancellations, have required sophisticated tools.

The variations alluded to above have applications in several areas of theoretical computer science, such as algebraic complexity (Polynomial Identity Testing and Reconstruction) and coding theory (Locally Correctable Codes). We now discuss some of these variations and their connections to TCS, and direct readers to [BM90] for more on classical Sylvester–Gallai problems.

Robust Sylvester-Gallai theorems: In this variation, one is given a constant $0 < \delta < 1$, and one requires the points $v_1, \dots, v_m \in \mathbb{C}^n$ to satisfy the following condition: for every v_i, there are δm many points v_j such that the line joining v_i, v_j contains a third point in the set. The robust Sylvester-Gallai theorem states that such configurations lie on a constant dimensional subspace.

These configurations were first studied in [ST83], where the above theorem was proved for all values of δ that are close to 1. Subsequently, in [BDYW11], the authors proved the theorem for all values of δ, and showed that such configurations have dimension $O(1/\delta^2)$. In [DSW14], this result was further improved, and the authors showed that such configurations have dimension $\mathcal{O}(1/\delta)$.

These configurations are useful in the study of locally correctable codes [BDYW11] and circuit reconstruction [Sin16].

High dimensional Sylvester-Gallai theorems: Another variation of the Sylvester-Gallai theorem involves considering higher dimensional linear spaces instead of lines. For example, suppose now that for any v_i, v_j, v_k that are not collinear, we require the 2-dimensional affine space spanned by v_i, v_j, v_k to contain a fourth point in the configuration. The higher dimensional Sylvester-Gallai theorem states that such configurations also lie in a constant dimensional affine subspace.

These configurations were first studied in [Han65], who proved the above theorem for affine spaces of all dimensions (the above is the case of dimension two). Further, in [BDWY13] the authors proved a robust version of the high dimensional Sylvester–Gallai theorem of [Han65].

These configurations have application in polynomial identity testing of depth three circuits ([KS09, SS13]). The authors show that the linear forms in any depth three circuit computing the zero polynomial satisfy a version of this Sylvester-Gallai theorem, and therefore have low rank.

Higher degree generalisations and PIT: Motivated by the application of Syvester-Gallai theorems for depth three PIT, Gupta ([Gup14]) introduced non-linear Sylvester-Gallai configurations and proposed Conjecture 1.1 below, generalizing the classical SG theorems to polynomials of higher degree, where the incidence condition is given by radical membership. [Gup14] shows that a positive solution to Conjecture 1.1 yields deterministic poly-time PIT algorithms for depth four circuits with bounded top and bottom fan-in (circuits of the form $\Sigma^k\Pi\Sigma\Pi^d$).

Gupta divides nonzero $\Sigma^k\Pi\Sigma\Pi^d$ circuits into two classes, namely non-SG circuits and SG circuits. Informally, non-SG circuits are those where there is not much cancellation among the low degree polynomials computed at the bottom addition gate. These circuits form the easy case for their PIT algorithm, and the author gives an unconditional polynomial time algorithm to test if such circuits are nonzero. The analysis for non-SG circuits was recently simplified in [Guo21].

The hard case for PIT is when there are non-trivial cancellations among the low-degree polynomials computed at the bottom addition gate. The author conjectures that such cancellations can only occur if this set of polynomials have constant transcendence degree. If this conjecture is true, then the Jacobian based method of [ASSS16] gives a poly-time deterministic PIT algorithm.

We now state the main conjecture of [Gup14]:

Conjecture 1.1 (Conjecture 1, [Gup14])**.** *Let $k, d, c \in \mathbb{N}^*$ be parameters, and let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be finite sets of irreducible polynomials of degree at most d satisfying*

- $\cap_i \mathcal{F}_i = \emptyset$,
- *for every $Q_1, \ldots, Q_{k-1}$, where each Q_j is from a distinct set $\mathcal{F}_{i_j}$, there are polynomials $P_1, \ldots, P_c$ in the remaining set such that $\prod P_i \in \operatorname{rad}(Q_1, \ldots, Q_{k-1})$.*

Then the transcendence degree of $\cup_i \mathcal{F}_i$ is a function of k, d, c, independent of the number of variables or the size of the sets $\mathcal{F}_i$.

In Conjecture 1.1, the division into k sets and the fact that the product of the forms in the remaining set are in the radical are both artefacts of the fact that the goal of the work was to solve $\Sigma^k\Pi\Sigma\Pi^d$ PIT. Since the conjecture above is a far-reaching non-linear generalization of Sylvester's conjecture, it is important to study simpler versions of this conjecture which are still wide open, just as was done in the linear case. With this in mind, towards the above conjecture, Gupta lists a series of conjectures regarding configurations that more closely resemble linear Sylvester-Gallai configurations, the first of which is the following.

Conjecture 1.2 (Conjecture 2, [Gup14])**.** *Let $Q_1, \ldots, Q_m \in \mathbb{C}[x_1, \ldots, x_n]$ be irreducible, homogeneous, and of degree at most d such that for every pair Q_i, Q_j there is $k \neq i, j$ such that $Q_k \in \operatorname{rad}(Q_i, Q_j)$. Then the transcendence degree of $Q_1, \ldots, Q_m$ is $O_d(1)$ (where the constant depends on the degree d).*

Conjecture 1.2 is a beautiful mathematical generalization of the classical SG theorem as well as a stepping stone towards a full resolution of the PIT problem. So far Conjecture 1.2 is known for degrees 2 and 3 [Shp20, OS22] and it is open in general.

Since Conjecture 1.1 deals with radical ideals generated by $k-1$ polynomials (and hence of potentially higher codimension), it is important to generalize Conjecture 1.2 to a conjecture about radical ideals generated by k elements. Just as in the linear case (see [Han65]), some care must be taken when defining higher-codimensional Sylvester-Gallai configurations, and we address this formally in Section 3. Now, we present an informal version of the higher-codimensional SG conjecture, which will be the main focus of this work.[1]

Conjecture 1.3 (Higher-codimensional SG conjecture)**.** *Let $\mathcal{F} \subset \mathbb{C}[x_1, \ldots, x_n]$ be a finite set of irreducible homogeneous forms of degree at most d. Suppose for every $F_1, \ldots, F_{k+1} \in \mathcal{F}$, either $F_{k+1} \in \operatorname{rad}(F_1, \ldots, F_k)$ or there exists $R \in \mathcal{F}$ such that $R \in \operatorname{rad}(F_1, \ldots, F_{k+1}) \setminus (\operatorname{rad}(F_1, \ldots, F_k) \cup (F_{k+1}))$. Then $\dim \operatorname{span}_{\mathbb{C}}\{\mathcal{F}\} = O_{d,k}(1)$ (where the constant depends on the degree d and the codimension parameter k).*

Note that the Sylvester-Gallai conditions in the above conjectures look different from the previous ones: we talk about membership in radical ideals as opposed to containment in affine spans. A discussion on why this is an appropriate generalisation of the linear Sylvester-Gallai condition can be found in [Gup14].

Our main result, a proof of Conjecture 1.3 in the case where $d = 2$, is a step towards Conjecture 1.1 for the parameters $(k, d, c) = (k, 2, c)$ for any choice of $k, c \in \mathbb{N}$.

[1] The conjecture stated here is implied by our formal conjecture in Section 3.

1.1 Main Result & Technical Contributions

In this subsection we informally state our main result, the higher codimensional analogue of the radical Sylvester–Gallai theorem. As is the case with the higher codimensional linear setting, the formal statement (Theorem 3.12) requires some additional definitions and is given in Section 3.[2]

Theorem 1.4 (Main theorem, informal)**.** *Let $\mathcal{F} \subset \mathbb{C}[x_1, \cdots, x_n]$ be a finite set of irreducible forms of degree at most 2. Suppose for every $F_1, \dots, F_{k+1} \in \mathcal{F}$, either $F_{k+1} \in \text{rad}(F_1, \dots, F_k)$ or there exists $R \in \mathcal{F}$ such that $R \in \text{rad}(F_1, \dots, F_{k+1}) \setminus (\text{rad}(F_1, \dots, F_k) \cup (F_{k+1}))$. Then $\dim \text{span}_{\mathbb{C}}\{\mathcal{F}\} = O_k(1)$.*

Remark 1.5. Note that our theorem, with $k = 1$, recovers the main theorem in [Shp20].

Geometrically, the above statement says that the algebraic set defined by every set of $k + 1$ forms in the configuration lies in the variety defined by another form. Since such algebraic sets have codimension at most $k + 1$, we call our configurations higher codimension Sylvester-Gallai configurations.

In previous works [Shp20, PS20, PS21, PS22, GOS22, OS22], which deal with (variants of) the case where $k = 1$, the approach used to prove a theorem of the above type required a structure theorem that would categorize ideals of the form (F_1, F_2) where each F_i is either a quadratic or a cubic form. These structure theorems used two main facts about ideals of the form (F_1, F_2):

1. they are complete intersections, and therefore Cohen-Macaulay (which implies unmixed).

2. they have small degree (four in the quadratic case and nine in the cubic case).

These two facts, along with properties of Hilbert-Samuel multiplicity, yield a list of special minimal primes and multiplicities such ideals can have, whenever they are not radical. Combined with existing literature and some new results on prime (and primary) ideals of codimension 2, the structure theorems are derived, and then used in the proof of their main theorem.

In our setting, neither of the above facts hold in general. The ideals we consider are generated by k quadratics, and therefore can have degree up to 2^k. Further, these ideals may no longer be complete intersections, and therefore can have embedded primes and even minimal primes of any codimension between 2 and k. This rules out the feasibility of using very fine-grained structure theorems as was done in previous works.

In a recent breakthrough, [AH20] proved that if one has quadratics $F_1, \dots, F_k$ which are "far enough apart," then the ideal $(F_1, \dots, F_k)$ is a complete intersection and prime (and hence radical). However, as discussed above, in our case this result alone is not enough for us to prove all we need: in many cases of interest, the forms in our configuration will not be far enough apart and the result from [AH20] will not apply.

To handle the remaining cases, we build on the techniques of [OS22] and prove a more general structural result on ideals generated by k quadratic forms. Our structural result (Lemma 6.6) states that given certain conditions on the quadratic forms $F_1, \dots, F_k$, even though they may not be far enough apart, one can still prove that the ideal $(F_1, \dots, F_k)$ is radical and has well-behaved minimal primes. The precise conditions of Lemma 6.6 are somewhat technical, and are developed in Section 6.1 with the definition of *integral sequences* of forms. An easier version of our structural lemma can be stated as follows:

Lemma 1.6 (Basic Lemma 6.6)**.** *Let $F_1, \dots, F_k \in \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_k]$ be irreducible quadratic forms such that $F_i \in \mathbb{C}[x_1, \dots, x_n, y_i]$ is monic in y_i. Then, the ideal $I := (F_1, \dots, F_k)$ is radical and for any minimal prime $\mathfrak{p} \supset I$, we have $\mathfrak{p} \cap \mathbb{C}[x_1, \dots, x_n] = (0)$.*

[2] Theorem 3.12 in fact implies the result that we are stating in this page.

Lemma 6.6, and the more basic version above, can be seen as general structural results, which say that either a given ideal is radical, or the generators are "related" (i.e. the "extra variables" $y_1, \dots, y_k$ must be related). This is a weaker structural result than the ones in the previous works, but holds in a more general setting, and is likely to generalise to higher degree configurations.

The proof of Lemma 6.6 involves tools from dimension theory, as well as the discriminant lemma, and the transfer principles from [OS22]. All of these concepts can be found in Section 4.

1.2 High level proof ideas

Our high level strategy is the that in order to bound $\dim \operatorname{span}_{\mathbb{C}}\{\mathcal{F}\}$, it is enough to prove that $\mathcal{F}$ is contained in a small graded algebra. To deal with the issues raised in the previous subsection, our strategy will be to prove that any such SG configuration $\mathcal{F}$ must be contained in a special ideal, which satisfies two properties:

1. the ideal is generated by a vector space $V := V_1 + V_2$ with $\dim V = O_k(1)$, where V_1 is a vector space of linear forms and V_2 is a vector space of quadratics
2. Any nonzero quadratic in V_2 is of very high rank (relative to $\dim V$).

With this result, we reduce the radical Sylvester-Gallai question to a linear, high-codimensional Sylvester-Gallai question, and apply the theorems from [BDYW11, DSW14, DGOS18] to obtain that $\mathcal{F}$ must be contained in a small algebra. This is done in Section 7.3.

To prove that such special ideals exist, we proceed in two steps, each guided by a different conceptual principle. In the first step, we construct a small graded vector space W such that all forms in $\mathcal{F}$ are "close to" the algebra $\mathbb{C}[W]$. That is, there exists a constant B such that for each form $F \in \mathcal{F}$, there exist constantly many linear forms $y_1, \dots, y_r$, where $r \leqslant B$, such that $F \in \mathbb{C}[W, y_1, \dots, y_r]$. Note that both the linear forms y_i and the constant r depend on the form F, and the point here is to obtain a global upper bound on the values that r can take. We name such algebra $\mathbb{C}[W]$ *core algebra* (see Section 7.1).

In the second step, given $\mathcal{F}$ and a core algebra, we want to construct the special ideal V satisfying properties 1 and 2 above such that $\mathcal{F} \subset (V)$. To do this, we use Lemma 6.6 to show that for any sequence $F_1, \dots, F_k$ such that $(F_1, \dots, F_k)$ is not a radical ideal, it must be the case that the "extra variables" of the forms $F_1, \dots, F_k$ must be (very) dependent. Thus we get a win-win type of result here: either the ideal $(F_1, \dots, F_k)$ is radical (which gives us some linear dependencies amongst the forms of $\mathcal{F}$), or the linear forms coming from the extra variables must have very strong linear dependencies (and hence we can control their total dimension).

We now give an overview of each step.

Step 1 - constructing core algebras (Section 7.1): given a quadratic form Q and a vector space W, we say that Q is B-close to $\mathbb{C}[W]$ if there is a vector space Y of linear forms with $\dim Y \leqslant B$ such that $Q \in \mathbb{C}[W, Y]$.[3] That is, Q is a polynomial in few (linear) variables whenever we are allowed to have coefficients in $\mathbb{C}[W]$. We say that $\mathcal{F}$ is B-close to $\mathbb{C}[W]$ if every form in $\mathcal{F}$ is B-close to $\mathbb{C}[W]$. A *core algebra* is an algebra $\mathbb{C}[W]$ such that $\mathcal{F}$ is B-close to $\mathbb{C}[W]$ for some constant B.

The key inspiration for constructing such core algebras comes from the work [AH20], where the authors prove that if the quadratic forms $F_1, \dots, F_{k+1}$ are "sufficiently far apart," then they form a *prime sequence* (which is a much stronger condition than complete intersection). Thus, either a given set of quadratic forms is a prime sequence, or one of the quadratics is "close" (that is, of low rank) to the vector space generated by the other quadratics.

[3] We extend this definition to linear forms by saying that any linear form is 1-close to any algebra.

One consequence of being a prime sequence is that the ideal $(F_1, \ldots, F_{k+1})$ will be a prime ideal (hence radical) and a complete intersection. If we have too many quadratic forms which are far apart, then the radical SG condition will imply that dependencies among the quadratics are linear dependencies, and therefore we can apply [BDYW11, DSW14] and construct our core algebra.

Here we get our first win-win: either many forms are far apart, in which case we will get linear dependencies (and thereby a vector space of low dimension) or we can construct a small vector space W such that $\mathcal{F}$ is close to $\mathbb{C}[W]$.

Since we want to control the quadratic forms of high rank (which we call strong forms), the proof of the construction of W requires an auxiliary SG configuration, dealing only with dependencies of high rank quadratics. We term these *strong SG configurations* (see Section 6.2 for details) and our proof is via a careful induction on the codimension of such configurations. Due to the fact that we are now dealing with both linear and quadratic forms, and our condition is a radical membership condition, the proof of this step is more involved and more delicate than the inductive approach used in [BDYW11, Section 5].

The technical reason why this step is more delicate than the induction on codimension done in [BDYW11, Section 5], is due to the fact that quotienting by a quadratic form will lead us to working with rings which are not necessarily polynomial rings, as well as the fact that we still have to handle non-linear radical dependencies and quadratic forms of low rank.

Step 2 - from core algebras to special ideals (Section 7.2): once we have constructed our core algebra $\mathbb{C}[W]$, we now have a global constant bound B such that all forms in $\mathcal{F}$ are B-close to $\mathbb{C}[W]$. In this setting, our structural lemma (Lemma 6.6) applies and we are able to prove that either the quadratic forms are a linear Sylvester-Gallai configuration (which happens if many ideals $(F_1, \ldots, F_{k+1})$ are radical), or the extra variables of the quadratic forms must be (very) dependent. The proof of the aforementioned fact (in Section 7.2) is done by an iterative process to construct our special ideal. We couple Lemma 6.6 with two potential functions to prove termination of the iterative process providing the special ideal, in a similar way that [Shp20, GOS22] use their potential functions.

Wide algebras: Both steps 1 and 2 use the notion of forms being close to an algebra. In Section 5, we make this notion clear, and establish what properties are needed from such algebras to make sure that we preserve the geometric properties of polynomial rings. Since we are dealing with quadratic forms, we need a slightly simpler version of the wide algebras introduced in [OS22].

1.3 Related work

As stated above, the main motivation for studying higher degree versions of the Sylvester-Gallai theorem comes from the relation established to depth four PIT in [Gup14]. The $d = 2$ case of Conjecture 1.2 was proved in [Shp20], which also kick started this line of work. Subsequently, in [PS20], the authors prove a product version of Conjecture 1.2 where the radical of the ideal generated by every pair of quadratics contains the product of all other quadratics. In [PS21], the authors strengthen this further, and prove Conjecture 1.1 in the case when $k = 3, d = 2, c = 4$. This also implies polynomial time PIT for $\Sigma^3\Pi\Sigma\Pi^2$ circuits. In [GOS22] and [PS22] the authors independently proved a robust version of Conjecture 1.2 in the case when $d = 2$.

In [OS22], the authors prove Conjecture 1.2 in the case when $d = 3$. Our current work develops techniques building upon the intermediate results proved in [OS22]. In particular, the wide vector spaces we use are special cases of the wide vector spaces used in [OS22]. Further, our "structure theorems" are proved using the discriminant lemma from [OS22].

Progress on depth four PIT: There has been some recent progress on the PIT problem for depth four circuits with bounded top and bottom fan-in, the same model that is the focus on [Gup14]. In [DDS21], the authors give a quasipolynomial time PIT algorithm for such circuits. The authors use the Jacobian method of [ASSS16] to find a variable reduction map that preserves the algebraic independence of the inputs to the top addition gate. They are able to construct this map explicitly by first massaging the input circuits to change them to easier models, and then showing that the Jacobian can be computed by a read once oblivious arithmetic branching program (ROABP), for which hitting sets are known. Their methods are analytic in nature, and rely on the logarithmic derivative and its power series expansion.

In [LST22], the authors combine their lower bounds for bounded depth circuits with the methods of [CKS18] to obtain subexponential time PIT algorithms for the same circuit families. Note that the methods of [CKS18] cannot give a polynomial time PIT algorithm no matter how strong the lower bound assumptions are. Even getting a quasipolyomial time PIT from these methods for depth four circuits requires much stronger lower bounds than are currently known. However, these methods are more general, and work for all constant depth circuits.

The Sylvester–Gallai approach to PIT is the only one so far that can yield a deterministic poly–time algorithm. In both the works above, the methods used are quite distinct from the methods based on the Sylvester-Gallai theorem. In particular, they avoid dealing with cancellations, and therefore are unable to exploit the global structure that many local cancellations give rise to.

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2 Preliminaries

In this section we establish notation and preliminary facts we will need for the rest of the paper. Let $S = \mathbb{C}[x_1, \ldots, x_N]$ denote the polynomial ring, graded by degree $S = \bigoplus_{i \geqslant 0} S_i$. Given a vector space $V \subset S$, we use V_i to denote the degree i piece, that is, $V_i = V \cap S_i$. We say that a vector space is graded if $V = \oplus V_i$.

We use *form* to refer to a homogeneous polynomial. Given two forms A, B we say that A, B are non-associate if $A \notin (B)$ and $B \notin (A)$. If A, B are of the same degree, this is equivalent to them being linearly independent.

2.1 Rank and linear spaces of quadratic forms

We now define a notion of the rank of quadratic forms, in accordance to [Shp20].

Definition 2.1 (Rank of a quadratic form)**.** Let Q be a quadratic form. The rank of Q, denoted $\operatorname{rank} Q$, is the smallest $s \in \mathbb{N}$ such that we can write $Q = \sum_{i=1}^{s} a_i b_i$ with $a_i, b_i \in S_1$. If $\operatorname{rank} Q = s$, then a decomposition $Q = \sum_{i=1}^{s} a_i b_i$ with $a_i, b_i \in S_1$ is called a minimal representation of Q.

Proposition 2.2. *If $\phi : S_1 \to S_1$ is an invertible linear map and $\psi : S \to S$ is the map extending ϕ, then for any $Q \in S_2$ we have $\operatorname{rank} Q = \operatorname{rank} \psi(Q)$. If $U \subseteq S_1$ is a vector space of dimension k, and $\overline{Q}$ is the image of Q in $S/(U)$, then $\operatorname{rank} \overline{Q} \geqslant \operatorname{rank} Q - k$.*

Proof. Suppose $\operatorname{rank} Q = r$ and $Q = \sum_{i=1}^{r} a_i b_i$. We have $\psi(Q) = \sum_{i=1}^{r} \psi(a_i)\psi(b_i)$ therefore $\operatorname{rank} \psi(Q) \leqslant r$. If $\operatorname{rank} \psi(Q) = r'$ and $\psi(Q) = \sum_{i=1}^{r'} c_i d_i$ then $Q = \sum_{i=1}^{r'} \psi^{-1}(c_i)\psi^{-1}(d_i)$, which shows that $\operatorname{rank} Q = \operatorname{rank} \psi(Q)$.

Suppose $u_1, \dots, u_k$ is a basis for U, and suppose $\overline{Q} = \sum_{i=1}^{r'} \overline{a_i}\overline{b_i}$. Then $Q = \sum_{i=1}^{r} a_i b_i + \sum_{j=1}^{k} u_i v_i$ for some $v_i \in S_1$. Therefore $\operatorname{rank} Q \leqslant \operatorname{rank} \overline{Q} + k$. □

Remark 2.3. Let $Q = \sum_i a_{ii} x_i^2 + \sum_{i<j} 2a_{ij} x_i x_j$ be a quadratic form in S. Recall that there is an one-to-one correspondence between quadratic forms $Q \in S_2$ and symmetric bilinear forms. Let M be the symmetric matrix corresponding to the symmetric bilinear form of Q. Note that the (i,j)-the entry of M is given by a_{ij}. If M is of rank r, then after a suitable linear change of variables, we can write $Q = x_1^2 + \cdots + x_r^2$. Since the rank of a quadratic form is invariant under a linear change of variables(Proposition 2.2), we have $\operatorname{rank}(Q) = \lceil r/2 \rceil$, if M is of rank r.

In the next sections, we will need to use the following notion of a vector space of a quadratic form, which is a slight modification on the definition first given in [Shp20]. The only modification that we make is that we preserve the quadratic form if its rank is high enough.

Definition 2.4 (Vector space of a quadratic form)**.** Let Q be a quadratic form of rank s, so that $Q = \sum_{i=1}^{s} a_i b_i$. Define the vector space $\operatorname{Lin}(Q) := \operatorname{span}_{\mathbb{C}} \{a_1, \dots, a_s, b_1, \dots, b_s\}$. Define $\mathbb{L}(Q)$ as:

$$\mathbb{L}(Q) = \begin{cases} \operatorname{span}_{\mathbb{C}}\{Q\}, & \text{if } s \geqslant 5 \\ \operatorname{Lin}(Q), & \text{otherwise.} \end{cases}$$

We also extend the definition of Lin to linear forms in the natural way as follows.

Definition 2.5. For a linear form $\ell \in S_1$ define $\mathbb{L}(\ell) := \operatorname{span}_{\mathbb{C}}\{\ell\}$.

Note that $\mathbb{L}(Q)$ is always a vector space of $\mathcal{O}(1)$ dimension (in fact, it is of dimension at most 10), while $\operatorname{Lin}(Q)$ can have non constant dimension. While a minimal representation $Q = \sum_{i=1}^{s} a_i b_i$ is not unique, the vector space $\operatorname{Lin}(Q)$ is unique and hence well-defined. The following lemma, which appears in [PS20, Fact 2.15] characterizes $\operatorname{Lin}(Q)$ as the smallest vector space of linear forms defining the algebras that contain Q.

Lemma 2.6. *If* $Q = \sum_{i=1}^{r} x_i y_i$ *with* $x_i, y_i \in S_1$ *then* $\operatorname{Lin}(Q) \subseteq \operatorname{span}_{\mathbb{C}} \{x_i, y_j | i, j \in [r]\}$.

Remark 2.7. The space $\operatorname{Lin}(Q)$ can also be defined as the space of first order partial derivatives of Q (see Lemma 2.10). However, we decided to not state this definition in this manner as this definition does not generalize well to forms of higher degree, as it is done in the works [AH20, OS22].

We now state some useful results related to the rank and linear spaces of quadratics, some of which appear in [PS20, GOS22].

Lemma 2.8. *Suppose* $Q \in S_2$ *is such that* $\operatorname{rank} Q = r$. *Then* $\dim \operatorname{Lin}(Q) = 2r$ *or* $\dim \operatorname{Lin}(Q) = 2r - 1$. *In the second case, we can write* $Q = a_r^2 + \sum_{i=1}^{r-1} a_i b_i$.

Proof. Suppose $v_1, \dots, v_d$ is a basis for $\operatorname{Lin}(Q)$ for some $d \leqslant 2r$. We then have $Q \in \mathbb{C}[v_1, \dots, v_d]$. By Remark 2.3, we can write $Q = \sum_{i=1}^{d'} u_i^2$ for some $d' \leqslant d$, where each $u_i \in \operatorname{span}_{\mathbb{C}}\{v_1, \dots, v_d\}$. By Lemma 2.6 we have $\operatorname{Lin}(Q) \subseteq \operatorname{span}_{\mathbb{C}}\{u_1, \dots, u_{d'}\}$ whence $d' = d$. If d is even then we get $d/2 \geqslant r$. Since we also have $d \leqslant 2r$ we get $d = 2r$. If d is odd, we must have $(d-1)/2 + 1 \geqslant r$. Since we also have $d \leqslant 2r$ we get $d = 2r - 1$. In this case, $u_d^2 + \sum_{j=1}^{d/2-1} (u_{2j-1} + u_{2j})(u_{2j-1} - u_{2j})$ is a minimal representation of Q, proving the last statement. □

Remark 2.9. By the above lemma, given any $Q \in S_2$ such that $\operatorname{rank} Q = r$ we can write $Q = \sum_{i=1}^{r} a_i b_i$ such that $a_1, \ldots, a_r, b_1, \ldots, b_{r-1}$ are linearly independent, and either $b_r = a_r$ or b_r is independent of $a_1, \ldots, a_r, b_1, \ldots, b_{r-1}$.

Lemma 2.10. *Let $Q \in S = \mathbb{C}[x_1, \ldots, x_N]$ be a quadratic form. Then $\operatorname{Lin}(Q) = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial Q}{\partial x_1}, \cdots, \frac{\partial Q}{\partial x_N} \right\}$ is the space of all first order partial derivatives of Q.*

Proof. Suppose $\operatorname{rank} Q = r$ and $\sum_{i=1}^{r} a_i b_i$ be a decomposition of Q as in Remark 2.9. Then note that $\frac{\partial Q}{\partial a_i} = b_i$ and $\frac{\partial Q}{\partial b_i} = a_i$ for all $i \leqslant r-1$. If $b_r = a_r$, then $\frac{\partial Q}{\partial a_r} = 2a_r$, and otherwise we have $\frac{\partial Q}{\partial a_r} = b_r$ and $\frac{\partial Q}{\partial b_r} = a_r$. Therefore $\operatorname{Lin}(Q) \subset \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial Q}{\partial x_1}, \cdots, \frac{\partial Q}{\partial x_N} \right\}$. Since $Q = \sum_{i=1}^{r} a_i b_i$, we have $\frac{\partial Q}{\partial x_j} \in \operatorname{Lin}(Q)$ for all $j \in [N]$. □

The following lemma from [PS20] shows that adding a product of new variables increases the rank of a quadratic. In Lemma 2.12, we extend this to sums of quadratics in distinct variables.

Lemma 2.11 ([PS20, Claim 2.7])**.** *Suppose $Q \in \mathbb{C}[x_1, \ldots, x_m]$ is a polynomial of rank r. If y, z are new variables then $\operatorname{rank}(Q + yz) = r + 1$. In particular, $\operatorname{Lin}(Q + yz) = \operatorname{Lin}(Q) + \operatorname{span}_{\mathbb{C}}\{y, z\}$.*

Lemma 2.12. *Suppose $P \in \mathbb{C}[x_1, \ldots, x_m]$ and $Q \in \mathbb{C}[y_1, \ldots, y_n]$ are two quadratics in distinct variables. Then $\operatorname{Lin}(P + Q) = \operatorname{Lin}(P) + \operatorname{Lin}(Q)$.*

Proof. Note that we have $\frac{\partial (P+Q)}{\partial x_i} = \frac{\partial P}{\partial x_i}$ and $\frac{\partial (P+Q)}{\partial y_j} = \frac{\partial Q}{\partial y_j}$ for all $i \in [m]$ and $j \in [n]$. Therefore, by Lemma 2.10, we have that $\operatorname{Lin}(P + Q) = \operatorname{Lin}(P) + \operatorname{Lin}(Q)$. □

Lemma 2.13. *Let $W \subseteq S_1$ be a vector space. Suppose $Q \in S_2$ is such that $\operatorname{rank} Q = r$ in S and $\operatorname{rank} \overline{Q} = r' < r$ where $\overline{Q}$ is the image of Q in $S/(W)$. Then $W \cap \operatorname{Lin}(Q) \neq \{0\}$. In particular if $Q \in (W)$ then $W \cap \operatorname{Lin}(Q) \neq 0$.*

Proof. Suppose $Q = \sum_{i=1}^{r} a_i b_i$ is the minimal representation guaranteed by Remark 2.9. Assume towards a contradiction that $\operatorname{Lin}(Q) \cap W = \{0\}$. Since $a_1, \ldots, a_r, b_1, \ldots, b_{r-1}$ are independent in S, and either $b_r = a_r$ or b_r is independent of $a_1, \ldots, a_r, b_1, \ldots, b_{r-1}$, by assumption the same holds in $S/(W)$. We can now repeatedly apply Lemma 2.11 to deduce that $\operatorname{rank}(\overline{a_r}\overline{b_r} + \sum_{i=1}^{r-1} \overline{a_i}\overline{b_i}) = r$, contradicting assumption. □

2.2 General Projections

We now recall the definition and properties of projection maps from [Shp20, PS20, OS22].

Definition 2.14 (Projection maps)**.** Let $S = \mathbb{C}[x_1, \cdots, x_n]$ be a polynomial ring. Let $W \subset S_1$ be a subspace of linear forms and $y_1, \cdots, y_t$ be a basis of W. Let $y_1, \cdots, y_n$ be a basis of S_1 that extends the basis $y_1, \cdots, y_t$ of W. Let z be a formal variable not in $\{y_1, \cdots, y_n\}$. For $\alpha = (\alpha_1, \cdots, \alpha_t) \in \mathbb{C}^t$, we define the projection map $\varphi_{\alpha,W}$ as the $\mathbb{C}$-algebra homomorphism $\varphi_{\alpha,W} : S \to \mathbb{C}[z, y_{t+1}, \cdots, y_n] = S[z]/(W)$ defined by

$$y_i \mapsto \begin{cases} \alpha_i z, \text{ if } 0 \leqslant i \leqslant t \\ y_i, \text{ otherwise} \end{cases}$$

For simplicity we will often drop the subscripts W or α, and write φ_α or φ for a projection map when there is no ambiguity about the vector space W or the vector α.

General projections. Fix a vector space $W \subset S_1$ as in Definition 2.14. We will say that a property holds for a *general projection* φ_α, if there exists a non-empty open subset $U \subset \mathbb{C}^t$ such that the property holds for all φ_α with $\alpha \in U$. Here $U \subset \mathbb{C}^t$ is open with respect to the Zariski topology, hence U is the complement of the zero set of finitely many polynomial functions on $\mathbb{C}^t$. The general choice of the element α defining a general projection φ_α allows us to say that such projection maps will avoid any finite set of polynomial constraints. As shown in [Shp20, PS20], general projection maps preserve several important properties of polynomials.

Proposition 2.15. *[OS22, Proposition 2.6] Let $F \in S$ be a polynomial and $W \subset S_1$ be a vector space of linear forms.*

(a) If $F \notin \mathbb{C}[W]$, then $\varphi(F) \notin \mathbb{C}[z]$ for a general projection $\varphi : S \to S[z]/(W)$.

(b) If $F \neq 0$, then $\varphi(F) \neq 0$ for a general projection.

(c) Suppose F is a form which does not have any multiple factors and $F \in (W)$. If $\varphi(F) = z^k G$ where $G \notin (z)$, then G does not have any mulitple factors.

The next proposition is from [PS20, Claim 2.23].

Proposition 2.16. *Let $F, G \in S$ be two polynomials which have no common factor and $W \subset S_1$ a subspace of linear forms. For a general projection $\varphi : S \to S[z]/(W)$, we have $\gcd(\varphi(F), \varphi(G)) \in \mathbb{C}[z]$. In particular, if F, G are homogeneous then $\gcd(\varphi(F), \varphi(G)) = z^k$ for some $k \in \mathbb{N}$.*

The following result shows that general projections preserve linear independence for polynomials outside the algebra generated by W.

Corollary 2.17. *[OS22, Corollary 2.8] Let $F, G \in S$ be linearly independent irreducible forms and $W \subset S_1$ be a vector space of linear forms. If $F, G \notin \mathbb{C}[W]$ then $\varphi(F), \varphi(G)$ are linearly independent, for a general projection $\varphi : S \to S[z]/(W)$.*

The next proposition follows from [PS20, Claim 2.26].

Proposition 2.18. *Let $W \subset S_1$ be a vector space of linear forms. Let $\mathcal{F} \subset S_2$ be a finite set of quadratic forms. Suppose there is an integer $D > 0$ such that $\dim \operatorname{span}_{\mathbb{C}} \left\{ \bigcup_{F \in \mathcal{F}} \mathbb{L}(\varphi(F)) \right\} \leqslant D$ for a general projection $\varphi : S \to S[z]/(W)$. Then $\dim \operatorname{span}_{\mathbb{C}} \left\{ \bigcup_{F \in \mathcal{F}} \mathbb{L}(F) \right\} \leqslant (D+1) \cdot \dim W$.*

The proposition above can be sharpened if we have extra information about the linear forms in $\mathcal{F}$. We state this sharpening in the next proposition

Proposition 2.19. *Let $W \subset S_1$ be a vector space of linear forms and $\mathcal{F} \subset S_2$ be a finite set of quadratic forms such that $\mathcal{F} \cap (W)$ and $s(F) < s$ for each $F \in \mathcal{F}$. Suppose there is an integer $D > 0$ such that $\dim \operatorname{span}_{\mathbb{C}} \left\{ \bigcup_{F \in \mathcal{F}} \mathbb{L}(\varphi(F)) \right\} \leqslant D$ for a general projection $\varphi : S \to S[z]/(W)$. Then we have $\dim \operatorname{span}_{\mathbb{C}} \left\{ \bigcup_{F \in \mathcal{F}} \mathbb{L}(F) \right\} \leqslant (D+1) \cdot s$.*

3 Sylvester–Gallai configurations

We now formally define the Sylvester-Gallai configurations that we deal with in this work. Before we do this, we state the current known bounds on dimensions of linear Sylvester–Gallai configurations, these will be useful in our proofs.

3.1 Linear Sylvester–Gallai configurations

For this subsection, we let $\mathcal{L}$ be a finite set of pairwise non-associate linear forms and $\delta \in (0, 1]$ be a constant. We begin by defining ordinary and elementary spaces, as was done in [Han65, BDYW11].

Definition 3.1 (Ordinary spaces)**.** Let $\ell_1, \ldots, \ell_k \in \mathcal{L}$, and let $V = \operatorname{span}_{\mathbb{C}}\{\ell_1, \ldots, \ell_k\}$. The space V is called ordinary with respect to $\mathcal{L}$ if there are $\ell'_1, \ldots, \ell'_{k-1} \in S_1$, and $\ell \in \mathcal{L}$ such that $V \cap \mathcal{L} \subseteq \operatorname{span}_{\mathbb{C}}\{\ell'_1, \ldots, \ell'_{k-1}\} \cup \{\ell\}$.

Definition 3.2 (Elementary spaces)**.** Let $\ell_1, \ldots, \ell_k \in \mathcal{L}$, and let $V = \operatorname{span}_{\mathbb{C}}\{\ell_1, \ldots, \ell_k\}$. The space V is called elementary with respect to $\mathcal{L}$ if $V \cap \mathcal{L} = \{\ell_1, \ldots, \ell_k\}$.

Definition 3.3. The set $\mathcal{L}$ is a $\delta - SG^*_k$ configuration if for every linearly independent $\ell_1, \ldots, \ell_k \in \mathcal{L}$, there are $\delta \cdot |\mathcal{L}|$ forms ℓ in $\mathcal{L}$ such that either

1. $\ell \in \operatorname{span}_{\mathbb{C}}\{\ell_1, \ldots, \ell_k\}$,
2. or the linear space $\operatorname{span}_{\mathbb{C}}\{\ell_1, \ldots, \ell_k, \ell\}$ contains a form in $\mathcal{L} \setminus (\operatorname{span}_{\mathbb{C}}\{\ell_1, \ldots, \ell_k\} \cup \{\ell\})$.

Definition 3.4. The set $\mathcal{L}$ is a $\delta - SG_k$ configuration if for every linearly independent $\ell_1, \ldots, \ell_k \in \mathcal{L}$ there are $\delta \cdot |\mathcal{L}|$ forms $\ell \in \mathcal{L}$ such that either

1. $\ell \in \operatorname{span}_{\mathbb{C}}\{\ell_1, \ldots, \ell_k\}$,
2. or the linear space $\operatorname{span}_{\mathbb{C}}\{\ell_1, \ldots, \ell_k, \ell\}$ is not elementary.

Given the above definitions, the following theorem was proved in [DSW14, Theorem 1.14], improving on [BDYW11].

Theorem 3.5. *If $\mathcal{L}$ is a $\delta - SG^*_k$ configuration then* $\dim \operatorname{span}_{\mathbb{C}}\{\mathcal{L}\} = \mathcal{O}(k/\delta)$*. If $\mathcal{L}$ is a $\delta - SG_k$ configuration then* $\dim \operatorname{span}_{\mathbb{C}}\{\mathcal{L}\} = \mathcal{O}\left(C^k/\delta\right)$ *where C is a universal constant independent of* k*.*

In the case when $k = 1$, Definition 3.3 and Definition 3.4 coincide, and match the usual notion of robust linear Sylvester–Gallai configurations. In this case, the constant C is explicit.

Theorem 3.6 ([DGOS18, Theorem 1.6])**.** *If $\mathcal{L}$ is a $\delta - SG_1$ configuration then* $\dim \operatorname{span}_{\mathbb{C}}\{\mathcal{L}\} \leqslant 4/\delta$*.*

Remark 3.7. Note that in [BDYW11, DSW14], the SG configurations are described in terms of points in $\mathbb{C}^n$, instead of linear forms in S. Both settings are equivalent via duality between points in $\mathbb{C}^n$ and linear forms in S_1.

3.2 Radical Sylvester-Gallai configurations

We now define the higher dimension analogues of the above configurations. Let $\mathcal{F}$ be a finite set of irreducible forms of degree at most d that are pairwise non-associate.

Definition 3.8 (Relevant sets)**.** Let $\mathcal{P} = \{P_1, \ldots, P_t\}$ be a set of forms in $S_{\leqslant d}$. We say that $\mathcal{P}$ is relevant if for every $1 \leqslant i \leqslant t$, $P_i \notin \operatorname{rad}(\mathcal{P} \setminus P_i)$.

A relevant set of forms of size k is called a k*-relevant* set.

Geometrically, a set $\mathcal{P}$ is relevant if no subset of $\mathcal{P}$ define the same variety as $\mathcal{P}$. We can now extend Definition 3.3 and Definition 3.4 to configurations with forms of higher degree.

Definition 3.9 (k-ordinary set)**.** Let $\mathcal{P} \subset \mathcal{F}$ be a k-relevant set. We say that $\mathcal{P}$ is k-ordinary with respect to $\mathcal{F}$ if there are forms $F_1, \ldots, F_k \in \mathcal{F}$ such that

$$\mathrm{rad}\,(\mathcal{P}) \cap \mathcal{F} \subset \mathrm{rad}\,(F_1, \ldots, F_{k-1}) \cup \{F_k\}.$$

Definition 3.10 (k-elementary set)**.** Let $\mathcal{P} \subset \mathcal{F}$ be a k-relevant set. We say that $\mathcal{P}$ is k-elementary with respect to $\mathcal{F}$ if $\mathrm{rad}\,(\mathcal{P}) \cap \mathcal{F} = \mathcal{P}$.

Definition 3.11 (Radical Sylvester Gallai condition for tuples)**.** Let $\mathcal{F} := \{F_1, \ldots, F_m\} \subset S_{\leqslant d}$ be a finite set of irreducible forms and $k \in \mathbb{N}$. We say that $\mathcal{F}$ is a $\delta - \mathrm{SG}_k^*(d)$ configuration if for every $i \neq j$ we have $F_i \notin (F_j)$ and for every k-relevant subset $\mathcal{P} \subset \mathcal{F}$, there are $\delta(m-k)$ many forms $F \in \mathcal{F} \setminus \mathcal{P}$ such that either

- $F \in \mathrm{rad}\,(\mathcal{P})$ or
- $\mathrm{rad}\,(F, \mathcal{P}) \cap \mathcal{F}$ contains a form R not in $\mathrm{rad}\,(\mathcal{P}) \cup \{F\}$.

Note that the robust SG problem from [PS22, GOS22] is the $\delta - \mathrm{SG}_1^*(2)$. The higher codimensional radical SG problem for quadratics that we address here can be stated as follows: what is the maximum vector space dimension of any $1 - \mathrm{SG}_k^*\,(2)$ configuration? Our main theorem, which we now formally state, gives an answer to this question.

Theorem 3.12 (Radical SG Theorem for tuples of quadratics)**.** *Let $\mathcal{F}$ be a $1 - \mathrm{SG}_k^*(2)$ configuration. There is a universal constant $c > 0$ such that $\dim(\mathrm{span}_{\mathbb{C}}\{\mathcal{F}\}) \leqslant 3^{c \cdot 4^k}$.*

4 Commutative algebraic preliminaries

4.1 Basic Definitions

In this section we recall the necessary definitions and results needed from commutative algebra and algebraic geometry [AM69, Eis95].

Definition 4.1 (Regular sequence)**.** Let R be a commutative ring with unity. A sequence of elements $f_1, f_2, \cdots f_n \in R$ is called a regular sequence if

(1) $(f_1, f_2, \cdots, f_n) \neq R$, and

(2) for all $i \in [n]$, we have that f_i is a non-zero divisor on $R/(f_1, \cdots, f_{i-1})R$.

Ideals generated by regular sequences are well-behaved. For example, if $f_1, \cdots, f_m$ is a regular sequence in $S = \mathbb{C}[x_1, \cdots, x_n]$, we know that the ideal $I = (f_1, \cdots, f_m)$ is Cohen-Macaulay [Eis95, Proposition 18.13]. Cohen-Macaulayness imposes a simple and well-behaved structure on the primary decomposition of I. In particular, every associated prime of I is a minimal prime and the height/codimension of every minimal prime of I is the same, i.e. Cohen-Macaulay ideals are unmixed and equidimensional [Eis95, Corollaries 18.11, 18.14].

We note that if $f_1, \cdots, f_m$ is a regular sequence of forms in S, then $f_1, \cdots, f_m$ are algebraically independent. Therefore the subalgebra generated by $f_1, \cdots, f_m$ is isomorphic to a polynomial ring. In particular, the ring homomorphism $\mathbb{C}[y_1, \cdots, y_m] \to S$ defined by $y_i \mapsto f_i$ is an isomorphism onto its image.

Even though the $\mathbb{C}$-algebra $\mathbb{C}[f_1, \ldots, f_m] \subset S$ is isomorphic to a polynomial ring, its elements may not behave well when seen as elements of S. We next present a sufficient condition which will ensure to us that the subalgebra is well behaved with respect to S, in a way which we formalize later in Section 5.

Definition 4.2 (R_η-property)**.** Let η be a non-negative integer. We say that a Noetherian ring R satisfies the R_η property if the local ring $R_\mathfrak{p}$ is a regular local ring for all prime ideals $\mathfrak{p} \subset R$ such that $\text{height}(\mathfrak{p}) \leqslant \eta$.

We recall the definition of an R_η-sequence below [AH20]. A subalgebra generated by an R_η-sequence has several inetresting properties such as intersection flatness, which were essential in [AH20, OS22].

Definition 4.3. Let $\eta \in \mathbb{N}$ and R a Noetherian ring. A sequence of elements $f_1, \ldots, f_n \in R$ is called a prime sequence (respectively an R_η-sequence) if

1. $f_1, \cdots, f_n$ is a regular sequence, and
2. $R/(f_1, \cdots, f_i)$ is an integral domain (respectively, satisfies the R_η property) for all $i \in [n]$.

Remark 4.4. Note that a prime sequence in a ring R is also an R-regular sequence. Further, if R is a polynomial ring and $\eta \geqslant 1$, then any R_η-sequence is also a prime sequence.

4.2 Discriminant lemma

The following result provides an elimination theoretic criterion for a complete intersection ideal to be radical. It is a direct application of [OS22, Lemma 3.22].

Lemma 4.5. *Let $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_r, y_1, \ldots, y_s]$, $\mathcal{B} := \mathbb{K}[y_1, \ldots, y_s]$. Let $F_1, \cdots, F_k, P$ be a regular sequence of irreducible forms in $\mathcal{A}$ where $F_1, \cdots, F_k \in \mathcal{B}$. Suppose $P \in \mathcal{A} \setminus (y_1, \ldots, y_s)$. If $I = (F_1, \cdots, F_k) \subset \mathcal{B}$ is radical and $\text{disc}_{x_1}(P) \notin \mathfrak{q} \cdot S$ where $\mathfrak{q}$ is any minimal prime of I in $\mathcal{B}$, then the ideal $(F_1, \cdots, F_k, P)$ is radical in $\mathcal{A}$*

Proof. Let $\mathfrak{p}$ be a minimal prime of the ideal $(F_1, \cdots, F_k, P)$ in $\mathcal{A}$. Since $F_1, \cdots, F_k, P$ is a regular sequence we have $\text{codim}(\mathfrak{p}) = r + s - k - 1$. Let $\mathfrak{q} = \mathfrak{p} \cap \mathcal{B}$. Note that $\mathfrak{q}$ is a prime ideal containing $F_1, \cdots, F_k$ in $\mathcal{B}$. Therefore $\text{codim}(\mathfrak{q}) \geqslant s - k$. If $\text{codim}(\mathfrak{q}) > s - k$, then $\text{codim}(\mathfrak{q} \cdot \mathcal{A}) > r + s - k$. Since $\mathfrak{q} \cdot \mathcal{A} \subset \mathfrak{p}$, we must have $\mathfrak{q} \cdot \mathcal{A} = \mathfrak{p}$, which is a contradiction as $P \in \mathfrak{p}$, whereas $P \notin (y_1, \cdots, y_s)$. Therefore we must have that $\text{codim}(\mathfrak{q}) = s - k$. Then $\mathfrak{q}$ is a minimal prime of $(F_1, \cdots, F_k)$ in $\mathcal{B}$ and by [OS22, Lemma 3.22] we conclude that the ideal $(F_1, \cdots, F_k, P)$ is radical in $\mathcal{A}$. $\square$

5 Wide vector spaces and relative linear spaces

5.1 Wide vector spaces and algebras

We now define the main object that we will use in order to prove that Sylvester-Gallai configurations are low dimensional: wide Ananyan-Hochster vector spaces. Such spaces were used in [OS22] to give a positive solution to the radical SG problem for cubic forms. Our definition is slightly simpler than the one from [OS22, Definition 4.8], as we don't need the multiplicative factor used there.

Definition 5.1 (Wide vector spaces)**.** A vector space $V = V_1 + V_2$ where $V_i \subset S_i$ is said to be r-wide if, for any nonzero $Q \in V_2$ we have $\text{rank}\, Q \geqslant \dim V + r$. In this case, we also say that $\mathbb{C}[V]$ is an r-wide algebra.

We note that an r-wide vector space is a special case of the (w, t)-wide AH vector spaces from [OS22]. An r-wide vector space is precisely a $(r, 1)$-wide AH vector space according to [OS22].

Proposition 5.2 ([OS22], Proposition 4.11)**.** *Suppose* $U = U_1 + U_2$ *is a vector space in* S *and suppose* $r \in \mathbb{N}$. *There exists an* r*-wide vector space* $V = V_1 + V_2$ *with* $\mathbb{C}[U] \subseteq \mathbb{C}[V]$ *such that* $\dim V_2 \leqslant \dim U_2$ *and* $\dim V_1 \leqslant 3^{\dim U_2 + 1} \cdot (r + \dim U)$.

We now list some basic properties regarding these spaces. The first three of these are algebraic properties that show how these spaces are useful, and the next three show how we can build and modify these spaces, and how they behave with respect to projection.

Theorem 5.3 ([AH20], Theorem 1.10)**.** *Let* $V \subset S_2$ *be a vector space of dimension* d *such that* $\operatorname{rank} Q \geqslant d - 1 + \lceil \eta/2 \rceil$. *Then every sequence of linearly independent elements of* V *is an* R_η *sequence.*

Corollary 5.4. *Suppose* $V = V_1 + V_2$ *is a* r*-wide vector space with* $r \geqslant 1$. *If* $\ell_1, \dots, \ell_a$ *is a linearly independent sequence in* V_1 *and* $Q_1, \dots, Q_b$ *is a linearly independent subset of* V_2, *then the sequence* $\ell_1, \dots, \ell_a, Q_1, \dots, Q_b$ *is a prime sequence. In particular, the ideal* $(Q_1, \dots, Q_b)$ *is a prime ideal in the quotient ring* $S/(\ell_1, \dots, \ell_a)$.

Proof. That $\ell_1, \dots, \ell_a$ form a prime sequence follows from the fact that they are independent linear forms. Let $U := \operatorname{span}_{\mathbb{C}}\{\overline{Q_1}, \dots, \overline{Q_b}\}$ be the vector space spanned by $Q_1, \dots, Q_b$ in $S/(\ell_1, \dots, \ell_a)$. Every nonzero form in U has rank at least $\dim V_1 + \dim V_2 + r - a$, which is greater than $\dim U$. Therefore, by Theorem 5.3, the forms $\ell_1, \dots, \ell_a, Q_1, \dots, Q_b$ form a R_1 sequence. By [AH20, Discussion 1.3], such a sequence is also a prime sequence. The last statement follows by the definition of prime sequences (Definition 4.3). $\square$

Claim 5.5. *Suppose* $V := V_1 + V_2$ *is* r*-wide with* $V_i \subset S_i$. *If* $Q \in \mathbb{C}[V]$ *is a quadratic form of rank less than* r, *then* $Q \in \mathbb{C}[V_1]$. *If* $P \in (V)$ *is a quadratic form of rank less than* r, *then* $P \in (V_1)$.

Proof. Suppose $Q = Q_2 + Q_1$ with $Q_i \in \mathbb{C}[V_i]$. We have $Q_2 = Q - Q_1$ whence $\operatorname{rank} Q_2 \leqslant r + \dim V_1$. Therefore $Q_2 = 0$. Similarly, suppose $P = P_1 + P_2$ with $P_2 \in V_2$ and $P_1 \in (V_1)$. We have $P_2 = P - P_1$ whence $\operatorname{rank} P_2 \leqslant r + \dim V_1$. Therefore $P_2 = 0$. $\square$

Remark 5.6. Suppose $V = V_1 + V_2$ is a r-wide vector space, and suppose $U \subset S_1$ is a vector space of dimension k. We have $\dim V + U \leqslant \dim V + k$. Further, we have $(V + U)_2 = V_2$. For every $Q \in (V + U)_2$ we therefore have $\operatorname{rank} Q \geqslant (r - k) + \dim(V + U)$. Therefore $V + U$ is a $r - k$-wide vector space.

Remark 5.7. Suppose $V = V_1 + V_2$ is a r-wide vector space and $\varphi := \varphi_{\alpha, V_1}$ is a projection mapping as defined in Definition 2.14. If $Q \in V_2$ is such that $\operatorname{rank} \varphi(Q) = a$ in $S[z]/(V_1)$ then $a - 1 \leqslant \operatorname{rank} Q \leqslant a$ in $S/(V_1)$. Since V is r-wide, this proves that $a \geqslant r + \dim V_2$. Since $\dim \phi(V_1) = 1$, and since $\dim \phi(V_2) \leqslant \dim V_2$, we get $a \geqslant r - 1 + (\dim \phi(V_1) + \dim \phi(V_2)$. This shows that $\phi(V)$ is at least $r - 1$ wide.

The following lemmas show that radical membership among linear forms and certain elements in the ideal (V) imply relationships between the "low rank" and "high rank" parts individually.

Lemma 5.8. *Let* $F_1, \dots, F_k \in S_{\leqslant 2}$ *be irreducible forms. Let* $V = V_1 + V_2$ *be* r*-wide with* $r \geqslant k + 2$ *and let* $z \in V_1$. *Suppose each* F_i *is either of the form* $F_i = x_i$ *with* $x_i \in S_1$ *or of the form* $F_i = Q_i + z x_i$ *with* $Q_i \in V_2$ *and* $x_i \in S_1$. *If*

$$F_k \in \operatorname{rad}(F_1, \dots, F_{k-1})$$

then $z x_k \in (x_1, \dots, x_{k-1})$ *and* $Q_k \in \operatorname{span}_{\mathbb{C}}\{Q_1, \dots, Q_{k-1}\}$ *where* $Q_i = 0$ *if* $F_i \in S_1$.

Proof. Let $U := (x_1, \dots, x_k)$. In the ring S/U, the vector space V is $(r-k)$-wide by Remark 5.6. By Corollary 5.4, $(Q_1, \dots, Q_{k-1})$ is a prime ideal in S/U. Therefore we have $Q_k = \sum_{i=1}^{k-1} \alpha_i Q$ in $S/(U)$ for $\alpha_i \in \mathbb{C}$. This implies $Q_k = \sum_{i=1}^{k-1} \alpha_i Q_i + E$ in S, where $E \in (U)$. Since rank $E \leqslant \dim U \leqslant k$, and since V is r-wide, we must have $E = 0$, proving the first required statement.

Let $I := (Q_1, \dots, Q_{k-1}, x_1, \dots, x_{k-1})$. Since (U) is prime, and $(Q_1, \dots, Q_{k-1})$ is prime in $S/(U)$, the ideal I is prime. Since $Q_k \in \operatorname{span}_{\mathbb{C}}\{Q_1, \dots, Q_{k-1}\}$ and since $F_i \in I$ for $i \leqslant k-1$, we have $zx_k \in I$. Since W is r-wide, this implies $zx_k \in (x_1, \dots, x_{k-1})$, completing the proof. □

Lemma 5.9. *Let $F_1, \dots, F_k \in S_{\leqslant 2}$ be irreducible forms. Let $V = V_1 + V_2$ be r-wide with $r \geqslant k+2$ and let $z \in V_1$. Suppose each F_i is either of the form $F_i = x_i$ with $x_i \in S_1$ or of the form $F_i = Q_i + zx_i$ with $Q_i \in V_2$ and $x_i \in S_1$. Suppose further that $z, x_1, \dots, x_{k-1}$ are linearly independent. If*

$$F_k \in \operatorname{rad}(F_1, \dots, F_{k-1}),$$

and if $x_k \in (x_1)$ in $S/(z)$, then $F_1 = F_k$.

Proof. First assume that $Q_1 \neq 0$. By relabelling $F_2, \dots, F_k$ we can assume that $\operatorname{span}_{\mathbb{C}}\{Q_1, \dots, Q_{k-1}\} = \operatorname{span}_{\mathbb{C}}\{Q_1, \dots, Q_t\}$ for some $t \leqslant k-1$. For each $i \in [t+1, k-1]$, suppose $Q_i = \sum_{j=1}^{t} \beta_{ij} Q_j$. For each such i, let $y_i := x_i - \sum_{j=1}^{t} \beta_{ij} x_j$. Note that $x_1, \dots, x_t, y_{t+1}, \dots, y_{k-1}$ are linearly independent in $S/(z)$. We have $(F_1, \dots, F_{k-1}) = (F_1, \dots, F_t, zy_{t+1}, \dots, zy_{k-1})$. Let $J = (y_{t+1}, \dots, y_{k-1})$. By Remark 5.6 the vector space V is $r-k$-wide in S/J, therefore $\operatorname{rank}(Q_1, \dots, Q_t) \geqslant t + r - k$, and consequently $\operatorname{rank}(F_1, \dots, F_t) \geqslant t + r - k - 1$. By Theorem 5.3, the ideal $(F_1, \dots, F_t)$ is prime in S/J, therefore $(F_1, \dots, F_t) + J$ is a prime ideal containing $\operatorname{rad}(F_1, \dots, F_{k-1})$.

Let $x_k = x_1 + \alpha z$. Suppose $F_k \in S_2$. By Lemma 5.8 we have $Q_k \in \operatorname{span}_{\mathbb{C}}\{Q_1, \dots, Q_t\}$, say $Q_k = \sum_{j=1}^{t} \gamma_i Q_i$. We have $F_k - \sum_{j=1}^{t} \gamma_i Q_i = z(\alpha z + x_1 - \sum_{j=1}^{t} \gamma_j x_j) \in (F_1, \dots, F_t) + J$. Since the latter ideal is a graded prime ideal, we have either $z \in J$ or $(\alpha z + x_1 - \sum_{j=1}^{t} \gamma_j x_j) \in J$. By the linear independence assumption on the x_i, this is only possible if $(\alpha z + x_1 - \sum_{j=1}^{t} \gamma_j x_j) = 0$. This implies $\alpha = 0$ and $\gamma_1 = 1$ and $\gamma_j = 0$ for $j \geqslant 2$. This implies $F_1 = F_k$ as required.

Suppose now that $F_k \in S_1$. We then have $F_k \in (F_1, \dots, F_t) + J$, and therefore $x_1 + \alpha z \in J$, which contradicts the linear independence assumption.

We are left with the case when $Q_1 = 0$. After rearranging the forms, let $Q_2, \dots, Q_t$ be such that $\operatorname{span}_{\mathbb{C}}\{Q_1, \dots, Q_{k-1}\} = \operatorname{span}_{\mathbb{C}}\{Q_2, \dots, Q_t\}$, and let y_i be defined as in the previous case. Note that $y_1 = x_1$. Let $J := (y_1, y_{t+1}, \dots, y_{k+1})$. The ideal $(F_2, \dots, F_t) + J$ is a prime ideal containing $\operatorname{rad}(F_1, \dots, F_{k-1})$. As before suppose $x_k = x_1 + \alpha z$.

Suppose $F_k \in S_2$ so $Q_k = \sum_{j=2}^{t} \gamma_i Q_i$. We have $F_k - \sum_{j=2}^{t} \gamma_i F_i = z(x_1 + \alpha z - \sum_{j=2}^{t} \gamma_i x_i \in (F_2, \dots, F_t) + J$. Therefore, either $z \in J$, or $x_1 + \alpha z - \sum_{j=2}^{t} \gamma_i x_i \in J$. By the linear independence assumption, this implies $\alpha_i = 0$ for $i = 2, \dots, t$ contradicting the fact that $F_k \in S_2$.

Suppose $F_k \in S_1$. In this case we have $F_1 - F_k = \alpha z \in (F_2, \dots, F_t) + J$, which implies $\alpha z \in J$. By the independence assuption, we must have $\alpha = 0$, whence $F_1 = F_k$ as required. □

5.2 Relative linear spaces

Now that we have proved some properties of wide vector spaces, we introduce the notion of relative linear spaces and establish some properties which will be useful to us in the next section. This notion of relative linear spaces was used in [GOS22] in their proof of the robust SG theorem for quadratics.

Definition 5.10 (Forms close to a vector space)**.** Given a vector space $V = V_1 + V_2$ where $V_i \subseteq S_i$, we say that a quadratic form P is s-*close* to V if there is a form $Q \in \mathbb{C}[V]$ such that $\operatorname{rank}(P - Q) \leqslant s$. If a form P is not r-close to V, for any $r \leqslant s$, we say that P is s-*far* from V.

Given a linear form ℓ, we say ℓ is 1-close to V if $\ell \notin V_1$.

Remark 5.11. Given a set of forms $\mathcal{F}$, we will say that $\mathcal{F}$ is s-close to V if all forms in $\mathcal{F}$ are at most s-close to V.

Proposition 5.12 (Quadratics close to wide vector spaces)**.** *Let* $V = V_1 + V_2$ *be an* r*-wide vector space and* $s < r/2$*. If* P *is* s*-close to* V*, then for any* $Q, Q' \in \mathbb{C}[V]$ *such that* $\operatorname{rank}(P - Q) = \operatorname{rank}(P - Q') = s$*, we have that*

$$\operatorname{Lin}(P - Q) + V_1 = \operatorname{Lin}\left(P - Q'\right) + V_1.$$

In other words, $(\operatorname{Lin}(P - Q) + V_1)/V_1 = (\operatorname{Lin}(P - Q') + V_1)/V_1$ *for any two decompositions.*

Proof. Let $R = P - Q$ and $R' = P - Q'$. Thus, we have that $R - R' = Q' - Q \in \mathbb{C}[V]$ and we have $\operatorname{rank}(Q' - Q) = \operatorname{rank}(R - R') \leqslant \operatorname{rank}(R) + \operatorname{rank}(R') \leqslant 2s < r$. Hence, by Remark 5.5, we have that $Q' - Q \in \mathbb{C}[V_1]$. Now, from $R = R' + (Q' - Q)$ and $Q' - Q \in \mathbb{C}[V_1]$, we have that $\operatorname{Lin}(R) \subseteq \operatorname{Lin}(R') + V_1$, and similarly, we have that $\operatorname{Lin}(R') \subseteq \operatorname{Lin}(R) + V_1$. ☐

Definition 5.13 (Relative space of linear forms)**.** Let r, B be integers such that $r > 2B + 1$. If V is an r-wide vector space and P is s-close to V for $s < r/2$ we can define

$$\mathbb{L}_V(P) := \begin{cases} \mathbb{L}(P) + V_1, \text{ if } P \in S_1 \\ \operatorname{Lin}(P - Q) + V_1, \text{ if } P \in S_2, s \leqslant B \\ \operatorname{span}_{\mathbb{C}}\{P\}, \text{ otherwise} \end{cases}$$

where $Q \in \mathbb{C}[V]$ is a form such that $\operatorname{rank}(P - Q) = s$. We also define the quotient space

$$\overline{\mathbb{L}}_V(P) := \begin{cases} \mathbb{L}_V(P)/V_1, \text{ if } s \leqslant B \\ 0, \text{ otherwise} \end{cases}$$

Further, we define P_V^H to be the unique polynomial in V_2 such that $P - P_V^H$ is s-close to V_1. Finally we define $P_V^L = P - P_V^H$. Note that $\mathbb{L}_V(P) = \mathbb{L}_V\left(P_V^L\right)$. The superscript H indicates that P_V^H is the high-rank part of P with respect to V and the superscript L indicates that P_V^L is the low-rank part of P with respect to V.

Note that while the definition of $\mathbb{L}_V(P)$ depends on the parameter B, we suppress this from the notation for brevity. It will be clear from context the value of the parameter B whenever we use $\mathbb{L}_V(P)$.

Here are some useful results about relative linear spaces, and how they change when V is modified. Lemma 5.16 characterises exactly when $\dim \overline{\mathbb{L}}_V(F)$ is unchanged when $\mathbb{L}_V(G)$ is added to V_1. As the lemma shows, this happens when F and G do not share any common variables other than those that occur in V.

Proposition 5.14. *Suppose* V *is a* r*-wide space and* P *is* s*-close to* V *for* $2s < r$*. If* $P \in (V)$ *then* $P_V^H \in V_2$ *and* $P_V^L \in (V_1)$*.*

Proof. Since P is s-close to V we can write $P = P_V^H + P_V^L$. Since $P_V^H \in V_2$, we have $P_V^L \in (V)$ by assumption. We can write $P_V^L = P_2 + P_1$ with $P_1 \in (V_1)$ and $P_2 \in V_2$. In $S/(V_1)$ we have $P_2 = 0$. Since V is r-wide, this implies $P_2 = 0$ in S. Therefore $P_V^L \in (V_1)$. ☐

Proposition 5.15. *Suppose $V = V_1 + V_2$ is a r-wide vector space with $r > 2B + 1$, and suppose $P \in S_2$ is B-close to V. Then $Y := \mathbb{L}_V(P) + V_2$ is a $r - 2B$ wide vector space. If further $r > 4B + 1$ then for any other polynomial Q that is also B close to V we have $Q^H_V = Q^H_Y$.*

Proof. The first statement follows since Y is obtained by adding at most $2B$ linear forms to a basis of V. We now have $Q = Q^H_V + Q^L_V = Q^H_Y + Q^L_Y$ whence $Q^H_V - Q^H_Y = Q^L_Y - Q^L_V$. Here, we use the fact that $B < 4r + 1$ to ensure that Q^H_Y, Q^L_Y are well defined. Since both Q^L_Y, Q^L_V have rank at most B in $S/(Y)$ we obtain that $Q^H_V = Q^H_Y$. □

Lemma 5.16. *Suppose $V = V_1 + V_2$ is a r-wide vector space with $r > 4B + 1$, and suppose $F, G \in S_2$ are both B close to V. Let $Y := \mathbb{L}_V(G) + V_2$. Then the following hold.*

1. *$\mathbb{L}_Y(F) = \mathbb{L}_V(G) + \mathbb{L}_V(F)$.*
2. *$\dim \overline{\mathbb{L}}_V(F) = \dim \overline{\mathbb{L}}_Y(F)$ if and only if $\overline{\mathbb{L}}_V(F) \cap \overline{\mathbb{L}}_V(G) = \{0\}$.*
3. *If $F \notin (V)$ and $\dim \overline{\mathbb{L}}_V(F) = \dim \overline{\mathbb{L}}_Y(F)$ then $F \notin (Y)$.*

Proof. By Proposition 5.15 we have $H := F^H_V = F^H_Y$. Let P, R be such that $F - H - P = R$ with $P \in \mathbb{C}[V_1]$ and $\mathbb{L}_V(F) = \mathrm{Lin}(R) + V_1$. Let P', R' be such that $F - H - P' = R'$ with $P' \in \mathbb{C}[Y_1]$ and $\mathbb{L}_Y(F) = \mathrm{Lin}(R') + Y_1$. We have the equation $R' + P' = R + P$, which implies that $\mathrm{Lin}(R') + Y_1 = \mathrm{Lin}(R) + Y_1$. Since $V_1 \subset Y_1$, we have

$$\mathrm{Lin}\left(R'\right) + Y_1 = \mathrm{Lin}(R) + V_1 + Y_1. \tag{1}$$

Substituting $\mathbb{L}_V(F), \mathbb{L}_Y(F)$ in Eq. (1) and using the fact that $Y_1 = \mathbb{L}_V(G)$ we get $\mathbb{L}_Y(F) = \mathbb{L}_V(G) + \mathbb{L}_V(F)$.

Eq. (1) also implies

$$\overline{\mathbb{L}}_Y(F) = \frac{\mathrm{Lin}(R') + Y_1}{Y_1} = \frac{\mathbb{L}_V(F) + Y_1}{Y_1} = \frac{\mathbb{L}_V(F)}{\mathbb{L}_V(F) \cap Y_1}$$

therefore

$$\begin{aligned}
\dim \overline{\mathbb{L}}_V(F) = \dim \overline{\mathbb{L}}_Y(F) &\iff \dim V_1 = \dim\left(\mathbb{L}_V(F) \cap Y_1\right) \\
&\iff V_1 = \mathbb{L}_V(F) \cap Y_1 && (\text{since } V_1 \subseteq \mathbb{L}_V(F) \cap Y_1) \\
&\iff \{0\} = \overline{\mathbb{L}}_V(F) \cap \overline{\mathbb{L}}_V(G),
\end{aligned}$$

proving the second item.

Assume now that $F \in (Y)$. By Proposition 5.14 we have $F - H \in (Y_1)$. Further, by assumption we have $F - H \notin (V_1)$. In $S/(V_1)$ we have $0 \neq \overline{F - H} = \overline{R} \in (Y_1)$ which in turn implies that $\mathrm{Lin}\left(\overline{R}\right) \cap Y_1 \neq \{0\}$ by Lemma 2.13. We have $R = \sum a_i b_i$ for linear forms $a_1, \ldots, a_t, b_1, \ldots, b_t$ where $\overline{a_i}, \overline{b_j}$ span $\overline{\mathbb{L}}_V(F)$. Therefore $\mathrm{Lin}\left(\overline{R}\right) = \sum \overline{a_i}\,\overline{b_i}$, whence $\mathrm{Lin}\left(\overline{R}\right) \subseteq \overline{\mathbb{L}}_V(F)$. This shows that $\overline{\mathbb{L}}_V(F) \cap \overline{\mathbb{L}}_V(G) \neq 0$, which by item 2 implies $\dim \overline{\mathbb{L}}_V(F) \neq \dim \overline{\mathbb{L}}_Y(F)$, contradicting the assumption. □

Note that the condition $\overline{\mathbb{L}}_V(F) \cap \overline{\mathbb{L}}_V(G) = \{0\}$ is symmetric in F and G. Therefore, we have that $\dim \overline{\mathbb{L}}_V(F) = \dim \overline{\mathbb{L}}_{\mathbb{L}_V(G)+V_2}(F)$ if and only if $\dim \overline{\mathbb{L}}_V(G) = \dim \overline{\mathbb{L}}_{\mathbb{L}_V(F)+V_2}(G)$. Further, in this case we have $F \notin (\mathbb{L}_V(G), V_2)$ and also $G \notin (\mathbb{L}_V(F), V_2)$ if $F, G \notin (V)$. In the next subsection, we introduces the notion of integral sequences that generalises the above.

6 Integral sequences and strong sequences

In this section we define two special types of sequences of forms, namely integral sequences and strong sequences. We will use the strong sequences to construct our core algebra, that is, to prove that there is a small algebra such that all quadratics are close to it. We will then use integral sequence to handle the case where all the quadratics are close to a core algebra. We will prove that the ideals generated by integral and strong sequences are always radical and prime, respectively.

6.1 Integral sequences

Item 2 of Lemma 5.16 gives us a condition for when the relative linear spaces of two linear forms are disjointed. Intuitively, this is equivalent to the forms depending on disjoint sets of variables, other than those occurring in V. This is made formal in Corollary 6.4. The notion of integral sequences extends this to more that two forms. As in Lemma 5.16, we will require the forms to be close to a wide vector space for the notion to be well defined.

Definition 6.1 (Integral Sequences)**.** Let r, B, t be integers with $r > 4tB+1$. Suppose $V = V_1 + V_2$ is a r-wide vector space. Let $F_1, \dots, F_t \in \mathcal{F}$ be a sequence of forms that are B-close to V. Let $U_0 := V$ and let $U_i := \mathbb{L}_{U_{i-1}}(F_i) + V_2$. The sequence $F_1, \dots, F_t$ is called an integral sequence over V if for each i we have

- $\dim \overline{\mathbb{L}}_V(F_i) = \dim \overline{\mathbb{L}}_{U_{i-1}}(F_i)$, and
- $F_i \notin (V)$

When V is clear from context we just call $F_1, \dots, F_t$ an integral sequence.

In the rest of this section, we will assume that $r > 4tB + 1$.

Proposition 6.2. *Suppose V is a r-wide vector space. Suppose $F_1, \dots, F_t$ are a sequence of forms, and suppose $U_i := \mathbb{L}_{U_{i-1}}(F_i) + V_2$ with $U_0 := V$. Then*

1. $U_t = \sum_{j=1}^{t} \mathbb{L}_V(F_j) + V_2$.
2. $\dim \overline{\mathbb{L}}_V(F_i) = \dim \overline{\mathbb{L}}_{U_{i-1}}(F_i)$ *for every* $2 \leqslant i \leqslant t$ *if and only if for every* $2 \leqslant i \leqslant t$ *we have*

$$\overline{\mathbb{L}}_V(F_i) \cap \left(\sum_{j=1}^{i-1} \overline{\mathbb{L}}_V(F_j) \right) = \{0\}.$$

3. *If additionally* $F_i \notin (V)$ *for every* $1 \leqslant i \leqslant t$*, then* $F_i \notin (U_{i-1})$ *for* $2 \leqslant i \leqslant t$.

Proof. We prove the statements by induction on t. We will prove the additional statement that $\mathbb{L}_{U_{t-1}}(F_t) = \sum_{i=1}^{t} \mathbb{L}_V(F_t)$. Each of the three items are true by definition when $t = 1$. Suppose the statements are true for $t - 1$.

Now the space U_{t-2} is $4B + 1$ wide by Remark 5.6. Applying Lemma 5.16 to U_{t-2}, F_t, and F_{t-1} we can deduce that $\mathbb{L}_{U_{t-1}}(F_t) = \mathbb{L}_{U_{t-2}}(F_t) + \mathbb{L}_{U_{t-2}}(F_{t-1})$. The space U_{t-3} is also $4B + 1$ wide, therefore applying Lemma 5.16 to U_{t-3}, F_t, and F_{t-2} we can deduce that $\mathbb{L}_{U_{t-2}}(F_t) = \mathbb{L}_{U_{t-3}}(F_t) + \mathbb{L}_{U_{t-3}}(F_{t-2})$. Repeating this and substituting, we deduce that $\mathbb{L}_{U_{t-1}}(F_t) = \mathbb{L}_V(F_t) + \sum_{i=2}^{t} \mathbb{L}_{U_{t-i}}(F_{t-i+1})$. By the induction hypothesis, we get that $\mathbb{L}_{U_{t-i}}(F_{t-i+1}) = \sum_{j=1}^{t-i} \mathbb{L}_V(F_j)$. Therefore we get $\mathbb{L}_{U_{t-1}}(F_t) = \sum_{i=1}^{t} \mathbb{L}_V(F_i)$. The first item now follows by adding V_2 to both sides.

Suppose now that $\dim \overline{\mathbb{L}}_V(F_i) = \dim \overline{\mathbb{L}}_{U_{i-1}}(F_i)$ for every $2 \leqslant i \leqslant t-1$. Suppose $\dim \overline{\mathbb{L}}_V(F_t) = \dim \overline{\mathbb{L}}_{U_{t-1}}(F_t)$. This implies $\dim \overline{\mathbb{L}}_{U_{t-1}}(F_t) = \dim \overline{\mathbb{L}}_{U_{t-2}}(F_t)$, since $V \subset U_{t-2} \subset U_{t-1}$. By item 2 of Lemma 5.16 applied to U_{t-2}, F_{t-1}, F_t we can deduce that $\overline{\mathbb{L}}_{U_{t-2}}(F_t) \cap \overline{\mathbb{L}}_{U_{t-2}}(F_{t-1}) = \{0\}$. Using the fact that $\mathbb{L}_{U_{t-1}}(F_t) = \sum_{i=1}^{t} \mathbb{L}_V(F_i)$, this is equivalent to $\overline{\mathbb{L}}_V(F_i) \cap \left(\sum_{j=1}^{i-1} \overline{\mathbb{L}}_V(F_j)\right) = \{0\}$. Conversely, starting with this assumption we can deduce that $\dim \overline{\mathbb{L}}_{U_{t-1}}(F_t) = \dim \overline{\mathbb{L}}_{U_{t-2}}(F_t)$. Note that $F_1, \ldots, F_{t-2}, F_t$ also satisfy the conditions of item 2. Therefore, by induction we can deduce that $\dim \overline{\mathbb{L}}_{U_{t-2}}(F_t) = \dim \overline{\mathbb{L}}_V(F_t)$. This completes the proof of item 2.

Applying the induction hypothesis to $F_1, \ldots, F_{t-2}, F_t$, we can deduce that $F_t \notin (U_{t-2})$. We can now apply Lemma 5.16 to U_{t-2}, F_t and F_{t-1} to deduce that $F_t \notin (U_{t-1})$, proving item 3. □

Corollary 6.3. *If $F_1, \ldots, F_t$ is a integral sequence, then so is any permutation of $F_1, \ldots, F_t$.*

Proof. The second condition for integral sequences holds irrespective of the order of the forms. By Proposition 6.2, the first condition for integral sequences is equivalent to

$$\overline{\mathbb{L}}_V(F_i) \cap \left(\sum_{j=1}^{i-1} \overline{\mathbb{L}}_V(F_j)\right) = \{0\}$$

for every $2 \leqslant i \leqslant t$. This in turn is equivalent to $\dim \sum_{j=1}^{t} \overline{\mathbb{L}}_V(F_j) = \sum_{j=1}^{t} \dim \overline{\mathbb{L}}_V(F_j)$, which is independent of the order of the forms. □

Corollary 6.4. *Let $F_1, \ldots, F_t$ be an integral sequence with respect to V and $\mathcal{A} := \mathbb{C}\left[V_2, \sum_{i=1}^{t} \mathbb{L}_V(F_i)\right]$. There exist vector spaces of linear forms $Y_1, \ldots, Y_t \subset \mathcal{A}$ such that $Y_i \cap (V + Y_1 + \cdots + Y_{i-1}) = (0)$ for all i and $F_i \in \mathbb{C}[V, Y_i]$. Furthermore, $F_i \notin (V, Y_1, \cdots, Y_{i-1}, Y_{i+1}, \cdots, Y_t)$.*

Proof. By Proposition 6.2 we can take $Y_j := \overline{\mathbb{L}}_V(F_j)$. By Corollary 6.3, we may switch F_i and F_t. Then by Proposition 6.2 part (3), we see that $F_i \notin (V, Y_1, \cdots, Y_{i-1}, Y_{i+1}, \cdots, Y_t)$. □

Lemma 6.5. *Suppose V is a r-wide vector space and $F_1, \ldots, F_t$ is an integral sequence with respect to V. Suppose $F_0 \in \mathbb{C}[V] \setminus \{0\}$. Then $F_0, F_1, \ldots, F_t$ is a regular sequence in S.*

Proof. Note that by Corollary 6.4 we may assume that there exist vector spaces of linear forms $Y_1, \ldots, Y_t$ of $\mathcal{A}$ such that $Y_i \cap (V + Y_1 + \cdots + Y_{i-1}) = (0)$ and $F_i \in \mathbb{C}[V, Y_i]$. Let $U = V + Y_1 + \cdots + Y_t$ and $\mathcal{A} = \mathbb{C}[U]$. Since V is r-wide and $r > 4Bt + 1$, we know that U is $2Bt + 1$-wide, and hence has a basis consisting of a prime sequence. Thus $\mathcal{A} \to S$ is a free extension (see [AH20, Section 2]) and hence any regular sequence in $\mathcal{A}$ is also a regular sequence in S (see [Sta15, Tag 00LM]). Therefore it is enough to prove that $F_0, F_1, \cdots, F_t$ is a regular sequence in $\mathcal{A} = \mathbb{C}[U]$.

Note that the element F_0 is a regular sequence in $\mathcal{A} = \mathbb{C}[V + Y_1 + \cdots + Y_t]$. We will prove by induction that if $F_0, \cdots, F_i$ is a regular sequence in $\mathcal{A}$, then so is $F_0, \cdots, F_{i+1}$. Suppose $F_0, \cdots, F_i$ is a regular sequence in $\mathcal{A}$ (and hence in $\mathcal{A}_i = \mathbb{C}[V + Y_1 + \cdots + Y_i]$). If F_{i+1} is a zero divisor in $\mathcal{A}/(F_0, \cdots, F_i)$, then F_{i+1} is in a minimal prime $\mathfrak{p}$ of $(F_0, \cdots, F_i)$ in $\mathcal{A}$. Since $\mathcal{A}_i \to \mathcal{A}$ is a free extension and $\mathcal{A}_i$ is generated by a prime sequence in S, we must have that $\mathfrak{p} = \mathfrak{q} \cdot \mathcal{A}$ for some minimal prime $(F_0, \cdots, F_i) \subset \mathfrak{q}$ in $\mathcal{A}_i$. Note that by Proposition 6.2 we know that $F_{i+1} \notin (V + Y_1 + \cdots + Y_i)$. This is a contradiction since $F_{i+1} \in \mathfrak{q} \cdot \mathcal{A}$ and $\mathfrak{q} \subset (V + Y_1 + \cdots + Y_i)$. □

Lemma 6.6. *Suppose V is a r-wide vector space and $F_1, \ldots, F_t$ is an integral sequence with respect to V. Then $(F_1, \ldots, F_t)$ is radical and for any minimal prime $\mathfrak{p} \supset (F_1, \ldots, F_t)$ we have that $\mathfrak{p} \cap \mathbb{C}[V] = (0)$.*

Proof. Note that by Corollary 6.4 we may assume that there exist vector spaces of linear forms $Y_1, \dots, Y_t$ of $\mathcal{A}$ such that $Y_i \cap (V + Y_1 + \dots + Y_{i-1}) = (0)$ and $F_i \in \mathbb{C}[V, Y_i]$. By Lemma 6.5, we know that $F_1, \cdots, F_t$ is a regular sequence. Hence $\mathrm{ht}(\mathfrak{p}) = t$ for any minimal prime $\mathfrak{p} \supset (F_1, \dots, F_t)$. Let F_0 be a non-zero element in $\mathbb{C}[V]$. Then $F_0, \cdots, F_t$ is again a regular sequence and hence $\mathrm{ht}(F_0, \cdots, F_t) = t + 1$. This implies $F_0 \notin \mathfrak{p}$, as $\mathrm{ht}(\mathfrak{p}) = t$ implies that $\mathfrak{p}$ contains no regular sequence of length $t + 1$. Therefore we must have that $\mathfrak{p} \cap \mathbb{C}[V] = (0)$.

Now we will show that $(F_1, \cdots, F_t)$ is a radical ideal in S. Let $\mathcal{A} = \mathbb{C}[V + Y_1 + \dots + Y_t]$. Since $\mathcal{A} \to S$ is a free extension and the generators of $\mathcal{A}$ form a prime sequence in S, it is enough to prove that $(F_1, \cdots, F_t)$ is radical in $\mathcal{A}$.

For each i, we assume that F_i is monic in $y_i \in Y_i$ after a possible change of coordinates in Y_i. There exists such a variable since $F_i \notin (V)$. Let $U_i := Y_i / \mathrm{span}_{\mathbb{C}}\{y_i\}$ and $Z = V + U_1 + \dots + U_t$. Then $\mathcal{A} = \mathbb{C}[Z, y_1, \cdots, y_t]$. We will show by induction that $(F_1, \cdots, F_t)$ is radical.

Note that (F_1) is prime. Assume the statement holds for $i - 1$. We have $\mathrm{disc}_{y_i}(F_i) \in \mathbb{C}[V, U_i]$. Note that $\mathfrak{p} \cap \mathbb{C}[V, U_i] = (0)$ for every minimal prime $\mathfrak{p}$ of $(F_1, \dots, F_{i-1})$, as $F_{i-1} \notin (V, Y_1, \cdots, Y_{i-2}, Y_i)$ by Corollary 6.4. Therefore Lemma 4.5 implies $(F_1, \dots, F_i)$ is radical. □

Corollary 6.7. *Suppose $F_1, \dots, F_t$ is an integral sequence with respect to V. Then $F_1, \dots, F_t$ form a t-relevant set.*

Proof. The sequence $F_1, \dots, F_{t-1}$ is an integral sequence, and therefore by Lemma 6.5 it is a regular sequence. Since any regular sequence is a relevant set, we are done. □

6.2 Strong sequences

Integral sequences are only defined when the forms are close to a wide vector space. One special case is when every form is of low rank, and therefore every form is close to the vector space $\{0\}$. To deal with forms that are not close to a vector space (which is the general case), we introduce the notion of strong sequences.

We first extend the notion of the rank of a quadratic form to vector spaces of quadratic forms.

Definition 6.8. Let $V_2 \subset S_2$ be a vector space. Define $\mathrm{minrank}(V)$ as $\min_{Q \in V_2, Q \neq 0} \mathrm{rank}\, Q$. If $Q_1, \dots, Q_t$ are quadratic forms then define $\mathrm{minrank}(Q_1, \dots, Q_t) = \mathrm{minrank}\left(\mathrm{span}_{\mathbb{C}}\{Q_1, \dots, Q_t\}\right)$.

Definition 6.9. Let $k, t \in \mathbb{N}$ be such that $t \leqslant k + 1$. Given forms $Q_1, \dots, Q_t \in S_2$ we say that $Q_1, \dots, Q_t$ is a k-strong sequence if $Q_1, \dots, Q_t$ are linearly independent and $\mathrm{minrank}(Q_1, \dots, Q_t) \geqslant k + 5$.

Remark 6.10. By Theorem 5.3, if $Q_1, \dots, Q_t$ is k-strong then $Q_1, \dots, Q_t$ is a R_3 sequence. By the discussion in [AH20, Discussion 1.3], the ideal $(Q_1, \dots, Q_t)$ is prime and the ring $S/(Q_1, \dots, Q_t)$ is a UFD.

Lemma 6.11. *Suppose $\mathcal{F}_2 \subset S_2$, and suppose $Q_1, \dots, Q_t$ is a maximal* k-strong *sequence in $\mathcal{F}_2$ with $t \leqslant k$. For any $r \geqslant 2(k + 5)$ there exists a r-wide vector space W with $\dim W_1 \leqslant 7 \cdot r \cdot 3^t, \dim W_2 \leqslant t$ such that every $Q \in \mathcal{F}_2$ is $k + 4$-close to W.*

Proof. Let $U := \mathrm{span}_{\mathbb{C}}\{Q_1, \dots, Q_t\}$. By Proposition 5.2, there exists r-wide vector space W such that $U \subset \mathbb{C}[W]$, $\dim W_1 \leqslant 3^{t+1} \cdot (r + t)$ and $\dim W_2 \leqslant t$. Let $Q \in \mathcal{F}_2$ be a form. Consider the sequence $Q_1, \dots, Q_t, Q$, which has length at most $k + 1$. By assumption, $Q_1, \dots, Q_t, Q$ is not a k-strong sequence. Therefore, we have either $\mathrm{minrank}(Q_1, \dots, Q_t, Q) \leqslant k + 4$ or $Q \in \mathrm{span}_{\mathbb{C}}\{Q_1, \dots, Q_t\}$.

Suppose $P = \beta Q + \sum \alpha_i Q_i$ is such that $\mathrm{rank}\, P = \mathrm{minrank}(Q_1, \dots, Q_t, Q) \leqslant k + 4$. Since $Q_1, \dots, Q_t$ is k-strong we have $\beta \neq 0$. Therefore after scalar multiple we have $Q = \sum \alpha_i Q_i + P$, and Q is $k+4$-close to W. If $Q \in \operatorname{span}_{\mathbb{C}}\{Q_1, \ldots, Q_t\}$ then $Q \in W$ and therefore Q is $k+4$-close to W. $\square$

We now define the notion of strong Sylvester-Gallai configurations. We show that a constant fraction of the forms in any such configuration is close to a vector space of constant dimension. [4]

Definition 6.12. Let $\mathcal{F}_2 \subset S_2$ be a finite set of forms. Let $0 < \epsilon \leqslant 1$ and $k, t \in \mathbb{N}$ with $t \leqslant k$. We say that $\mathcal{F}_2$ is a strong $(\epsilon, k) - \mathrm{SG}^*_t(2)$ configuration if for every k-strong sequence $Q_1, \ldots, Q_t$ with $Q_i \in \mathcal{F}_2$, there are $\epsilon(|\mathcal{F}_2| - 1)$ forms $Q_{t+1} \in \mathcal{F}_2$ such that either:

1. $Q_1, \ldots, Q_t, Q_{t+1}$ is not a k-strong sequence, or
2. there is a form $R \in \mathcal{F}_2$ such that $R \in (Q_1, \ldots, Q_{t+1}) \setminus (Q_1, \ldots, Q_t) \cup (Q_{t+1})$.

Lemma 6.13. *Let $\mathcal{F}_2 \subset S_2$ finite, with $m := |\mathcal{F}_2|$. Let $0 < \epsilon \leqslant 1$ and $k, t \in \mathbb{N}$ with $2 \leqslant t \leqslant k$. If $\mathcal{F}_2$ is a* strong $(\epsilon, k) - \mathrm{SG}^*_t(2)$ *configuration then either*

1. *$\mathcal{F}_2$ is a* strong $(\epsilon/4, k) - \mathrm{SG}^*_{t-1}(2)$ *configuration, or*
2. *there exist a vector space W with $\dim W_1 \leqslant 7 \cdot r \cdot 3^{t+1+16/\epsilon}, \dim W_2 \leqslant t + 1 + 16/\epsilon$ such that at least $\epsilon m/4$ forms in $\mathcal{F}_2$ are $k+4$ close to W.*

Proof. Let $\epsilon' := \epsilon/4$. Suppose $\mathcal{F}_2$ is not a strong $(\epsilon', k) - \mathrm{SG}^*_{t-1}(2)$ configuration. If there exist no k-strong sequences of length $t-1$, then there exists some maximal k-strong sequence of length at most $t-2$, and the required space W exists by Lemma 6.11. We can therefore assume that there exists a k-strong sequence $Q_1, \ldots, Q_{t-1}$, and a set $\mathcal{B} \subset \mathcal{F}_2$ of size at least $(1-\epsilon')m$ such that for every $Q \in \mathcal{B}$ we have that $Q_1, \ldots, Q_{t-1}, Q$ is a k-strong sequence, and

$$\mathcal{F}_2 \cap (Q_1, \ldots, Q_{t-1}, Q) \setminus (Q_1, \ldots, Q_{t-1}) = \{Q\}. \tag{2}$$

Let $V := \operatorname{span}_{\mathbb{C}}\{Q_1, \ldots, Q_{t-1}\}$. Forms $P_1, P_2 \in \mathcal{B}$ are pairwise independent over S_2/V, since if $(P_1) = (P_2)$ in S_2/V, then $P_2 \in (Q_1, \ldots, Q_{t-1}, P_1) \setminus (Q_1, \ldots, Q_{t-1}) \cup (P_1)$, contradicting $P_1 \in \mathcal{B}$.

Let $P \in \mathcal{B}$. The sequence $Q_1, \ldots, Q_{t-1}, P$ is k-strong by definition of $\mathcal{B}$. Since $\mathcal{F}_2$ is a strong $(\epsilon, k) - \mathrm{SG}^*_t(2)$ configuration, there are $P_1, \ldots, P_s \in \mathcal{F}_2$ with $s \geqslant \epsilon m$ such that either $Q_1, \ldots, Q_{t-1}, P, P_i$ is not k-strong, or there is $R_i \in \mathcal{F}_2$ such that $R_i \in (Q_1, \ldots, Q_{t-1}, P, P_i) \setminus (Q_1, \ldots, Q_{t-1}, P) \cup (P_i)$.

Let $\mathcal{G} := \{P_i \mid Q_1, \ldots, Q_{t-1}, P, P_i \text{ is not a k-strong sequence}\}$. Let W be the r-wide vector space obtained by applying Proposition 5.2 to $V + \operatorname{span}_{\mathbb{C}}\{P\}$, we have $\dim W_1 \leqslant 7 \cdot r \cdot 3^t, \dim W_2 \leqslant t$. Every form in $\mathcal{G}$ is $k+4$-close to W. Hence, if $|\mathcal{G}| \geqslant \epsilon' m$ then we are done.

We are left with the case that $|\mathcal{G}| \leqslant \epsilon' m$. After relabelling, let $P_1, \ldots, P_{s'}$ be the forms that are in $\mathcal{B} \setminus \mathcal{G}$. Since $|\mathcal{B}| \geqslant (1-\epsilon')m$ and $|\mathcal{G}| \leqslant \epsilon' m$ we have $s' \geqslant (\epsilon - 2\epsilon')m$.

Now for each $i \leqslant s'$, there is a form $R_i \in \mathcal{F}_2 \cap (Q_1, \ldots, Q_{t-1}, P, P_i) \setminus (Q_1, \ldots, Q_{t-1}, P) \cup (P_i)$ say $R_i = \sum \alpha_j Q_j + \beta P + P_i$. Since $P_i \in \mathcal{B}$ we have $\beta \neq 0$. Suppose $P_1, \ldots, P_{s''}$ are such that

$$\mathcal{B} \cap ((Q_1, \ldots, Q_{t-1}, P, P_i) \setminus (Q_1, \ldots, Q_{t-1}, P)) = \{P_i\}. \tag{3}$$

If $R_i = \alpha R_j$ with $\alpha \neq 0$ for $i, j \leqslant s''$, then we have $\alpha P_j = \sum \alpha'_i Q_i + P_i + \beta' P$, contradicting Eq. (3) for P_i. Therefore we have $s'' \leqslant |\mathcal{F}_2 \setminus \mathcal{B}| \leqslant \epsilon' m$. Hence, there are at least $\epsilon' m$ forms P_i such that $\left|\operatorname{span}_{\mathbb{C}}\{P, P_i\} \cap \mathcal{B}\right| \geqslant 3$ in S_2/V. Since this holds for every $P \in \mathcal{B}$, the set $\mathcal{B}$ is a $(\epsilon', 2)$-linear-SG

[4] As we mentioned in Section 1, we need this notion of strong SG configurations since in our setting we cannot quotient by quadratic forms, as the quotient ring will not be a polynomial ring and the previous results on SG configurations may not apply. In particular, this is where our approach is more complex than [BDYW11], as in their case their quotients were all isomorphic to polynomial rings.

configuration in S_2/V. By Theorem 3.6 we have that $\dim \operatorname{span}_{\mathbb{C}}\{\mathcal{B}\} \leqslant 4/\epsilon'$ in S_2/V and that $\dim \operatorname{span}_{\mathbb{C}}\{\mathcal{B}\} + V \leqslant t+1+4/\epsilon'$. Applying Proposition 5.2 to $\operatorname{span}_{\mathbb{C}}\{\mathcal{B}\}+V$ gives us a r-wide vector space W with $\dim W_1 \leqslant 7 \cdot r \cdot 3^{t+1+4/\epsilon'}, \dim W_2 \leqslant t+1+4/\epsilon'$ and $\mathcal{B} \subset W$. ☐

Lemma 6.14. *Let $\mathcal{F}_2 \subset S_2$ finite, with $m := |\mathcal{F}_2|$. Suppose $\mathcal{F}_2$ is a* strong $(\epsilon, k) - SG_1^*(2)$ *configuration. Then there is a r-wide vector space W with $\dim W_1 \leqslant 7 \cdot r \cdot 3^{2+16/\epsilon}, \dim W_2 \leqslant 2+16/\epsilon$ such that at least $\epsilon m/4$ forms in $\mathcal{F}_2$ are $k+5$ close to W.*

Proof. Let $\epsilon' := \epsilon/4$. Let $\mathcal{B}$ be the set of forms in $\mathcal{F}_2$ of rank at least $k+5$. If $|\mathcal{B}| \leqslant (1-\epsilon')m$, then there are at least $\epsilon' m$ forms that are $k+5$ close to the zero vector space and we are done with $W=0$. We are left with the case when $|\mathcal{B}| \geqslant (1-\epsilon')m$.

Let $P \in \mathcal{B}$. Let $\mathcal{G} := \{P_i \mid P, P_i \text{ is not a k-strong sequence}\}$. Let W be the r-wide vector space obtained by applying Proposition 5.2 to $\operatorname{span}_{\mathbb{C}}\{P\}$, we have $\dim W_1 \leqslant 21 \cdot r, \dim W_2 \leqslant 1$. Every form in $\mathcal{G}$ is $k+4$ close to W. If therefore $|\mathcal{G}| \geqslant \epsilon' m$ then we are done. We are left with the case that $|\mathcal{G}| \leqslant \epsilon' m$.

Suppose $P_1, \ldots, P_{r'}$ are the forms in $\mathcal{B} \setminus \mathcal{G}$ such that P, P_i is a k-strong sequence and there exist $R_i \in (P, P_i) \setminus (P) \cup (P_i)$. We have $r' \geqslant 2\epsilon' m$. Suppose $P_1, \ldots, P_{r''}$ are such that $(P, P_i) \cap \mathcal{B} = \{P, P_i\}$. If $R_i = \beta R_j$ for $i, j \leqslant r''$ then $P_j \in \operatorname{span}_{\mathbb{C}}\{P, P_i\}$, contradicting choice of P_i. Therefore there are at least $\epsilon' m$ many forms P_i such that $|(P, P_i) \cap \mathcal{B}| \geqslant 3$. Since this holds for every P, we have that $\mathcal{B}$ is a $(\epsilon', 2)$-linear-SG, and by Theorem 3.6 we have that $\dim \operatorname{span}_{\mathbb{C}}\{\mathcal{B}\} \leqslant 4/\epsilon'$. If W is the r-wide space obtained by applying Proposition 5.2 to $\mathcal{B}$ then $\dim W_1 \leqslant 7 \cdot r \cdot 3^{4/\epsilon'}, \dim W_2 \leqslant 2+4/\epsilon'$ and $\mathcal{B} \subset W$, completing the proof. ☐

Corollary 6.15. *Suppose $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2$ be a $1 - SG_k^*(2)$ configuration with $|\mathcal{F}_2| = m_2$. Then there exist a r-wide vector space W with $\dim W_1 \leqslant 7 \cdot r \cdot 3^{k+1+16 \cdot 4^{k-1}}, \dim W_2 \leqslant k+1+16 \cdot 4^{k-1}$ such that at least $m_2/4^k$ forms in $\mathcal{F}_2$ are $k+5$ close to W.*

Proof. We first show that $\mathcal{F}_2$ is a strong $(1,k) - SG_k^*(2)$ configuration. Suppose $Q_1, \ldots, Q_k$ is a k-strong sequence. Every subset of $Q_1, \ldots, Q_k$ is also a k-strong sequence, and hence generates a prime ideal by Remark 6.10. By definition $Q_1, \ldots, Q_k$ are linearly independent, therefore $Q_1, \ldots, Q_k$ form a k-relavent set. For every $Q_{k+1} \in \mathcal{F}_2$, if $Q_1, \ldots, Q_{k+1}$ is a k-strong sequence, then $(Q_1, \ldots, Q_{k+1})$ is prime and $Q_{k+1} \notin \operatorname{rad}(Q_1, \ldots, Q_k)$. Therefore there exists $R \in \mathcal{F}$ such that $R \in (Q_1, \ldots, Q_{k+1}) \setminus (Q_1, \ldots, Q_k) \cup (Q_{k+1})$. Since $Q_i \in \mathcal{F}_2$ it must be that $R \in \mathcal{F}_2$. This shows that $\mathcal{F}_2$ is a strong $(1,k) - SG_k^*(2)$ configuration.

Now let $t \geqslant 1$ be the smallest number such that $\mathcal{F}_2$ is a strong $\left(4^{k-t}, k\right) - SG_t^*(2)$ configuration. By the previous paragraph, we have $t \leqslant k$. If $t = 1$, the required vector space exists by Lemma 6.14. If $t > 1$, we apply Lemma 6.13. Since $\mathcal{F}_2$ is not a strong $\left(4^{k-t-1}, k\right) - SG_{t-1}^*(2)$ configuration, case 1 of Lemma 6.13 does not hold, Therefore there exists a vector space W with $\dim W_1 \leqslant 7 \cdot r \cdot 3^{t+1+16 \cdot 4^{k-t}}, \dim W_2 \leqslant t+1+16 \cdot 4^{k-t}$ such that at least $4^{k-t-1} \cdot m_2$ forms in $\mathcal{F}_2$ are $k+4$ close to W. ☐

7 Proof of Sylvester-Gallai Theorem

In this section, we prove our main theorem: $1 - SG_k^*(2)$ configurations have constant vector space dimension. Throughout this section we denote our $1 - SG_k^*(2)$ configuration by $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2$ where $\mathcal{F}_d$ is the set of forms of degree d in our configuration. Additionally, we define $m := |\mathcal{F}|$, $m_1 := |\mathcal{F}_1|$ and $m_2 := |\mathcal{F}_2|$.

Our proof has three main steps. In Section 7.1 we show that given $\mathcal{F}$, we can find a constant dimensional wide vector space W such that $\mathcal{F}$ is close to W. We call any such $\mathbb{C}$-algebra $\mathbb{C}[W]$ a *core algebra* of our configuration $\mathcal{F}$. This step uses the notion of strong sequences. In Section 7.2 we show that given such a vector space W, we can extend it to obtain a constant dimensional wide vector space $W \subset V$ such that $\mathcal{F}_2 \subset (V)$. This step uses the notion of integral sequences. In Section 7.3 we show that our main theorem follows given such a vector space V. This step uses general projections and the bound for linear SG configurations from [BDYW11, DSW14].

Define functions $\lambda_2(r,k) := k+1+16\cdot 4^{k-1}$, $\lambda_1(r,k) := 7\cdot r\cdot 3^{\lambda_2(r,k)}$ and $B(k) := 3k+15$. For the rest of this section, we set the parameter B in the definition of $\mathbb{L}_V(P)$ to $B(k)$. Note that while this parameter depends on k, it is independent of $|\mathcal{F}|$.

7.1 Constructing core algebras

We begin by showing that, to put all forms close to a wide algebra, it is enough to construct a small wide algebra which contains a constant fraction of the quadratics. More precisely, the next lemma allows us to increase the fraction of forms close to a given vector space without increasing the size of the vector space too much, so long as we start with a wide vector space which contains a constant fraction of the quadratics.[5]

Before we state and prove the lemma, the following notation will be very useful in this subsection: if $\gamma \in \mathbb{N}$, $\mathcal{G}$ is a set of forms and W is a graded vector space, we let

$$\mathcal{G}(\gamma, W) := \{P \in \mathcal{G} \mid P \text{ is } \gamma\text{-close to } W\}.$$

Lemma 7.1 (Increasing algebra intersection). *Let $0 < \delta \leqslant 1$, $r, \gamma, k \in \mathbb{N}$ be such that $r > 2\gamma \geqslant k+5$ and W be a r-wide vector space. If $|\mathcal{F}(\gamma, W)| \geqslant \delta m$ then there is a r-wide vector space Y with $\dim Y_1 \leqslant 3^k\cdot(\dim W + r)$, $\dim Y_2 \leqslant \dim W_2 + k$ such that either $|\mathcal{F}(\gamma, Y)| \geqslant 3\delta m/2$, or $\mathcal{F} = \mathcal{F}(3\gamma, Y)$.*

Proof. Note that $\mathcal{F}_1 \subseteq \mathcal{F}(\gamma, W)$. Let $\mathcal{H} := \mathcal{F}_2 \setminus \mathcal{F}(3\gamma, W)$. In other words, $\mathcal{H}$ is the set of forms that are 3γ-far from W. Let $H_1, \dots, H_t \in \mathcal{H}$ be the longest sequence of linearly independent forms such that

1. $\text{minrank}(H_1, \dots, H_t) \geqslant k+5$, and
2. No nonzero form in $\text{span}_{\mathbb{C}}\{H_1, \dots, H_t\}$ is 2γ-close to W.

Suppose $t < k$. Let Y be the r-wide vector space obtained by applying Proposition 5.2 to $W + \text{span}_{\mathbb{C}}\{H_1, \dots, H_t\}$. Since $H_1, \dots, H_t$ is the longest linearly independent sequence that satisfies the above conditions, for every other $H \in \mathcal{H}$, it must be that

- either $H \in \text{span}_{\mathbb{C}}\{H_1, \dots, H_t\}$, or
- $\text{minrank}(H, H_1, \dots, H_t) \leqslant k+4$, or
- there exists $R \in \text{span}_{\mathbb{C}}\{H, H_1, \dots, H_t\} \setminus \text{span}_{\mathbb{C}}\{H_1, \dots, H_t\}$ such that R is 2γ-close to W.

In each of these three cases, it follows that $H \in \mathcal{F}(3\gamma, Y)$. Therefore in this case, Y is the required vector space.

We are now in the case where $t \geqslant k$. Consider the k elements $H_1, \dots, H_k$. Note that $H_1, \dots, H_k$ are linearly independent, and also satisfy $\text{minrank}(H_1, \dots, H_k) \geqslant k+5$. Therefore, $H_1, \dots, H_k$ is a k-strong sequence. By Remark 6.10, the ideal $(H_1, \dots, H_k)$ is prime and k-relevant, and $S/(H_1, \dots, H_k)$ is a UFD. Let Y be the r-wide vector space obtained by applying Proposition 5.2 to $W + \text{span}_{\mathbb{C}}\{H_1, \dots, H_k\}$, so $\dim Y_1 \leqslant 3^k\cdot(\dim W + r)$, $\dim Y_2 \leqslant \dim W_2 + k$.

[5] This is similar in spirit to [BDYW11, Proposition 7.11] and [GOS22, Lemma 5.15].

Now for each $G_i \in \mathcal{F}(\gamma, W)$ we have $G_i \notin (H_1, \ldots, H_k)$. In the graded UFD $S/(H_1, \ldots, H_k)$, the image of G_i must be irreducible: if not then $G_i = ab + \sum \alpha_j H_j$ in S, with $a, b \in S_1$, contradicting the fact that $\mathrm{span}_{\mathbb{C}}\{H_1, \ldots, H_k\}$ does not contain forms 2γ-close to W. The ideal $(H_1, \ldots, H_k, G_i)$ is therefore prime, and we have $R_i \in (H_1, \ldots, H_k, G_i) \setminus (H_1, \ldots, H_k)$ since $\mathcal{F}$ is a $1 - \mathrm{SG}_k^*(2)$ configuration. We have $R_i \in \mathcal{F}(\gamma, Y)$.

If $R_i \in \mathcal{F}_1$ then we must have $R_i \in (G_i)$, contradicting the pairwise independence of elements of $\mathcal{F}_2$, therefore $R_i \in \mathcal{F}_2$. After scaling we have either $R_i - G_i \in \mathrm{span}_{\mathbb{C}}\{H_1, \ldots, H_k\}$ (if $G_i \in \mathcal{F}_2$) or $R_i - aG_i \in \mathrm{span}_{\mathbb{C}}\{H_1, \ldots, H_k\}$ (if $G_i \in \mathcal{F}_1$). Therefore $R_i \notin \mathcal{F}(\gamma, W)$ since otherwise $\mathrm{span}_{\mathbb{C}}\{H_1, \ldots, H_k\}$ contains a form 2γ-close to W. If G_j is another form such that $R_i = R_j$, then $R_i - G_j$ or $R_i - bG_j$ is in $\mathrm{span}_{\mathbb{C}}\{H_1, \ldots, H_t\}$, and it must be that $G_i, G_j \in \mathcal{F}_1$ and $aG_i = bG_j$ so $G_j \in (a), G_i \in (b)$. This shows that $|\{R_i\}_i| \geqslant \delta m/2$. Since $\mathcal{F}(\gamma, W) \cup \{R_i\}_i \subseteq \mathcal{F}(\gamma, Y)$, we are done. □

We are now ready to prove the main lemma of this subsection.

Lemma 7.2 (Constructing core algebras)**.** *Suppose $\mathcal{F}$ is a $1 - \mathrm{SG}_k^*(2)$ configuration. For any r there exists a r-wide vector space W with $\dim W_1 \leqslant 2 \cdot 3^{k^2} \cdot \lambda_1(r, k)$ and $\dim W_2 \leqslant 4k^2 + \lambda_2(r, k)$ such that $\mathcal{F} = \mathcal{F}(B(k), W)$.*

Proof of Lemma 7.2. We build a sequence of vector spaces $W^{(i)}$ such that either $\mathcal{F} = \mathcal{F}(B(k), W^{(i)})$ or $\left|\mathcal{F}(k+5, W^{(i)})\right| \geqslant (3/2)^i \cdot m/4^k$.

Set $W^{(0)}$ to be the r-wide vector space obtained by Corollary 6.15. By Corollary 6.15, at least $m_2/4^k$ forms in $\mathcal{F}_2$ are $k+5$ close to $W^{(0)}$. Further, every form in $\mathcal{F}_1$ is 1-close to $W^{(0)}$. Since $m_1 + m_2/4^k \geqslant m/4^k$, we have $\left|\mathcal{F}(k+5, W^{(0)}\right| \geqslant m/4^k$. Therefore, $W^{(0)}$ satisfies the above property. We have $\dim W_i^{(0)} \leqslant \lambda_i(r, k)$.

Given $W^{(i)}$, if $\mathcal{F} = \mathcal{F}(B(k), W^{(i)})$ then terminate. If not, then apply Lemma 7.1 to $W^{(i)}$ with $\gamma = k+5$ and $\delta = (3/2)^i \cdot 1/4^k$ to obtain $W^{(i+1)}$. By Lemma 7.1, either $\mathcal{F} = \mathcal{F}(B(k), W^{(i+1)} = \mathcal{F}$ or $\left|\mathcal{F}(k+5, W^{(i+1)})\right| \geqslant (3/2)^{i+1} \cdot m/4^k$. Therefore $W^{(i+1)}$ also has the required property.

The above process must terminate when $(3/2)^i \cdot 1/4^k \geqslant 1$, which holds when $i > 4k$. Further, by induction we have $\dim W_1^{(4k)} \leqslant 3^{k^2}\lambda_1(r,k) + 3^{k^2} \cdot 2^k \cdot r \leqslant 2 \cdot 3^{k^2} \cdot \lambda_1(r, k)$ and $\dim W_2 \leqslant 4k^2 + \lambda_2(r, k)$. Therefore $W^{(4k)}$ is the required space. □

7.2 Finding small ideal containing the quadratic forms

In this section we show that all quadratics in any $1 - \mathrm{SG}_k^*(2)$ must be contained in an ideal generated by a small number of forms. The main idea is that given any wide vector space, there exist short maximal integral sequences with respect to the vector space. Recall that the we set the parameter B in the definition of relative linear spaces to $B(k) := 3k + 15$.

Lemma 7.3. *Suppose $r \geqslant 4(k+2)B(k) + 1$. Suppose $\mathcal{F}$ is a $1 - \mathrm{SG}_k^*(2)$ configuration, and suppose W is a r-wide vector space such that every $F \in \mathcal{F}$ is $B(k)$-close to W. Then there exists a maximal integral sequence with respect to inclusion of length at most k with respect to W.*

Proof. For each $F \in \mathcal{F}$ let $\overline{F_W^L}$ be the image of F_W^L in $S/(W_1)$. Define potential function Φ on integral sequences as

$$\Phi(G_1, \ldots, G_c) := \sum_{i=1}^{c} \dim \mathrm{Lin}\left(\overline{(G_i)_W^L}\right).$$

If $\mathcal{F} \subset (W)$ then there are no integral sequences with respect to W, and the statement holds vacuously, therefore we can assume that $\mathcal{F} \setminus (W) \neq \emptyset$. Combined with the fact that W is r-wide, and that every form in $\mathcal{F}$ is $B(k)$-close to W, there exists nonempty integral sequences with respect to W. Among all integral sequences of length at most $k+1$, pick $F_1, \ldots, F_c$ such that the above potential function is maximised. If $c \leqslant k$, then $F_1, \ldots, F_c$ is maximal: if not, and if $F_1, \ldots, F_{c+1}$ is an integral sequence that extends $F_1, \ldots, F_c$ then $\Phi(F_1, \ldots, F_{c+1}) > \Phi(F_1, \ldots, F_c)$, contradicting maximality.

We are left with the case where $c = k+1$. We will find an integral sequence of length at most k with the same potential function value, and therefore the new integral sequence will be maximal. The sequence $F_1, \ldots, F_k$ is an integral sequence, therefore by Lemma 6.6 we have that $(F_1, \ldots, F_k)$ is a radical ideal. Further, by Corollary 6.7 we have that $F_1, \ldots, F_k$ is a k-relevant set. Similarly, $F_1, \ldots, F_{k+1}$ is a $k+1$-relevant set, therefore $F_{k+1} \notin (F_1, \ldots, F_k)$.

Since $\mathcal{F}$ is a $1 - SG_2^*(2)$, we have $R \in (F_1, \ldots, F_{k+1}) \setminus (F_1, \ldots, F_k)$, that is, $R = \sum_{j=1}^{k+1} \alpha_j F_j$ with $\alpha_{k+1} \neq 0$. Without loss of generality, suppose $\alpha_j = 0$ for $j = 1, \ldots, b$ and $\alpha_j \neq 0$ for $j = b + 1, \ldots, k+1$. Since the polynomials in $\mathcal{F}$ are pairwise linearly independent we have $b < k$.

Now $R = R_W^H + R_W^L = \sum_{j>b} \alpha_i F_i = \sum_{j>b} \alpha_i ((F_i)_W^H + (F_i)_W^L)$. Since the space W is r-wide, we have $R_W^H = \sum_{j>b} \alpha_i (F_i)_W^H$ and $R_W^L = \sum_{j>b} \alpha_i (F_i)_W^L$. By Corollary 6.4, after a change of basis we can assume that there are disjoint sets of variables $Y, Y_1, \ldots, Y_k$ such that W_1 is spanned by Y and $(F_i)_W^L \in \mathbb{C}[Y, Y_i]$. We have $\mathrm{Lin}\left(\overline{R_W^L}\right) \subseteq \mathbb{C}[Y, \overline{Y_{b+1}, \ldots, Y_{k+1}}]$, whence $F_1, \ldots, F_b, R$ is an integral sequence by Proposition 6.2. Further $\overline{R_W^L} = \sum_{j>b} \alpha_j \overline{(F_j)_W^L}$, and since $\overline{(F_i)_W^L} \in \mathbb{C}[Y_i]$, by Lemma 2.12 we can deduce that $\dim \mathrm{Lin}\left(\overline{R_W^L}\right) = \sum_{j>b} \dim \mathrm{Lin}\left(\overline{(F_j)_W^L}\right)$. Therefore $F_1, \ldots, F_b, R$ is an integral sequence of length at most k with $\Phi(F_1, \ldots, F_b, R) = \Phi(F_1, \ldots, F_{k+1})$. This proves that $F_1, \ldots, F_b, R$ is a maximal integral sequence. $\square$

Lemma 7.4. *Suppose $\mathcal{F}$ is a $1 - SG_k^*(2)$ configuration. Suppose $r \geqslant 8(k+2)B(k)^2 + 1$. There exists a $\left(r - 4kB(k)^2\right)$-wide vector space W with $\dim W_1 \leqslant 3 \cdot 3^{k^2} \cdot \lambda_1(r,k)$ and $\dim W_2 \leqslant 4k^2 + \lambda_2(r,k)$ such that $\mathcal{F}_2 \subset (W)$.*

Proof. For any r-wide vector space U such that every polynomial $F \in \mathcal{F}$ is $B(k)$ close to U, define potential function Ψ as

$$\Psi(U) = \max_{F \in \mathcal{F}_2 \setminus (U)} \dim \overline{\mathbb{L}}_U(F).$$

If $\Psi(U) = 0$ for some such U, then $\mathcal{F}_2 \subset (U)$.

We now construct W iteratively. Let $W^{(0)}$ be the r-wide vector space whose existence is guaranteed by Lemma 7.2. Since every $F \in \mathcal{F}$ is $B(k)$-close to $W^{(0)}$ we have $\Psi(W^{(0)}) \leqslant 2B(k)$. The vector space $W^{(0)}$ is r-wide, and $r \geqslant 4(k+2)B(k)+1$, therefore by Lemma 7.3 we can find a maximal integral sequence $F_1, \ldots, F_c$ with respect to $W^{(0)}$ with $c \leqslant k$. Set $W^{(1)} = \sum_{i=1}^{c} \mathbb{L}_{W^{(0)}}(F_i) + W_2^{(0)}$. That $F_1, \ldots, F_c$ is maximal implies the following: for every $G \in \mathcal{F}_2 \setminus \left(W^{(0)}\right)$ we have $\dim \overline{\mathbb{L}}_{W^{(0)}}(G) > \dim \overline{\mathbb{L}}_{W^{(1)}}(G)$. Therefore $\Psi(W^{(1)}) < \Psi(W^{(0)})$. By Lemma 5.16, the vector space $W^{(1)}$ is $r - 2kB(k)$-wide since $W_1^{(1)}$ is obtained by adding at most $2kB(k)$ linear forms to $W_1^{(0)}$. In general, given $W^{(i)}$ we use Lemma 7.3 to find a maximal integral sequence $F_{i1}, \ldots, F_{ic}$, and set $W^{(i+1)} := \sum_{j=1}^{c} \mathbb{L}_{W^{(i)}}\left(F_{ij}\right) + W_2^{(i)}$. Maximality of the sequence implies $\Psi(W^{(i+1)}) < \Psi(W^{(i)})$. By the bound on r, at every step $W^{(i)}$ is at least $4(k+2)B(k)+1$-wide. After at most $2B(k)$ steps we find a t such that $\Psi(W^{(t)}) = 0$.

By Lemma 7.2 we have $\dim W_1^{(0)} \leqslant 2 \cdot 3^{k^2} \lambda_1(r,k)$. Since $W^{(i+1)}$ is obtained by adding $2B(k)k$ linear forms to $W^{(i)}$ we get $\dim W_1^{(t)} \leqslant 2 \cdot 3^{k^2} \lambda_1(r,k) + 4B(k)^2 k \leqslant 3 \cdot 3^{k^2} \lambda_1(r,k)$. Further we have $\dim W_2^{(i)} = \dim W_2^{(i-1)}$ for all i, therefore $\dim W_2^{(t)} = \dim W_0^{(t)} \leqslant 4k^2 + \lambda_2(r,k)$. This completes the proof. $\square$

7.3 Basic configuration

In this section we prove Theorem 3.12 for the special case where all the quadratics are in the ideal generated by an r-wide algebra.

Lemma 7.5. *Suppose $\mathcal{F}$ is a $1-\mathrm{SG}_k^*(2)$ configuration. Suppose there is an r-wide linear subspace W with $r \geqslant k+5$ such that $\mathcal{F}_2 \subset (W)$. Then there is linear subspace W_1' with $\dim(W_1') = \left(C'^k\right) \cdot \dim W_1$, such that $\mathcal{F} \subseteq W_2 + \mathbb{C}[W_1']$.*

Proof. Let $\varphi := \varphi_{\alpha, W_1}$ be a projection mapping as defined in Definition 2.14. By Remark 5.7, the space $\varphi(W)$ is a $r-1$-wide vector space. Let $\Delta := \dim W_1$. As $\mathcal{F}_2 \subseteq (W)$, every $F \in \mathcal{F}_2$ satisfies $\varphi(F) = \alpha\varphi(F_W^H) + z \cdot \ell$ for some linear form $\ell \in S\,[z]_1$.

Let $\mathcal{L}$ be the union of all the linear forms that occur in the above way, and all the linear forms in $\mathcal{F}$. Formally, $\mathcal{L} := \left\{\ell \mid \varphi(F) = \alpha\varphi(F_W^H) + z \cdot \ell, F \in \mathcal{F}_2\right\} \cup \varphi(\mathcal{F}_1)$. Let $\mathcal{L}/\,(z)$ denote the image of $\mathcal{L}$ in the vector space $(S\,[z]\,/\,(z))_1$, that is, the linear forms modulo z. We show that $\mathcal{L}/(z)$ is a $1-\mathrm{SG}_k(1)$ configuration.

Let $\overline{\ell_1}, \ldots, \overline{\ell_k} \in \mathcal{L}/(z)$ be independent. Let $\overline{\ell_{k+1}} \in \mathcal{L}/(z)$. We need to show that one of the following cases holds:

1. $\bar{\ell}_{k+1} \in \mathrm{span}_{\mathbb{C}}\left\{\overline{\ell_1}, \ldots, \overline{\ell_k}\right\}$.
2. there is $\bar{g} \in \mathrm{span}_{\mathbb{C}}\left\{\overline{\ell_1}, \ldots, \overline{\ell_{k+1}}\right\} \setminus \{\overline{\ell_1}, \ldots, \overline{\ell_k}\}$ with $\bar{g} \in \mathcal{L}/\,(z)$.

Consider the corresponding $F_1, \ldots, F_{k+1} \in \mathcal{F}$ such that $\varphi(F_i) = \alpha_i\pi({F_i}_W^H) + z \cdot \ell_i$, with $\ell_i/(z) = \bar{\ell}_i$, or, if $F_i \in \mathcal{F}_1$ then $F_i = \ell_i$.

The first step is to show that $F_1, \ldots, F_k$ form a k-relevant set. Without loss of generality, assume that $F_1 \in \mathrm{rad}\,(F_2, \ldots, F_k)$. We have $\varphi(F_1) \in \mathrm{rad}\,(\varphi(F_2), \ldots, \varphi(F_k))$, and by Lemma 5.8 we have $z\ell_1 \in (\ell_2, \ldots, \ell_k)$. Since the ideal $\ell_2, \ldots, \ell_k$ is prime, and since $\overline{\ell_2}, \ldots, \overline{\ell_k}$ are independent, we get $\ell_1 \in \mathrm{span}_{\mathbb{C}}\{\ell_2, \ldots, \ell_k\}$ contradicting choice of $\ell_1, \ldots, \ell_k$. Therefore $F_1, \ldots, F_k$ is k-relevant.

By the same argument, if $F_{k+1} \in \mathrm{rad}\,(F_1, \ldots, F_k)$ then $\overline{\ell_{k+1}} \in \mathrm{span}_{\mathbb{C}}\left\{\overline{\ell_1}, \ldots, \overline{\ell_k}\right\}$. We are left with the case when $F_{k+1} \notin \mathrm{rad}\,(F_1, \ldots, F_k)$. Since $\mathcal{F}$ is a $1-\mathrm{SG}_2^*(2)$ configuration, there exists $R \in \mathrm{rad}\,(F_1, \ldots, F_{k+1}) \setminus \mathrm{rad}\,(F_1, \ldots, F_k)$. Let $\overline{g}$ be such that $\varphi(R) = \alpha_i\varphi(R_W^H) + z \cdot g$ if $R \in \mathcal{F}_2$, and $R = g$ otherwise. We have $\varphi(R) \in \mathrm{rad}\,(\varphi(F_1), \ldots, \varphi(F_{k+1}))$. By Lemma 5.8, we have $z\overline{g} \in \mathrm{span}_{\mathbb{C}}\left\{\overline{\ell_1}, \ldots, \overline{\ell_{k+1}}\right\}$ which implies $\overline{g} \in \mathrm{span}_{\mathbb{C}}\left\{\overline{\ell_1}, \ldots, \overline{\ell_{k+1}}\right\}$. Finally, by Lemma 5.9, we have that $\overline{g} \notin \left(\overline{\ell_i}\right)$ for any i. This completes the proof that $\mathcal{L}/\,(z)$ is a $1-\mathrm{SG}_k(1)$ configuration.

By Theorem 3.5 we have

$$\dim(\mathbb{L}_W\,(\varphi(\mathcal{F})) = \dim(\mathcal{L}/(z)) + 1 \leqslant C'^k,$$

for some universal constant C'. Applying Proposition 2.18 it follows that $\dim(\mathbb{L}_W\,(\mathcal{F})) \leqslant C'^k \cdot \Delta$. In particular, it follows that there is a linear space of linear forms W_1', with $\dim(W_1') \leqslant C'^k \cdot \Delta$, satisfying $\mathcal{F} \subseteq W_2 + \mathbb{C}[W_1']$, completing the proof. ∎

7.4 Proof of main theorem

We now prove the main theorem, which we restate for convenience.

Theorem 3.12 (Radical SG Theorem for tuples of quadratics)**.** *Let $\mathcal{F}$ be a $1-\mathrm{SG}_k^*(2)$ configuration. There is a universal constant $c > 0$ such that $\dim(\mathrm{span}_{\mathbb{C}}\{\mathcal{F}\}) \leqslant 3^{c \cdot 4^k}$.*

Proof. Let $r := 8(k+2)B(k)^2 + k + 6$. By Lemma 7.4, there exists a $k+5$-wide vector space W with $\dim W_1 \leqslant 3 \cdot 3^{k^2} \cdot \lambda_1(r,k)$ and $\dim W_2 \leqslant 4k^2 + \lambda_2(r,k)$ such that $\mathcal{F}_2 \subseteq (W)$. Applying Lemma 7.5 with this W, we obtain a vector space $W_1' \subseteq S_1$ with $\dim W_1' \leqslant 3 \cdot 3^{k^2} \cdot \lambda_1(r,k) \cdot C'^k$ such that $\mathcal{F} \subseteq W_2 + \mathbb{C}\left[W_1'\right]$. If $Y \subseteq S_2$ is the space spanned by pairwise products of forms in W_1', then $\mathcal{F} \subseteq W_2 + Y$ and $\dim Y \leqslant 9 \cdot 9^{k^2} \cdot \lambda_1^2(r,k) \cdot C'^{2k}$. Substituting for λ_1, λ_2 gives us the required result. □

Remark 7.6. Suppose the set $\mathcal{F}$ does not have any k-relevant sequences. In this case, $\mathcal{F}$ is vacuously an $1 - \mathrm{SG}_k^*(2)$ configuration. There are no k-strong sequences in $\mathcal{F}$ of length k, since any such sequence is a k-relevant set. Therefore every form in such a configuration is $k+5$-close to a r-wide vector space W of dimension $7 \cdot r \cdot 3^k$ by Lemma 6.11. Further, such a configuration has no integral sequences of length $k+1$. Therefore, by the arguments in Lemma 7.3 and Lemma 7.4, by adding $4kB(k)^2$ linear forms to W, we get a wide vector space Y such that $\mathcal{F} \subseteq Y$. If we project to Y_1 and pick out the linear forms corresponding to each element of $\mathcal{F}$ as in Lemma 7.5, then there are no set of $k+1$ linearly independent forms by Lemma 5.8. Therefore, we can deduce by the properties of the projection map that $\dim \operatorname{span}_{\mathbb{C}} \{\mathcal{F}\} = 2^{O(k)}$ in this case.

8 Conclusion

In this work, we prove a higher codimension analogue of the quadratic Sylvester–Gallai theorem, generalising the results of [Shp20, Han65]. Our ability to handle ideals of higher codimension shows our approach is a promising one towards a full derandomisation of PIT for $\Sigma^k\Pi\Sigma\Pi^2$ circuits.

To prove our main theorem, we build upon the results of [AH20, OS22] and use the wide algebras developed in these works to control the cancellations in SG configurations. One key difference between this work and previous works [Shp20, PS20, PS21, PS22, GOS22, OS22] is that we prove our Sylvester-Gallai theorem without a fine classification of the ideals we deal with.

Our work leaves several open questions which are of interest to combinatorialists, algebraic geometers, and complexity theorists. On the combinatorial and geometric side, understanding the different generalizations of Sylvester's problems to higher codimension (such as the elementary SG configurations defined in [Han65] and also studied in [BDYW11]) is a problem of independent interest, as well as the generalization to higher codimension of the "product" version of Sylvester's question, defined in [Gup14, PS20]. And of course, fully derandomizing PIT for $\Sigma^k\Pi\Sigma\Pi^2$ is still a major open question.

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A Alternative proof of Lemma 6.6

We give here an alternative proof suggested by an anonymous reviewer (with proper formalizations).

Alternative Proof of Lemma 6.6. Let $I := (F_1, \ldots, F_t)$. Since the inclusion $\mathcal{A} := \mathbb{C}[V, Y_1, \ldots, Y_t] \to S$ is a free extension, it is enough to prove that I is radical in $\mathcal{A}$. Moreover, since $\mathcal{A}$ is isomorphic to a polynomial ring, by Corollary 6.4 we can assume that our polynomial ring is $\mathcal{A} := \mathbb{C}[Z, y_1, y_2, \ldots, y_t]$ where $\mathcal{F}_i \in \mathbb{C}[Z, y_i]$. By Lemma 6.5, we know that $F \in \mathbb{C}[Z] \setminus \{0\}$ is regular with $F_1, \ldots, F_t$, and hence it is not in any minimal prime of I. Thus, F is not a zero divisor over $\mathcal{A}/I$.

Since $\mathcal{B} := \mathbb{C}(Z)[y_1, \ldots, y_t]$ is the localization of $\mathcal{A}$ over $\mathbb{C}[Z] \setminus \{0\}$, by the above, we have that I is radical in $\mathcal{A}$ iff $I \cdot \mathcal{B}$ is radical in $\mathcal{B}$. Let $R := \overline{\mathbb{C}(Z)}[y_1, \ldots, y_t]$. It is easy to see that $I \cdot \mathcal{B}$ is radical in $\mathcal{B}$ if $I \cdot R$ is radical over R. To see that $I \cdot R$ is a radical ideal, note that $F_i \in \mathbb{C}[Z, y_i]$ irreducible implies that $\mathrm{disc}_{y_i}(F_i) \in \mathbb{C}[Z] \setminus \{0\}$ and hence $F_i = (y_i - \alpha_i)(y_i - \beta_i)$ over R, with $\alpha_i \neq \beta_i$. Thus, $I \cdot R$ is the intersection of maximal ideals and therefore radical. $\square$

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