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New Explicit Constant-Degree Lossless Expanders

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Abstract

We present a new explicit construction of onesided bipartite lossless expanders of constant degree, with arbitrary constant ratio between the sizes of the two vertex sets. Our construction is simpler to state and analyze than the only prior construction of Capalbo, Reingold, Vadhan, and Wigderson (2002), and achieves improvements in some parameters.

We construct our lossless expanders by imposing the structure of a constant-sized lossless expander "gadget" within the neighborhoods of a large bipartite spectral expander; similar constructions were previously used to obtain the weaker notion of unique-neighbor expansion. Our analysis simply consists of elementary counting arguments and an application of the expander mixing lemma.

1 Introduction

We construct infinite families of constant-degree onesided lossless bipartite expanders with arbitrary constant ratio between the sizes of the left and right vertex sets. A lossless expander is defined as a graph for which for all sufficiently small vertex sets, most of the outgoing edges lead to distinct vertices. These objects are applicable to various areas of computer science, including networks and distributed algorithms [PU89, ALM96, BFU99, MMP20], compressed sensing [XH07, JXHC09, IR08], error-correcting codes [SS96, LH22a, LH22b], and proof complexity [BSW01, ABSRW04, AR01], among others.

While constant-degree random graphs give lossless expanders with high probability, the only previously known explicit construction[1] was obtained by Capalbo, Reingold, Vadhan, and Wigderson [CRVW02], using a fairly involved form of the zigzag product [RVW02]. Therefore given the numerous applications of lossless expanders described above, it is desirable to have additional, simpler explicit constructions. A new such construction, which simplifies the construction of [CRVW02], is the main result of this paper.

Unlike lossless expanders, there are numerous known explicit constructions of spectral expanders (e.g. [Mar73, LPS88, Mor94, RVW02, BATS11, KO18]), which are defined as graphs with no large nontrivial eigenvalues of the adjacency matrix. Yet Kahale [Kah95] showed that even optimal

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[1]Shortly after our paper was posted online, a similar result obtained independently and concurrently was posted in [CRTS23]; see Remark 3.

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spectral expanders can fail to exhibit lossless expansion. Hence constructions of lossless expanders must rely on different techniques.

We now formally define lossless expansion.

Definition 1. For real numbers $0 \leq \mu, \epsilon \leq 1$, a bipartite graph $G = (L(G) \sqcup R(G), E(G))$ with left-degree d is a (onesided) (μ, ϵ)-**lossless expander** if for every $S \subseteq L(G)$ with $|S| \leq \mu|L(G)|$, it holds that $|N_G(S)| \geq (1-\epsilon)d|S|$.

The main result of [CRVW02], which we recover with a simpler construction and analysis, is stated below.

Theorem 2. *For every open interval $\beta = (\beta^{(1)}, \beta^{(2)}) \subseteq \mathbb{R}_+$ and every $\epsilon > 0$, there exists a sufficiently large $D = D(\beta, \epsilon) \in \mathbb{N}$ and a sufficiently small $\mu = \mu(\beta, \epsilon) > 0$ such that there is an infinite explicit family of (μ, ϵ)-lossless expanders G with left-degree D and with $|R(G)|/|L(G)| \in (\beta^{(1)}, \beta^{(2)})$.*

For fixed β, ϵ, our construction achieves a smaller degree D than [CRVW02], but [CRVW02] achieves a larger expansion cutoff μ (see Remark 19). Thus the two constructions have incomparable parameters. For instance, for a fixed constant β, we obtain $D = \tilde{O}(1/\epsilon^2)$ and $\mu = \tilde{\Omega}(\epsilon^{10})$, whereas [CRVW02] obtains $D = \tilde{O}(1/\epsilon^3)$ and $\mu = \tilde{\Omega}(\epsilon^4)$. Neither construction achieves the optimal dependencies $D = \tilde{O}(1/\epsilon)$ and $\mu = \tilde{\Omega}(\epsilon^2)$ (see Proposition 12 below).

Reducing the degree D as a function of ϵ is useful in recent applications of lossless expanders [GMM22, DMOZ23]. For instance, by Corollary 10 of [GMM22], our improved dependence $D = \tilde{O}(1/\epsilon^2)$ immediately yields explicit constructions of matrices with the ℓ_p-restricted isometry property for new values of p; such matrices have applications to compressed sensing (see e.g. [AZGR15]).

We remark that while Theorem 2 provides explicit onesided lossless expansion, there remains no known explicit construction of twosided lossless expanders, which require lossless expansion from right to left as well as from left to right.

Theorem 2 provides constant-degree lossless expanders with any constant ratio $|R(G)|/|L(G)| = \Theta(1)$ between the sizes of the left and right vertex sets. Highly unbalanced lossless expanders, where $|R(G)|/|L(G)|$ decays polynomially and the left-degree grows poly-logarithmically with the number of vertices, were constructed by [TSUZ07, GUV09, KTS22] to obtain randomness extractors.

Our construction follows the same framework as the unique-neighbor expander constructions of [AC02, Bec16, AD23, HMMP23] in that we begin with a good spectral expander and then impose the structure of a smaller "gadget graph" locally in the neighborhoods of vertices in the spectral expander. In fact, our construction is essentially the same as the onesided unique neighbor expanders of [AD23], though we provide a new analysis in order to obtain the stronger object of onesided lossless expanders. Note that unlike lossless expanders, which require most vertices in the neighborhood of every sufficiently small set S to be connected to S by a unique edge, unique-neighbor expanders only require the existence of a single such vertex[2] in the neighborhood of S. Furthermore, whereas the analysis of [AD23] requires the construction to be instantiated with an unbalanced bipartite Ramanujan graph with the optimal twosided expansion (of which few constructions are currently known), our analysis is more robust in that near-Ramanujan onesided expansion suffices (see Remark 11 for details).

[2] Some definitions of unique-neighbor expansion make the stronger requirement that at least a small constant fraction of the neighbors of S are connected to S by a single edge. However, this requirement is still weaker than lossless expansion.

We construct our lossless expanders G by combining a large unbalanced bipartite spectral expander X with a constant-sized lossless expander (a "gadget") G_0 as follows. We let $L(G) = L(X)$ and $R(G) = R(X) \times R(G_0)$, and then let G be the union of $|R(X)|$ copies of the gadget G_0. Specifically, we add to G a copy of G_0 for each $v \in R(X)$ by associating neighbors of v with left-vertices of G_0, and elements of $\{v\} \times R(G_0)$ with right-vertices of G_0. Note that this construction requires the right-degree of X to equal $|L(G_0)|$.

Whereas our construction combines a large unbalanced spectral expander with a small lossless expander, the construction of [CRVW02] combines a large balanced spectral expander with two small gadgets, namely one lossless expander and one more sophisticed object called a "buffer conductor." The unbalanced spectral expander in our construction essentially serves the same purpose as the combination of the balanced spectral expander and buffer conductor in [CRVW02].

We remark that both our construction and that of [CRVW02] yield graphs permitting a free group action on the vertices and edges, with group size linear in the number of vertices. Onesided lossless expanders permitting such group actions in turn give asymptotically good locally testable codes by [LH22a]. Thus our construction implies a new family of good locally testable codes.

Similarly, [LH22b] show that *twosided* lossless expanders permitting a group action imply asymptotically good quantum LDPC codes with linear-time decoders. While no such expanders are currently known, it is an interesting question whether our techniques could be extended to obtain twosided expanders that can instantiate these codes. However, we note that there are other unconditional constructions of good quantum LDPC codes [PK21, LZ22, DHLV23] with linear-time decoders [LZ23, GPT23, DHLV23].

Remark 3. In independent and concurrent work, Cohen, Roth, and Ta-Shma [CRTS23] obtained a similar construction of lossless expanders. Specifically, [CRTS23] and our work both use the same framework described above of combining a large unbalanced bipartite spectral expander with a constant-sized lossless expander. However, [CRTS23] construct the large bipartite spectral expander using the hyper-regular high-dimensional expanders (HDXs) of [FI20]. In contrast, we show that it suffices to use HDXs with weaker regularity properties, or to simply use bipartite Ramanujan graphs.

2 Preliminaries

This section introduces basic notions and known results.

For a graph $G = (V(G), E(G))$ and a set of vertices $S \subseteq V(G)$, we let $N_G(S)$ denote the set of neighbors of S in G. For $v, v' \in V(G)$, we let $w_G(v, v')$ denote the weight of the edge from v to v' (which for a simple graph is always 0 or 1). Similarly, for $S, S' \subseteq V(G)$, we let $w_G(S, S') = \sum_{(v,v') \in S \times S'} w_G(v, v')$ denote the sum of the weights of edges from vertices in S to vertices in S'. For a vertex $v \in V(G)$, the degree $\deg(v) = w_G(v, V(G))$ equals the sum of the weights of the edges incident to that vertex. For a bipartite graph G, we let $V(G) = L(G) \sqcup R(G)$ denote the decomposition into the left and right vertex sets.

The main focus of our paper is to construct lossless expanders satisfying the following standard notion of explicitness.

Definition 4. A family of graphs is **explicit** if there exists a $\mathrm{poly}(n)$-time algorithm that takes as input an integer n, and outputs an n-vertex graph in the family, if one exists.

Our analysis will rely heavily on the notion of spectral expansion, defined below.

Definition 5. For an n-vertex graph G, the **(onesided) spectral expansion** $\lambda_2(G)$ is defined as the second largest eigenvalue of the random walk matrix of G. Formally, letting W_G denote the random walk matrix, so that $(W_G)_{v,v'} = w_G(v,v')/\deg(v)$, if the eigenvalues of W_G are $1 = \lambda_1(W_G) \geq \lambda_2(W_G) \geq \cdots \geq \lambda_n(W_G)$, then $\lambda_2(G) := \lambda_2(W_G)$ is the onesided spectral expansion.

We will make use of the following well known property of spectral expanders.

Lemma 6 (Expander Mixing Lemma; see for instance Lemma 4.15 of [Vad12])**.** *For a D-regular graph G, it holds for every subset of vertices $S \subseteq V(G)$ that*

$$w_G(S,S) \leq \left(\lambda_2(G) + \frac{|S|}{|V(G)|}\right) D|S|.$$

We will make use of unbalanced bipartite graphs for which the "nonlazy" or "nonbacktracking" length-2 walk (that is, the square) has good spectral expansion.

Definition 7. For a bipartite graph G, the **nonlazy square** G' is the graph on vertex set $V(G') = R(G)$, with edge weights given for $v,v' \in R(G)$ by $w_{G'}(v,v') = 0$ if $v = v'$ and $w_{G'}(v,v') = \sum_{w \in L(G)} w_G(v,w) w_G(w,v')$ if $v \neq v'$.

Proposition 8. *For every integer $k \geq 2$ and for every $\lambda_2 \geq 0$, it holds for infinitely many $D \in \mathbb{N}$ that there exists an infinite explicit family of (k,D)-biregular bipartite graphs whose nonlazy square has (onesided) spectral expansion $\leq \lambda_2$.*

Proof. We describe two different known constructions that each prove the proposition:

1. If X is a $(k-1)$-dimensional simplicial complex for $k \in \mathbb{N}$ and G is the incidence graph between $(k-1)$-dimensional faces $X(k-1) = L(G)$ and vertices $X(0) = R(G)$, then the nonlazy square G' of G is the 1-skeleton of X. For any fixed $k \in \mathbb{N}$ and $\lambda_2 > 0$, Ramanujan complexes [LSV05b, LSV05a] as well as the coset complexes of [KO18, OP22] provide examples of explicit such $(k-1)$-dimensional simplicial complexes X with constant degree and arbitrarily good spectral expansion $\lambda_2(G') \leq \lambda_2$.

2. Let G be a (k,D)-biregular bipartite Ramanujan graph, for instance as constructed explicitly in [GM21], so that every nontrivial eigenvalue of the unnormalized adjacency matrix M_G of G is at most $\lambda_2(M_G) \leq \sqrt{D-1} + \sqrt{k-1}$. We emphasize that here $(M_G)_{v,v'} = w_G(v,v')$ is unnormalized, so $\lambda_2(M_G) \neq \lambda_2(G) = \lambda_2(W_G)$. Then every nontrivial eigenvalue of the unnormalized adjacency matrix M_{G^2} of the (ordinary) square G^2 is at most $\lambda_2(M_G)^2 \leq (\sqrt{D-1}+\sqrt{k-1})^2 = (D-1)+(k-1)+2\sqrt{(D-1)(k-1)}$. Here we have used the fact that the spectrum of M_G is symmetric about 0 as G is bipartite. Furthermore, M_{G^2} is block diagonal with blocks $L(G) \times L(G)$ and $R(G) \times R(G)$ both having spectrum equal to the square of the spectrum of M_G, up to zero-eigenvalues. Therefore $\lambda_2(M_{G^2}|_{R(G)}) = \lambda_2(M_G)^2$. Thus every nontrivial eigenvalue of the unnormalized adjacency matrix $M_{G'}$ of the nonlazy square G' is at most $\lambda_2(M_{G'}) = \lambda_2(M_{G^2}|_{R(G)}) - D \leq (k-1) + 2\sqrt{(D-1)(k-1)}$. As G' is $D(k-1)$-regular, it holds that $W_{G'} = M_{G'}/D(k-1)$, so $\lambda_2(G') = \lambda_2(W_{G'}) \leq 1/D + 2/\sqrt{D(k-1)}$. Thus $\lambda_2(G') \leq \lambda_2$ if D is sufficiently large.

□

Remark 9. As Ramanujan complexes are Cayley complexes, the construction in the 1st proof of Proposition 8 has the added benefit of permitting a free group action on the vertices and faces that acts transitively on the vertices. Our entire construction can be made to respect this group action, and the orbits have linear size with respect to the number of vertices. Hence our construction can be used to instantiate the asymptotically good locally testable codes of [LH22a].

Remark 10. We prove Proposition 8 for the ordinary notion of (weak) explicitness given in Definition 4. However, we could instead consider *strong explicitness*, which requires the family of graphs to have a $\text{poly}(\log n)$-time algorithm that takes as input integers n, i, j, and outputs the jth neighbor of the ith vertex of the n-vertex graph in the family, if one exists. Some of our constructions proving Proposition 8, such as the construction using the high-dimensional expanders of [KO18, OP22], satisfy this notion of strong explicitness (see [OP22] for a proof). With such an instantiation, our entire construction of lossless expanders becomes strongly explicit. However, for simplicity in this paper we primarily discuss ordinary (weak) explicitness.

Remark 11. The 2nd proof of Proposition 8 is robust in the sense that it will still work for onesided near-Ramanujan bipartite graphs, that is, for (k, D)-biregular bipartite graphs for which the nontrivial eigenvalues of the adjacency matrix have absolute value $\leq \sqrt{D-1} + \sqrt{k-1} + \alpha$, as long as $\alpha < o(\sqrt{D})$. It for instance follows that this argument works for random bipartite graphs, which achieve $\alpha = o(1)$ [BDH22]. In constrast, the analysis of unique-neighbor expansion in [AD23] requires exactly Ramanujan bipartite graphs with twosided expansion, meaning they require all nontrivial eigenvalues to have absolute value lying in the interval $[\sqrt{D-1} - \sqrt{k-1}, \sqrt{D-1} + \sqrt{k-1}]$.

We will make use of the following bound showing lossless expansion of random bipartite graphs.

Proposition 12 (Well known; see for instance Theorem 11.2.8 of [GRS22])**.** *For all constants $\beta, \epsilon > 0$, there exists an integer $d = d(\beta, \epsilon) = \Theta(\log(1/\epsilon\beta)/\epsilon)$, a sufficiently large integer $n_0 = n_0(\beta, \epsilon)$, and a sufficiently small real number $\mu = \mu(\beta, \epsilon) = \Theta(\epsilon\beta/d)$ such that for all $n \geq n_0$, there exists a bipartite graph G with left-degree d and with $|L(G)| = n$, $|R(G)| = \lfloor \beta n \rfloor$ such that G is a (μ, ϵ)-lossless expander.*

3 Construction

In this section, we present our construction that we use to prove Theorem 2. We will subsequently prove that this construction has lossless expansion in Section 4. Our construction uses essentially the same framework as the unique neighbor expanders of [AD23], though we analyze it differently to obtain the stronger object of lossless expanders.

3.1 General framework

We first describe the general framework for constructing our lossless expanders G, and then present precise parameters. Throughout this section we fix a constant interval $\beta = (\beta^{(1)}, \beta^{(2)}) \subseteq \mathbb{R}_+$ inside which we want the $|R(G)|/|L(G)|$ to lie, and we fix $\epsilon > 0$ denoting the desired expansion.

To construct G, we begin by taking a (k, D_0)-biregular graph X for which the nonlazy square X' has good spectral expansion $\lambda_2(X')$. We choose X using Proposition 8, so we think of $|V(X)|$ growing arbitrarily large for fixed $k \ll D_0$ and fixed $\lambda_2(X') = \text{poly}(1/k, \beta^{(2)})$.

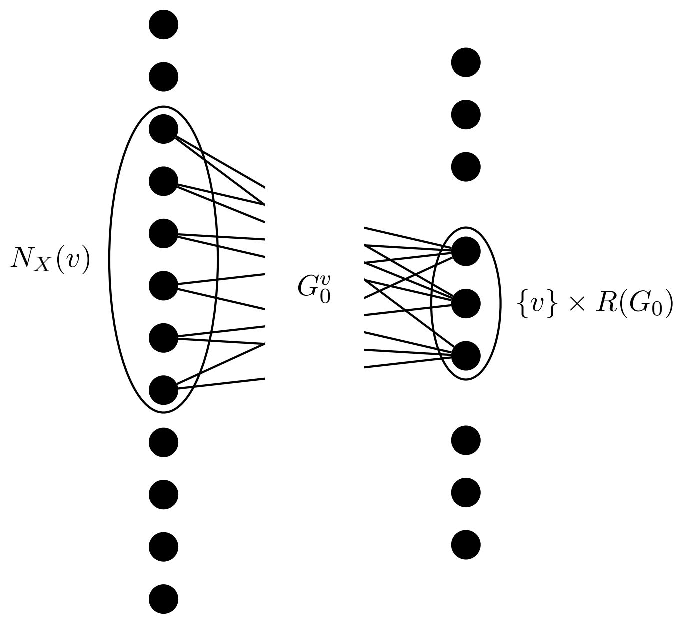

Figure 1: An illustration of how we construct our lossless expanders $G = G(X, G_0)$ as the union over $v \in R(X)$ of the gadgets $G_0^v \cong G_0$. A single such gadget is shown above, connecting the left vertices $L(G_0^v) = N_X(v) \cong L(G_0)$ to the right vertices $R(G_0^v) = \{v\} \times R(G_0) \cong R(G_0)$.

We also choose a constant-sized bipartite "gadget" graph G_0 with $|L(G_0)| = D_0$ and $|R(G_0)| = \lfloor D_0\beta^{(2)}/k \rfloor$ that is a lossless expander, as guaranteed to exist by Proposition 12.

We now define our desired lossless expander $G = G(X, G_0)$ as follows. An illustration is provided in Figure 1.

- The left vertex set $L(G) = L(X)$ of G equals the left vertex set of X.
- The right vertex set $R(G) = R(X) \times R(G_0)$ of G is obtained by replacing each right vertex of X with a cluster of $|R(G_0)|$ vertices.
- The edge set $E(G)$ is defined as follows. For each $v \in R(X)$, as $|N_X(v)| = D_0 = |L(G_0)|$, we may associate the neighborhood $N_X(v)$ with $L(G_0)$. We may similarly associate $\{v\} \times R(G_0) \subseteq R(G)$ with $R(G_0)$. Therefore we may introduce a copy G_0^v of G_0 with left vertex set $L(G_0^v) := N_X(v) \subseteq L(G)$ and right vertex set $R(G_0^v) := \{v\} \times R(G_0) \subseteq R(G)$. We then let G be the union of the graphs G_0^v over all $v \in R(X)$. That is, $(w, (v, v_0)) \in E(G)$ if and only if $(w, (v, v_0)) \in E(G_0^v)$.

The resulting graph $G = G(X, G_0) := \sum_{v \in L(X)} G_0^v$ satisfies the following basic properties.

Claim 13. *If G_0 has left-degree d_0, then G has left-degree $D = d_0 k$.*

Proof. Each vertex $w \in L(G) = L(X)$ has k X-neighbors $v \in N_X(w) \subseteq R(X)$, for each of which the graph G_0^v contributes d_0 edges to w in G. □

Claim 14. *If $D_0 \geq k/(\beta^{(2)} - \beta^{(1)})$, then $|R(G)|/|L(G)| \in (\beta^{(1)}, \beta^{(2)})$.*

Proof. By construction

$$\frac{|R(G)|}{|L(G)|} = \frac{|R(X)| \cdot |R(G_0)|}{|L(X)|} = \frac{k}{D_0} \cdot \left\lfloor \frac{D_0}{k} \cdot \beta^{(2)} \right\rfloor,$$

which is at most $\beta^{(2)}$ and at least $k/D_0 \cdot (D_0/k \cdot \beta^{(2)} - 1) = \beta^{(2)} - k/D_0$. Thus the claim follows. □

3.2 Choosing the parameters

Formally, our construction uses the following parameters and components, for fixed $\beta = (\beta^{(1)}, \beta^{(2)}) \subseteq \mathbb{R}_+$ and $\epsilon > 0$:

- Let $k = \lceil 10/\epsilon \rceil$.
- For balance constant $\beta_0 = \beta^{(2)}/k$ and loss constant $\epsilon_0 = \epsilon/10$, let $d_0 = d_0(\beta_0, \epsilon_0) = \Theta(\log(k/\beta^{(2)}\epsilon)/\epsilon)$, $n_0 = n_0(\beta_0, \epsilon_0)$, and $\mu_0 = \mu_0(\beta_0, \epsilon_0) = \Theta(\epsilon\beta^{(2)}/d_0 k)$ be the degree, size bound, and relative set size bound respectively given by Proposition 12.
- Let $\lambda_2 = \mu_0/10k^3$.
- Let $D_0 \geq \max\{n_0, k/(\beta^{(2)} - \beta^{(1)})\}$ be an integer such that there exists an infinite explicit family $\mathcal{X}$ of (k, D_0)-biregular bipartite graphs $X \in \mathcal{X}$ for which the nonlazy square X' has $\lambda_2(X') \leq \lambda_2$. Such a D_0 exists by Proposition 8.
- Let G_0 be a bipartite graph with left-degree d_0 and with $|L(G_0)| = D_0$, $|R(G_0)| = \lfloor \beta_0 D_0 \rfloor$ that is a (μ_0, ϵ_0)-lossless expander, as given by Proposition 12.

Our desired family $\mathcal{G}$ of graphs is then defined as $\mathcal{G} = \{G(X, G_0) : X \in \mathcal{X}\}$, where $G = G(X, G_0)$ is constructed from X and G_0 as described above.

The following explicitness claim is immediate from our construction, as $G = G(X, G_0)$ is by definition obtained from X by inserting vertices and edges locally within the (constant-sized) neighborhoods of vertices in X.

Claim 15. *If the family $\mathcal{X}$ is (strongly) explicit, then the family $\mathcal{G}$ is (strongly) explicit.*

Note that $\mathcal{X}$ can be made strongly explicit by using a strongly explicit family of high-dimensional expanders (e.g. [KO18]) as in the 1st proof of Proposition 8.

4 Proof of lossless expansion

We now show that the graphs $G \in \mathcal{G}$ defined in Section 3 have lossless expansion. Specifically, we show the following result, which when combined with Claim 13, Claim 14, and Claim 15 directly implies Theorem 2.

Proposition 16. *Defining all variables as in Section 3, then for every $X \in \mathcal{X}$, the bipartite graph $G = G(X, G_0) \in \mathcal{G}$ is a (μ, ϵ)-lossless expander for $\mu = k^2\lambda_2^2$.*

Proof. Fix any set $S \subseteq L(G) = L(X)$ of size $|S| \leq \mu|L(G)|$. Define the "heavy vertices" $H = \{v \in R(X) : |N_X(v) \cap S| \geq \mu_0 D_0\} \subseteq R(X)$ to be those vertices in $R(X)$ incident to $\geq \mu_0 D_0$ vertices in S. Below we present the key claim for our proof, which states that most vertices in S are incident to ≤ 1 heavy vertices, and therefore to $\geq k-1$ non-heavy vertices. We prove this claim with an application of the expander mixing lemma.

We first need the following notation. For $0 \leq i \leq k$, let $S_{=i} = \{w \in L(X) : |N_X(w) \cap H| = i\} \subseteq L(X)$ be the set of vertices in $L(X)$ incident to exactly i heavy vertices. Similary define $S_{\geq i} = \bigcup_{j \geq i} S_{=j}$ and $S_{\leq i} = \bigcup_{j \leq i} S_{=j}$.

Claim 17. *It holds that*

$$\frac{|S_{\leq 1}|}{|S|} \geq 1 - \frac{1}{5k}.$$

Proof of Claim 17. By definition

$$|H| \leq \frac{k|S|}{\mu_0 D_0} = \frac{|S|}{10k^2 \lambda_2 D_0} \leq \frac{\mu|L(G)|}{10k^2 \lambda_2 D_0} = \frac{\lambda_2 |R(X)|}{10},$$

where the equalities above apply the definitions of λ_2 and μ respectively. Thus letting X' be the nonlazy square of X, Lemma 6 implies that

$$w_{X'}(H, H) \leq \frac{11}{10} \lambda_2 D_0 (k-1)|H|.$$

For each vertex $w \in S_{\geq 2}$, we may choose two distinct heavy vertices $v, v' \in N_X(w) \cap H$, and let $e(w) = \{v, v'\} \in E(X')$ be the edge in X' induced by the path $v \to w \to v'$ in X. By definition all edges $e(w)$ for $w \in S_{\geq 2}$ are distinct,[3] and both endpoints of each $e(w)$ lie in H, so

$$|S_{\geq 2}| = |\{e(w) : w \in S_{\geq 2}\}| \leq w_{X'}(H, H) \leq \frac{11}{10} \lambda_2 D_0 (k-1)|H|.$$

Meanwhile, as each vertex $w \in S_{\geq 1}$ is incident to at least one but at most k heavy vertices, we have that

$$|S_{\geq 1}| \geq \frac{\mu_0 D_0 |H|}{k}.$$

Thus

$$\begin{aligned}
|S_{\leq 1}| &\geq |S_{=1}| \\
&= |S_{\geq 1}| - |S_{\geq 2}| \\
&\geq \left(\frac{\mu_0}{k} - \frac{11}{10} \lambda_2 (k-1) \right) D_0 |H| \\
&\geq 8k^2 \lambda_2 D_0 |H| \\
&\geq 5k |S_{\geq 2}|.
\end{aligned}$$

where the third inequality above applies the definition of λ_2. Thus

$$\frac{|S_{\leq 1}|}{|S|} = \frac{|S_{\leq 1}|}{|S_{\leq 1}| + |S_{\geq 2}|} \geq 1 - \frac{|S_{\geq 2}|}{|S_{\leq 1}|} \geq 1 - \frac{1}{5k}.$$

□

[3] Here we view $E(X')$ as a multiset where the edge (v, v') has a distinct copy for every path $v \to w \to v'$ in X.

We now prove the proposition using this fact that most vertices $w \in S$ are incident to at most one heavy vertex. By definition G_0 is a (μ_0, ϵ_0)-lossless expander, so the intersection of S with the X-neighborhood $N_X(v) \subseteq L(X) = L(G)$ of every non-heavy vertex $v \in R(X) \setminus H$ exhibits expansion $(1-\epsilon_0)d_0$ in the subgraph $G_0^v \cong G_0$ of G. Then it follows that most vertices in $L(G)$ contribute $(k-1)(1-\epsilon_0)d_0$ to the expansion of S in G, which implies the proposition.

Formally, for $v \in R(X) \setminus H$ then by definition $|N_X(v) \cap S| \leq \mu_0 D_0$, so

$$|N_{G_0^v}(N_X(v) \cap S)| \geq (1-\epsilon_0)d_0|N_X(v) \cap S|.$$

Thus

$$\begin{aligned}
|N_G(S)| &= \sum_{v \in R(X)} |N_{G_0^v}(N_X(v) \cap S)| \\
&\geq \sum_{v \in R(X) \setminus H} (1-\epsilon_0)d_0|N_X(v) \cap S| \\
&= \sum_{w \in S} \sum_{v \in N_X(w) \setminus H} (1-\epsilon_0)d_0 \\
&\geq \sum_{w \in S_{\leq 1}} (k-1)(1-\epsilon_0)d_0 \\
&\geq |S| \left(1 - \frac{1}{5k}\right)(k-1)(1-\epsilon_0)d_0 \\
&\geq |S| \left(1 - \frac{\epsilon}{50}\right)\left(1 - \frac{\epsilon}{10}\right) k \left(1 - \frac{\epsilon}{10}\right) d_0 \\
&\geq |S|(1-\epsilon)D,
\end{aligned}$$

where the third inequality holds by Claim 17, and the final inequality holds because $D = kd_0$ by Claim 13. Thus we have shown that G is a (μ, ϵ)-lossless expander, as desired. □

Remark 18. One could hope that our construction $G = G(X, G_0)$ happens to expand losslessly from right to left as well. However, if we fix any $v \in R(X)$, then the set $T = \{v\} \times R(G_0) \subseteq R(G)$ consisting of a single cluster of right vertices in G has neighborhood of size $|N_G(T)| = |N_X(v)| = D_0$, whereas $(\Omega(1), \epsilon)$-lossless right-to-left expansion would require the much larger neighborhood size $|N_G(T)| \geq (1-\epsilon)D_0 d_0$. Thus a new approach is needed for twosided expansion.

Remark 19. Given $\epsilon > 0$ and $\beta^{(2)} < 1/2$, by tracing through the parameters in Section 3.2, we find that our (μ, ϵ)-lossless expanders in Proposition 16 have degree

$$D = O\left(\frac{\log \frac{1}{\epsilon} + \log \frac{1}{\beta^{(2)}}}{\epsilon^2}\right)$$

and exhibit expansion up to the cutoff

$$\mu = \Omega\left(\frac{\epsilon^{10} {\beta^{(2)}}^2}{\left(\log \frac{1}{\epsilon} + \log \frac{1}{\beta^{(2)}}\right)^2}\right).$$

In comparison, the construction of [CRVW02] has degree

$$D' = O\left(\frac{\left(\log\frac{1}{\epsilon} + \log\frac{1}{\beta^{(2)}}\right)^2}{\epsilon^3}\right)$$

and exhibits expansion up to the cutoff

$$\mu' = \Theta\left(\frac{\epsilon\beta^{(2)}}{D'}\right) = \Omega\left(\frac{\epsilon^4\beta^{(2)}}{\left(\log\frac{1}{\epsilon} + \log\frac{1}{\beta^{(2)}}\right)^2}\right).$$

Thus the two constructions achieve incomparable parameters; we achieve a smaller degree $D < D'$, whereas [CRVW02] achieves a larger expansion cutoff $\mu' > \mu$.

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References

[ABSRW04] Michael Alekhnovich, Eli Ben-Sasson, Alexander A. Razborov, and Avi Wigderson. Pseudorandom Generators in Propositional Proof Complexity. *SIAM Journal on Computing*, 34(1):67–88, January 2004. Publisher: Society for Industrial and Applied Mathematics.

[AC02] N. Alon and M. Capalbo. Explicit unique-neighbor expanders. In *The 43rd Annual IEEE Symposium on Foundations of Computer Science, 2002. Proceedings.*, pages 73–79, November 2002. ISSN: 0272-5428.

[AD23] Ron Asherov and Irit Dinur. Bipartite unique-neighbour expanders via Ramanujan graphs, January 2023. arXiv:2301.03072 [cs, math].

[ALM96] Sanjeev Arora, F. T. Leighton, and Bruce M. Maggs. On-Line Algorithms for Path Selection in a Nonblocking Network. *SIAM Journal on Computing*, 25(3):600–625, June 1996. Publisher: Society for Industrial and Applied Mathematics.

[AR01] M. Alekhnovich and A.A. Razborov. Lower bounds for polynomial calculus: non-binomial case. In *Proceedings 42nd IEEE Symposium on Foundations of Computer Science*, pages 190–199, October 2001. ISSN: 1552-5244.

[AZGR15] Zeyuan Allen-Zhu, Rati Gelashvili, and Ilya Razenshteyn. Restricted Isometry Property for General p-Norms. In Lars Arge and János Pach, editors, *31st International*

Symposium on Computational Geometry (SoCG 2015), volume 34 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 451–460, Dagstuhl, Germany, 2015. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. ISSN: 1868-8969.

[BATS11] Avraham Ben-Aroya and Amnon Ta-Shma. A Combinatorial Construction of Almost-Ramanujan Graphs Using the Zig-Zag Product. *SIAM Journal on Computing*, 40(2):267–290, January 2011.

[BDH22] Gerandy Brito, Ioana Dumitriu, and Kameron Decker Harris. Spectral gap in random bipartite biregular graphs and applications. *Combinatorics, Probability and Computing*, 31(2):229–267, March 2022. Publisher: Cambridge University Press.

[Bec16] Oren Becker. Symmetric unique neighbor expanders and good LDPC codes. *Discrete Applied Mathematics*, 211:211–216, October 2016.

[BFU99] Andrei Z. Broder, Alan M. Frieze, and Eli Upfal. Static and Dynamic Path Selection on Expander Graphs: A Random Walk Approach. *Random Structures and Algorithms*, 14(1):87–109, January 1999.

[BSW01] Eli Ben-Sasson and Avi Wigderson. Short proofs are narrow—resolution made simple. *Journal of the ACM*, 48(2):149–169, March 2001.

[CRTS23] Itay Cohen, Roy Roth, and Amnon Ta-Shma. HDX Condensers. *ECCC*, 2023. TR23-090.

[CRVW02] Michael Capalbo, Omer Reingold, Salil Vadhan, and Avi Wigderson. Randomness conductors and constant-degree lossless expanders. In *Proceedings of the thiry-fourth annual ACM symposium on Theory of computing*, STOC '02, pages 659–668, New York, NY, USA, May 2002. Association for Computing Machinery.

[DHLV23] Irit Dinur, Min-Hsiu Hsieh, Ting-Chun Lin, and Thomas Vidick. Good Quantum LDPC Codes with Linear Time Decoders. In *Proceedings of the 55th Annual ACM Symposium on Theory of Computing*, STOC 2023, pages 905–918, New York, NY, USA, June 2023. Association for Computing Machinery.

[DMOZ23] Dean Doron, Dana Moshkovitz, Justin Oh, and David Zuckerman. Almost Chor-Goldreich Sources and Adversarial Random Walks. In *Proceedings of the 55th Annual ACM Symposium on Theory of Computing*, STOC 2023, pages 1–9, New York, NY, USA, June 2023. Association for Computing Machinery.

[FI20] Ehud Friedgut and Yonatan Iluz. Hyper-regular graphs and high dimensional expanders. *arXiv:2010.03829 [math]*, October 2020. arXiv: 2010.03829.

[GM21] Aurelien Gribinski and Adam W. Marcus. Existence and polynomial time construction of biregular, bipartite Ramanujan graphs of all degrees, August 2021. arXiv:2108.02534 [cs, math].

[GMM22] Venkatesan Guruswami, Peter Manohar, and Jonathan Mosheiff. lp-Spread and Restricted Isometry Properties of Sparse Random Matrices. In Shachar Lovett, editor, *37th Computational Complexity Conference (CCC 2022)*, volume 234 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 7:1–7:17, Dagstuhl, Germany, 2022. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. ISSN: 1868-8969.

[GPT23] Shouzhen Gu, Christopher A. Pattison, and Eugene Tang. An Efficient Decoder for a Linear Distance Quantum LDPC Code. In *Proceedings of the 55th Annual ACM Symposium on Theory of Computing*, STOC 2023, pages 919–932, New York, NY, USA, June 2023. Association for Computing Machinery.

[GRS22] Venkatesan Guruswami, Atri Rudra, and Madhu Sudan. Essential coding theory. *Draft available at http://www. cse. buffalo. edu/ atri/courses/coding-theory/book*, 2022.

[GUV09] Venkatesan Guruswami, Christopher Umans, and Salil Vadhan. Unbalanced expanders and randomness extractors from Parvaresh–Vardy codes. *Journal of the ACM*, 56(4):20:1–20:34, July 2009.

[HMMP23] Jun-Ting Hsieh, Theo McKenzie, Sidhanth Mohanty, and Pedro Paredes. Explicit two-sided unique-neighbor expanders, February 2023. arXiv:2302.01212 [cs, math].

[IR08] Piotr Indyk and Milan Ruzic. Near-Optimal Sparse Recovery in the L1 Norm. In *2008 49th Annual IEEE Symposium on Foundations of Computer Science*, pages 199–207, October 2008. ISSN: 0272-5428.

[JXHC09] Sina Jafarpour, Weiyu Xu, Babak Hassibi, and Robert Calderbank. Efficient and Robust Compressed Sensing Using Optimized Expander Graphs. *IEEE Transactions on Information Theory*, 55(9):4299–4308, September 2009. Conference Name: IEEE Transactions on Information Theory.

[Kah95] Nabil Kahale. Eigenvalues and expansion of regular graphs. *Journal of the ACM*, 42(5):1091–1106, September 1995.

[KO18] Tali Kaufman and Izhar Oppenheim. Construction of new local spectral high dimensional expanders. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, STOC 2018, pages 773–786, New York, NY, USA, June 2018. Association for Computing Machinery.

[KTS22] Itay Kalev and Amnon Ta-Shma. Unbalanced Expanders from Multiplicity Codes. Technical Report 073, 2022.

[LH22a] Ting-Chun Lin and Min-Hsiu Hsieh. c^3-Locally Testable Codes from Lossless Expanders, January 2022. arXiv:2201.11369 [cs, math].

[LH22b] Ting-Chun Lin and Min-Hsiu Hsieh. Good quantum LDPC codes with linear time decoder from lossless expanders. *arXiv:2203.03581 [quant-ph]*, March 2022. arXiv: 2203.03581.

[LPS88] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, September 1988.

[LSV05a] Alexander Lubotzky, Beth Samuels, and Uzi Vishne. Explicit constructions of Ramanujan complexes of type $\tilde{A}_d$. *European Journal of Combinatorics*, 26(6):965–993, August 2005.

[LSV05b] Alexander Lubotzky, Beth Samuels, and Uzi Vishne. Ramanujan complexes of type $\tilde{A}_d$. *Israel Journal of Mathematics*, 149(1):267–299, December 2005.

[LZ22] Anthony Leverrier and Gilles Zémor. Quantum Tanner codes. In *2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 872–883. IEEE Computer Society, October 2022.

[LZ23] Anthony Leverrier and Gilles Zémor. Efficient decoding up to a constant fraction of the code length for asymptotically good quantum codes. In *Proceedings of the 2023 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, Proceedings, pages 1216–1244. Society for Industrial and Applied Mathematics, January 2023.

[Mar73] Grigorii Aleksandrovich Margulis. Explicit constructions of expanders. *Problemy Peredachi Informatsii*, 9(4):71–80, 1973. Publisher: Russian Academy of Sciences, Branch of Informatics, Computer Equipment and ....

[MMP20] Anuran Makur, Elchanan Mossel, and Yury Polyanskiy. Broadcasting on Random Directed Acyclic Graphs. *IEEE Transactions on Information Theory*, 66(2):780–812, February 2020. Conference Name: IEEE Transactions on Information Theory.

[Mor94] M. Morgenstern. Existence and Explicit Constructions of q + 1 Regular Ramanujan Graphs for Every Prime Power q. *Journal of combinatorial theory. Series B*, 62(1):44–62, 1994. Place: SAN DIEGO Publisher: Elsevier Inc.

[OP22] Ryan O'Donnell and Kevin Pratt. High-Dimensional Expanders from Chevalley Groups. In Shachar Lovett, editor, *37th Computational Complexity Conference (CCC 2022)*, volume 234 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 18:1–18:26, Dagstuhl, Germany, 2022. Schloss Dagstuhl – Leibniz-Zentrum für Informatik. ISSN: 1868-8969.

[PK21] Pavel Panteleev and Gleb Kalachev. Asymptotically Good Quantum and Locally Testable Classical LDPC Codes. *arXiv:2111.03654 [quant-ph]*, November 2021. arXiv: 2111.03654.

[PU89] D. Peleg and E. Upfal. Constructing disjoint paths on expander graphs. *Combinatorica*, 9(3):289–313, September 1989.

[RVW02] Omer Reingold, Salil Vadhan, and Avi Wigderson. Entropy Waves, the Zig-Zag Graph Product, and New Constant-Degree Expanders. *The Annals of Mathematics*, 155(1):157, January 2002.

[SS96] M. Sipser and D.A. Spielman. Expander codes. *IEEE Transactions on Information Theory*, 42(6):1710–1722, November 1996.

[TSUZ07] Amnon Ta-Shma*, Christopher Umans†, and David Zuckerman‡. Lossless Condensers, Unbalanced Expanders, And Extractors. *Combinatorica*, 27(2):213–240, March 2007.

[Vad12] Salil P. Vadhan. Pseudorandomness. *Foundations and Trends® in Theoretical Computer Science*, 7(1–3):1–336, December 2012.

[XH07] Weiyu Xu and Babak Hassibi. Efficient Compressive Sensing with Deterministic Guarantees Using Expander Graphs. In *2007 IEEE Information Theory Workshop*, pages 414–419, September 2007.

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