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Border Complexity of Symbolic Determinant under Rank One Restriction

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Abstract

VBP is the class of polynomial families that can be computed by the determinant of a symbolic matrix of the form $A_0 + \sum_{i=1}^{n} A_i x_i$ where the size of each A_i is polynomial in the number of variables (equivalently, computable by polynomial-sized algebraic branching programs (ABP)). A major open problem in geometric complexity theory (GCT) is to determine whether VBP is closed under *approximation* i.e. whether $\mathsf{VBP} \stackrel{?}{=} \overline{\mathsf{VBP}}$. The power of approximation is well understood for some restricted models of computation, e.g. the class of depth-two circuits, read-once oblivious ABPs (ROABP), monotone ABPs, depth-three circuits of bounded top fan-in, and width-two ABPs. The former three classes are known to be closed under approximation [BIM+20], whereas the approximative closure of the last one captures the entire class of polynomial families computable by polynomial-sized formulas [BIZ17].

In this work, we consider the subclass of VBP computed by the determinant of a symbolic matrix of the form $A_0 + \sum_{i=1}^{n} A_i x_i$ where for each $1 \leq i \leq n$, A_i is of rank one. This class has been studied extensively [Edm68, Edm79, Mur93] and efficient identity testing algorithms are known for it [Lov89, GT20]. We show that this class is closed under approximation. In the language of algebraic geometry, we show that the set obtained by taking coordinatewise products of pairs of points from (the Plücker embedding of) a Grassmannian variety is closed.

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1 Introduction

The determinant polynomial plays a central role in the study of complexity theory. It is known to be a *complete polynomial* i.e. every polynomial can be computed by some affine projection of the determinant of a symbolic matrix. More precisely, for any polynomial $f \in \mathbb{F}[x_1, \ldots, x_n]$, there is some m and $A_0, A_1, \ldots, A_n$ in $\mathbb{F}^{m\times m}$ such that $f = \det_m(A_0 + \sum_{i=1}^{n} A_i x_i)$. VBP is defined as the class of polynomial families for which the size of such determinantal representation is polynomially bounded in the number of variables (equivalently, such polynomial families can be computed by polynomial-size *algebraic branching programs* (ABP)).

The other polynomial of significant interest is the permanent polynomial, a close cousin of the determinant polynomial. The permanent polynomial is also known to be a *complete polynomial*. VNP is defined as the class of polynomial families for which the size of the permanental representation is polynomially bounded in the number of variables. It is known that $\mathsf{VBP} \subseteq \mathsf{VNP}$. The goal of algebraic complexity theory is to separate VBP and VNP, equivalently, to show a super-polynomial lower bound on the determinantal representation of the permanent polynomial.

Even though we have witnessed some outstanding progress in our understanding of the lower bound problem on various restricted models of computation in the last few years, the fundamental problem in the general setting remains elusive. *Geometric Complexity Theory* (GCT) was proposed as a possible approach to settle this question by showing $\mathsf{VNP} \not\subseteq \overline{\mathsf{VBP}}$ [MS01] where $\overline{\mathsf{VBP}}$ denotes the *approximative closure* of VBP. Let $\mathcal{C}$ be a circuits class over $\mathbb{F}$, $\mathbb{F}[\varepsilon]$ be the polynomial ring and $\mathbb{F}(\varepsilon)$ be the fraction field of $\mathbb{F}[\varepsilon]$. We can define $\overline{\mathcal{C}}$, the (approximative) closure of the circuit class $\mathcal{C}$ in the following equivalent ways.
(a) Approximative closure. A polynomial family $\{f_n\}$ is in the approximative closure of $\mathcal{C}$ over $\mathbb{F}$ if there is a polynomial family $\{g_n\}$ in $\mathbb{F}[\varepsilon][x_1, \ldots, x_n]$ computable in $\mathcal{C}$ over $\mathbb{F}(\varepsilon)$, such that for every n,

$$g_n(x_1, \ldots, x_n) = f_n(x_1, \ldots, x_n) + \varepsilon \cdot h_n(x_1, \ldots, x_n)$$

for some polynomial h_n in $\mathbb{F}[\varepsilon][x_1, \ldots, x_n]$. We say, the polynomial family $\{f_n\}$ is approximated by the family $\{g_n\}$.
(b) Euclidean closure. A polynomial family $\{f_n\}$ is in the Euclidean closure of $\mathcal{C}$ over $\mathbb{F}$ if, for every n, there exists an infinite sequence of polynomials $\{g_{n,i}\}$ in $\mathcal{C}$ over $\mathbb{F}$ such that the limit point of the sequence of coefficient vectors corresponding to $\{g_{n,i}\}$ is the coefficient vector of f_n. This definition is known to be equivalent to the previous definition when $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$ [Bö4].
(c) Zariski closure. Another equivalent way is to define the approximative closure as a *Zariski closure* [Mum95]. For a circuit class $\mathcal{C}$, consider the system of all polynomial equations which are satisfied by the coefficient vector corresponding to each polynomial in $\mathcal{C}$. Then, the Zariski closure $\overline{\mathcal{C}}$ consists of the polynomials such that the corresponding coefficient vectors are satisfying assignments of the system of polynomial equations.

As all these definitions are equivalent, without loss of generality, we define $\overline{\mathcal{C}}$ to be the approximative closure of $\mathcal{C}$. If $\mathcal{C} = \overline{\mathcal{C}}$, we say $\mathcal{C}$ is closed under approximation.

One of the main objectives of geometric complexity theory is to decide whether VBP is closed under approximation or not. Showing $\mathsf{VBP} = \overline{\mathsf{VBP}}$ would imply that showing $\mathsf{VBP} \neq \mathsf{VNP}$ is equivalent to showing $\mathsf{VNP} \not\subseteq \overline{\mathsf{VBP}}$. Though the complexity of $\overline{\mathsf{VBP}}$ is not well-understood, the power of approximation has been successfully studied for various restricted models of computation. For example, it is known that the following classes are closed under approximation: (a) $\Sigma\Pi$ i.e. the sparse polynomials, (b) Monotone ABPs [BIM+20], and (c) Read-once oblivious ABPs (ROABP). Recently, the approximative closure of the depth three circuits of bounded top fan-in is shown to be contained in VBP [DDS22]. Surprisingly, even a restricted circuit class can efficiently compute a much larger class under approximation. For example, consider VBP_2, the class of polynomials computed by the width-two ABPs. Even though, there are families of polynomials that cannot be expressed by this class [AW16] , the approximative closure of this class contains VF, the class of polynomials computed by a small formula. Indeed, it is known that $\overline{\mathsf{VBP}_2} = \overline{\mathsf{VF}}$ [BIZ17].

It is interesting to notice that, for the circuit classes for which the approximative closure is well-understood, we also know efficient identity testing algorithms. It motivates us to study the class VBP under some natural restriction for which we already have an efficient identity testing algorithm. The class of our interest is the symbolic determinant under rank one restriction. Recall that any n-variate polynomial in VBP can be computed as $\det(A_0 + \sum_{i=1}^n A_i x_i)$ where the size of each A_i is polynomially bounded in n. We consider the class of polynomials of form $\det(A_0 + \sum_{i=1}^n A_i x_i)$ where for each $1 \leq i \leq n$, $\mathrm{rank}(A_i) = 1$. This class has been studied extensively in contexts of polynomial identity testing, combinatorial optimization, and matrix completion (see, for example [Edm67, Lov89, Mur93]). It admits a deterministic polynomial-time identity testing algorithm in the white-box setting [Lov89] and a deterministic quasi-polynomial-time algorithm in the black-box setting [GT20]. This class is equivalent to the class of polynomial families computed by the determinant of symbolic matrices with each variable occurring at most once, also known as read-once determinants [ASV16] (as cited in [GT20, Lemma 4.3]). The expressive power of this class has also been studied. It strictly contains some well-studied classes like the polynomials computed by a small read-once formula (see, for example [AJ15]). However, it is known that for large enough n, n-variate elementary symmetric polynomials and the permanent polynomial cannot be expressed as $\det(A_0 + \sum_{i=1}^n A_i x_i)$ with $\mathrm{rank}(A_i) = 1$ for each $i \in [n]$ [AJ15].

Another motivation to study the approximative closure of this class is the fact that the approximative closure of the orbit of this class under the action of the general linear group contains VBP [MS21, ST21]. Therefore, understanding the approximative closure of this class may shed new light on the $\mathsf{VBP} \stackrel{?}{=} \overline{\mathsf{VBP}}$ question.

Our Results. The main result of this paper is that the class of the determinant of symbolic matrices under rank one restriction is closed under approximation. More precisely, we show the following theorem, where we use $\mathbb{F}$ to denote $\mathbb{R}$ or $\mathbb{C}$.

Theorem 1.1. *Given $A_0, A_1, A_2, \ldots, A_n \in \mathbb{F}(\varepsilon)^{r \times r}$ such that for each $1 \leq i \leq n$,* $\mathrm{rank}(A_i) =$

1 over $\mathbb{F}(\varepsilon)$. Let $f = \lim_{\varepsilon \to 0} \det(A_0 + \sum_{i=1}^{n} A_i x_i)$ be defined. Then, there exists $B_0, B_1, B_2, \ldots, B_n$ in $\mathbb{F}^{(n+r)\times(n+r)}$ such that $f = \det(B_0 + \sum_{i=1}^{n} B_i x_i)$ and $\operatorname{rank}(B_i) = 1$ over $\mathbb{F}$ for each $i \in [n]$. Moreover, if $A_0 = 0$, then the matrices $B_1, B_2, \ldots, B_n$ lie in $\mathbb{F}^{r\times r}$.

Since this class is closed under approximation, the known hitting set and non-expressibility results for this class also hold for its approximative closure.

Remark 1.2. *By using formal power series, we can extend this result to any arbitrary field. For the sake of simplicity, we only work with $\mathbb{C}$ or $\mathbb{R}$.*

An algebraic geometry perspective on the result. Consider the simpler case of Theorem 1.1, when $A_0 = 0$. Using known techniques, the statement can be reduced to this simpler case. Now, suppose $A_1, A_2, \ldots, A_n$ are $r \times r$ matrices of rank 1. Let us write $A_i = \mathbf{u}^i \cdot \mathbf{v}^{i^T}$ for some vectors $\mathbf{u}^i, \mathbf{v}^i \in \mathbb{F}^r$ and define matrices $U, V \in \mathbb{F}^{r\times n}$ whose ith columns are $\mathbf{u}^i$ and $\mathbf{v}^i$, respectively. It can be verified that

$$\det\left(\sum_i A_i x_i\right) = \sum_S \det(U_S)\det(V_S)\prod_{j\in S} x_j,$$

where the sum is over all size-r subsets S of $[n]$ and U_S (or V_S) denotes the submatrix of U (or V) obtained by taking columns with indices in the set S. Hence, essentially our main result says that the image of the map

$$(\mathbb{F}^{r\times n})^2 \to \mathbb{F}^{\binom{n}{r}}, \quad (U, V) \mapsto (\det(U_S) \times \det(V_S))_S$$

is Euclidean closed (and hence, Zariski closed). A closely related map

$$\mathbb{F}^{r\times n} \to \mathbb{F}^{\binom{n}{r}}, t \quad U \mapsto (\det(U_S))_S$$

has been well-studied in algebraic geometry, which gives the Plücker coordinates of elements in the Grassmannian variety. And hence, the image of this map is known to be a closed set. Putting it another way, our result says that the set obtained by taking coordinatewise products of pairs of points in the Grassmannian variety is closed.

Note that this is not a general phenomenon. It is easy to construct varieties where the set obtained by taking coordinatewise products of pairs of points from the variety is not closed. To see a simple example, consider the projective variety in $\mathbb{P}^2$ defined by

$$\{[x : y : z] \mid xz + y^2 - x^2 = 0\}.$$

Now, observe that the point $(0, 1, 0)$ cannot be obtained as a coordinatewise product of two points in the variety. On the other hand, it can be obtained as a limit of the product of two points $(\varepsilon, 1, \varepsilon - 1/\varepsilon)$ and $(1, 1, 0)$. See [BC22] for a related notion called Hadamard power of varieties.

Closure of a principal minor map. Our main result also implies the closure of the image of a principal minor map, as defined below. The *affine principal minor map* $\phi : \mathbb{C}^{n^2} \to \mathbb{C}^{2^n}$ is defined as

$$\phi(A) = (\det(A_I))_{I \subseteq [n]}$$

where is A_I is the principal submatrix of A with rows and columns indexed by I. Lin and Sturmfels [LS09] showed that for any $n > 0$, the image of ϕ on $n \times n$ matrices is closed. Our result implies the closure result for a closely related map, which we refer to as the size k principal minor map. For any $k \leq n$, let us define the map $\phi_k : \mathbb{C}^{n^2} \to \mathbb{C}^{\binom{n}{k}}$ as

$$\phi_k(A) = (\det(A_I))_{I \in \binom{[n]}{k}}$$

where $\binom{[n]}{k}$ is the set of all size-k subsets of $[n]$. We show that the image of ϕ_k on $n \times n$ rank-k matrices is closed. Formally,

Corollary 1.3. *For any $n > 0$ and $k \leq n$, the image of the size k principal minor map on $n \times n$ matrices with rank at most k is closed in $\mathbb{C}^{\binom{n}{k}}$.*

One can define another similar map, where a rank-at-most-k matrix is mapped to the tuple of its size-at-most-k principal minors. Note that the closure of the image of this map follows easily from the result of Lin and Sturmfels [LS09]. However, to the best of our knowledge, Corollary 1.3 does not follow from their result.

Proof idea of the main result. As we said, our goal is to show that the image of the map

$$(U, V) \mapsto (\det(U_S) \times \det(V_S))_S$$

is closed under approximation. The idea is to start with any two given matrices $U, V \in \mathbb{F}(\varepsilon)^{r \times n}$ and construct matrices $\widehat{U}, \widehat{V} \in \mathbb{F}^{r \times n}$ such that for each size-r subset $S \subseteq [n]$, we have

$$\lim_{\varepsilon \to 0} (\det(U_S) \det(V_S)) = \det(\widehat{U}_S) \det(\widehat{V}_S).$$

Of course, we can hope to construct such matrices only when the limit exists for every S. Note that one cannot simply apply the limit operation on the matrix entries because the matrix U and V can have rational functions in ε as entries.

We can view each term like $\det(U_S)$ as a Laurent series in ε. For any Laurent series f in ε, one can define $\mathsf{val}(f)$ as the minimum exponent of ε appearing in f. Clearly, $\lim_{\varepsilon \to 0} f$ exists if and only if $\mathsf{val}(f) \geq 0$. So let us assume that $\mathsf{val}(\det(U_S) \det(V_S)) \geq 0$ for every S. In other words,

$$\min_S \{\mathsf{val}(\det(U_S) \det(V_S))\} = \min_S \{\mathsf{val}(\det(U_S)) + \mathsf{val}(\det(V_S))\} = 0.$$

Observe that only those sets S which achieve this minimum will give a nonzero term in the limit. It would have been convenient if min operator was distributive over the sum, i.e.,

$$\min_S \{\mathsf{val}(\det(U_S)) + \mathsf{val}(\det(V_S))\} = \min_S \{\mathsf{val}(\det(U_S))\} + \min_S \{\mathsf{val}(\det(V_S))\},$$

but that is of course not true. Amazingly, it turns out that in the case of val function, it is almost true. This comes from the fact that the val function satisfies a matroid like exchange property: for any two distinct $S, T \subseteq [n]$ of size r and any $j \in T \setminus S$, there exists a $k \in S \setminus T$ such that

$$\mathsf{val}(\det(U_S)) + \mathsf{val}(\det(U_T)) \geq \mathsf{val}(\det(U_{S-k+j})) + \mathsf{val}(\det(U_{T-j+k})).$$

Based on this property, Dress and Wenzel [DW90] defined the so-called valuated matroids. More interestingly, Murota [Mur96] proved the valuated matroid splitting theorem, which says that the min operator indeed distributes over the sum of two val functions, but with a "correction" term which is a linear function. To be more precise, there is a tuple $\mathbf{z} \in \mathbb{Z}^n$ such that

$$\begin{aligned}\min_S\{\mathsf{val}(\det(U_S)) + \mathsf{val}(\det(V_S))\} = &\min_S\{\mathsf{val}(\det(U_S)) + \sum_{i \in S} \mathbf{z}_i\} \\ &+ \min_S\{\mathsf{val}(\det(V_S)) - \sum_{i \in S} \mathbf{z}_i\}.\end{aligned}$$

The correction term is easy to handle because of linearity. Then basically, the problem breaks into two independent problems on U and V. That is, given any two matrices $U, V \in \mathbb{F}(\varepsilon)^{r \times n}$, construct matrices $\widehat{U}, \widehat{V} \in \mathbb{F}^{r \times n}$ such that for each size-r subset $S \subseteq [n]$, we have

$$\lim_{\varepsilon \to 0} \det(U_S) = \det(\widehat{U}_S) \text{ and } \lim_{\varepsilon \to 0} \det(V_S) = \det(\widehat{V}_S).$$

The problem now becomes tractable essentially because the image of the map $U \mapsto (\det(U_S))_S$ is known to be closed.

Discussion. As discussed earlier, showing that a class of polynomials is closed under approximation also implies that it is Zariski closed. That is, the class of polynomials must be characterized by a set of polynomial equations (in the coefficients of the polynomials in the class). It would be interesting to find the set of characterizing equations for the class of determinant of symbolic matrices under rank one restriction. Another natural class of polynomials for which we can study the closure question is that of symbolic determinant under rank 2 (or higher) restriction.

2 Preliminaries and Notations

We use $\mathbb{N}$ to denote the set of natural numbers, $\mathbb{R}$ to denote the set of real numbers, $\mathbb{C}$ to denote the set of complex numbers, $\mathbb{Z}$ to denote the set of integers, and $\mathbb{F}$ to denote field $\mathbb{R}$ or $\mathbb{C}$, respectively. For a field $\mathbb{F}$ and an indeterminate ε, $\mathbb{F}(\varepsilon)$ denotes the fractional field. For a positive integer n, $[n]$ denotes the set $\{1, 2, \ldots n\}$. For a set E, 2^E denotes the family of all possible subsets of E. For a subset S of E and an element $a \in E$, $S - a$ and $S + a$ denote the set $S \setminus \{a\}$ and $S \cup \{a\}$, respectively. For any subset S of $[n]$, $\mathbf{1}_S \in \mathbb{F}^n$ denotes the characteristic vector of the subset S. For a set E and a non-negative integer r, $\binom{E}{r}$ denotes the set family consisting of all subsets of E of size r.

Every element f in the fractional field $\mathbb{F}(\varepsilon)$ is of the form g/h where $g, h \in \mathbb{F}[\varepsilon]$ with $h \neq 0$. For a nonzero polynomial $p \in \mathbb{F}[\varepsilon]$, let $\mathsf{mindeg}(p)$ be the degree of the minimum degree term in p. The function val from $\mathbb{F}(\varepsilon)$ to $\mathbb{Z}$ is defined as

$$\mathsf{val}(f) := \begin{cases} \mathsf{mindeg}(g) - \mathsf{mindeg}(h) & \text{if } f \neq 0 \\ +\infty & \text{otherwise} \end{cases}$$

Proposition 2.1. *The* val *function satisfies the following properties.*

- *For any* $f, g \in \mathbb{F}(\varepsilon)$, $\mathsf{val}(fg) = \mathsf{val}(f) + \mathsf{val}(g)$.
- *For any* $f, g \in \mathbb{F}(\varepsilon)$, $\mathsf{val}(f+g) \geq \min\{\mathsf{val}(f), \mathsf{val}(g)\}$.
- *For any* $g \in \mathbb{F}(\varepsilon) \setminus \{0\}$, $\mathsf{val}(1/g) = -\mathsf{val}(g)$.
- *For an* $f \in \mathbb{F}(\varepsilon)$, $\lim_{\varepsilon\to 0} f$ *exists if and only if* $\mathsf{val}(f) \geq 0$. *Furthermore,* $\lim_{\varepsilon\to 0} f = 0$ *if and only if* $\mathsf{val}(f) > 0$.

For a polynomial $P \in \mathbb{F}(\varepsilon)[X]$ where $X = \{x_1, x_2, \ldots, x_n\}$ is the set of variables, we say $\lim_{\varepsilon\to 0} P$ exists if coefficient wise limit exists for every monomial of P at $\varepsilon = 0$. In other words, for any coefficient $f \in \mathbb{F}(\varepsilon)$ of a monomial of P, $\mathsf{val}(f) \geq 0$.
For a matrix $U \in \mathbb{F}^{r\times n}$, $i \in [r]$ and $j \in [n]$, $U[i, j]$ denotes the entry at ith row and jth column of U. For a matrix $U \in \mathbb{F}^{r\times n}$ and a subset $S \subseteq [n]$, U_S denotes the submatrix of U with columns indexed by S.

Next, we describe the Cauchy-Binet formula, which is an identity for the determinant of the product of two rectangular matrices of transposed shape.

Lemma 2.2 (Cauchy-Binet formula, [Zen93])**.** *Let* $n \geq r$ *be two positive integers. Let* A *and* B *are two* $r \times n$ *and* $n \times r$ *matrices over* $\mathbb{F}$, *respectively. Then*

$$\det(AB) = \sum_{S \in \binom{[n]}{r}} \det(A_S) \cdot \det(B_S),$$

where B_S *denotes the submatrix of* B *with rows indexed by* S.

Now we describe the Grassmann-Plücker identity.

Lemma 2.3 (Equation 1.3 [DW92])**.** *Let* $n \in \mathbb{N}$. *Let* $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b}_2, \mathbf{b}_3, \ldots, \mathbf{b}_n$ *be* $2n$ *vectors in* $\mathbb{F}^n$. *For all* $i \in \{0, 1, \ldots, n\}$, *let* U_i *and* V_i *be the matrices* $(\mathbf{a}_0, \ldots, \mathbf{a}_{i-1}, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n)$ *and* $(\mathbf{a}_i, \mathbf{b}_2, \ldots, \mathbf{b}_n)$, *respectively. Then,*

$$\sum_{i=0}^{n} \det(U_i) \cdot \det(V_i) = 0.$$

Matroids. A matroid M is a set family $\mathcal{I}$ defined on a ground set E such that $\mathcal{I}$ satisfies the following two properties:

1. *Closure under subsets:* If $X \in \mathcal{I}$ and $Y \subset X$, then $Y \in \mathcal{I}$.
2. *Augmentation Property:* If $|X| > |Y|$ and $X, Y \in \mathcal{I}$, then there exists $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

The set family $\mathcal{I}$ is called the *independent set family* M. The augmentation property ensures that all the maximal independent sets of M have the same size. The collection $\mathcal{B}$ of all the maximal independent sets is called the *base family* of M. The base family $\mathcal{B}$ satisfies the following property:

Base exchange property: Let $B, B' \in \mathcal{B}$. Then for all $a \in B \setminus B'$ there exists a $b \in B' \setminus B$ such that $B - a + b$ is in $\mathcal{B}$.

Given the base family $\mathcal{B}$ of a matroid M, its independent set family $\mathcal{I} = \{I \mid \exists B \in \mathcal{B},\ I \subseteq B\}$. Therefore, a matroid M can be represented as $(E, \mathcal{I})$ or $(E, \mathcal{B})$. In this work, we mostly use $M = (E, \mathcal{B})$ to represent a matroid. Every matroid M is associated with a function, rank : $2^E \to \mathbb{N}$, defined as

$$\text{rank}(S) = \max\{|Y| \mid Y \subseteq S,\ Y \in \mathcal{I}\}.$$

The rank of the ground set E is called the rank of the matroid M. It is equal to the cardinality of the bases. For more details on matroids, one can see some excellent textbooks like [Oxl06, Sch03].

Linear Matroids. A well-known example of matroids is the *linear matroids*. A linear matroid over a field $\mathbb{F}$ is represented by an $r \times n$ matrix U over the field $\mathbb{F}$ with the full row rank. Assume that the columns are indexed by $[n]$, which is the ground set of the matroid. Let $\mathcal{B} = \{B \subseteq [n] \mid |B| = r,\ \det(U_B) \neq 0\}$. It is not hard to prove that $M = ([n], \mathcal{B})$ is a matroid with $\mathcal{B}$ as the base family.

Matroid Intersection. Let $M_1 = (E, \mathcal{B}_1)$ and $M_2 = (E, \mathcal{B}_2)$ be two matroids defined on the same ground set E. The problem of finding a common base is called matroid intersection problem. The problem of perfect matching for bipartite graphs and many other problems can be formulated in the language of the matroid intersection problem.

In this paper, we study symbolic matrix $M = \sum_{i=1}^n A_i x_i$ with each A_i having rank one. Next, we give an alternate representation of such symbolic matrices.

Observation 2.4. *Let $M = \sum_{i=1}^n A_i x_i$ where each A_i is a $r \times r$ rank one matrix over $\mathbb{F}$. Then, there exist $U, V \in \mathbb{F}^{r \times n}$ such that $M = UXV^T$ where X is the $n \times n$ diagonal matrix with x_i as its ith diagonal entry.*

Proof. Since A_i is a rank one matrix over $\mathbb{F}$, there exist $\mathbf{u}^i, \mathbf{v}^i \in \mathbb{F}^r$ such that $A_i = \mathbf{u}^i \cdot {\mathbf{v}^i}^T$. Let U and V be two $r \times n$ matrices such that the ith column of U and V are $\mathbf{u}^i$ and $\mathbf{v}^i$, respectively, for all $i \in [n]$. Then, for any $p, q \in [r]$, $UXV^T[p,q] = \sum_{i=1}^n u_p^i v_q^i x_i = \sum_{i=1}^n A_i[p,q] x_i$. This implies that $UXV^T = \sum_{i=1}^n A_i x_i$. □

Valuated Matroid. Dress and Wenzel [DW90, DW92] introduced the notion of valuated matroid. Here, we discuss it as described by Murota [Mur96]. Suppose that $M = (E, \mathcal{B})$ is a matroid with rank r. A *valuation* on a matroid M is a function ω from $\mathcal{B}$ to $\mathbb{Z} \cup \{+\infty\}$ such that for all $B, B' \in \mathcal{B}$ and $a \in B - B'$ there exists $b \in B' - B$ such that $B - a + b \in \mathcal{B}$, $B' - b + a \in \mathcal{B}$ and

$$\omega(B) + \omega(B') \geq \omega(B - a + b) + \omega(B' - b + a) \tag{1}$$

A matroid M with a valuation function ω on it is called a *valuated matroid*, and we denote it by the 3-tuple $(E, \mathcal{B}, \omega)$. The definition of the valuated matroids by Dress and Wenzel [DW90, DW92] and in a subsequent work by Murota [Mur96] consider the inequality in Equation 1 in the reverse direction. The reason is that their work talks about maximization problems over valuated matroids, but for our convenience, we describe their results in terms of minimization.

For a matrix, $U \in \mathbb{F}(\varepsilon)^{r \times n}$, the following lemma defines a valuation on the linear matroid represented by U. A very similar valuation has already been studied by Dress and Wenzel [DW92] and by Murota [Mur96, Example 3.2]. For an $f \in \mathbb{F}(\varepsilon)$ with $g, h \in \mathbb{F}[\varepsilon]$ and $f = g/h$, they consider $\deg_\varepsilon(f)$ instead of $\mathsf{val}(f)$ which is defined as the difference of degree of p and q.

Lemma 2.5. *Let U be an $r \times n$ matrix in $\mathbb{F}(\varepsilon)^{r \times n}$ with full row rank. Let $\mathcal{B}$ be the base family of the linear matroid representable by U. Let ω be a function from $\mathcal{B}$ to $\mathbb{Z} \cup \{+\infty\}$, defined as follows: for all $B \in \mathcal{B}$,*

$$\omega(B) = \mathsf{val}(\det(U_B)).$$

Then $([n], \mathcal{B}, \omega)$ forms a valuated matroid.

Proof. We prove the above lemma using the Grassmann-Plücker identity based technique used in [DW92]. From Grassmann-Plücker identity (Lemma 2.3), for any two distinct $S, T \subseteq [n]$ of size r and any $j \in T \setminus S$,

$$\det(U_S) \cdot \det(U_T) = \sum_{i \in S \setminus T} \mu_{i,j} \det(U_{S-i+j}) \cdot \det(U_{T-j+i}),$$

where $\mu_{i,j} \in \{1, -1\}$. Then, from Proposition 2.1, there exists a $k \in S \setminus T$ such that

$$\mathsf{val}(\det(U_S)) + \mathsf{val}(\det(U_T)) \geq \mathsf{val}(\det(U_{S-k+j})) + \mathsf{val}(\det(U_{T-j+k})).$$

This implies that if $S, T \in \mathcal{B}$, then for any $j \in T \setminus S$ there exists $k \in S \setminus T$ such that both $T - j + k$ and $S - k + j$ are in $\mathcal{B}$ and

$$\omega(S) + \omega(T) \geq \omega(S - k + j) + \omega(T - j + k).$$

Therefore, $([n], \mathcal{B}, \omega)$ forms a valuated matroid. $\square$

Suppose that $U_1 = (E, \mathcal{B}_1, \omega_1)$ and $U_2 = (E, \mathcal{B}_2, \omega_2)$ are two valuated matroids over the same ground set E. Let $\mathbf{w} : E \to \mathbb{Z}$ be a weight function. For any weight function,

$\mathbf{w}$ on the ground set E, it naturally extends to all the subsets of E as follows: for any $S \subseteq E$, $\mathbf{w}(S) = \sum_{a \in S} \mathbf{w}(a)$. Then, the *valuated matroid intersection* problem asks to find a common base $B \in \mathcal{B}_1 \cap \mathcal{B}_2$ that minimizes $\mathbf{w}(B) + \omega_1(B) + \omega_2(B)$. Like Frank's weight splitting theorem for weighted matroid intersection [Fra81], Murota [Mur96, Theorem 4.2] gave a weight splitting theorem for the valuated matroid intersection. Here, we describe the result on the minimization version of valuated matroid intersection whose proof can be deduced from the result on the maximization version in a natural way.

Lemma 2.6 (Weight-splitting). *Let $U_1 = (E, \mathcal{B}_1, \omega_1)$ and $U_2 = (E, \mathcal{B}_2, \omega_2)$ be two valuated matroids and $\mathbf{w}$ be a function from E to $\mathbb{Z}$. Then, there exist $\mathbf{w}^1, \mathbf{w}^2 : E \to \mathbb{Z}$ such that a common base B minimizes $\mathbf{w}(B) + \omega_1(B) + \omega_2(B)$ if and only if the following holds:*

1. $\mathbf{w}(e) = \mathbf{w}^1(e) + \mathbf{w}^2(e)$ *for all* $e \in E$.
2. *B is a minimum weight base for the matroid $U_1 = (E, \mathcal{B}_1)$ with respect to $\omega_1 + \mathbf{w}^1$.*
3. *B is a minimum weight base for the matroid $U_2 = (E, \mathcal{B}_2)$ with respect to $\omega_2 + \mathbf{w}^2$.*

3 Proof of our closure results

In this section, we prove Theorem 1.1 and Corollary 1.3. First, we discuss some lemmas that we use in the proof of our results. One of the ingredients of our proof is the fact that the maximal minors of $r \times n$ matrices parameterize a variety (Plücker embedding of the Grassmannian). Since a variety is Euclidean closed, we get that for any $r \times n$ matrix U over $\mathbb{F}(\varepsilon)$ whose $r \times r$ minors approach a vector $\mathbf{u} \in \mathbb{F}^{\binom{n}{r}}$ as $\varepsilon \to 0$, there exists an $r \times n$ matrix $\widehat{U}$ over $\mathbb{F}$ whose $r \times r$ minors equal to $\mathbf{u}$. The next lemma shows how such a matrix $\widehat{U}$ can be constructed. For notations, see Section 2.

Lemma 3.1. *Let U be a matrix in $\mathbb{F}(\varepsilon)^{r \times n}$ such that for every $S \subseteq [n]$ of size r, $\lim_{\varepsilon \to 0} \det(U_S)$ exists. Then, we can construct $\widehat{U}$ in $\mathbb{F}^{r \times n}$ such that for every $S \subseteq [n]$ of size r the following holds:*

$$\lim_{\varepsilon \to 0} \det(U_S) = \det(\widehat{U}_S).$$

Proof. First consider the trivial case when $\lim_{\varepsilon \to 0} \det(U_S)$ is zero for every $S \subseteq [n]$ of size r. In that case, $\widehat{U}$ can be defined as the matrix with all entries being zero. Now, we assume that there exists a $S \subseteq [n]$ of size r such that $\lim_{\varepsilon \to 0} \det(U_S)$ is non-zero. Without loss of generality, assume that $\lim_{\varepsilon \to 0} \det(U_{[r]})$ is nonzero. Let

$$U' = U_{[r]}^{-1} \cdot U.$$

Claim 3.2. *For every $S \subseteq [n]$ of size r, $\lim_{\varepsilon \to 0} \det(U'_S)$ exists.*

Proof. Since $U' = U_{[r]}^{-1} \cdot U$, for any $S \subseteq [n]$ of size r,

$$\det(U'_S) = \det(U_{[r]}^{-1}) \cdot \det(U_S).$$

Since $\det(U_{[r]}^{-1}) = 1/\det(U_{[r]})$ and $\mathsf{val}(\det(U_{[r]})) = 0$, from Proposition 2.1, $\mathsf{val}(\det(U_{[r]}^{-1}))$ is also zero. Therefore, applying Proposition 2.1, we get that $\lim_{\varepsilon \to 0} \det(U_{[r]}^{-1})$ is non-zero. The hypothesis of the lemma ensures that $\lim_{\varepsilon \to 0} \det(U_S)$ exists. Therefore,

$$\lim_{\varepsilon \to 0} \det(U'_S) = \lim_{\varepsilon \to 0} \det(U_{[r]}^{-1}) \cdot \lim_{\varepsilon \to 0} \det(U_S).$$

This implies that $\lim_{\varepsilon \to 0} \det(U'_S)$ exists. $\square$

Claim 3.3. *For every $i \in [r]$ and $j \in [n]$, $\lim_{\varepsilon \to 0} U'[i,j]$ exists.*

Proof. From the definition, $U' = [I_r | A]$ where I_r is the $r \times r$ identity matrix. The claim trivially follows for $i, j \in [r]$. For an $i \in [r]$ and $j \in [n] - [r]$, let $T = [r] - \{i\} + \{j\}$, and U'_T be the matrix obtained by replacing the ith column of I_r by the jth column of U'. This implies that the matrix U'_T is of the following form:

$$U'_T = \begin{bmatrix} 1 & 0 & 0 & \dots & U'[1,j] & \dots & 0 \\ 0 & 1 & 0 & \dots & U'[2,j] & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & U'[i,j] & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & U'[r,j] & \dots & 1 \end{bmatrix}.$$

Therefore, $\det(U'_T) = U'[i,j]$. From the hypothesis of the lemma combined with Proposition 2.1, we know that $\mathsf{val}(\det(U'_T)) \geq 0$. Hence, $\mathsf{val}(U'[i,j]) \geq 0$. Now applying Proposition 2.1, $\lim_{\varepsilon \to 0} U'[i,j]$ exists. $\square$

Now we define the matrix $\widetilde{U} \in \mathbb{F}^{r \times n}$ as follows: for all $i \in [r]$ and $j \in [n]$,

$$\widetilde{U}[i,j] := \lim_{\varepsilon \to 0} U'[i,j].$$

From Claim 3.3, the entries of the matrix $\widetilde{U}$ are well defined. Since determinant is a continuous function,

$$\lim_{\varepsilon \to 0} \det(U'_S) = \det(\widetilde{U}_S). \tag{2}$$

Let $\lim_{\varepsilon \to 0} det(U_{[r]}) = \alpha$. Consider the matrix $\widehat{U} \in \mathbb{F}^{r \times n}$ which exists by multiplying the first row of $\widetilde{U}$ by α, that is for all $i \in [r]$ and $j \in [n]$,

$$\widehat{U}[i,j] = \begin{cases} \alpha \cdot \widetilde{U}[i,j] & \text{if } i = 1 \\ \widetilde{U}[i,j] & \text{otherwise.} \end{cases}$$

The definition of $\widehat{U}$ implies that for any $S \subseteq [n]$ of size r,

$$\det(\widehat{U}_S) = \alpha \cdot \det(\widetilde{U}_S). \tag{3}$$

From the definition of U',

$$\lim_{\varepsilon\to 0}\det(U_S) = \lim_{\varepsilon\to 0}(\det(U_{[r]}\cdot\det(U'_S)).$$

Applying Claim 3.2, $\lim_{\varepsilon\to 0}\det(U'_S)$ exists. Therefore,

$$\begin{aligned}
\lim_{\varepsilon\to 0}\det(U_S) &= \lim_{\varepsilon\to 0}\det(U_{[r]})\cdot\lim_{\varepsilon\to 0}\det(U'_S) \\
&= \alpha\cdot\det(\widetilde{U}_S) && \text{[from Equation 2]} \\
&= \det(\widehat{U}_S) && \text{[from Equation 3].}
\end{aligned}$$

This completes the proof of our lemma. □

Suppose that U, V are two matrices in $\mathbb{F}(\varepsilon)^{r\times n}$ with full row rank. Let $\lim_{\varepsilon\to 0}(\det(U_S)\cdot\det(V_S))$ exists for all $S\subseteq[n]$ of size r. However, the limit value of $\det(U_S)$ and $\det(V_S)$ at $\varepsilon = 0$ individually may not exist for all S. Our next lemma shows that there exists two $r\times n$ matrices $\widetilde{U}$ and $\widetilde{V}$ such that the limit value of both $\det(U_S)\cdot\det(V_S)$ and $\det(\widetilde{U}_S)\cdot\det(\widetilde{V}_S)$ at $\varepsilon = 0$ are same and also the limit value of $\det(\widetilde{U}_S)$ and $\det(\widetilde{V}_S)$ at $\varepsilon = 0$ individually exists.

For a matrix $U\in\mathbb{F}(\varepsilon)^{r\times n}$ with full row rank, let us define

$$\mathsf{minval}(U) := \min_{S\in\binom{[n]}{r}}\mathsf{val}(\det(U_S)).$$

Lemma 3.4. *Let U, V in $\mathbb{F}(\varepsilon)^{r\times n}$ with full row rank. Let $\lim_{\varepsilon\to 0}\det(U_S)\cdot\det(V_S)$ exists for all $S\subseteq[n]$ of size r. Then, there exist $\widetilde{U},\widetilde{V}$ in $\mathbb{F}(\varepsilon)^{r\times n}$ such that for every $S\subseteq[n]$ of size r the following holds:*

$$\lim_{\varepsilon\to 0}\det(U_S)\det(V_S) = \left(\lim_{\varepsilon\to 0}\det(\widetilde{U}_S)\right)\cdot\left(\lim_{\varepsilon\to 0}\det(\widetilde{V}_S)\right).$$

Proof. When $\lim_{\varepsilon\to 0}\det(U_S)\det(V_S) = 0$ for all $S\subseteq[n]$ of size r, the lemma is trivial to prove. Now we consider the case when there exists an $S\subseteq[n]$ of size r such that $\lim_{\varepsilon\to 0}\det(U_S)\det(V_S)\neq 0$. Next, we show that there exists a vector $\mathbf{z}\in\mathbb{Z}^n$ such that

$$\mathsf{minval}(U\cdot\mathrm{Diag}(\varepsilon^{\mathbf{z}})) + \mathsf{minval}(V\cdot\mathrm{Diag}(\varepsilon^{-\mathbf{z}})) = 0,$$

where $\mathrm{Diag}(\varepsilon^{\mathbf{z}})$ is the diagonal matrix with (i,i)th entry as ε^{z_i}. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the base families for the linear matroid represented by U and V, respectively. Let ω_1 be a function from $2^{[n]}$ to $\mathbb{Z}\cup\{+\infty\}$ defined as follows: for all $B\in 2^{[n]}$,

$$\omega_1(B) = \begin{cases}\mathsf{val}(\det(U_B)) & \text{if } B\in\mathcal{B}_1\\ +\infty & \text{otherwise}\end{cases}$$

Similarly, we can define $\omega_2 : 2^{[n]}\to\mathbb{Z}\cup\{+\infty\}$ for the matrix V. Now, from Lemma 2.5, both $([n],\mathcal{B}_1,\omega_1)$ and $([n],\mathcal{B}_2,\omega_2)$ are valuated matroids. Therefore, applying Lemma 2.6 with $\mathbf{w}$ as the zero function on $[n]$, there exists a weight function $\mathbf{z} : [n]\to\mathbb{Z}$ such that a common base $B\in\mathcal{B}_1\cap\mathcal{B}_2$ minimizes $\omega_1(B)+\omega_2(B)$ if and only if the following holds:

1. B is a minimum weight base for the matroid $([n], \mathcal{B}_1)$ with respect to $\omega_1 + \mathbf{z}$.

2. B is a minimum weight base for the matroid $([n], \mathcal{B}_2)$ with respect to $\omega_2 - \mathbf{z}$.

Abusing notation, let $\mathbf{z}$ also denote a vector in $\mathbb{Z}^n$ with ith coordinate as $\mathbf{z}(i)$. Let $U' = U \cdot \mathrm{Diag}(\varepsilon^{\mathbf{z}})$ and $V' = V \cdot \mathrm{Diag}(\varepsilon^{-\mathbf{z}})$. From the definitions, $\mathsf{minval}(U')$ is the minimum weight of a base of $([n], \mathcal{B}_1)$ with respect to $\omega_1 + \mathbf{z}$. Similarly, $\mathsf{minval}(V')$ is the minimum weight of a base of $([n], \mathcal{B}_2)$ with respect to $\omega_2 - \mathbf{z}$. Since for every $S \subseteq [n]$ of size r, $\lim_{\varepsilon \to 0} \det(U_S) \det(V_S)$ exists, for all $B \in \mathcal{B}_1 \cap \mathcal{B}_2$, $\mathsf{val}(\det(U_B)) + \mathsf{val}(\det(V_B)) \geq 0$. On the other hand, from our assumption, there exists an $S \subseteq [n]$ of size r such that $\lim_{\varepsilon \to 0} \det(U_S) \det(V_S) \neq 0$. Therefore,

$$\min_{B \in \mathcal{B}_1 \cap \mathcal{B}_2} \mathsf{val}(\det(U_B)) + \mathsf{val}(\det(V_B)) = 0.$$

This implies that

$$\begin{aligned}
\mathsf{minval}(U') + \mathsf{minval}(V') &= \min_{B \in \mathcal{B}_1} (\omega_1 + \mathbf{z})(B) + \min_{B \in \mathcal{B}_2} (\omega_2 - \mathbf{z})(B) \\
&= \min_{B \in \mathcal{B}_1 \cap \mathcal{B}_2} \omega_1(B) + \omega_2(B) \\
&= \min_{B \in \mathcal{B}_1 \cap \mathcal{B}_2} \mathsf{val}(\det(U_B)) + \mathsf{val}(\det(V_B)) \\
&= 0.
\end{aligned}$$

Let $c = \mathsf{minval}(U') = -\mathsf{minval}(V')$. Let $\widetilde{U}$ and $\widetilde{V}$ be the matrix obtained by multiplying the first row of U' and V' by ε^{-c} and ε^{c}, respectively. Thus, for all $S \subseteq [n]$ of size r, we have that

$$\det(U_S) \cdot \det(V_S) = \det(U'_S) \cdot \det(V'_S) = \det(\widetilde{U}_S) \cdot \det(\widetilde{V}_S),$$

and $\mathsf{minval}(\widetilde{U}) = \mathsf{minval}(U') - c = 0$. Similarly, $\mathsf{minval}(\widetilde{V}) = 0$. This implies that for all $S \subseteq [n]$ of size r,

$$\lim_{\varepsilon \to 0} \det(U_S) \cdot \det(V_S) = \left(\lim_{\varepsilon \to 0} \det(\widetilde{U}_S) \right) \cdot \left(\lim_{\varepsilon \to 0} \det(\widetilde{V}_S) \right).$$

□

3.1 Proof of Theorem 1.1

In this subsection, we give the proof of Theorem 1.1. First, we prove for the case when $A_0 = 0$. From Observation 2.4, we get $U, V \in \mathbb{F}(\varepsilon)^{r \times n}$ such that $\sum_{i=1}^{n} A_i x_i = UXV^T$ where X is the diagonal matrix with x_i as its ith diagonal entry. Abusing notation, we use X_S to denote $\prod_{i \in S} x_i$. Therefore,

$$
\begin{aligned}
f &= \lim_{\varepsilon \to 0} \det\left(\sum_{i=1}^{n} A_i x_i\right) = \lim_{\varepsilon \to 0} \det(UXV^T) \\
&= \lim_{\varepsilon \to 0} \sum_{S \subseteq [n], |S| = r} \det(U_S)\det(V_S) X_S \qquad\qquad \text{[from Lemma 2.2]} \\
&= \sum_{S \subseteq [n], |S| = r} \left(\lim_{\varepsilon \to 0} \det(U_S)\det(V_S)\right) X_S.
\end{aligned}
$$

In the last equality above, we can take the limit inside as f is defined if and only if the limit exists for the coefficient of every monomial. Applying Lemma 3.4,

$$
f = \sum_{S \subseteq [n],\, |S| = r} \left(\lim_{\varepsilon \to 0} \det(\widetilde{U}_S)\right)\left(\lim_{\varepsilon \to 0} \det(\widetilde{V}_S)\right) X_S.
$$

From Lemma 3.1, we have two $r \times n$ matrices $\widehat{U}$ and $\widehat{V}$ in $\mathbb{F}^{r \times n}$ such that

$$
\begin{aligned}
f &= \sum_{S \subseteq [n],\, |S| = r} \det(\widehat{U}_S)\det(\widehat{V}_S) X_S \\
&= \det(\widehat{U} X \widehat{V}^T).
\end{aligned}
$$

For all $i \in [n]$, let B_i be the $r \times r$ rank one matrix defined as $\widehat{U}[i] \cdot \widehat{V}[i]^T$, where $\widehat{U}[i]$ and $\widehat{V}[i]$ are the ith columns of $\widehat{U}$ and $\widehat{V}$ respectively. Then,

$$
f = \det(\widehat{U} X \widehat{V}^T) = \det(\sum_{i=1}^{n} B_i x_i).
$$

This completes the proof of Theorem 1.1 where $A_0 = 0$.

For the case when $A_0 \neq 0$, we first give the following lemma that essentially reduces it to the previous case. The proof idea comes from Anderson, Shpilka, and Volk [ASV16] (as cited in [GT20, Lemma 4.3]).

For positive integers m and n, let I_n denote the $n \times n$ identity matrix and $0_{m,n}$ denote the $m \times n$ rectangular matrix with all zeros.

Lemma 3.5. *Let $P = \det\left(A_0 + UXV^T\right)$ for some U, V in $\mathbb{F}(\varepsilon)^{r \times n}$, $A_0 \in \mathbb{F}(\varepsilon)^{r \times r}$ and X is an $n \times n$ diagonal matrix with $x_1, x_2, \ldots, x_n$ in the diagonal. Let X' be a $(2n + r) \times (2n + r)$ diagonal matrix with $x_1, x_2, \ldots x_{2n+r}$ in the diagonal. Then, there exist rectangular matrices $U', V' \in \mathbb{F}(\varepsilon)^{(n+r) \times (2n+r)}$ such that the following holds:*

- *Let Q be the polynomial in $x_1, x_2, \ldots x_n$ obtained by putting $x_{n+1}, \ldots, x_{2n+r}$ equal to 1 in $det(U'X'V'^T)$. Then, $P = Q$.*
- *If $\lim_{\varepsilon \to 0} P$ exists, then $\lim_{\varepsilon \to 0} \det(U'X'V'^T)$ also exists.*

Proof. Let us define

$$U' = \left[\begin{array}{c|c|c} 0_{n,n} & I_n & V^T \\ \hline -U & 0_{r,n} & A_0 \end{array}\right] \quad \text{and, } V' = \left[\begin{array}{c|c|c} I_n & I_n & 0_{n,r} \\ \hline 0_{r,n} & 0_{r,n} & I_r \end{array}\right].$$

Let X_1 be a $n \times n$ diagonal matrix with $x_{n+1}, \dots, x_{2n}$ in the diagonal and X_2 be a $r \times r$ diagonal matrix with $x_{2n+1}, \dots, x_{2n+r}$ in the diagonal. We now consider $U'X'V'^T$. Notice that,

$$U'X'V'^T = \left[\begin{array}{c|c|c} 0_{n,n} & X_1 & V^T X_2 \\ \hline -UX & 0_{r,n} & A_0 X_2 \end{array}\right] \cdot \left[\begin{array}{c|c} I_n & 0_{n,r} \\ \hline I_n & 0_{n,r} \\ \hline 0_{r,n} & I_r \end{array}\right] = \begin{bmatrix} X_1 & V^T X_2 \\ -UX & A_0 X_2 \end{bmatrix}.$$

Let A, B, C, D be matrices where A and D are square matrices and A is invertible. Then, we have

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A). \det(D - CA^{-1}B)$$

Therefore,

$$\begin{aligned} \det(U'X'V'^T) &= \det(X_1). \det(A_0 X_2 + UXX_1^{-1} V^T X_2) \\ &= \det(X_1). \det(A_0 + UXX_1^{-1} V^T). \det(X_2). \end{aligned}$$

It is easy to see that if we put the value of 1 to $x_{n+1}, \dots, x_{2n+r}$, we get $\det(A_0 + UXV^T)$. Also,

$$\lim_{\varepsilon \to 0} \det(U'X'V'^T) = \det(X_1). \det(X_2). \lim_{\varepsilon \to 0} \det(A_0 + UXX_1^{-1} V^T).$$

The second part of the lemma follows from the fact that if $\lim_{\varepsilon \to 0} P$ exists, then $\lim_{\varepsilon \to 0} \det(A_0 + UXX_1^{-1} V^T)$ also exists as XX_1^{-1} can be treated as a diagonal matrix with a different set of indeterminates. $\square$

Now we prove for the case when $A_0 \neq 0$. Let $f = \det(A_0 + UXV^T)$ and $f' = \det(U'X'V'^T)$. From the Lemma 3.5, $\lim_{\varepsilon \to 0} f'$ exists as it is given that $\lim_{\varepsilon \to 0} f$ exists. Just like we discussed above for the case of $A_0 = 0$, we can get $\widehat{U}', \widehat{V}' \in \mathbb{F}^{(n+r) \times (2n+r)}$ such that $\lim_{\varepsilon \to 0} f' = \det(\widehat{U}' X' \widehat{V}'^T)$. For all $i \in [2n + r]$, let B_i be the $(n + r) \times (n + r)$ rank one matrix defined as $\widehat{U}'[i] \cdot \widehat{V}'[i]^T$, where $\widehat{U}'[i]$ and $\widehat{V}'[i]$ are the ith columns of $\widehat{U}'$ and $\widehat{V}'$ respectively. Hence, $\lim_{\varepsilon \to 0} f' = \det(\sum_{i=1}^{2n+r} B_i x_i)$. Let $\sum_{i=n+1}^{2n+r} B_i = B_0$. From the first part of Lemma 3.5, $\lim_{\varepsilon \to 0} f = \det(B_0 + \sum_{i=1}^{n} B_i x_i)$.

3.2 Proof of Corollary 1.3

We will show the following lemma which directly implies Corollary 1.3.

Lemma 3.6. *Let $A \in \mathbb{C}(\varepsilon)^{n\times n}$ be a matrix of rank at most k and $A[S]$ denote the minor of A whose rows and columns are indexed by $S \subseteq [n]$. Let $\lim_{\varepsilon\to 0} A[S]$ exist for all subset $S \subset [n]$ of size k. Then, there exists $B \in \mathbb{C}^{n\times n}$ such that for all $S \subset [n]$ of size k,*

$$\lim_{\varepsilon\to 0} A[S] = B[S]$$

Proof. The claim is trivial when $\mathrm{rank}(A) < k$ as all the minors are zero. Hence, we assume that $\mathrm{rank}(A) = k$. Let $U, V \in \mathbb{C}(\varepsilon)^{k\times n}$ such that U^T, V is a rank-factorization of A. This implies that $A = U^T.V$ and for any subset $S \subset [n]$, $A[S] = \det(U_S^T.V_S)$. Since $\lim_{\varepsilon\to 0} A[S] = \lim_{\varepsilon\to 0} \det(U_S)\det(V_S)$ exists for all $S \subset [n]$ of size k, from Lemma 3.4 there exists $\widetilde{U}, \widetilde{V}$ in $\mathbb{C}(\varepsilon)^{r\times n}$ such that for every $S \subseteq [n]$ of size k the following holds:

$$\lim_{\varepsilon\to 0} A[S] = \left(\lim_{\varepsilon\to 0} \det(\widetilde{U}_S)\right) \cdot \left(\lim_{\varepsilon\to 0} \det(\widetilde{V}_S)\right).$$

From Lemma 3.1, there exist two $k \times n$ matrices $\widehat{U}$ and $\widehat{V} \in \mathbb{C}^{k\times n}$ such that for all $S \subset [n]$, the following holds:

$$\lim_{\varepsilon\to 0} \det(\widetilde{U}_S) = \det(\widehat{U}_S) \text{ and } \lim_{\varepsilon\to 0} \det(\widetilde{V}_S) = \det(\widehat{V}_S)$$

Let $B = \widehat{U}^T.\widehat{V}$. Hence, for all $S \subset [n]$ of size k,

$$\lim_{\varepsilon\to 0} A[S] = \det(\widehat{U}_S^T).\det(\widehat{V}_S) = \det(\widehat{U}_S^T.\widehat{V}_S) = B[S].$$

□

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