Electronic Colloquium on Computational Complexity, Revision 1 of Report No. 117 (2023)

Space-bounded quantum state testing
via space-efficient quantum singular value transformation

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Abstract

Driven by exploring the power of quantum computation with a limited number of qubits, we present a novel complete characterization for space-bounded quantum computation, which encompasses settings with one-sided error (unitary coRQL) and two-sided error (BQL), approached from a quantum (mixed) state testing perspective:

- The *first* family of natural complete problems for unitary coRQL, namely *space-bounded quantum state certification* for trace distance and Hilbert-Schmidt distance;
- A new family of (arguably simpler) natural complete problems for BQL, namely *space-bounded quantum state testing* for trace distance, Hilbert-Schmidt distance, and (von Neumann) entropy difference.

In the space-bounded quantum state testing problem, we consider two logarithmic-qubit quantum circuits (devices) denoted as Q_0 and Q_1, which prepare quantum states ρ_0 and ρ_1, respectively, with access to their "source code". Our goal is to decide whether ρ_0 is ϵ_1-close to or ϵ_2-far from ρ_1 with respect to a specified distance-like measure. Interestingly, unlike time-bounded state testing problems, which exhibit computational hardness depending on the chosen distance-like measure (either QSZK-complete or BQP-complete), our results reveal that the space-bounded state testing problems, considering all three measures, are computationally as easy as preparing quantum states. Furthermore, our algorithms on such problems with respect to the trace distance inspire an *algorithmic* Holevo-Helstrom measurement, implying QSZK is in QIP(2) with a quantum linear-space honest prover.

Our results primarily build upon *a space-efficient variant* of the quantum singular value transformation (QSVT) introduced by Gilyén, Su, Low, and Wiebe (STOC 2019), which is of independent interest. Our technique provides a unified approach for designing space-bounded quantum algorithms. Specifically, we show that implementing QSVT for any bounded polynomial that approximates a piecewise-smooth function incurs only a constant overhead in terms of the space required for (special forms of) the projected unitary encoding.

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ISSN 1433-8092

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1 Introduction

In recent years, exciting experimental advancements in quantum computing have been achieved, but concerns about their scalability persist. It thus becomes essential to characterize the computational power of feasible models of quantum computation that operate under restricted resources, such as *time* (i.e., the number of gates in the circuit) and *space* (i.e., the number of qubits on which the circuit acts). This paper specifically focuses on the latter aspect: What is the computational power of quantum computation with a limited number of qubits?

Previous studies on complete problems of space-bounded quantum computation [Wat99, Wat03, vMW12] have primarily focused on well-conditioned versions of standard linear algebraic problems [TS13, FL18, FR21] and have been limited to the two-sided error scenario. In contrast, we propose a novel family of complete problems that not only characterize the *one-sided error scenario (and extend to the two-sided scenario)* but also arise from a quantum property testing perspective. Our new complete problems are arguably more natural and simpler, driven by recent intriguing challenges of verifying the intended functionality of quantum devices.

Consider the situation where a quantum device is designed to prepare a quantum (mixed) state ρ_0, but a possibly malicious party could provide another quantum device that outputs a different n-qubit (mixed) state ρ_1, claiming that $\rho_0 \approx_\epsilon \rho_1$. The problem of testing whether ρ_0 is ϵ_1-close to or ϵ_2-far from ρ_1 with respect to a specified distance-like measure, given the ability to produce copies of ρ_0 and ρ_1, is known as *quantum state testing* [MdW16, Section 4]. Quantum state testing (resp., distribution testing) typically involves utilizing sample accesses to quantum states ρ_0 and ρ_1 (resp., distributions D_0 and D_1) and determining the number of samples required to test the closeness between quantum states (resp., distributions). This problem is a quantum (non-commutative) generalization of classical property testing, which is a fundamental problem in theoretical computer science (see [Gol17]), specifically (tolerant) distribution testing (see [Can20]). Moreover, this problem is an instance of the emerging field of quantum property testing (see [MdW16]), which aims at designing quantum testers for the properties of quantum objects.

In this paper, we investigate quantum state testing problems where quantum states ρ_0 and ρ_1 are preparable by *computationally constrained resources*, specifically state-preparation circuits (viewed as the "source code" of devices) that are *(log)space-bounded*. Our main result conveys a conceptual message that testing quantum states prepared in bounded space is (computationally) as *easy* as preparing these states in a space-bounded manner. Consequently, we can introduce the first family of natural $\mathsf{coRQ_UL}$-complete promise problems since Watrous [Wat01] introduced unitary RQL and coRQL (known as $\mathsf{RQ_UL}$ and $\mathsf{coRQ_UL}$, respectively) in 2001, as well as a new family of natural BQL-complete promise problems.

Our main technique is a *space-efficient variant* of the quantum singular value transformation (QSVT) [GSLW19], distinguishing itself from prior works primarily focused on time-efficient QSVT. As time-efficient QSVT provides a unified framework for designing time-efficient quantum algorithms [GSLW19, MRTC21], we believe our work indicates a unified approach to designing space-bounded quantum algorithms, potentially facilitating the discovery of new complete problems for BQL and its one-sided error variants. Subsequently, we will first state our main results and then provide justifications for the significance of our results from various perspectives.

1.1 Main results

We will commence by providing definitions for time- and space-bounded quantum circuits. We say that a quantum circuit Q is *(poly)time-bounded* if Q is polynomial-size and acts on $\mathrm{poly}(n)$ qubits. Likewise, we say that a quantum circuit Q is *(log)space-bounded* if Q is polynomial-size and acts on $O(\log n)$ qubits. It is worthwhile to note that primary complexity classes, e.g., BQL, $\mathsf{coRQ_UL}$, and BPL, mentioned in this paper correspond to *promise problems*.

Complete characterizations of quantum logspace from state testing. While prior works [TS13, FL18, FR21] on BQL-complete problems have mainly focused on well-conditioned versions of standard linear algebraic problems (in DET^*), our work takes a different perspective by exploring quantum property testing. Specifically, we investigate the problem of *space-bounded quantum state testing*, which aims to test the closeness between two quantum states that are preparable by (log)space-bounded quantum circuits (devices), with access to the corresponding "source code" of these devices.

We begin by considering a computational problem that serves as a "white-box" space-bounded counterpart of *quantum state certification* [BOW19], equivalent to quantum state testing with one-sided error. Our first main theorem (Theorem 1.1) demonstrates the *first* family of natural $\mathsf{coRQ_UL}$-complete problems in the context of space-bounded quantum state certification with respect to the trace distance (td) and the squared Hilbert-Schmidt distance (HS^2).

Theorem 1.1 (Informal of Theorem 4.5)**.** *The following (log)space-bounded quantum state certification problems are* $\mathsf{coRQ_UL}$*-complete: for any* $\alpha(n) \geq 1/\operatorname{poly}(n)$*, decide whether*

(1) $\overline{\textsc{CertQSD}}_{\log}$: $\rho_0 = \rho_1$ *or* $\mathrm{td}(\rho_0, \rho_1) \geq \alpha(n)$*;*

(2) $\overline{\textsc{CertQHS}}_{\log}$: $\rho_0 = \rho_1$ *or* $\mathrm{HS}^2(\rho_0, \rho_1) \geq \alpha(n)$*.*

By extending the error requirement from one-sided to two-sided, we broaden the scope of space-bounded quantum state testing to include two more distance-like measures: the quantum entropy difference, denoted by $\mathrm{S}(\rho_0) - \mathrm{S}(\rho_1)$, and the quantum Jensen-Shannon divergence (QJS_2). As a result, we establish our second main theorem, introducing a new family of natural BQL-complete problems:[1]

Theorem 1.2 (Informal of Theorem 4.6)**.** *The following (log)space-bounded quantum state testing problems are* BQL*-complete: for any* $\alpha(n)$ *and* $\beta(n)$ *such that* $\alpha(n) - \beta(n) \geq 1/\operatorname{poly}(n)$*, or for any* $g(n) \geq 1/\operatorname{poly}(n)$*, decide whether*

(1) $\textsc{GapQSD}_{\log}$: $\mathrm{td}(\rho_0, \rho_1) \geq \alpha(n)$ *or* $\mathrm{td}(\rho_0, \rho_1) \leq \beta(n)$*;*

(2) $\textsc{GapQED}_{\log}$: $\mathrm{S}(\rho_0) - \mathrm{S}(\rho_1) \geq g(n)$ *or* $\mathrm{S}(\rho_1) - \mathrm{S}(\rho_0) \geq g(n)$*;*

(3) $\textsc{GapQJS}_{\log}$: $\mathrm{QJS}_2(\rho_0, \rho_1) \geq \alpha(n)$ *or* $\mathrm{QJS}_2(\rho_0, \rho_1) \leq \beta(n)$*;*

(4) $\textsc{GapQHS}_{\log}$: $\mathrm{HS}^2(\rho_0, \rho_1) \geq \alpha(n)$ *or* $\mathrm{HS}^2(\rho_0, \rho_1) \leq \beta(n)$*.*

Algorithmic Holevo-Helstrom measurement and its implication. The celebrated Holevo-Helstrom bound [Hol73a, Hel69] states that the maximum success probability to discriminate quantum states ρ_0 and ρ_1 is given by $\frac{1}{2} + \frac{1}{2}\mathrm{td}(\rho_0, \rho_1)$. There is then an optimal two-outcome measurement $\{\Pi_0, \Pi_1\}$, referred to as the Holevo-Helstrom measurement, such that $\mathrm{td}(\rho_0, \rho_1) = \mathrm{Tr}(\Pi_0\rho_0) - \mathrm{Tr}(\Pi_0\rho_1)$. Interestingly, by leveraging the BQL containment in Theorem 1.2(1), we can obtain an (approximately) explicit implementation of the Holevo-Helstrom measurement, called *algorithmic Holevo-Helstrom measurement*:

Theorem 1.3 (Informal of Theorem 5.3)**.** *For quantum states* ρ_0 *and* ρ_1 *specified in* GapQSD *such that their purification can be prepared by* n*-qubit polynomial-size quantum circuits* Q_0 *and* Q_1*, we can approximately implement the Holevo-Helstrom measurement* $\{\Pi_0, \Pi_1\}$ *in quantum single-exponential time and linear space with additive error* 2^{-n}*.*

As an implication, we provide a slightly improved upper bound for the class QSZK by inspecting "distance test" in [Wat02] since GapQSD is QSZK-hard:

[1] It is noteworthy that our algorithm for $\textsc{GapQSD}_{\log}$ in Theorem 1.2(1) exhibits a *polynomial advantage* in space over the best known classical algorithms [Wat02]. Watrous implicitly showed in [Wat02, Proposition 21] that $\textsc{GapQSD}_{\log}$ is contained in the class NC, which corresponds to (classical) poly-logarithmic space.

Theorem 1.4 (Informal of Theorem 5.4)**.** GapQSD *is in* QIP(2) *with a quantum single-exponential-time and linear-space honest prover.*

The best known (quantum) upper bound for the class QSZK is QIP(2) [Wat02, Wat09b, JUW09], where the computational power of the honest prover is *unbounded*. It is noteworthy that Theorem 1.4 also applies to GapQSD instances that are not known to be in QSZK.[2]

Space-efficient quantum singular value transformation. Proving our main theorems mentioned above poses a significant challenge: establishing the containment in the relevant class (BQL or coRQ$_{\mathrm{U}}$L), which is also the difficult direction for showing the known family of BQL-complete problems [TS13, FL18, FR21].

Proving the containment for the one-sided error scenario is not an effortless task: such a task is not only already relatively complicated for $\overline{\text{CertQHS}}_{\log}$, but also additionally requires novel techniques for $\overline{\text{CertQSD}}_{\log}$. On the other hand, for two-sided error scenarios, while showing the containment is straightforward for GapQHS$_{\log}$, it still demands sophisticated techniques for all other problems, such as GapQSD$_{\log}$, GapQED$_{\log}$, and GapQJS$_{\log}$.

As explained in Section 1.4, our primary technical contribution and proof technique involve developing a space-efficient variant of the quantum singular value transformation (QSVT), which constitutes our last main theorem (Theorem 1.6).

1.2 Background on space-bounded quantum computation

Watrous [Wat99, Wat03] initiated research on space-bounded quantum computation and showed that fundamental properties, including closure under complement, hold for BQSPACE$[s(n)]$ with $s(n) \geq \Omega(\log n)$. Watrous also investigated classical simulations of space-bounded quantum computation (with unbounded error), presenting deterministic simulations in $O(s^2(n))$ space and unbounded-error randomized simulations in $O(s(n))$ space. A decade later, van Melkebeek and Watson [vMW12] provided a simultaneous $\tilde{O}(t(n))$ time and $O(s(n) + \log t(n))$ space unbounded-error randomized simulation for a bounded-error quantum algorithm in $t(n)$ time and $s(n)$ space. The complexity class corresponding to space-bounded quantum computation with $s(n) = \Theta(\log(n))$ is known as BQL, or BQ$_{\mathrm{U}}$L if only *unitary* gates are permitted.

Significantly, several developments over the past two decades have shown that BQL is well-defined, independent of the following factors in chronological order:

- **The choice of gateset**. The Solovey-Kitaev theorem [Kit97] establishes that most quantum classes are gateset-independent, given that the gateset is closed under adjoint and all entries in gates have reasonable precision. The work of [vMW12] presented a space-efficient counterpart of the Solovay-Kitaev theorem, implying that BQL is also *gateset-independent*.
- **Error reduction**. Repeating BQ$_{\mathrm{U}}$L sequentially necessitates reusing the workspace, making it unclear how to reduce errors for BQ$_{\mathrm{U}}$L as intermediate measurements are not allowed. To address this issue, the work of [FKL+16] adapted the witness-preserving error reduction for QMA [MW05] with several other ideas to the space-efficient setting.
- **Intermediate measurements**. In the space-bounded scenario, the principle of deferred measurement is not applicable since this approach leads to an exponential increase in space complexity. Initially, BQL appeared to be seemingly more powerful than BQ$_{\mathrm{U}}$L since we cannot directly demonstrate that BPL $\subseteq$ BQ$_{\mathrm{U}}$L. Recently, Fefferman and Remscrim [FR21] (as well as [GRZ21, GR22]) proved the equivalence between BQL and BQ$_{\mathrm{U}}$L, indicating a space-efficient approach to eliminating intermediate measurements.

BQL-complete problems. Identifying natural complete problems for the class BQL (or BQ$_{\mathrm{U}}$L) is a crucial and intriguing question. Ta-Shma [TS13] proposed the first candidate BQL-complete

[2] See Section 1.3.1, particularly Footnote 7, for the details.

problem, building upon the work of Harrow, Hassidim, and Lloyd [HHL09] which established a BQP-complete problem for inverting a (polynomial-size) well-conditioned matrix. Specifically, Ta-Shma showed that inverting a well-conditioned matrix with polynomial precision is in BQL. Similarly, computing eigenvalues of an Hermitian matrix is also in BQL. These algorithms offer a quadratic space advantage over the best-known classical algorithms that saturate the classical simulation bound [Wat99, Wat03, vMW12]. Fefferman and Lin [FL18] later improved upon this result to obtain the first natural $\mathsf{BQ_UL}$-complete problem by ingeniously utilizing amplitude estimation to avoid intermediate measurements.

More recently, Fefferman and Remscrim [FR21] further extended this natural $\mathsf{BQ_UL}$-complete problem (or BQL-complete, equivalently) to a *family* of natural BQL-complete problems. They showed that a well-conditioned version of standard DET^*-complete problems is BQL-complete, where DET^* denotes the class of problems that are NC^1 (Turing) reducible to INTDET, including well-conditioned integer determinant (DET), well-conditioned matrix powering (MATPOW), and well-conditioned iterative matrix product (ITMATPROD), among others.

$\mathsf{RQ_UL}$- and $\mathsf{coRQ_UL}$-complete problems. Watrous [Wat01] introduced the one-sided error counterpart of $\mathsf{BQ_UL}$, namely $\mathsf{RQ_UL}$ and $\mathsf{coRQ_UL}$, and developed error reduction techniques. Moreover, Watrous proved that the undirected graph connectivity problem (USTCON) is in $\mathsf{RQ_UL} \cap \mathsf{coRQ_UL}$ whereas Reingold [Rei08] demonstrated that USTCON is in L several years later. It is noteworthy that the question of whether intermediate measurements offer computational advantages in one-sided error scenarios, specifically $\mathsf{RQ_UL}$ vs. RQL and $\mathsf{coRQ_UL}$ vs. coRQL, remains open. Recently, Fefferman and Remscrim [FR21] proposed a "verification" version of the well-conditioned iterative matrix product problem (vITMATPROD) as a *candidate* coRQL-complete problem. However, although this problem is known to be coRQL-hard, its containment remains *unresolved*. Specifically, vITMATPROD requires to decide whether a single entry in the product of polynomially many well-conditioned matrices is equal to zero.

1.3 Time-bounded and space-bounded distribution and state testing

We summarize prior works and our main results for time-bounded[3] and space-bounded distribution and state testing with respect to ℓ_1 norm, entropy difference, and ℓ_2 norm in Table 1.

Interestingly, the sample complexity of testing the closeness of quantum states (resp., distributions) depends on the choice of distance-like measures,[4] including the one-sided error counterpart known as *quantum state certification* [BOW19]. In particular, for distance-like measures such as the ℓ_1 norm, called total variation distance in the case of distributions [CDVV14] and trace distance in the case of states [BOW19], as well as classical entropy difference [JVHW15, WY16] and its quantum analog [AISW20, OW21], the sample complexity of distribution and state testing is polynomial in the dimension N. However, for distance-like measures such as the ℓ_2 norm, called Euclidean distance in the case of distributions [CDVV14] and Hilbert-Schmidt distance in the case of states [BOW19], the sample complexity is *independent* of dimension N.

As depicted in Table 1, this phenomenon that the required sample complexity for distribution and state testing, with polynomial precision and exponential dimension, depends on the choice of distance-like measure has reflections on time-bounded quantum state testing:

- For ℓ_1 norm and entropy difference, the time-bounded scenario is *seemingly much harder than* preparing states or distributions since $\mathsf{QSZK} \subseteq \mathsf{BQP}$ and $\mathsf{SZK} \subseteq \mathsf{BPP}$ are unlikely.
- For ℓ_2 norm, the time-bounded scenario is *as easy as* preparing states or distributions.

[3] The problem of *time-bounded distribution (resp., state) testing* aims to test the closeness between two distributions (resp., states) that are preparable by (poly)time-bounded circuits (devices), with access to the corresponding "source code" of these devices.

[4] It is noteworthy that the quantum entropy difference is not a distance.

	ℓ_1 norm	ℓ_2 norm	Entropy
Classical Time-bounded	SZK-complete[6] [SV03, GSV98]	BPP-complete Folklore	SZK-complete [GV99, GSV98]
Quantum Time-bounded	QSZK-complete[7] [Wat02, Wat09b]	BQP-complete [BCWdW01, RASW23]	QSZK-complete [BASTS10, Wat09b]
Quantum Space-bounded	BQL-complete Theorem 1.2(1)	BQL-complete [BCWdW01] and Theorem 1.2(4)	BQL-complete Theorem 1.2(2)

Table 1: Time- and space-bounded distribution or state testing.

However, interestingly, a similar phenomenon *does not appear* for space-bounded quantum state testing. Although no direct classical counterpart has been investigated before in a complexity-theoretic fashion, namely space-bounded distribution testing, there is another closely related model (a version of streaming distribution testing) that does not demonstrate an analogous phenomenon either, as we will discuss in Section 1.3.2.

1.3.1 Time-bounded distribution and state testing

We review prior works on time-bounded state (resp., distribution) testing, with a particular focus on testing the closeness between states (resp., distributions) are preparable by (poly)time-bounded quantum (resp., classical) circuits (device), with access to the "source code" of corresponding devices. For time-bounded distribution testing, we also recommend a brief survey [GV11] by Goldreich and Vadhan.

ℓ_1 norm scenarios. Sahai and Vadhan [SV03] initiated the study of the time-bounded distribution testing problem, where distributions D_0 and D_1 are *efficiently samplable*, and the distance-like measure is the total variation distance. Their work named this problem Statistical Difference (SD). In particular, the promise problem (α, β)-SD asks whether D_0 is α-far from or β-close to D_1 with respect to $\|D_0 - D_1\|_{\mathrm{TV}}$. Although sampling from the distribution is in BPP,[5] testing the closeness between these distributions is SZK-complete [SV03, GSV98], where SZK is the class of promise problems possessing statistical zero-knowledge proofs. It is noteworthy that the SZK containment of (α, β)-SD for any $\alpha(n) - \beta(n) \geq 1/\operatorname{poly}(n)$ is currently unknown.[6] In addition, we note that SZK is contained in AM $\cap$ coAM [For87, AH91].

Following the pioneering work [SV03], Watrous [Wat02] introduced the time-bounded quantum state testing problem, where two quantum states ρ_0 and ρ_1 that are preparable by time-bounded quantum circuits Q_0 and Q_1, respectively, as well as the distance-like measure is the trace distance. This problem is known as the Quantum State Distinguishability (QSD), specifically, (α, β)-QSD asks whether ρ_0 is α-far from or β-close to ρ_1 with respect to $\mathrm{td}(\rho_0, \rho_1)$. Analogous to its classical counterpart, QSD is QSZK-complete [Wat02, Wat09b], whereas the QSZK containment for any $\alpha(n) - \beta(n) \geq 1/\operatorname{poly}(n)$ remains an open question.[7]

Entropy difference scenarios. Beyond ℓ_1 norm, another distance-like measure commonly con-

[5] Rigorously speaking, as an instance in SD, sample-generating circuits are not necessarily (poly)time-uniform.

[6] The works of [SV03, GSV98] demonstrated that (α, β)-SD is in SZK for any constant $\alpha^2 - \beta > 0$. The same technique works for the parameter regime $\alpha^2(n) - \beta(n) \geq 1/O(\log n)$. However, further improvement of the parameter regime requires new ideas, as clarified in [Gol19]. Recently, the work of [BDRV19] improved the parameter regime to $\alpha^2(n) - \beta(n) \geq 1/\operatorname{poly}(n)$ by utilizing a series of tailor-made reductions. Currently, we only know that (α, β)-SD for $\alpha(n) - \beta(n) \geq 1/\operatorname{poly}(n)$ is also in AM $\cap$ coAM [BL13].

[7] Like SD and SZK, the techniques in [Wat02, Wat09b] show that (α, β)-QSD is in QSZK for $\alpha^2(n) - \beta(n) \geq 1/O(\log n)$, and the same limitation also applies to the quantum settings. A recent result [Liu23] following the line of work of [BDRV19] improved the parameter regime to $\alpha^2(n) - \sqrt{2 \ln 2}\beta(n) \geq 1/\operatorname{poly}(n)$, but the differences between classical and quantum distances make it challenging to push the bound further.

sidered in time-bounded quantum state testing (or distribution testing) is the (quantum) entropy difference, which also corresponds to the (quantum) Jensen-Shannon divergence. The promise problem Entropy Difference (ED), first introduced by Goldreich and Vadhan [GV99] following the work of [SV03], asks whether efficiently samplable distributions D_0 and D_1 satisfy $\mathrm{H}(D_0) - \mathrm{H}(D_1) \geq g$ or $\mathrm{H}(D_1) - \mathrm{H}(D_0) \geq g$ for $g = 1$. They demonstrated that ED is SZK-complete. Ben-Aroya, Schwartz, and Ta-Shma [BASTS10] further investigated the promise problem Quantum Entropy Difference (QED), which asks whether $\mathrm{S}(\rho_0) - \mathrm{S}(\rho_1) \geq g$ or $\mathrm{S}(\rho_1) - \mathrm{S}(\rho_0) \geq g$, for efficiently preparable quantum states ρ_0 and ρ_1 and $g = 1/2$. They showed that QED is QSZK-complete. Moreover, the SZK (resp., QSZK) containment for ED (resp., QED) automatically holds for any $g(n) \geq 1/\operatorname{poly}(n)$.

Furthermore, Berman, Degwekar, Rothblum, and Vasudevan [BDRV19] demonstrated that the Jensen-Shannon divergence problem (JSP), asking whether $\mathrm{JS}(D_0, D_1) \geq \alpha$ or $\mathrm{JS}(D_0, D_1) \leq \beta$ for efficiently samplable distributions D_0 and D_1, is SZK-complete. Their work accomplished this result by reducing the problem to ED, and this containment applies to $\alpha(n) - \beta(n) \geq 1/\operatorname{poly}(n)$. Recently, Liu [Liu23] showed a quantum counterpart, referred to as the Quantum Jensen-Shannon Divergence Problem (QJSP), is QSZK-complete. Notably, the quantum Jensen-Shannon divergence is a special instance of the Holevo χ quantity [Hol73b].[8]

ℓ_2 norm scenarios. For the quantum setting, it is straightforward that applying the SWAP test [BCWdW01] to efficiently preparable quantum states ρ_0 and ρ_1 can lead to a BQP containment, in particular, additive-error estimations of $\mathrm{Tr}(\rho_0^2)$, $\mathrm{Tr}(\rho_1^2)$, and $\mathrm{Tr}(\rho_0\rho_1)$ with polynomial precision. Recently, the work of [RASW23] observed that time-bounded quantum state testing with respect to the squared Hilbert-Schmidt distance is BQP-complete. For the classical setting, namely the squared Euclidean distance, the BPP-completeness is relatively effortless.[9]

1.3.2 Space-bounded distribution and state testing

To the best of our knowledge, no prior work has specifically focused on space-bounded distribution testing from a complexity-theoretic perspective. Instead, we will review prior works that are (closely) related to this computational problem. Afterward, we will delve into space-bounded quantum state testing, which constitutes the main contribution of our work.

Space-bounded distribution testing and related works. We focus on a computational problem involving two $\operatorname{poly}(n)$-size classical circuits C_0 and C_1, which generate samples from the distributions D_0 and D_1 respectively. Each circuit contains a read-once polynomial-length random-coins tape.[10] The input length and output length of the circuits are $O(\log n)$. The task is to decide whether D_0 is α-far from or β-close to D_1 with respect to some distance-like measure. Additionally, we can easily observe that space-bounded distribution testing with respect to the squared Euclidean distance (ℓ_2 norm) is BPL-complete, much like its time-bounded counterpart.

Several models related to space-bounded distribution testing have been investigated previously. Earlier streaming-algorithmic works [FKSV02, GMV06] utilize *entries* of the distribution as the data stream, with entries given in different orders for different models. On the other hand, a later work [CLM10] considered a data stream consisting of a sequence of i.i.d. samples drawn from distributions and studied low-space streaming algorithms for distribution testing.

Regarding (Shannon) entropy estimation, previous streaming algorithms considered worst-case ordered samples drawn from N-dimensional distributions and required $\operatorname{polylog}(N/\epsilon)$ space,

[8] In particular, the quantum Jensen-Shannon divergence coincides with the Holevo χ quantity on size-2 ensembles with a uniform distribution, which arises in the Holevo bound [Hol73b]. See [NC02, Theorem 12.1].

[9] Specifically, we achieve BPP containment by following the approach in [BCH+19, Theorem 7.1]. On the other hand, the BPP hardness owes to the fact that the squared Euclidean distance between the distribution $(p_{\mathrm{acc}}, 1 - p_{\mathrm{acc}})$ from the output bit of any BPP algorithm and the distribution $(1, 0)$ is $(1 - p_{\mathrm{acc}})^2$.

[10] It is noteworthy that random coins are provided as *input* to classical circuits C_0 and C_1 for generating samples from the corresponding distributions in the time-bounded scenario, such as SD and ED.

where ϵ is the additive error. Recently, Acharya, Bhadane, Indyk, and Sun [ABIS19] addressed the entropy estimation problem with i.i.d. samples drawn from distributions as the data stream and demonstrated the first $O(\log(N/\epsilon))$ space streaming algorithm. The sample complexity, viewed as the time complexity, was subsequently improved in [AMNW22].

However, for the total variation distance (ℓ_1 norm), previous works focused on the trade-off between the sample complexity and the space complexity (memory constraints), achieving only a nearly-log-squared space streaming algorithm [DGKR19].

Notably, the main differences between the computational and streaming settings lie in how we access the sampling devices.[11] In the computational problem, we have access to the "source code" of the devices and can potentially use them for purposes like "reverse engineering". Conversely, the streaming setting utilizes the sampling devices in a "black-box" manner, obtaining i.i.d. samples. As a result, a logspace streaming algorithm will result in a BPL containment.[12]

Space-bounded quantum state testing. Among the prior works on streaming distribution testing, particularly entropy estimation, the key takeaway is that the space complexity of the corresponding computational problem is $O(\log(N/\epsilon))$. This observation leads to a conjecture that the computational hardness of space-bounded distribution and state testing is *independent* of the choice of commonplace distance-like measures. Our work, in turn, provides a positive answer for space-bounded quantum state testing.

Space-bounded state testing with respect to the squared Hilbert-Schmidt distance (ℓ_2 norm) is BQL-complete, as shown in Theorem 1.2(4). Specifically, the BQL containment follows from the SWAP test [BCWdW01], similar to the time-bounded scenario. Moreover, proving BQL hardness, as well as $\mathsf{coRQ_UL}$-hardness for state certification, is not challenging.[13]

Regarding space-bounded state testing with respect to the trace distance (ℓ_1 norm), we note that [Wat02, Proposition 21] implicitly established an NC containment. The BQL-hardness, as well as $\mathsf{coRQ_UL}$-hardness for state certification, is adapted from [RASW23]. Similarly, we derive the BQL-hardness for space-bounded state testing with respect to the quantum Jensen-Shannon divergence and the quantum entropy difference from the previous work [Liu23].

Finally, we devote the remainder of this section to our main technique (Theorem 1.6), and consequently, we present BQL (resp., $\mathsf{coRQ_UL}$) containment for state testing (resp., certification) problems for other distance-like measures beyond the squared Hilbert-Schmidt distance.

1.4 Proof technique: Space-efficient quantum singular value transformation

The quantum singular value transformation (QSVT) [GSLW19] is a powerful and efficient framework for manipulating the singular values $\{\sigma_i\}_i$ of a linear operator A, using a corresponding projected unitary encoding U of $A = \tilde{\Pi}U\Pi$ for projections $\tilde{\Pi}$ and Π.[14] The singular value decomposition is $A = \sum_i \sigma_i |\tilde{\psi}_i\rangle\langle\psi_i|$ where $|\tilde{\psi}_i\rangle$ and $|\psi_i\rangle$ are left and right singular vectors, respectively. QSVT has numerous applications in quantum algorithm design, and is even considered a grand unification of quantum algorithms [MRTC21]. To implement the transformation $f^{(\mathrm{SV})}(A) = f^{(\mathrm{SV})}(\tilde{\Pi}U\Pi)$, we require a degree-$d$ polynomial P_d that satisfies two conditions. Firstly, P_d well-approximates f on the interval of interest $\mathcal{I}$, with $\max_{x\in\mathcal{I}\setminus\mathcal{I}_\delta} |P_d(x) - f(x)| \leq \epsilon$, where $\mathcal{I}_\delta \subseteq \mathcal{I} \subseteq [-1,1]$ and typically $\mathcal{I}_\delta := (-\delta,\delta)$. Secondly, P_d is bounded with $\max_{x\in[-1,1]} |P_d(x)| \leq 1$. The degree of P_d depends on the precision parameters δ and ϵ, with $d = O(\delta^{-1}\log\epsilon^{-1})$, and all coefficients of P_d can be computed efficiently.

According to [GSLW19], we can use alternating phase modulation to implement $P_d^{(\mathrm{SV})}(\tilde{\Pi}U\Pi)$,[15]

[11] Of course, not all distributions can be described as a polynomial-size circuit (i.e., a succinct description).

[12] In particular, the sample-generating circuits C_0 and C_1 in space-bounded distribution testing can produce the i.i.d. samples in the data stream.

[13] See Lemma 4.17 for details.

[14] Regardless of QSVT, it is noteworthy that the concept of block-encoding, specifically a unitary dilation U of a contraction A (see Footnote 25), is already used in quantum logspace for powering contraction matrices [GRZ21].

[15] This procedure is a generalization of quantum signal processing, as explained in [MRTC21, Section II.A].

which requires a sequence of rotation angles $\Phi \in \mathbb{R}^d$. For instance, consider $P_d(x) = T_d(x)$ where $T_d(x)$ is the d-th Chebyshev polynomial (of the first kind), then we know that $\phi_1 = (1-d)\pi/2$ and $\phi_j = \pi/2$ for all $j \in \{2, 3, \cdots, d\}$. QSVT techniques, including the pre-processing and quantum circuit implementation, are generally *time-efficient*. Additionally, the quantum circuit implementation of QSVT is already *space-efficient* because implementing QSVT with a degree-d bounded polynomial for any $s(n)$-qubit projected unitary encoding requires $O(s(n))$ qubits, where $s(n) \geq \Omega(\log n)$. However, the pre-processing in the QSVT techniques is typically not space-efficient. Indeed, prior works on the pre-processing for QSVT, specifically angle-finding algorithms in [Haa19, CDG+20, DMWL21], which have time complexity polynomially dependent on the degree d, do not consider the space-efficiency. Therefore, the use of previous angle-finding algorithms may lead to an *exponential* increase in space complexity. This raises a fundamental question on making the pre-processing space-efficient as well:

Problem 1.5 (Space-efficient QSVT)**.** Can we implement a degree-d QSVT for any $s(n)$-qubit projected unitary encoding with $d \leq 2^{O(s(n))}$, using only $O(s(n))$ space in both the pre-processing and quantum circuit implementation?

QSVT via averaged Chebyshev truncation. A space-efficient QSVT associated with Chebyshev polynomials is implicitly shown in [GSLW19], as the angles for any Chebyshev polynomial $T_k(x)$ are explicitly known. This insight sheds light on Problem 1.5 and suggests an alternative pre-processing approach for QSVT: Instead of finding rotation angles, it seems suffice to find projection coefficients of Chebyshev polynomials.

Recently, Metger and Yuen [MY23] realized this approach and constructed bounded polynomial approximations of the sign and *shifted* square root functions with exponential precision in polynomial space by utilizing Chebyshev truncation, which offers a partial solution to Problem 1.5.[16] The key ingredient behind their approach is the degree-d Chebyshev truncation $\tilde{P}_d(x) = \frac{c_0}{2} + \sum_{k=1}^{d} c_k T_k$ where T_k is the k-th Chebyshev polynomial (of the first kind) and $c_k := \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}\mathrm{d}x$. This provides a *nearly best* uniform approximation compared to the best degree-d polynomial approximation with error $\varepsilon_d(f)$ for the function $f\colon [-1,1] \to \mathbb{R}$. In particular, $\tilde{P}_d$ satisfies $\max_{x\in[-1,1]} |\tilde{P}_d(x) - f(x)| \leq O(\log d)\varepsilon_d(f)$.

Our construction achieves an error bound *independent* of d via a carefully chosen *average* of the Chebyshev truncation, known as the *de La Vallée Poussin partial sum*, $\hat{P}_{d'}(x) = \frac{1}{d}\sum_{l=d}^{d'} \tilde{P}_d(x) = \frac{\hat{c}_0}{2} + \sum_{k=1}^{d'} \hat{c}_k T_k(x)$, with a slightly larger degree $d' = 2d-1$. The degree-d averaged Chebyshev truncation $\hat{P}_{d'}$ satisfies $\max_{x\in[-1,1]} |\hat{P}_{d'}(x) - f(x)| \leq 4\varepsilon_d(f)$.

Once we have a space-efficient polynomial approximation for the function f (pre-processing), we can establish a space-efficient QSVT associated with f for *bitstring indexed encodings* that additionally require projections $\tilde{\Pi}$ and Π spanning the corresponding subset of $\{|0\rangle, |1\rangle\}^{\otimes s}$,[17] as stated in Theorem 1.6: With the space-efficient QSVT associated with Chebyshev polynomials $T_k(x)$, it suffices to implement the averaged Chebyshev truncation polynomial by LCU techniques [BCC+15] and to renormalize the bitstring indexed encoding by robust oblivious amplitude amplification (if necessary and applicable).

A refined analysis indicates that applying an averaged Chebyshev truncation to a bitstring indexed encoding for any $d' \leq 2^{O(s(n))}$ and $\epsilon \geq 2^{-O(s(n))}$ requires $O(s(n))$ qubits and deterministic $O(s(n))$ space, provided that an evaluation oracle Eval_{P_d} estimates coefficients $\{\hat{c}_k\}_{k=0}^{d'}$ of the averaged Chebyshev truncation with $O(\log(\epsilon^2/d))$ precision. Nevertheless, our approach causes a *quadratic* dependence of the degree d in the query complexity to U.

[16] To clarify, we can see from [MY23] that directly adapting their construction shows that implementing QSVT for any $s(n)$-qubit block-encoding with $O(s(n))$-bit precision requires $\mathrm{poly}(s(n))$ classical and quantum space for any $s(n) \geq \Omega(\log n)$. However, Problem 1.5 (space-efficient QSVT) seeks to reduce the dependence of $s(n)$ in the space complexity from *polynomial* to *linear*.

[17] To ensure that $\tilde{\Pi}U\Pi$ admits a matrix representation, we require the basis of projections $\tilde{\Pi}$ and Π to have a well-defined order, leading us to focus exclusively on bitstring indexed encoding. Additionally, for simplicity, we assume no ancillary qubits are used here, and refer to Definition 3.1 for a formal definition.

Theorem 1.6 (Space-efficient QSVT, informal of Theorem 3.2)**.** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function bounded on $\mathcal{I} \subseteq [-1,1]$. If there exists a degree-d polynomial P_d^* that approximates $h\colon [-1,1] \to \mathbb{R}$, where h approximates f only on $\mathcal{I}$ with additive error at most ϵ, such that $\max_{x\in[-1,1]} |h(x) - P_d^*(x)| \leq \epsilon$, then the degree-$d$ averaged Chebyshev truncation yields another degree-d' polynomial $P_{d'}$, with $d' = 2d-1$, satisfying the following conditions:*

$$\max_{x\in\mathcal{I}} |f(x) - P_{d'}(x)| \leq O(\epsilon) \text{ and } \max_{x\in[-1,1]} |P_{d'}(x)| \leq 1.$$

Furthermore, we have an algorithm $\mathcal{A}_f$ that computes any coefficient $\{\hat{c}_k\}_{k=0}^{d'}$ of the averaged Chebyshev truncation polynomial $P_{d'}$ space-efficiently. The algorithm is deterministic for continuously bounded f, and bounded-error randomized for piecewise-smooth f. Additionally, for any $s(n)$-qubit bitstring indexed encoding U of $A = \tilde{\Pi}U\Pi$ with $d' \leq 2^{O(s(n))}$, we can implement the quantum singular value transformation $P_{d'}^{(\mathrm{SV})}(A)$ using $O(d^2\|\hat{\mathbf{c}}\|_1)$ queries[18] to U with $O(s(n))$ qubits. It is noteworthy that $\|\hat{\mathbf{c}}\|_1$ is bounded by $O(\log d)$ in general, and we can further improve to a constant norm bound for twice continuously differentiable functions.

Our techniques in Theorem 1.6 offer three advantages over the techniques proposed by [MY23]. Firstly, our techniques can handle any *piecewise-smooth function*, such as the (normalized) logarithmic function $\ln(1/x)$, the multiplicative inverse function $1/x$, and the square-root function $\sqrt{x}$;[19] whereas the techniques from [MY23] are restricted to continuously bounded functions whose second derivative of the integrand in $\{\hat{c}_k\}_{k=1}^{d'}$ is at most $\mathrm{poly}(d)$ on the interval $\mathcal{I} = [-1,1]$, such as the sign function and the *shifted* square-root function $\sqrt{(x+1)/2}$.[20] Secondly, our techniques are *constant overhead* in terms of the space complexity of the bitstring indexed encoding U, while the techniques from [MY23] are only *poly-logarithmic overhead*. Thirdly, our techniques have an error bound independent of d, unlike the $\log d$ factor in [MY23], simplifying parameter trade-offs for applying the space-efficient QSVT to concrete problems.

In addition, it is noteworthy that applying the space-efficient QSVT with the sign function will imply a unified approach to error reduction for the classes $\mathsf{BQ_UL}$, $\mathsf{coRQ_UL}$, and $\mathsf{RQ_UL}$.

Computing the coefficients. We will implement the evaluation oracle Eval_{P_d} to prove Theorem 1.6. To estimate the coefficients $\{\hat{c}_k\}_{k=0}^{d'}$ in the averaged Chebyshev truncation for any function f that is bounded on the interval $\mathcal{I} = [-1,1]$, we can use standard numerical integral techniques,[21] given that the integrand's second derivative in $\{\hat{c}_k\}_{k=0}^{d'}$ is bounded by $\mathrm{poly}(d)$.

However, implementing the evaluation oracle for piecewise-smooth functions f on an interval $\mathcal{I} \subsetneq [-1,1]$ is relatively complicated. We cannot simply apply averaged Chebyshev truncation to f. Instead, we consider a low-degree Fourier approximation g resulting from implementing smooth functions to Hamiltonians [vAGGdW20, Appendix B]. We then make the error vanish outside $\mathcal{I}$ by multiplying with a Gaussian error function, resulting in h which approximates f *only* on $\mathcal{I}$. Therefore, we can apply averaged Chebyshev truncation and our algorithm for bounded functions to h through a somewhat complicated calculation.

Finally, we need to compute the coefficients of the low-degree Fourier approximation g. Interestingly, this step involves the *stochastic matrix powering problem*, which lies at the heart of space-bounded derandomization, e.g., [SZ99, CDSTS23, PP23]. We utilize space-efficient random walks on a directed graph to estimate the power of a stochastic matrix. Consequently, we can only develop a bounded-error randomized algorithm $\mathcal{A}_f$ for piecewise-smooth functions.[22]

[18] The dependence of $\|\hat{\mathbf{c}}\|_1$ arises from renormalizing the bitstring indexed encoding via amplitude amplification.

[19] Our technique can imply a better norm bound $\|\hat{\mathbf{c}}\| \leq O(1)$. See Remark 3.5 for the details.

[20] Specifically, the second derivative $|f''(x)|$ of the shifted square-root function $f(x) \coloneqq \sqrt{(x+1)/2}$ is unbounded at $x = -1$. Nevertheless, we can circumvent this point by instead considering $g_\delta(x) = \sqrt{(1-\delta)(x+1)/2 + \delta}$ with the second derivative $|g_\delta''(-1)| = O(\delta^{-3/2})$, as shown in [MY23, Lemma 2.11].

[21] We remark that using a more efficient numerical integral technique, such as the exponentially convergent trapezoidal rule, may improve the required space complexity for computing coefficients by a constant factor.

[22] The (classical) pre-processing in space-efficient QSVT is *not* part of the deterministic Turing machine producing the quantum circuit description in the BQL model (Definition 2.8). Instead, we treat it as a component

1.5 Proof overview: A general framework for quantum state testing

Our framework enables space-bounded quantum state testing, specifically for proving Theorem 1.1 and Theorem 1.2, and is based on the one-bit precision phase estimation [Kit95], also known as the *Hadamard test* [AJL09]. Prior works [TS13, FL18] have employed (one-bit precision) phase estimation in space-bounded quantum computation.

To address quantum state testing problems, we reduce them to estimating $\mathrm{Tr}(P_{d'}(A)\rho)$, where ρ is a (mixed) quantum state prepared by a quantum circuit Q_ρ, A is an Hermitian operator block-encoded in a unitary operator U_A, and $P_{d'}$ is a space-efficiently computable degree-d' polynomial obtained from some degree-d averaged Chebyshev truncation with $d' = 2d - 1$. Similar approaches have been applied in *time-bounded* quantum state testing, including fidelity estimation [GP22] and subsequently trace distance estimation [WZ24].

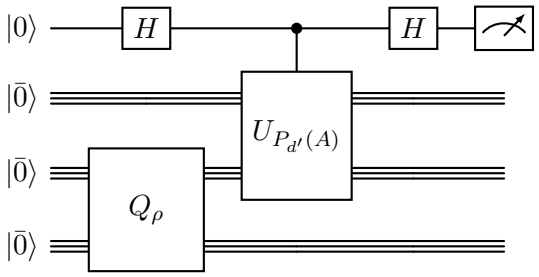

Figure 1: General framework for quantum state testing $\mathcal{T}(Q_\rho, U_A, P_{d'})$.

To implement a unitary operator $U_{P_{d'}(A)}$ that (approximately) block-encodes $P_{d'}(A)$ in a space-efficient manner, we require $P_{d'}$ to meet the conditions specified in Theorem 1.6. As illustrated in Figure 1, we denote the quantum circuit as $\mathcal{T}(Q_\rho, U_A, P_{d'})$, where we exclude the precision for simplicity. The measurement outcome of $\mathcal{T}(Q_\rho, U_A, P_{d'})$ will be 0 with a probability close to $\frac{1}{2}\big(1 + \mathrm{Tr}(P_{d'}(A)\rho)\big)$. This property allows us to estimate $\mathrm{Tr}(P_{d'}(A)\rho)$ within an additive error ϵ using $O(1/\epsilon^2)$ sequential repetitions, resulting in a BQL containment.

As an example of the application, $\mathcal{T}(Q_i, U_{\frac{\rho_0-\rho_1}{2}}, P^{\mathrm{sgn}}_{d'})$ is utilized in GapQSD, where $U_{\frac{\rho_0-\rho_1}{2}}$ is a block-encoding of $\frac{\rho_0-\rho_1}{2}$, and $P^{\mathrm{sgn}}_{d'}$ is a space-efficient polynomial approximation of the sign function. Notably, this algorithm can be viewed as a two-outcome measurement $\{\hat{\Pi}_0, \hat{\Pi}_1\}$ where $\hat{\Pi}_0 = \frac{1}{2}I + \frac{1}{2}P^{\mathrm{sgn}}_{d'}\big(\frac{\rho_0-\rho_1}{2}\big)$, which is essentially the algorithmic Holevo-Helstrom measurement in Theorem 1.3. Similarly, $\mathcal{T}(Q_i, U_{\rho_i}, P^{\mathrm{ln}}_{d'})$ is utilized in GapQED, where U_{ρ_i} is a block-encoding of ρ_i for $i \in \{0,1\}$, and $P^{\mathrm{ln}}_{d'}$ is a space-efficient polynomial approximation of the normalized logarithmic function. Both $P^{\mathrm{sgn}}_{d'}$ and $P^{\mathrm{ln}}_{d'}$ can be obtained by employing Theorem 1.6.

Making the error one-sided. The main challenge is constructing a unitary U of interest, such as $\mathcal{T}(Q_\rho, U_A, P_{d'})$, that accepts with a *certain fixed* probability p for *yes* instances ($\rho_0 = \rho_1$), while having a probability that polynomially deviates from p for *no* instances. As an example, we consider $\overline{\textsc{CertQHS}}_{\log}$ and express $\mathrm{HS}^2(\rho_0, \rho_1)$ as a linear combination of $\mathrm{Tr}(\rho_0^2)$, $\mathrm{Tr}(\rho_1^2)$, and $\mathrm{Tr}(\rho_0\rho_1)$. We can then design a unitary quantum algorithm that satisfies the requirement for *yes* instances based on the SWAP test [BCWdW01], and consequently, we can achieve perfect completeness by applying the exact amplitude amplification [BBHT98, BHMT02]. The analysis demonstrates that the acceptance probability polynomially deviates from 1 for *no* instances. By applying error reduction for $\mathsf{coRQ_UL}$, the resulting algorithm is indeed in $\mathsf{coRQ_UL}$.

Moving on to $\overline{\textsc{CertQSD}}_{\log}$, we consider the quantum circuit $U_i = \mathcal{T}(Q_i, U_{\frac{\rho_0-\rho_1}{2}}, P^{\mathrm{sgn}}_{d'})$ for $i \in \{0,1\}$. Since our space-efficient QSVT *preserves parity*, specifically, the approximation

of quantum computation, allowing the use of randomized algorithms since $\mathsf{BPL} \subseteq \mathsf{BQL}$ [FR21].

polynomial $P_{d'}^{\mathrm{sgn}}$ satisfies $P_{d'}^{\mathrm{sgn}}(0) = 0$,[23] the requirement for *yes* instances is satisfied. Then we can similarly achieve the $\mathsf{coRQ_UL}$ containment for $\overline{\textsc{CertQSD}}_{\log}$.

1.6 Discussion and open problems

Since space-efficient quantum singular value transformation (QSVT) offers a unified framework for designing quantum logspace algorithms, it suggests a new direction to find applications of space-bounded quantum computation. An intriguing candidate is solving positive semidefinite programming (SDP) programs with constant precision [JY11, AZLO16]. A major challenge in achieving a BQL containment for this problem is that iteratively applying the space-efficient QSVT super-constantly many times may lead to a bitstring indexed encoding requiring $\omega(\log n)$ ancillary qubits, raising the question:

(i) Is it possible to have an approximation scheme (possibly under certain conditions) that introduces merely $O(1)$ additional ancillary qubits in the bitstring indexed encoding per iteration, such that applying space-efficient QSVT $\log n$ times results in a bitstring indexed encoding with at most $O(\log n)$ ancillary qubits?

Furthermore, as quantum distances investigated in this work are all instances of a quantum analog of symmetric f-divergence, there is a natural question on other instances:

(ii) Can we demonstrate that space-bounded quantum state testing problems with respect to other quantum distance-like measures are also BQL-complete?

In addition, there is a question on improving the efficiency of the space-efficient QSVT:

(iii) Recently, a query complexity lower bound $\Omega(d)$ for matrix functions [MS23] implies that time-efficient QSVT [GSLW19] is time-optimal. Can we improve the query complexity of U and $U^\dagger$ in space-efficient QSVT for smooth functions from $O(d^2)$ to $O(d)$? This improvement would make QSVT optimal for both (quantum) time and space.

Notably, the pre-processing in QSVT techniques, which is not necessarily classical in general, usually involves finding the sequence of z-axis rotation angles. Our approach, however, uses averaged Chebyshev truncation and the LCU technique. A general solution thus seems to involve developing a space-efficient (quantum) angle-finding algorithm.

1.7 Related works: More on quantum state testing problems

Testing the spectrum of quantum states was studied in [OW21]: for example, whether a quantum state is maximally mixed or ϵ-far away in trace distance from mixed states can be tested using $\Theta(N/\epsilon^2)$ samples. Later, it was generalized in [BOW19] to quantum state certification with respect to fidelity and trace distance. Estimating distinguishability measures of quantum states [RASW23] is another topic, including the estimation of fidelity [FL11, WZC+23, GP22] and trace distance [WGL+24, WZ24].

Entropy estimation of quantum states has been widely studied in the literature. Given quantum purified access, it was shown in [GL20] that the von Neumann entropy $\mathrm{S}(\rho)$ can be estimated within additive error ϵ with query complexity $\tilde{O}(N/\epsilon^{1.5})$. If we know the reciprocal κ of the minimum non-zero eigenvalue of ρ, then $\mathrm{S}(\rho)$ can be estimated with query complexity $\tilde{O}(\kappa^2/\epsilon)$ [CLW20]. We can estimate $\mathrm{S}(\rho)$ within multiplicative error ϵ with query complexity $\tilde{O}(n^{\frac{1}{2}+\frac{1+\eta}{2\epsilon^2}})$ [GHS21], provided that $\mathrm{S}(\rho) = \Omega(\epsilon + 1/\eta)$. If ρ is of rank r, then $\mathrm{S}(\rho)$ can be estimated with query complexity $\tilde{O}(r/\epsilon^2)$ [WGL+24]. Estimating the Rényi entropy $S_\alpha(\rho)$ given quantum purified access was first studied in [GHS21], and then was improved in [WGL+24, WZL24]. In addition, the work of [GH20] investigates the (conditional) hardness of GapQED with logarithmic depth or constant depth.

[23] Let f be any odd function such that space-efficient QSVT associated with f can be implemented by Theorem 1.6. It follows that the corresponding approximation polynomial $P_{d'}^{(f)}$ is also odd. See Remark 3.11.

Paper organization. Our paper begins by introducing key concepts and useful tools in Section 2. In Section 3, we demonstrate our space-efficient variant of quantum singular value transformation (Theorem 1.6) and offer examples of continuously bounded functions and piecewise-smooth functions. We also provide a simple proof of space-efficient error reduction for unitary quantum computation. Then, in Section 4, we formally define space-bounded quantum state testing problems with four distance-like measures, and present the first family of natural $\mathsf{coRQ_UL}$-complete problems (Theorem 1.1), as well as a novel family of natural BQL-complete problems (Theorem 1.2). Lastly, in Section 5, we establish an (approximately) explicit implementation of the Holevo-Helstrom measurement (Theorem 1.3), called algorithmic Holevo-Helstrom measurement, implying QSZK is in $\mathsf{QIP}(2)$ with a quantum linear-space honest prover (Theorem 1.4).

2 Preliminaries

We assume that the reader is familiar with quantum computation and the theory of quantum information. For an introduction, the textbooks by [NC02] and [dW19] provide a good starting point, while for a more comprehensive survey on quantum complexity theory, refer to [Wat09a].

In addition, we adopt the convention that the logarithmic function log has a base of 2, denoted by $\log(x) \coloneqq \log_2(x)$ for any $x \in \mathbb{R}^+$. Moreover, we define $\tilde{O}(f) \coloneqq O(f \operatorname{polylog}(f))$. Lastly, for the sake of simplicity, we utilize the notation $|\bar{0}\rangle$ to represent $|0\rangle^{\otimes a}$ with $a > 1$.

2.1 Singular value decomposition and transformation

We recommend [Bha96, HJ12] for comprehensive textbooks on matrix analysis and linear algebra. For any $\tilde{d} \times d$ (complex) matrix A, there is a *singular value decomposition* of A such that $A = \sum_{i=1}^{\min\{d,\tilde{d}\}} \sigma_i |\tilde{\psi}_i\rangle\langle\psi_i|$, where:

- The *singular values* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\min\{d,\tilde{d}\}} \geq 0$, where non-zero singular values σ_i are the square roots of non-zero the eigenvalues of $AA^\dagger$ or $AA^\dagger$.
- $|\tilde{\psi}_1\rangle, \cdots, |\tilde{\psi}_{\tilde{d}}\rangle$ form an orthonormal basis and are eigenvectors of $AA^\dagger$.
- $|\psi_1\rangle, \cdots, |\psi_d\rangle$ form an orthonormal basis and are eigenvectors of $A^\dagger A$.

Notably, the largest singular value of A coincides with the operator norm of A, specifically $\|A\| \coloneqq \|A\|_{2\to 2} = \sigma_1(A)$. Let $\||\psi\rangle\|_2 \coloneqq \sqrt{\langle\psi|\psi\rangle}$ be the Euclidean norm of a vector $|\psi\rangle$. Next, we list the families of matrices that are commonly used in this work. It is noteworthy that they all admit the singular value decomposition:

- **Hermitian matrices**. $H^\dagger = H$, if and only if $\langle\psi|H|\psi\rangle \in \mathbb{R}$ for all $|\psi\rangle$ such that $\||\psi\rangle\|_2 = 1$, if and only if the absolute values of the eigenvalues of H coincide with its singular value.
- **Isometries and unitary matrices**. $GG^\dagger = I_{\tilde{d}}$ and $G^\dagger G = I_d$, if and only if $\|U|\psi\rangle\|_2 = \|U^\dagger|\psi'\rangle\|_2 = 1$ for all $|\psi\rangle, |\psi'\rangle$ such that $\||\psi\rangle\|_2 = \||\psi'\rangle\|_2 = 1$, if and only if the $\operatorname{rank}(G)$ largest singular values of G are all 1 and the remained singular values are all zero.[24] Furthermore, *unitary* U is a special case of isometry with $\tilde{d} = d$.
- **Positive semi-definite matrices**. $P = CC^\dagger$ for some matrix C, if and only if $\langle\psi|P|\psi\rangle \geq 0$ for all $|\psi\rangle$ such that $\||\psi\rangle\|_2 = 1$, if and only if all eigenvalues of P are non-negative.
- **Orthogonal projection matrices**. $\Pi = GG^\dagger$ for some isometry $G^\dagger G = I$, if and only if $\Pi^2 = \Pi$ and $\|\Pi|\psi\rangle\|_2 \leq 1$ for all $|\psi\rangle$ such that $\||\psi\rangle\|_2 = 1$, if and only if all eigenvalues of Π are either 0 or 1 (see [HJ12, Corollary 3.4.3.3]).

[24] There are several definitions of isometry, and our definition coincides with [HJ12, 5.4.P11(c)].

For any matrix A satisfying $\|A\| \leq 1$, there is a unitary U with orthogonal projections $\tilde{\Pi}$ and Π such that $A = \tilde{\Pi}U\Pi$.[25] With these definitions in place, we can view the singular value decomposition as the *projected unitary encoding* (see Definition 3.1):

Definition 2.1 (Singular value decomposition of a projected unitary, adapted from Definition 7 in [GSLW19])**.** Given a projected unitary encoding of A, denoted by U, associated with orthogonal projections Π and $\tilde{\Pi}$ on a finite-dimensional Hilbert space $\mathcal{H}_U$: $A = \tilde{\Pi}U\Pi$. Then the singular value decomposition of A ensures that orthonormal bases of Π and $\tilde{\Pi}$ such that:

- Π: $\{|\psi_i\rangle : i \in [d]\}$, where $d \coloneqq \mathrm{rank}(\Pi)$, of a subspace $\mathrm{Img}(\Pi) = \mathrm{span}\{|\psi_i\rangle\}$;
- $\tilde{\Pi}$: $\{|\tilde{\psi}_i\rangle : i \in [\tilde{d}]\}$, where $\tilde{d} \coloneqq \mathrm{rank}(\tilde{\Pi})$, of a subspace $\mathrm{Img}(\tilde{\Pi}) = \mathrm{span}\{|\tilde{\psi}_i\rangle\}$.

We say that a function $f\colon \mathbb{R} \to \mathbb{C}$ is *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$, and that it is *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Next, we define the *singular value transformation* of matrices:

Definition 2.2 (Singular value transformation by even or odd functions, adapted from Definition 9 in [GSLW19])**.** Let $f\colon \mathbb{R} \to \mathbb{C}$ be an even or odd function. We consider a linear operator $A \in \mathbb{C}^{\tilde{d}\times d}$ satisfying the singular value decomposition $A = \sum_{i=1}^{\min\{d,\tilde{d}\}} \sigma_i |\tilde{\psi}_i\rangle\langle\psi_i|$. We define the *singular value transformation* corresponding to f as follows:

$$f^{(\mathrm{SV})}(A) \coloneqq \begin{cases} \sum_{i=1}^{\min\{d,\tilde{d}\}} f(\sigma_i)|\tilde{\psi}_i\rangle\langle\psi_i|, & \text{for odd } f, \\ \sum_{i=1}^{d} f(\sigma_i)|\psi_i\rangle\langle\psi_i|, & \text{for even } f. \end{cases}$$

Here, $\sigma_i \coloneqq 0$ for $i \in \{\min\{d,\tilde{d}\}+1, \cdots, d-1, d\}$. For any Hermitian matrix A, $f^{(\mathrm{SV})}(A) = f(A)$.

Finally, for any $d \times d$ Hermitian matrix A, there is a *spectral decomposition* of A such that $A = \sum_{i=1}^{d} \lambda_i |\psi_i\rangle\langle\psi_i|$ where all eigenvalues $\{\lambda_i\}_{i=1}^{d}$ are real and $\{|\psi_i\rangle\}_{i=1}^{d}$ is an orthonormal basis. As a consequence, if f is an even or odd function, $f(A) = \sum_{i=1}^{d} f(\lambda_i)|\psi_i\rangle\langle\psi_i| = f^{(\mathrm{SV})}(A)$ can be achieved by singular value transformation defined in Definition 2.2.

2.2 Distances and divergences for quantum states

We will provide an overview of relevant quantum distances and divergences, along with useful inequalities among different quantum distance-like measures. Additionally, we recommend [BOW19, Section 3.1] for a nice survey on quantum distance and divergences. We say that a square matrix ρ is a quantum state if ρ is positive semi-definite and $\mathrm{Tr}(\rho) = 1$.

Definition 2.3 (Quantum distances and divergences)**.** For any quantum states ρ_0 and ρ_1, we define several distance-like measures and relevant quantities:

- **Trace distance**. $\mathrm{td}(\rho_0, \rho_1) \coloneqq \frac{1}{2}\mathrm{Tr}|\rho_0 - \rho_1| = \frac{1}{2}\mathrm{Tr}(((\rho_0 - \rho_1)^\dagger(\rho_0 - \rho_1))^{1/2})$.
- **(Uhlmann) Fidelity**. $\mathrm{F}(\rho_0, \rho_1) \coloneqq \mathrm{Tr}|\sqrt{\rho_0}\sqrt{\rho_1}|$.
- **Squared Hilbert-Schmidt distance**. $\mathrm{HS}^2(\rho_0, \rho_1) \coloneqq \frac{1}{2}\mathrm{Tr}(\rho_0 - \rho_1)^2$.
- **von Neumann entropy**. $\mathrm{S}(\rho) \coloneqq -\mathrm{Tr}(\rho \ln \rho)$ for any quantum state ρ.
- **Quantum Jensen-Shannon divergence**. $\mathrm{QJS}(\rho_0, \rho_1) \coloneqq \mathrm{S}\left(\frac{\rho_0+\rho_1}{2}\right) - \frac{\mathrm{S}(\rho_0)+\mathrm{S}(\rho_1)}{2}$.

The trace distance and the squared Hilbert-Schmidt distance reach the minimum of 0 when ρ_0 equals ρ_1, while the fidelity attains a maximum value of 1. Additionally, there are two equalities when at least one of the two states is a pure state – A quantum state ρ is a pure state if and only if $\mathrm{Tr}(\rho^2) = 1$, equivalently $\rho = |\psi\rangle\langle\psi|$ for some $|\psi\rangle$ satisfying $\||\psi\rangle\|_2 = 1$:

[25] As indicated in [HJ12, 2.7.P2], such a matrix U is called a *unitary dilation* of A. This unitary dilation U exists if and only if A is a contraction, namely $\|A\| \leq 1$.

- For a pure state ρ_0 and a mixed state ρ_1, $\mathrm{F}^2(\rho_0, \rho_1) = \mathrm{Tr}(\rho_0\rho_1)$.
- For two pure states ρ_0 and ρ_1, $\mathrm{Tr}(\rho_0\rho_1) = 1 - \mathrm{HS}^2(\rho_0, \rho_1)$.

Moreover, we have $\mathrm{HS}^2(\rho_0, \rho_1) = \frac{1}{2}(\mathrm{Tr}(\rho_0^2) + \mathrm{Tr}(\rho_1^2)) - \mathrm{Tr}(\rho_0\rho_1)$. Additionally, Fuchs and van de Graaf [FvdG99] showed a well-known inequality between the trace distance and the fidelity:

Lemma 2.4 (Trace distance vs. fidelity, adapted from [FvdG99])**.** *For any states ρ_0 and ρ_1,*

$$1 - \mathrm{F}(\rho_0, \rho_1) \leq \mathrm{td}(\rho_0, \rho_1) \leq \sqrt{1 - \mathrm{F}^2(\rho_0, \rho_1)}.$$

The joint entropy theorem (Lemma 2.5) enhances our understanding of entropy in classical-quantum states and is necessary for our usages of the von Neumann entropy.

Lemma 2.5 (Joint entropy theorem, adapted from Theorem 11.8(5) in [NC02])**.** *Suppose p_i are probabilities corresponding to a distribution D, $|i\rangle$ are orthogonal states of a system A, and $\{\rho_i\}_i$ is any set of density operators for another system B. Then $\mathrm{S}\big(\sum_i p_i|i\rangle\langle i| \otimes \rho_i\big) = \mathrm{H}(D) + \sum_i p_i \mathrm{S}(\rho_i)$.*

Let us now turn our attention to the quantum Jensen-Shannon divergence, which is defined in [MLP05]. For simplicity, we define $\mathrm{QJS}_2(\rho_0, \rho_1) \coloneqq \mathrm{QJS}(\rho_0, \rho_1)/\ln 2$ using the base-2 (matrix) logarithmic function. Notably, when considering size-2 ensembles with a uniform distribution, the renowned Holevo bound [Hol73b] (see Theorem 12.1 in [NC02]) indicates that the *quantum Shannon distinguishability* studied in [FvdG99] is at most the quantum Jensen-Shannon divergence. Consequently, this observation yields inequalities between the trace distance and the quantum Jensen-Shannon divergence.[26]

Lemma 2.6 (Trace distance vs. quantum Jensen-Shannon divergence, adapted from [FvdG99, Hol73b, BH09])**.** *For any quantum states ρ_0 and ρ_1, we have*

$$1 - \mathrm{H}_2\left(\tfrac{1-\mathrm{td}(\rho_0,\rho_1)}{2}\right) \leq \mathrm{QJS}_2(\rho_0, \rho_1) \leq \mathrm{td}(\rho_0, \rho_1).$$

Here, the binary entropy $\mathrm{H}_2(p) \coloneqq -p\log(p) - (1-p)\log(1-p)$.

2.3 Space-bounded quantum computation

We say that a function $s(n)$ is *space-constructible* if there exists a deterministic space $s(n)$ Turing machine that takes 1^n as an input and output $s(n)$ in the unary encoding. Moreover, we say that a function $f(n)$ is $s(n)$-*space computable* if there exists a deterministic space $s(n)$ Turing machine that takes 1^n as an input and output $f(n)$. Our definitions of space-bounded quantum computation are formulated in terms of *quantum circuits*, whereas many prior works focused on *quantum Turing machines* [Wat09b, Wat03, vMW12]. For a discussion on the equivalence between space-bounded quantum computation using *quantum circuits* and *quantum Turing machines*, we refer readers to [FL18, Appendix A] and [FR21, Section 2.2].

We begin by defining time-bounded and space-bounded quantum circuit families, and then proceed to the corresponding complexity class $\mathsf{BQ_USPACE}[s(n)]$. It is worth noting that we use the abbreviated notation C_x to denote that the circuit $C_{|x|}$ takes input x.

Definition 2.7 (Time- and space-bounded quantum circuit families)**.** A (unitary) quantum circuit is a sequence of quantum gates, each of which belongs to some fixed gateset that is universal for quantum computation, such as $\{\textsc{Hadamard}, \mathrm{CNOT}, \mathrm{T}\}$. For a promise problem $\mathcal{L} = (\mathcal{L}_{\mathrm{yes}}, \mathcal{L}_{\mathrm{no}})$, we say that a family of quantum circuits $\{C_x : x \in \mathcal{L}\}$ is $t(n)$-time-bounded if there is a deterministic Turing machine that, on any input $x \in \mathcal{L}$, runs in time $O(t(|x|))$, and outputs a description of C_x such that C_x accepts (resp., rejects) if $x \in \mathcal{L}_{\mathrm{yes}}$ (resp., $x \in \mathcal{L}_{\mathrm{no}}$). Similarly, we say that a family of quantum circuits $\{C_x : x \in \mathcal{L}\}$ is $s(n)$-space-bounded if there

[26] For a detailed proof of these inequalities, please refer to [Liu23, Appendix B].

is a deterministic Turing machine that, on any input $x \in \mathcal{L}$, runs in space $O(s(|x|))$ (and hence time $2^{O(s(|x|))}$), and outputs a description of C_x such that C_x accepts (resp., rejects) if $x \in \mathcal{L}_{\text{yes}}$ (resp., $x \in \mathcal{L}_{\text{no}}$), as well as C_x is acting on $O(s(|x|))$ qubits and has $2^{O(s(|x|)}$ gates..

Definition 2.8 ($\mathsf{BQ_USPACE}[s(n), a(n), b(n)]$, adapted from Definition 5 in [FR21])**.** Let $s\colon \mathbb{N} \to \mathbb{N}$ be a space-constructible function such that $s(n) \geq \Omega(\log n)$. Let $a(n)$ and $b(n)$ be functions that are computable in deterministic space $s(n)$. A promise problem $(\mathcal{L}_{\text{yes}}, \mathcal{L}_{\text{no}})$ is in $\mathsf{BQ_USPACE}[s(n), a(n), b(n)]$ if there exists a family of $s(n)$-space-bounded (unitary) quantum circuits $\{C_x\}_{x\in\mathcal{L}}$, where $n = |x|$, satisfying the following:

- The output qubit is measured in the computational basis after applying C_x. We say that C_x *accepts* x if the measurement outcome is 1, whereas C_x *rejects* x if the outcome is 0.
- $\Pr[C_x \text{ accepts } x] \geq a(|x|)$ if $x \in \mathcal{L}_{\text{yes}}$, whereas $\Pr[C_x \text{ accepts } x] \leq b(|x|)$ if $x \in \mathcal{L}_{no}$.

We remark that Definition 2.8 is *gateset-independent*, given that the gateset is closed under adjoint and all entries in chosen gates have reasonable precision. This property is due to the space-efficient Solovay-Kitaev theorem presented in [vMW12]. Moreover, we can achieve error reduction for $\mathsf{BQ_USPACE}[s(n), a(n), b(n)]$ as long as $a(n) - b(n) \geq 2^{-O(s(n))}$, which follows from [FKL+16] or our space-efficient QSVT-based construction in Section 3.4. We thereby define $\mathsf{BQ_USPACE}[s(n)] \coloneqq \mathsf{BQ_USPACE}[s(n), 2/3, 1/3]$ to represent (two-sided) bounded-error unitary quantum space, and $\mathsf{BQ_UL} \coloneqq \mathsf{BQ_USPACE}[O(\log n)]$ to denote unitary quantum logspace.

We next consider general space-bounded quantum computation, which allows *intermediate quantum measurements*. As indicated in [AKN98, Section 4.1], for any quantum channel Φ mapping from density matrices on k_1 qubits to density matrices on k_2 qubits, we can exactly simulate this quantum channel Φ by a unitary quantum circuit acting on $2k_1 + k_2$ qubits. Therefore, we extend Definition 2.7 to *general quantum circuits*, which allows local operations, such as intermediate measurements in the computational basis, resetting qubits to their initial states, and tracing out qubits. Now we proceed with a definition on $\mathsf{BQSPACE}[s(n)]$.

Definition 2.9 ($\mathsf{BQSPACE}[s(n), a(n), b(n)]$, adapted from Definition 7 in [FR21])**.** Let $s\colon \mathbb{N} \to \mathbb{N}$ be a space-constructible function such that $s(n) \geq \Omega(\log n)$. Let $a(n)$ and $b(n)$ be functions that are computable in deterministic space $s(n)$. A promise problem $(\mathcal{L}_{\text{yes}}, \mathcal{L}_{\text{no}})$ is in $\mathsf{BQSPACE}[s(n), a(n), b(n)]$ if there exists a family of $s(n)$-space-bounded general quantum circuits $\{\Phi_x\}_{x\in\mathcal{L}}$, where $n = |x|$, satisfying the following holds:

- The output qubit is measured in the computational basis after applying Φ_x. We say that Φ_x *accepts* x if the measurement outcome is 1, whereas Φ_x *rejects* x if the outcome is 0.
- $\Pr[\Phi_x \text{ accepts } x] \geq a(|x|)$ if $x \in \mathcal{L}_{\text{yes}}$, whereas $\Pr[\Phi_x \text{ accepts } x] \leq b(|x|)$ if $x \in \mathcal{L}_{no}$.

It is noteworthy that unitary quantum circuits, which correspond to unitary channels, are a specific instance of general quantum circuits that correspond to quantum channels. we thus infer that $\mathsf{BQ_USPACE}[s(n)] \subseteq \mathsf{BQSPACE}[s(n)]$ for any $s(n) \geq \Omega(\log n)$. However, the opposite direction was a long-standing open problem. Recently, Fefferman and Remscrim [FR21] demonstrated a remarkable result that $\mathsf{BQSPACE}[s(n)] \subseteq \mathsf{BQ_USPACE}[O(s(n))]$. In addition, it is evident that $\mathsf{BQSPACE}[s(n)]$ can achieve error reduction since it admits sequential repetition simply by resetting working qubits. Therefore, we define $\mathsf{BQSPACE}[s(n)] \coloneqq \mathsf{BQSPACE}[s(n), 2/3, 1/3]$ to represent (two-sided) bounded-error general quantum space, and denote general quantum logspace by $\mathsf{BQL} \coloneqq \mathsf{BQSPACE}[O(\log n)]$.

We now turn our attention to *one-sided* bounded-error unitary quantum space $\mathsf{RQ_USPACE}[s(n)]$ and $\mathsf{coRQ_USPACE}[s(n)]$ for $s(n) \geq \Omega(\log n)$. These complexity classes were first introduced by Watrous [Wat01] and have been further discussed in [FR21]. We proceed with the definitions:

- $\mathsf{RQ_USPACE}[s(n), a(n)] \coloneqq \mathsf{BQ_USPACE}[s(n), a(n), 0]$;

- $\mathsf{coRQ_USPACE}[s(n), b(n)] \coloneqq \mathsf{BQ_USPACE}[s(n), 1, b(n)]$.

Note that $\mathsf{RQ_USPACE}[s(n), a(n)]$ and $\mathsf{coRQ_USPACE}[s(n), b(n)]$ can achieve error reduction, as shown in [Wat01] or our space-efficient QSVT-based construction in Section 3.4. We define

$$\mathsf{RQ_USPACE}[s(n)] \coloneqq \mathsf{BQ_USPACE}\left[s(n), \tfrac{1}{2}, 0\right] \text{ and } \mathsf{coRQ_USPACE}[s(n)] \coloneqq \mathsf{BQ_USPACE}\left[s(n), 1, \tfrac{1}{2}\right]$$

to represent one-sided bounded-error unitary quantum space, as well as logspace counterparts

$$\mathsf{RQ_UL} \coloneqq \mathsf{RQ_USPACE}[O(\log n)] \text{ and } \mathsf{coRQ_UL} \coloneqq \mathsf{coRQ_USPACE}[O(\log n)].$$

Remark 2.10 ($\mathsf{RQ_UL}$ and $\mathsf{coRQ_UL}$ are gateset-dependent). We observe that changing the gateset in space-efficient Solovay-Kitaev theorem [vMW12] can cause errors, revealing the *gateset-dependence* of unitary quantum space classes with one-sided bounded-error. To address this issue, we adopt a larger gateset $\mathcal{G}$ for $\mathsf{RQ_USPACE}[s(n)]$ and $\mathsf{coRQ_USPACE}[s(n)]$, which includes any single-qubit gates whose amplitudes can be computed in deterministic $O(s(n))$ space.

2.4 Polynomial approximation via averaged Chebyshev truncation

We begin by defining Chebyshev polynomials, and then introduce Chebyshev truncation and averaged Chebyshev truncation, with the latter being known as the *de La Vallée Poussin partial sum*. These concepts are essential to our space-efficient quantum singular value transformation techniques (space-efficient QSVT, Section 3). We recommend [Riv90, Chapter 3] for a comprehensive review of Chebyshev series and Chebyshev expansion.

Definition 2.11 (Chebyshev polynomials)**.** The Chebyshev polynomials (of the first kind) $T_k(x)$ are defined via the following recurrence relation: $T_0(x) \coloneqq 1$, $T_1(x) \coloneqq x$, and $T_{k+1}(x) \coloneqq 2xT_k(x) - T_{k-1}(x)$. For $x \in [-1, 1]$, an equivalent definition is $T_k(\cos\theta) = \cos(k\theta)$.

To use Chebyshev polynomials (of the first kind) for Chebyshev expansion, we first need to define an inner product between two functions, f and g, as long as the following integral exists:

$$\langle f, g\rangle \coloneqq \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}}\mathrm{d}x = \frac{2}{\pi}\int_{-\pi}^{0} f(\cos\theta)g(\cos\theta)\mathrm{d}\theta. \tag{2.1}$$

The Chebyshev polynomials form an orthonormal basis in the inner product space induced by $\langle\cdot,\cdot\rangle$ defined in Equation (2.1). As a result, any continuous and integrable function $f : [-1, 1] \to \mathbb{R}$ whose Chebyshev coefficients satisfy $\lim_{k\to\infty} c_k = 0$, where c_k is defined in Equation (2.2), has a Chebyshev expansion given by:

$$f(x) = \frac{1}{2}c_0T_0(x) + \sum_{k=1}^{\infty} c_kT_k(x) \text{ where } c_k \coloneqq \langle T_k, f\rangle. \tag{2.2}$$

A natural approach to approximating functions with a Chebyshev expansion is to consider the truncated version of the Chebyshev expansion $\tilde{P}_d = c_0/2 + \sum_{k=1}^{d} c_kT_k$, denoted as *Chebyshev truncation*. Remarkably, $\tilde{P}_d$ provides a *nearly best* uniform approximation to f:

Lemma 2.12 (Nearly best uniform approximation by Chebyshev truncation, adapted from Theorem 3.3 in [Riv90])**.** *For any continuous and integrable function $f\colon [-1, 1] \to \mathbb{R}$, let $\varepsilon_d(f)$ be the truncation error that corresponds to the degree-d best uniform approximation on $[-1, 1]$ to f, then the degree-d Chebyshev truncation polynomial $\tilde{P}_d$ satisfies*

$$\varepsilon_d(f) \le \max_{x\in[-1,1]} |f(x) - \tilde{P}_d(x)| \le \left(4 + \frac{4}{\pi^2}\log d\right)\varepsilon_d(f)$$

Consequently, if there is a degree-d polynomial $P_d^* \in \mathbb{R}[x]$ such that $\max_{x\in[-1,1]} |f(x) - P_d^*(x)| \le \epsilon$, then the degree-$d$ Chebyshev truncation polynomial $\tilde{P}_d$ satisfies

$$\max_{x\in[-1,1]} |f(x) - \tilde{P}_d(x)| \le O(\epsilon \log d).$$

It is noteworthy that the proof of Lemma 2.12 in [Riv90] relies only on the linear decay of Chebyshev coefficients c_k for any Chebyshev expansion. However, for functions with a Chebyshev expansion whose Chebyshev coefficients decay almost exponentially, Chebyshev truncation is "asymptotically" as good as the best uniform approximation:

Lemma 2.13 (A sufficient condition that Chebyshev truncation is "asymptotically" best, adapted from Equation (3.44) in [Riv90])**.** *For any function f that admits a Chebyshev expansion, consider a degree-d Chebyshev truncation polynomial $\tilde{P}_d$, and let $\varepsilon_d(f)$ be the truncation error corresponds to the degree-d best uniform approximation on $[-1,1]$ to f. If the Chebyshev coefficients of $\tilde{P}_d$ satisfy $\sum_{k=2}^{\infty} |c_{d+j}| \le \eta |c_{d+1}|$, then*

$$\varepsilon_d(f) \le \max_{x\in[-1,1]} |f(x) - \tilde{P}_d(x)| \le \frac{4}{\pi}(1+\eta)\varepsilon_d(f).$$

Although Lemma 2.13 improves the truncation error in Lemma 2.12 from $O(\epsilon \log d)$ to $O(\epsilon)$, it only applies to a fairly narrow range of functions, such as sine and cosine functions. Using an average of Chebyshev truncations, known as the de La Vallée Poussin partial sum, we obtain the degree-d *averaged Chebyshev truncation* $\hat{P}_{d'}$, which is a polynomial of degree $d' = 2d-1$:

$$\hat{P}_{d'}(x) \coloneqq \frac{1}{d}\sum_{l=d}^{d'} \tilde{P}_l(x) = \frac{\hat{c}_0}{2} + \sum_{k=1}^{d'} \hat{c}_k T_k(x) \text{ where } \hat{c}_k = \begin{cases} c_k, & 0 \le k \le d' \\ \frac{2d-k}{d} c_k, & k > d \end{cases}, \tag{2.3}$$

we can achieve the truncation error 4ϵ for any function that admits Chebyshev expansion.

Lemma 2.14 (Asymptotically best approximation by averaged Chebyshev truncation, adapted from Exercise 3.4.7 in [Riv90])**.** *For any function f that has a Chebyshev expansion, consider the degree-d averaged Chebyshev truncation $\hat{P}_{d'}$ defined in Equation (2.3). Let $\varepsilon_d(f)$ be the truncation error corresponds to the degree-d best uniform approximation on $[-1,1]$ to f. If there is a degree-d polynomial $P_d^* \in \mathbb{R}[x]$ such that $\max_{x\in[-1,1]} |f(x) - P_d^*(x)| \le \epsilon$, then*

$$\max_{x\in[-1,1]} \left|f(x) - \hat{P}_{d'}(x)\right| \le 4\varepsilon_d(f) \le 4 \max_{x\in[-1,1]} |f(x) - P_d^*(x)| \le 4\epsilon.$$

Lastly, since the ℓ_1 norm of the coefficient vector corresponding to the polynomial approximation plays a key role in our space-efficient QSVT (Section 3), we provide upper bounds for the coefficient vector $\hat{\mathbf{c}} \coloneqq (\hat{c}_0, \cdots, \hat{c}_d)$ in Lemma 2.15. Interestingly, Chebyshev coefficients c_k (and so do $\hat{c}_k$) decay a bit faster if the function f becomes a bit smoother.

Lemma 2.15 (ℓ_1-norm bounds on the averaged truncated Chebyshev coefficient vector)**.** *For any function f that admits a Chebyshev expansion and is bounded with $\max_{x\in[-1,1]} |f(x)| \le B$ for some constant $B > 0$, we have the following ℓ_1-norm bounds for the coefficient vector $\hat{\mathbf{c}}$ corresponds to the degree-d averaged Chebyshev truncation $\hat{P}_{d'}$ with $d' = 2d-1$:*

- *For any function f satisfying our conditions, we have $\|\mathbf{c}\|_1 \le O(B \log d)$;*
- *If the function f additionally is (at least) twice continuously differentiable, $\|\mathbf{c}\|_1 \le O(B)$.*

Proof. By substituting $\cos\theta$ for x and calculating a direct integral, we obtain:

$$\begin{aligned} c_k &= \frac{2}{\pi}\int_{-\pi}^{0} \cos(k\theta) f(\cos\theta) \mathrm{d}\theta \\ &\le \frac{2}{\pi}\int_{-\pi}^{0} \cos(k\theta) \Big(\max_{x\in\{-1,1\}} |f(x)| \Big) \mathrm{d}\theta \\ &\le \frac{2B}{\pi}\int_{-\pi}^{0} \cos(k\theta) \mathrm{d}\theta \\ &= \frac{2B}{\pi} \cdot \frac{\sin(k\pi)}{k}. \end{aligned} \tag{2.4}$$

Here, the third line follows from the fact that f is bounded with $\max_{x\in[-1,1]} |f(x)| \leq B$ for some constant $B > 0$. Hence, by combining Equation (2.4) and the Euler-Maclaurin formula, we know that the coefficient vector $\hat{\mathbf{c}}$ satisfies

$$\|\hat{\mathbf{c}}\|_1 = \sum_{k=0}^{d} |c_k| + \sum_{k=d+1}^{2d-1} \frac{2d-k}{d}|c_k| \leq \frac{2B}{\pi} \sum_{k=0}^{2d-1} \frac{1}{k} \leq O(B \log d).$$

For any function f that exhibits better smoothness, we can derive a sharper bound by considering $c_k = \frac{2}{\pi}\int_{-\pi}^{0} \cos(k\theta) f(\cos\theta) \mathrm{d}\theta$ as Fourier coefficients of the function $f(\cos\theta)$. For any function f that is (at least) twice continuously differentiable, the decay properties of its Fourier coefficients (e.g., [SS03, Exercise 3.18(a)]) imply that $|c_k| \leq O(B/k^2)$. Hence, we obtain an improved norm bound $\|\hat{\mathbf{c}}\|_1 \leq \sum_{k=0}^{2d-1} O(B/k^2) \leq O(B)$ as per the Euler-Maclaurin formula. □

2.5 Tools for space-bounded randomized and quantum algorithms

Our convention assumes that for any algorithm $\mathcal{A}$ in bounded-error randomized time $t(n)$ and space $s(n)$, $\mathcal{A}$ outputs the correct value with probability at least 2/3 (viewed as "success probability"). We first proceed with space-efficient success probability estimation.

Lemma 2.16 (Space-efficient success probability estimation by sequential repetitions)**.** *Let $\mathcal{A}$ be a randomized (resp., quantum) algorithm that outputs the correct value with probability p, has time complexity $t(n)$, and space complexity $s(n)$. We can obtain an additive-error estimation $\hat{p}$ such that $|p - \hat{p}| \leq \epsilon$, where $\epsilon \geq 2^{-O(s(n))}$. Moreover, this estimation can be computed in bounded-error randomized (resp., quantum) time $O(\epsilon^{-2}t(n))$ and space $O(s(n))$.*

Proof. Consider a m-time sequential repetition of the algorithm $\mathcal{A}$, and let X_i be a random variable indicating whether the i-th repetition succeeds, then we obtain a random variable $X = \frac{1}{m}\sum_{i=1}^{m} X_i$ such that $\mathbb{E}[X] = p$. Now let $\hat{X} = \frac{1}{m}\sum_{i=1}^{m} \hat{X}_i$ be the additive-error estimation, where $\hat{X}_i$ is the outcome of $\mathcal{A}$ in the i-th repetition. By the Chernoff-Hoeffding bound (e.g., Theorem 4.12 in [MU17]), we know that $\Pr\left[|\hat{X} - p| \geq \epsilon\right] \leq 2\exp(-2m\epsilon^2)$. By choosing $m = 2\epsilon^{-2}$, this choice of m ensures that this procedure based on $\mathcal{A}$ succeeds with probability at least 2/3.

Furthermore, the space complexity of our algorithm is $O(s(n))$ since we can simply reuse the workspace. Also, the time complexity is $m \cdot t(n) = O(\epsilon^{-2}t(n))$ as desired. □

Notably, when applying Lemma 2.16 to a quantum algorithm, we introduce intermediate measurements to retain space complexity through reusing working qubits. While space-efficient success probability estimation without intermediate measurements is possible,[27] we will use Lemma 2.16 for convenience, given that $\mathsf{BQL} = \mathsf{BQ_UL}$ [FR21].

The SWAP test was originally proposed for pure states in [BCWdW01]. Subsequently, in [KMY09], it was demonstrated that the SWAP test can also be applied to mixed states.

Lemma 2.17 (SWAP test for mixed states, adapted from [KMY09, Proposition 9])**.** *Suppose ρ_0 and ρ_1 are two n-qubit mixed quantum states. There is a $(2n+1)$-qubit quantum circuit that outputs 0 with probability $\frac{1+\mathrm{Tr}(\rho_0\rho_1)}{2}$, using 1 sample of each ρ_0 and ρ_1 and $O(n)$ one- and two-qubit quantum gates.*

A matrix B is said to be *sub-stochastic* if all its entries are non-negative and the sum of entries in each row (respectively, column) is strictly less than 1. Moreover, a matrix B is *row-stochastic* if all its entries are non-negative and the sum of entries in each row is equal to 1.

[27] Fefferman and Lin [FL18] noticed that one can achieve space-efficient success probability estimation for quantum algorithms without intermediate measurements via quantum amplitude estimation [BHMT02].

Lemma 2.18 (Sub-stochastic matrix powering in bounded space)**.** *Let B be an $l \times l$ upper-triangular sub-stochastic matrix, where each entry of B requires at most ℓ-bit precision. Then, there exists an explicit randomized algorithm that computes the matrix power $B^k[s,t]$ in $\log(l+1)$ space and $O(\ell k)$ time. Specifically, the algorithm accepts with probability $B^k[s,t]$.*

Proof. Our randomized algorithm leverages the equivalence between space-bounded randomized computation and Markov chains, see [Sak96, Section 2.4] for a detailed introduction.

First, we construct a row-stochastic matrix $\hat{B}$ from B by adding an additional column and row. Let $\hat{B}[i,j]$ denote the entry at the i-th column and the j-th row of $\hat{B}$. Specifically,

$$\hat{B}[i,j] \coloneqq \begin{cases} B[i,j], & \text{if } 1 \le i,j \le l; \\ 1 - \sum_{s=j}^{l} B[s,j], & \text{if } i = l+1 \text{ and } 1 \le j \le l+1; \\ 0, & \text{if } 1 \le i \le l \text{ and } j = l+1. \end{cases}$$

Next, we view $\hat{B}$ as a transition matrix of a Markov chain since $\hat{B}$ is row-stochastic. We consequently have a random walk on the directed graph $G = (V, E)$ where $V = \{1, 2, \cdots, l\} \cup \{\perp\}$ and $(u,v) \in E$ iff $\hat{B}(u,v) > 0$. In particular, the probability that a k-step random walk starting at node s and ending at node t is exactly $\hat{B}^k[s,t] = B^k[s,t]$. This is because the walker who visits the dummy node $\perp$ will not reach other nodes.

Finally, note that $\hat{B}$ is a $(l+1) \times (l+1)$ matrix, the matrix powering of $\hat{B}^k$ can be computed in $\log(l)$ space. In addition, the overall time complexity is $O(\ell k)$ since we simulate the dyadic rationals (with ℓ-bit precision) of a single transition exactly by ℓ coin flips. □

3 Space-efficient quantum singular value transformations

We begin by defining the *projected unitary encoding* and its special forms, viz. the bitstring indexed encoding and the block-encoding.

Definition 3.1 (Projected unitary encoding and its special forms, adapted from [GSLW19])**.** Let U be an (α, a, ϵ)-projected unitary encoding of a linear operator A if $\|A - \alpha\tilde{\Pi}U\Pi\| \le \epsilon$, where U and orthogonal projections $\tilde{\Pi}$ and Π act on $s+a$ qubits, and both $\mathrm{rank}(\tilde{\Pi})$ and $\mathrm{rank}(\Pi)$ are at least 2^a (a is viewed as the number of ancillary qubits). Furthermore, we are interested in two special forms of the projected unitary encoding:

- **Bitstring indexed encoding.** We say that a projected unitary encoding is a *bitstring indexed encoding* if both orthogonal projections $\tilde{\Pi}$ and Π span on $\tilde{S}, S \subseteq \{|0\rangle, |1\rangle\}^{\otimes(a+s)}$, respectively.[28] In particular, for any $|\tilde{s}_i\rangle \in \tilde{S}$ and $|s_j\rangle \in S$, we have a matrix representation $A_{\tilde{S},S}(i,j) \coloneqq \langle \tilde{s}_i | U | s_j \rangle$ of A.
- **Block encoding.** We say that a projected unitary encoding is a block-encoding if both orthogonal projections are of the form $\Pi = \tilde{\Pi} = |0\rangle\langle 0|^{\otimes a} \otimes I_s$. We use the shorthand $A = (\langle \bar{0}| \otimes I_s) U (|\bar{0}\rangle \otimes I_s)$ for convenience.

See Section 2.1 for definitions of singular value decomposition and transformation. With these definitions in place, we present the main (informal) theorem in this section:

Theorem 3.2 (Space-efficient QSVT)**.** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function bounded on the closed interval of interest $\mathcal{I} \subseteq [-1,1]$. If there exists a degree-d polynomial P_d^* that approximates $h\colon [-1,1] \to \mathbb{R}$, where h approximates f only on $\mathcal{I}$ with additive error at most ϵ, such that $\max_{x\in[-1,1]} |h(x) - P_d^*(x)| \le \epsilon$, then degree-$d$ averaged Chebyshev truncation yields another degree-d' polynomial $P_{d'}$, with $d' = 2d-1$, satisfying the following conditions:*

$$\max_{x\in\mathcal{I}} |f(x) - P_{d'}(x)| \le O(\epsilon) \text{ and } \max_{x\in[-1,1]} |P_{d'}(x)| \le 1.$$

[28] Typically, to ensure these orthogonal projections coincide with space-bounded quantum computation, we additionally require the corresponding subsets $\tilde{S}$ and S admit space-efficient set membership, namely deciding the membership of these subsets is in deterministic $O(s+a)$ space.

Moreover, there is a space-efficient classical algorithm for computing any entry in the coefficient vector $\hat{\mathbf{c}}$ of the averaged Chebyshev truncation polynomial $P_{d'}$:

- *If f is a continuously bounded function with $\max_{x\in[-1,1]} |f''(x)| \leq \mathrm{poly}(d)$,[29] then any entry in the coefficient vector $\hat{\mathbf{c}}$ can be computed in deterministic $O(\log d)$ space;*
- *If f is a piecewise-smooth function, then any entry in the coefficient vector $\hat{\mathbf{c}}$ can be computed in bounded-error randomized $O(\log d)$ space.*

Furthermore, for any $(1, a, 0)$-bitstring indexed encoding U of $A = \tilde{\Pi}U\Pi$, acting on $s + a$ qubits where $a(n) \leq s(n)$, and any $P_{d'}$ with $d' \leq 2^{O(s(n))}$, we can implement an $(\alpha, a+\log d+O(1), \epsilon_\alpha)$-bitstring indexed encoding of the quantum singular value transformation $P_{d'}^{(\mathrm{SV})}(A)$ that acts on $O(s(n))$ qubits using $O(d^2\eta_\alpha)$ queries to U, where ϵ_α is specified in Theorem 3.10. Here, $\alpha = \|\hat{\mathbf{c}}\|_1$ with $\eta_\alpha = 1$ in general, and particularly $\alpha = 1$ with $\eta_\alpha = \|\hat{\mathbf{c}}\|_1$ if $P_{d'}^{(\mathrm{SV})}(A)$ is an isometry. It is noteworthy that $\|\hat{\mathbf{c}}\|_1$ is bounded by $O(\log d)$ in general, and can be improved to a constant bound for twice continuously differentiable functions.

We remark that we can apply Theorem 3.2 to general forms of the projected unitary encoding U with orthogonal projections Π and $\tilde{\Pi}$, as long as such an encoding meets the conditions: (1) The basis of Π and $\tilde{\Pi}$ admits a well-defined order; (2) Both controlled-Π and controlled-$\tilde{\Pi}$ admit computationally efficient implementation. We note that bitstring indexed encoding defined in Definition 3.1 trivially meets the first condition, and a sufficient condition for the second condition is that the corresponding subsets S and $\tilde{S}$ have space-efficient set membership.

Next, we highlight the main technical contributions leading to our space-efficient quantum singular value transformations (Theorem 3.2). To approximately implement a space-efficient QSVT $f^{(\mathrm{SV})}(A)$, we require *the pre-processing* to find a space-efficient polynomial approximation $P_{d'}^{(f)} \approx f$ on $\mathcal{I}$. These polynomial approximations are detailed in Section 3.1:

- We provide deterministic space-efficient polynomial approximations for *continuously bounded* functions (Lemma 3.3) using averaged Chebyshev truncation (see Section 2.4), including the sign function (Corollary 3.6).
- We present bounded-error randomized space-efficient polynomial approximations for *piecewise-smooth* functions (Theorem 3.7), such as the normalized logarithmic function (Corollary 3.9). To achieve this, we adapt the time-efficient technique in [vAGGdW20, Lemma 37] to the space-efficient scenario by leveraging space-efficient random walks (Lemma 3.8).

With an appropriate polynomial approximation $P_{d'}^{(f)}$, we can implement the space-efficient QSVT $P_{f,d'}^{(\mathrm{SV})}(A)$, as established in Section 3.2 (specifically Theorem 3.10). Note that a space-efficient QSVT for Chebyshev polynomials is implicitly shown in [GSLW19] (Lemma 3.12). We establish Theorem 3.10 by combining this result with the LCU technique (Lemma 3.13) and the renormalization procedure (Lemma 3.14, if necessary and applicable).

In addition to these general techniques, we provide explicit space-efficient QSVT examples in Section 3.3, including those for the sign function (Corollary 3.15) and the normalized logarithmic function (Corollary 3.16). Notably, the former leads to a simple proof of space-efficient error reduction for unitary quantum computations (Section 3.4).

3.1 Space-efficient bounded polynomial approximations

We provide a systematic approach for constructing *space-efficient* polynomial approximations of real-valued piecewise-smooth functions, which is a space-efficient counterpart of Corollary 23 in [GSLW19]. Notably, our algorithm (Lemma 3.3) is *deterministic* for continuous functions

[29] This conclusion also applies to a linear combination of bounded functions, provided that the coefficients are bounded and can be computed deterministically and space-efficiently.

that are bounded on the interval $[-1,1]$. However, for general piecewise-smooth functions, we only introduce a *randomized* algorithm (Theorem 3.7). In addition, please refer to Section 2.4 as a brief introduction to Chebyshev polynomial and (averaged) Chebyshev truncation.

3.1.1 Continuously bounded functions

We propose a space-efficient algorithm for computing the coefficients of a polynomial approximation with high accuracy for continuously bounded functions. Our approach leverages the averaged Chebyshev truncation, specifically *the de La Vallée Poussin partial sum*, in conjunction with numerical integration, namely *the composite trapezium rule*.

Lemma 3.3 (Space-efficient polynomial approximations for bounded functions)**.** *For any continuous function f that f is bounded with $\max_{x\in[-1,1]}|f(x)| \le B$ for some known constant $B > 0$. Let $P^*_{f,d}$ be a degree-d polynomial with the same parity as f satisfying $\max_{x\in[-1,1]}|f(x)-P^*_{f,d}(x)| \le \epsilon$. By employing the degree-d averaged Chebyshev truncation, we can obtain a degree-d' polynomial $P^{(f)}_{d'}$ that has the same parity as $P^*_{f,d}$ and satisfies $\max_{x\in[-1,1]}|f(x)-P^{(f)}_{d'}(x)| \le 4\epsilon$.*[30] *This polynomial $P^{(f)}_{d'}$ is defined as a linear combination of Chebyshev polynomials $T_k(\cos\theta) = \cos(k\theta)$ with $d' = 2d-1$ and the integrand $F_k(\theta) \coloneqq \cos(k\theta)f(\cos\theta)$:*

$$P^{(f)}_{d'} = \frac{\hat{c}_0}{2} + \sum_{k=1}^{d'} \hat{c}_k T_k \text{ where } c_k = \frac{2}{\pi}\int_{-\pi}^{0} F_k(\theta)\mathrm{d}\theta \text{ and } \hat{c}_k = \begin{cases} c_k, & 0 \le k \le d' \\ \frac{2d-k}{d}c_k, & k > d \end{cases}. \tag{3.1}$$

If the integrand $F_k(\theta)$ satisfies $\max_{\xi\in[-\pi,0]}|F_k''(\theta)| \le O(d^\gamma)$ for some constant γ, then any entry of the coefficient vector $\hat{\mathbf{c}} = (\hat{c}_0,\cdots,\hat{c}_{d'})$, up to additive error ϵ for $\|\hat{\mathbf{c}}\|_1$, can be computed in deterministic time $O(d^{(\gamma+1)/2}\epsilon^{-1/2}t(\ell))$ and space $O(\log(d^{(\gamma+3)/2}\epsilon^{-3/2}B))$, where $\ell = O(\log(d^{(\gamma+3)/2}\epsilon^{-3/2}))$ and evaluating $F(\theta)$ in ℓ-bit precision is in deterministic time $t(\ell)$ and space $O(\ell)$. Furthermore, the coefficient vector $\hat{\mathbf{c}}$ has the following ℓ_1 norm bound:

- *For any function f satisfying our conditions, we have $\|\mathbf{c}\|_1 \le O(B\log d)$;*
- *If the function f is additionally (at least) twice continuously differentiable, $\|\mathbf{c}\|_1 \le O(B)$.*

Proof. We begin with the polynomial approximation $P^{(f)}_{d'}$ obtained from the degree-d averaged Chebyshev truncation expressed in Equation (3.1). The degree d' is $2d-1$ if f is odd, and $2d-2$ if f is even. To bound the truncation error of $P^{(f)}_{d'}$, we require a degree-d polynomial $P^*_{f,d}$ such that $\max_{x\in[-1,1]}|f(x)-P^*_{f,d}(x)| \le \epsilon$. By utilizing Lemma 2.14, we obtain the desired error bound $\max_{x\in[-1,1]}|f(x)-P^{(f)}_{d'}(x)| \le 4\epsilon$.

Computing the coefficients. To compute the coefficients $\hat{c}_k$ for $0 \le k \le d'$, it suffices to compute the Chebyshev coefficients c_k for $0 \le k \le 2d-1$. Note that $c_k = \frac{2}{\pi}\int_{-\pi}^{0} F_k(\theta)\mathrm{d}\theta$ where $F_k(\theta) \coloneqq \cos(k\theta)f(\cos\theta)$, we can estimate the numerical integration using the composite trapezium rule, e.g., [SM03, Section 7.5]. The application of this method yields the following:

$$\int_{-\pi}^{0} F_k(\theta)\mathrm{d}\theta \approx \frac{\pi}{m}\Big(\frac{F_k(\theta_0)}{2} + \sum_{l=1}^{m} F_k(\theta_l) + \frac{F_k(\theta_m)}{2}\Big) \text{ where } \theta_l \coloneqq \frac{\pi l}{m} - \pi \text{ for } l = 0,1,\cdots,m. \tag{3.2}$$

Moreover, we know the upper bound on the numerical errors for computing the coefficient c_k:

$$\varepsilon^{(f)}_{d',k} \coloneqq \sum_{l=1}^{m}\Big|\int_{x_{i-1}}^{x_i} F_k(\theta)\mathrm{d}\theta - \frac{\pi}{2m}\cdot(F_k(\theta_{i-1})+F_k(\theta_i))\Big| \le \frac{\pi^3}{12m^2}\max_{\xi\in[-\pi,\pi]}\big|F_k''(\xi)\big|. \tag{3.3}$$

[30] It is noteworthy that for any even function f, the degree of $P^{(f)}_{d'}$ is $2d-2$ rather than $2d-1$. Nevertheless, for the sake of convenience, we continue to choose $d' = 2d-1$.

To obtain an upper bound on the number of intervals m, we need to ensure that the error of the numerical integration is within

$$\varepsilon_{d'}^{(f)} = \sum_{k=0}^{d} \varepsilon_{d',k}^{(f)} + \sum_{k=d+1}^{d'} \frac{2d-k}{d} \varepsilon_{d',k}^{(f)} \leq \sum_{k=0}^{d'} \varepsilon_{d',k}^{(f)} \leq \epsilon.$$

Plugging the assumption $|F_k''(x)| \leq O(d^\gamma)$ into Equation (3.3), by choosing an appropriate value of $m = \Theta(\epsilon^{-1/2} d^{(\gamma+1)/2})$, we establish that $\varepsilon_{d'}^{(f)} \leq O(d^{\gamma+1})/m^2 \leq \epsilon$. Moreover, to guarantee that the accumulated error is $O(\epsilon/d)$ in Equation (3.2), we need to evaluate the integrand $F(\theta)$ with ℓ-bit precision, where $\ell = O(\log(dm/\epsilon)) = O(\log(\epsilon^{-3/2} d^{(\gamma+3)/2}))$. Lastly, the desired ℓ_1 norm bound of the coefficient vector $\hat{\mathbf{c}}$ directly follows from Lemma 2.15.

Analyzing time and space complexity. The presented numerical integration algorithm is deterministic, and therefore, the time complexity for computing the integral is $O(mt(\ell))$, where $t(\ell)$ is the time complexity for evaluating the integrand $F_k(\theta)$ within $2^{-\ell}$ accuracy (i.e., ℓ-bit precision) in $O(\ell)$ space. The space complexity required for computing the numerical integration is the number of bits required to index the integral intervals and represent the resulting coefficients. To be specific, the space complexity is

$$\begin{aligned} \max\left\{O(\log m), O(\ell), \log \|\hat{\mathbf{c}}\|_\infty\right\} &\leq O\big(\max\big\{\log\big(\epsilon^{-\frac{3}{2}} d^{\frac{\gamma+3}{2}}\big), \log B\big\}\big) \\ &\leq O\big(\log\big(\epsilon^{-\frac{3}{2}} d^{\frac{\gamma+3}{2}} B\big)\big). \end{aligned}$$

Here, $\|\hat{\mathbf{c}}\|_\infty = \max\limits_{0\leq k\leq d'} \frac{2}{\pi} \left| \int_{-\pi}^{0} \cos(k\theta) f(\cos\theta) \mathrm{d}\theta \right| \leq \max\limits_{-\pi\leq\theta\leq 0} O(|f(\cos\theta)|) \leq O(B)$, and the last inequality is due to the fact that $\Theta(\max\{\log A, \log B\}) = \Theta(\log(AB))$ for any $A, B > 0$. □

It is worth noting that evaluating a large family of functions, called holonomic functions, with ℓ-bit precision requires only *deterministic* $O(\ell)$ space:

Remark 3.4 (Space-efficient evaluation of holonomic functions). Holonomic functions encompass several commonly used functions,[31] such as polynomials, rational functions, sine and cosine functions (but not other trigonometric functions such as tangent or secant), exponential functions, logarithms (to any base), the Gaussian error function, and the normalized binomial coefficients. In [CGKZ05, Mez12], these works have demonstrated that evaluating a holonomic function with ℓ-bit precision is achievable in deterministic time $\tilde{O}(\ell)$ and space $O(\ell)$. Prior works achieved the same time complexity, but with a space complexity of $O(\ell \log \ell)$.

In addition, we provide an example in Remark 3.5 that achieves only a logarithmically weaker bound on $\|\hat{\mathbf{c}}\|_1$ using Lemma 3.3, whereas a constant norm bound can be achieved by leveraging Theorem 3.7 for piecewise-smooth functions.

Remark 3.5 (On the norm bound of the square-root function's polynomial approximation). We consider a function $\mathrm{Sqrt}_\delta(x)$ that coincides with $\sqrt{x}$ on the interval $[\delta, 1]$.[32] Specifically, $\mathrm{Sqrt}_\delta(x)$ is defined as $\sqrt{x}$ for $x \geq \delta$, $-\sqrt{-x}$ for $x \leq -\delta$, and $1/\sqrt{\delta}$ for $x \in (-\delta, \delta)$. $\mathrm{Sqrt}_\delta(x)$ is continuously bounded on $[-1, 1]$ and satisfies $|\mathrm{Sqrt}_\delta''(x)| \leq \delta^{-3/2}/4$ with the maximum at $x = \pm\delta$. As $\mathrm{Sqrt}_\delta''(x)$ is not continuous, its polynomial approximation via Lemma 3.3 achieves only $\|\mathbf{c}\|_1 \leq O(\log d)$.

We now present an example of bounded functions, specifically the sign function.

Corollary 3.6 (Space-efficient approximation to the sign function)**.** *For any $\delta > 0$ and $\epsilon > 0$, there is an explicit odd polynomial $P_{d'}^{\mathrm{sgn}}(x) = \hat{c}_0/2 + \sum_{k=1}^{d'} \hat{c}_k T_k(x) \in \mathbb{R}[x]$ of degree $d' \leq$*

[31] For a more detailed introduction, please refer to [BZ10, Section 4.9.2].

[32] Since the second derivative of the square-root function $\sqrt{x}$ is unbounded at $x = 0$, we cannot directly apply Lemma 3.3 to $\sqrt{x}$.

$\tilde{C}_{\mathrm{sgn}}\delta^{-1}\log\epsilon^{-1}$, *where* $d' = 2d-1$ *and* $\tilde{C}_{\mathrm{sgn}}$ *is a universal constant. Any entry of the coefficient vector* $\hat{\mathbf{c}} := (\hat{c}_0, \cdots, \hat{c}_{d'})$ *can be computed in deterministic time* $\tilde{O}\big(\epsilon^{-1/2}d^2\big)$ *and space* $O(\log(\epsilon^{-3/2}d^3))$. *Furthermore, the polynomial* $P^{\mathrm{sgn}}_{d'}$ *satisfies the following conditions:*

$$\forall x \in [-1,1]\setminus[-\delta,\delta], \left|\mathrm{sgn}(x) - P^{\mathrm{sgn}}_{d'}(x)\right| \le C_{\mathrm{sgn}}\epsilon, \text{ where } C_{\mathrm{sgn}} = 5;$$
$$\forall x \in [-1,1], \left|P^{\mathrm{sgn}}_{d'}(x)\right| \le 1.$$

Additionally, the coefficient vector $\hat{\mathbf{c}}$ *has a norm bounded by* $\|\hat{\mathbf{c}}\|_1 \le \hat{C}_{\mathrm{sgn}}$, *where* $\hat{C}_{\mathrm{sgn}}$ *is another universal constant. Without loss of generality, we assume that* $\hat{C}_{\mathrm{sgn}}$ *and* $\tilde{C}_{\mathrm{sgn}}$ *are at least* 1.

Proof. We start from a degree-d polynomial $\tilde{P}^{\mathrm{sgn}}_d$ that well-approximates $\mathrm{sgn}(x)$:

Proposition 3.6.1 (Polynomial approximation of the sign function, adapted from Lemma 10 and Corollary 4 in [LC17])**.** *For any* $\delta > 0$, $x \in \mathbb{R}$, $\epsilon \in (0, \sqrt{2e\pi})$. *Let* $\kappa = \frac{2}{\delta}\log^{1/2}\left(\frac{\sqrt{2}}{\sqrt{\pi}\epsilon}\right)$, *Then*

$$g_{\delta,\epsilon}(x) := \mathrm{erf}(\kappa x) \text{ satisfies that } |g_{\delta,\epsilon}(x)| \le 1 \text{ and } \max_{|x|\ge\delta/2}|g_{\delta,\epsilon}(x) - \mathrm{sgn}(x)| \le \epsilon.$$

Moreover, there is an explicit odd polynomial $\tilde{P}^{\mathrm{sgn}}_d \in \mathbb{R}[x]$ *of degree* $d = O(\sqrt{(\kappa^2+\log\epsilon^{-1})\log\epsilon^{-1}})$ *such that* $\max_{x\in[-1,1]}\left|\tilde{P}^{\mathrm{sgn}}_d(x) - \mathrm{erf}(\kappa x)\right| \le \epsilon$

By applying Proposition 3.6.1, we obtain a degree-d polynomial $\tilde{P}^{\mathrm{sgn}}_d$ that well approximates the function $\mathrm{erf}(\kappa x)$ where $\kappa = O(\delta^{-1}\sqrt{\log\epsilon^{-1}})$.

To utilize Lemma 3.3, it suffices to upper bound the second derivative $\max_{\xi\in[-\pi,0]}|F_k''(\xi)|$ for any $0 \le k \le d'$, as specified in Fact 3.6.2. The proof is deferred to Appendix A.1.1.

Fact 3.6.2. *Let* $F_k(\theta) = \mathrm{erf}(\kappa\cos\theta)\cos(k\theta)$, $\max\limits_{0\le k\le d'}\max\limits_{\xi\in[-\pi,0]}|F_k''(\xi)| \le \frac{2}{\sqrt{\pi}}\kappa + k^2 + \frac{4}{\sqrt{\pi}}\kappa^3 + \frac{4}{\sqrt{\pi}}k\kappa$.

Note that both κ and k are at most $O(d)$. By Fact 3.6.2, we have $\max_{\xi\in[-\pi,0]}|F_k''(\xi)| \le O(d^3)$ for any $0 \le k \le d'$. Utilizing Lemma 3.3, we obtain a polynomial approximation $P^{\mathrm{sgn}}_{d'}(x) = \hat{c}_0/2 + \sum_{k=1}^{d'}\hat{c}_k T_k(x)$ with a degree of $d' = 2d-1 \le \tilde{C}_{\mathrm{sgn}}\delta^{-1}\log\epsilon^{-1}$, where $\tilde{C}_{\mathrm{sgn}}$ is a universal constant. This polynomial satisfies $\max_{x\in[-1,1]}|\mathrm{erf}(\kappa x) - P^{\mathrm{sgn}}_{d'}(x)| \le 4\epsilon$. Then we can derive:

$$\max_{x\in[-1,1]}|\mathrm{sgn}(x) - P^{\mathrm{sgn}}_{d'}(x)| \le \epsilon + \max_{x\in[-1,1]}|\mathrm{erf}(\kappa x) - P^{\mathrm{sgn}}_{d'}(x)| \le C_{\mathrm{sgn}}\epsilon, \text{ where } C_{\mathrm{sgn}} = 5.$$

Moreover, to bound the norm $\|\hat{\mathbf{c}}\|_1$, it suffices to consider the function $\mathrm{erf}(\kappa x)$ due to Proposition 3.6.1. We observe that the first and second derivatives of $\mathrm{erf}(\kappa x)$, namely $2\kappa e^{-\kappa^2x^2}/\sqrt{\pi}$ and $-4\kappa^3 x e^{-\kappa^2x^2}/\sqrt{\pi}$, respectively, are continuous, making $\mathrm{erf}(\kappa x)$ is twice continuously differentiable. Hence, according to Lemma 3.3, $\|\hat{\mathbf{c}}\|_1 \le \hat{C}_{\mathrm{sgn}}$ for some universal constant $\hat{C}_{\mathrm{sgn}}$.[33]

For the complexity of computing coefficients $\{\hat{c}_k\}_{k=1}^{d'}$, note that the evaluation of the integrand $F(\theta)$ requires ℓ-bit precision, where $\ell = O(\log(\epsilon^{-3/2}d^3))$. Following $t(\ell) = \tilde{O}(\ell)$ specified in Remark 3.4, any entry of the coefficient vector $\hat{\mathbf{c}}$ can be computed in deterministic time $O(\epsilon^{-1/2}d^2t(\ell)) = \tilde{O}(\epsilon^{-1/2}d^2)$ and space $O(\log(\epsilon^{-3/2}d^3))$.

Finally, we note that $\max_{x\in[-1,1]}|P^{\mathrm{sgn}}_{d'}(x)| \le 1+\epsilon$ due to numerical errors in computing the coefficients $\{\hat{c}_k\}_{k=1}^{d'}$. We finish the proof by normalizing $\hat{P}^{\mathrm{sgn}}_{d'}$. In particular, we consider $P^{\mathrm{sgn}}_{d'}(x) := (1+\epsilon)^{-1}\hat{P}^{\mathrm{sgn}}_{d'}$ and adjust the coefficient vector $\hat{\mathbf{c}}$ of $P^{\mathrm{sgn}}_{d'}$ accordingly. □

3.1.2 Piecewise-smooth functions

We present a randomized algorithm for constructing bounded polynomial approximations of piecewise-smooth functions, offering a *space-efficient* alternative to Corollary 23 in [GSLW19], as described in Theorem 3.7. Our algorithm leverages Lemma 3.3 and Lemma 3.8.

[33] Let $c'_k := \langle T_k, \mathrm{sgn}\rangle$ be the coefficients corresponding to the sign function. A direct calculation, as shown in [MY23, Lemma 2.10], yields $\|\mathbf{c}'\|_1 = O(\log d)$. Our improved norm bound arises from utilizing smoother functions like $\mathrm{erf}(\kappa x)$, instead of relying on the sign function which is discontinuous at $x = 0$.

Since this subsection mostly focuses on polynomial approximations, we introduce some notation for convenience. For a function $f\colon \mathcal{I} \to \mathbb{R}$ and an interval $\mathcal{I}' \subseteq \mathcal{I}$, we define $\|f\|_{\mathcal{I}'} \coloneqq \sup\{|f(x)|\colon x \in \mathcal{I}'\}$ to denote the supremum of the function f on the interval $\mathcal{I}'$.

Theorem 3.7 (Taylor series based space-efficient bounded polynomial approximations)**.** *Consider a real-valued function $f\colon [-x_0-r-\delta, x_0+r+\delta] \to \mathbb{R}$ such that $f(x_0+x) = \sum_{l=0}^{\infty} a_l x^l$ for all $x \in [-r-\delta, r+\delta]$, where $x_0 \in [-1,1]$, $r \in (0,2]$, $\delta \in (0,r]$. Assume that $\sum_{l=0}^{\infty}(r+\delta)^l|a_l| \le B$ where $B > 0$. Let $\epsilon \in (0, \frac{1}{2B}]$ such that $B > \epsilon$, then there is a polynomial $P_{d'} \in \mathbb{R}[x]$ of degree $d' = 2d-1 \le O(\delta^{-1}\log(\epsilon^{-1}B))$, corresponding to some degree-d averaged Chebyshev truncation, such that any entry of the coefficient vector $\hat{\mathbf{c}}$ can be computed in bounded-error randomized time $\tilde{O}(\max\{(\delta')^{-5}\epsilon^{-2}B^2, d^2\epsilon^{-1/2}\})$ and space $O(\log(d^3(\delta')^{-4}\epsilon^{-3/2}B))$ where $\delta' \coloneqq \frac{\delta}{2(r+\delta)}$, such that*

$$\begin{aligned}\|f(x) - P(x)\|_{[x_0-r,x_0+r]} &\le O(\epsilon),\\ \|P(x)\|_{[-1,1]} &\le O(\epsilon) + \|f(x)\|_{[x_0-r-\delta/2,x_0+r+\delta/2]} \le O(\epsilon) + B,\\ \|P(x)\|_{[-1,1]\setminus[x_0-r-\delta/2,x_0+r+\delta/2]} &\le O(\epsilon).\end{aligned}$$

Furthermore, the coefficient vector $\hat{\mathbf{c}}$ of $P_{d'}$ has a norm bounded by $\|\hat{\mathbf{c}}\|_1 \le O(B)$.

The main ingredient, and the primary challenge, for demonstrating Theorem 3.7 is to construct a low-weight approximation using Fourier series, as shown in Lemma 37 of [vAGGdW20], which requires computing the powers of sub-stochastic matrices in bounded space (Lemma 2.18).

Lemma 3.8 (Space-efficient low-weight approximation by Fourier series)**.** *Let $0 < \delta, \epsilon < 1$ and $f\colon \mathbb{R} \to \mathbb{R}$ be a real-valued function such that $|f(x) - \sum_{k=0}^{K} a_k x^k| \le \epsilon/4$ for all $x \in \mathcal{I}_\delta$, the interval $\mathcal{I}_\delta \coloneqq [-1+\delta, 1-\delta]$ and $\|\mathbf{a}\|_1 \le O(\max\{\epsilon^{-1}, \delta^{-1}\})$. Then there is a coefficient vector $\mathbf{c} \in \mathbb{C}^{2M+1}$ such that*

- *For even functions, $\left|f(x) - \sum_{m=-M}^{M} c_m^{(\mathrm{even})}\cos(\pi x m)\right| \le \epsilon$ for any $x \in \mathcal{I}_\delta$;*
- *For odd functions, $\left|f(x) - \sum_{m=-M}^{M} c_m^{(\mathrm{odd})}\sin\left(\pi x\left(m+\frac{1}{2}\right)\right)\right| \le \epsilon$ for any $x \in \mathcal{I}_\delta$;*
- *Otherwise, $\left|f(x) - \sum_{m=-M}^{M}\left(c_m^{(\mathrm{even})}\cos(\pi x m) + c_m^{(\mathrm{odd})}\sin\left(\pi x\left(m+\frac{1}{2}\right)\right)\right)\right| \le \epsilon$ for any $x \in \mathcal{I}_\delta$.*

Here $M \coloneqq \max\left(2\lceil\delta^{-1}\ln(4\|a\|_1\epsilon^{-1})\rceil, 0\right)$ and $\|\mathbf{c}\|_1 \le \|\mathbf{a}\|_1$. Moreover, the coefficient vector $\mathbf{c}$ can be computed in bounded-error randomized time $\tilde{O}(\delta^{-5}\epsilon^{-2})$ and space $O(\log(\delta^{-4}\epsilon^{-1}))$.

Proof. We begin by noticing that the truncation error of $\sum_{k=0}^{K} a_k x^k$, as shown in [SM03, Theorem A.4], is $(1-\delta)^{k+1} \le e^{-\delta(k+1)} \le \epsilon$, implying that $K \ge \Omega(\delta^{-1}\ln\epsilon^{-1})$. Without loss of generality, we can assume that $\|\mathbf{a}\|_1 \ge \epsilon/2$.[34]

Construction of polynomial approximations. Our construction involves three approximations, as described in Lemma 37 of [vAGGdW20]. We defer the detailed proofs of all three approximations to Appendix A.1.2.

The first approximation combines the assumed $\sum_{k=0}^{K} a_k x^k$ with $\arcsin(x)$'s Taylor series.

Proposition 3.8.1 (First approximation)**.** *Let $\hat{f}_1(x) \coloneqq \sum_{k=0}^{K} a_k x^k$ such that $\|f - \hat{f}_1\|_{\mathcal{I}_\delta} \le \epsilon/4$. Then we know that $\hat{f}_1(x) = \sum_{k=0}^{K} a_k \sum_{l=0}^{\infty} b_l^{(k)} \sin^l\left(\frac{x\pi}{2}\right)$ where the coefficients $b_l^{(k)}$ satisfy that*

$$b_l^{(k+1)} = \sum_{l'=0}^{l} b_{l'}^{(k)} b_{l-l'}^{(1)} \text{ where } b_l^{(1)} = \begin{cases} 0 & \text{if } l \text{ is even,} \\ \binom{l-1}{\frac{l-1}{2}}\frac{2^{-l+1}}{l} \cdot \frac{2}{\pi} & \text{if } l \text{ is odd.} \end{cases} \tag{3.4}$$

Furthermore, the coefficients $\{b_l^{(k)}\}$ satisfies the following: (1) $\|\mathbf{b}^{(k)}\|_1 = 1$ for all $k \ge 1$; (2) $\mathbf{b}^{(k)}$ is entry-wise non-negative for all $k \ge 1$; (3) $b_l^{(k)} = 0$ if l and k have different parities.

[34] This is because if $\|\mathbf{a}\|_1 < \epsilon/2$, then $\|f\|_{\mathcal{I}_\delta} \le \|f(x) - \sum_{k=0}^{K} a_k x^k\|_{\mathcal{I}_\delta} + \|\sum_{k=0}^{K} a_k x^k\|_{\mathcal{I}_\delta} \le \epsilon/4 + \|\mathbf{a}\|_1 < \epsilon$, implying that $M = 0$ and $\mathbf{c} = 0$.

The second approximation truncates the series at $l = L$, and bounds the truncation error.

Proposition 3.8.2 (Second approximation)**.** *Let* $\hat{f}_2(x) \coloneqq \sum_{k=0}^{K} a_k \sum_{l=0}^{L} b_l^{(k)} \sin^l\left(\frac{x\pi}{2}\right)$ *where* $L \coloneqq \lceil \delta^{-2} \ln(4\|\mathbf{a}\|_1 \epsilon^{-1}) \rceil$*, then we have that* $\|\hat{f}_1 - \hat{f}_2\|_{\mathcal{I}_\delta} \le \epsilon/4$.

The third approximation approximates the functions $\sin^l(x)$ in $\hat{f}_2(x)$ using a tail bound of the binomial distribution. Notably, this construction not only quadratically improves the dependence on δ, but also ensures that the integrand's second derivative is *bounded* when combined with Lemma 3.3.

Proposition 3.8.3 (Third approximation)**.** *Let* $\hat{f}_3(x)$ *be polynomial approximations of* f *that depends on the parity of* f *such that* $\|\hat{f}_2 - \hat{f}_3\|_{\mathcal{I}_\delta} \le \epsilon/2$ *and* $M = \lfloor \delta^{-1} \ln(4\|\mathbf{a}\|_1 \epsilon^{-1}) \rfloor$*, then we have*

$$\hat{f}_3^{(\text{even})}(x) \coloneqq \sum_{k=0}^{K} a_k \sum_{\hat{l}=0}^{L/2} (-1)^{\hat{l}} 2^{-2\hat{l}} b_{2\hat{l}}^{(k)} \sum_{m'=\hat{l}-M}^{\hat{l}+M} (-1)^{m'} \binom{2\hat{l}}{m'} \cos(\pi x(m' - \hat{l})),$$

$$\hat{f}_3^{(\text{odd})}(x) \coloneqq \sum_{k=0}^{K} a_k \sum_{\hat{l}=0}^{(L-1)/2} (-1)^{\hat{l}+1} 2^{-2\hat{l}-1} b_{2\hat{l}+1}^{(k)} \sum_{m'=\hat{l}+1-M}^{\hat{l}+1+M} (-1)^{m'} \binom{2\hat{l}+1}{m'} \sin\left(\pi x\left(m' - \hat{l} - \tfrac{1}{2}\right)\right).$$

Therefore, we have that $\hat{f}_3(x) \coloneqq \hat{f}_3^{(\text{even})}(x)$ *if* f *is even, whereas* $\hat{f}_3(x) \coloneqq \hat{f}_3^{(\text{odd})}(x)$ *if* f *is odd. In addition, if* f *is neither even or odd, then* $\hat{f}_3(x) \coloneqq \hat{f}_3^{(\text{even})}(x) + \hat{f}_3^{(\text{odd})}(x)$.

We adopt the third approximation as our construction by rearranging the summations and introducing a new parameter m. The value of m is defined as $m \coloneqq m' - \hat{l}$ if f is even and $m \coloneqq m' - \hat{l} - 1$ if f is odd. Moreover, the definition of m depends on the parity of $l = 2\hat{l} + 1$[35] if f is neither even nor odd. By applying this approach, we can derive the following:

$$\begin{aligned}
&\hat{f}_3^{(\text{even})}(x) = \sum_{m=-M}^{M} c_m^{(\text{even})} \cos(\pi x m) \text{ where } c_m^{(\text{even})} \coloneqq (-1)^m \sum_{k=0}^{K} a_k \sum_{\hat{l}=0}^{L/2} b_{2\hat{l}}^{(k)} \binom{2\hat{l}}{m+\hat{l}} 2^{-2\hat{l}};\\
&\hat{f}_3^{(\text{odd})}(x) = \sum_{m=-M}^{M} c_m^{(\text{odd})} \sin\left(\pi x\left(m + \tfrac{1}{2}\right)\right) \text{ where } c_m^{(\text{odd})} \coloneqq (-1)^m \sum_{k=0}^{K} a_k \sum_{\hat{l}=0}^{(L-1)/2} b_{2\hat{l}+1}^{(k)} \binom{2\hat{l}+1}{m+\hat{l}+1} 2^{-2\hat{l}-1}.
\end{aligned} \tag{3.5}$$

We then notice that the rearrangement of terms in Equation (3.5) can be directly applied to the definition of $\hat{f}_3(x)$ in Proposition 3.8.3. As a consequence, we obtain the following bound on the accumulative error: $\|f - \hat{f}_3\|_{\mathcal{I}_\delta} \le \|f - \hat{f}_1\|_{\mathcal{I}_\delta} + \|\hat{f}_1 - \hat{f}_2\|_{\mathcal{I}_\delta} + \|\hat{f}_2 - \hat{f}_3\|_{\mathcal{I}_\delta} \le \epsilon$. Additionally, we remark that $\|\mathbf{c}\|_1 \le \|\mathbf{a}\|_1$, since $\|\mathbf{b}^{(k)}\|_1 = 1$ (see Proposition 3.8.1) and $\sum_{m=0}^{l} \binom{l}{m} = 2^l$.

Analyzing time and space complexity. To evaluate the bounded polynomial approximation $\hat{f}_3(x)$ with ϵ accuracy, it is necessary to approximate the summand with ℓ-bit precision, where $\ell = O(\log(KLM\epsilon^{-1})) = O(\log(\delta^{-4}\epsilon^{-1}))$. Since the summand is a product of a constant number of holonomic functions, approximating $b_l^{(k)}$ with ℓ-bit precision is sufficient. Other quantities in the summand can be evaluated with the desired accuracy in deterministic time $\tilde{O}(\ell)$ and space $O(\ell)$ as stated in Remark 3.4.

We now present a bounded-error randomized algorithm for estimating $b_l^{(k)}$. As $\mathbf{b}^{(1)}$ is entry-wise non-negative and $\sum_{i=1}^{l} b_i^{(1)} < \|\mathbf{b}^{(1)}\|_1 = 1$ following Proposition 3.8.1, we can express the recursive formula in Equation (3.4) as the matrix powering of a sub-stochastic matrix B_1:

$$B_1^k \coloneqq \begin{pmatrix} b_1^{(1)} & b_2^{(1)} & \cdots & b_{l-1}^{(1)} & b_l^{(1)} \\ 0 & b_1^{(1)} & \cdots & b_{l-2}^{(1)} & b_{l-1}^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_1^{(1)} & b_2^{(1)} \\ 0 & 0 & \cdots & 0 & b_1^{(1)} \end{pmatrix}^k = \begin{pmatrix} b_1^{(k)} & b_2^{(k)} & \cdots & b_{l-1}^{(k)} & b_l^{(k)} \\ 0 & b_1^{(k)} & \cdots & b_{l-2}^{(k)} & b_{l-1}^{(k)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_1^{(k)} & b_2^{(k)} \\ 0 & 0 & \cdots & 0 & b_1^{(k)} \end{pmatrix} \coloneqq B_k.$$

[35] In particular, the summand in $\hat{f}_3(x)$ is $c_m^{(\text{even})} \cos(\pi x m) + c_m^{(\text{odd})} \sin\left(\pi x\left(m + \frac{1}{2}\right)\right)$ if f is neither even nor odd.

In addition, we approximate the sub-stochastic matrix B_1 by dyadic rationals with ℓ-bit precision, denoted as $\hat{B}_1$. Utilizing Lemma 2.18, we can compute any entry $\hat{B}_1^k[s,t]$ with a randomized algorithm that runs in $O(\ell k)$ time and $\log(l+1)$ space with acceptance probability $\hat{B}_1^k[s,t]$. To evaluate $\hat{B}_1^k[s,t]$ with an additive error of ϵ, we use the sequential repetitions outlined in Lemma 2.16. Specifically, we repeat the algorithm $m = 2\epsilon^{-2}\ln(KLM) = O(\epsilon^{-2}\log(\delta^{-4}))$ times, and each turn succeeds with probability at least $1 - 1/(3KLM)$. Note that the number of the evaluation of $b_l^{(k)}$ for computing $\hat{f}_3(x)$ is $O(KLM)$, and by the union bound, we can conclude that the success probability of evaluating all coefficients in $\mathbf{c}$ is at least $2/3$.

Finally, we complete the proof by analyzing the overall computational complexity. It is evident that our algorithm utilizes $O(\ell + \log m) = O(\log(\delta^{-4}\epsilon^{-3}))$ space because indexing m repetitions requires additional $O(\log m)$ bits. Moreover, since there are $O(KLM)$ summands in $\hat{f}_3(x)$, and evaluating $b_l^{(k)}$ takes m repetitions with time complexity $O(\ell K)$ for a single turn, the overall time complexity is $O(KLM \cdot \ell K \cdot \epsilon^{-2}\log(KLM)) = \tilde{O}(\delta^{-5}\epsilon^{-2})$. □

Now we present the proof of Theorem 3.7, which is a space-efficient and randomized algorithm for constructing bounded polynomial approximations for piecewise-smooth functions.

Proof of Theorem 3.7. Our approach is based on Theorem 40 in [vAGGdW20] and Corollary 23 in [GSLW19]. Firstly, we obtain a Fourier approximation $\hat{f}(x)$ of the given function $f(x)$ by truncating it using Lemma 3.8. Next, we ensure that $\hat{f}(x)$ is negligible outside the interval $[-x_0 - r, x_0 + r]$ by multiplying it with a suitable rectangle function, denoted as $h(x)$. Finally, we derive a space-efficient polynomial approximation $\hat{h}(x)$ of $h(x)$ by applying Lemma 3.3.

Construction of a bounded function. Let us begin by defining a linear transformation $L(x) \coloneqq \frac{x-x_0}{r+\delta}$ that maps $[x_0 - r - \delta, x_0 + r + \delta]$ to $[-1, 1]$. For convenience, we denote $g(y) \coloneqq f(L^{-1}(y))$ and $b_l \coloneqq a_l(r+\delta)^l$, then it is evident that $g(y) \coloneqq \sum_{l=0}^{\infty} b_l y^l$ for any $y \in [-1, 1]$.

To construct a Fourier approximation by Lemma 3.8, we need to bound the truncation error $\varepsilon_J^{(g)}$. We define $\delta' \coloneqq \frac{\delta}{2(r+\delta)}$ and $J \coloneqq \lceil (\delta')^{-1}\log(12B\epsilon^{-1})\rceil$. This ensures that the truncation error $\varepsilon_J^{(g)} \coloneqq \left|g(y) - \sum_{j=0}^{J-1} b_j y^j\right|$ for any $y \in [-1+\delta', 1-\delta']$ satisfies the following:

$$\varepsilon_J^{(g)} = \left|\sum_{j=J}^{\infty} b_j y^j\right| \le \sum_{j=J}^{\infty} \left|b_j(1-\delta')^j\right| \le (1-\delta')^J \sum_{j=J}^{\infty} |b_j| \le (1-\delta')^J B \le e^{-\delta' J} B \le \frac{\epsilon}{12} \coloneqq \frac{\epsilon'}{4}.$$

Afterward, let $\hat{\mathbf{b}} \coloneqq (b_0, b_1, \cdots, b_{J-1})$, then we know that $\|\hat{\mathbf{b}}\|_1 \le \|\mathbf{b}\|_1 \le B$ by the assumption. Now we utilize Lemma 3.8 and obtain the Fourier approximation $\hat{g}(y)$:

$$\hat{g}(y) \coloneqq \begin{cases} \sum_{m=-M}^{M} c_m^{(\text{even})} \cos(\pi y m), & \text{if } f \text{ is even} \\ \sum_{m=-M}^{M} c_m^{(\text{odd})} \sin\left(\pi y\left(m+\frac{1}{2}\right)\right), & \text{if } f \text{ is odd} \\ \sum_{m=-M}^{M} \left(c_m^{(\text{even})}\cos(\pi y m) + c_m^{(\text{odd})}\sin\left(\pi y\left(m+\frac{1}{2}\right)\right)\right), & \text{otherwise} \end{cases}. \tag{3.6}$$

By appropriately choosing $M = O\big((\delta')^{-1}\log\left(\|\hat{\mathbf{b}}\|_1/\epsilon'\right)\big) = O\big(r\delta^{-1}\log\left(B/\epsilon\right)\big)$, we obtain that the vectors of coefficients $\mathbf{c}^{(\text{even})}$ and $\mathbf{c}^{(\text{odd})}$ satisfy $\|\mathbf{c}^{(\text{even})}\|_1 \le \|\hat{\mathbf{b}}\|_1 \le \|\mathbf{b}\|_1 \le B$ and similarly $\|\mathbf{c}^{(\text{odd})}\|_1 \le B$. Plugging $f(x) = g(L(x))$ into Equation (3.6), we conclude that $\hat{f}(x) = \hat{g}(L(x))$ is a Fourier approximation of f with an additive error of $\epsilon/3$ on the interval $[x_0 - r - \delta/2, x_0 + r + \delta/2]$:

$$\hat{f}(x) = \hat{g}\Big(\frac{x-x_0}{r+\delta}\Big) = \begin{cases} \sum\limits_{m=-M}^{M} c_m^{(\text{even})}\cos\big(\pi m\big(\frac{x-x_0}{r+\delta}\big)\big), & \text{if } f \text{ is even} \\ \sum\limits_{m=-M}^{M} c_m^{(\text{odd})}\sin\big(\pi\big(m+\frac{1}{2}\big)\big(\frac{x-x_0}{r+\delta}\big)\big), & \text{if } f \text{ is odd} \\ \sum\limits_{m=-M}^{M} c_m^{(\text{even})}\cos\big(\pi m\big(\frac{x-x_0}{r+\delta}\big)\big) + c_m^{(\text{odd})}\sin\big(\pi\big(m+\frac{1}{2}\big)\big(\frac{x-x_0}{r+\delta}\big)\big), & \text{otherwise} \end{cases}.$$

Making the error negligible outside the interval. Subsequently, we define the function $h(x) = \hat{f}(x) \cdot R(x)$ such that it becomes negligible outside the interval of interest, i.e., $[x_0 - r - \delta/2, x_0 + r + \delta/2]$. Here, the approximate rectangle function $R(x)$ is $\tilde{\epsilon}$-close to 1 on the interval $[x_0 - r, x_0 + r]$, and is $\tilde{\epsilon}$-close to 0 on the interval $[-1, 1] \setminus [x_0 - r - 2\tilde{\delta}, x_0 + r + 2\tilde{\delta}]$, where $\tilde{\epsilon} \coloneqq \epsilon/(3B)$ and $\tilde{\delta} \coloneqq \delta/4$. Moreover, $|R(x)| \leq 1$ for any $x \in [-1, 1]$. Similar to Lemma 29 in [GSLW19], $R(x)$ can be expressed as a linear combination of Gaussian error functions:

$$R(x) \coloneqq \tfrac{1}{2}\Big[\operatorname{erf}\big(\kappa(x - x_0 + r + \delta')\big) - \operatorname{erf}\big(\kappa(x - x_0 - r - \delta')\big)\Big] \text{ where } \kappa \coloneqq \tfrac{2}{\delta'}\log^{\frac{1}{2}}\tfrac{\sqrt{2}}{\sqrt{\pi}\epsilon'} = \tfrac{8}{\delta}\log^{\frac{1}{2}}\tfrac{\sqrt{18}B}{\sqrt{\pi}\epsilon}. \quad (3.7)$$

Bounded polynomial approximation via averaged Chebyshev truncation. We here present an algorithmic, space-efficient, randomized polynomial approximation method using averaged Chebyshev truncation to approximate the function $h(x) \coloneqq \hat{f}(x) \cdot R(x)$. As suggested in Proposition 3.7.1, we use an explicit polynomial approximation $P_d^*(x)$ of the bounded function $h(x)$ of degree $d = O(\delta^{-1}\log(B\epsilon^{-1}))$ that satisfies the conditions specified in Equation (3.8).

Proposition 3.7.1 (Bounded polynomial approximations based on a local Taylor series, adapted from [GSLW19, Corollary 23])**.** *Let $x_0 \in [-1, 1]$, $r \in (0, 2]$, $\delta \in (0, r]$ and let $f\colon [-x_0 - r - \delta, x_0 + r + \delta] \to \mathbb{R}$ and be such that $f(x_0 + x) \coloneqq \sum_{l=0}^{\infty} a_l x^l$ for all $x \in [-r-\delta, r+\delta]$. Suppose $B > 0$ is such that $\sum_{l=0}^{\infty}(r + \delta)^l |a_l| \leq B$. Let $\epsilon \in \big(0, \frac{1}{2B}\big]$, there is a $\epsilon/3$-precise Fourier approximation $\tilde{f}(x)$ of $f(x)$ on the interval $[x_0 - r + \delta/2, x_0 + r + \delta/2]$, where $\hat{f}(x) \coloneqq \sum_{m=-M}^{M} \operatorname{Re}\Big[\tilde{c}_m e^{-\frac{\mathrm{i}\pi m}{2(r+\delta)}x_0} e^{\frac{\mathrm{i}\pi m}{2(r+\delta)}x}\Big]$ and $\|\tilde{\mathbf{c}}\|_1 \leq B$. We have an explicit polynomial $P_d^* \in \mathbb{R}[x]$ of degree $d = O(\delta^{-1}\log(B\epsilon^{-1}))$ s.t.*

$$\begin{aligned} \|\hat{f}(x)R(x) - P_d^*(x)\|_{[x_0 - r, x_0 + r]} &\leq \epsilon, \\ \|P_d^*(x)\|_{[-1,1]} &\leq \epsilon + \|\hat{f}(x)R(x)\|_{[x_0 - r - \delta/2, x_0 + r + \delta/2]} \leq \epsilon + B, \\ \|P_d^*(x)\|_{[-1,1] \setminus [x_0 - r - \delta/2, x_0 + r + \delta/2]} &\leq \epsilon. \end{aligned} \quad (3.8)$$

To utilize Lemma 3.3, we need to bound the second derivative $\max_{\xi \in [-\pi, 0]} |F_k''(\xi)|$, where the integrand $F_k(\cos\theta) \coloneqq \cos(k\theta)h(\cos\theta)$ for any $0 \leq k \leq d'$ with $d' = 2d - 1$. We will calculate this upper bound directly in Fact 3.7.2, and the proof is deferred to Appendix A.1.3.

Fact 3.7.2. *Consider the integrand $F_k(\theta) = \sum_{m=-M}^{M} \frac{c_m}{2}\big(H_{k,m}^{(+)} - H_{k,m}^{(-)}\big)$ for any function f which is either even or odd. If f is even, we have that $c_m = c_m^{(\mathrm{even})}$ defined in Lemma 3.8, and*

$$H_{k,m}^{(\pm)}(\theta) \coloneqq \cos\Big(\pi m\Big(\frac{\cos\theta - x_0}{r + \delta}\Big)\Big) \cdot \cos(k\theta) \cdot \operatorname{erf}\Big(\kappa\Big(\cos\theta - x_0 \pm r \pm \frac{\delta}{4}\Big)\Big). \quad (3.9)$$

Likewise, if f is odd, we know that $c_m = c_m^{(\mathrm{odd})}$ defined in Lemma 3.8, and

$$H_{k,m}^{(\pm)}(\theta) \coloneqq \sin\Big(\pi\Big(m + \frac{1}{2}\Big)\Big(\frac{\cos\theta - x_0}{r + \delta}\Big)\Big) \cdot \cos(k\theta) \cdot \operatorname{erf}\Big(\kappa\Big(\cos\theta - x_0 \pm r \pm \frac{\delta}{4}\Big)\Big). \quad (3.10)$$

Moreover, the integrand is $F_k(\theta) = \sum_{m=-M}^{M}\Big(\frac{c_m^{(\mathrm{even})}}{2}\big(\hat{H}_{k,m}^{(+)} - \hat{H}_{k,m}^{(-)}\big) + \frac{c_m^{(\mathrm{odd})}}{2}\big(\tilde{H}_{k,m}^{(+)} - \tilde{H}_{k,m}^{(-)}\big)\Big)$ when f is neither even nor odd, where $\hat{H}_{k,m}^{(\pm)}$ and $\tilde{H}_{k,m}^{(\pm)}$ follow from Equation (3.9) and Equation (3.10), respectively. Regardless of the parity of f, we have that the second derivative $F_k''(\theta) \leq O(Bd^3)$.

Together with Fact 3.7.2, we are ready to apply Lemma 3.3 to $h(x) = \hat{f}(x)R(x)$, resulting in a degree-d' polynomial $P_{d'} = \hat{c}_0/2 + \sum_{k=1}^{d'} \hat{c}_k T_k$ where $d' = 2d - 1$ and $\hat{c}_k$ is defined as in Equation (3.1). Since $P_{d'}$ is the degree-d averaged Chebyshev truncation of the function h and satisfies Equation (3.8), we define intervals $\mathcal{I}_{\mathrm{int}} \coloneqq [x_0 - r, x_0 + r]$ and $\mathcal{I}_{\mathrm{ext}} \coloneqq [-1, 1] \setminus [x_0 - r - \delta/2, x_0 + r + \delta/2]$ to obtain:

$$\begin{aligned} \|f(x) - P_{d'}(x)\|_{\mathcal{I}_{\mathrm{int}}} &\leq \|f(x) - h(x)\|_{\mathcal{I}_{\mathrm{int}}} + \|h(x) - P_{d'}(x)\|_{\mathcal{I}_{\mathrm{int}}} \leq \epsilon + 4\epsilon = O(\epsilon), \\ \|P_{d'}(x) - 0\|_{\mathcal{I}_{\mathrm{ext}}} &\leq \|P_{d'}(x) - h(x)\|_{\mathcal{I}_{\mathrm{ext}}} + \|h(x) - 0\|_{\mathcal{I}_{\mathrm{ext}}} \leq 4\epsilon + \epsilon/3B \leq O(\epsilon). \end{aligned} \quad (3.11)$$

We can achieve the desired error bound by observing Equation (3.11) implies:

$$\|P_{d'}(x)\|_{[-1,1]} \le \|P_{d'}(x)\|_{\mathcal{I}_{\text{ext}}} + \|P_{d'}(x)\|_{[-1,1]\setminus\mathcal{I}_{\text{ext}}} \le O(\epsilon) + B.$$

Moreover, it is not too hard to see that the first and the second derivatives of the function $h(\cos\theta)$ are continuous, implying that $h(\cos\theta)$ is twice continuously differentiable. By using Lemma 3.3, we deduce that the norm of the coefficient vector $\hat{\mathbf{c}}$ of the polynomial $P_{d'}$ is bounded by $\|\hat{\mathbf{c}}\|_1 \le O(B)\cdot(1+O(\epsilon)) = O(B)$.

Analyzing time and space complexity. The construction of $\hat{f}(x)$ can be implemented in bounded-error randomized time $\tilde{O}((\delta')^{-5}\epsilon^{-2}B^2)$ and space $O(\log((\delta')^{-4}\epsilon^{-1}B))$, given that this construction uses Lemma 3.8 with $\delta' = \frac{\delta}{2(r+\delta)} \in (0, \frac{1}{2}]$ and $\epsilon' = \frac{\epsilon}{3B}$. Having $\hat{f}(x)$, we can construct a bounded polynomial approximation $\hat{h}(x)$ deterministically using Lemma 3.3. This construction can be implemented in deterministic time $O(d^{(\gamma+1)/2}\epsilon^{-1/2}t(\ell)) \le \tilde{O}(d^2\epsilon^{-1/2})$ and space $O(\log(d^{(\gamma+3)/2}\epsilon^{-3/2}B)) \le O(\log(d^3\epsilon^{-3/2}B))$ since the integrand $F_k(\theta)$ is a product of a constant number of (compositions of) holonomic functions (Remark 3.4). Therefore, our construction can be implemented in bounded-error randomized time $\tilde{O}(\max\{(\delta')^{-5}\epsilon^{-2}B^2, d^2\epsilon^{-1/2}\})$ and space $O(\max\{\log((\delta')^{-4}\epsilon^{-1}B), \log(d^3\epsilon^{-3/2}B)\}) \le O(\log(d^3(\delta')^{-4}\epsilon^{-3/2}B))$. □

With the aid of Theorem 3.7, we can provide a space-efficient polynomial approximation to the normalized logarithmic function utilized in Lemma 11 of [GL20].

Corollary 3.9 (Space-efficient polynomial approximation to the normalized logarithmic function)**.** *Let $\beta \in (0,1]$ and $\epsilon \in (0, 1/2)$, there is an even polynomial $P^{\ln}_{d'}$ of degree $d' = 2d-1 \le \tilde{C}_{\ln}\beta^{-1}\log\epsilon^{-1}$, where $P^{\ln}_{d'}$ corresponds to some degree-d averaged Chebyshev truncation and $\tilde{C}_{\ln}$ is a universal constant, such that*

$$\forall x \in [\beta, 1], \left|P^{\ln}_{d'}(x) - \tfrac{\ln(1/x)}{2\ln(2/\beta)}\right| \le C_{\ln}\epsilon, \text{ where } C_{\ln} \text{ is a universal constant,}$$
$$\forall x \in [-1,1], |P^{\ln}_{d'}(x)| \le 1.$$

Moreover, the coefficient vector $\mathbf{c}^{\ln}$ of $P^{\ln}_{d'}$ has a norm bounded by $\|\mathbf{c}^{\ln}\|_1 \le \hat{C}_{\ln}$, where $\hat{C}_{\ln}$ is another universal constant. In addition, any entry of the coefficient vector $\mathbf{c}^{\ln}$ can be computed in bounded-error randomized time $\tilde{O}(\max\{\beta^{-5}\epsilon^{-2}, d^2\epsilon^{-1/2}\})$ and space $O(\log(d^3\beta^{-4}\epsilon^{-3/2}))$. Without loss of generality, we assume that all constants $C_{\ln}$, $\hat{C}_{\ln}$, and $\tilde{C}_{\ln}$ are at least 1.

Proof. Consider the function $f(x) \coloneqq \frac{\ln(1/x)}{2\ln(2/\beta)}$. We apply Theorem 3.7 to f by choosing the same parameters as in Lemma 11 of [GL20], specifically $\epsilon' = \epsilon/2$, $x_0 = 1$, $r = 1-\beta$, $\delta = \beta/2$, and $B = 1/2$.[36] This results in a space-efficient randomized polynomial approximation $\tilde{P}_{d'} \in \mathbb{R}[x]$ of degree $d' = 2d-1 = O(\delta^{-1}\log(\epsilon^{-1}B)) \le \tilde{C}_{\ln}\beta^{-1}\log\epsilon^{-1}$, where $\tilde{P}_{d'}$ corresponds to some degree-d averaged Chebyshev truncation and $\tilde{C}_{\ln}$ is a universal constant. By appropriately choosing $\eta \le 1/2$ such that $C'_{\ln}\epsilon = \eta/4$ for a universal constant $C'_{\ln}$, this polynomial approximation $\tilde{P}_{d'}$ satisfies the following inequalities:

$$\begin{aligned} \|f(x) - \tilde{P}_{d'}(x)\|_{[\beta,2-\beta]} &\le C'_{\ln}\epsilon = \tfrac{\eta}{4} \\ \|\tilde{P}_{d'}(x)\|_{[-1,1]} &\le B + C'_{\ln}\epsilon \le \tfrac{1}{2} + C'_{\ln}\epsilon = \tfrac{1}{2} + \tfrac{\eta}{4} \\ \|\tilde{P}_{d'}(x)\|_{[-1,\beta/2]} &\le C'_{\ln}\epsilon = \tfrac{\eta}{4}. \end{aligned} \tag{3.12}$$

Additionally, the coefficient vector $\mathbf{c}^{(\tilde{P})}$ of $\tilde{P}_{d'}$ satisfies that $\|\mathbf{c}^{(\tilde{P})}\|_1 \le O(B) \le \hat{C}_{\ln}$ where $\hat{C}_{\ln}$ is a universal constant. Notice that $\delta' = \frac{\delta}{2(r+\delta)} = \frac{\beta/2}{2(1-\beta+\beta/2)} = \frac{\beta}{4(1-\beta/2)} = \Theta(\beta)$, our utilization of Theorem 3.7 yields a bounded-error randomized algorithm that requires $O(\log(d^3(\delta')^{-4}\epsilon^{-3/2}B)) = O(\log(d^3\beta^{-4}\epsilon^{-3/2}))$ space and $\tilde{O}(\max\{(\delta')^{-5}\epsilon^{-2}B^2, d^2\epsilon^{-1/2}\}) = \tilde{O}(\max\{\beta^{-5}\epsilon^{-2}, d^2\epsilon^{-1/2}\})$ time.

[36] As indicated in Lemma 11 of [GL20], since the Taylor series of $f(x)$ at $x=1$ is $\frac{1}{2\ln(2/\beta)}\sum_{l=1}^{\infty}\frac{(-1)^l x^l}{l}$, we obtain that $B = f(\frac{\beta}{2}-1) = \frac{1}{2\ln(2/\beta)}\sum_{l=1}^{\infty}\frac{(1-\beta/2)^l}{l} = -\frac{1}{2\ln(2/\beta)}\sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{l}(\beta/2-1)^l = -\frac{1}{2\ln(2/\beta)}\ln\frac{\beta}{2} = \frac{1}{2}$.

Furthermore, note that the real-valued function $f(x)$ only defines when $x > 0$, then $\tilde{P}(x)$ is not an even polynomial in general. Instead, we consider $P^{\ln}_{d'}(x) \coloneqq (1+\eta)^{-1}(\tilde{P}_{d'}(x) + \tilde{P}_{d'}(-x))$ for all $x \in [-1,1]$. Together with Equation (3.12), we have derived that:

$$\begin{aligned}
&\|f(x) - P^{\ln}_{d'}(x)\|_{[\beta,1]} \\
\leq &\left\|f(x) - \tfrac{1}{1+\eta}\tilde{P}_{d'}(x)\right\|_{[\beta,1]} + \left\|\tfrac{1}{1+\eta}\tilde{P}_{d'}(-x)\right\|_{[\beta,1]} \\
\leq &\left\|f(x) - \tilde{P}_{d'}(x)\right\|_{[\beta,1]} + \left\|\tilde{P}_{d'}(x) - \tfrac{1}{1+\eta}\tilde{P}_{d'}(x)\right\|_{[\beta,1]} + \left\|\tfrac{1}{1+\eta}\tilde{P}_{d'}(-x)\right\|_{[\beta,1]} \\
\leq &\tfrac{\eta}{4} + \tfrac{\eta}{1+\eta}\cdot\left(\tfrac{1}{2} + \tfrac{\eta}{4}\right) + \tfrac{1}{1+\eta}\cdot\tfrac{\eta}{4} \\
= &\tfrac{\eta}{4} + \tfrac{\eta}{1+\eta}\cdot\tfrac{1+\eta}{4} + \tfrac{1}{1+\eta}\cdot\tfrac{\eta}{2} \\
\leq &\eta.
\end{aligned} \tag{3.13}$$

Here, the last line owes to the fact that $\eta > 0$. Consequently, Equation (3.13) implies that $\|f(x) - P^{\ln}_{d'}(x)\|_{[\beta,1]} \leq 4C'_{\ln}\epsilon \coloneqq C_{\ln}\epsilon$ for another universal constant $C_{\ln}$. Notice $P^{\ln}_{d'}$ is an even polynomial with $\deg(P^{\ln}_{d'}) \leq \tilde{C}_{\ln}\beta^{-1}\log\epsilon^{-1}$, Equation (3.12) yields that:

$$\|P^{\ln}_{d'}(x)\|_{[-1,1]} = \|P^{\ln}_{d'}(x)\|_{[0,1]} \leq \|\tfrac{1}{1+\eta}\tilde{P}_{d'}(x)\|_{[0,1]} + \|\tfrac{1}{1+\eta}\tilde{P}_{d'}(x)\|_{[-1,0]} \leq \tfrac{1}{1+\eta}\cdot\tfrac{1+\eta}{2} + \tfrac{1}{1+\eta}\cdot\tfrac{\eta}{2} \leq 1.$$

Here, the last inequality is due to $\eta \leq 1/2$.

Following the coefficient vector of $\tilde{P}_{d'}$ obtained by applying Theorem 3.7 to f, we complete the proof by noting the coefficient vector $\mathbf{c}^{\ln}$ of $P^{\ln}_{d'}$ satisfies all the desired properties. □

3.2 Applying averaged Chebyshev truncation to bitstring indexed encodings

With space-efficient bounded polynomial approximations of piecewise-smooth functions, it suffices to implement averaged Chebyshev truncation on bitstring indexed encodings, as specified in Theorem 3.10. The proof combines Lemma 3.12, Lemma 3.13, and Lemma 3.14.

Theorem 3.10 (Averaged Chebyshev truncation applied to bitstring indexed encodings)**.** *Let A be an Hermitian matrix acting on s qubits, and let U be a $(1, a, \epsilon_1)$-bitstring indexed encoding of A that acts on $s + a$ qubits. For any degree-d averaged Chebyshev truncation $P_{d'}(x) = \hat{c}_0/2 + \sum_{k=1}^{d'}\hat{c}_k T_k(x)$ where $d' = 2d - 1 \leq 2^{O(s(n))}$ and T_k is the k-th Chebyshev polynomial (of the first kind), equipped with an evaluation oracle* Eval *that returns $\tilde{c}_k$ with precision $\varepsilon \coloneqq O(\epsilon_2^2/d')$, we have the following bitstring indexed encoding of $P_{d'}(A)$ depending on whether $P_{d'}(A)$ is an isometry (up to a normalization factor):*[37]

- ***Isometry*** *$P_{d'}(A)$:* *We obtain a $(1, a', 144d'\sqrt{\epsilon_1}\|\hat{\mathbf{c}}\|_1^2 + 36\epsilon_2\|\hat{\mathbf{c}}\|_1)$-bitstring indexed encoding V_{normed} of $P_{d'}(A)$ that acts on $s + a'$ qubits where $a' \coloneqq a + \lceil\log d'\rceil + 3$.*
- ***Non-isometry*** *$P_{d'}(A)$:* *We obtain a $(\|\hat{\mathbf{c}}\|_1, \hat{a}, 4d'\sqrt{\epsilon_1}\|\hat{\mathbf{c}}\|_1^2 + \epsilon_2\|\hat{\mathbf{c}}\|_1)$-bitstring indexed encoding V_{unnorm} of $P_{d'}(A)$ that acts on $s + a'$ qubits where $\hat{a} \coloneqq a + \lceil\log d'\rceil + 1$.*

Let V be the bitstring indexed encoding of $P_{d'}(A)$. The implementation of V requires $O(d^2\eta_V)$ uses of U, $U^\dagger$, $C_\Pi NOT$, $C_{\tilde{\Pi}}NOT$, and multi-controlled single-qubit gates.[38] *The description of the resulting quantum circuit of V can be computed in deterministic time $\tilde{O}(d^2\eta_V\log(d/\epsilon_2))$, space $O(\max\{s(n), \log(d/\epsilon_2^2)\})$, and $O(d^2\eta_V)$ oracle calls to* Eval *with precision ε. Here, $\eta_V = \|\hat{\mathbf{c}}\|_1$ if $V = V_{\text{normed}}$ whereas $\eta_V = 1$ if $V = V_{\text{unnorm}}$.*
Furthermore, our construction straightforwardly extends to any linear (possibly non-Hermitian) operator A by simply replacing $P_{d'}(A)$ with $P^{(\text{SV})}_{d'}(A)$ defined in Definition 2.2.

Remark 3.11 (QSVT implementations of averaged Chebyshev truncation preserve the parity). As shown in Proposition 3.12.1, we can implement the quantum singular value transformation

[37] This condition differs from the one that A is an isometry. Specifically, $P_{d'}(A)$ is an isometry (up to a normalization factor) if A is an isometry, whereas $\text{sgn}^{(\text{SV})}(A)$ is an isometry for any A.

[38] As indicated in Figure 3(c) of [GSLW19] (see also Lemma 19 in [GSLW18]), we replace the single-qubit gates used in Lemma 3.12 with multi-controlled (or "multiply controlled") single-qubit gates.

$T_k^{(\mathrm{SV})}(A)$ *exactly* for any linear operator A that admits a bitstring indexed encoding, because the rotation angles corresponding to the k-th Chebyshev polynomials are either $\pi/2$ or $(1-k)\pi/2$, indicating that $T_k(0) = 0$ for any odd k. We then implement the QSVT corresponding to the averaged Chebyshev truncation polynomial $P_{d'}(x) = \sum_{l=0}^{(d'-1)/2} \hat{c}_{2l+1} T_{2l+1}(x)$, as described in Corollary 3.15, although the actual implementation results in a slightly different polynomial, $\tilde{P}_{d'}(x) = \sum_{l=0}^{(d'-1)/2} \tilde{c}_{2l+1} T_{2l+1}(x)$. However, we still have $\tilde{P}_{d'}(0) = 0 = P_{d'}(0)$, indicating that the implementations in Theorem 3.10 preserve the parity.

We first demonstrate an approach, based on Lemma 3.12 in [MY23], that constructs Chebyshev polynomials of bitstring indexed encodings in a space-efficient manner.

Lemma 3.12 (Chebyshev polynomials applied to bitstring indexed encodings)**.** *Let A be a linear operator acting on s qubits, and let U be a $(1, a, \epsilon)$-bitstring indexed encoding of A that acts on $s+a$ qubits. Then, for the k-th Chebyshev polynomial (of the first kind) $T_k(x)$ of degree $k \leq 2^{O(s)}$, there exists a new $(1, a+1, 4k\sqrt{\epsilon})$-bitstring indexed encoding V of $T_k^{(\mathrm{SV})}(A)$ that acts on $s+a+1$ qubits. This implementation requires k uses of U, $U^\dagger$, $C_\Pi NOT$, $C_{\tilde{\Pi}} NOT$, and k single-qubit gates. Moreover, we can compute the description of the resulting quantum circuit in deterministic time k and space $O(s)$.*
Furthermore, consider $A' \coloneqq \tilde{\Pi} U \Pi$, where $\tilde{\Pi}$ and Π are the corresponding orthogonal projections of the bitstring indexed encoding U. If A and A' satisfy the conditions $\|A - A'\| + \left\|\frac{A+A'}{2}\right\|^2 \leq 1$ and $\left\|\frac{A+A'}{2}\right\|^2 \leq \zeta$, then V is a $\left(1, a+1, \frac{\sqrt{2}}{\sqrt{1-\zeta}} k\epsilon\right)$-bitstring indexed encoding of $T_k^{(\mathrm{SV})}(A)$.

Proof. As specified in Proposition 3.12.1, we first notice that we can derive the sequence of rotation angles corresponding to Chebyshev polynomials $T_k(x)$ by directly factorizing them.

Proposition 3.12.1 (Chebyshev polynomials in quantum signal processing, adapted from Lemma 6 in [GSLW19])**.** *Let $T_k \in \mathbb{R}[x]$ be the k-th Chebyshev polynomial (of the first kind). Consider the corresponding sequence of rotation angles $\Phi \in \mathbb{R}^k$ such that $\phi_1 \coloneqq (1-k)\pi/2$, and $\phi_j \coloneqq \pi/2$ for all $j \in [k] \setminus \{1\}$, then we know that $\prod_{j=1}^{k} \left[\begin{pmatrix} \exp(i\phi_j) & 0 \\ 0 & \exp(-i\phi_j) \end{pmatrix} \begin{pmatrix} x & \sqrt{1-x^2} \\ \sqrt{1-x^2} & -x \end{pmatrix} \right] = \begin{pmatrix} T_k & \cdot \\ \cdot & \cdot \end{pmatrix}$.*

Then we implement the quantum singular value transformation $T_k^{(\mathrm{SV})}(A)$, utilizing an alternating phase modulation (Proposition 3.12.2) with the aforementioned sequence of rotation angles, denoted by V.

Proposition 3.12.2 (QSVT by alternating phase modulation, adapted from Theorem 10 and Figure 3 in [GSLW19])**.** *Suppose $P \in \mathbb{C}[x]$ is a polynomial, and let $\Phi \in \mathbb{R}^n$ be the corresponding sequence of rotation angles. We can construct $P^{(\mathrm{SV})}(\tilde{\Pi} U \Pi) = \begin{cases} \tilde{\Pi} U_\Phi \Pi, & \text{if } n \text{ is odd} \\ \Pi U_\Phi \Pi, & \text{if } n \text{ is even} \end{cases}$ with a single ancillary qubit. Moreover, this implementation in [GSLW19, Figure 3] makes k uses of U, $U^\dagger$, $C_\Pi NOT$, $C_{\tilde{\Pi}} NOT$, and single-qubit gates.*

Owing to the robustness of QSVT (Lemma 22 in [GSLW18], full version of [GSLW19]), we have that $\left\|T_k^{(\mathrm{SV})}(U) - T_k^{(\mathrm{SV})}(U')\right\| \leq 4k\sqrt{\|A - A'\|} = 4k\sqrt{\epsilon}$, where U' is a $(1, a, 0)$-bitstring indexed encoding of A. Moreover, with a tighter bound for A and A', namely $\|A - A'\| + \left\|\frac{A+A'}{2}\right\|^2 \leq 1$, we can deduce that $\|T_k^{(\mathrm{SV})}(U) - T_k^{(\mathrm{SV})}(U')\| \leq k\frac{\sqrt{2}}{\sqrt{1-\|(A+A')/2\|^2}}\|A - A'\| \leq \frac{\sqrt{2}}{\sqrt{1-\zeta}} k\epsilon$ following [GSLW18, Lemma 23], indicating an improved dependence of ϵ. Finally, we can compute the description of the resulting quantum circuits in $O(\log k) = O(s(n))$ space and $O(k)$ times because of the implementation specified in Proposition 3.12.2. □

We then proceed by presenting a linear combination of bitstring indexed encodings, which adapts the LCU technique proposed by Berry, Childs, Cleve, Kothari, and Somma in [BCC+15],

and incorporates a space-efficient state preparation operator. We say that $P_{\mathbf{y}}$ is an ϵ-state preparation operator for $\mathbf{y}$ if $P_{\mathbf{y}}|\bar{0}\rangle := \sum_{i=1}^{m} \sqrt{\hat{y}_i}|i\rangle$ for some $\hat{\mathbf{y}}$ such that $\|\mathbf{y}/\|\mathbf{y}\|_1 - \hat{\mathbf{y}}\|_1 \leq \epsilon$.

Lemma 3.13 (Linear combinations of bitstring indexed encodings, adapted from Lemma 29 in [GSLW19])**.** *Given a matrix $A = \sum_{i=0}^{m-1} y_i A_i$ such that each linear operator A_i $(1 \leq i \leq m)$ acts on s qubits with the corresponding $(\|\mathbf{y}\|_1, a, \epsilon_1)$-bitstring indexed encoding U_i acting on $s+a$ qubits associated with projections $\tilde{\Pi}_i$ and Π_i. Also each y_i $(1 \leq i \leq m)$ can be expressed in $O(s(n))$ bits with an evaluation oracle* Eval *that returns $\hat{y}_i$ with precision $\varepsilon := O(\epsilon_2^2/m)$. Then utilizing an ϵ_2-state preparation operator $P_{\mathbf{y}}$ for $\mathbf{y}$ acting on $O(\log m)$ qubits, and a $(s + a + \lceil \log m \rceil)$-qubit unitary $W = \sum_{i=0}^{m-1} |i\rangle\langle i| \otimes U_i + \left(I - \sum_{i=0}^{m-1} |i\rangle\langle i|\right) \otimes I$, we can implement a $(\|\mathbf{y}\|_1, a+\lceil \log m \rceil, \epsilon_1\|\mathbf{y}\|_1^2+\epsilon_2\|\mathbf{y}\|_1)$-bitstring indexed encoding of A acting on $s+a+\lceil \log m \rceil$ qubits with a single use of W, $P_{\mathbf{y}}$, $P_{\mathbf{y}}^\dagger$. In addition, the (classical) pre-processing can be implemented in deterministic time $\tilde{O}(m^2 \log(m/\epsilon_2))$ and space $O(\log(m/\epsilon_2^2))$, as well as m^2 oracle calls to* Eval *with precision ε.*

Proof. For the ϵ_2-state preparation operator $P_{\mathbf{y}}$ such that $P_{\mathbf{y}}|\bar{0}\rangle = \sum_{i=1}^{m} \sqrt{\hat{y}_i}|i\rangle$, we utilize a scheme introduced by Zalka [Zal98] (also independently rediscovered in [GR02] and [KM01]). We make an additional analysis of the required classical computational complexity, and the proof can be found in Appendix A.2.

Proposition 3.13.1 (Space-efficient state preparation, adapted from [Zal98, KM01, GR02])**.** *Given an l-qubit quantum state $|\psi\rangle := \sum_{i=1}^{m} \sqrt{\hat{y}_i}|i\rangle$, where $l = \lceil \log m \rceil$ and $\hat{y}_i$ are real amplitudes associated with an evaluation oracle* Eval(i, ε) *that returns $\hat{y}_i$ up to accuracy ε we can prepare $|\psi\rangle$ up to accuracy ϵ in deterministic time $\tilde{O}(m^2 \log(m/\epsilon))$ and space $O(\log(m/\epsilon^2))$, together with m^2 evaluation oracle calls with precision $\varepsilon := O(\epsilon^2/m)$.*

Now consider the bitstring indexed encoding $\left(P_{\mathbf{y}}^\dagger \otimes I_s\right) W \left(P_{\mathbf{y}} \otimes I_s\right)$ of A acting on $s+a+\lceil \log m \rceil$ qubits. Let $y_i' := y_i/\|\mathbf{y}\|_1$, then we obtain the implementation error:

$$
\begin{aligned}
&\left\| A - \|\mathbf{y}\|_1 \left(|\bar{0}\rangle\langle\bar{0}| \otimes \tilde{\Pi}\right) \left(P_y^\dagger \otimes I_s\right) W \left(P_y \otimes I_s\right) \left(|\bar{0}\rangle\langle\bar{0}| \otimes \Pi\right) \right\| \\
=&\left\| A - \|\mathbf{y}\|_1 \textstyle\sum_{i=0}^{m-1} \hat{y}_i \tilde{\Pi}_i U_i \Pi_i \right\| \\
\leq&\left\| A - \|\mathbf{y}\|_1 \textstyle\sum_{i=0}^{m-1} y_i' \tilde{\Pi}_i U_i \Pi_i \right\| + \|\mathbf{y}\|_1 \textstyle\sum_{i=0}^{m-1} (y_i' - \hat{y}_i) \|\tilde{\Pi}_i U_i \Pi_i\| \\
\leq&\|\mathbf{y}\|_1 \textstyle\sum_{i=0}^{m-1} y_i' \|A_i - \tilde{\Pi}_i U_i \Pi_i\| + \epsilon_2 \|\mathbf{y}\|_1 \\
\leq&\epsilon_1 \|\mathbf{y}\|_1^2 + \epsilon_2 \|\mathbf{y}\|_1.
\end{aligned}
$$

Here, the third line is due to the triangle inequality, the fourth line owes to Proposition 3.13.1, and the fifth line is because U_i is a $(1, a, \epsilon_1)$-bitstring indexed encoding of A_i for $0 \leq i < m$. □

To make the resulting bitstring indexed encoding from Lemma 3.13 with $\alpha = 1$, we need to perform *a renormalization procedure* to construct a new encoding with the desired α. We achieve this by extending the proof strategy outlined by Gilyen [Gil19, Page 52] for block-encodings to bitstring indexed encodings, which works merely for *isometries* (up to a normalization factor).[39] The renormalization procedure is provided in Lemma 3.14, and the complete proof is available in Appendix A.2. Additionally, a similar result has been established in [MY23, Lemma 7.10].

Lemma 3.14 (Renormalizing bitstring indexed encoding)**.** *Let U be an (α, a, ϵ)-bitstring indexed encoding of A, where $\alpha > 1$ and $0 < \epsilon < 1$, and A is an isometry acting on $s(n)$ qubits. We can implement a quantum circuit V, serving as a normalization of U, such that V is a $(1, a+2, 36\epsilon)$-bitstring indexed encoding of A. This implementation requires $O(\alpha)$ uses of U, $U^\dagger$, $C_\Pi NOT$,*

[39] Renormalizing bitstring indexed encodings of *non-isometries* for space-efficient QSVT seems achievable by mimicking [GSLW19, Theorem 17]. This approach cleverly uses space-efficient QSVT with the sign function (Corollary 3.15), where the corresponding encoding can be re-normalized by carefully using Lemma 3.14. Nevertheless, since this renormalization procedure is not required in this paper, we leave it for future work.

$C_{\tilde{\Pi}}NOT$, *and* $O(\alpha)$ *single-qubit gates. Moreover, the description of the resulting quantum circuit can be computed in deterministic time* $O(\alpha)$ *and space* $O(s)$.

Finally, we combine Lemma 3.12, Lemma 3.13, and Lemma 3.14 to proceed with the proof of Theorem 3.10.

Proof of Theorem 3.10. By using Lemma 3.12, we obtain $(1, a+1, 4k\sqrt{\epsilon_1})$-bitstring indexed encodings V_k corresponding to $T_k(A)$, where $1 \le k \le d' = 2d-1$. The descriptions of quantum circuits $\{V_k\}_{k=0}^{d'}$ can be computed in $O(s(n))$ space and $\sum_{k=0}^{d'} k = O(d^2)$ time. Employing Lemma 3.13, we obtain a $(\|\hat{\mathbf{c}}\|_1, \hat{a}, 4k\sqrt{\epsilon_1}\|\hat{\mathbf{c}}\|_1^2 + \epsilon_2\|\hat{\mathbf{c}}\|_1)$-bitstring indexed encoding V_{unnorm} for $P_{d'}(A) = \hat{c}_0/2 + \sum_{k=1}^{d'} \hat{c}_k T_k(A)$, where $\hat{a} \coloneqq a + \lceil \log d \rceil + 1$. The remaining analysis depends on whether $P_{d'}(A)$ is an isometry (up to a normalization factor):

- **Isometry** $P_{d'}(A)$**:** We can renormalize V_{unnorm} by utilizing Lemma 3.14 and obtain a $(1, a', 144k\sqrt{\epsilon_1}\|\hat{\mathbf{c}}\|_1^2 + 36\epsilon_2\|\hat{\mathbf{c}}\|_1)$-bitstring indexed encoding V_{normed} that acts on $s + a'$ qubits, where $a' \coloneqq \hat{a} + 2 = a + \lceil \log d' \rceil + 3$. A direct calculation shows that the implementation of V_{normed} makes $\sum_{k=1}^{d'} k \cdot O(\|\hat{\mathbf{c}}\|_1) = O(d^2\|\hat{\mathbf{c}}\|_1)$ uses of U, $U^\dagger$, $C_\Pi NOT$, $C_{\tilde{\Pi}}NOT$, and multi-controlled single-qubit gates. The description of the quantum circuit V_{normed} thus can be computed in deterministic time
$$\max\{\tilde{O}\big((d')^2\|\hat{\mathbf{c}}\|_1 \log(d'/\epsilon_2)\big), O((d')^2\|\hat{\mathbf{c}}\|_1)\} = \tilde{O}(d^2\|\hat{\mathbf{c}}\|_1 \log(d/\epsilon_2))$$
and space $O(\max\{s(n), d'/\epsilon_2^2\}) = O(\max\{s(n), d/\epsilon_2^2\})$, as well as $O\big((d')^2\|\hat{\mathbf{c}}\|_1\big) = O(d^2\|\hat{\mathbf{c}}\|_1)$ oracle calls to Eval with precision ε.
- **Non-isometry** $P_{d'}(A)$**:** We simply use the bitstring indexed encoding V_{unnorm} without renormalizing it. Similarly, the implementation of V_{unnorm} makes $O(d^2)$ uses of U, $U^\dagger$, $C_\Pi NOT$, $C_{\tilde{\Pi}}NOT$, and multi-controlled single-qubit gates. Therefore, the description of the quantum circuit V_{unnorm} can be computed in deterministic time $\tilde{O}(d^2 \log(d/\epsilon_2))$ and space $O(\max\{s(n), d/\epsilon_2^2\})$, as well as $O(d^2)$ oracle calls to Eval with precision ε.

Finally, we can extend our construction to any linear operator A by replacing $P_{d'}(A)$ with $P_{d'}^{(\text{SV})}$ as defined in Definition 2.2, taking into account that the Chebyshev polynomial (of the first kind) T_k is either an even or an odd function. □

3.3 Examples: the sign function and the normalized logarithmic function

In this subsection, we provide explicit examples that illustrate the usage of the space-efficient quantum singular value transformation (QSVT) technique. We define two functions:
$$\operatorname{sgn}(x) \coloneqq \begin{cases} 1, & x > 0 \\ -1, & x < 0 \\ 0, & x = 0 \end{cases} \quad \text{and} \quad \ln_\beta(x) \coloneqq \frac{\ln(1/x)}{2\ln(2/\beta)}.$$

In particular, the sign function is a bounded function, and we derive the corresponding bitstring indexed encoding with *deterministic* space-efficient (classical) pre-processing in Corollary 3.6. On the other hand, the logarithmic function is a piecewise-smooth function that is bounded by 1, and we deduce the corresponding bitstring indexed encoding with *randomized* space-efficient (classical) pre-processing in Corollary 3.9.

Corollary 3.15 (Sign polynomial with space-efficient coefficients applied to bitstring indexed encodings)**.** *Let* A *be an Hermitian matrix that acts on* s *qubits, where* $s(n) \ge \Omega(\log(n))$*. Let* U *be a* $(1, a, \epsilon_1)$*-bitstring indexed encoding of* A *that acts on* $s+a$ *qubits. Then, for any* $d' \le 2^{O(s(n))}$ *and* $\epsilon_2 \ge 2^{-O(s(n))}$*, we have an* $\big(1, a + \lceil \log d' \rceil + 3, 144\hat{C}_{\text{sgn}}^2 d\epsilon_1^{1/2} + (36\hat{C}_{\text{sgn}} + 37)\epsilon_2\big)$*-bitstring indexed encoding* V *of* $P_{d'}^{\text{sgn}}(A)$*, where* $P_{d'}^{\text{sgn}}$ *is a space-efficient bounded polynomial approximation of the sign function (corresponding to some degree-*d *averaged Chebyshev truncation) specified*

in Corollary 3.6, and $\hat{C}_{\mathrm{sgn}}$ is a universal constant. This implementation requires $O(d^2)$ uses of U, $U^\dagger$, $C_\Pi NOT$, $C_{\tilde{\Pi}}NOT$, and $O(d^2)$ multi-controlled single-qubit gates. The description of V can be computed in deterministic time $\tilde{O}(\epsilon_2^{-1}d^{9/2})$ and space $O(s(n))$.
Furthermore, our construction directly extends to any non-Hermitian (but linear) matrix A by simply replacing $P_d^{\mathrm{sgn}}(A)$ with $P_{\mathrm{sgn},d}^{(\mathrm{SV})}(A)$ defined in the same way as Definition 2.2.

Proof. Following Corollary 3.6, we have $P_{d'}^{\mathrm{sgn}}(x) = \hat{c}_0/2 + \sum_{k=1}^{d'} \hat{c}_k T_k(x)$, where $d' = 2d-1$ and $d' = O(\delta^{-1}\log\epsilon^{-1})$. The approximation error is given by:

$$\forall x \in [-1,1] \setminus [-\delta,\delta],\ |\mathrm{sgn}(x) - P_{d'}^{\mathrm{sgn}}(x)| \le C_{\mathrm{sgn}}\epsilon \coloneqq \epsilon_2. \tag{3.14}$$

To implement Eval with precision $\varepsilon = O(\epsilon_2^2/d')$, we can compute the corresponding entry $\hat{c}_k$ of the coefficient vector, which requires deterministic time $\tilde{O}\big(\varepsilon^{-1/2}(d')^2\big) = \tilde{O}(\epsilon_2^{-1}d^{5/2})$ and space $O\big(\log(\varepsilon^{-3/2}(d')^3)\big) = O(\log(\epsilon_2^{-3}d^{9/2}))$.

Note that $P_{d'}^{\mathrm{sgn}}(A)$ is a non-isometry (up to a normalization factor) and $\|\hat{\mathbf{c}}\|_1 \le \hat{C}_{\mathrm{sgn}}$. Using Theorem 3.10, we have a $(\hat{C}_{\mathrm{sgn}}, a_{\mathrm{un}}, 4d'\hat{C}_{\mathrm{sgn}}^2\epsilon_1^{1/2} + \hat{C}_{\mathrm{sgn}}\epsilon_2)$-bitstring indexed encoding V_{un}, with projections $\tilde{\Pi}$ and Π, that acts on $s + a_{\mathrm{un}}$ qubits and $a_{\mathrm{un}} \coloneqq a + \lceil \log d' \rceil + 1$.

Renormalizing the encoding of $P_{d'}^{\mathrm{sgn}}(A)$. Notably, the renormalization procedure (Lemma 3.14) is *still applicable* when $P_{d'}^{\mathrm{sgn}}(A)$ is restricted to appropriately chosen subspaces Γ_L and Γ_R. Let $A = \sum_{i=1}^{\mathrm{rank}(A)} \sigma_i \mathbf{u}_i \mathbf{v}_i^\dagger$ be the singular value decomposition of A. We define $A_{>\delta} \coloneqq \sum_{i:\sigma_i>\delta} \sigma_i \mathbf{u}_i \mathbf{v}_i^\dagger$, as well as subspaces $\Gamma_L \coloneqq \mathrm{span}\{\mathbf{u}_i | \sigma_i > \delta\}$ and $\Gamma_R \coloneqq \mathrm{span}\{\mathbf{v}_i^\dagger | \sigma_i > \delta\}$. Consequently, we obtain the following for the bitstring indexed encoding V_{un} with projections $\tilde{\Pi}$ and Π:

$$\begin{aligned}
&\|\mathrm{sgn}^{(\mathrm{SV})}(A_{>\delta}) - \hat{C}_{\mathrm{sgn}}\tilde{\Pi}|_{\Gamma_L} V_{\mathrm{nu}} \Pi|_{\Gamma_R}\| \\
\le &\|\mathrm{sgn}^{(\mathrm{SV})}(A_{>\delta}) - P_{d'}^{\mathrm{sgn}}(A_{>\delta})\| + \|P_{d'}^{\mathrm{sgn}}(A_{>\delta}) - \hat{C}_{\mathrm{sgn}}\tilde{\Pi}|_{\Gamma_L} V_{\mathrm{nu}} \Pi|_{\Gamma_R}\| \\
\le &\epsilon_2 + \|P_{d'}^{\mathrm{sgn}}(A_{>\delta}) - \hat{C}_{\mathrm{sgn}}\tilde{\Pi} V_{\mathrm{nu}} \Pi\| \\
\le &\epsilon_2 + 4d'C_{\mathrm{sgn}}^2\epsilon_1^{1/2} + \hat{C}_{\mathrm{sgn}}\epsilon_2.
\end{aligned} \tag{3.15}$$

Here, the second line owes to the triangle inequality, the third line is due to Equation (3.14), and the last line is because V_{nu} is a $(\hat{C}_{\mathrm{sgn}}, a_{\mathrm{un}}, 4d'C_{\mathrm{sgn}}^2\epsilon_1^{1/2} + \hat{C}_{\mathrm{sgn}}\epsilon_2)$-bitstring indexed encoding of $P_{d'}^{\mathrm{sgn}}(A_{>\delta})$. Note that $\mathrm{sgn}^{(\mathrm{SV})}(A_{>\delta})$ is an isometry, and Equation (3.15) implies that V_{nu} is a projected unitary encoding of $\mathrm{sgn}^{(\mathrm{SV})}(A_{>\delta})$. By applying Lemma 3.14 to V_{nu} with projections $\tilde{\Pi}|_{\Gamma_L}$ and $\Pi|_{\Gamma_R}$, we can obtain a $(1, a_{\mathrm{un}} + 2, 144d'C_{\mathrm{sgn}}^2\epsilon_1^{1/2} + 36(\hat{C}_{\mathrm{sgn}} + 1)\epsilon_2)$-projected unitary encoding of $\mathrm{sgn}^{(\mathrm{SV})}(A_{>\delta})$, denoted as V. Consequently, we can derive that:

$$\begin{aligned}
&\|P_{d'}^{\mathrm{sgn}}(A_{>\delta}) - \tilde{\Pi}|_{\Gamma_L} V_{\mathrm{nu}} \Pi|_{\Gamma_R}\| \\
\le &\|P_{d'}^{\mathrm{sgn}}(A_{>\delta}) - \mathrm{sgn}^{(\mathrm{SV})}(A_{>\delta})\| + \|\mathrm{sgn}^{(\mathrm{SV})}(A_{>\delta}) - \tilde{\Pi}|_{\Gamma_L} V_{\mathrm{nu}} \Pi|_{\Gamma_R}\| \\
\le &\epsilon_2 + 144d'C_{\mathrm{sgn}}^2\epsilon_1^{1/2} + 36(\hat{C}_{\mathrm{sgn}} + 1)\epsilon_2.
\end{aligned} \tag{3.16}$$

Here, the second line follows from the triangle inequality, and the third line additionally owes to Equation (3.14). Note that Lemma 3.14 essentially applies a Chebyshev polynomial to V_{nu} and preserves the projections $\tilde{\Pi}|_{\Gamma_L}$ and $\Pi|_{\Gamma_R}$, then we have $\|\tilde{\Pi} V \Pi\| \le 1$. Therefore, following Equation (3.16), we conclude that $P_{d'}^{\mathrm{sgn}}(A)$ has a $\big(1, a', 144d'C_{\mathrm{sgn}}^2\epsilon_1^{1/2} + (36\hat{C}_{\mathrm{sgn}} + 37)\epsilon_2\big)$-bitstring indexed encoding V that acts on $s + a'$ qubits, where $a' \coloneqq a + \lceil \log d \rceil + 3$.[40]

Lastly, we can complete the remained analysis similar to Theorem 3.10 with an isometry $P_{d'}(A)$. Since $\|\hat{\mathbf{c}}\|_1 \le \hat{C}_{\mathrm{sgn}} \le O(1)$, the quantum circuit of V makes $O(d^2)$ uses of U, $U^\dagger$, $\mathrm{C}_\Pi\mathrm{NOT}$, and $\mathrm{C}_{\tilde{\Pi}}\mathrm{NOT}$ as well as $O(d^2)$ multi-controlled single-qubit gates. We note that $d \le 2^{O(s(n))}$ and $\epsilon_2 \ge 2^{-O(s(n))}$. Moreover, we can compute the description of V in $O(s(n))$

[40] We are somewhat abusing notations – strictly speaking, V corresponds $\tilde{P}_{d'}^{\mathrm{sgn}}(A)$, where $\tilde{P}_{d'}^{\mathrm{sgn}}$ is another polynomial satisfying all requirements in Corollary 3.6 but does not necessarily exactly coincide with $P_{d'}^{\mathrm{sgn}}$.

space since each oracle call to Eval with precision ε can be computed in $O(\log(\epsilon_2^{-3}d^{9/2}))$ space. Additionally, the time complexity for computing the description of V is

$$\max\{\tilde{O}(d^2\log(d/\epsilon_2)), O(d^2)\cdot\tilde{O}(\epsilon_2^{-1}d^{5/2})\} = \tilde{O}(\epsilon_2^{-1}d^{9/2}). \qquad \square$$

Corollary 3.16 (Log polynomial with space-efficient coefficients applied to bitstring indexed encodings)**.** *Let A be an Hermitian matrix that acts on s qubits, where $s(n) \geq \Omega(\log(n))$. Let U be a $(1, a, \epsilon_1)$-bitstring indexed encoding of A that acts on $s+a$ qubits. Then, for any $d' = 2d-1 \leq 2^{O(s(n))}$, $\epsilon_2 \geq 2^{-O(s(n))}$, and $\beta \geq 2^{-O(s(n))}$, we have a $(\hat{C}_{\ln}, a+\lceil\log d\rceil+1, 4d\hat{C}_{\ln}^2\epsilon_1^{1/2} + \hat{C}_{\ln}\epsilon_2)$-bitstring indexed encoding V of $P_{d'}^{\ln}(A)$, where $P_{d'}^{\ln}$ is a space-efficient bounded polynomial approximation of the normalized log function (corresponding to some degree-d averaged Chebyshev truncation) specified in Corollary 3.9, and $\hat{C}_{\ln}$ is a universal constant. This implementation requires $O(d^2)$ uses of U, $U^\dagger$, $C_\Pi NOT$, $C_{\tilde{\Pi}}NOT$, and multi-controlled single-qubit gates. Moreover, we can compute the description of the resulting quantum circuit in bounded-error randomized time $\tilde{O}(\max\{\beta^{-5}\epsilon_2^{-4}d^4, \epsilon_2^{-1}d^{9/2}\})$ and space $O(s(n))$.*

Proof. Following Corollary 3.9, we have $P_{d'}^{\ln}(x) = c_0^{\ln}/2 + \sum_{k=1}^{d'} c_k^{\ln}T_k(x)$, where $P_{d'}^{\ln}$ corresponds to some degree-d averaged Chebyshev truncation and $d' = 2d-1 \leq O(\delta^{-1}\log\epsilon^{-1})$. For any $\ln_\beta(x)$, we have $|\ln_\beta(x) - P_{d'}^{\ln}(x)| \leq C_{\ln}\epsilon \coloneqq \epsilon_2$ for all $x \in [\beta, 1]$. To implement Eval with precision $\varepsilon = O(\epsilon_2^2/d)$, we can compute the corresponding entry $c_k^{\ln}$ of the coefficient vector by a bounded-error randomized algorithm. This requires $O(\log(\beta^{-4}\varepsilon^{-3/2}d^3)) = O(\log(\beta^{-4}\epsilon_2^{-3}d^{9/2}))$ space and $\tilde{O}(\max\{\beta^{-5}\varepsilon^{-2}, \varepsilon^{-1/2}d^2\}) = \tilde{O}(\max\{\beta^{-5}\epsilon_2^{-4}d^2, \epsilon_2^{-1}d^{5/2}\})$ time. Applying Theorem 3.10 with $\|\mathbf{c}^{\ln}\|_1 \leq \hat{C}_{\ln}$, we conclude that $P_{d'}^{\ln}$ has a $(\hat{C}_{\ln}, \hat{a}, 4d\hat{C}_{\ln}^2\epsilon_1^{1/2} + \hat{C}_{\ln}\epsilon_2)$-bitstring indexed encoding V that acts on $s+\hat{a}$ qubits, where $\hat{a} \coloneqq a + \lceil\log d'\rceil + 1$.

Furthermore, the quantum circuit of V makes $O((d')^2) = O(d^2)$ uses of U, $U^\dagger$, $\mathrm{C}_\Pi\mathrm{NOT}$, $\mathrm{C}_{\tilde{\Pi}}\mathrm{NOT}$, and multi-controlled single-qubit gates. We note that $d' = 2d-1 \leq 2^{O(s(n))}$, $\epsilon_2 \geq 2^{-O(s(n))}$, and $\beta \geq 2^{-O(s(n))}$. Additionally, we can compute the description of V in $O(s(n))$ space since each oracle call to Eval with precision ε can be computed in $O(\log(\beta^{-4}\epsilon_2^{-3}d^{9/2}))$ space. The time complexity for computing the description of V is given by:

$$\max\{\tilde{O}(d^2\log(d/\epsilon_2)), O(d^2)\tilde{O}(\max\{\beta^{-5}\epsilon_2^{-4}d^2, \epsilon_2^{-1}d^{5/2}\}) = \tilde{O}(\max\{\beta^{-5}\epsilon_2^{-4}d^4, \epsilon_2^{-1}d^{9/2}\}). \quad (3.17)$$

Finally, to guarantee that the probability that all $O((d')^2) = O(d^2)$ oracle calls to Eval succeed is at least $2/3$, we use a $(4\ln d')$-time sequential repetition of Eval for each oracle call. Together with the Chernoff-Hoeffding bound and the union bound, the resulting randomized algorithm succeeds with probability at least $1 - (d')^2 \cdot 2\exp(-4\ln d') \geq 2/3$. We further note that the time complexity specified in Equation (3.17) only increases by a $4\ln d'$ factor. $\square$

3.4 Application: space-efficient error reduction for unitary quantum computations

We provide a unified space-efficient error reduction for unitary quantum computations. In particular, one-sided error scenarios (e.g., $\mathsf{RQ_UL}$ and $\mathsf{coRQ_UL}$) have been proven in [Wat01], and the two-sided error scenario (e.g., $\mathsf{BQ_UL}$) has been demonstrated in [FKL+16].

Theorem 3.17 (Space-efficient error reduction for unitary quantum computations)**.** *Let $s(n)$ be a space-constructible function, and let $a(n)$, $b(n)$, and $l(n)$ be deterministic $O(s(n))$ space computable functions such that $a(n) - b(n) \geq 2^{-O(s(n))}$, we know that for any $l(n) \leq O(s(n))$, there is $d \coloneqq l(n)/\max\{\sqrt{a}-\sqrt{b}, \sqrt{1-b}-\sqrt{1-a}\}$ such that*

$$\mathsf{BQ_USPACE}[s(n), a(n), b(n)] \subseteq \mathsf{BQ_USPACE}\big[s(n) + \lceil\log d\rceil + 1, 1 - 2^{-l(n)}, 2^{-l(n)}\big].$$

Furthermore, for one-sided error scenarios, we have that for any $l(n) \leq 2^{O(s(n))}$:

$$\mathsf{RQ_USPACE}[s(n), a(n)] \subseteq \mathsf{RQ_USPACE}\big[s(n) + \lceil\log d_0\rceil + 1, 1 - 2^{-l(n)}\big] \text{ where } d_0 \coloneqq \frac{l(n)}{\max\{\sqrt{a}, 1-\sqrt{1-a}\}},$$

$$\mathsf{coRQ_USPACE}[s(n), b(n)] \subseteq \mathsf{coRQ_USPACE}\big[s(n) + \lceil\log d_1\rceil + 1, 2^{-l(n)}\big] \text{ where } d_1 \coloneqq \frac{l(n)}{\max\{1-\sqrt{b}, \sqrt{1-b}\}}.$$

By choosing $s(n) = \Theta(\log(n))$, we derive error reduction for logarithmic-space quantum computation in a unified approach:

Corollary 3.18 (Error reduction for $\mathsf{BQ_UL}$, $\mathsf{RQ_UL}$, and $\mathsf{coRQ_UL}$)**.** *For deterministic logspace computable functions $a(n)$, $b(n)$, and $l(n)$ satisfying $a(n) - b(n) \geq 1/\operatorname{poly}(n)$ and $l(n) \leq O(\log n)$, we have the following inclusions:*

$$\mathsf{BQ_UL}[a(n), b(n)] \subseteq \mathsf{BQ_UL}[1 - 2^{-l(n)}, 2^{-l(n)}],$$
$$\mathsf{RQ_UL}[a(n)] \subseteq \mathsf{RQ_UL}[1 - 2^{-l(n)}],$$
$$\mathsf{coRQ_UL}[b(n)] \subseteq \mathsf{coRQ_UL}[2^{-l(n)}].$$

The construction specified in Theorem 3.17 crucially relies on Lemma 3.19. And the proof of Lemma 3.19 directly follows from Theorem 20 in [GSLW19], which is deferred to Appendix A.3.

Lemma 3.19 (Space-efficient singular value discrimination)**.** *Let $0 \leq \alpha < \beta \leq 1$ and U be a $(1, 0, 0)$-bitstring indexed encoding of $A \coloneqq \tilde{\Pi} U \Pi$, where U acts on s qubits and $s(n) \geq \Omega(\log n)$. Consider an unknown quantum state $|\psi\rangle$, with the promise that it is a right singular vector of A with a singular value either above α or below β. There is a degree-d' polynomial P, where $d' = O(\delta^{-1} \log \varepsilon^{-1})$ and $\delta \coloneqq \max\{\beta - \alpha, \sqrt{1-\alpha^2} - \sqrt{1-\beta^2}\}/2$, such that there is a singular value discriminator U_P that distinguishes the two cases with error probability at most ε. Moreover, the discriminator U_P achieves one-sided error when $\alpha = 0$ or $\beta = 1$.*
Furthermore, the quantum circuit implementation of U_P requires $O(d^2)$ uses of U, $U^\dagger$, $C_\Pi NOT$, $C_{\tilde{\Pi}} NOT$, and multi-controlled single-qubit gates. In addition, the description of the implementation can be computed in deterministic time $\tilde{O}(\varepsilon^{-1}\delta^{-9/2})$ and space $O(s(n))$.

Finally, we provide the proof of Theorem 3.17, which closely relates to Theorem 38 in [GSLW18] (the full version of [GSLW19]).

Proof of Theorem 3.17. It suffices to amplify the promise gap by QSVT. Note that the probability that a $\mathsf{BQ_USPACE}[s(n)]$ circuit C_x accepts is $\Pr[C_x \text{ accepts}] = \| |1\rangle\langle 1|_{\text{out}} C_x |0^{k+m}\rangle \|_2^2 \geq a$ for *yes* instances, whereas $\Pr[C_x \text{ accepts }] = \| |1\rangle\langle 1|_{\text{out}} C_x |0^{k+m}\rangle \|_2^2 \leq b$ for *no* instances. Then consider a $(1, 0, 0)$-bitstring indexed encoding $M_x \coloneqq \Pi_{\text{out}} C_x \Pi_{\text{in}}$ such that $\|M_x\| \geq \sqrt{a}$ for *yes* instances while $\|M_x\| \leq \sqrt{b}$ for *no* instances, where $\Pi_{\text{in}} \coloneqq |0\rangle\langle 0|^{\otimes k+m}$ and $\Pi_{\text{out}} \coloneqq |1\rangle\langle 1|_{\text{out}} \otimes I_{m+k-1}$. Since $\|M_x\| = \sigma_{\max}(M_x)$ where $\sigma_{\max}(M_x)$ is the largest singular value of M_x, it suffices to distinguish the largest singular value of M_x are either above $\sqrt{a}$ or below $\sqrt{b}$. By setting $\alpha \coloneqq \sqrt{a}$, $\beta \coloneqq \sqrt{b}$ and $\varepsilon \coloneqq 2^{-l(n)}$, this task is a direct corollary of Lemma 3.19. □

4 Space-bounded quantum state testing

We begin by defining the problem of quantum state testing in a space-bounded manner:

Definition 4.1 (Space-bounded Quantum State Testing)**.** Given polynomial-size quantum circuits (devices) Q_0 and Q_1 that act on $O(\log n)$ qubits and have a succinct description (the "source code" of devices), with $r(n)$ specified output qubits, where $r(n)$ is a deterministic logspace computable function such that $0 < r(n) \leq O(\log(n))$. For clarity, n represents the (total) number of gates in Q_0 and Q_1.[41] Let ρ_i denote the mixed state obtained by running Q_i on the all-zero state $|\bar{0}\rangle$ and tracing out the non-output qubits.
We define a *space-bounded quantum state testing* problem, with respect to a specified distance-like measure, to decide whether ρ_0 and ρ_1 are easily distinguished or almost indistinguishable. Likewise, we also define a *space-bounded quantum state certification* problem to decide whether ρ_0 and ρ_1 are easily distinguished or *exactly* indistinguishable.

[41] It is noteworthy that in the time-bounded scenario, the input length of circuits, the size of circuit descriptions, and the number of gates in circuits are polynomially equivalent. However, in the space-bounded scenario, only the last two quantities are polynomially equivalent, and their dependence on the first quantity may be exponential.

We remark that space-bounded quantum state certification, defined in Definition 4.1, represents a "white-box" (log)space-bounded counterpart of quantum state certification [BOW19].

Remark 4.2 (Lifting to exponential-size instances by succinct encodings). For $s(n)$ space-uniform quantum circuits Q_0 and Q_1 acting on $O(s(n))$ qubits, if these circuits admit a succinct encoding,[42] namely there is a deterministic $O(s(n))$-space Turing machine with time complexity $\mathrm{poly}(s(n))$ can uniformly generate the corresponding gate sequences, then Definition 4.1 can be extended to any $s(n)$ satisfying $\Omega(\log n) \leq s(n) \leq \mathrm{poly}(n)$.[43]

Next, we define space-bounded quantum state testing problems, based on Definition 4.1, with respect to four commonplace distance-like measures.

Definition 4.3 (Space-bounded Quantum State Distinguishability Problem, $\textsc{GapQSD}_{\log}$)**.** Consider deterministic logspace computable functions $\alpha(n)$ and $\beta(n)$, satisfying $0 \leq \beta(n) < \alpha(n) \leq 1$ and $\alpha(n) - \beta(n) \geq 1/\mathrm{poly}(n)$. Then the promise is that one of the following holds:

- *Yes* instances: A pair of quantum circuits (Q_0, Q_1) such that $\mathrm{td}(\rho_0, \rho_1) \geq \alpha(n)$;
- *No* instances: A pair of quantum circuits (Q_0, Q_1) such that $\mathrm{td}(\rho_0, \rho_1) \leq \beta(n)$.

Moreover, we also define the certification counterpart of $\textsc{GapQSD}_{\log}$, referred to as $\textsc{CertQSD}_{\log}$, given that $\beta = 0$. Specifically, $\textsc{CertQSD}_{\log}[\alpha(n)] \coloneqq \textsc{GapQSD}_{\log}[\alpha(n), 0]$.

Likewise, we can define $\textsc{GapQJS}_{\log}$ and $\textsc{GapQHS}_{\log}$, also the certification version $\overline{\textsc{CertQHS}}_{\log}$, in a similar manner to Definition 4.3 by replacing the distance-like measure accordingly:

- $\textsc{GapQJS}_{\log}[\alpha(n), \beta(n)]$: Decide whether $\mathrm{QJS}_2(\rho_0, \rho_1) \geq \alpha(n)$ or $\mathrm{QJS}_2(\rho_0, \rho_1) \leq \beta(n)$;
- $\textsc{GapQHS}_{\log}[\alpha(n), \beta(n)]$: Decide whether $\mathrm{HS}^2(\rho_0, \rho_1) \geq \alpha(n)$ or $\mathrm{HS}^2(\rho_0, \rho_1) \leq \beta(n)$.

Furthermore, we use the notation $\overline{\textsc{CertQSD}}_{\log}$ to indicate the *complement* of $\textsc{CertQSD}_{\log}$ with respect to the chosen parameter $\alpha(n)$, and so does $\overline{\textsc{CertQHS}}_{\log}$.

Definition 4.4 (Space-bounded Quantum Entropy Difference Problem, $\textsc{GapQED}_{\log}$)**.** Consider a deterministic logspace computable function $g : \mathbb{N} \to \mathbb{R}^+$, satisfying $g(n) \geq 1/\mathrm{poly}(n)$. Then the promise is that one of the following cases holds:

- *Yes* instance: A pair of quantum circuits (Q_0, Q_1) such that $\mathrm{S}(\rho_0) - \mathrm{S}(\rho_1) \geq g(n)$;
- *No* instance: A pair of quantum circuits (Q_0, Q_1) such that $\mathrm{S}(\rho_1) - \mathrm{S}(\rho_0) \geq g(n)$.

Novel complete characterizations for space-bounded quantum computation. We now present the main theorems in this section and the paper. Theorem 4.5 establishes the first family of natural $\mathsf{coRQ_UL}$-complete problems. By relaxing the error requirement from one-sided to two-sided, Theorem 4.6 identifies a new family of natural BQL-complete problems on space-bounded quantum state testing. It is noteworthy that Theorem 4.5 and Theorem 4.6 also have natural exponential-size up-scaling counterparts.[44]

Theorem 4.5. *The computational hardness of the following (log)space-bounded quantum state certification problems, for any deterministic logspace computable $\alpha(n) \geq 1/\mathrm{poly}(n)$, is as follows:*

(1) $\overline{\textsc{CertQSD}}_{\log}[\alpha(n)]$ *is* $\mathsf{coRQ_UL}$*-complete;*

(2) $\overline{\textsc{CertQHS}}_{\log}[\alpha(n)]$ *is* $\mathsf{coRQ_UL}$*-complete.*

[42] For instance, the construction in [FL18, Remark 11], or [PY86, BLT92] in general.

[43] It is noteworthy that Definition 4.1 (mostly) coincides with the case of $s(n) = \Theta(\log n)$ and directly takes the corresponding gate sequence of Q_0 and Q_1 as an input.

[44] We can naturally extend Theorem 4.5 and Theorem 4.6 to their exponential-size up-scaling counterparts with $2^{-O(s(n))}$-precision, employing the extended version of Definition 4.1 outlined in Remark 4.2, thus achieving the complete characterizations for $\mathsf{coRQ_USPACE}[s(n)]$ and $\mathsf{BQPSPACE}[s(n)]$, respectively.

Theorem 4.6. *The computational hardness of the following (log)space-bounded quantum state testing problems, where $\alpha(n) - \beta(n) \geq 1/\operatorname{poly}(n)$ or $g(n) \geq 1/\operatorname{poly}(n)$ as well as $\alpha(n)$, $\beta(n)$, $g(n)$ can be computed in deterministic logspace, is as follows:*

(1) $\textsc{GapQSD}_{\log}[\alpha(n), \beta(n)]$ *is* BQL*-complete;*

(2) $\textsc{GapQED}_{\log}[g(n)]$ *is* BQL*-complete;*

(3) $\textsc{GapQJS}_{\log}[\alpha(n), \beta(n)]$ *is* BQL*-complete;*

(4) $\textsc{GapQHS}_{\log}[\alpha(n), \beta(n)]$ *is* BQL*-complete.*

To establish Theorem 4.5 and Theorem 4.6, we introduce a general framework for space-bounded quantum state testing in Section 4.1. Interestingly, BQL and $\mathsf{coRQ_UL}$ containments for these problems with respect to different distance-like measures, utilizing our general framework, correspond to approximate implementations of distinct two-outcome measurements. The main technical challenges then mostly involve *parameter trade-offs* when using our space-efficient QSVT to construct these approximate measurement implementations. We summarize this correspondence in Table 2 and the associated subsection, which provides the detailed proof.

Distance-like measure	State testing	State certification	Two-outcome measurement Π_b for $b \in \{0,1\}$
Trace distance	$\textsc{GapQSD}_{\log}$ Section 4.2	$\overline{\textsc{CertQSD}}_{\log}$ Section 4.4	$\frac{I}{2} + \frac{(-1)^b}{2}\operatorname{sgn}^{(\mathrm{SV})}\left(\frac{\rho_0-\rho_1}{2}\right)$ for ρ_0 and ρ_1
Quantum entropy difference Quantum JS divergence	$\textsc{GapQED}_{\log}$ $\textsc{GapQJS}_{\log}$ Section 4.3	None	$\frac{I}{2} - \frac{(-1)^b}{2} \cdot \frac{\ln(\rho_i)}{2\ln(2/\beta)}$ for ρ_i where $i \in \{0,1\}$ and $\lambda(\rho_i) \in [-\beta, \beta]$
Hilbert-Schmidt distance	$\textsc{GapQHS}_{\log}$ Appendix B	$\overline{\textsc{CertQHS}}_{\log}$ Section 4.4	$\frac{I}{2} + \frac{(-1)^b}{2}\mathrm{SWAP}$ for $\rho_0 \otimes \rho_1$

Table 2: The correspondence between the distance-like measures and measurements.

Notably, the measurement corresponding to the trace distance in Table 2 can be viewed as an *algorithmic Holevo-Helstrom measurement*, as discussed further in Section 5. Lastly, the corresponding hardness proof for all these problems is provided in Section 4.5.

4.1 Space-bounded quantum state testing: a general framework

In this subsection, we introduce a general framework for quantum state testing that utilizes a quantum tester $\mathcal{T}$. Specifically, the space-efficient tester $\mathcal{T}$ succeeds (outputting the value "0") with probability x, which is linearly dependent on some quantity closely related to the distance-like measure of interest. Consequently, we can obtain an additive-error estimation $\widetilde{x}$ of x with high probability through sequential repetition (Lemma 2.16).

To construct $\mathcal{T}$, we combine the one-bit precision phase estimation [Kit95], commonly known as the Hadamard test [AJL09], for block-encodings (Lemma 4.9), with our space-efficient quantum singular value transformation (QSVT) technique, which we describe in Section 3.

Constructing a space-efficient quantum tester. We now provide a formal definition and the detailed construction of the quantum tester $\mathcal{T}$. The quantum circuit shown in Figure 2 defines the quantum tester $\mathcal{T}(Q, U_A, P_{d'}, \epsilon)$ using the following parameters with $s(n) = \Theta(\log n)$:

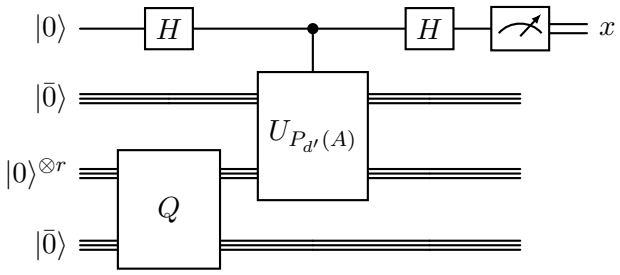

Figure 2: Quantum tester $\mathcal{T}(Q, U_A, P_{d'}, \epsilon)$: the circuit implementation.

- A $s(n)$-qubit quantum circuit Q prepares the purification of an $r(n)$-qubit quantum state ρ where ρ is the quantum state of interest;
- U_A is a $(1, s(n) - r(n), 0)$-block-encoding of an $r(n)$-qubit Hermitian operator A where A relates to the quantum states of interest and $r(n) \leq s(n)$;
- $P_{d'}$ is the space-efficiently computable degree-d' polynomial defined by Equation (2.3) obtained from some degree-d averaged Chebyshev truncation $P_{d'}$ with $d' = 2d - 1$, where $P_{d'}(x) = \hat{c}_0/2 + \sum_{k=1}^{d'} \hat{c}_k T_k(x) \in \mathbb{R}[x]$ and T_k is the k-th Chebyshev polynomial, with $d' \leq 2^{O(s(n))}$, such that the coefficients $\hat{\mathbf{c}} := (\hat{c}_0, \cdots, \hat{c}_{d'})$ can be computed in bounded-error randomized space $O(s(n))$;
- ϵ is the precision parameter used in the estimation of x, with $\epsilon \geq 2^{-O(s(n))}$.

Leveraging our space-efficient QSVT, we assume that there is an $(\alpha, *, *)$-block-encoding of $P_{d'}(A)$, which is an approximate implementation of $U_{P_{d'}(A)}$ in Figure 2. Now, we can define the corresponding estimation procedure, $\hat{\mathcal{T}}(Q, U_A, P_{d'}, \epsilon, \epsilon_H, \delta)$, namely a quantum algorithm that computes an additive-error estimate $\alpha\widetilde{x}$ of $\mathrm{Re}(\mathrm{Tr}(P_{d'}(A)\rho))$ from the tester $\mathcal{T}(Q, U_A, P_{d'}, \epsilon)$. Technically speaking, $\hat{\mathcal{T}}$ outputs $\widetilde{x}$ such that $|\alpha\widetilde{x} - \mathrm{Re}(\mathrm{Tr}(P_{d'}(A)\rho))| \leq \|\hat{\mathbf{c}}\|_1\epsilon + \alpha\epsilon_H$ with probability at least $1 - \delta$. Now we will demonstrate that both the tester $\mathcal{T}$ and the corresponding estimation procedure $\hat{\mathcal{T}}$ are space-efficient:

Lemma 4.7 (Quantum tester $\mathcal{T}$ and estimation procedure $\hat{\mathcal{T}}$ are space-efficient)**.** *Assume that there is an $(\alpha, *, *)$-block-encoding of $P_{d'}(A)$ that approximately implements $U_{P_{d'}}(A)$, where α is either $\|\hat{\mathbf{c}}\|_1$ or 1 based on conditions of $P_{d'}$ and A. The quantum tester $\mathcal{T}(Q, U_A, P_{d'}, \epsilon)$, as specified in Figure 2, accepts (outputting the value "0") with probability $\frac{1}{2}\big(1 + \frac{1}{\alpha}\mathrm{Re}(\mathrm{Tr}(P_{d'}(A)\rho))\big) \pm \frac{1}{2\alpha}\|\hat{\mathbf{c}}\|_1\epsilon$. In addition, $\hat{\mathcal{T}}(Q, U_A, P_{d'}, \epsilon, \epsilon_H, \delta)$ outputs $\widetilde{x}$ such that, with probability at least $1 - \delta$, it holds that $\|\alpha\widetilde{x} - \mathrm{Re}(\mathrm{Tr}(P_{d'}(A)\rho))\| \leq \|\hat{\mathbf{c}}\|_1\epsilon + \alpha\epsilon_H$.*
Moreover, we can compute the quantum circuit description of $\mathcal{T}$ in deterministic space $O(s + \log(1/\epsilon))$ given the coefficient vector $\hat{\mathbf{c}}$ of $P_{d'}$. Furthermore, we can implement the corresponding estimation procedure $\hat{\mathcal{T}}$ in bounded-error quantum space $O(s+\log(1/\epsilon)+\log(1/\epsilon_H)+\log\log(1/\delta))$.

We first provide two useful lemmas for implementing our quantum tester $\mathcal{T}$. It is noteworthy that Lemma 4.8 originates from [LC19], as well as Lemma 4.9 is a specific version of one-bit precision phase estimation (or the Hadamard test) [Kit95, AJL09].

Lemma 4.8 (Purified density matrix, [GSLW19, Lemma 25])**.** *Suppose ρ is an s-qubit density operator and U is an $(a + s)$-qubit unitary operator such that $U|0\rangle^{\otimes a}|0\rangle^{\otimes s} = |\rho\rangle$ and $\rho = \mathrm{Tr}_a(|\rho\rangle\langle\rho|)$. Then, we can construct an $O(a+s)$-qubit quantum circuit $\widetilde{U}$ that is an $(O(a+s), 0)$-block-encoding of ρ, using $O(1)$ queries to U and $O(a + s)$ one- and two-qubit quantum gates.*

Lemma 4.9 (Hadamard test for block-encodings, adapted from [GP22, Lemma 9])**.** *Suppose U is an $(a + s)$-qubit unitary operator that is a block-encoding of $s(n)$-qubit operator A. We can implement an $O(a + s)$-qubit quantum circuit that, on input $s(n)$-qubit quantum state ρ, outputs 0 with probability $\frac{1+\mathrm{Re}(\mathrm{Tr}(A\rho))}{2}$.*

Finally, we proceed with the actual proof of Lemma 4.7.

Proof of Lemma 4.7. Note that U_A is a $(1, a, 0)$-block-encoding of A, where $a = s - r$.

We first consider the case where $\alpha = \|\hat{\mathbf{c}}\|_1$, which holds for any $P_{d'}$ and A. By Theorem 3.10 with $\epsilon_1 := 0$ and $\epsilon_2 := \epsilon$, we can implement an $O(s)$-qubit quantum circuit U' that is a $(\|\hat{\mathbf{c}}\|_1, \hat{a}, \epsilon\|\hat{\mathbf{c}}\|_1)$-block-encoding of $P_{d'}(A)$, using $O(d^2)$ queries to U_A, where $\hat{a} = a + \lceil \log d' \rceil + 1$. Assume that U' is a $(1, \hat{a}, 0)$-block-encoding of A', then $\|\|\hat{\mathbf{c}}\|_1 A' - P_{d'}(A)\| \leq \|\hat{\mathbf{c}}\|_1 \epsilon$. Additionally, we can compute the quantum circuit description of U' in deterministic space $O(s + \log(1/\epsilon))$ given the coefficient vector $\hat{\mathbf{c}}$ of $P_{d'}$. As the quantum tester $\mathcal{T}(Q, U_A, P_{d'}, \epsilon)$ is mainly based on the Hadamard test, by employing Lemma 4.9, we have that $\mathcal{T}$ outputs 0 with probability

$$\Pr[x = 0] = \frac{1}{2}\big(1 + \mathrm{Re}(\mathrm{Tr}(A'\rho))\big) = \frac{1}{2}\left(1 + \mathrm{Re}\left(\mathrm{Tr}\Big(\frac{P_{d'}(A)}{\|\hat{\mathbf{c}}\|_1}\rho\Big)\right)\right) \pm \frac{1}{2}\epsilon.$$

It is left to construct the estimation procedure $\hat{\mathcal{T}}$. As detailed in Lemma 2.16, we can obtain an estimation $\widetilde{x}$ by sequentially repeating the quantum tester $\mathcal{T}(Q, U_A, P_{d'}, \epsilon)$ for $O(1/\epsilon_H^2)$ times. This repetition ensures that $|\widetilde{x} - \mathrm{Re}(\mathrm{Tr}(A'\rho))| \leq \epsilon_H$ holds with probability at least $\Omega(1)$, and derives an further implication on $P_{d'}(A)$:

$$\Pr[|\|\hat{\mathbf{c}}\|_1 \widetilde{x} - \mathrm{Re}(\mathrm{Tr}(P_{d'}(A)\rho))| \leq (\epsilon + \epsilon_H)\|\hat{\mathbf{c}}\|_1] \geq \Omega(1).$$

We thus conclude that construction of the estimation procedure $\hat{\mathcal{T}}(Q, U_A, P_{d'}, \epsilon, \epsilon_H, \delta)$ by utilizing $O(\log(1/\delta)/\epsilon_H^2)$ sequential repetitions of $\mathcal{T}(Q, U_A, P_{d'}, \epsilon)$. Similarly following Lemma 2.16, $\hat{\mathcal{T}}(Q, U_A, P_d, \epsilon, \epsilon_H, \delta)$ outputs an estimation $\widetilde{x}$ satisfies the following condition:

$$\Pr[|\|\hat{\mathbf{c}}\|_1 \widetilde{x} - \mathrm{Re}(\mathrm{Tr}(P_{d'}(A)\rho))| \leq (\epsilon + \epsilon_H)\|\hat{\mathbf{c}}\|_1] \geq 1 - \delta.$$

In addition, a direct calculation indicates that we can implement $\hat{\mathcal{T}}(Q, U_A, P_{d'}, \epsilon, \epsilon_H, \delta)$ in quantum space $O(s + \log(1/\epsilon) + \log(1/\epsilon_H) + \log\log(1/\delta))$ as desired.

Next, we move to the case where $\alpha = 1$, applicable to certain $P_{d'}$ and A, namely when $P_{d'}(A)$ is an isometry in Theorem 3.10 or $P_{d'} = P_{d'}^{\mathrm{sgn}}$ in Corollary 3.15. The proof is similar, and we just sketch the key points as follows. Using Theorem 3.10 with $\epsilon_1 := 0$ and $\epsilon_2 := \epsilon/36$, we can implement an $O(s)$-qubit quantum circuit U' that is a $(1, a', \epsilon\|\hat{\mathbf{c}}\|_1)$-block-encoding of $P_{d'}(A)$, using $O(d^2\|\hat{\mathbf{c}}\|_1)$ queries to U_A, where $a' = a + \lceil \log d' \rceil + 3$. Assume that U' is a $(1, a', 0)$-block-encoding of A', then $\|A' - P_{d'}(A)\| \leq \|\hat{\mathbf{c}}\|_1 \epsilon$. Similarly, $\mathcal{T}$ outputs 0 with probability

$$\Pr[x = 0] = \frac{1}{2}\big(1 + \mathrm{Re}(\mathrm{Tr}(A'\rho))\big) = \frac{1}{2}\left(1 + \mathrm{Re}\left(\mathrm{Tr}\left(P_{d'}(A)\rho\right)\right)\right) \pm \frac{1}{2}\|\hat{\mathbf{c}}\|_1 \epsilon. \tag{4.1}$$

Therefore, we can obtain an estimate $\widetilde{x}$ such that

$$\Pr[|\widetilde{x} - \mathrm{Re}(\mathrm{Tr}(P_{d'}(A)\rho))| \leq \|\hat{\mathbf{c}}\|_1 \epsilon + \epsilon_H] \geq 1 - \delta. \tag{4.2}$$

Analogously, using Corollary 3.15 with $\epsilon_1 := 0$ and $\epsilon_2 := \epsilon/(36\hat{C}_{\mathrm{sgn}} + 37)$ when $P_{d'} = P_{d'}^{\mathrm{sgn}}$, we can also obtain the corresponding formulas of Equation (4.1) and Equation (4.2) by substituting $\|\hat{\mathbf{c}}\|_1$ with $\hat{C}_{\mathrm{sgn}}$, where $\|\hat{\mathbf{c}}^{\mathrm{sgn}}\| \leq \hat{C}_{\mathrm{sgn}}$ is a constant defined in Corollary 3.6. □

4.2 GapQSD$_{\log}$ is in BQL

In this subsection, we demonstrate Theorem 4.10 by constructing a quantum algorithm that incorporates testers $\mathcal{T}(Q_i, U_{\frac{\rho_0 - \rho_1}{2}}, P_d^{\mathrm{sgn}}, \epsilon)$ for $i \in \{0, 1\}$, where the construction of testers utilizes the space-efficient QSVT associated with the sign function.

Theorem 4.10. *For any functions $\alpha(n)$ and $\beta(n)$ that can be computed in deterministic logspace and satisfy $\alpha(n) - \beta(n) \geq 1/\operatorname{poly}(n)$, we have that $\mathrm{GapQSD}_{\log}[\alpha(n), \beta(n)]$ is in BQL.*

Proof. Inspired by time-efficient algorithms for the low-rank variant of GapQSD [WZ24], we devise a space-efficient algorithm for GapQSD$_{\log}$, presented formally in Algorithm 1.

Algorithm 1: Space-efficient algorithm for $\textsc{GapQSD}_{\log}$.

Input : Quantum circuits Q_i that prepare the purification of ρ_i for $i \in \{0,1\}$.
Output: An additive-error estimation of $\mathrm{td}(\rho_0, \rho_1)$.
Params: $\varepsilon := \frac{\alpha-\beta}{4}$, $\delta := \frac{\varepsilon}{2^{r+3}}$, $\epsilon := \frac{\varepsilon}{2(36\hat{C}_{\mathrm{sgn}}+2C_{\mathrm{sgn}}+37)}$, $d' := \tilde{C}_{\mathrm{sgn}} \cdot \frac{1}{\delta} \log \frac{1}{\epsilon} = 2d-1$, $\varepsilon_H := \frac{\varepsilon}{4}$.

1. Construct block-encodings of ρ_0 and ρ_1, denoted by U_{ρ_0} and U_{ρ_1}, respectively, using $O(1)$ queries to Q_0 and Q_1 and $O(s(n))$ ancillary qubits by Lemma 4.8;
2. Construct a block-encoding of $\frac{\rho_0-\rho_1}{2}$, denoted by $U_{\frac{\rho_0-\rho_1}{2}}$, using $O(1)$ queries to U_{ρ_0} and U_{ρ_1} and $O(s(n))$ ancillary qubits by Lemma 3.13;

Let $P_{d'}^{\mathrm{sgn}}$ be the degree-d' polynomial specified in Corollary 3.6 with parameters δ and ϵ, and its coefficients $\{\hat{c}_k\}_{k=0}^{d'}$ are computable in deterministic space $O(\log(d/\epsilon))$;

3. Set $x_0 := \hat{\mathcal{T}}(Q_0, U_{\frac{\rho_0-\rho_1}{2}}, P_{d'}^{\mathrm{sgn}}, \epsilon, \epsilon_H, 1/10)$, $x_1 := \hat{\mathcal{T}}(Q_1, U_{\frac{\rho_0-\rho_1}{2}}, P_{d'}^{\mathrm{sgn}}, \epsilon, \epsilon_H, 1/10)$;
4. Compute $x = (x_0 - x_1)/2$. Return "yes" if $x > (\alpha+\beta)/2$, and "no" otherwise.

Let us demonstrate the correctness of Algorithm 1 and analyze the computational complexity. We focus on the setting with $s(n) = \Theta(\log n)$. We set $\varepsilon := (\alpha-\beta)/4 \geq 2^{-O(s)}$ and assume that Q_0 and Q_1 are $s(n)$-qubit quantum circuits that prepare the purifications of ρ_0 and ρ_1, respectively. According to Lemma 4.8, we can construct $O(s)$-qubit quantum circuits U_{ρ_0} and U_{ρ_1} that encode ρ_0 and ρ_1 as $(1, O(s), 0)$-block-encodings, using $O(1)$ queries to Q_0 and Q_1 as well as $O(1)$ one- and two-qubit quantum gates. Next, we apply Lemma 3.13 to construct a $(1, O(s), 0)$-block-encoding $U_{\frac{\rho_0-\rho_1}{2}}$ of $\frac{\rho_0-\rho_1}{2}$, using $O(1)$ queries to Q_{ρ_0} and Q_{ρ_1}, as well as $O(1)$ one- and two-qubit quantum gates.

Let $\delta := \frac{\varepsilon}{2^{r+3}}$, $\epsilon := \frac{\varepsilon}{2(36\hat{C}_{\mathrm{sgn}}+2C_{\mathrm{sgn}}+37)}$, and $d' := \tilde{C}_{\mathrm{sgn}} \cdot \frac{1}{\delta} \log \frac{1}{\epsilon} \leq 2^{O(s(n))}$, where $\tilde{C}_{\mathrm{sgn}}$ comes from Corollary 3.6. Let $P_{d'}^{\mathrm{sgn}} \in \mathbb{R}[x]$ be the polynomial specified in Corollary 3.6 with $d' = 2d-1$. Let $\epsilon_H = \varepsilon/4$. By employing Corollary 3.15 (with $\epsilon_1 := 0$ and $\epsilon_2 := \epsilon$) and the corresponding estimation procedure $\hat{\mathcal{T}}(Q_i, U_{\frac{\rho_0-\rho_1}{2}}, P_{d'}^{\mathrm{sgn}}, \Theta(\epsilon), \epsilon_H, 1/10)$ from Lemma 4.7, we obtain the values x_i for $i \in \{0,1\}$, ensuring the following inequalities:

$$\Pr\left[\left|x_i - \mathrm{Tr}\left(P_{d'}^{\mathrm{sgn}}\left(\frac{\rho_0-\rho_1}{2}\right)\rho_i\right)\right| \leq (36\hat{C}_{\mathrm{sgn}}+37)\epsilon + \epsilon_H\right] \geq 0.9 \text{ for } i \in \{0,1\}. \tag{4.3}$$

Here, the implementation uses $O(d^2)$ queries to $U_{\frac{\rho_0-\rho_1}{2}}$ and $O(d^2)$ multi-controlled single-qubit gates. Moreover, the circuit descriptions of $\hat{\mathcal{T}}(Q_i, U_{\frac{\rho_0-\rho_1}{2}}, P_{d'}^{\mathrm{sgn}}, \epsilon, \epsilon_H, 1/10)$ can be computed in deterministic time $\tilde{O}(d^{9/2}/\epsilon)$ and space $O(s(n))$.

Now let $x := (x_0 - x_1)/2$. We will finish the correctness analysis of Algorithm 1 by showing $\Pr[|x - \mathrm{td}(\rho_0, \rho_1)| \leq \varepsilon] > 0.8$ through Equation (4.3). By considering the approximation error of $P_{d'}^{\mathrm{sgn}}$ in Corollary 3.6 and the QSVT implementation error in Corollary 3.15, we derive the following inequality in Proposition 4.10.1, and the proof is deferred to Appendix B.1:

Proposition 4.10.1. $\Pr\left[|x - \mathrm{td}(\rho_0, \rho_1)| \leq (36\hat{C}_{\mathrm{sgn}}+2C_{\mathrm{sgn}}+37)\epsilon + \epsilon_H + 2^{r+1}\delta\right] > 0.8.$

Under the aforementioned choice of δ, ϵ, and ϵ_H, we have $\epsilon_H = \varepsilon/4$, $2^{r+1}\delta = \varepsilon/4$, and $(36\hat{C}_{\mathrm{sgn}}+2C_{\mathrm{sgn}}+37)\epsilon \leq \varepsilon/2$, and thus $\Pr[|x - \mathrm{td}(\rho_0, \rho_1)|] > 0.8$.

Finally, we analyze the computational resources required for Algorithm 1. According to Lemma 4.7, we can compute x in BQL, with the resulting algorithm requiring $O(d^2/\epsilon_H^2) = \tilde{O}(2^{2r}/\varepsilon^4)$ queries to Q_0 and Q_1. In addition, its circuit description can be computed in deterministic time $\tilde{O}(d^{9/2}/\varepsilon) = \tilde{O}(2^{4.5r}/\varepsilon^{5.5})$. □

4.3 $\text{GapQED}_{\log}$ and $\text{GapQJS}_{\log}$ are in BQL

In this subsection, we will demonstrate Theorem 4.11 by devising a quantum algorithm that encompasses testers $\mathcal{T}(Q_i, U_{\rho_i}, P_{d'}^{\ln}, \epsilon)$ for $i \in \{0,1\}$, where the construction of testers employs the space-efficient QSVT associated with the normalized logarithmic function. Consequently, we can deduce that $\text{GapQJS}_{\log}$ is in BQL via a reduction from $\text{GapQJS}_{\log}$ to $\text{GapQED}_{\log}$.

Theorem 4.11. *For any deterministic logspace computable function $g(n)$ that satisfies $g(n) \geq 1/\operatorname{poly}(n)$, we have that $\text{GapQED}_{\log}[g(n)]$ is in BQL.*

Proof. We begin with a formal algorithm in Algorithm 2.

Algorithm 2: Space-efficient algorithm for $\text{GapQED}_{\log}$.

Input : Quantum circuits Q_i that prepare the purification of ρ_i for $i \in \{0,1\}$.
Output: An additive-error estimation of $\mathrm{S}(\rho_0) - \mathrm{S}(\rho_1)$.
Params: $\varepsilon := \frac{g}{4}$, $\beta := \min\{\frac{\varepsilon}{2^{r+6}\ln(2^{r+6}/\varepsilon)}, \frac{1}{4}\}$, $d' := \tilde{C}_{\ln} \cdot \frac{1}{\beta} \log \frac{1}{\epsilon} = 2d-1$,
$\epsilon := \frac{\varepsilon}{4\ln(2/\beta)(\hat{C}_{\ln}+C_{\ln})}$, $\epsilon_H := \frac{\varepsilon}{8\ln(2/\beta)}$.

1. Construct block-encodings of ρ_0 and ρ_1, denoted by U_{ρ_0} and U_{ρ_1}, respectively, using $O(1)$ queries to Q_0 and Q_1 and $O(s(n))$ ancillary qubits by Lemma 4.8;
Let $P_{d'}^{\ln}$ be the degree-d' polynomial specified in Corollary 3.9 with parameters β and ϵ, and its coefficients $\{c_k^{\ln}\}_{k=0}^{d'}$ are computable in bounded-error randomized space $O(\log(d/\epsilon))$;
2. Set $x_0 := \hat{\mathcal{T}}(Q_0, U_{\rho_0}, P_{d'}^{\ln}, \epsilon, \epsilon_H, 1/10)$, $x_1 := \hat{\mathcal{T}}(Q_1, U_{\rho_1}, P_{d'}^{\ln}, \epsilon, \epsilon_H, 1/10)$;
3. Compute $x = 2\ln(\frac{2}{\beta})(x_0 - x_1)$. Return "yes" if $x > 0$, and "no" otherwise.

Let us now demonstrate the correctness and computational complexity of Algorithm 2. We concentrate on the scenario with $s(n) = \Theta(\log n)$ and $\varepsilon = g/4 \geq 2^{-O(s)}$. Our strategy is to estimate the entropy of each of ρ_0 and ρ_1, respectively. We assume that Q_0 and Q_1 are s-qubit quantum circuits that prepare the purifications of ρ_0 and ρ_1, respectively. By Lemma 4.8, we can construct $(1, O(s), 0)$-block-encodings U_{ρ_0} and U_{ρ_1} of ρ_0 and ρ_1, respectively, using $O(1)$ queries to Q_0 and Q_1 as well as $O(1)$ one- and two-qubit quantum gates.

Let $\beta = \min\{\frac{\varepsilon}{2^{r+6}\ln(2^{r+6}/\varepsilon)}, \frac{1}{4}\}$, $\epsilon := \frac{\varepsilon}{4\ln(2/\beta)(\hat{C}_{\ln}+C_{\ln})}$ and $d' := \tilde{C}_{\ln} \cdot \frac{1}{\beta} \log \frac{1}{\epsilon} = 2^{O(s(n))}$ where $\tilde{C}_{\ln}$ comes from Corollary 3.9. Let $P_{d'}^{\ln} \in \mathbb{R}[x]$ be the polynomial specified in Corollary 3.9 with $d' = 2d-1$. Let $\epsilon_H := \frac{\varepsilon}{8\ln(2/\beta)}$. By utilizing Corollary 3.16 (with $\epsilon_1 := 0$ and $\epsilon_2 := \epsilon$) and the corresponding estimation procedure $\hat{\mathcal{T}}(Q_i, U_{\rho_i}, P_{d'}^{\ln}, \epsilon, \epsilon_H, 1/10)$ from Lemma 4.7, we obtain the values x_i for $i \in \{0,1\}$, ensuring the following inequalities:

$$\Pr\left[\left|x_i - \operatorname{Tr}\left(P_{d'}^{\ln}(\rho_i)\,\rho_i\right)\right| \leq \hat{C}_{\ln}\epsilon + \epsilon_H\right] \geq 0.9 \text{ for } i \in \{0,1\}. \tag{4.4}$$

Here, the implementation uses $O(d^2)$ queries to U_{ρ_i} and $O(d^2)$ multi-controlled single-qubit gates. Moreover, the circuit descriptions of $\hat{\mathcal{T}}(Q_i, U_{\rho_i}, P_{d'}^{\ln}, \epsilon, \epsilon_H, 1/10)$ can be computed in bounded-error time $\tilde{O}(d^9/\epsilon^4)$ and space $O(s(n))$.

We will finish the correctness analysis of Algorithm 2 by demonstrating $\Pr[|x_i - \mathrm{S}(\rho_i)| \leq \varepsilon] \geq 0.9$ through Equation (4.4). By considering the approximation error of $P_{d'}^{\ln}$ in Corollary 3.9 and the QSVT implementation error in Corollary 3.16, we derive the following inequality in Proposition 4.11.1, and the proof is deferred to Appendix B.1:

Proposition 4.11.1. *The following inequality holds for $i \in \{0,1\}$:*

$$\Pr\left[\left|2\ln\left(\tfrac{2}{\beta}\right)x_i - \mathrm{S}(\rho_i)\right| \leq 2\ln\left(\tfrac{2}{\beta}\right)\left((\hat{C}_{\ln}+C_{\ln})\epsilon + \epsilon_H + 2^{r+1}\beta\right)\right] \geq 0.9.$$

Consequently, it is left to show that $2\ln\left(\frac{2}{\beta}\right)\left((\hat{C}_{\ln}+C_{\ln})\epsilon+\epsilon_H+2^{r+1}\beta\right)\leq\varepsilon$ for the specified value of β, ϵ, and ϵ_H. This can be seen by noting that $2\ln(2/\beta)\epsilon_H=\varepsilon/4$, $2\ln(2/\beta)(\hat{C}_{\ln}+C_{\ln})\epsilon=\varepsilon/2$, and $2\ln(2/\beta)\cdot 2^{r+1}\beta\leq\varepsilon/4$. The first two inequalities are trivial. For the third inequality, we state it in Proposition 4.11.2 and its proof is deferred to Appendix B.1.

Proposition 4.11.2. $2\ln(\frac{2}{\beta})\cdot 2^{r+1}\beta\leq\frac{\varepsilon}{4}$.

Finally, we analyze the computational resources required for Algorithm 2. As per Lemma 4.7, we can compute x in BQL, with the resulting algorithm requiring $O(d^2/\epsilon_H^2)=\tilde{O}(2^{2r}/\varepsilon^4)$ queries to Q_0 and Q_1. Furthermore, its circuit description can be computed in bounded-error randomized time $\tilde{O}(d^9/\varepsilon^4)=\tilde{O}(2^{9r}/\varepsilon^{13})$. □

$\textsc{GapQJS}_{\log}$ is in BQL. It is noteworthy that we can achieve $\textsc{GapQJS}_{\log}\in\mathsf{BQL}$ by employing the estimation procedure $\hat{\mathcal{T}}$ in Algorithm 2 for *three according states*, given that the quantum Jensen-Shannon divergence $\mathrm{QJS}(\rho_0,\rho_1)$ is a linear combination of $\mathrm{S}(\rho_0),\mathrm{S}(\rho_1)$, and $\mathrm{S}\left(\frac{\rho_0+\rho_1}{2}\right)$. Nevertheless, the logspace Karp reduction from $\textsc{GapQJS}_{\log}$ to $\textsc{GapQED}_{\log}$ (Corollary 4.12) allows us to utilize $\hat{\mathcal{T}}$ for only *two* states. Furthermore, our construction is adapted from the time-bounded scenario [Liu23, Lemma 4.4].

Corollary 4.12. *For any functions $\alpha(n)$ and $\beta(n)$ that can be computed in deterministic logspace and satisfy $\alpha(n)-\beta(n)\geq 1/\operatorname{poly}(n)$, we have that $\textsc{GapQJS}_{\log}[\alpha(n),\beta(n)]$ is in BQL.*

Proof. Let Q_0 and Q_1 be the given $s(n)$-qubit quantum circuits where $s(n)=\Theta(\log n)$. Consider a classical-quantum mixed state on a classical register B and a quantum register Y, denoted by $\rho_1'\coloneqq\frac{1}{2}|0\rangle\langle 0|\otimes\rho_0+\frac{1}{2}|1\rangle\langle 1|\otimes\rho_1$, where ρ_0 and ρ_1 are the state obtained by running Q_0 and Q_1, respectively, and tracing out the non-output qubits. We utilize our reduction to output classical-quantum mixed states ρ_0' and ρ_1', which are the output of $(s+2)$-qubit quantum circuits Q_0' and Q_1',[45] respectively, where $\rho_0'\coloneqq(p_0|0\rangle\langle 0|+p_1|1\rangle\langle 1|)\otimes(\frac{1}{2}\rho_0+\frac{1}{2}\rho_1)$ and $\mathsf{B}'\coloneqq(p_0,p_1)$ is an independent random bit with entropy $\mathrm{H}(\mathsf{B}')=1-\frac{1}{2}[\alpha(n)+\beta(n)]$. Let $\mathrm{S}_2(\rho)\coloneqq\mathrm{S}(\rho)/\ln 2$ for any quantum state ρ, we then have derived that:

$$\begin{aligned}
\mathrm{S}_2(\rho_0')-\mathrm{S}_2(\rho_1') &= \mathrm{S}_2(\mathsf{B}',\mathsf{Y})_{\rho_0'}-\mathrm{S}_2(\mathsf{B},\mathsf{Y})_{\rho_1'}\\
&= [\mathrm{H}(\mathsf{B}')+\mathrm{S}_2(\mathsf{Y}|\mathsf{B}')_{\rho_0'}]-[\mathrm{H}(\mathsf{B})+\mathrm{S}_2(\mathsf{Y}|\mathsf{B})_{\rho_1'}]\\
&= \mathrm{S}_2(\mathsf{Y})_{\rho_0'}-\mathrm{S}_2(\mathsf{Y}|\mathsf{B})_{\rho_1'}+\mathrm{H}(\mathsf{B}')-\mathrm{H}(\mathsf{B})\\
&= \mathrm{S}_2(\mathsf{Y})_{\rho_0'}-\mathrm{S}_2(\mathsf{Y}|\mathsf{B})_{\rho_1'}-\tfrac{1}{2}[\alpha(n)+\beta(n)]\\
&= \mathrm{S}_2\left(\tfrac{1}{2}\rho_0+\tfrac{1}{2}\rho_1\right)-\tfrac{1}{2}(\mathrm{S}_2(\rho_0)+\mathrm{S}_2(\rho_1))-\tfrac{1}{2}[\alpha(n)+\beta(n)]\\
&= \mathrm{QJS}_2(\rho_0,\rho_1)-\tfrac{1}{2}[\alpha(n)+\beta(n)].
\end{aligned}\tag{4.5}$$

Here, the second line derives from the definition of quantum conditional entropy and acknowledges that both B and B' are classical registers. The third line owes to the independence of B' as a random bit. Furthermore, the fifth line relies on the Joint entropy theorem (Lemma 2.5).

By plugging Equation (4.5) into the promise of $\textsc{GapQJS}_{\log}[\alpha(n),\beta(n)]$, we can define $g(n')\coloneqq\frac{\ln 2}{2}\big(\alpha(n)-\beta(n)\big)$ and conclude that:

- If $\mathrm{QJS}_2(\rho_0,\rho_1)\geq\alpha(n)$, then $\mathrm{S}(\rho_0')-\mathrm{S}(\rho_1')\geq\frac{\ln 2}{2}\big(\alpha(n)-\beta(n)\big)=g(n')$;
- If $\mathrm{QJS}_2(\rho_0,\rho_1)\leq\beta(n)$, then $\mathrm{S}(\rho_0')-\mathrm{S}(\rho_1')\leq-\frac{\ln 2}{2}\big(\alpha(n)-\beta(n)\big)=-g(n')$.

[45] To construct Q_1', we follow these steps: We start by applying a $\textsc{Hadamard}$ gate on B followed by a $\mathrm{CNOT}_{\mathsf{B}\to\mathsf{R}}$ gate where B and R are single-qubit quantum registers initialized on $|0\rangle$. Next, we apply the controlled-Q_1 gate on the qubits from B to S, where $\mathsf{S}=(\mathsf{Y},\mathsf{Z})$ is an $s(n)$-qubit register initialized on $|\bar{0}\rangle$. We then apply X gate on B followed by the controlled-Q_0 gate on the qubits from B to S, and we apply X gate on B again. Finally, we obtain ρ_1' by tracing out R and the qubits in Z. In addition, we can construct Q_0' similarly.

As ρ'_1 and ρ'_0 are $r'(n')$-qubit states where $n' := 3n$[46] and $r'(n') := r(n)+1$, the output length of the corresponding space-bounded quantum circuits Q'_0 and Q'_1 is $r'(n)$. Hence, $\mathrm{GAPQJS}_s[\alpha,\beta]$ is logspace Karp reducible to $\mathrm{GAPQED}_{s+3}[g]$ by mapping (Q_0,Q_1) to (Q'_0,Q'_1). □

4.4 $\overline{\mathrm{CERTQSD}}_{\log}$ and $\overline{\mathrm{CERTQHS}}_{\log}$ are in $\mathsf{coRQ_UL}$

To make the error one-sided, we adapt the Grover search when the number of solutions is one quarter [BBHT98], also known as the exact amplitude amplification [BHMT02].

Lemma 4.13 (Exact amplitude amplification, adapted from [BHMT02, Equation 8])**.** *Suppose U is a unitary of interest such that $U|\bar{0}\rangle = \sin(\theta)|\psi_0\rangle + \cos(\theta)|\psi_1\rangle$, where $|\psi_0\rangle$ and $|\psi_1\rangle$ are normalized pure states and $\langle\psi_0|\psi_1\rangle = 0$. Let $G = -U(I-2|\bar{0}\rangle\langle\bar{0}|)U^\dagger(I-2|\psi_0\rangle\langle\psi_0|)$ be the Grover operator. Then, for every integer $j \geq 0$, we have $G^jU|\bar{0}\rangle = \sin((2j+1)\theta)|\psi_0\rangle+\cos((2j+1)\theta)|\psi_1\rangle$. In particular, with a single application of G, we obtain $GU|\bar{0}\rangle = \sin(3\theta)|\psi_0\rangle + \cos(3\theta)|\psi_1\rangle$, signifying that $GU|\bar{0}\rangle = |\psi_0\rangle$ when $\sin(\theta) = 1/2$.*

Notably, when dealing with the unitary of interest with the property specified in Lemma 4.13, which is typically a quantum algorithm with acceptance probability linearly dependent on the chosen distance-like measure (e.g., a tester $\mathcal{T}$ from Lemma 4.7), Lemma 4.13 guarantees that the resulting algorithm $\mathcal{A}$ accepts with probability exactly 1 for *yes* instances ($\rho_0 = \rho_1$). However, achieving $\mathcal{A}$ to accept with probability polynomially deviating from 1 for *no* instances requires additional efforts, leading to the $\mathsf{coRQ_UL}$ containment established through error reduction for $\mathsf{coRQ_UL}$ (Corollary 3.18). In a nutshell, demonstrating $\mathsf{coRQ_UL}$ containment entails satisfying the desired property, which is achieved differently for $\overline{\mathrm{CERTQSD}}_{\log}$ and $\overline{\mathrm{CERTQHS}}_{\log}$.

4.4.1 $\overline{\mathrm{CERTQSD}}_{\log}$ is in $\mathsf{coRQ_UL}$

Our algorithm in Theorem 4.14 relies on the quantum tester $\mathcal{T}(Q_i, U_{\frac{\rho_0-\rho_1}{2}}, P_d^{\mathrm{sgn}}, \epsilon)$ specified in Algorithm 1. Note that the exact implementation of the space-efficient QSVT associated with odd polynomials preserves the original point (Remark 3.11). Consequently, $\mathcal{T}(Q_i, U_{\frac{\rho_0-\rho_1}{2}}, P_d^{\mathrm{sgn}}, \epsilon)$ outputs 0 with probability exactly $1/2$ when $\rho_0 = \rho_1$, enabling us to derive the $\mathsf{coRQ_UL}$ containment through a relatively involved analysis for cases when $\mathrm{td}(\rho_0,\rho_1) \geq \alpha$:

Theorem 4.14. *For any deterministic logspace computable function $\alpha(n) \geq 1/\operatorname{poly}(n)$, we have that $\overline{\mathrm{CERTQSD}}_{\log}[\alpha(n)]$ is in $\mathsf{coRQ_UL}$.*

Proof. We first present a formal algorithm in Algorithm 3:

Constructing the unitary of interest via the space-efficient QSVT. We consider the setting with $s(n) = \Theta(\log n)$ and $\varepsilon = \alpha/2$. Suppose Q_0 and Q_1 are $s(n)$-qubit quantum circuits that prepare the purifications of ρ_0 and ρ_1, respectively. Similar to Algorithm 1, we first construct an $O(s)$-qubit quantum circuit $U_{\frac{\rho_0-\rho_1}{2}}$ that is a $(1, O(s), 0)$-block-encoding of $\frac{\rho_0-\rho_1}{2}$, using $O(1)$ queries to Q_0 and Q_1 and $O(1)$ one- and two-qubit quantum gates.

Let $\delta = \frac{\varepsilon}{2^{r+3}}$, $\epsilon := \frac{\varepsilon}{2(36\hat{C}_{\mathrm{sgn}}+2C_{\mathrm{sgn}}+37)}$ and $d' := \tilde{C}_{\mathrm{sgn}} \cdot \frac{1}{\delta}\log\frac{1}{\epsilon} = 2^{O(s(n))}$ where $\tilde{C}_{\mathrm{sgn}}$ comes from Corollary 3.6. Let $P_{d'}^{\mathrm{sgn}} \in \mathbb{R}[x]$ be the odd polynomial specified in Corollary 3.6. Let $U_i := \mathcal{T}(Q_i, U_{\frac{\rho_0-\rho_1}{2}}, P_{d'}^{\mathrm{sgn}}, \epsilon)$ for $i \in \{0,1\}$, then we have the following equalities with $0 \leq p_0, p_1 \leq 1$:

$$U_0|0\rangle|\bar{0}\rangle = \sqrt{p_0}|0\rangle|\psi_0\rangle + \sqrt{1-p_0}|1\rangle|\psi_1\rangle,$$
$$U_1|0\rangle|\bar{0}\rangle = \sqrt{p_1}|0\rangle|\phi_0\rangle + \sqrt{1-p_1}|1\rangle|\phi_1\rangle.$$

[46] By inspecting the circuit description of Q'_0 and Q'_1 (see [Liu23, Appendix C] for details), the maximum number of gates in Q'_0 and Q'_1 is $2n+9+\operatorname{polylog}(1/\epsilon) \leq 3n$ for large enough n. Specifically, the implementation of R_θ in [Liu23, Figure 1] requires $\operatorname{polylog}(1/\epsilon) = \operatorname{polylog}(n)$ gates due to the space-efficient Solovay-Kitaev theorem [vMW12, Theorem 4.3].

Algorithm 3: Space-efficient algorithm for $\overline{\text{CERTQSD}}_{\log}$.

Input : Quantum circuits Q_i that prepare the purification of ρ_i for $i \in \{0,1\}$.
Output: Return "yes" if $\rho_0 = \rho_1$, and "no" otherwise.
Params: $\varepsilon := \frac{\alpha}{2}$, $\delta := \frac{\varepsilon}{2^{r+3}}$, $\epsilon := \frac{\varepsilon}{2(36\hat{C}_{\text{sgn}}+2C_{\text{sgn}}+37)}$, $d' := \tilde{C}_{\text{sgn}} \cdot \frac{1}{\delta} \log \frac{1}{\epsilon} = 2d - 1$.

1. Construct block-encodings of ρ_0 and ρ_1, denoted by U_{ρ_0} and U_{ρ_1}, respectively, using $O(1)$ queries to Q_0 and Q_1 and $O(s(n))$ ancillary qubits by Lemma 4.8;
2. Construct a block-encoding of $\frac{\rho_0-\rho_1}{2}$, denoted by $U_{\frac{\rho_0-\rho_1}{2}}$, using $O(1)$ queries to U_{ρ_0} and U_{ρ_1} and $O(s(n))$ ancillary qubits by Lemma 3.13;

Let $P^{\text{sgn}}_{d'}$ be the degree-d' odd *polynomial specified in Corollary 3.6 with parameters δ and ϵ, and its coefficients $\{\hat{c}_k\}_{k=0}^{d'}$ are computable in deterministic space $O(\log(d/\epsilon))$;*

3. Let $U_0 := \mathcal{T}(Q_0, U_{\frac{\rho_0-\rho_1}{2}}, P^{\text{sgn}}_{d'}, \epsilon)$ and $U_1 := \mathcal{T}(Q_1, U_{\frac{\rho_0-\rho_1}{2}}, P^{\text{sgn}}_{d'}, \epsilon)$;
4. Let $G_i := -(H \otimes U_i)(I - 2|\bar{0}\rangle\langle\bar{0}|)(H \otimes U_i^\dagger)(I - 2\Pi_0)$ for $i \in \{0,1\}$, where Π_0 is the projection onto the subspace spanned by $\{|0\rangle|0\rangle|\varphi\rangle\}$ over all $|\varphi\rangle$;
5. Measure the first two qubits of $G_i(H \otimes U_i)|0\rangle|0\rangle|\bar{0}\rangle$, and let x_{i0} and x_{i1} be the outcomes, respectively. Return "yes" if $x_{00} = x_{01} = x_{10} = x_{11} = 0$, and "no" otherwise.

Let H be the HADAMARD gate, then we derive the following equality for $i \in \{0,1\}$:

$$(H \otimes U_i)|0\rangle|0\rangle|\bar{0}\rangle = \sqrt{\frac{p_i}{2}}|0\rangle|0\rangle|\psi_0\rangle + \underbrace{\sqrt{\frac{p_i}{2}}|0\rangle|1\rangle|\psi_0\rangle + \sqrt{\frac{1-p_i}{2}}|1\rangle|0\rangle|\psi_1\rangle + \sqrt{\frac{1-p_i}{2}}|1\rangle|1\rangle|\psi_1\rangle}_{\sqrt{1-\frac{p_i}{2}}|\perp_i\rangle}.$$

Making the error one-sided by exact amplitude amplification. Consider the Grover operator $G_i := -(H \otimes U_i)(I - 2|\bar{0}\rangle\langle\bar{0}|)(H \otimes U_i^\dagger)(I - 2\Pi_0)$, where Π_0 is the projection onto the subspace spanned by $\{|0\rangle|0\rangle|\varphi\rangle\}$ over all $|\varphi\rangle$. By employing the exact amplitude amplification (Lemma 4.13), we can obtain that:

$$G_i(H \otimes U_i)|0\rangle|0\rangle|\bar{0}\rangle = \sin(3\theta_i)|0\rangle|0\rangle|\psi_0\rangle + \cos(3\theta_i)|\perp_i\rangle \text{ where } \sin^2(\theta_i) = \tfrac{p_i}{2} \text{ when } \theta_i \in \left[0, \tfrac{\pi}{4}\right]. \quad (4.6)$$

Let x_{i0} and x_{i1} be the measurement outcomes of the first two qubits of $G_i(H \otimes U_i)|0\rangle|0\rangle|\bar{0}\rangle$ for $i \in \{0,1\}$. Algorithm 3 returns "yes" if $x_{00} = x_{01} = x_{10} = x_{11} = 0$, and "no" otherwise. Let $U_{P^{\text{sgn}}_{d'}\left(\frac{\rho_0-\rho_1}{2}\right)}$ be the unitary operator being controlled in the implementation of $U_i := \mathcal{T}(Q_i, U_{\frac{\rho_0-\rho_1}{2}}, P^{\text{sgn}}_{d'}, \epsilon)$, and note that by Corollary 3.15, $U_{P^{\text{sgn}}_{d'}\left(\frac{\rho_0-\rho_1}{2}\right)}$ is a $(1, O(s), (36\hat{C}_{\text{sgn}}+37)\epsilon)$-block-encoding of $P^{\text{sgn}}_{d'}\left(\frac{\rho_0-\rho_1}{2}\right)$. We will show the correctness of our algorithm as follows:

- For *yes* instances ($\rho_0 = \rho_1$), $U_{P^{\text{sgn}}_{d'}\left(\frac{\rho_0-\rho_1}{2}\right)}$ is a $(1, O(s), 0)$-block-encoding of the zero operator, following from Remark 3.11. Consequently, $\mathcal{T}(Q_i, U_{\frac{\rho_0-\rho_1}{2}}, P^{\text{sgn}}_{d'}, \epsilon)$ outputs 0 with probability $1/2$ for $i \in \{0,1\}$, i.e., $p_0 = p_1 = 1/2$. As a result, we have $\theta_0 = \theta_1 = \pi/6$ and $\sin^2(3\theta_0) = \sin^2(3\theta_1) = 1$. Substituting these values into Equation (4.6), we can conclude that $x_{00} = x_{01} = x_{10} = x_{11} = 0$ with certainty, which completes the analysis.
- For *no* instances ($\text{td}(\rho_0, \rho_1) \geq \alpha$), $U_{P^{\text{sgn}}_{d'}\left(\frac{\rho_0-\rho_1}{2}\right)}$ is a $(1, O(s), 0)$-block-encoding of A satisfying $\left\|A - P^{\text{sgn}}_{d'}\left(\frac{\rho_0-\rho_1}{2}\right)\right\| \leq (36\hat{C}_{\text{sgn}}+37)\epsilon$. Let p_i be the probability that $\mathcal{T}(Q_i, U_{\frac{\rho_0-\rho_1}{2}}, P^{\text{sgn}}_{d'}, \epsilon)$ outputs 0 for $i \in \{0,1\}$, then $p_i = \frac{1}{2}\left(1 + \text{Re}(\text{Tr}(\rho_i A))\right)$ following from Lemma 4.7. A direct calculation similar to Proposition 4.10.1 indicates that:

  $$|(p_0 - p_1) - \text{td}(\rho_0, \rho_1)| \leq (36\hat{C}_{\text{sgn}} + 2C_{\text{sgn}} + 37)\epsilon + 2^{r+1}\delta.$$

  Under the choice of δ and ϵ (the same as in the proof of Theorem 4.11), we obtain that

$|(p_0 - p_1) - \mathrm{td}(\rho_0, \rho_1)| \leq \varepsilon$ which yields that $\max\{|p_0 - 1/2|, |p_1 - 1/2|\} \geq \varepsilon/2$.[47]

Note that $\Pr[x_{i0} = x_{i1} = 0] = \sin^2(3\theta_i)$ for $i \in \{0,1\}$, Algorithm 3 will return "yes" with probability $p_{\text{yes}} = \sin^2(3\theta_0)\sin^2(3\theta_1)$. We provide an upper bound for p_{yes} in Proposition 4.14.1, with the proof deferred to Appendix B.2:

Proposition 4.14.1. *Let $f(\theta_0, \theta_1) \coloneqq \sin^2(3\theta_0)\sin^2(3\theta_1)$ be a function such that $\sin^2(\theta_i) = p_i/2$ for $i \in \{0,1\}$ and $\max\{|p_0 - 1/2|, |p_1 - 1/2|\} \geq \varepsilon/2$, then $f(\theta_0, \theta_1) \leq 1 - \varepsilon^2/4$.*

Consequently, we finish the analysis by noticing $p_{\text{yes}} = f(\theta_0, \theta_1) \leq 1 - \varepsilon^2/4 = 1 - \alpha^2/16$.

Now we analyze the complexity of Algorithm 3. Following Lemma 4.7, we can compute $x_{00}, x_{01}, x_{10}, x_{11}$ in BQL. The quantum circuit that computes $x_{00}, x_{01}, x_{10}, x_{11}$ takes $O(d^2) = \tilde{O}(2^{2r}/\alpha^2)$ queries to Q_0 and Q_1, and its circuit description can be computed in deterministic time $\tilde{O}(d^{9/2}/\alpha) = \tilde{O}(2^{4.5r}/\alpha^{5.5})$. Finally, we conclude the $\mathsf{coRQ_UL}$ containment of $\overline{\textsc{CertQSD}}_{\log}$ by applying error reduction for $\mathsf{coRQ_UL}$ (Corollary 3.18) to Algorithm 3. □

4.4.2 $\overline{\textsc{CertQHS}}_{\log}$ is in $\mathsf{coRQ_UL}$

Our algorithm in Theorem 4.15 is based on the observation that by expressing $\mathrm{HS}^2(\rho_0, \rho_1)$ as a summation of $\frac{1}{2}\mathrm{Tr}(\rho_0^2)$, $\frac{1}{2}\mathrm{Tr}(\rho_1^2)$, and $-\mathrm{Tr}(\rho_0\rho_1)$, we can devise a hybrid algorithm with *two random coins* using the SWAP test. However, to ensure *unitary*, we design another algorithm employing the LCU technique, which serves as the unitary of interest with the desired property.

Theorem 4.15. *For any deterministic logspace computation function $\alpha(n) \geq 1/\operatorname{poly}(n)$, we have that $\overline{\textsc{CertQHS}}_{\log}[\alpha(n)]$ is in $\mathsf{coRQ_UL}$.*

Proof. We first provide a formal algorithm in Algorithm 4.

Algorithm 4: Space-efficient algorithm for $\overline{\textsc{CertQHS}}_{\log}$.

Input : Quantum circuits Q_i that prepare the purification of ρ_i for $i \in \{0,1\}$.
Output: Return "yes" if $\rho_0 = \rho_1$, and "no" otherwise.

1. Construct subroutines $T_{ij} \coloneqq \mathrm{SWAP}(\rho_i, \rho_j)$ for $(i,j) \in \{(0,0),(1,1),(0,1)\}$, which output 0 with probability p_{ij}. The subroutine $\mathrm{SWAP}(\rho_i, \rho_j)$ involves applying Q_i and Q_j to prepare quantum states ρ_i and ρ_j, respectively, and then employing the SWAP test (Lemma 2.17) on these states ρ_i and ρ_j;
2. Construct a block-encoding of $\varrho\big(\frac{1}{2} + \frac{\mathrm{HS}^2(\rho_0,\rho_1)}{4}\big)$ where $\varrho(p) \coloneqq p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$, denoted by U, using $O(1)$ queries to T_{00}, T_{11}, and T_{01} by Lemma 3.13;
3. Let $G \coloneqq -U(I - 2|\bar{0}\rangle\langle\bar{0}|)U^\dagger(I - 2|\bar{0}\rangle\langle\bar{0}|)$;
4. Measure all qubits of $GU|\bar{0}\rangle$ in the computational basis. Return "yes" if the measurement outcome is an all-zero string, and "no" otherwise.

Constructing the unitary of interest via the SWAP test. We consider the setting with $s(n) = \Theta(s(n))$. Our main building block is the circuit implementation of the SWAP test (Lemma 2.17). Specifically, we utilize the subroutine $\mathrm{SWAP}(\rho_i, \rho_j)$ for $i, j \in \{0,1\}$, which involves applying Q_i and Q_j to prepare quantum states ρ_i and ρ_j, respectively, and then employing the SWAP test on these states ρ_i and ρ_j. We denote by p_{ij} the probability that $\mathrm{SWAP}(\rho_i, \rho_j)$ outputs 0 based on the measurement outcome of the control qubit in the SWAP test. Following Lemma 2.17, we have $p_{ij} = \frac{1}{2}\big(1 + \mathrm{Tr}(\rho_i\rho_j)\big)$ for $i, j \in \{0,1\}$.

We define $T_{ij} \coloneqq \mathrm{SWAP}(\rho_i, \rho_j)$ for $(i,j) \in \mathcal{I} \coloneqq \{(0,0), 1, 1, 0, 1\}$, with the control qubit in $\mathrm{SWAP}(\rho_i, \rho_j)$ serving as the output qubit of T_{ij}. By introducing another ancillary qubit, we

[47] This inequality is because $|p_0 - p_1| \geq \mathrm{td}(\rho_0, \rho_1) - \varepsilon \geq 2\varepsilon - \varepsilon = \varepsilon$.

construct $T'_{ij} \coloneqq \mathrm{CNOT}(I \otimes T_{ij})$ for $(i,j) \in \mathcal{I}$, where CNOT is controlled by the output qubit of T_{ij} and targets on the new ancillary qubit. It is effortless to see that T'_{ij} prepares the purification of $\varrho(p_{ij})$ with $\varrho(p_{ij}) \coloneqq p_{ij}|0\rangle\langle 0| + (1-p_{ij})|1\rangle\langle 1|$ for $(i,j) \in \mathcal{I}$.

By applying Lemma 4.8, we can construct quantum circuits T''_{ij} for $(i,j) \in \mathcal{I}$ that serve as $(1, O(s), 0)$-block-encoding of $\varrho(p_{ij})$, using $O(1)$ queries to T'_{ij} and $O(1)$ one- and two-qubit quantum gates. Notably, $(X \otimes I)T''_{01}$, with X acting on the qubit of $\varrho(p_2)$, prepares the purification of $X\varrho(p_{01})X^\dagger = p_{01}|1\rangle\langle 1| + (1-p_{01})|0\rangle\langle 0| = \varrho(1-p_{01})$, leading to the equality:

$$\varrho(\rho_0,\rho_1) \coloneqq \frac{1}{4}\varrho(p_{00}) + \frac{1}{4}\varrho(p_{11}) + \frac{1}{2}\varrho(1-p_{01}) = \varrho\left(\frac{1}{2} + \frac{\mathrm{HS}^2(\rho_0,\rho_1)}{4}\right).$$

Consequently, we employ Lemma 3.13 to construct a unitary quantum circuit U that is a $(1,m,0)$-block-encoding of $\varrho\big(\frac{1}{2} + \frac{\mathrm{HS}^2(\rho_0,\rho_1)}{4}\big)$ using $O(1)$ queries to T''_{00}, T''_{11}, $(X\otimes I)T''_{01}$, and $O(1)$ one- and two-qubit quantum gates, where $m \coloneqq O(s)$. The construction ensures the following:

$$U|0\rangle|0\rangle^{\otimes m} = \underbrace{\left(\frac{1}{2} + \frac{\mathrm{HS}^2(\rho_0,\rho_1)}{4}\right)}_{\sin(\theta)}|0\rangle|0\rangle^{\otimes m} + \cos(\theta)|\perp\rangle, \text{ where } \langle 0|\langle 0|^{\otimes m}|\perp\rangle = 0. \tag{4.7}$$

Making the error one-sided. Let us consider the Grover operator $G \coloneqq -U(I-2|\bar{0}\rangle\langle\bar{0}|)U^\dagger(I-2|\bar{0}\rangle\langle\bar{0}|)$. By applying Lemma 4.13, we derive that $GU|0\rangle|0\rangle^{\otimes m} = \sin(3\theta)|0\rangle|0\rangle^{\otimes m} + \cos(3\theta)|\perp\rangle$. Subsequently, we measure all qubits of $GU|0\rangle|0\rangle^{\otimes m}$ in the computational basis, represented as $x \in \{0,1\}^{m+1}$. Hence, Algorithm 4 returns "yes" if the outcome x is 0^{m+1} and "no" otherwise. Algorithm 4 accepts with probability $\sin^2(3\theta)$. Now we analyze the correctness of the algorithm:

- For *yes* instances ($\rho_0 = \rho_1$), we have $\mathrm{HS}^2(\rho_0,\rho_1) = 0$. Following Equation (4.7), we obtain $\sin(\theta) = 1/2$ and thus $\sin^2(3\theta) = 1$. We conclude that Algorithm 4 will always return "yes".
- For *no* instances, we have $\mathrm{HS}^2(\rho_0,\rho_1) \geq \alpha$. According to Equation (4.7), we derive that:

$$\sin(\theta) = \frac{1}{2} + \frac{\mathrm{HS}^2(\rho_0,\rho_1)}{4} \geq \frac{1}{2} + \frac{\alpha}{4} \text{ and } \frac{1}{4} \leq \sin^2(\theta) = \left(\frac{1}{2} + \frac{\mathrm{HS}^2(\rho_0,\rho_1)}{4}\right)^2 \leq \left(\frac{1}{2} + \frac{1}{4}\right)^2 = \frac{9}{16}. \tag{4.8}$$

As a result, considering the fact that $\sin^2(3\theta) = f(\sin^2(\theta))$ where $f(x) \coloneqq 16x^3 - 24x^2 + 9x$, we require Proposition 4.15.1 and the proof is deferred to Appendix B.2:

Proposition 4.15.1. *The polynomial function $f(x) \coloneqq 16x^3 - 24x^2 + 9x$ is monotonically decreasing in $[1/4, 9/16]$. Moreover, we have $f\big(\big(\frac{1}{2} + \frac{\alpha}{4}\big)^2\big) \leq 1 - \frac{\alpha^2}{2}$ for any $0 \leq \alpha \leq 1$.*

Combining Equation (4.8) and Proposition 4.15.1, we have that $\sin^2(3\theta) = f(\sin^2(\theta)) \leq f\big(\big(\frac{1}{2} + \frac{\alpha}{4}\big)^2\big) \leq 1 - \frac{\alpha^2}{2}$. Hence, Algorithm 4 will return "no" with probability at least $\alpha^2/2$.

Regarding the computational complexity of Algorithm 4, this algorithm requires $O(s(n))$ qubits and performs $O(1)$ queries to Q_0 and Q_1. Finally, we finish the proof by applying error reduction from $\mathsf{coRQ_UL}$ (Corollary 3.18) to Algorithm 3. □

4.5 BQL- and $\mathsf{coRQ_UL}$-hardness for space-bounded state testing problems

We will prove that space-bounded state testing problems mentioned in Theorem 4.6 are $\mathsf{BQ_UL}$-hard, which implies their BQL-hardness since $\mathsf{BQL}=\mathsf{BQ_UL}$ [FR21]. Similarly, all space-bounded state certification problems mentioned in Theorem 4.5 are $\mathsf{coRQ_UL}$-hard.

4.5.1 Hardness results for $\mathrm{GapQSD}_{\log}$, $\mathrm{GapQHS}_{\log}$, and their certification version

Employing analogous constructions, we can establish the $\mathsf{BQ_UL}$-hardness of both $\mathrm{GapQSD}_{\log}$ and $\mathrm{GapQHS}_{\log}$. The former involves a single-qubit pure state and a single-qubit mixed state, while the latter involves two pure states.

Lemma 4.16 ($\overline{\text{GAPQSD}}_{\log}$ is $\mathsf{BQ_UL}$-hard)**.** *For any deterministic logspace computable functions $a(n)$ and $b(n)$ such that $a(n)-b(n) \geq 1/\operatorname{poly}(n)$, we have that $\overline{\text{GAPQSD}}_{\log}[1-\sqrt{a(n)}, \sqrt{1-b(n)}]$ is $\mathsf{BQ_UL}[a(n), b(n)]$-hard.*

Proof. Consider a promise problem $(\mathcal{L}_{yes}, \mathcal{L}_{no}) \in \mathsf{BQ_UL}[a(n), b(n)]$, then we know that the acceptance probability $\Pr[C_x \text{ accepts}] \geq a(n)$ if $x \in \mathcal{L}_{yes}$, whereas $\Pr[C_x \text{ accepts}] \leq b(n)$ if $x \in \mathcal{L}_{no}$. Now we notice that the acceptance probability is the fidelity between a single-qubit pure state ρ_0 and a single-qubit mixed state ρ_1 that is prepared by two logarithmic-qubit quantum circuits Q_0 and Q_1, respectively:

$$\begin{aligned}\Pr[C_x \text{ accepts}] =& \| |1\rangle\langle 1|_{\text{out}} C_x |\bar{0}\rangle \|_2^2 \\ =& \operatorname{Tr}\left(|1\rangle\langle 1|_{\text{out}} \operatorname{Tr}_{\overline{\text{out}}}\left(C_x |\bar{0}\rangle\langle\bar{0}| C_x^\dagger \right) \right) \\ =& \mathrm{F}^2\left(|1\rangle\langle 1|_{\text{out}}, \operatorname{Tr}_{\overline{\text{out}}}\left(C_x |\bar{0}\rangle\langle\bar{0}| C_x^\dagger \right) \right) \\ :=& \mathrm{F}^2(\rho_0, \rho_1).\end{aligned} \tag{4.9}$$

In particular, the corresponding Q_0 is simply flipping the designated output qubit, as well as the corresponding Q_1 is exactly the circuit C_x, then we prepare ρ_0 and ρ_1 by tracing out all non-output qubits. By utilizing Lemma 2.4, we have derived that:

- For *yes* instances, $\mathrm{F}^2(\rho_0, \rho_1) \geq a(n)$ deduces that $\operatorname{td}(\rho_0, \rho_1) \leq 1 - \sqrt{a(n)}$;
- For *no* instances, $\mathrm{F}^2(\rho_0, \rho_1) \leq b(n)$ yields that $\operatorname{td}(\rho_0, \rho_1) \geq \sqrt{1 - b(n)}$

Therefore, we demonstrate that $\overline{\text{GAPQSD}}_{\log}[1-\sqrt{a(n)}, \sqrt{1-b(n)}]$ is $\mathsf{BQL}[a(n), b(n)]$-hard. □

To construct pure states, adapted from the construction in Lemma 4.16, we replace the final measurement in the BQL circuit C_x with a quantum gate (CNOT) and design a new algorithm based on C_x with the final measurement on *all* qubits in the computational basis.

Lemma 4.17 ($\overline{\text{GAPQHS}}_{\log}$ is $\mathsf{BQ_UL}$-hard)**.** *For any deterministic logspace computable functions $a(n)$ and $b(n)$ such that $a(n) - b(n) \geq 1/\operatorname{poly}(n)$, we have that $\overline{\text{GAPQHS}}_{\log}[1 - a^2(n), 1 - b^2(n)]$ is $\mathsf{BQ_UL}[a(n), b(n)]$-hard.*

Proof. For any promise problem $(\mathcal{L}_{yes}, \mathcal{L}_{no}) \in \mathsf{BQ_UL}[a(n), b(n)]$, we have that the acceptance probability $\Pr[C_x \text{ accepts}] \geq a(n)$ if $x \in \mathcal{L}_{yes}$, whereas $\Pr[C_x \text{ accepts}] \leq b(n)$ if $x \in \mathcal{L}_{no}$. For convenience, let the output qubit be the register O. Now we construct a new quantum circuit C'_x with an additional ancillary qubit on the register F initialized to zero:

$$C'_x := C_x^\dagger X_{\mathsf{O}}^\dagger \text{CNOT}_{\mathsf{O}\to\mathsf{F}} X_{\mathsf{O}} C_x.$$

And we say that C'_x accepts if the measurement outcome of all qubits (namely the working qubit of C_x and F) are all zero. Through a direct calculation, we obtain:

$$\begin{aligned}\Pr\left[C'_x \text{ accepts}\right] &= \left\| (|\bar{0}\rangle\langle\bar{0}| \otimes |0\rangle\langle 0|_{\mathsf{F}}) C_x^\dagger X_{\mathsf{O}} \text{CNOT}_{\mathsf{O}\to\mathsf{F}} X_{\mathsf{O}} C_x (|\bar{0}\rangle \otimes |0\rangle_{\mathsf{F}}) \right\|_2^2 \\ &= \left| (\langle\bar{0}| \otimes \langle 0|_{\mathsf{F}}) C_x^\dagger (|1\rangle\langle 1|_{\mathsf{O}} \otimes I_{\mathsf{F}} + |0\rangle\langle 0|_{\mathsf{O}} \otimes X_{\mathsf{F}}) C_x (|\bar{0}\rangle \otimes |0\rangle_{\mathsf{F}}) \right|^2 \\ &= \left| \langle\bar{0}| C_x^\dagger |1\rangle\langle 1|_{\mathsf{O}} C_x |\bar{0}\rangle \right|^2 \\ &= \Pr^2\left[C_x \text{ accepts}\right].\end{aligned} \tag{4.10}$$

Here, the second line owes to $\text{CNOT}_{\mathsf{O}\to\mathsf{F}} = |0\rangle\langle 0|_{\mathsf{O}} \otimes I_{\mathsf{F}} + |1\rangle\langle 1|_{\mathsf{O}} \otimes X_{\mathsf{F}}$, and the last line is because of Equation (4.9). Interestingly, by defining two pure states $\rho_0 := |\bar{0}\rangle\langle\bar{0}| \otimes |0\rangle\langle 0|_{\mathsf{F}}$ and $\rho_1 := C'_x(|\bar{0}\rangle\langle\bar{0}| \otimes |0\rangle\langle 0|_{\mathsf{F}})C'^\dagger_x$ corresponding to $Q_0 = I$ and $Q_1 = C'_x$, respectively, we deduce the following from Equation (4.10):

$$\Pr\left[C'_x \text{ accepts}\right] = \operatorname{Tr}(\rho_0\rho_1) = 1 - \mathrm{HS}^2(\rho_0, \rho_1). \tag{4.11}$$

Combining Equation (4.10) and Equation (4.11), we conclude that:

- For *yes* instances, $\Pr[C_x \text{ accepts}] \geq a(n)$ implies that $\mathrm{HS}^2(\rho_0, \rho_1) \leq 1 - a^2(n)$;

- For *no* instances, $\Pr[C_x \text{ accepts}] \leq b(n)$ yields that $\mathrm{HS}^2(\rho_0, \rho_1) \geq 1 - b^2(n)$.

We thus complete the proof of $\overline{\textsc{GapQHS}}_{\log}[1 - a^2(n), 1 - b^2(n)]$ is $\mathsf{BQ_UL}[a(n), b(n)]$-hard. □

Our constructions in the proof of Lemma 4.16 and Lemma 4.17 are somewhat analogous to Theorem 12 and Theorem 13 in [RASW23]. Then we proceed with a few direct corollaries of Lemma 4.16 and Lemma 4.17.

Corollary 4.18 ($\mathsf{BQ_UL}$- and $\mathsf{coRQ_UL}$-hardness)**.** *For any functions $a(n)$ and $b(n)$ are computable in deterministic logspace such that $a(n) - b(n) \geq 1/\operatorname{poly}(n)$, the following holds for some polynomial $p(n)$ which can be computed in deterministic logspace:*

(1) $\textsc{GapQSD}_{\log}[\alpha(n), \beta(n)]$ *is* $\mathsf{BQ_UL}$*-hard for* $\alpha \leq 1 - 1/p(n)$ *and* $\beta \geq 1/p(n)$*;*

(2) $\overline{\textsc{CertQSD}}_{\log}[\gamma(n)]$ *is* $\mathsf{coRQ_UL}$*-hard for* $\gamma \leq 1 - 1/p(n)$*;*

(3) $\textsc{GapQHS}_{\log}[\alpha(n), \beta(n)]$ *is* $\mathsf{BQ_UL}$*-hard for* $\alpha \leq 1 - 1/p(n)$ *and* $\beta \geq 1/p(n)$*;*

(4) $\overline{\textsc{CertQHS}}_{\log}[\gamma(n)]$ *is* $\mathsf{coRQ_UL}$*-hard for* $\gamma \leq 1 - 1/p(n)$.

Proof. Firstly, it is important to note that $\mathsf{BQ_UL}$ is closed under complement, as demonstrated in [Wat99, Corollary 4.8]. By combining error reduction for $\mathsf{BQ_UL}$ (Corollary 3.18) and Lemma 4.16 (resp., Lemma 4.17), we can derive the first statement (resp., the third statement).

Moreover, to obtain the second statement (resp., the fourth statement), we can utilize error reduction for $\mathsf{coRQ_UL}$ (Corollary 3.18) and set $a = 1$ in Lemma 4.16 (resp., Lemma 4.17). □

4.5.2 Hardness results for $\textsc{GapQJS}_{\log}$ and $\textsc{GapQED}_{\log}$

We demonstrate the $\mathsf{BQ_UL}$-hardness of $\textsc{GapQJS}_{\log}$ by reducing $\textsc{GapQSD}_{\log}$ to $\textsc{GapQJS}_{\log}$, following a similar approach as shown in [Liu23, Lemma 4.12].

Lemma 4.19 ($\textsc{GapQJS}_{\log}$ is $\mathsf{BQ_UL}$-hard)**.** *For any functions $\alpha(n)$ and $\beta(n)$ are computable in deterministic logspace, we have $\textsc{GapQJS}_{\log}[\alpha(n), \beta(n)]$ is $\mathsf{BQ_UL}$-hard for $\alpha(n) \leq 1 - \sqrt{2}/\sqrt{p(n)}$ and $\beta(n) \geq 1/p(n)$, where $p(n)$ is some deterministic logspace computable polynomial.*

Proof. By employing Corollary 4.18, it suffices to reduce $\textsc{GapQSD}_{\log}[1 - 1/p(n), 1/p(n)]$ to $\textsc{GapQJS}_{\log}[\alpha(n), \beta(n)]$. Consider logarithmic-qubit quantum circuits Q_0 and Q_1, which is a $\textsc{GapQSD}_{\log}$ instance. We can obtain ρ_k for $k \in \{0, 1\}$ by performing Q_k on $|0^n\rangle$ and tracing out the non-output qubits. We then have the following:

- If $\mathrm{td}(\rho_0, \rho_1) \geq 1 - 1/p(n)$, then Lemma 2.6 yields that
$$\mathrm{QJS}_2(\rho_0, \rho_1) \geq 1 - \mathrm{H}_2\left(\tfrac{1-\mathrm{td}(\rho_0,\rho_1)}{2}\right) \geq 1 - \mathrm{H}_2\left(\tfrac{1}{2p(n)}\right) \geq 1 - \tfrac{\sqrt{2}}{\sqrt{p(n)}} \geq \alpha(n),$$
where the third inequality owing to $\mathrm{H}_2(x) \leq 2\sqrt{x}$ for all $x \in [0, 1]$.
- If $\mathrm{td}(\rho_0, \rho_1) \leq 1/p(n)$, then Lemma 2.6 indicates that
$$\mathrm{QJS}_2(\rho_0, \rho_1) \leq \mathrm{td}(\rho_0, \rho_1) \leq \tfrac{1}{p(n)} \leq \beta(n).$$

Therefore, we can utilize the same quantum circuits Q_0 and Q_1, along with their corresponding quantum states ρ_0 and ρ_1, respectively, to establish a logspace Karp reduction from $\textsc{GapQSD}_{\log}[1 - 1/p(n), 1/p(n)]$ to $\textsc{GapQJS}_{\log}[\alpha(n), \beta(n)]$, as required. □

By combining the reduction from $\textsc{GapQSD}_{\log}$ to $\textsc{GapQJS}_{\log}$ (Lemma 4.19) and the reduction from $\textsc{GapQJS}_{\log}$ to $\textsc{GapQED}_{\log}$ (Corollary 4.12), we will demonstrate that the $\mathsf{BQ_UL}$-hardness for $\textsc{GapQED}_{\log}$ through reducing $\textsc{GapQSD}_{\log}$ to $\textsc{GapQED}_{\log}$. This proof resembles the approach outlined in [Liu23, Corollary 4.3].

Corollary 4.20 ($\textsc{GapQED}_{\log}$ is $\mathsf{BQ_UL}$-hard)**.** *For any function $g(n)$ are computable in deterministic logspace, we have $\textsc{GapQED}_{\log}[g(n)]$ is $\mathsf{BQ_UL}$-hard for $g(n) \leq \frac{\ln 2}{2}\big(1-\frac{\sqrt{2}}{\sqrt{p(n/3)}}-\frac{1}{p(n/3)}\big)$, where $p(n)$ is some polynomial that can be computed in deterministic logspace.*

Proof. By combining Corollary 4.18 and Lemma 4.19, we establish that $\textsc{GapQJS}_{\log}[\alpha(n),\beta(n)]$ is $\mathsf{BQ_UL}$-hard for $\alpha(n) \leq 1-\sqrt{2}/\sqrt{p(n)}$ and $\beta(n) \geq 1/p(n)$, where $p(n)$ is some deterministic logspace computable polynomial. The $\textsc{GapQSD}_{\log}$-hard (and simultaneously $\textsc{GapQJS}_{\log}$-hard) instances, as specified in Corollary 4.18, consist of $s(n)$-qubit quantum circuits Q_0 and Q_1 that prepare a purification of $r(n)$-qubit quantum (mixed) states ρ_0 and ρ_1, respectively, where $1 \leq r(n) \leq s(n) = \Theta(\log n)$.

Subsequently, by employing Corollary 4.12, we construct $(s+3)$-qubit quantum circuits Q_0' and Q_1' that prepare a purification of $(r+1)$-qubit quantum states $\rho_0' = \big(p|0\rangle\langle 0|+(1-p)|1\rangle\langle 1|\big)\otimes(\frac{1}{2}\rho_0+\frac{1}{2}\rho_1)$ satisfying $\mathrm{H}_2(p) = 1-\frac{\ln 2}{2}\big(\alpha(n)+\beta(n)\big)$ and $\rho_1' = \frac{1}{2}|0\rangle\langle 0|\otimes\rho_0+\frac{1}{2}|1\rangle\langle 1|\otimes\rho_1$, respectively. Following Corollary 4.12, $\textsc{GapQED}_{\log}[g(n)]$ is $\mathsf{BQ_UL}$-hard as long as

$$g(n) = \tfrac{\ln 2}{2}\big(\alpha(n/3)-\beta(n/3)\big) \leq \tfrac{\ln 2}{2}\Big(1-\tfrac{\sqrt{2}}{\sqrt{p(n/3)}}-\tfrac{1}{p(n/3)}\Big).$$

Therefore, $\textsc{GapQSD}_s$ is logspace Karp reducible to $\textsc{GapQED}_{s+1}$ by mapping (Q_0,Q_1) to (Q_0',Q_1'). □

5 Algorithmic Holevo-Helstrom measurement and its implication

In this section, we introduce an *algorithmic* Holevo-Helstrom measurement that achieves the optimal probability (with an additive error) for discriminating between quantum states ρ_0 and ρ_1, as outlined in Theorem 5.2. We assume knowledge of the corresponding polynomial-size quantum circuits, viewed as "source codes" for quantum devices, used to prepare (purifications of) these states. We now define the Computational Quantum Hypothesis Testing Problem:

Problem 5.1 (Computational Quantum Hypothesis Testing Problem)**.** Given polynomial-size quantum circuits Q_0 and Q_1 acting on n qubits and having r designated output qubits. Let ρ_b denote the quantum state obtained by performing Q_b on the initial state $|0^n\rangle$ and tracing out the non-output qubits for $b \in \{0,1\}$. Now, consider the following computational task:

- **Input:** A quantum state ρ, either ρ_0 or ρ_1, is chosen uniformly at random.
- **Output:** A bit b indicates that $\rho = \rho_b$.

For the Quantum Hypothesis Testing Problem analogous to Problem 5.1, where ρ_0 and ρ_1 are not necessarily efficiently preparable, the maximum success probability to discriminate between quantum states ρ_0 and ρ_1 is given by the celebrated Holevo-Helstrom bound:

Theorem 5.2 (Holevo-Helstrom bound, [Hol73a, Hel69])**.** *Given a quantum (mixed) state ρ, either ρ_0 or ρ_1, that is chosen uniformly at random, the maximum success probability to discriminate between quantum states ρ_0 and ρ_1 is given by $\frac{1}{2}+\frac{1}{2}\mathrm{td}(\rho_0,\rho_1)$.*

Next, we specify the optimal two-outcome measurement $\{\Pi_0,\Pi_1\}$ that achieves the maximum discrimination probability in Theorem 5.2:

$$\Pi_0 = \frac{I}{2}+\frac{1}{2}\mathrm{sgn}^{(\mathrm{SV})}\Big(\frac{\rho_0-\rho_1}{2}\Big) \text{ and } \Pi_1 = \frac{I}{2}-\frac{1}{2}\mathrm{sgn}^{(\mathrm{SV})}\Big(\frac{\rho_0-\rho_1}{2}\Big). \tag{5.1}$$

It is straightforward to see that $\mathrm{td}(\rho_0,\rho_1) = \frac{1}{2}\mathrm{Tr}|\rho_0-\rho_1| = \mathrm{Tr}(\Pi_0\rho_0)-\mathrm{Tr}(\Pi_0\rho_1)$.

By leveraging our space-efficient quantum singular value transformation in Section 3, we can approximately implement the Holevo-Helstrom measurement specified in Equation (5.1) in quantum *single-exponential* time and *linear* space. We refer to this explicit implementation of the Holevo-Helstrom measurement as the *algorithmic Holevo-Helstrom measurement*:

Theorem 5.3 (Algorithmic Holevo-Helstrom measurement)**.** *Let ρ_0 and ρ_1 be quantum states prepared by n-qubit quantum circuits Q_0 and Q_1, respectively, as defined in Problem 5.1. An approximate version of the Holevo-Helstrom measurement Π_0 specified in Equation (5.1), denoted as $\tilde{\Pi}_0$, can be implemented such that*

$$|\mathrm{td}(\rho_0,\rho_1) - \big(\mathrm{Tr}(\tilde{\Pi}_0\rho_0) - \mathrm{Tr}(\tilde{\Pi}_0\rho_1)\big)| \leq 2^{-n}.$$

The quantum circuit implementation of $\tilde{\Pi}_0$, acting on $O(n)$ qubits, requires $2^{O(n)}$ queries to the quantum circuits Q_0 and Q_1, as well as $2^{O(n)}$ one- and two-qubit quantum gates. Moreover, the circuit description can be computed in deterministic time $2^{O(n)}$ and space $O(n)$.

Additionally, we demonstrate an implication of our algorithmic Holevo-Helstrom measurement in Theorem 5.3. By inspecting the (honest-verifier) quantum statistical zero-knowledge protocol ("distance test") for (α,β)-QSD where $\alpha^2 > \beta$ in [Wat02], we established a slightly improved upper bound for the class QSZK since GapQSD is QSZK-hard:

Theorem 5.4 (GapQSD is in $\mathsf{QIP}(2)$ with a quantum linear-space honest prover)**.** *There is a two-message quantum interactive proof system for $\mathrm{GapQSD}[\alpha(n),\beta(n)]$ with completeness $c(n) = \big(1+\alpha(n)-2^{-n}\big)/2$ and soundness $s(n) = (1+\beta(n))/2$. Moreover, the optimal prover strategy for this protocol can be implemented in quantum single-exponential time and linear space. Consequently, for any $\alpha(n)$ and $\beta(n)$ satisfying $\alpha(n)-\beta(n) \geq 1/\mathrm{poly}(n)$, $\mathrm{GapQSD}[\alpha(n),\beta(n)]$ is in $\mathsf{QIP}(2)$ with a quantum $O(n')$ space honest prover, where n' is the total input length of the quantum circuits that prepare the corresponding tuple of quantum states.*[48]

In the rest of this section, we provide the proof of Theorem 5.3 and the proof of Theorem 5.4 in Section 5.1 and Section 5.2, respectively.

5.1 Algorithmic Holevo-Helstrom measurement: Proof of Theorem 5.3

Our algorithmic Holevo-Helstrom measurement primarily utilizes the space-efficient quantum state tester (see Figure 2) in Section 4. By leveraging the space-efficient polynomial approximation $P^{\mathrm{sgn}}_{d'}$ of the sign function (Corollary 3.6), it suffices to implement another two-outcome measurement $\{\hat{\Pi}_0,\hat{\Pi}_1\}$:

$$\hat{\Pi}_0 = \frac{I}{2} + \frac{1}{2}P^{\mathrm{sgn}}_{d'}\Big(\frac{\rho_0-\rho_1}{2}\Big) \text{ and } \hat{\Pi}_1 = \frac{I}{2} - \frac{1}{2}P^{\mathrm{sgn}}_{d'}\Big(\frac{\rho_0-\rho_1}{2}\Big).$$

By applying the space-efficient QSVT associated with the polynomial $P^{\mathrm{sgn}}_{d'}$ to the block-encoding of $(\rho_0-\rho_1)/2$ (Corollary 3.15), we obtain the unitary U_{HH} which is a block-encoding of $A_{\mathrm{HH}} :\approx P^{\mathrm{sgn}}_{d'}\big(\frac{\rho_0-\rho_1}{2}\big)$. We now instead implement two-outcome measurement $\{\tilde{\Pi}_0,\tilde{\Pi}_1\}$ where $\tilde{\Pi}_0 = (I+A_{\mathrm{HH}})/2$, and the difference between $\{\hat{\Pi}_0,\hat{\Pi}_1\}$ and $\{\tilde{\Pi}_0,\tilde{\Pi}_1\}$ is caused by the implementation error of our space-efficient QSVT. Then we proceed to the actual proof.

Proof of Theorem 5.3. Our algorithmic Holevo-Helstrom measurement is inspired by Algorithm 1 in the proof of Theorem 4.10 ($\mathrm{GapQSD}_{\log}$ is in BQL), as presented in Figure 3.

Note that the input state ρ to the circuit specified in Figure 3 is an $r(n)$-qubit quantum state, either ρ_0 or ρ_1, prepared by an n-qubit polynomial-size quantum circuit (Q_0 or Q_1) after tracing out all $n-r$ non-output qubits, where Q_0 and Q_1 are defined in Problem 5.1. The key ingredient in Figure 3 is to implement the unitary U_{HH}, which can be achieved as follows:

(1) Applying Lemma 4.8, we can construct n-qubit quantum circuits U_{ρ_0} and U_{ρ_1} that encode ρ_0 and ρ_1 as $(1,n-r,0)$-block-encodings, using $O(1)$ queries to Q_0 and Q_1, as well as $O(1)$ one- and two-qubit quantum gates.

[48] This tuple of quantum states results from a standard parallel repetition of the two-message quantum interactive proof system for $\mathrm{GapQSD}[\alpha(n),\beta(n)]$ with $c(n)-s(n) \geq 1/poly(n)$.

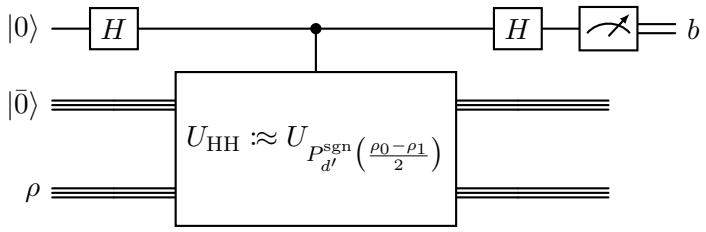

Figure 3: Algorithmic Holevo-Helstrom measurement.

(2) Applying Lemma 3.13, we can construct a $(1, n-r+1, 0)$-block-encoding $U_{\frac{\rho_0-\rho_1}{2}}$ of $\frac{\rho_0-\rho_1}{2}$, using $O(1)$ queries to Q_0 and Q_1, as well as $O(1)$ one- and two-qubit quantum gates.

(3) Let $P_{d'}^{\mathrm{sgn}} \in \mathbb{R}[x]$ be the degree-d' polynomial obtained from some degree-d averaged Chebyshev truncation, with $d' = 2d-1$, as specified in Corollary 3.6. We choose parameters $\varepsilon := 2^{-n}$, $\delta := \frac{\varepsilon}{2^{r+3}}$, $\epsilon := \frac{\varepsilon}{2(36\hat{C}_{\mathrm{sgn}}+2C_{\mathrm{sgn}}+37)}$, and $d' := \tilde{C}_{\mathrm{sgn}} \cdot \frac{1}{\delta} \log \frac{1}{\epsilon} = 2^{O(n)}$ where $\tilde{C}_{\mathrm{sgn}}$ comes from Corollary 3.6. Applying the space-efficient QSVT associated with the sign function (Corollary 3.15 with $\epsilon_1 := 0$ and $\epsilon_2 := \epsilon$), we obtain the unitary U_{HH}.

Error analysis. We first bound the error caused by space-efficient polynomial approximation in Corollary 3.6. Consider the spectral decomposition $\frac{\rho_0-\rho_1}{2} = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$, where $\{|\psi_j\rangle\}$ is an orthonormal basis. We can define index sets $\Lambda_- := \{j : \lambda_j < -\delta\}$, $\Lambda_0 := \{j : -\delta \le \lambda_j \le \delta\}$, and $\Lambda_+ := \{j : \lambda_j > \delta\}$. Next, we have derived that:

$$
\begin{aligned}
&\left|\mathrm{td}(\rho_0,\rho_1) - \left(\mathrm{Tr}(\hat{\Pi}_0\rho_0) - \mathrm{Tr}(\hat{\Pi}_0\rho_1)\right)\right| \\
=&\left|\mathrm{Tr}\left(\mathrm{sgn}\left(\frac{\rho_0-\rho_1}{2}\right)\frac{\rho_0-\rho_1}{2}\right) - \mathrm{Tr}\left(P_{d'}^{\mathrm{sgn}}\left(\frac{\rho_0-\rho_1}{2}\right)\frac{\rho_0-\rho_1}{2}\right)\right| \\
\le& \sum_{j\in\Lambda_-} \left|\lambda_j \mathrm{sgn}(\lambda_j) - \lambda_j P_{d'}^{\mathrm{sgn}}(\lambda_j)\right| + \sum_{j\in\Lambda_0} \left|\lambda_j \mathrm{sgn}(\lambda_j) - \lambda_j P_{d'}^{\mathrm{sgn}}(\lambda_j)\right| + \sum_{j\in\Lambda_+} \left|\lambda_j \mathrm{sgn}(\lambda_j) - \lambda_j P_{d'}^{\mathrm{sgn}}(\lambda_j)\right| \\
\le& \sum_{j\in\Lambda_-} |\lambda_j| \cdot \left|-1 - P_{d'}^{\mathrm{sgn}}(\lambda_j)\right| + \sum_{j\in\Lambda_0} \left|\lambda_j \mathrm{sgn}(\lambda_j) - \lambda_j P_{d'}^{\mathrm{sgn}}(\lambda_j)\right| + \sum_{j\in\Lambda_+} |\lambda_j| \cdot \left|1 - P_{d'}^{\mathrm{sgn}}(\lambda_j)\right| \\
\le& \sum_{j\in\Lambda_-} |\lambda_j| C_{\mathrm{sgn}}\epsilon + \sum_{j\in\Lambda_0} 2|\lambda_j| + \sum_{j\in\Lambda_+} |\lambda_j| C_{\mathrm{sgn}}\epsilon \\
\le& 2C_{\mathrm{sgn}}\epsilon + 2^{r+1}\delta.
\end{aligned}
$$

Here, the third line owes to the triangle inequality, the fourth line applies the sign function, the fifth line is guaranteed by Corollary 3.6, and the last line is because $\sum_j |\lambda_j| = \mathrm{td}(\rho_0,\rho_1) \le 1$ and $\mathrm{rank}\left(\frac{\rho_0-\rho_1}{2}\right)$ is at most 2^r.

We then bound the error caused by space-efficient QSVT implementation in Corollary 3.15:

$$
\begin{aligned}
&\left|\left(\mathrm{Tr}(\hat{\Pi}_0\rho_0) - \mathrm{Tr}(\hat{\Pi}_0\rho_1)\right) - \left(\mathrm{Tr}(\tilde{\Pi}_0\rho_0) - \mathrm{Tr}(\tilde{\Pi}_0\rho_1)\right)\right| \\
=&\left|\mathrm{Tr}\left(P_{d'}^{\mathrm{sgn}}\left(\frac{\rho_0-\rho_1}{2}\right)\frac{\rho_0-\rho_1}{2}\right) - \mathrm{Tr}\left((\langle\bar{0}|\otimes I_r)U_{\mathrm{HH}}(|\bar{0}\rangle\otimes I_r)\frac{\rho_0-\rho_1}{2}\right)\right| \\
\le&\left\|P_{d'}^{\mathrm{sgn}}\left(\frac{\rho_0-\rho_1}{2}\right) - (\langle\bar{0}|\otimes I_r)U_{\mathrm{HH}}(|\bar{0}\rangle\otimes I_r)\right\| \cdot \mathrm{td}(\rho_0,\rho_1) \\
\le& (36\hat{C}_{\mathrm{sgn}} + 37)\epsilon \cdot 1.
\end{aligned}
$$

Here, the third line is due to a matrix Hölder inequality (e.g., Corollary IV.2.6 in [Bha96]) and the last line is guaranteed by Corollary 3.15.

Combining the above error bounds caused by Corollary 3.6 and Corollary 3.15, respectively,

we obtain the following under the aforementioned choice of parameters:

$$\begin{aligned}
&\left|\mathrm{td}(\rho_0,\rho_1) - \left(\mathrm{Tr}(\tilde{\Pi}_0\rho_0) - \mathrm{Tr}(\tilde{\Pi}_0\rho_1)\right)\right| \\
\leq &\left|\mathrm{td}(\rho_0,\rho_1) - \left(\mathrm{Tr}(\hat{\Pi}_0\rho_0) - \mathrm{Tr}(\hat{\Pi}_0\rho_1)\right)\right| + \left|\left(\mathrm{Tr}(\hat{\Pi}_0\rho_0) - \mathrm{Tr}(\hat{\Pi}_0\rho_1)\right) - \left(\mathrm{Tr}(\tilde{\Pi}_0\rho_0) - \mathrm{Tr}(\tilde{\Pi}_0\rho_1)\right)\right| \\
\leq &2C_{\mathrm{sgn}}\epsilon + 2^{r+1}\delta + (36\hat{C}_{\mathrm{sgn}} + 37)\epsilon \cdot 1 \\
\leq &\varepsilon.
\end{aligned}$$

Complexity analysis. We complete the proof by analyzing the computational complexity of our algorithm. According to Corollary 3.15, our algorithm specified in Figure 3 requires $O(n)$ qubits and $O(d^2) \leq \tilde{O}(2^{2r}/\varepsilon^2) \leq 2^{O(n)}$ queries to Q_0 and Q_1. In addition, the circuit description of our algorithm can be computed in deterministic time $\tilde{O}(d^{9/2}/\varepsilon) = \tilde{O}(2^{4.5r}/\varepsilon^{5.5}) \leq 2^{O(n)}$. □

5.2 A slightly improved upper bound for QSZK: Proof of Theorem 5.4

We start by presenting the quantum interactive proof protocol used in Theorem 5.4, as shown in Protocol 5. This protocol draws inspiration from [Wat02, Figure 2], and the honest prover now employs the algorithmic Holevo-Helstrom measurement $\{\tilde{\Pi}_0, \tilde{\Pi}_1\}$ from Theorem 5.3, rather than the optimal measurement $\{\Pi_0, \Pi_1\}$ in Equation (5.1) as per Theorem 5.2.

Protocol 5: Two-message protocol for GapQSD with a quantum linear-space prover.

1. The verifier $\mathcal{V}$ first chooses $b \in \{0,1\}$ uniformly at random. Subsequently, $\mathcal{V}$ applies Q_b to $|0^n\rangle$, and trace out all non-output qubits. The resulting state ρ_b in the remaining qubits is then sent to the prover $\mathcal{P}$;
2. The prover $\mathcal{P}$ measures the received state ρ using the algorithmic Holevo-Helstrom measurement $\{\tilde{\Pi}_0, \tilde{\Pi}_1\}$ specified in Theorem 5.3. Let $\hat{b}$ be the measurement outcome, specifically, the outcome is $\hat{b}$ if the measurement indicates ρ is $\rho_{\hat{b}}$, with $\hat{b} \in \{0,1\}$. $\mathcal{P}$ then sends $\hat{b}$ to $\mathcal{V}$;
3. The verifier $\mathcal{V}$ accepts if $b = \hat{b}$; otherwise $\mathcal{V}$ rejects.

Following that, we delve into the analysis of Protocol 5:

Proof of Theorem 5.4. Note that $\Pr\left[\hat{b} = a' | b = a\right]$ denotes the probability that the prover $\mathcal{P}$ uses a two-outcome measurement $\{\Pi'_0, \Pi'_1\}$, which is arbitrary in general, to measure the state ρ_a, resulting in the measurement outcome a' for $a, a' \in \{0,1\}$. We then derive the corresponding acceptance probability of Protocol 5:

$$\Pr\left[b = \hat{b}\right] = \frac{1}{2}\Pr\left[\hat{b} = 0 | b = 0\right] + \frac{1}{2}\Pr\left[\hat{b} = 1 | b = 1\right] = \frac{1}{2} + \frac{1}{2}\left(\mathrm{Tr}(\Pi'_0\rho_0) - \mathrm{Tr}(\Pi'_0\rho_1)\right). \tag{5.2}$$

For *yes* instances where $\mathrm{td}(\rho_0,\rho_1) \geq \alpha(n)$, considering that the prover $\mathcal{P}$ is honest, we have

$$\begin{aligned}
\Pr\left[b = \hat{b}\right] &= \tfrac{1}{2} + \tfrac{1}{2}\left(\mathrm{Tr}(\tilde{\Pi}_0\rho_0) - \mathrm{Tr}(\tilde{\Pi}_0\rho_1)\right) \\
&\geq \tfrac{1}{2} + \tfrac{1}{2}\left(\mathrm{Tr}(\Pi_0\rho_0) - \mathrm{Tr}(\Pi_0\rho_1)\right) - \left|\tfrac{1}{2}\left(\mathrm{Tr}(\Pi_0\rho_0) - \mathrm{Tr}(\Pi_0\rho_1)\right) - \tfrac{1}{2}\left(\mathrm{Tr}(\tilde{\Pi}_0\rho_0) - \mathrm{Tr}(\tilde{\Pi}_0\rho_1)\right)\right| \\
&= \tfrac{1}{2} + \tfrac{1}{2}\mathrm{td}(\rho_0,\rho_1) - \left|\tfrac{1}{2}\mathrm{td}(\rho_0,\rho_1) - \tfrac{1}{2}\left(\mathrm{Tr}(\tilde{\Pi}_0\rho_0) - \mathrm{Tr}(\tilde{\Pi}_0\rho_1)\right)\right| \\
&\geq \tfrac{1}{2} + \tfrac{1}{2}\left(\alpha(n) - 2^{-n}\right).
\end{aligned}$$

Here, the first line follows Equation (5.2), the second line owes to the triangle equality and the fact that $\mathrm{Tr}(\tilde{\Pi}_0\rho_0) - \mathrm{Tr}(\tilde{\Pi}_0\rho_1) > 0$,[49] the third line is because of Theorem 5.2 and Equation (5.1), and the last line uses Theorem 5.3. Hence, we have the completeness $c(n) = \frac{1}{2} + \frac{1}{2}\left(\alpha(n) - 2^{-n}\right)$.

[49] This is because the difference between $\mathrm{Tr}(\Pi_0\rho_0) - \mathrm{Tr}(\Pi_0\rho_1)$ and $\mathrm{Tr}(\tilde{\Pi}_0\rho_0) - \mathrm{Tr}(\tilde{\Pi}_0\rho_1)$ is much smaller than $\mathrm{Tr}(\Pi_0\rho_0) - \mathrm{Tr}(\Pi_0\rho_1) = \mathrm{td}(\rho_0,\rho_1) \geq \alpha(n)$, guaranteeing by the parameters chosen in Theorem 5.3.

For *no* instances where $\mathrm{td}(\rho_0, \rho_1) \leq \beta(n)$, we obtain the following from Equation (5.2):

$$\Pr\left[b = \hat{b}\right] = \frac{1}{2} + \frac{1}{2}\left(\mathrm{Tr}(\Pi'_0\rho_0) - \mathrm{Tr}(\Pi'_0\rho_1)\right) \leq \frac{1}{2} + \frac{1}{2}\mathrm{td}(\rho_0, \rho_1) \leq \frac{1}{2}\left(1 + \beta(n)\right) \coloneqq s(n).$$

Here, the first inequality is guaranteed by the Holevo-Helstrom bound (Theorem 5.2).

Therefore, since the honest prover (for *yes* instances) utilizes the algorithmic Holevo-Helstrom measurement $\{\tilde{\Pi}_0, \tilde{\Pi}_1\}$, the optimal prover strategy aligned with Protocol 5 is indeed implementable in quantum single-exponential time and linear space due to Theorem 5.3.

Error reduction for Protocol 5. Note that the class $\mathsf{QIP}(2)$ consists of two-message quantum interactive proof systems with completeness $c \geq 2/3$ and soundness $s \leq 1/3$ [JUW09, Section 3.1]. We aim to reduce the completeness and soundness errors in Protocol 5.

Following [JUW09, Section 3.2], we can achieve this task by a standard parallel repetition of Protocol 5. Specifically, we define new verifier $\mathcal{V}'$ and honest prover $\mathcal{P}'$ such that for any polynomial-bounded function $l(n)$, the resulting two-message quantum interactive proof system has completeness $c'(n) \geq 1 - 2^{-l(n)}$ and soundness $s'(n) \leq 2^{-l(n)}$. Let $c(n) - s(n) \geq 1/q(n)$ for some polynomially-bounded function q and define $[l] \coloneqq \{1, \cdots, l\}$, a description of V' follows:

(1) Let $s \coloneqq 2lq$ and $t \coloneqq 8lq^2s$. Run st independent and parallel executions of Protocol 5 for $\mathcal{V}'$, one for each pair (i, j) with $i \in [s]$ and $j \in [t]$. Measure the output qubit for each execution, and let the measurement outcome for execution (i, j) be denoted by $y_{i,j} \in \{0, 1\}$.

(2) For each $i \in [s]$, set $z_i \coloneqq \begin{cases} 1, & \text{if } \sum_{j=1}^{t} y_{i,j} \geq t \cdot \frac{a+b}{2} \\ 0, & \text{otherwise .} \end{cases}$

(3) $\mathcal{V}'$ accepts if $\wedge_{i=1}^{s} z_i = 1$; otherwise, it rejects.

The correctness of $\mathcal{V}'$ directly follows from [JUW09, Section 3.2].

We now analyze the complexity of the honest prover $\mathcal{P}'$. Since st independent and parallel executions of Protocol 5 can be viewed as discriminating st pairs of quantum states $(\rho_0^{(j)}, \rho_1^{(j)})$ for $1 \leq j \leq st$, the total input length of the quantum circuits to independently and parallelly prepare the states $\rho_b^{(1)}, \cdots, \rho_b^{(st)}$ for $b \in \{0, 1\}$ is $n \cdot st = 16nl^2(n)q^3(n) \leq O(n^c) \coloneqq n'$ for some constant c. Replacing n with n', the space complexity of the honest prover $\mathcal{P}'$ is still $O(n')$.

Lastly, we completely the proof by choosing an appropriate $r(n)$ such that the completeness $c(n') \geq 1 - 2^{-r^{1/c}(n')} \geq 2/3$ and the soundness $s(n') \leq 2^{-r^{1/c}(n')} \leq 1/3$. □

Acknowledgments

This work was partially supported by MEXT Q-LEAP grant No. JPMXS0120319794. FLG was also supported by JSPS KAKENHI grants Nos. JP19H04066, JP20H05966, JP20H00579, JP20H04139, and JP21H04879. YL was also supported by JST, the establishment of University fellowships towards the creation of science technology innovation, Grant No. JPMJFS2125. We express our gratitude to anonymous reviewers for providing detailed suggestions on the space-efficient quantum singular value transformation, particularly improved norm bounds for the coefficient vector in Lemma 2.15 (and consequently Lemma 3.3) by leveraging the smoothness property of functions, and for suggesting to add discussion on space-bounded distribution testing. Circuit diagrams were drawn by the Quantikz package [Kay18].

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A Omitted proofs in space-efficient QSVT

In this section, we will present all Omitted proofs in Section 3.

A.1 Space-efficient bounded polynomial approximations

A.1.1 Omitted proofs in Corollary 3.6

Fact 3.6.2. *Let* $F_k(\theta) = \operatorname{erf}(\kappa\cos\theta)\cos(k\theta)$, $\max_{0\le k\le d'}\max_{\xi\in[-\pi,0]}|F_k''(\xi)| \le \frac{2}{\sqrt{\pi}}\kappa + k^2 + \frac{4}{\sqrt{\pi}}\kappa^3 + \frac{4}{\sqrt{\pi}}k\kappa$.

Proof. Through a straightforward calculation, we have derived that

$$\begin{aligned}|F_k''(\theta)| =& \frac{2}{\sqrt{\pi}}\left|\kappa\exp(-\kappa^2\cos^2\theta)\cos\theta\cos(\kappa\theta)\right| + \left|k^2\cos(k\theta)\operatorname{erf}(\kappa\cos(\theta))\right| \\ &+ \frac{4}{\sqrt{\pi}}\left|\kappa^3\exp(-\kappa^2\cos^2\theta)\cos\theta\cos(k\theta)\sin^2\theta\right| \\ &+ \frac{4}{\sqrt{\pi}}\left|k\kappa\exp(-\kappa^2\cos^2\theta)\sin\theta\sin(k\theta)\right| \\ \le& \frac{2}{\sqrt{\pi}}\kappa + k^2 + \frac{4}{\sqrt{\pi}}\kappa^3 + \frac{4}{\sqrt{\pi}}k\kappa.\end{aligned} \tag{A.1}$$

The last line owes to the facts that $|\operatorname{erf}(x)| \le 1$, $\exp(-x^2) \le 1$, $|\sin x| \le 1$, and $|\cos x| \le 1$ for any x. We complete the proof by noting that Equation (A.1) holds for any $0 \le k \le d'$. □

A.1.2 Omitted proofs in Lemma 3.8

Proposition 3.8.1 (First approximation)**.** *Let* $\hat{f}_1(x) \coloneqq \sum_{k=0}^{K} a_k x^k$ *such that* $\|f - \hat{f}_1\|_{\mathcal{I}_\delta} \le \epsilon/4$. *Then we know that* $\hat{f}_1(x) = \sum_{k=0}^{K} a_k \sum_{l=0}^{\infty} b_l^{(k)} \sin^l\left(\frac{x\pi}{2}\right)$ *where the coefficients* $b_l^{(k)}$ *satisfy that*

$$b_l^{(k+1)} = \sum_{l'=0}^{l} b_{l'}^{(k)} b_{l-l'}^{(1)} \text{ where } b_l^{(1)} = \begin{cases} 0 & \text{if } l \text{ is even,} \\ \binom{l-1}{\frac{l-1}{2}}\frac{2^{-l+1}}{l}\cdot\frac{2}{\pi} & \text{if } l \text{ is odd.} \end{cases} \tag{3.4}$$

Furthermore, the coefficients $\{b_l^{(k)}\}$ *satisfies the following:* (1) $\|\mathbf{b}^{(k)}\|_1 = 1$ *for all* $k \ge 1$; (2) $\mathbf{b}^{(k)}$ *is entry-wise non-negative for all* $k \ge 1$; (3) $b_l^{(k)} = 0$ *if* l *and* k *have different parities.*

Proof. We construct a Fourier series by a linear combination of the power of sines. We first note that $x = \frac{2}{\pi}\cdot\arcsin\left(\sin\left(\frac{x\pi}{2}\right)\right)$ for all $x \in [-1,1]$, and plug it into $\hat{f}_1(x) \coloneqq \sum_{k=0}^{K} a_k x^k$, which deduces that $\|f - \hat{f}_1\|_{\mathcal{I}_\delta} \le \epsilon/4$ by the assumption. Let $\mathbf{b}^{(k)}$ be the coefficients of $\left(\frac{\arcsin y}{\pi/2}\right)^k = \sum_{l=0}^{\infty} b_l^{(k)} y^l$ for all $y \in [-1,1]$, then we result in our first approximation. Moreover, we observe that $\frac{\pi}{2}\cdot\mathbf{b}^{(1)}$ is exactly the Taylor series of arcsin, whereas we know that $\left(\frac{\arcsin y}{\pi/2}\right)^{k+1} = \left(\frac{\arcsin y}{\pi/2}\right)^k \cdot \left(\sum_{l=0}^{\infty} b_l^{(1)} y^l\right)$ for $k > 1$, which derives Equation (3.4) by comparing the coefficients. In addition, notice that $\|\mathbf{b}^{(k)}\|_1 = \sum_{l=0}^{\infty} b_l^{(k)} 1^l = \left(\frac{\arcsin 1}{\pi/2}\right)^k = 1$, together with straightforward reasoning follows from Equation (3.4), we deduce the desired property for $\{b_l^{(k)}\}$. □

Proposition 3.8.2 (Second approximation)**.** *Let* $\hat{f}_2(x) \coloneqq \sum_{k=0}^{K} a_k \sum_{l=0}^{L} b_l^{(k)} \sin^l\left(\frac{x\pi}{2}\right)$ *where* $L \coloneqq \lceil \delta^{-2}\ln(4\|\mathbf{a}\|_1\epsilon^{-1})\rceil$, *then we have that* $\|\hat{f}_1 - \hat{f}_2\|_{\mathcal{I}_\delta} \le \epsilon/4$.

Proof. We truncate the summation over l in $f_1(x)$ at $l = L$, and it suffices to bound the truncation error. For all $k \in \mathbb{N}$ and $x \in [-1+\delta, 1-\delta]$, we obtain the error bound:

$$\left|\sum_{l=\lfloor L\rfloor}^{\infty} b_l^{(k)} \sin^l\left(\tfrac{x\pi}{2}\right)\right| \le \sum_{l=\lfloor L\rfloor}^{\infty} b_l^{(k)}\left|\sin^l\left(\tfrac{x\pi}{2}\right)\right| \le \sum_{l=\lfloor L\rfloor}^{\infty} b_l^{(k)}|1-\delta^2|^l \le (1-\delta^2)^L \sum_{l=\lfloor L\rfloor}^{\infty} b_l^{(k)} \le (1-\delta^2)^L.$$

Here, the second inequality owing to $\forall\delta \in [0,1]$, $\sin\left((1-\delta)\frac{\pi}{2}\right) \le 1-\delta^2$, and the last inequality is due to $\|\mathbf{b}^{(k)}\|_1 = 1$ in Proposition 3.8.1. By appropriately choosing $L \coloneqq \delta^{-2}\ln(4\|\mathbf{a}\|_1\epsilon^{-1})$, we obtain that $\|\hat{f}_1 - \hat{f}_2\|_{\mathcal{I}_\delta} \le \sum_{k=0}^{K} a_k(1-\delta^2)^L \le \|\mathbf{a}\|_1 \cdot \exp(-\delta^2 L) \le \epsilon/4$. □

Proposition 3.8.3 (Third approximation)**.** *Let $\hat{f}_3(x)$ be polynomial approximations of f that depends on the parity of f such that $\|\hat{f}_2-\hat{f}_3\|_{\mathcal{I}_\delta}\le\epsilon/2$ and $M=\lfloor\delta^{-1}\ln(4\|\mathbf{a}\|_1\epsilon^{-1})\rfloor$, then we have*

$$\hat{f}_3^{(\text{even})}(x) \coloneqq \sum_{k=0}^{K} a_k \sum_{\hat{l}=0}^{L/2} (-1)^{\hat{l}} 2^{-2\hat{l}} b_{2\hat{l}}^{(k)} \sum_{m'=\hat{l}-M}^{\hat{l}+M} (-1)^{m'} \binom{2\hat{l}}{m'} \cos(\pi x(m'-\hat{l})),$$

$$\hat{f}_3^{(\text{odd})}(x) \coloneqq \sum_{k=0}^{K} a_k \sum_{\hat{l}=0}^{(L-1)/2} (-1)^{\hat{l}+1} 2^{-2\hat{l}-1} b_{2\hat{l}+1}^{(k)} \sum_{m'=\hat{l}+1-M}^{\hat{l}+1+M} (-1)^{m'} \binom{2\hat{l}+1}{m'} \sin\big(\pi x\big(m'-\hat{l}-\tfrac{1}{2}\big)\big).$$

Therefore, we have that $\hat{f}_3(x)\coloneqq\hat{f}_3^{(\text{even})}(x)$ if f is even, whereas $\hat{f}_3(x)\coloneqq\hat{f}_3^{(\text{odd})}(x)$ if f is odd. In addition, if f is neither even or odd, then $\hat{f}_3(x)\coloneqq\hat{f}_3^{(\text{even})}(x)+\hat{f}_3^{(\text{odd})}(x)$.

Proof. We upper-bound $\sin^l(x)$ in $\hat{f}_2(x)$ defined in Proposition 3.8.2 using a tail bound of binomial coefficients. We obtain that $\sin^l(z)=\big(\frac{e^{-\mathrm{i}z}-e^{\mathrm{i}z}}{-2\mathrm{i}}\big)^l=\big(\frac{\mathrm{i}}{2}\big)^l\sum_{m=0}^{l}\exp(\mathrm{i}z(2m-l))$ by a direct calculation, which implies the counterpart for real-valued functions:

$$\sin^l(z)=\begin{cases}2^{-l}(-1)^{(l+1)/2}\sum_{m'=0}^{l}(-1)^{m'}\binom{l}{m'}\sin(z(2m'-l)), & \text{if } l \text{ is odd;}\\ 2^{-l}(-1)^{l/2}\sum_{m'=0}^{l}(-1)^{m'}\binom{l}{m'}\cos(z(2m'-l)), & \text{if } l \text{ is even.}\end{cases} \tag{A.2}$$

Recall that the Chernoff bound (e.g., Corollary A.1.7 [AS16]) which corresponds a tail bound of binomial coefficients, and assume that $l\le L$, we have derived that:

$$\sum_{m'=0}^{\lfloor l/2\rfloor-M} 2^{-l}\binom{l}{m'} = \sum_{m'=\lceil l/2\rceil+M}^{l} 2^{-l}\binom{l}{m'} \le e^{-\frac{2M^2}{l}} \le e^{-\frac{2M^2}{L}} \le \Big(\frac{\epsilon}{4\|a\|_1}\Big)^2 \le \frac{\epsilon}{4\|a\|_1}. \tag{A.3}$$

Here, we choose $M=\lceil\delta^{-1}\ln(4\|\mathbf{a}\|_1\epsilon^{-1})\rceil$, and the last inequality is because of the assumption $\epsilon\le 2\|\mathbf{a}\|_1$. As stated in Proposition 3.8.1, $b_l^{(k)}=0$ if k and l have different parities. Consequently, we only need to consider all odd (resp., even) $l\le L$ for odd (resp., even) functions. If the function f is neither even nor odd, we must consider all $l\le L$. Plugging Equation A.3 into Equation A.2, we can derive that:

$$\begin{aligned}&\text{If } l \text{ is odd, } \Big\|\sin^l(z)-2^{-l}(-1)^{(l+1)/2}\sum_{m'=(l+1)/2-M}^{(l+1)/2+M}(-1)^{m'}\tbinom{l}{m'}\sin(z(2m'-l))\Big\|_{\mathcal{I}_\delta}\le\frac{\epsilon}{2\|\mathbf{a}\|_1};\\ &\text{If } l \text{ is even, } \Big\|\sin^l(z)-2^{-l}(-1)^{l/2}\sum_{m'=l/2-M}^{l/2+M}(-1)^{m'}\tbinom{l}{m'}\cos(z(2m'-l))\Big\|_{\mathcal{I}_\delta}\le\frac{\epsilon}{2\|\mathbf{a}\|_1};\end{aligned} \tag{A.4}$$

Plugging Equation (A.4) into $\hat{f}_2(x)$, and substituting $z=x\pi/2$, this equation leads to $\hat{f}_3(x)$ as desired. In addition, combining $\sum_{k=0}^{K}|a_k|\sum_{l=0}^{\lfloor L\rfloor}|b_l^{(k)}|\le\sum_{k=0}^{K}|a_k|=\|\mathbf{a}\|_1$ with Equation (A.4), we achieve that $\|\hat{f}_2-\hat{f}_3\|_{\mathcal{I}_\delta}\le\epsilon/2$. □

A.1.3 Omitted proof in Theorem 3.7

Fact 3.7.2. *Consider the integrand $F_k(\theta)=\sum_{m=-M}^{M}\frac{c_m}{2}\big(H_{k,m}^{(+)}-H_{k,m}^{(-)}\big)$ for any function f which is either even or odd. If f is even, we have that $c_m=c_m^{(\text{even})}$ defined in Lemma 3.8, and*

$$H_{k,m}^{(\pm)}(\theta)\coloneqq\cos\Big(\pi m\Big(\frac{\cos\theta-x_0}{r+\delta}\Big)\Big)\cdot\cos(k\theta)\cdot\operatorname{erf}\Big(\kappa\Big(\cos\theta-x_0\pm r\pm\frac{\delta}{4}\Big)\Big). \tag{3.9}$$

Likewise, if f is odd, we know that $c_m=c_m^{(\text{odd})}$ defined in Lemma 3.8, and

$$H_{k,m}^{(\pm)}(\theta)\coloneqq\sin\Big(\pi\Big(m+\frac{1}{2}\Big)\Big(\frac{\cos\theta-x_0}{r+\delta}\Big)\Big)\cdot\cos(k\theta)\cdot\operatorname{erf}\Big(\kappa\Big(\cos\theta-x_0\pm r\pm\frac{\delta}{4}\Big)\Big). \tag{3.10}$$

Moreover, the integrand is $F_k(\theta)=\sum_{m=-M}^{M}\Big(\frac{c_m^{(\text{even})}}{2}\big(\hat{H}_{k,m}^{(+)}-\hat{H}_{k,m}^{(-)}\big)+\frac{c_m^{(\text{odd})}}{2}\big(\tilde{H}_{k,m}^{(+)}-\tilde{H}_{k,m}^{(-)}\big)\Big)$ when f is neither even nor odd, where $\hat{H}_{k,m}^{(\pm)}$ and $\tilde{H}_{k,m}^{(\pm)}$ follow from Equation (3.9) and Equation (3.10),

respectively. Regardless of the parity of f, we have that the second derivative $F_k''(\theta) \le O(Bd^3)$.

Proof. We begin by deriving an upper bound of the second derivative of the integrand $F_k(\theta)$:

$$|F_k''(\theta)| \le \sum_{m=-M}^{M} \frac{c_m}{2} \left| \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} H_{k,m}^{(+)}(\theta) - \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} H_{k,m}^{(-)}(\theta) \right| \le \frac{\|\mathbf{c}\|}{2} \max_{-\pi \le \theta \le 0} \left(\left| \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} H_{k,m}^{(+)}(\theta) \right| + \left| \frac{\mathrm{d}^2}{\mathrm{d}\theta^2} H_{k,m}^{(-)}(\theta) \right| \right). \quad \text{(A.5)}$$

By a straightforward calculation, we have the second derivatives of $H_{k,m}^{\pm}(\theta)$ if f is even:

$$\begin{aligned}
\tfrac{\mathrm{d}^2}{\mathrm{d}\theta^2} H_{k,m}^{(\pm)}(\theta) = &- k^2 \cos(k\theta) \cos\big(\tfrac{\pi m(\cos\theta - x_0)}{\delta + r}\big) \operatorname{erf}\big(\kappa\big(\cos\theta - x_0 \mp r \mp \tfrac{\delta}{4}\big)\big) \\
&- \tfrac{\pi^2 m^2}{(\delta+r)^2} \sin^2(\theta) \cos(k\theta) \cos\big(\tfrac{\pi m(\cos\theta - x_0)}{\delta + r}\big) \operatorname{erf}\big(\kappa\big(\cos\theta - x_0 \mp r \mp \tfrac{\delta}{4}\big)\big) \\
&+ \tfrac{\pi m}{\delta + r} \cos\theta \cos(k\theta) \sin\big(\tfrac{\pi m(\cos\theta - x_0)}{\delta + r}\big) \operatorname{erf}\big(\kappa\big(\cos\theta - x_0 \mp r \mp \tfrac{\delta}{4}\big)\big) \\
&- \tfrac{2\pi k m}{\delta + r} \sin(\theta) \sin(k\theta) \sin\big(\tfrac{\pi m(\cos\theta - x_0)}{\delta + r}\big) \operatorname{erf}\big(\kappa\big(\cos\theta - x_0 \mp r \mp \tfrac{\delta}{4}\big)\big) \\
&- \tfrac{2\kappa}{\sqrt{\pi}} \cos\theta \cos(k\theta) \cos\big(\tfrac{\pi m(\cos\theta - x_0)}{\delta + r}\big) e^{-\kappa^2 \left(\cos\theta - x_0 \mp r \mp \frac{\delta}{4}\right)^2} \\
&- \tfrac{4\sqrt{\pi}\kappa m}{\delta + r} \sin^2(\theta) \cos(k\theta) \sin\big(\tfrac{\pi m(\cos\theta - x_0)}{\delta + r}\big) e^{-\kappa^2 \left(\cos\theta - x_0 \mp r \mp \frac{\delta}{4}\right)^2} \\
&+ \tfrac{4\kappa k}{\sqrt{\pi}} \sin(\theta) \sin(k\theta) \cos\big(\tfrac{\pi m(\cos\theta - x_0)}{\delta + r}\big) e^{-\kappa^2 \left(\cos\theta - x_0 \mp r \mp \frac{\delta}{4}\right)^2} \\
&- \tfrac{4\kappa^3}{\sqrt{\pi}} \sin^2(\theta) \cos(k\theta) \cos\big(\tfrac{\pi m(\cos\theta - x_0)}{\delta + r}\big) \big(\cos\theta - x_0 \mp r \mp \frac{\delta}{4}\big) e^{-\kappa^2 \left(\cos\theta - x_0 \mp r \mp \frac{\delta}{4}\right)^2}.
\end{aligned}$$

Note that all functions appear in $\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} H_{k,m}^{(\pm)}(\theta)$, viz. $\sin x$, $\cos x$, $\exp(-x^2)$, and $\operatorname{erf}(x)$, are at most 1, as well as $|x_0 \pm r \pm \delta/4| \le 7/2$, then we obtain that

$$\begin{aligned}
\left|\tfrac{\mathrm{d}^2}{\mathrm{d}\theta^2} H_{k,m}^{(\pm)}(\theta)\right| &\le k^2 + \tfrac{2\kappa}{\sqrt{\pi}} + \tfrac{4\kappa k}{\sqrt{\pi}} + \tfrac{18\kappa^3}{\sqrt{\pi}} + m \cdot \big(\tfrac{\pi}{\delta + r} + \tfrac{2\pi k}{\delta + r} + \tfrac{4\sqrt{\pi}\kappa}{\delta + r}\big) + m^2 \cdot \tfrac{\pi^2}{(\delta+r)^2} \\
&\le (d')^2 + O(d) + O(d^2) + O(d^3) + \tfrac{M}{\delta + r} \cdot (O(1) + O(d) + O(d)) + M^2 \cdot \tfrac{O(1)}{(\delta+r)^2} \quad \text{(A.6)} \\
&= O(d^3).
\end{aligned}$$

Here, the second line according to $k \le d' = 2d - 1$ and $\kappa \le O(d)$, also the last line is due to facts that $M \le O(rd)$ and $1/2 \le r/(\delta + r) \le 1$ if $0 < \delta \le r$ and $0 < r \le 2$. Additionally, a similar argument shows that the upper bound in Equation (A.6) applies to odd functions and functions that are neither even nor odd as well. This is because a direct computation yields the second derivatives of $H_{k,m}^{(\pm)}(\theta)$ when f is odd:

$$\begin{aligned}
\tfrac{\mathrm{d}^2}{\mathrm{d}\theta^2} H_{k,m}^{(\pm)}(\theta) = &- k^2 \cos(kx) \sin\big(\tfrac{\pi\left(m + \frac{1}{2}\right)(\cos(x) - x_0)}{\delta + r}\big) \operatorname{erf}\big(\kappa\big(\cos(x) - x_0 \mp r \mp \tfrac{\delta}{4}\big)\big) \\
&- \tfrac{\pi\left(m + \frac{1}{2}\right)}{\delta + r} \cos(x) \cos(kx) \cos\big(\tfrac{\pi\left(m + \frac{1}{2}\right)(\cos(x) - x_0)}{\delta + r}\big) \operatorname{erf}\big(\kappa\big(\cos(x) - x_0 \mp r \mp \tfrac{\delta}{4}\big)\big) \\
&- \tfrac{\pi^2\left(m + \frac{1}{2}\right)^2}{(\delta + r)^2} \sin^2(x) \cos(kx) \sin\big(\tfrac{\pi\left(m + \frac{1}{2}\right)(\cos(x) - x_0)}{\delta + r}\big) \operatorname{erf}\big(\kappa\big(\cos(x) - x_0 \mp r \mp \tfrac{\delta}{4}\big)\big) \\
&+ \tfrac{2\pi k\left(m + \frac{1}{2}\right)}{\delta + r} \sin(x) \sin(kx) \cos\big(\tfrac{\pi\left(m + \frac{1}{2}\right)(\cos(x) - x_0)}{\delta + r}\big) \operatorname{erf}\big(\kappa\big(\cos(x) - x_0 \mp r \mp \tfrac{\delta}{4}\big)\big) \\
&+ \tfrac{4\sqrt{\pi}\kappa\left(m + \frac{1}{2}\right)}{\delta + r} \sin^2(x) \cos(kx) \cos\big(\tfrac{\pi\left(m + \frac{1}{2}\right)(\cos(x) - x_0)}{\delta + r}\big) e^{-\kappa^2 \left(\cos(x) - x_0 \mp r \mp \frac{\delta}{4}\right)^2} \\
&- \tfrac{2\kappa}{\sqrt{\pi}} \cos(x) \cos(kx) \sin\big(\tfrac{\pi\left(m + \frac{1}{2}\right)(\cos(x) - x_0)}{\delta + r}\big) e^{-\kappa^2 \left(\cos(x) - x_0 \mp r \mp \frac{\delta}{4}\right)^2} \\
&+ \tfrac{4\kappa k}{\sqrt{\pi}} \sin(x) \sin(kx) \sin\big(\tfrac{\pi\left(m + \frac{1}{2}\right)(\cos(x) - x_0)}{\delta + r}\big) e^{-\kappa^2 \left(\cos(x) - x_0 \mp r \mp \frac{\delta}{4}\right)^2} \\
&- \tfrac{4\kappa^3}{\sqrt{\pi}} \sin^2(x) \cos(kx) \big(\cos(x) - x_0 \mp r \mp \tfrac{\delta}{4}\big) \sin\big(\tfrac{\pi\left(m + \frac{1}{2}\right)(\cos(x) - x_0)}{\delta + r}\big) e^{-\kappa^2 \left(\cos(x) - x_0 \mp r \mp \frac{\delta}{4}\right)^2}.
\end{aligned}$$

Substituting Equation (A.6) into Equation (A.5), and noticing that the coefficient vector $\|\mathbf{c}^{(\text{even})} + \mathbf{c}^{(\text{odd})}\|_1 \le B$ regardless of the parity of f, we conclude that $|F_k''(\theta)| \le O(Bd^3)$. □

A.2 Applying arbitrary polynomials of bitstring indexed encodings

Proposition 3.13.1 (Space-efficient state preparation, adapted from [Zal98, KM01, GR02])**.** *Given an l-qubit quantum state $|\psi\rangle \coloneqq \sum_{i=1}^{m} \sqrt{\hat{y}_i}|i\rangle$, where $l = \lceil \log m \rceil$ and $\hat{y}_i$ are real amplitudes associated with an evaluation oracle $\mathrm{Eval}(i, \varepsilon)$ that returns $\hat{y}_i$ up to accuracy ε we can prepare $|\psi\rangle$ up to accuracy ϵ in deterministic time $\tilde{O}(m^2 \log(m/\epsilon))$ and space $O(\log(m/\epsilon^2))$, together with m^2 evaluation oracle calls with precision $\varepsilon \coloneqq O(\epsilon^2/m)$.*

Proof. We follow the analysis presented in [MP16, Section III.A], with a particular focus on the classical computational complexity required for this state preparation procedure. The algorithm for preparing the state $|\psi\rangle$ expresses the weight W_x as a telescoping product, given by

$$\forall x \in \{0,1\}^l,\ W_x = W_{x_1} \cdot \frac{W_{x_1x_2}}{W_{x_1}} \cdot \frac{W_{x_1x_2x_3}}{W_{x_1x_2}} \cdots \frac{W_x}{W_{x_1\cdots x_{n-1}}} \text{ where } W_x \coloneqq \sum_{y\in\{0,1\}^{l-|x|}} |\langle xy|\psi\rangle|^2. \tag{A.7}$$

To estimate $|\psi\rangle$ up to accuracy ϵ in the ℓ_2 norm, it suffices to approximate each weight W_x up to additive error $\varepsilon \coloneqq O(\epsilon^2/m)$, as indicated in [MP16, Section III.A]. To compute $W_{x'}$, we need $2^{l-|x'|}$ oracle calls to $\mathrm{Eval}(\cdot, \varepsilon)$. Evaluating all terms in Equation (A.7) requires computing $W_{x_1}, W_{x_1x_2}, \cdots, W_x$ for any $x \in \{0,1\}^l$, which can be achieved by $2^{l-1} + 2^{l-2} + \cdots + 1 = 2^l$ oracle calls to $\mathrm{Eval}(\cdot, \varepsilon)$. As we need to compute Equation (A.7) for all $x \in \{0,1\}^l$, the overall number of oracle calls to $\mathrm{Eval}(\cdot, \varepsilon)$ is $2^{2l} = m^2$. The remaining computation can be achieved in deterministic time $\tilde{O}(m^2 \log(m/\epsilon))$ and space $O(\log(m/\epsilon))$ where the time complexity is because of the iterated integer multiplication. □

Lemma 3.14 (Renormalizing bitstring indexed encoding)**.** *Let U be an (α, a, ϵ)-bitstring indexed encoding of A, where $\alpha > 1$ and $0 < \epsilon < 1$, and A is an isometry acting on $s(n)$ qubits. We can implement a quantum circuit V, serving as a normalization of U, such that V is a $(1, a+2, 36\epsilon)$-bitstring indexed encoding of A. This implementation requires $O(\alpha)$ uses of U, $U^\dagger$, $C_\Pi NOT$, $C_{\tilde{\Pi}} NOT$, and $O(\alpha)$ single-qubit gates. Moreover, the description of the resulting quantum circuit can be computed in deterministic time $O(\alpha)$ and space $O(s)$.*

Proof. Following Definition 3.1, we have $\|A - \alpha\tilde{\Pi}U\Pi\| \leq \epsilon$, where $\tilde{\Pi}$ and Π are the corresponding orthogonal projections. Because U is a $(1, a, \epsilon/\alpha)$-bitstring indexed encoding A/α, we obtain that $\|A/\alpha\| \leq \|U\| + \epsilon/\alpha = 1 + \epsilon/\alpha$, equivalently $\|A\| \leq \alpha + \epsilon$.

Adjusting the encoding through a single-qubit rotation. Consider an odd integer $k \coloneqq 2\lceil \pi(\alpha+1)/2 \rceil + 1 \leq 9\alpha = O(\alpha)$ and $\gamma \coloneqq (\alpha + \epsilon)\sin(\pi/2k) \leq 1$. We define new orthogonal projections $\tilde{\Pi}' \coloneqq \tilde{\Pi} \otimes |0\rangle\langle 0|$ and $\Pi' \coloneqq \Pi \otimes |0\rangle\langle 0|$, and combine them with $U' = U \otimes R_\gamma$, where $R_\gamma = \begin{pmatrix} \gamma & -\sqrt{1-\gamma^2} \\ \sqrt{1-\gamma^2} & \gamma \end{pmatrix}$. By noting that $\tilde{\Pi}'U'\Pi' = \gamma\tilde{\Pi}U\Pi \otimes |0\rangle\langle 0|$, we deduce that U' is a $(1, a+1, \gamma\epsilon/\alpha)$-bitstring indexed encoding of $\gamma A/\alpha \otimes |0\rangle\langle 0|$, which is consequently a $(1, a+1, 2\gamma\epsilon/\alpha)$-bitstring indexed encoding of $\sin(2\pi/k) \cdot (A \otimes |0\rangle\langle 0|)$. An error bound follows:

$$\left\| \frac{\gamma}{\alpha} A - \sin\left(\frac{\pi}{2k}\right) A \right\| = \left\| \frac{\epsilon}{\alpha} \sin\left(\frac{\pi}{2k}\right) A \right\| \leq \frac{\epsilon}{\alpha} \sin\left(\frac{\pi}{2k}\right)(\alpha + \epsilon) = \frac{\gamma\epsilon}{\alpha}.$$

Renormalizing the encoding via robust oblivious amplitude amplification. We follow the construction in [GSLW18, Theorem 28], the full version of [GSLW19], and perform a meticulous analysis of the complexity. We observe that it suffices to consider $k \geq 3$, as for U' is already a $(1, a+1, 2\gamma\epsilon/\alpha)$-bitstring indexed encoding of $A \otimes |0\rangle\langle 0|$ when $k = 1$. Let $\varepsilon \coloneqq 2\gamma\epsilon/\alpha$, and for simplicity, we first start by considering the case with $\varepsilon = 0$. By Definition 3.1, we have $\tilde{\Pi}'U'\Pi' = \alpha \sin\left(\frac{\pi}{2k}\right)\tilde{\Pi}U\Pi \otimes |0\rangle\langle 0|$. Let $T_k \in \mathbb{R}[x]$ be the degree-k Chebyshev polynomial (of the first kind). By employing Lemma 3.12, we can apply the QSVT associated with T_k to the bitstring indexed encoding U', yielding:

$$\tilde{\Pi}'T_k^{(\mathrm{SV})}(U')\Pi' = \alpha T_k\Big(\sin\left(\frac{\pi}{2k}\right)\Big)\tilde{\Pi}U\Pi \otimes |0\rangle\langle 0| = \cos\left(\frac{k-1}{2}\pi\right) A \otimes |0\rangle\langle 0| = A \otimes |0\rangle\langle 0|.$$

Here, the second equality is due to $T_k\big(\sin\big(\frac{\pi}{2k}\big)\big) = T_k\big(\cos\big(\frac{\pi}{2}-\frac{\pi}{2k}\big)\big) = \cos\big(\frac{k-1}{2}\pi\big)$, and the last equality holds because k is odd.

Next, we move on the case with $\varepsilon > 0$ and restrict it to $\varepsilon \le 1/3$.[50] Let $A' \coloneqq \tilde{\Pi}'U'\Pi'$ and $\hat{A} \coloneqq \gamma A \otimes |0\rangle\langle 0|$, then we have $\|A'-\hat{A}\| \le \varepsilon$, indicating that $\big\|\frac{A'+\hat{A}}{2}\big\|^2 \le \frac{4}{9} \coloneqq \zeta$[51] and $\|A'-\hat{A}\| + \big\|\frac{A'+\hat{A}}{2}\big\|^2 \le \frac{1}{3}+\frac{4}{9} < 1$. By employing Lemma 3.12, as well as the facts that $\frac{\sqrt{2}}{\sqrt{1-\zeta}} < 2$ and $2k\varepsilon = 4k\gamma\epsilon/\alpha \le 36\epsilon$, we can construct a $(1, a+2, 36\epsilon)$-bitstring indexed encoding of A, denoted by V.

Finally, we provide the computational resources required for implementing V. As shown in Lemma 3.12, the implementation of V requires $O(\alpha)$ uses of U, $U^\dagger$, $\mathrm{C}_\Pi\mathrm{NOT}$, $\mathrm{C}_{\tilde{\Pi}}\mathrm{NOT}$, and $O(\alpha)$ single-qubit gates. Furthermore, the description of the resulting quantum circuit can be computed in deterministic time $O(\alpha)$ and space $O(s)$. □

A.3 Application: space-efficient error reduction for unitary quantum computations

Lemma 3.19 (Space-efficient singular value discrimination)**.** *Let $0 \le \alpha < \beta \le 1$ and U be a $(1,0,0)$-bitstring indexed encoding of $A \coloneqq \tilde{\Pi}U\Pi$, where U acts on s qubits and $s(n) \ge \Omega(\log n)$. Consider an unknown quantum state $|\psi\rangle$, with the promise that it is a right singular vector of A with a singular value either above α or below β. There is a degree-d' polynomial P, where $d' = O(\delta^{-1}\log\varepsilon^{-1})$ and $\delta \coloneqq \max\{\beta-\alpha, \sqrt{1-\alpha^2}-\sqrt{1-\beta^2}\}/2$, such that there is a singular value discriminator U_P that distinguishes the two cases with error probability at most ε. Moreover, the discriminator U_P achieves one-sided error when $\alpha = 0$ or $\beta = 1$.*
Furthermore, the quantum circuit implementation of U_P requires $O(d^2)$ uses of U, $U^\dagger$, $\mathrm{C}_\Pi NOT$, $\mathrm{C}_{\tilde{\Pi}}NOT$, and multi-controlled single-qubit gates. In addition, the description of the implementation can be computed in deterministic time $\tilde{O}(\varepsilon^{-1}\delta^{-9/2})$ and space $O(s(n))$.

Proof of Lemma 3.19. Let the singular value decomposition of A be $A = W\Sigma V^\dagger = \sum_i \sigma_i |\tilde{\psi}_i\rangle\langle\psi_i|$. Note that U is a $(1,0,0)$-bitstring indexed encoding, with projections $\tilde{\Pi}$ and Π, of A. Let singular value threshold projectors $\Pi_{\ge\delta}$ and $\Pi'_{\ge\delta}$ be defined as $\Pi_{\ge\delta} \coloneqq \Pi V\Sigma_{\ge\delta}V^\dagger\Pi$ and $\Pi'_{\ge\delta} \coloneqq \Pi' W\sigma_{\ge\delta}W^\dagger\Pi'$, respectively, with similar definitions for $\Pi_{\le\delta}$ and $\Pi'_{\le\delta}$.

To discriminate whether the singular value corresponding to a given right singular vector of A exceeds a certain threshold, we need an *ε-singular value discriminator U_P*. Specifically, it suffices to construct a $(1, a, \tilde{\epsilon})$-bitstring indexed encoding U_P of A, associated with an appropriate odd polynomial P, that satisfies Equation (A.8). The parameters a and $\tilde{\epsilon}$ will be specified later.

$$\begin{aligned} \Big\|\big(\langle 0|^{\otimes a}\otimes\Pi'_{\ge t+\delta}\big)U_P\big(|0\rangle^{\otimes a}\otimes\Pi_{\ge t+\delta}\big) - \textstyle\sum_{i\in\Lambda}|\tilde{\psi}_i\rangle\langle\psi_i|\Big\| &\le \varepsilon, \\ \Big\|\big(\langle 0|^{\otimes a}\Pi'_{\le t-\delta}\big)U_P\big(|0\rangle^{\otimes a}\otimes\Pi_{\le t-\delta}\big) - 0\Big\| &\le \varepsilon. \end{aligned} \tag{A.8}$$

Here, the index set $\Lambda \coloneqq \{i\colon \sigma_i \ge t+\delta\}$. Additionally, following the proof in [GSLW19, Theorem 20], Π' is defined as $\tilde{\Pi}$ if $\beta-\alpha \ge \sqrt{1-\alpha^2}-\sqrt{1-\beta^2}$, and as $I-\tilde{\Pi}$ otherwise.[52]

With the construction of this bitstring indexed encoding U_P, we can apply an ε-singular value discriminator with $\Pi' = \tilde{\Pi}$ by choosing $t \coloneqq (\alpha+\beta)/2$ and $\delta \coloneqq (\beta-\alpha)/2$. Next, we measure $|0\rangle\langle 0|^{\otimes a}\otimes\Pi'$: If the final state is in $\mathrm{Img}\big(|0\rangle\langle 0|^{\otimes a}\otimes\Pi'\big)$, there exists a singular value σ_i above α (resp., $\sqrt{1-\beta^2}$); otherwise, all singular value σ_i must be below β (resp., $\sqrt{1-\alpha^2}$).

[50] If $\varepsilon > 1/3$, then $\|\tilde{\Pi}'U'\Pi' - A\otimes|0\rangle\langle 0|\| \le 2 = 2\cdot 3\cdot\frac{1}{3}$ always holds, implying that we can directly use U' as V.

[51] This is because $\|A'+\hat{A}\| \le \|A'\| + \|A'\| + \|A'-\hat{A}\| \le 2\sin(\pi/2k) + \varepsilon \le 2\sin(\pi/6) + 1/3 = 4/3$.

[52] By applying [GSLW18, Definition 12] (the full version of [GSLW19]) to $\Pi' \coloneqq I-\tilde{\Pi}$, we know that $|\psi\rangle$ is a right singular vector of $\Pi'U\Pi$ with a singular value of at least $\sqrt{1-a^2}$ in the first case, or with a singular value of at most $\sqrt{1-b^2}$ in the second case. Additionally, in one-sided error scenarios, if $a = 0$, then $b-a = b \ge 1-\sqrt{1-b^2} = \sqrt{1-a^2}-\sqrt{1-b^2}$; while if $b = 1$, then $b-a = 1-a \le \sqrt{1-a^2} = \sqrt{1-a^2}-\sqrt{1-b^2}$.

Furthermore, we can make the error one-sided when $\alpha = 0$ or $\beta = 1$, since a space-efficient QSVT associated with an *odd* polynomial always preserves 0 singular values (see Remark 3.11).

It remains to implement an ε-singular value discriminator U_P for some odd polynomial P.

Implementing ε-singular value discriminator. We begin by considering the following odd function $Q(x)$ such that $Q(A) \approx U_P$ and $Q(A)$ satisfies Equation (A.8):

$$Q(x) \coloneqq \tfrac{1}{2}\left[\left(1-\tfrac{\varepsilon}{2}\right)\cdot \mathrm{sgn}(x+t) + \left(1-\tfrac{\varepsilon}{2}\right)\cdot \mathrm{sgn}(x-t) + \varepsilon \cdot \mathrm{sgn}(x)\right].$$

Let $\epsilon \coloneqq \frac{\varepsilon}{2(C_{\mathrm{sgn}}+36\hat{C}_{\mathrm{sgn}}+37)}$. Using the space-efficient polynomial approximation $P^{\mathrm{sgn}}_{d'}$ of the sign function (Corollary 3.6 with $\epsilon_1 \coloneqq 0$ and $\epsilon_2 \coloneqq \epsilon$), we obtain the following degree-d' polynomial P associated with some degree-d averaged Chebyshev truncation:

$$P(x) = \tfrac{1}{2}\left[\left(1-\tfrac{\varepsilon}{2}\right)\cdot P^{\mathrm{sgn}}_{d'}(x+t) + \left(1-\tfrac{\varepsilon}{2}\right)\cdot P^{\mathrm{sgn}}_{d'}(x-t) + \varepsilon \cdot P^{\mathrm{sgn}}_{d'}(x)\right].$$

Note that $P(x)$ is a convex combination of $P^{\mathrm{sgn}}_{d'}(x+t)$, $P^{\mathrm{sgn}}_{d'}(x-t)$, and $P^{\mathrm{sgn}}_{d'}(x)$, the constants $\tilde{C}_{\mathrm{sgn}}$ and $\hat{C}_{\mathrm{sgn}}$ specified in Corollary 3.6 remain the same. Hence, P is a polynomial of degree $d' = 2d-1 \le \tilde{C}_{\mathrm{sgn}}\frac{1}{\delta}\log\frac{1}{\epsilon}$, and the coefficient vector $\hat{\mathbf{c}}^{(P)}$ satisfies $\|\hat{\mathbf{c}}^{(P)}\|_1 \le \hat{C}_{\mathrm{sgn}}$.

Recall the notation $\|f\|_{\mathcal{I}}$ defined in Section 3.1.2, namely $\|f\|_{\mathcal{I}} \coloneqq \sup\{|f(x)| : x \in \mathcal{I}\}$. Let $D(x) \coloneqq \mathrm{sgn}(x) - P^{\mathrm{sgn}}_{d'}$, $\mathcal{I}_0 \coloneqq (0, t-\delta]$, and $\mathcal{I}_1 \coloneqq [t+\delta, 1]$. Following Corollary 3.6, we obtain:

$$\begin{aligned}
&\|P(x)-Q(x)\|_{[\delta-t,0)} = \|P(x)-Q(x)\|_{\mathcal{I}_0}\\
\le&\tfrac{2-\varepsilon}{4}\|D(x+t)\|_{\mathcal{I}_0} + \tfrac{2-\varepsilon}{4}\|D(x-t)\|_{\mathcal{I}_0} + \tfrac{\varepsilon}{2}\|D(x)\|_{\mathcal{I}_0} \le \left(1-\tfrac{\varepsilon}{2}\right)C_{\mathrm{sgn}}\epsilon + \tfrac{\varepsilon}{2},\\
&\|P(x)-Q(x)\|_{[-1,-t-\delta]} = \|P(x)-Q(x)\|_{\mathcal{I}_1}\\
\le&\tfrac{2-\varepsilon}{4}\|D(x+t)\|_{\mathcal{I}_1} + \tfrac{2-\varepsilon}{4}\|D(x-t)\|_{\mathcal{I}_1} + \tfrac{\varepsilon}{2}\|D(x)\|_{\mathcal{I}_1} \le \left(1-\tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{2}\right)C_{\mathrm{sgn}}\epsilon.
\end{aligned} \tag{A.9}$$

Here, the equalities hold because both P and Q are odd functions.

Using Corollary 3.15 with P, we obtain a $(1, a, \tilde{\epsilon})$-bitstring indexed encoding U_P of A, with $a \coloneqq \lceil \log d' \rceil + 3$ and $\tilde{\epsilon} \coloneqq (36\hat{C}_{\mathrm{sgn}}+37)\epsilon$. Together with Equation (A.9), we obtain:

$$\begin{aligned}
\left\|\left(\langle 0|^{\otimes a}\otimes \Pi'_{\ge t+\delta}\right)U_P\left(|0\rangle^{\otimes a}\otimes \Pi_{\ge t+\delta}\right) - \textstyle\sum_{i\in\Lambda}|\tilde{\psi}_i\rangle\langle\psi_i|\right\| &\le \left(1-\tfrac{\varepsilon}{2}+\tfrac{\varepsilon}{2}\right)C_{\mathrm{sgn}}\epsilon + (36\hat{C}_{\mathrm{sgn}}+37)\epsilon \le \varepsilon,\\
\left\|\left(\langle 0|^{\otimes a}\Pi'_{\le t-\delta}\right)U_P\left(|0\rangle^{\otimes a}\otimes \Pi_{\le t-\delta}\right) - 0\right\| &\le C_{\mathrm{sgn}}\epsilon + \tfrac{\varepsilon}{2} + (36\hat{C}_{\mathrm{sgn}}+37)\epsilon \le \varepsilon.
\end{aligned}$$

Hence, we conclude that our construction of U_P indeed satisfies Equation (A.8).

Finally, we analyze the complexity of this ε-singular value discriminator U_P. Following Corollary 3.15, the quantum circuit implementation of U_P requires $O(d^2)$ uses of U, $U^\dagger$, $\mathrm{C}_\Pi\mathrm{NOT}$, $\mathrm{C}_{\tilde{\Pi}}\mathrm{NOT}$, and multi-controlled single-qubit gates. Moreover, we can compute the description of the circuit implementation in deterministic time $\tilde{O}(\epsilon^{-1}d^{9/2}) = \tilde{O}(\varepsilon^{-1}\delta^{-9/2})$ and space $O(s(n))$, where $\delta = \max\{\beta-\alpha, \sqrt{1-\alpha^2}-\sqrt{1-\beta^2}\}/2$. □

B Omitted proofs in space-bounded quantum state testing

Theorem B.1. *For any functions $\alpha(n)$ and $\beta(n)$ that can be computed in deterministic logspace and satisfy $\alpha(n)-\beta(n) \ge 1/\mathrm{poly}(n)$, we have that $\textsc{GapQHS}_{\log}[\alpha(n),\beta(n)]$ is in BQL.*

Proof. Note that $\mathrm{HS}^2(\rho_0,\rho_1) = \frac{1}{2}\left(\mathrm{Tr}(\rho_0^2)+\mathrm{Tr}(\rho_1^2)\right) - \mathrm{Tr}(\rho_0\rho_1)$. Let $\varepsilon \coloneqq (\alpha-\beta)/100$. According to Lemma 2.17, we can use the SWAP test to estimate $\mathrm{Tr}(\rho_0^2)$, $\mathrm{Tr}(\rho_1^2)$, and $\mathrm{Tr}(\rho_0\rho_1)$, and hence $\mathrm{HS}^2(\rho_0,\rho_1)$, within additive error ε with high probability by performing $O(1/\varepsilon^2)$ sequential repetitions. Therefore, we can conclude that $\textsc{GapQHS}_{\log}[\alpha(n),\beta(n)]$ is in BQL. □

B.1 Omitted proofs in BQL containments

Proposition 4.10.1. $\Pr\left[|x - \mathrm{td}(\rho_0,\rho_1)| \le (36\hat{C}_{\mathrm{sgn}}+2C_{\mathrm{sgn}}+37)\epsilon + \epsilon_H + 2^{r+1}\delta\right] > 0.8.$

Proof of Proposition 4.10.1. Using the triangle inequality, we obtain the following:

$$\begin{aligned}\left|\frac{x_0-x_1}{2}-\mathrm{td}(\rho_0,\rho_1)\right| &= \left|\frac{x_0-x_1}{2}-\mathrm{Tr}\Big(\frac{\rho_0-\rho_1}{2}\mathrm{sgn}\Big(\frac{\rho_0-\rho_1}{2}\Big)\Big)\right| \\ &\le \left|\frac{x_0-x_1}{2}-\mathrm{Tr}\Big(\frac{\rho_0-\rho_1}{2}P_{d'}^{\mathrm{sgn}}\Big(\frac{\rho_0-\rho_1}{2}\Big)\Big)\right| \\ &\quad+\left|\mathrm{Tr}\Big(\frac{\rho_0-\rho_1}{2}P_{d'}^{\mathrm{sgn}}\Big(\frac{\rho_0-\rho_1}{2}\Big)\Big)-\mathrm{Tr}\Big(\frac{\rho_0-\rho_1}{2}\mathrm{sgn}\Big(\frac{\rho_0-\rho_1}{2}\Big)\Big)\right|.\end{aligned}$$

For the first term, by noting the QSVT implementation error in Corollary 3.15, we know by Equation (4.3) that, with probability at least $0.9^2 > 0.8$, it holds that

$$\left|\frac{x_0-x_1}{2}-\mathrm{Tr}\Big(\frac{\rho_0-\rho_1}{2}P_{d'}^{\mathrm{sgn}}\Big(\frac{\rho_0-\rho_1}{2}\Big)\Big)\right| \le (36\hat{C}_{\mathrm{sgn}}+37)\epsilon+\epsilon_H. \tag{B.1}$$

For the second term, let $\frac{\rho_0-\rho_1}{2}=\sum_j \lambda_j|\psi_j\rangle\langle\psi_j|$, where $\{|\psi_j\rangle\}$ is an orthonormal basis. Then,

$$\left|\mathrm{Tr}\Big(\frac{\rho_0-\rho_1}{2}P_{d'}^{\mathrm{sgn}}\Big(\frac{\rho_0-\rho_1}{2}\Big)\Big)-\mathrm{Tr}\Big(\frac{\rho_0-\rho_1}{2}\mathrm{sgn}\Big(\frac{\rho_0-\rho_1}{2}\Big)\Big)\right| \le \sum_j \left|\lambda_j P_{d'}^{\mathrm{sgn}}(\lambda_j)-\lambda_j\mathrm{sgn}(\lambda_j)\right|. \tag{B.2}$$

We split the summation over j into three separate summations: $\sum_j = \sum_{\lambda_j<-\delta}+\sum_{\lambda_j>\delta}+\sum_{-\delta\le\lambda_j\le\delta}$. By noticing the approximation error of $P_{d'}^{\mathrm{sgn}}$ in Corollary 3.6 and $\sum_j|\lambda_j| = \mathrm{td}(\rho_0,\rho_1)\le 1$, we can then obtain the following results for each of the three summations:

$$\sum_{\lambda_j>\delta}\left|\lambda_j P_{d'}^{\mathrm{sgn}}(\lambda_j)-\lambda_j\mathrm{sgn}(\lambda_j)\right| = \sum_{\lambda_j>\delta}|\lambda_j|\left|P_{d'}^{\mathrm{sgn}}(\lambda_j)-1\right| \le \sum_{\lambda_j>\delta}|\lambda_j|C_{\mathrm{sgn}}\epsilon \le C_{\mathrm{sgn}}\epsilon,$$

$$\sum_{\lambda_j<-\delta}\left|\lambda_j P_{d'}^{\mathrm{sgn}}(\lambda_j)-\lambda_j\mathrm{sgn}(\lambda_j)\right| = \sum_{\lambda_j<-\delta}|\lambda_j|\left|P_{d'}^{\mathrm{sgn}}(\lambda_j)+1\right| \le \sum_{\lambda_j<-\delta}|\lambda_j|C_{\mathrm{sgn}}\epsilon \le C_{\mathrm{sgn}}\epsilon,$$

$$\sum_{-\delta\le\lambda_j\le\delta}\left|\lambda_j P_{d'}^{\mathrm{sgn}}(\lambda_j)-\lambda_j\mathrm{sgn}(\lambda_j)\right| \le \sum_{-\delta\le\lambda_j\le\delta}2|\lambda_j| \le 2^{r+1}\delta.$$

Hence, we derive the following inequality by summing over the aforementioned three inequalities:

$$\sum_j\left|\lambda_j P_{d'}^{\mathrm{sgn}}(\lambda_j)-\lambda_j\mathrm{sgn}(\lambda_j)\right| \le 2^{r+1}\delta+2C_{\mathrm{sgn}}\epsilon. \tag{B.3}$$

By combining Equation (B.1), Equation (B.2), and Equation (B.3), we conclude that

$$\left|\frac{x_0-x_1}{2}-\mathrm{td}(\rho_0,\rho_1)\right| \le (36\hat{C}_{\mathrm{sgn}}+37)\epsilon+\epsilon_H+2C_{\mathrm{sgn}}\epsilon+2^{r+1}\delta. \qquad \square$$

Proposition 4.11.1. *The following inequality holds for $i\in\{0,1\}$:*

$$\Pr\Big[\left|2\ln\big(\tfrac{2}{\beta}\big)x_i-\mathrm{S}(\rho_i)\right| \le 2\ln\big(\tfrac{2}{\beta}\big)\left((\hat{C}_{\ln}+C_{\ln})\epsilon+\epsilon_H+2^{r+1}\beta\right)\Big] \ge 0.9.$$

Proof of Proposition 4.11.1. We only prove the case with $i=0$ while the case with $i=1$ follows straightforwardly. By applying the triangle inequality with $i=0$, we have:

$$\left|2\ln\big(\tfrac{2}{\beta}\big)x_0-\mathrm{S}(\rho_0)\right| = \left|2\ln\big(\tfrac{2}{\beta}\big)x_0-2\ln\big(\tfrac{2}{\beta}\big)\mathrm{Tr}\left(P_{d'}^{\ln}(\rho_0)\rho_0\right)\right|+\left|2\ln\big(\tfrac{2}{\beta}\big)\mathrm{Tr}\left(P_{d'}^{\ln}(\rho_0)\rho_0\right)-\mathrm{S}(\rho_0)\right|.$$

For the first term, by noting the QSVT implementation error in Corollary 3.16, we have by Equation (4.4) that with probability at least 0.9, it holds that

$$\left|2\ln\big(\tfrac{2}{\beta}\big)x_0-2\ln\big(\tfrac{2}{\beta}\big)\mathrm{Tr}\left(P_{d'}^{\ln}(\rho_0)\rho_0\right)\right| \le 2\ln\big(\tfrac{2}{\beta}\big)\left(\hat{C}_{\ln}\epsilon+\epsilon_H\right). \tag{B.4}$$

For the second term, let $\rho_0=\sum_j\lambda_j|\psi_j\rangle\langle\psi_j|$, where $\{|\psi_j\rangle\}$ is an orthonormal basis. Then,

$$\left|2\ln\big(\tfrac{2}{\beta}\big)\mathrm{Tr}\left(P_{d'}^{\ln}(\rho_0)\rho_0\right)-\mathrm{S}(\rho_0)\right| \le \sum_j\left|2\ln\big(\tfrac{2}{\beta}\big)\lambda_j P_{d'}^{\ln}(\lambda_j)-\lambda_j\ln(1/\lambda_j)\right|. \tag{B.5}$$

We split the summation over j into two separate summations: $\sum_j = \sum_{\lambda_j>\beta}+\sum_{\lambda\le\beta}$. By

noticing the approximation error of $P_{d'}^{\ln}$ in Corollary 3.9 and $\sum_j |\lambda_j| = \mathrm{Tr}(\rho) \leq 1$, we can then obtain the following results for each of the two summations:

$$\begin{aligned}\sum_{\lambda_j>\beta} \left|2\ln\left(\tfrac{2}{\beta}\right)\lambda_j P_{d'}^{\ln}(\lambda_j) - \lambda_j \ln(1/\lambda_j)\right| &= \sum_{\lambda_j>\beta} |\lambda_j| \cdot \left|2\ln\left(\tfrac{2}{\beta}\right)P_{d'}^{\ln}(\lambda_j) - \ln(1/\lambda_j)\right| \\ &\leq \sum_{\lambda_j>\beta} |\lambda_j| \cdot 2\ln\left(\tfrac{2}{\beta}\right) C_{\ln}\epsilon \\ &\leq 2\ln\left(\tfrac{2}{\beta}\right) C_{\ln}\epsilon, \\ \sum_{\lambda_j\leq\beta} \left|2\ln\left(\tfrac{2}{\beta}\right)\lambda_j P_{d'}^{\ln}(\lambda_j) - \lambda_j \ln(1/\lambda_j)\right| &\leq \sum_{\lambda_j\leq\beta} \left(2\ln\left(\tfrac{2}{\beta}\right)|\lambda_j| + |\lambda_j|\ln(1/\beta)\right) \\ &\leq 2\ln\left(\tfrac{2}{\beta}\right) 2^{r+1}\beta.\end{aligned}$$

Hence, we have derived the following inequality by summing over the aforementioned inequalities:

$$\sum_j \left|2\ln\left(\tfrac{2}{\beta}\right)\lambda_j P_{d'}^{\ln}(\lambda_j) - \lambda_j \ln(1/\lambda_j)\right| \leq 2\ln\left(\tfrac{2}{\beta}\right) C_{\ln}\epsilon + 2\ln\left(\tfrac{2}{\beta}\right) 2^{r+1}\beta. \tag{B.6}$$

By combining Equation (B.4), Equation (B.5), and Equation (B.6), we conclude that

$$\left|2\ln\left(\tfrac{2}{\beta}\right)x_0 - \mathrm{S}(\rho_0)\right| \leq 2\ln\left(\tfrac{2}{\beta}\right)\left(\hat{C}_{\ln}\epsilon + \epsilon_H + C_{\ln}\epsilon + 2^{r+1}\beta\right). \qquad \square$$

Proposition 4.11.2. $2\ln(\frac{2}{\beta}) \cdot 2^{r+1}\beta \leq \frac{\varepsilon}{4}$.

Proof of Proposition 4.11.2. Note that the choice of β is given by $\beta := \frac{\varepsilon}{2^{r+6}\ln(\frac{2^{r+6}}{\varepsilon})}$. Then, to demonstrate the inequality $2\ln(\frac{2}{\beta}) \cdot 2^{r+1}\beta \leq \frac{\varepsilon}{4}$, it suffices to prove that

$$2\ln\left(\frac{2^{r+7}\ln(\frac{2^{r+6}}{\varepsilon})}{\varepsilon}\right) \cdot \frac{\varepsilon}{2^5\ln(\frac{2^{r+6}}{\varepsilon})} \leq \frac{\varepsilon}{4}. \tag{B.7}$$

Let $x := 2^{-r-6}\varepsilon \in (0,1)$, then Equation (B.7) becomes $\ln\left(\frac{2}{x}\ln\left(\frac{1}{x}\right)\right) \leq 4\ln\left(\frac{1}{x}\right)$. This simplifies further to $2x^3\ln\left(\frac{1}{x}\right) \leq 1$.

To complete the proof, let $f(x) = 2x^3\ln(\frac{1}{x})$, then its first derivative is $f'(x) = 2x^2\left(3\ln\left(\frac{1}{x}\right) - 1\right)$. Note that $f'(x) > 0$ for $x \in (0, e^{-1/3})$ and $f'(x) < 0$ for $x \in (e^{-1/3}, 1)$. Thus $f(x)$ is monotonically increasing for $x \in (0, e^{-1/3})$ and monotonically decreasing for $x \in (e^{-1/3}, 1)$. Therefore, $f(x)$ takes the maximum value at $x = e^{-1/3}$, and consequently, $f(x) \leq f(e^{-1/3}) = \frac{2}{3e} \leq 1$. $\square$

B.2 Omitted proofs in coRQ$_{\mathrm{U}}$L containments

Proposition 4.14.1. *Let $f(\theta_0, \theta_1) := \sin^2(3\theta_0)\sin^2(3\theta_1)$ be a function such that $\sin^2(\theta_i) = p_i/2$ for $i \in \{0,1\}$ and $\max\{|p_0 - 1/2|, |p_1 - 1/2|\} \geq \varepsilon/2$, then $f(\theta_0, \theta_1) \leq 1 - \varepsilon^2/4$.*

Proof. We begin by stating the facts that $\sin^2(\theta_i) = p_i/2$ for $i \in \{0,1\}$ and $\sin^2(3\theta) = \sin^6(\theta) - 6\cos^2(\theta)\sin^4(\theta) + 9\cos^4(\theta)\sin^2(\theta)$. Then we notice that $0 \leq p_0, p_1 \leq 1$ and complete the proof by a direct calculation:

$$\begin{aligned}f(\theta_0, \theta_1) &= \left(2p_0^3 - 6p_0^2 + \tfrac{9}{2}p_0\right)\left(2p_1^3 - 6p_1^2 + \tfrac{9}{2}p_1\right) \\ &\leq \left(1 - \left(p_0 - \tfrac{1}{2}\right)^2\right)\left(1 - \left(p_1 - \tfrac{1}{2}\right)^2\right) \\ &\leq 1 - \left(\max\left\{\left|p_0 - \tfrac{1}{2}\right|, \left|p_1 - \tfrac{1}{2}\right|\right\}\right)^2 \\ &\leq 1 - \tfrac{\varepsilon^2}{4}.\end{aligned}$$

$\square$

Proposition 4.15.1. *The polynomial function $f(x) := 16x^3 - 24x^2 + 9x$ is monotonically decreasing in $[1/4, 9/16]$. Moreover, we have $f\left(\left(\frac{1}{2} + \frac{\alpha}{4}\right)^2\right) \leq 1 - \frac{\alpha^2}{2}$ for any $0 \leq \alpha \leq 1$.*

Proof. Through a direct calculation, we have $f'(x) = 48x^2 - 48x + 9 \leq 0$ for $x \in [1/4, 3/4]$, then $f(x)$ is monotonically decreasing in $[1/4, 9/16] \subseteq [1/4, 3/4]$. Moreover, it is left to show that:

$$f\left(\left(\tfrac{1}{2} + \tfrac{\alpha}{4}\right)^2\right) = \tfrac{\alpha^6}{256} + \tfrac{3\alpha^5}{64} + \tfrac{9\alpha^4}{64} - \tfrac{\alpha^3}{8} - \tfrac{3\alpha^2}{4} + 1 \leq 1 - \tfrac{\alpha^2}{2}.$$

Equivalently, it suffices to show that $g(x) \coloneqq -\frac{x^4}{256} - \frac{3x^3}{64} - \frac{9x^2}{64} + \frac{x}{8} + \frac{1}{4} \geq 0$ for $0 \leq x \leq 1$. We first compute the first derivative of $g(x)$, which is $g'(x) = -\frac{x^3}{64} - \frac{9x^2}{64} - \frac{9x}{32} + \frac{1}{8}$. Setting $g'(x)$ equal to zero, we obtain three roots: $x_1 = -4$, $x_2 = \frac{1}{2}(-\sqrt{33} - 5) < 0$, and $x_3 = \frac{1}{2}(\sqrt{33} - 5) \in (0, 1)$.

Since $g'(0) = 1/8 > 0$ and $g'(1) = -5/16 < 0$, we conclude that $g(x)$ is monotonically increasing in $[0, x_3]$ and monotonically decreasing in $[x_3, 1]$. Therefore, we can determine the minimum value of $g(x)$ by evaluating $g(0) = \frac{1}{4}$ and $g(1) = \frac{47}{256}$. Since both values are greater than zero, we conclude that $\min\{g(0), g(1)\} = \left\{\frac{1}{4}, \frac{47}{256}\right\} > 0$, as desired. □

ECCC ISSN 1433-8092

https://eccc.weizmann.ac.il