

Product Mixing in Compact Lie Groups

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Abstract

If G is a group, we say a subset S of G is *product-free* if the equation $xy = z$ has no solutions with $x, y, z \in S$. For $D \in \mathbb{N}$, a group G is said to be *D -quasirandom* if the minimal dimension of a nontrivial complex irreducible representation of G is at least D . Gowers showed that in a D -quasirandom finite group G , the maximal size of a product-free set is at most $|G|/D^{1/3}$. This disproved a longstanding conjecture of Babai and Sós from 1985.

For the special unitary group, $G = \mathrm{SU}(n)$, Gowers observed that his argument yields an upper bound of $n^{-1/3}$ on the measure of a measurable product-free subset. In this paper, we improve Gowers' upper bound to $\exp(-cn^{1/3})$, where $c > 0$ is an absolute constant. In fact, we establish something stronger, namely, *product-mixing* for measurable subsets of $\mathrm{SU}(n)$ with measure at least $\exp(-cn^{1/3})$; for this product-mixing result, the $n^{1/3}$ in the exponent is sharp.

Our approach involves introducing novel hypercontractive inequalities, which imply that the non-Abelian Fourier spectrum of the indicator function of a small set concentrates on high-dimensional irreducible representations. Our hypercontractive inequalities are obtained via methods from representation theory, harmonic analysis, random matrix theory and differential geometry. We generalize our hypercontractive inequalities from $\mathrm{SU}(n)$ to an arbitrary D -quasirandom compact connected Lie group for D at least an absolute constant, thereby extending our results on product-free sets to such groups.

We also demonstrate various other applications of our inequalities to geometry (viz., non-Abelian Brunn-Minkowski type inequalities), mixing times, and the theory of growth in compact Lie groups. A subsequent work due to Arunachalam, Girish and Lifshitz uses our inequalities to establish new separation results between classical and quantum communication complexity.

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1 Introduction

A subset \mathcal{A} of a group G is said to be *product-free* if $gh \notin \mathcal{A}$ for all $g, h \in \mathcal{A}$. The study of product-free subsets of groups has attracted significant attention over the past three decades. In 1985, Babai and Sós [3] considered the problem of determining the largest size of a product-free set in a finite group G . They conjectured that exists an absolute positive constant $c_0 > 0$ such that any finite group G has a product-free set of size at least $c_0|G|$. In the Abelian case, this is quite easy to see, and had previously been observed by Erdős, in an unpublished communication to Babai and Sós. (In the cyclic case $(\mathbb{Z}_n, +)$, one can take a ‘middle-third’ construction, viz., $\{x \in \mathbb{Z}_n : n/3 < x \leq 2n/3\}$, as a large product-free set, and one can reduce to the cyclic case by observing that any finite Abelian group has a nontrivial cyclic quotient, and that the preimage of a product-free set under a quotient map is also product-free and of the same measure.) The exact answer in the Abelian case was given by Green and Ruzsa [15] in 2003: the largest product-free subset of a finite Abelian group G has size $c|G|$, where the function $c = c(G) \in [2/7, 1/2]$ was explicitly determined by Green and Ruzsa. The general Babai-Sós conjecture was disproved in 2008 by Gowers [14], who showed that if G is a finite group such that the minimal dimension of a nontrivial irreducible complex representation of G is equal to D , then any product-free subset of G has size at most $D^{-1/3}|G|$. It remains to observe that the quantity $D = D(G)$ is unbounded over finite non-Abelian groups G . For example, for the projective special

linear group $\mathrm{PSL}_2(\mathbb{F}_q)$ (for q an odd prime power), we have $D(\mathrm{PSL}_2(\mathbb{F}_q)) = (q - 1)/2$, so the measure of a product-free subset of $\mathrm{PSL}_2(\mathbb{F}_q)$ is at most $O(q^{-1/3})$, which tends to zero as q tends to infinity.

Gowers observed that his argument also implies that if G is an (infinite) compact group for which the minimal dimension of a nontrivial irreducible complex continuous representation is equal to D , then the maximal Haar measure of a measurable, product-free set in G is at most $D^{-1/3}$. For $\mathrm{SU}(n)$ we have $D(\mathrm{SU}(n)) = n$, implying an upper bound of $n^{-1/3}$ on the measure of a measurable product-free subset of $\mathrm{SU}(n)$. However, Gowers conjectured that for $\mathrm{SU}(n)$, the true answer is exponentially small in n . Indeed, as Gowers states, it seems difficult to come up with an example better than the following. Recall that group $\mathrm{SU}(n)$ acts on the complex unit sphere $\{v \in \mathbb{C}^n : \|v\|_2 = 1\}$, and take \mathcal{A} to be the set of all matrices $A \in \mathrm{SU}(n)$ such that the real part of $\langle Ae_1, e_1 \rangle$ is less than $-1/2$. As noted by Gowers, it follows from the triangle inequality that this set is product-free, and it is easy to check that the measure of \mathcal{A} is $2^{-\Omega(n)}$.

In this work, we make progress towards proving Gowers' conjecture. Specifically, we improve Gowers' upper bound by a stretched exponential factor, viz., from $n^{-1/3}$ to $e^{-cn^{1/3}}$.

Theorem 1.1. *There exists an absolute constant $c > 0$ such that the following holds. Let $n \in \mathbb{N}$ and let $\mathcal{A} \subset \mathrm{SU}(n)$ be Haar-measurable and product-free. Then $\mu(\mathcal{A}) \leq \exp(-cn^{1/3})$.*

1.1 Quasirandomness for groups, and mixing.

Gowers' bound for product-free sets relies on a relationship between spectral gaps and dimensions of irreducible representations, a relationship which was first discovered by Sarnak and Xue [40]. In fact, Gowers' proof uses a beautiful connection between the problem and a purely representation-theoretic notion that Gowers called *quasirandomness* (due to a rough equivalence with the graph-quasirandomness of certain Cayley graphs, an equivalence which we shall explain below). For a group G we denote by $D(G)$ the minimal dimension of a non-trivial complex irreducible continuous representation of G . (Henceforth, for brevity, we will use the term *representation* to mean continuous representation.) For $d \in \mathbb{N}$, we say that a group G is *d-quasirandom* if $D(G) \geq d$.¹ Denoting by $\alpha(G)$ the largest possible density $\frac{|A|}{|G|}$ of a product-free set $A \subseteq G$ (if G is a finite group), Gowers showed that for any finite group G , $\alpha(G) \leq D(G)^{-1/3}$. Since $D(G)$ can be arbitrarily large (as is the case for the alternating groups, which have $D(A_n) = n - 1$ for all $n \geq 7$, and the groups $\mathrm{PSL}_2(\mathbb{F}_q)$ as mentioned above, and for many other natural infinite families of finite groups), this disproved the conjecture of Babai and Sós.

For finite groups, the quasirandomness parameter gives an almost complete description of the maximal size of a product free set. Pyber (see [14]) used the Classification of Finite Simple Groups, together with a construction of Kedlaya, showing that $\alpha(G) \geq D(G)^{-C}$ where $C > 0$ is an absolute constant. Nikolov and Pyber [34] later improved this to

¹To avoid confusion with the quasirandomness parameter for graphs, it might have been less ambiguous to call this notion '*d*-group-quasirandomness', but as the latter is rather cumbersome we have opted for the above shorter formulation; we hope that this will not cause the reader confusion, in the sequel.

$\alpha(G) \geq \frac{1}{cD(G)}$. This established a remarkable fact, namely that the purely representation theoretic quasirandomness parameter $D(G)$ is polynomially related to the combinatorial quantity $\alpha(G)$.

$$(cD(G))^{-1} \leq \alpha(G) \leq D(G)^{-1/3} \tag{1}$$

For compact connected Lie groups we obtain the following general variant of Theorem 1.1, which gives an upper bound on the size of a product-free set in the group.

Theorem 1.2. *There exists an absolute constant $c > 0$ such that the following holds. Let G be a compact connected Lie group, and let \tilde{G} be its universal cover. Let $\mathcal{A} \subset G$ be Haar-measurable and product-free. Let μ denote the Haar probability measure on G . Then $\mu(\mathcal{A}) \leq \exp(-cD(\tilde{G})^{1/3})$.*

An elegant argument of Gowers [14] (proof of Theorem 4.6, therein) for finite groups, which generalises very easily to the case of compact groups, shows that if G is a compact group then it has a measurable product-free subset of measure at least $\exp(-\Omega(D(G)))$. In Section 2 we show that $D(G) = O(D(\tilde{G})^2)$. These two facts combine with Theorem 1.2 to give the following analogue of (1) for compact connected Lie groups.

Corollary 1.3. *There exists an absolute constant $c > 0$ such that the following holds. For every compact connected Lie group G ,*

$$cD(G)^{1/6} \leq \log(1/\alpha(G)) \leq \frac{1}{c}D(G).$$

(We remark that our logs will always be taken with respect to the natural basis.) Corollary 1.3 says that, as with finite groups, the maximal measure of a measurable product-free set in a compact connected Lie group is controlled by the quasirandomness parameter, but this time the control moves to the exponent.

Quasirandomness (for groups) was a crucial ingredient in the ‘Bourgain–Gamburd expansion machine’, which is a three-step method for obtaining spectral gaps for Cayley graphs (see e.g. Tao [41], for an exposition). Briefly, this ‘machine’ proceeds as follows: one first shows that the graph has high girth, then one shows that there are no ‘approximate subgroups’ in which a random walk could be entrapped, and then quasirandomness is used (together with the trace method) to finally obtain a spectral gap. Quasirandomness (for groups) has many other applications, such as in bounding the diameters of Cayley graphs (see e.g. the survey of Helfgott [19]).

The term ‘quasirandomness’ was used (for groups) by Gowers, due to the following connection with the (now classical) notion of quasirandomness for graphs. (There are, of course, now notions of quasirandomness for a huge variety of combinatorial and algebraic structures; roughly speaking, these say the structure behaves in a random-like way, in an appropriate sense.) We now need some more terminology. The *normalized adjacency matrix* $A_H \in \mathbb{R}^{V \times V}$ of a d -regular graph $H = (V, E)$ has (i, j) -th entry equal to $1/d$ if $\{i, j\} \in E$, and equal to zero otherwise. The graph H is said to be ε -*quasirandom* if all the nontrivial eigenvalues of A_H are at most ε in absolute value (here, ‘nontrivial’ means having an eigenvector orthogonal to the constant functions).

One of the striking consequences of d -quasirandomness for a finite group G , is that it implies that Cayley graphs of the form $\text{Cay}(G, S)$ are $(1/\text{poly}(d))$ -quasirandom, whenever S is a dense subset of G . The fact that this only relies on density considerations and does not require any assumption on the structure of S , makes the notion of quasirandomness for groups rather powerful.

More generally, applications of quasirandomness for a group G can often be (re)phrased as follows. Suppose that G is d -quasirandom, and that we have a linear operator $T : L^2(G) \rightarrow L^2(G)$ whose nontrivial eigenvalues we want to bound (in absolute value) from above; suppose further that T commutes with either the left or the right action of G on $L^2(G)$. (In Gowers' proof, slightly rephrased, the operator T could be viewed as B^*B , where B is the bipartite adjacency matrix of the bipartite Cayley graph with vertex-classes consisting of two disjoint copies of G , and where the edges are all pairs of the form (g, sg) for $g \in G$ and $s \in S$, S being a product-free set in G .) Then by the commuting property, each eigenspace of T is a nontrivial representation of G , and therefore has dimension at least d ; it follows that each nontrivial eigenvalue of G has multiplicity at least d . But the sum of the squares of the eigenvalues of T is equal to $\text{Trace}(T^2)$, and this yields the bound $d|\lambda|^2 \leq \text{Trace}(T^2)$ for all nontrivial eigenvalues λ of T . This is often called the Sarnak-Xue trick, as it was first employed in [40]

Bourgain and Gamburd used their 'expansion machine' (alluded to above) to show that taking two uniformly random elements $a, b \in \text{SL}_2(\mathbb{F}_p)$ is sufficient for the Cayley graph $\text{Cay}(\text{SL}_2(\mathbb{F}_p), \{a, b, a^{-1}, b^{-1}\})$ to be an expander with high probability, p tending to infinity. It is a major open problem in the theory of Cayley graphs to obtain a similar result in the unbounded-rank case, for example for $\text{SL}_n(\mathbb{F}_p)$ where p is fixed and n tends to infinity. One of the properties that breaks down when one attempts to use the Bourgain–Gamburd expansion machine in the case of unbounded rank, is the dependence of the quasirandomness parameter on the cardinality of the group. Specifically, in order for the Bourgain–Gamburd expansion machine to work effectively for a group G , the quasirandomness parameter $D(G)$ needs to be polynomial in the cardinality of G . In the unbounded rank case, this no longer holds. For example, $D(\text{SL}_n(\mathbb{F}_p)) \leq p^n$ (consider the representation of dimension p^n induced by the natural action of $\text{SL}_n(\mathbb{F}_p)$ on \mathbb{F}_p^n). The situation is even worse for the alternating group A_n , as $D(A_n) = n - 1$ for $n \geq 7$, and $n - 1$ is less than logarithmic in the cardinality of the group.

1.2 Ideas and techniques

To improve on the upper bound of Gowers, we need to find methods for 'dealing with' the low-dimensional irreducible representations (more precisely, for dealing with the corresponding parts of the Fourier transform). In this paper, we develop some new techniques for this in the case of compact connected Lie groups. These techniques turn out also to be useful for finite groups; for example, in [28], analogues of some of our methods are developed for the alternating group A_n (where the idea of mixing is replaced by a refined notion, referred to therein as a 'mixing property for global sets').

Below we give indications of the new techniques that are used to obtain our improved

bounds, and the various areas of mathematics from which they originate.

Level d inequalities and hypercontractivity

One of our key ideas is motivated by the (now well-developed) theory of the analysis of Boolean functions. A function

$$f: \{-1, 1\}^n \rightarrow \mathbb{R}$$

has a *Fourier expansion* $f = \sum_{S \subseteq [n]} \hat{f}(S) \chi_S$, where $\chi_S: \{-1, 1\}^n \rightarrow \{-1, 1\}$ is defined by $\chi_S(x) := \prod_{i \in S} x_i$ for each $x \in \{-1, 1\}^n$ and $S \subseteq [n]$. The functions χ_S , known as the *Fourier-Walsh functions* or *characters*, are orthonormal (with respect to the natural inner product on $\mathbb{R}[\{-1, 1\}^n]$ induced by the uniform measure). The Fourier expansion gives rise to a coarser orthogonal decomposition, $f = \sum_{d=0}^n f^{=d}$, where

$$f^{=d} := \sum_{|S|=d} \hat{f}(S) \chi_S.$$

This is known as the *degree decomposition* (as each function $f^{=d}$ is a homogeneous polynomial of total degree d in the x_i 's).

The *level d inequality* for the Boolean cube (essentially due to Kahn–Kalai–Lindal [27] and Benjamini–Kalai–Schramm [6]) states that there exists an absolute constant $C > 0$, such that for a set $A \subseteq \{-1, 1\}^n$ of density $\frac{|A|}{2^n} = \alpha$, if $d \leq \log(1/\alpha)$ then the characteristic function $f = 1_A$ satisfies $\|f^{=d}\|_2^2 \leq \alpha^2 \left(\frac{C \log(1/\alpha)}{d} \right)^d$. Roughly speaking, the level d inequality says that indicators of small sets are very much uncorrelated with low degree polynomials. One of our key ideas in this paper is to generalize the level d inequality from the Boolean cube to the setting of compact connected Lie groups.

The main tool in the proof of the Boolean level d inequality is the Bonami–Gross–Beckner hypercontractivity theorem. It states that the *noise operator* $T_\rho f := \sum_{d=0}^n \rho^d f^{=d}$ is a contraction as an operator from L_q to L_p , for all $q > p \geq 1$ provided $0 \leq \rho \leq \sqrt{\frac{q-1}{p-1}}$. This immediately implies that $\|f^{=d}\|_q \leq \rho^{-d} \|f^{=d}\|_p$ for any function f . Roughly speaking, this last inequality says that L_p -norm of a low-degree function does not change too drastically with p . This is in stark contrast with the behaviour of indicator functions of small sets, $f = 1_A$. These satisfy $\|f\|_p = \alpha^{1/p}$, which does change rapidly with p . This difference in behaviours can be used to prove the level d -inequality, stating that indicators of small sets are essentially orthogonal to the low degree functions.

The same proof-concept works hand in hand with the representation theory of compact simple Lie groups. For simplicity, let us restrict our attention (at first) to the group $G = \text{SO}(n)$. For each $d \in \mathbb{N} \cup \{0\}$, we let $V_{\leq d} \subseteq L^2(G)$ denote the subspace of $L^2(G)$ spanned by the polynomials of degree at most d in the matrix entries of $X \in G = \text{SO}(n)$; so, for example, $X_{11}X_{22} \in V_{\leq 2}$. We also let $V_{=d} := V_{\leq d} \cap (V_{\leq d-1})^\perp$, for each $d \in \mathbb{N}$. Given $f \in L^2(G)$, we let $f^{\leq d}$ denote the orthogonal projection of f onto $V_{\leq d}$, and we let $f^{=d}$ denote the orthogonal projection of f onto $V_{=d}$, so that $f^{=d} = f^{\leq d} - f^{\leq d-1}$. The subspaces

$V_{\leq d}$ and $V_{=d}$ are two-sided ideals of $L^2(G)$ (i.e., they are closed under both left and right actions of G on $L^2(G)$). Now, if J is a two-sided ideal of $L^2(G)$ and $T : L^2(G) \rightarrow L^2(G)$ is a linear operator that commutes with either the left or the right action of G (as will be the case with all the operators we will work with), it follows from the classical representation theory of compact groups (viz., the Peter-Weyl theorem and Schur's lemma) that T has J as an invariant subspace. Hence, such an operator T has each $V_{=d}$ as an invariant subspace, so each eigenspace of T can be taken to be within one of the $V_{=d}$'s. It therefore makes sense to consider quasirandomness relative to the degree decomposition. For each $d \in \mathbb{N}$, we let D_d be the smallest dimension of a subrepresentation of the G -representation $V_{=d}$. The obvious adaptation of the Sarnak-Xue trick, described above, then yields that for any eigenvalue λ of T with eigenspace within $V_{=d}$, we have $D_d|\lambda|^2 \leq \text{Trace}(T^2)$. It turns out that D_d grows very fast with d , yielding very strong upper bounds on the corresponding $|\lambda|$ for large d .

On the other hand, an ideal level d inequality would imply that if A is an indicator of a small set, then most of its mass lies on the high degrees. This combines with the fast growth of D_d (with d) to give a much more powerful form of quasirandomness, one that takes into account the fact that f is $\{0, 1\}$ -valued, and gives much better bounds.

We remark that the above degree decomposition can be easily extended to all compact linear Lie groups $G \leq GL_n(\mathbb{C})$ by letting $V_{\leq d}$ be the space of degree $\leq d$ polynomials in the real and imaginary parts of the matrix entries of $X \in G$. (In fact, this notion generalizes fairly easily to arbitrary compact simple Lie groups, even when they are not linear.) As in the $SO(n)$ case, we let $f^{\leq d}$ denote the orthogonal projection of f onto $V_{\leq d}$.

We obtain the following level d inequality.

Theorem 1.4. *There exists absolute constants $c, C > 0$ such that the following holds. Let G be a simple compact Lie group equipped with its Haar probability measure μ . Suppose that $D(G) \geq C$. Let $A \subseteq G$ be a measurable set with $\alpha := \mu(A) \geq e^{-cD(G)}$. Then for each $d \in \mathbb{N} \cup \{0\}$ with $d \leq \log(1/\alpha)$, we have $\|f^{\leq d}\|_2^2 \leq \alpha^2 \left(\frac{2 \log(1/\alpha)}{d} \right)^{Cd}$.*

When G is simply connected and $d \leq c\sqrt{n}$ we are able to obtain an even stronger level d inequality, which is similar to the one on the Boolean cube without the extra C factor in the exponent. This leads to the following.

Theorem 1.5. *There exists absolute constants $C, c > 0$ such that the following holds. Let G be a compact connected Lie group, let \tilde{G} denote its universal cover, and write $n = D(\tilde{G})$. Suppose that $n \geq C$. Let $A \subseteq G$ be a measurable set with $\alpha := \mu(A) \geq \exp(-cn^{1/2})$. Then for each $d \in \mathbb{N} \cup \{0\}$ with $d \leq \log(1/\alpha)$, we have $\|f^{\leq d}\|_2^2 \leq \alpha^2 \left(\frac{C \log(1/\alpha)}{d} \right)^d$.*

It is this second level d -inequality that is responsible for the $1/3$ in the exponent of Theorem 1.2. Unfortunately, one would not be able to improve our $1/3$ in the exponent to the (conjectural) right one, merely by strengthening this level d -inequality. Indeed, our second level d inequality can be easily seen to be sharp up to the value of the absolute

constant C , by considering sets of the form $\{A \in \text{SO}(n) : \langle Ae_1, e_1 \rangle > 1 - t\}$ for appropriate values of t , when $G = \text{SO}(n)$, for example.

Both of our level d inequalities are inspired by the same ideas from the Boolean setting, together with an extra representation theoretic ideas. Namely, in order to show a level d inequality, we upper-bound q -norms of low degree polynomials in terms of their 2-norms, and then use Hölder's inequality. In the Boolean cube, such upper bounds follow from two facts. The first is that the noise operator T_ρ is hypercontractive. The second is that all the eigenvalues of the restriction of T_ρ to $V_{\leq d}$ are bounded from below by ρ^d . Our approach is to construct operators on $L^2(G)$ that satisfy the same two properties.

Differential geometry and Markov diffusion processes

Our level d inequalities stem from two techniques for obtaining hypercontractivity. Our first level d inequality, Theorem 1.4, is obtained via the following method. First, we observe that we may assume without loss of generality that our group G is simply connected. (This is because every compact simple Lie group is a quotient of its universal cover by a discrete subgroup of its centre.) We then make use of classical lower bounds on the Ricci curvature of our (simply connected) compact simple Lie group. The Bakry-Emery criterion [4] translates such lower bounds on the Ricci curvature into log-Sobolev inequalities for the Laplace-Beltrami operator L . We then apply an inequality of Gross [16] to deduce a hypercontractive inequality for the operator e^{-tL} from the log-Sobolev inequality. This inequality then allows us to prove our first level d inequality. The operator e^{-tL} is the one corresponding to Brownian motion on G . In order to deduce our level d -inequality we rely on a formula for the eigenvalues of the Laplacian in terms of a step vector corresponding to each eigenspace. This formula is well-known in the theory of Lie groups; it is given for example in Berti and Procesi [7].

Random walks on bipartite graphs

There are two mutually adjoint linear operators that correspond to a random walk on a d -regular bipartite graph $B \subseteq L \times R$. We denote those by $T: L^2(L) \rightarrow L^2(R)$ and $T^*: L^2(R) \rightarrow L^2(L)$ and they are given by taking expectations over a random neighbour; explicitly, $(Tf)(x) = \mathbb{E}_{y \sim x} f(y)$ for $f \in L^2(L)$ and $x \in R$, and $(T^*g)(y) = \mathbb{E}_{x \sim y} g(x)$ for $g \in L^2(R)$ and $y \in L$. It is easy to see that both operators are contractions with respect to any norm. It turns out that given such a bipartite graph and given a hypercontractive operator S on R one gets for free that the operator T^*ST is hypercontractive. Filmus et al [11] used this idea to obtain a ‘non-Abelian’ hypercontractive estimate for ‘global’ functions on the symmetric group, from an ‘Abelian’ hypercontractive result for ‘global’ functions on $(\mathbb{Z}_n)^n$. (Informally, a ‘global’ function is one where one cannot increase the expectation very much by restricting the values of a small number of coordinates.)

In this work, we extend this idea to the continuous domain, by replacing a bipartite graph by a coupling of two probability distributions. Specifically, we consider the probability space $(\mathbb{R}^{n \times n}, \gamma)$ of n by n Gaussian matrices (i.e., $\mathbb{R}^{n \times n}$ with each entry being an independent

standard Gaussian), and the Haar measure on $O(n)$. For $(\mathbb{R}^{n \times n}, \gamma)$, the Ornstein–Uhlenbeck operator U_ρ is a hypercontractive analogue of the noise operator from the Boolean case. We couple $(\mathbb{R}^{n \times n}, \gamma)$ with $SO(n)$ by applying the Gram–Schmidt operation on the columns of a given Gaussian matrix (flipping the sign of the last column, if necessary, so as to ensure that the determinant is equal to one). We note that essentially the same coupling has been used before, e.g. by Jiang [23]; however, it has not been used before (to our knowledge) to analyse the distribution of high-degree polynomials in the matrix-entries, which is crucial in our work.

This coupling gives rise to operators T_{col} and T_{col}^* , similar to the ones in the discrete case. The hypercontractive inequality for the Ornstein–Uhlenbeck operator U_ρ , together with our coupling implies a hypercontractive inequality for the operator $T'_\rho := T_{\text{col}}^* U_\rho T_{\text{col}}$. We then use a symmetrization trick to obtain an operator $T_\rho := \mathbb{E}_{B \sim \mu} R_B^* T'_\rho R_B$, where R_B corresponds to right multiplication by B . The symmetrization does not change the hypercontractive properties, which are the same as for U_ρ (see Theorem 8.8), but it has the advantage of allowing us to analyse more easily the eigenvalues of the operator.

Representation theory

The hypercontractive inequality for the operator T_ρ is useful due to the fact that it immediately gives bounds on the norms of eigenfunctions of T_ρ . Because of the symmetrization, T_ρ commutes with the action of G from both sides. Therefore, the Peter-Weyl theorem implies that every isotypical² component of $L^2(G)$ is contained in an eigenspace of T_ρ .

We eventually show that the eigenvalues of the restriction of T_ρ to V_d are at least $(c\rho)^d$, for some absolute constant $c > 0$. This implies that T_ρ is indeed a good analogue of the noise operator on the Boolean cube, and of the Ornstein–Uhlenbeck operator U_ρ , i.e. the noise operator on Gaussian space. We obtain this lower bound by showing that each isotypical component contains certain functions that are nice to deal with, functions we call the *comfortable juntas*.

The latter are defined as follows. We define a *d-junta* to be a function in the matrix entries of $X \in SO(n)$ that depends only upon the upper-left d by d minor of X . Such a *d-junta* is said to be *comfortable d-junta* if it is contained in the linear span of the monomials $\{m_\sigma : \sigma \in S_d\}$, where $m_\sigma : SO(n) \rightarrow \mathbb{R}$ is defined by $m_\sigma(X) = \prod_{i=1}^d X_{i,\sigma(i)}$ for each $X \in SO(n)$, for each permutation $\sigma \in S_d$.

Random matrix theory

One of the main discoveries of random matrix theory is that the entries of a random orthogonal matrix behave (in an appropriate sense) like independent Gaussians of the same expectation and variance: at least, when one restricts minors of the matrix that are not too large. (In fact, this holds for many different models of random matrices, not just the orthogonal ensemble.) The power of this discovery is of course that a Gaussian random

²If ρ is an irreducible representation of G and V is a G -module, the ρ -isotypical component of V is the sum of all subrepresentations of V that are isomorphic to ρ .

matrix is *a priori* much easier to analyse than e.g. the random matrix given by the Haar measure on a group.

One way to test that two distributions are similar is to apply a continuous ‘test function’ and take expectations. Usually, for applications in random matrix theory, the test function can be taken to be an arbitrary fixed polynomial.

When computing the eigenvalues of our operator T_ρ we need to show a similarity in distribution between the upper $d \times d$ -minor of $O(n)$ and the $d \times d$ minor of a random Gaussian matrix. For us, however, it is not sufficient to look at a single polynomial of fixed degree. Instead, we need to show a similarity in the distribution with respect to our comfortable d -juntas (where d may be as large as \sqrt{n} , rather than an absolute constant). Hence, while the philosophy is similar to that of random matrix theory, we require new techniques enabling us to deal with the distributions of polynomials whose degrees may be a function of n , indeed up to \sqrt{n} .

1.3 Applications

In this section we list several applications of our hypercontractive theory: to some problems in group theory, in geometry, and in probability.

To state some of our results, we need some more terminology. If G is a compact connected Lie group, we define $n(G) := D(\tilde{G})$, where \tilde{G} denotes the universal cover of G . It is well-known that, for each $m \in \mathbb{N}$, we have $D(\mathrm{SU}(m)) = D(\mathrm{Spin}(m)) = m$ and $D(\mathrm{Sp}(m)) = 2m$ (and all these groups are simply connected except for $\mathrm{Spin}(2)$); we also have $D(\mathrm{SO}(m)) = m$. Since $\mathrm{Spin}(m)$ is the universal cover of $\mathrm{SO}(m)$ for all $m > 2$, we have $n(\mathrm{SO}(m)) = m$ for all $m > 2$. As we will see in the next section, any compact connected semisimple Lie group G with $D(G)$ at least an absolute constant, can be written in the form $(\prod_{i=1}^r K_i)/F$ where each K_i is one of $\mathrm{SU}(n_i)$, $\mathrm{Spin}(n_i)$ or $\mathrm{Sp}(n_i)$ for some $n_i \geq 3$, and F is a finite subgroup of the centre of $\prod_{i=1}^r K_i$; the universal cover of such is $\prod_{i=1}^r K_i$, and $D(\prod_{i=1}^r K_i) = \Theta(\min_i n_i)$. Hence, the quantity $n(G)$ has a very explicit description in terms of the structure of the Lie group G .

Growth in groups: the diameter problem

The theory of growth in groups has been a very active area of study in recent decades, and an important class of problem in this area is to determine the diameter of a metric space defined by a group (e.g., the diameter of a Cayley graph of the group). For a compact group G equipped with its Haar probability measure, and a measurable generating set $\mathcal{A} \subseteq G$ of measure μ , it is natural to consider the metric space on G where the distance between x and y is defined to be the minimal length of a word in the elements of \mathcal{A} and their inverses which is equal to xy^{-1} . The diameters of such metric spaces in the case where G is finite have become a focus of intense study in the last two decades: see e.g. the works of Liebeck and Shalev [31], Helfgott [18], Helfgott and Seress [20], Pyber and Szabo [36] and Breuillard, Green and Tao [8].

For a subset \mathcal{A} of a group G and $t \in \mathbb{N}$, we define

$$\mathcal{A}^t := \{a_1 \cdot a_2 \cdots a_t \mid a_1, a_2, \dots, a_t \in \mathcal{A}\}.$$

The *diameter problem for G with respect to \mathcal{A}* asks for the smallest positive integer t for which $\mathcal{A}^t = G$. For a compact group G and a real number $0 < \alpha \leq 1$, the *diameter problem for sets of measure α in G* asks for the minimum possible diameter of a measurable set in G of measure α .

In the case where G is a compact and connected group, we note that the diameter of G with respect to any subset \mathcal{A} of positive measure is finite. This follows (almost) immediately from Kemperman's theorem [29], which states that for any compact connected group G (equipped with its Haar probability measure μ) and any measurable $\mathcal{A}, \mathcal{B} \subset G$, we have $\mu(\mathcal{A}\mathcal{B}) \geq \min\{\mu(\mathcal{A}) + \mu(\mathcal{B}), 1\}$.

We make the following conjecture, concerning the diameter of large sets.

Conjecture 1.6. *Let G be one of $\mathrm{SU}(n), \mathrm{SO}(n), \mathrm{Spin}(n)$ or $\mathrm{Sp}(n)$, and let $\mathcal{A} \subseteq G$ be a measurable subset of measure ν . Then the diameter of G with respect to \mathcal{A} is $O(\nu^{-1/(\ell n)})$, where $\ell = 1$ in the case of $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$, $\ell = 2$ in the case of $\mathrm{SU}(n)$, and $\ell = 4$ in the case of $\mathrm{Sp}(n)$. In particular, if $\nu \geq e^{-cn}$, then the diameter of G with respect to \mathcal{A} is at most $O_c(1)$.*

We note that if true, the conjecture is essentially tight, as can be seen for $\mathrm{SO}(n)$ by considering the set

$$\mathcal{S}_\varepsilon := \{X \in \mathrm{SO}(n) : \text{the angle between } Xe_1 \text{ and } e_1 \text{ is at most } \varepsilon\},$$

For $\varepsilon \leq 1/2$, we have $\mu(\mathcal{S}_\varepsilon) = (\Theta(\varepsilon))^n$, and the diameter of $\mathrm{SO}(n)$ with respect to \mathcal{S}_ε is $\Theta(1/\varepsilon)$. If $\pi : \mathrm{Spin}(n) \rightarrow \mathrm{SO}(n)$ is the usual (double) covering homomorphism, then the lift $\pi^{-1}(\mathcal{S}_\varepsilon)$ is a subset of $\mathrm{Spin}(n)$ of the same measure as \mathcal{S}_ε (using, of course, the Haar probability measure on both groups), and the diameter of $\mathrm{Spin}(n)$ with respect to $\pi^{-1}(\mathcal{S}_\varepsilon)$ is the same the diameter of $\mathrm{SO}(n)$ with respect to \mathcal{S}_ε , since $\pi(\mathcal{A}^t) = (\pi(\mathcal{A}))^t$ for any subset $\mathcal{A} \subset \mathrm{Spin}(n)$ and any $t \in \mathbb{N}$. Hence, $\pi^{-1}(\mathcal{S}_\varepsilon)$ demonstrates tightness for $\mathrm{Spin}(n)$. The group $\mathrm{SU}(n)$ acts transitively on the unit sphere in \mathbb{C}^n , which can be identified with S^{2n-1} , and the group $\mathrm{Sp}(n)$ acts transitively on the unit sphere in \mathbb{H}^n , which can be identified with S^{4n-1} ; both actions are angle-preserving (in S^{2n-1} and S^{4n-1} respectively). So our above construction for $\mathrm{SO}(n)$ (which comes from the action of $\mathrm{SO}(n)$ on S^{n-1}) has the obvious analogues for $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$, which we conjecture are sharp for those groups.

We show that for a compact connected Lie group G with $n(G) = n$, for all $\delta > 0$ and all measurable subsets \mathcal{A} of G with measure at least $2^{-cn^{1-\delta}}$, the diameter of G with respect to \mathcal{A} is at most $O_\delta(1)$.

Theorem 1.7. *For each $\delta > 0$ there exist $n_0, k > 0$ such that the following holds. Let $n > n_0$ and let G a compact connected Lie group with $n(G) = n$. If $\mathcal{A} \subset G$ is a Haar-measurable set, and $\mu(\mathcal{A}) \geq 2^{-n^{1-\delta}}$, then $\mathcal{A}^k = G$.*

Doubling inequalities for groups

Theorem 1.7 follows from a new lower bound on $\mu(\mathcal{A}^2)$, where $\mathcal{A} \subseteq G$ is a measurable subset of the compact connected Lie group G . We prove the following ‘doubling inequality’.

Theorem 1.8. *There exists absolute constants $C, c > 0$ such that the following holds. Let G be a compact connected Lie group with $n(G) = n \geq C$. Let $\mathcal{A} \subseteq G$ be a measurable set with $\mu(\mathcal{A}) \geq e^{-cn}$. Then $\mu(\mathcal{A}^2) \geq \mu(\mathcal{A})^{0.1}$.*

The problem of giving a lower bound on $\mu(\mathcal{A}^2)$ in terms of $\mu(\mathcal{A})$, for \mathcal{A} a measurable subset of a compact group G , dates back to the work of Henstock and Macbeath [21] from 1953, the aforementioned bound of Kemperman [29] from 1964, and the work of Jenkins [22] from 1973. Several recent works of Jing, Tran and Zhang have introduced some powerful new methods into the field. For instance, in [26], Jing, Tran and Zhang generalized the Brunn-Minkowskii inequality from \mathbb{R}^n to an arbitrary connected Lie group, using the Iwasawa decomposition to facilitate an inductive approach; their result is essentially sharp for helix-free Lie groups. In [25], they used techniques from O -minimal geometry to show that $\mu(\mathcal{A}^2) \geq 3.99\mu(\mathcal{A})$ for all measurable subsets $\mathcal{A} \subseteq \mathrm{SO}(3)$ of sufficiently small measure. In a forthcoming paper [24] they prove that there exists a function $\delta = \delta(n)$ and an absolute constant $c > 0$, such that if $\mathcal{A} \subseteq \mathrm{SO}(n)$ is a measurable set of measure at most $\delta(n)$, then $\mu(\mathcal{A}^2) \geq 2^{cn^{1/10}}\mu(\mathcal{A})$; the function $\delta(n)$ satisfies $\delta(n) \leq 2^{-n^{1+c'}}$ where $c' > 0$ is an absolute constant. (For comparison, we note that Theorem 1.8, in conjunction with Theorem 1.10 below, imply the existence of an absolute constant $c > 0$ such that $\mu(\mathcal{A}^2) \geq \min\{2^{cn^{1/2}}\mu(\mathcal{A}), 0.99\}$ for all measurable subsets $\mathcal{A} \subseteq \mathrm{SO}(n)$ of measure at least 2^{-cn} , so our result and that of Jing, Tran and Zhang leave a ‘gap’ between them.) It remains an open problem to determine whether $\mu(\mathcal{A}^2) \geq \min\{2^{cn^{1/10}}\mu(\mathcal{A}), 0.99\}$ for all subsets $\mathcal{A} \subseteq \mathrm{SO}(n)$.

Spectral gaps

We also give the following upper bound on the spectral gaps of the operator corresponding to convolution by $\frac{1_A}{\mu(A)}$. If G is a compact group equipped with its (unique) Haar probability measure μ , for a measurable set $\mathcal{A} \subseteq G$ we write $x \sim \mathcal{A}$ to mean that x is chosen (conditionally) according to the Haar measure μ , conditional on the event that $x \in \mathcal{A}$.

Theorem 1.9. *There exist absolute constants $c, C > 0$ such that the following holds. Let G be compact connected Lie group and suppose $n := n(G) \geq C$. Let $\mathcal{A} = \mathcal{A}^{-1}$ be a symmetric, measurable set in G and suppose that $\mu(\mathcal{A}) \geq e^{-cn^{1/2}}$. Then the nontrivial spectrum of the operator T defined by $Tf(x) = \mathbb{E}_{a \sim \mathcal{A}}[f(ax)]$ is contained in the interval*

$$\left[-\sqrt{\frac{C \log 1/\alpha}{n}}, \sqrt{\frac{C \log 1/\alpha}{n}} \right].$$

Mixing times

Let G be a compact group, equipped with its (unique) Haar probability measure; then every measurable subset $\mathcal{A} \subseteq G$ of positive Haar measure corresponds to a random walk

on G . Indeed, we may define a (discrete-time) random walk $R_{\mathcal{A}} = (X_t)_{t \in \mathbb{N} \cup \{0\}}$ on G , by letting $X_0 = \text{Id}$, and for each $t \in \mathbb{N}$, if $X_{t-1} = x$ then $X_t = ax$, where a is chosen uniformly at random from \mathcal{A} . In the case where G is finite and \mathcal{A} is closed under taking inverses, this is the usual random walk associated to the Cayley graph $\text{Cay}(G, \mathcal{A})$. One of the fundamental problems associated to such random walks is to determine their *mixing time*. (Following Larsen and Shalev [30], we say that the *mixing time* of a Markov chain $(X_t)_{t \in \mathbb{N} \cup \{0\}}$ is the minimal non-negative integer T such that the total variation distance between the distribution of X_T and the uniform distribution, is at most $1/e$. We note that $1/e$ could be replaced by any other absolute constant $c \in (0, 1)$, without materially altering the definition; Larsen and Shalev use the constant $1/e$ as it makes the statement of certain results concerning S_n and A_n more elegant.)

Larsen and Shalev [30] considered the case where \mathcal{A} is a normal set, i.e. a set closed under conjugation, and G is the alternating group A_n . They showed that for each $\varepsilon > 0$, if $\mathcal{A} \subseteq A_n$ of density $\frac{2|\mathcal{A}|}{n!} \geq \exp(-n^{1/2-\varepsilon})$, then the mixing time of $R_{\mathcal{A}}$ is 2, provided that $n \geq n_0(\varepsilon)$ is sufficiently large depending on ε . Their proof was based upon a heavy use of character theory. Their result is almost sharp, in the sense the number $1/2$ cannot be replaced by any number smaller than $1/2$. We show that a similar phenomenon holds for compact connected Lie groups, even when \mathcal{A} is not a normal set.

Theorem 1.10. *There exist absolute constants $c, n_0 > 0$, such that the following holds. Let G be a compact connected Lie group with $n := n(G) > n_0$. Let $\mathcal{A} \subseteq G$ be a measurable set with Haar measure at least $e^{-cn^{1/2}}$. Then the mixing time of the random walk $R_{\mathcal{A}}$ is 2.*

This result is essentially best possible. For instance, taking $G = \text{SO}(n)$, we may take $\mathcal{A} = \{X \in \text{SO}(n) : X_{11} > 10/n^{1/4}\}$. It is easy to see that the mixing time of $R_{\mathcal{A}}$ is 3, while $\mu(\mathcal{A}) = \exp(-\Theta(n^{1/2}))$.

Product mixing

Gowers' proof of his upper bound on the sizes of product-free sets actually establishes a stronger phenomenon, known as *product mixing*. We say that a compact group G (equipped with its Haar probability measure μ) is an (α, ε) -*mixer* if for all sets $A, B, C \subseteq G$ of Haar probability measures $\geq \alpha$, when choosing independent uniformly random elements $a \sim A$ and $b \sim B$, the probability that $ab \in C$ lies in the interval $(\mu(C)(1 - \varepsilon), \mu(C)(1 + \varepsilon))$. Gowers' proof actually yields the following statement: there exists an absolute constant $C > 0$, such that if G is a D -quasirandom compact group, then it is a $(CD^{-1/3}, 0.01)$ -mixer. (The proof is given only for finite groups, but it generalises easily to all compact groups.) For finite groups, Gowers' product-mixing result is sharp up to the value of the constant C . Here, we obtain an analogous result for compact connected Lie groups, where the $n^{-1/3}$ moves to the exponent.

Theorem 1.11. *For any $\varepsilon > 0$, there exist $c, n_0 > 0$ such that the following holds. Let $n > n_0$ and let G be a compact connected Lie group with $n := n(G) > n_0$. Set $\alpha = \exp(-cn^{-1/3})$. Then G is an (α, ε) -mixer.*

This result is sharp up to the dependence of the constants $c = c(\varepsilon)$ and $n_0 = n_0(\varepsilon)$ upon ε . Indeed, we may take $G = \text{SO}(n)$ and let $\mathcal{A} = \mathcal{B} = \{X \in \text{SO}(n) : X_{11} > 10/n^{1/3}\}$ and $\mathcal{C} = \{X \in \text{SO}(n) : X_{11} < -10/n^{1/3}\}$, to obtain a triple of sets each of measure $e^{-\Theta(n^{1/3})}$, such that when choosing $a \sim \mathcal{A}$ and $b \sim \mathcal{B}$ independently, the probability that $ab \in \mathcal{C}$ is smaller than $\frac{1}{2}\mu(\mathcal{C})$.

Homogeneous dynamics and equidistribution

Suppose that a compact Lie group G acts on a topological space X . The space X is said to be G -homogeneous if G acts transitively and continuously on X (the latter meaning that the action map from $G \times X$ to X is continuous); in this case, X has a unique G -invariant probability measure, which is called the Haar probability measure. We obtain the following equidistribution result for homogeneous spaces.

Theorem 1.12. *For each $\varepsilon > 0$ there exist $c, n_0 > 0$ such that the following holds. Let G be a compact connected Lie group with $n(G) =: n > n_0$. Let X be a G -homogeneous topological space, and let μ_X denotes its unique G -invariant (Haar) probability measure. Suppose that $\mathcal{A} \subseteq G$ and $\mathcal{B} \subseteq X$ are both measurable sets of Haar probability measures $\geq e^{-cn^{1/2}}$. Let ν be the probability measure on X which is given by the distribution of $a(b)$, for a uniform random $a \sim \mathcal{A}$ and an (independent) uniform random $b \sim \mathcal{B}$. Then the total variation distance between μ and ν is less than ε .*

L^q -norms of low degree polynomials

We obtain the following upper bounds on the q -norms of low degree polynomials (we state the result for $\text{SO}(n)$, for simplicity).

Theorem 1.13. *There exist absolute constants $c, C > 0$ such that the following holds. Let $q > 2$ and let $f \in L^2(\text{SO}(n))$ be a polynomial of degree d in the matrix entries of $X \in \text{SO}(n)$. If $d \leq cn$, then*

$$\|f\|_{L^q(\mu)} \leq q^{Cd} \|f\|_{L^2(\mu)}.$$

If moreover, $d \leq c\sqrt{n}$, then

$$\|f\|_{L^q(\mu)} \leq (C\sqrt{q})^d \|f\|_{L^2(\mu)}.$$

Separation of quantum and classical communication complexity

Starting with their introduction to computer science in the seminal paper of Kahn Kalai and Linial [27], hypercontractive inequalities have found a huge number of applications in various branches of computer science and related fields (see e.g. [13, 32, 10, 37], to name but a few). While these applications have generally required hypercontractive inequalities for functions on discrete sets, some applications require continuous domains. For example, in the paper of Klartag and Regev [38], a hypercontractive inequality for functions on the n -sphere is used to obtain a lower bound on the number of (classical) communication

bits required for two parties to jointly compute a certain function. While with quantum communication, the value of that function can be computed by one party transmitting only $O(\log n)$ quantum-bits to the other, it was shown in [38] that classical communication requires at least $\Omega(n^{1/3})$ bits of communication to be sent, even if parties are allowed to send bits both ways, thereby showing an exponential separation between the power of classical communication and that of one-way quantum communication.

In the field of quantum communication, establishing a significant separation between classical communication and practically implemented modes of quantum communication remains a major open problem. In a forthcoming work, Arunachalam, Girish, and Lifshitz [2] apply our hypercontractive inequality for $SU(n)$ to make a substantial step towards this goal. They used it to obtain an exponential separation between classical communication and a more realistic version of quantum communication, namely the one-clean-qubit model.

2 Preliminaries: Quasirandomness and min-rank

In this section we show that, for D at least an absolute constant, the universal cover \tilde{G} of a D -quasirandom compact connected Lie group G is a product of ‘classical’ (compact, simple, simply connected) Lie groups of the form $\text{Spin}(n), \text{SU}(n), \text{Sp}(n)$. We then make use of this to determine $D(\tilde{G})$, and we show that $D(\tilde{G}) \leq 4D(G)^2$.

In what follows, as usual, a *compact group* G is a Hausdorff topological group for which the group operations (or equivalently, the map $(g, h) \mapsto gh^{-1}$) are continuous. We recall that a compact group has a unique left-multiplication-invariant probability measure (called the Haar measure), which is also the unique right-multiplication-invariant probability measure. As usual, if G is a compact group, we let

$$L^2(G) = \{f : G \rightarrow \mathbb{C} : \mathbb{E}_\mu[|f|^2] < \infty\} / \sim,$$

where the expectation is with respect to the Haar probability measure μ on G and the equivalence relation \sim is defined by $f \sim g$ iff $f = g$ μ -almost-everywhere, and we view $L^2(G)$ as a Hilbert space, with the natural inner product,

$$\langle f, g \rangle := \mathbb{E}_\mu[\bar{f}g].$$

We make use of the following fact, appearing for example in [35] Chapter 10, Section 7.2, Theorem 4, page 380. (Note that the word ‘Lie’ is missing from the statement of this theorem; this omission is clearly just a typographical error.)

Fact 2.1. *Every compact connected Lie group is Lie-isomorphic to a group of the form $(\prod_{i=1}^r K_i \times T)/F$, where each K_i is a simply connected simple compact Lie group (equivalently, K_i is one of $\text{Sp}(n_i)$ for some $n_i \geq 1$, $\text{Spin}(n_i)$ for some $n_i \geq 3$, $\text{SU}(n_i)$ for some $n_i \geq 2$, or the compact form of one of the five exceptional Lie groups, for each $i \in [r]$), T is a finite-dimensional torus (i.e. $T = (\mathbb{R}/\mathbb{Z})^m$ for some integer m), and F is a finite group contained in the center of $\prod_{i=1}^r K_i \times T$, with $F \cap T = \{1\}$.*

Lemma 2.2. *Suppose that $D > 1$ and that G is a D -quasirandom compact connected Lie group. Then G is semisimple.*

Proof. Write $G = (\prod_{i=1}^r K_i \times T)/F$, where each K_i is a simply connected simple compact Lie group, T is a finite-dimensional torus (i.e. $T = (\mathbb{R}/\mathbb{Z})^m$ for some integer m), and F is a finite group contained in the center of $\prod_{i=1}^r K_i \times T$, with $F \cap T = \{1\}$, as in the above fact. Semisimplicity of G is equivalent to $T = \{1\}$. Suppose on the contrary that $T \neq \{1\}$. Let π be the projection map from $\prod_{i=1}^r K_i \times T$ onto the T component. Since $F \cap T = \{1\}$, we have $\pi(F) = \{1\}$, so the projection π induces a (surjective) group homomorphism $\tilde{\pi}$ from G to T , and therefore G has a quotient isomorphic to $(\mathbb{R}/\mathbb{Z})^m$ for some integer $m \geq 1$; any nontrivial complex one-dimension irreducible representation of the latter quotient lifts to one of G , contradicting the D -quasirandomness of G (for any $D > 1$) and proving the lemma. \square

We also recall the following standard fact.

Fact 2.3. *Every compact semisimple Lie group has finite centre.*

We now show that if G is sufficiently quasirandom, then the exceptional groups do not make an appearance as some K_i when writing $G = (\prod_{i=1}^r K_i)/F$.

Lemma 2.4. *Set $D_0 = 248$. Let G be a compact connected Lie group, and suppose that it is D -quasirandom for some $D > D_0$. Then $G = (\prod_{i=1}^r K_i)/F$, where each K_i is one of $\mathrm{Sp}(n_i)$, $\mathrm{Spin}(n_i)$, $\mathrm{SU}(n_i)$ for some $n_i \geq \sqrt{D}/2$ and F is a subgroup of the (finite) centre of $\prod_{i=1}^r K_i$.*

Proof. By the previous lemma, provided $D_0 > 1$, G is semisimple. Hence, we may write $G = (\prod_{i=1}^r K_i)/F$, where each K_i is one of $\mathrm{Sp}(n_i)$, $\mathrm{Spin}(n_i)$, $\mathrm{SU}(n_i)$ or the compact form of one of the five exceptional Lie groups, for each i , and F is a finite group contained in the (finite) center of $\prod_{i=1}^r K_i$. As the quotient of a D -quasirandom group is D -quasirandom, we may project to any one of the components and still obtain a D -quasirandom group K_i/F' . (In detail, let π_i denote projection of $\prod_{j=1}^r K_j$ onto the K_i factor; π_i induces a surjective homomorphism from G onto $K_i/\pi_i(F)$, and since F is a subgroup of the centre of $\prod_{j=1}^r K_j$, F_i is a subgroup of the centre of K_i . The group $K_i/\pi_i(F)$ is therefore a quotient of G , and so inherits its D -quasirandomness.) It is therefore sufficient to consider the case where $G = K_1/F'$. We now note that the adjoint representation of K_1 factors through K_1/F' (since F' is contained in the centre of K_1), so it can also be viewed as a representation of K_1/F' . As the adjoint representation of K_1 is not a sum of copies of the trivial representation (this follows from the fact that K_1 is non-Abelian), its dimension (which is the same as the dimension of the Lie group K_1) is at least D . The five exceptional Lie groups, E_6 , E_7 , E_8 , F_4 and G_2 , have dimensions 78, 133, 248, 52 and 14 respectively, so K_1 cannot equal any of these (since $D > D_0 = 248$). Hence, K_1 is one of $\mathrm{Sp}(n_1)$, $\mathrm{Spin}(n_1)$ or $\mathrm{SU}(n_1)$. The dimensions of these Lie groups are $n_1(2n_1 + 1)$, $n_1(n_1 - 1)/2$ and $n_1^2 - 1$ respectively, so we obtain $n_1(2n_1 + 1) \geq D$, which implies that $n_1 \geq \sqrt{D}/2$. This completes the proof of the lemma. \square

The following (non-standard) definition will be convenient for us.

Definition 2.5. Let G be a compact, connected, semisimple Lie group and write $G = (\prod_{i=1}^r K_i)/F$, where, as above, K_i is one of $\mathrm{Sp}(n_i)$, $\mathrm{Spin}(n_i)$ or $\mathrm{SU}(n_i)$ for each i , and F is a finite subgroup of the (finite) centre of G . We define the min-rank of G to be $\min\{n_1, \dots, n_r\}$.

Using this terminology, the above lemma can be restated by saying that if a compact connected Lie group G is D -quasirandom for large enough D , then it has min-rank at least $\sqrt{D}/2$.

We remark that the *rank* of a Lie group is defined to be the dimension of any one of its Cartan subgroups, so the ranks of $\mathrm{Sp}(n_i)$, $\mathrm{Spin}(n_i)$ and $\mathrm{SU}(n_i)$ are respectively n_i , $\lfloor n_i/2 \rfloor$ and $n_i - 1$, so in particular are all $\Theta(n_i)$; hence, while the min-rank of G is not exactly the minimum of the ranks of the K_i 's (where the K_i 's are as above), it is within an absolute constant factor thereof. (We hope this slight abuse of terminology will not cause confusion.)

To establish that $D(\tilde{G}) \leq 4D(G)^2$ we also need the following.

Lemma 2.6. Let G be a simply compact semisimple Lie group of min-rank m . Then its universal cover \tilde{G} satisfies $D(\tilde{G}) \in \{m, 2m\}$.

Proof. Write $G = \prod_{i=1}^r K_i/F$. As the projection map from $\prod_{i=1}^r K_i$ to G is a cover map, and since $\prod_{i=1}^r K_i$ is simply connected, we obtain that $\tilde{G} = \prod_{i=1}^r K_i$. The lemma now follows from the fact that the complex irreducible representations of a product $\prod_{i=1}^r K_i$ of finitely many compact groups are tensor products of complex irreducible representations, of the form $\rho_1 \otimes \dots \otimes \rho_r$ where ρ_i is an complex irreducible representation of ρ_i , for each i together with the fact that $D(\mathrm{SU}(n)) = D(\mathrm{SO}(n)) = n$, $D(\mathrm{Sp}(n)) = 2n$. \square

Lemma 2.6 shows that, when proving the theorems in the introduction, we may replace $D(\tilde{G})$ with the min-rank of G .

3 Good groups and fine groups

3.1 Graded groups and ‘good’ groups.

In this section we define some basic properties of compact connected groups, which we later use to prove various growth properties. We define *graded* and *strongly quasirandom* groups, and *hypercontractive* groups; we say that groups satisfying all these properties are *good*. We also define a somewhat weaker (or, technically, incomparable) notion of a *fine* group.

The compact, simple, simply connected real Lie groups of large enough rank, i.e $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$ and $\mathrm{Spin}(n)$, are indeed good (this is proved in Section 8). We show that goodness is preserved when taking products and quotients (quotients, that is, by closed normal subgroups, as usual), thereby showing that every D -quasirandom group is a good graded group, provided D is sufficiently large.

Definition 3.1 (Graded groups). For $n \in \mathbb{N}$, we say a compact connected group G is n -graded if there exists an orthogonal direct sum,

$$L^2(G) = \bigoplus_{d=0}^{\lceil n/2 \rceil - 1} V_{=d} \oplus V_{\geq n/2},$$

such that the spaces $V_{=d}$ are invariant under the action of G from both sides, and where $V_{=0}$ contains only the constant functions. For an n -graded group and an integer $0 \leq d_0 < n/2$,

we denote by $V_{>d_0}$ the direct sum $V_{>d_0} := \bigoplus_{d=d_0+1}^{\lceil n/2 \rceil - 1} V_{=d} \oplus V_{\geq n/2}$. (Note that we will sometimes

write $V_{=d}^G$ in place of $V_{=d}$, when we want to stress that the group in question is G , e.g. if there are several groups involved in our argument.)

Remark 3.2. Note that it follows from the definition of an n -grading that $V_{\geq n/2}$ is also invariant under the action of G from both sides.

Grading for the compact simply connected simple Lie groups (of large enough rank). We note in this section that $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$ and $\mathrm{Spin}(n)$ are all n -graded, for $n \geq 3$. For the group $\mathrm{SO}(n)$, for each integer $0 \leq d < n/2$ we define $V_{\leq d}$ to be the subspace of $L^2(\mathrm{SO}(n))$ spanned by degree $\leq d$ multivariate polynomials in the matrix entries of $X \in \mathrm{SO}(n)$ (so, for example, $V_{\leq 2}$ contains the polynomial $X_{11}X_{12}$). For notational convenience, for $0 < r < d/2$ we define $V_{<r}$ to be $V_{\leq d}$, where d is the maximal integer less than r . We then set $V_{=0} := V_{\leq 0}$ and $V_{=d} := V_{\leq d} \cap (V_{\leq d-1})^\perp$ for each integer $1 \leq d < n/2$, and we define $V_{\geq n/2} := (V_{<n/2})^\perp$. Note that each $V_{=d}$ is finite-dimensional, but $V_{\geq n/2}$ is infinite-dimensional.

For the special unitary group we perform a similar construction, except that one views the complex entries of the input matrix as a pair of real numbers. More precisely, we define $V_{\leq d}$ to consist of the functions f that are degree $\leq d$ multivariate polynomials in the real and imaginary parts of the matrix entries of $X \in \mathrm{SU}(n)$ (so, for example, $V_{\leq 3}$ contains the polynomial $\mathrm{Re}(X_{11})\mathrm{Im}(X_{11})\mathrm{Re}(X_{12})$). As in the $\mathrm{SO}(n)$ case, we then set $V_{=0} := V_{\leq 0}$ and $V_{=d} := V_{\leq d} \cap (V_{\leq d-1})^\perp$ for each integer $1 \leq d < n/2$, and we set $V_{\geq n/2} := (V_{<n/2})^\perp$.

For the compact symplectic group, $\mathrm{Sp}(n)$, we view it as the group of n by n unitary matrices over the field of quaternions (see Section 6.5 for more details), and we define $V_{\leq d}$ to be the vector subspace of $L^2(\mathrm{Sp}(n))$ spanned by degree $\leq d$ multivariate polynomials in the real parts, the \mathbf{i} -parts, the \mathbf{j} -parts and the \mathbf{k} -parts of the matrix entries; we then proceed as in the previous two cases.

For the spin group $\mathrm{Spin}(n)$ things are a little more involved, as it has no straightforward description as a linear group. In order to define the grading we make use of the double covering homomorphism $\pi: \mathrm{Spin}(n) \rightarrow \mathrm{SO}(n)$ (recall that $\mathrm{Spin}(n)$ is the universal covering group of $\mathrm{SO}(n)$, and that this cover is a double cover, for each $n \geq 3$). The covering homomorphism π gives rise to an embedding $i: L^2(\mathrm{SO}(n)) \rightarrow L^2(\mathrm{Spin}(n))$ given by $if =$

$f \circ \pi$. We then take the grading of the spin group to be

$$\mathrm{Spin}(n) = \bigoplus_{d < n/2} i(V_{=d}^{\mathrm{SO}(n)}) \oplus (i(V_{<n/2}^{\mathrm{SO}(n)}))^{\perp},$$

i.e. $V_{=d}^{\mathrm{Spin}(n)} = i(V_{=d}^{\mathrm{SO}(n)})$ for each $0 \leq d < n/2$, and $V_{\geq n/2}^{\mathrm{Spin}(n)} = (i(V_{<n/2}^{\mathrm{SO}(n)}))^{\perp}$.

We remark that the above arguments also imply that $\mathrm{SO}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$ and $\mathrm{Spin}(n)$ are m -graded for all $m \in \mathbb{N}$ and all $n \geq 3$, but we will only require the n -grading, in the sequel.

Next we define a notion of ‘strong quasirandomness’ for graded compact groups. In section 6 we show that the simply connected n -graded groups are all c -strongly-quasirandom for some absolute constant $c > 0$.

Definition 3.3 (Strongly quasirandom graded group). *We say an n -graded compact group G is $((Q_d)_{d=0}^{\lceil n/2 \rceil - 1}, Q)$ -strongly-quasirandom if the minimal dimension of a subrepresentation of $V_{=d}$ (as a left G -module) is $\geq Q_d$ for all integers $0 \leq d \leq \lceil n/2 \rceil - 1$, and is $\geq Q$ for $V_{\geq n/2}$.*

For $c > 0$ we say that the n -graded compact group G is c -strongly-quasirandom if it is $((Q_d)_{d=1}^{\lceil n/2 \rceil - 1}, Q)$ -strongly-quasirandom when we set $Q_d := (\frac{cn}{d})^d$ for $d < cn/(1+c)$, and $Q, Q_d := (1+c)^{cn/(1+c)}$ for $d \geq cn/(1+c)$.

In Section 3.4, we show that all the (infinite families of) compact simply connected simple Lie groups are c -strongly quasirandom for some absolute constant $c > 0$.

Theorem 3.4. *The n -graded compact groups $\mathrm{SU}(n), \mathrm{Sp}(n), \mathrm{Spin}(n)$ (for $n \geq 3$) are all c -strongly quasirandom for some absolute constant $c > 0$, when equipped with our chosen n -grading.*

Definition 3.5 (Beckner operator for graded groups). *Let G be an n -graded compact group, let r be an integer with $0 \leq r < n/2$, and let $0 \leq \delta \leq 1$. We define the Beckner operator $T_{\delta,r} : L^2(G) \rightarrow L^2(G)$ by $T_{\delta,r}(f) := \sum_{i=0}^r \delta^i f^{=i}$, for all $f \in L^2(G)$.*

Definition 3.6 (Hypercontractive group). *Let $C > 0$ and let r be an integer with $0 \leq r < n/2$. We say that an n -graded group compact G is (r, C) -hypercontractive if for every $q \geq 2$ and every $0 \leq \delta \leq 1/(C\sqrt{q})$, we have $\|T_{\delta,r}\|_{2 \rightarrow q} \leq 1$.*

The following is an easy consequence of hypercontractivity.

Lemma 3.7. *Let G be an n -graded (r, C) -hypercontractive group, where $0 \leq r < n/2$. Then for every integer $d \leq r$, every $q \geq 2$ and function $f \in V_{=d}$, we have*

$$\|f\|_q \leq (C^2 q)^{d/2} \|f\|_2.$$

Proof. Let $f \in V_{=d}$. Then $\delta^d \|f\|_q = \|T_{\delta,r} f\|_q \leq \|f\|_2$ for all $\delta \leq 1/(C\sqrt{q})$; setting $\delta = 1/(C\sqrt{q})$ completes the proof. \square

In Section 8 we show that the compact simple simply connected n -graded Lie groups $\mathrm{Sp}(n)$, $\mathrm{SU}(n)$ and $\mathrm{Spin}(n)$ are $(c\sqrt{n}, C)$ -hypercontractive for some positive absolute constants C and c , when they are equipped with our chosen n -grading.

Theorem 3.8. *The groups $\mathrm{Sp}(n)$, $\mathrm{SU}(n)$, $\mathrm{Spin}(n)$ are $(c\sqrt{n}, C)$ -hypercontractive, when equipped with our chosen n -grading, for some positive absolute constants C and c .*

Definition 3.9 (Good groups). *An n -graded compact group G is said to be (C, c) -good if it is $(cn^{1/2}, C)$ -hypercontractive and c -strongly quasirandom.*

The next theorem follows from Theorem 3.8 and Theorem 3.4.

Theorem 3.10. *For each $n \geq 3$, the n -graded groups $\mathrm{Sp}(n)$, $\mathrm{SU}(n)$, $\mathrm{Spin}(n)$ (equipped with our choice of n -grading) are (C, c) -good, for some absolute positive constants C and c .*

We prove Theorem 3.10 in Section 8.

3.2 Fine groups

We now introduce the closely related notion of a ‘fine’ group. It involves a weaker form of hypercontractivity than with a ‘good’ group, but we manage to prove it for higher values of d , viz., up to linear in n .

Definition 3.11 (Weakly hypercontractive group). *Let $C > 1$ and $1 \leq r < n/2$. An n -graded compact group G is (r, C) -weakly hypercontractive if for every function $f \in L^2(G)$ and every $q \geq 2$ and $0 \leq \delta \leq 1/q^C$ we have*

$$\|T_{\delta, r} f\|_q \leq \|f\|_2.$$

The following lemma follows similarly to Lemma 3.7.

Lemma 3.12. *Let G be an n -graded (r, C) -weakly hypercontractive compact group, where $C > 1$ and $1 \leq r < n/2$. Then for any integer $d \leq r$, every $q \geq 2$, and any function $f \in V_{=d}$, we have*

$$\|f\|_q \leq q^{Cd} \cdot \|f\|_2$$

Definition 3.13 (Fine groups). *If $c > 0$ and $C > 1$, an n -graded compact group G is said to be (C, c) -fine if it is both (cn, C) -weakly hypercontractive, and c -strongly-quasirandom.*

In Section 7 we show that the (infinite families of) compact simply connected simple groups, viz. $\mathrm{Sp}(n)$, $\mathrm{SU}(n)$, and $\mathrm{Spin}(n)$, are all fine, as n -graded groups equipped with our chosen gradings.

Theorem 3.14. *For $n \geq 3$, the n -graded groups $\mathrm{Sp}(n)$, $\mathrm{SU}(n)$, and $\mathrm{Spin}(n)$ are (C, c) -fine, for some absolute constants $C > 1$ and $c > 0$, when equipped with our chosen n -gradings.*

3.3 Goodness and fineness are preserved under taking products and quotients.

In this subsection, we show that goodness and fineness are preserved under taking products, and quotients (quotients, that is, by closed normal subgroups) — more precisely, that suitable gradings can be defined on products and quotients. This, together with Theorems 3.10 and 3.14, implies that all compact Lie groups of large-enough min-rank are both good and fine.

Lemma 3.15. *Assuming Theorem 3.10, the following holds. There exist positive constants c, C and n_0 such that if $n > n_0$, then every compact connected Lie group of min-rank n can be equipped with an n -grading that makes it (C, c) -good, as an n -graded group.*

Lemma 3.16. *Assuming Theorem 3.14, the following holds. There exist positive constants c, C and n_0 such that if $n > n_0$, then every compact connected Lie group of min-rank n can be equipped with an n -grading that makes it (C, c) -fine, as an n -graded group.*

Grading for group products. Let G and H be compact groups, and let f and g be functions on G and H respectively. We write $f \otimes g$ for the function on $G \times H$ given by $(x, y) \mapsto f(x)g(y)$. If U is a closed linear subspace of $L^2(G)$ and V is a closed linear subspace of $L^2(H)$, then we denote by $U \otimes V$ be the Hilbert-space tensor product of U and V , i.e. the linear subspace of $L^2(G \times H)$ consisting of the closure of the linear span of the set of functions $\{f \otimes g : f \in U, g \in V\}$. We recall that if $\{u_i\}_{i=1}^\infty$ is a Hilbert-space basis for U and $\{v_j\}_{j=1}^\infty$ is a Hilbert-space basis for V , then $\{u_i \otimes v_j\}_{i,j=1}^\infty$ is a Hilbert-space basis for $U \otimes V$.

Definition 3.17. *Let $n \geq m$, let G be an n -graded compact group and H be an m -graded compact group. We give $G \times H$ the structure of an m -graded compact group by setting*

$$V_{=d}^{G \times H} = \bigoplus_{d_1+d_2=d} V_{=d_1}^G \otimes V_{=d_2}^H$$

for $d < m/2$, and $V_{\geq m/2}^{G \times H} = (V_{\leq [m/2]-1}^{G \times H})^\perp$.

For products of more than two compact groups, $G_1 \times G_2 \times \dots \times G_\ell$ say, we simply iterate the above definition (noting associativity). Viz., if G_i is n_i -graded for $i = 1, 2, \dots, \ell$, then we let $n = \min\{n_1, \dots, n_\ell\}$ and we give $G_1 \times \dots \times G_\ell$ the structure of an n -graded compact group by setting

$$V_{=d}^{G_1 \times \dots \times G_\ell} = \bigoplus_{d_1+\dots+d_\ell=d} V_{=d_1}^{G_1} \otimes \dots \otimes V_{=d_\ell}^{G_\ell}$$

for $d < n/2$, and $V_{\geq n/2}^{G_1 \times \dots \times G_\ell} = (V_{\leq [n/2]-1}^{G_1 \times \dots \times G_\ell})^\perp$.

Our goal is now to show that the product of good groups is good. We make use of the following lemma of Beckner [5].

Lemma 3.18. *Let $X_1, \dots, X_r, Y_1, \dots, Y_r$ be probability spaces. Let T_1, \dots, T_r be linear operators such that $T_i: L^2(X_i) \rightarrow L^q(Y_i)$ for all $i \in [r]$. Then $\|T_1 \otimes \dots \otimes T_r\|_{2 \rightarrow q} \leq \prod_{i=1}^r \|T_i\|_{2 \rightarrow q}$.*

Lemma 3.19. *Let G_1, \dots, G_l be compact groups and suppose that each G_i is n_i -graded. Write $n = \min\{n_1, \dots, n_l\}$. Suppose that each G_i is (r, C) -hypercontractive, where $r \in \mathbb{N}$ with $r < n/2$; then $G := \prod_{i=1}^l G_i$ (with the above n -grading) is also (r, C) -hypercontractive.*

Proof. Let $q \geq 2$, and let $\delta = \frac{1}{C\sqrt{q}}$. Let $T_i: L^2(G_i) \rightarrow L^2(G_i)$, $T: L^2(G) \rightarrow L^2(G)$ be the appropriate $T_{\delta, r}$ Beckner operators. Then by hypothesis $\|T_i\|_{2 \rightarrow q} \leq 1$ for all $i \in [r]$; our goal is to show that $\|T\|_{2 \rightarrow q} \leq 1$. This now follows from Lemma 3.18 and the fact that T can be decomposed as $T = T_1 \otimes \dots \otimes T_l \circ S$, where S is given by $f \mapsto f^{\leq r}$. We have

$$\|Tf\|_q \leq \|T_1 \otimes \dots \otimes T_l\|_{2 \rightarrow q} \|Sf\|_2 \leq \|f\|_2$$

for each f . □

A similar argument works for weakly hypercontractive groups, yielding the following lemma.

Lemma 3.20. *Let G_1, \dots, G_l be compact groups and suppose that each G_i is n_i -graded. Write $n = \min\{n_1, \dots, n_l\}$. Suppose that each G_i is (r, C) -weakly hypercontractive, where $0 \leq r < n/2$. Then the n -graded group $G := \prod_{i=1}^l G_i$ (with the above n -grading) is also (r, C) -weakly hypercontractive.*

Grading for quotients. We now define a grading for quotients of graded groups, and show that hypercontractivity (and weak hypercontractivity) is preserved under taking quotients, using this ‘induced’ grading.

Definition 3.21. *Let G be an n -graded compact group, let H be a closed normal subgroup of G , and let $\pi: G \rightarrow G/H$ be the quotient map. The n -graded structure on $L^2(G/H)$ induced from that on $L^2(G)$ is given by letting $V_{=d}^{G/H}$ consist of all functions f in $L^2(G/H)$ such that $f \circ \pi$ is in $V_{=d}^G$. The space $V_{\geq n/2}^{G/H}$ is defined similarly.*

We note that this definition is consistent with our choices of the gradings of $\text{Spin}(n)$ and of $\text{SO}(n)$.

Lemma 3.22. *The subspaces $V_{=d}^{G/H}$ constitute an n -grading of the compact group G/H .*

Proof. We first note that if

$$L^2(G) = \left(\bigoplus_{d=0}^{\lceil n/2 \rceil - 1} V_{=d}^G \right) \oplus V_{\geq n/2}$$

is a grading of $L^2(G)$, then each $V_{=d}^G$ is closed (being an orthogonal complement of a subspace), as is $V_{\geq n/2}^G$.

Let $i: L^2(G/H) \rightarrow L^2(G)$ be given by $i(f) = f \circ \pi$. We let i^* be its adjoint, which is given explicitly by $i^*(f)(xH) = \mathbb{E}_{h \in H}[f(xh)]$, where the expectation is taken with respect to the Haar probability measure on H . Now i^* commutes with the action of G (from either the left or the right) and therefore the spaces $i^*(V_{=d}^G)$ and $i^*(V_{\geq n/2}^G)$ are invariant under both the left and the right actions of G . Since i^* preserves orthogonality, these spaces are also pairwise orthogonal. Since $i^* \circ i$ is the identity (so $i^*(L^2(G)) = L^2(G/H)$), we obtain that the spaces $i^*(V_{=d}^G), i^*(V_{>n/2}^G)$ constitute a grading of $L^2(G/H)$. The fact that $i^* \circ i$ is the identity also implies that $V_{=d}^{G/H} \subseteq i^*(V_{=d}^G)$ for each $d < n/2$, and that $V_{\geq n/2}^{G/H} \subseteq i^*(V_{\geq n/2}^G)$. We now claim that $i^*(V_{=d}^G) = V_{=d}^{G/H}$ for all $d < n/2$, and that $i^*(V_{\geq n/2}^G) = V_{\geq n/2}^{G/H}$. To prove this, it suffices to show that $V_{=d}^G$ is invariant under $i \circ i^*$ for each $d < n/2$, and that $V_{\geq n/2}^G$ is invariant under $i \circ i^*$. (Indeed, the latter implies that $i^*(V_{=d}^G) \subseteq V_{=d}^{G/H}$ for each $d < n/2$, and that $i^*(V_{\geq n/2}^G) \subseteq V_{\geq n/2}^{G/H}$.) Now, $i \circ i^*$ is given by $f \mapsto (x \mapsto \mathbb{E}_{h \sim H}[f(xh)])$. Suppose that $f \in V_{=d}^G$. By the right-invariance, each function $f(xh)$ is then in the space $V_{=d}^G$, and as $V_{=d}^G$ is closed, it follows that the average $i \circ i^*$ also lies in $V_{=d}^G$. Exactly the same argument works with $V_{\geq n/2}^G$, proving the claim. This completes the proof of the lemma. \square

Lemma 3.23. *Let G be an n -graded compact group, and let H be a closed normal subgroup of G . Suppose that G is (r, C) -(weakly) hypercontractive, where $0 \leq r < n/2$. Then G/H is also (r, C) -(weakly) hypercontractive, when equipped with the induced n -grading defined above.*

Proof. Let $i: L^2(G/H) \rightarrow L^2(G)$ be given by $i(f) = f \circ \pi$, as before. We notice that by definition of the grading and of the Beckner operator, it holds that $T_{\delta, r}^G \circ i = i \circ T_{\delta, r}^{G/H}$. Noting that the map i is an L^p -isometric embedding for all p , we have that for any $f \in L^2(G/H)$,

$$\left\| T^{G/H} f \right\|_q = \left\| i \circ T^{G/H} f \right\|_q = \left\| T^G(i(f)) \right\|_q \leq \|i(f)\|_2 = \|f\|_2.$$

This completes the proof of the lemma. \square

If the map i from earlier induces an isomorphism between the spaces $V_{=d}^{G/H}$ and $V_{=d}^G$ for all $d \leq r$, then the converse of Lemma 3.23 also holds; this will be useful for going from $\text{SO}(n)$ to $\text{Spin}(n)$.

Lemma 3.24. *Let G be an n -graded compact group, and let H be a closed normal subgroup of G . Equip G/H with the induced (quotient) grading, defined above. Suppose that G/H is (r, C) -(weakly) hypercontractive as an n -graded group, where $0 \leq r < n/2$. Suppose further that for all $d \leq r$, the gradings satisfy $i(V_{=d}^{G/H}) = V_{=d}^G$. Then G is also (r, C) -(weakly) hypercontractive, as an n -graded group.*

Proof. Let us denote the Beckner operator $T_{\delta, r}$ on $L^2(G)$ by T , and the corresponding operator on $L^2(G/H)$ by T' . Then we may write $T \circ i = i \circ T'$. Composing with i^* we obtain

$$T \circ i \circ i^* = i \circ T' \circ i^*.$$

We now claim that $T \circ i \circ i^* = T$. First note that each space $V_{=d}^G, V_{>r}^G$ is $i \circ i^*$ -invariant. We can therefore use fact that the operator T annihilates $V_{>r}^G$ to deduce that the operator $T \circ i \circ i^*$ agrees with T on $V_{>r}^G$. Our claim will follow once we show that $i \circ i^*$ is the identity on $V_{\leq r}^G$. To accomplish that we note that i is injective, and by the hypothesis the restriction of i to the corresponding $V_{=d}$ spaces is also surjective, and thus so is its restriction to $V_{\leq r}^G$. As the operator $i^* \circ i$ is the identity we obtain that the restriction of $i \circ i^*$ to $V_{\leq r}^G$ is the identity as well.

We can therefore write $T = i \circ T' \circ i^*$, and using the fact that i is an L^q -isometric embedding and that i^* contracts 2-norms we obtain:

$$\|T\|_{2 \rightarrow q} = \|T' \circ i^*\|_{2 \rightarrow q} \leq \|T'\|_{2 \rightarrow q} \|i^*\|_{2 \rightarrow 2} \leq 1.$$

□

Conclusion. We have shown that (weak) hypercontractivity is preserved under taking products and quotients. It is easy to check that c -strong-quasirandomness is also preserved under taking products or quotients (using the gradings above), so we immediately obtain Lemma 3.15 and Lemma 3.16.

4 Growth in good groups

In this section we prove Theorems 1.2, 1.5, 1.9, 1.10 1.11 and 1.12. For now, the reader may consider the objective of proving Theorem 1.2 as motivation for what follows.

Let G be a compact group, and let μ be the Haar probability measure on G . We would like to bound $\mu(A)$ for a set $A \subseteq G$ that is product free. We first note that the property of being product free can be stated in terms of convolutions.

Definition 4.1. For two functions $f, g \in L^2(G)$, we define their convolution $f * g \in L^2(G)$ by

$$f * g(x) := \int f(xy^{-1})g(y)d\mu(y).$$

For $f \in L^2(G)$, we write T_f for the linear operator from $L^2(G)$ to itself defined by $g \mapsto g * f$. Observe that if $A \subset G$ is a product-free set of density $\mu(A) = \alpha$, and $f = \frac{1_A}{\alpha}$, then $\langle T_f 1_A, 1_A \rangle = 0$. If G is an n -graded group, we can decompose $g := 1_A$ into its orthogonal projections onto the $V_{=d}$'s, and write $g = \sum_{d=0}^{\lceil n/2 \rceil - 1} g^{=d} + g^{\geq n/2}$, where $g^{=d}$ is the orthogonal projection of g onto $V_{=d}$. Noting that $g^{=0} \equiv \alpha$, this allows us to expand

$$\langle T_f 1_A, 1_A \rangle = \langle T_f g, g \rangle$$

as a sum of a main term, α^2 , and other terms of the form $\langle T_f g^{=d}, g^{=d} \rangle$ or $\langle T_f g^{\geq n/2}, g^{\geq n/2} \rangle$. We upper-bound each term by using Cauchy-Schwarz:

$$|\langle T_f g^{=d}, g^{=d} \rangle| \leq \|T_f\|_{V_{=d}} \|g^{=d}\|_2^2,$$

where for a closed subspace $M \leq L^2(G)$ and a linear operator $T : L^2(G) \rightarrow L^2(G)$ we write $\|T\|_M$ for the supremum of $\frac{\|Tv\|_2}{\|v\|_2}$ over all nonzero $v \in M$.

Our goal will be to show that these other terms make a negligible contribution to the sum compared to the main term α^2 . We accomplish that by observing that the space $V_{=d}$ is T_f -invariant. This shows that the operator $T_f^*T_f$ can be diagonalized inside $V_{=d}$. It also implies that $\|T_f\|_{V_{=d}}^2$ is equal to the maximal eigenvalue of $T_f^*T_f$. We then upper bound the maximal eigenvalue of $T_f^*T_f$ inside $V_{=d}$, showing that these eigenvalues get smaller and smaller as the degree gets larger. Finally, we combine our upper bound on the eigenvalues of T_f with a level d -inequality which shows that the L^2 -mass of g is concentrated on the high degrees. Together, we obtain that the sum of terms $\langle T_f g^{=d}, g^{=d} \rangle$ is indeed negligible.

Our upper bound on the ‘degree d ’ eigenvalues of $T_f^*T_f$ follows by combining a level d inequality with a lower bound on the dimension of each eigenspace of $T_f^*T_f$. We use the fact that each such eigenspace is a subrepresentation of $V_{=d}$ and therefore by strong quasirandomness must have dimension $\geq Q_d$, for the appropriate quasirandomness parameter Q_d . We upper bound $|\langle T_f g^{\geq n/2}, g^{\geq n/2} \rangle|$ in a similar fashion.

4.1 Level d inequalities and the eigenvalues of convolution operators

Recall that the *Hilbert-Schmidt norm* of a linear operator T on a separable Hilbert space H is defined by

$$\|T\|_{\text{HS}}^2 := \sum_{i=1}^{\infty} \|T(e_i)\|_2^2,$$

where $\{e_i\}_{i=1}^{\infty}$ is any Hilbert-space basis for H ; if T is a compact operator, then $\|T\|_{\text{HS}}$ is the square root of the sum of the eigenvalues of T^*T (counted with multiplicity). One standard fact (see e.g. [9], page 267) is the following.

Fact 4.2. *Let $f \in L^2(G)$ and let T_f be the linear operator from $L^2(G)$ to itself defined by $g \mapsto g * f$. Then T_f is a compact operator, and the Hilbert-Schmidt norm of T_f is equal to the 2-norm of f :*

$$\|T_f\|_{\text{HS}} = \|f\|_2.$$

We recall our notation for the norm of an operator on a subspace of $L^2(G)$.

Definition 4.3. *Let $M \leq L^2(G)$ be a closed subspace. Let $T : L^2(G) \rightarrow L^2(G)$ be a linear operator; then we write $\|T\|_M := \sup\{\|Tf\|_2/\|f\|_2 : f \in M \setminus \{0\}\}$.*

We now give our upper bound on the level d eigenvalues of T_f .

Lemma 4.4. *Let G be an n -graded $((Q_d)_{d=1}^{n/2}, Q)$ -quasirandom group. Let $f \in L^2(G)$. Then the spaces $V_{=d}, V_{\geq n/2}$ are T_f -invariant. Moreover, $\|T_f\|_{V_{=d}} \leq \frac{\|f^{=d}\|_2}{\sqrt{Q_d}}$ and $\|T_f\|_{V_{\geq n/2}} \leq \frac{\|f\|_2}{\sqrt{Q}}$.*

Proof. We first claim that the subspaces $V_{=d}, V_{\geq n/2}$ are all T_f invariant. To see this, observe that if $U \leq L^2(G)$ is a closed subspace that is invariant under the right-action of G , then

for every $g \in U$, we have $g * f \in U$ as well. (Indeed, let $h \in U^\perp$; then

$$\langle g * f, h \rangle = \int \int g(xy^{-1})f(y)h(x)d\mu(y)d\mu(x) = \int \int g(xy^{-1})f(y)h(x)d\mu(x)d\mu(y) = 0,$$

using Fubini and the fact that for each fixed $y \in G$ the function $x \mapsto g(xy^{-1})f(y)$ lies in U . Hence, $g * f \in (U^\perp)^\perp = U$.) Applying this with $U = V_{=d}$, which is a closed subspace invariant under the right action of G , we see that the spaces $V_{=d}$ are indeed T_f -invariant. Similarly, applying it with $U = V_{\geq n/2}$ (which is also a closed subspace invariant under the right action of G), we see that $V_{\geq n/2}$ is also T_f -invariant.

Fix $d < n/2$, and let us orthogonally decompose f as $f = f^{=d} + f'$. Then $T_f = T_{f^{=d}} + T_{f'}$, by the linearity of convolution. We now assert that T_f agrees with $T_{f^{=d}}$ on $V_{=d}$. Essentially the same argument as that above shows that if $U \leq L^2(G)$ is a closed subspace that is invariant under the left-action of G , then $g * f' \in U$ for every $f' \in U$ and $g \in L^2(G)$; applying this with $U = V_{=d}^\perp$, we obtain that $T_{f'}g \in V_{=d}^\perp$ for every $g \in L^2(G)$. Hence, if $g \in V_{=d}$, then $T_{f'}g = 0$, proving our assertion. It follows that $\|T_f\|_{V_{=d}}^2$ is the maximal eigenvalue of the operator $T_{f^{=d}}^*T_{f^{=d}}$. On the other hand, Fact 4.2 implies that $\|f^{=d}\|_2^2$ is the sum of the eigenvalues of $T_{f^{=d}}^*T_{f^{=d}}$ counted with multiplicity. To complete the proof that $\|T_f\|_{V_{=d}} \leq \frac{\|f^{=d}\|_2}{\sqrt{Q_d}}$ we show that each such multiplicity is $\geq Q_d$ and so

$$Q_d \|T_f\|_{V_{=d}}^2 \leq \|f^{=d}\|_2^2.$$

To lower-bound the multiplicities of the eigenvalues of $T_{f^{=d}}^*T_{f^{=d}}$ inside $V_{=d}$ we note that $T_{f^{=d}}^*T_{f^{=d}}$ commutes with the left-action of G , since $T_{f^{=d}}$ does. This implies that the eigenspaces of $T_{f^{=d}}^*T_{f^{=d}}$ are left G -submodules of $V_{=d}$. Each such submodule contains an irreducible representation, which has dimension $\geq Q_d$ by hypothesis. The proof that $\|T_f\|_{V_{\geq n/2}} \leq \frac{\|f\|_2}{\sqrt{Q}}$ is similar. \square

We now upper-bound $\|T_f\|_{V_{=d}}$ by proving a corresponding level d -inequality.

Theorem 4.5. *Let G be an (r, C) -hypercontractive group. Let $f: G \rightarrow \{0, 1\}$ and write $\alpha := \mathbb{E}[f]$. Let $d \in \mathbb{N}$ be such that $0 < d \leq \min\{\frac{1}{2} \log(1/\alpha), r\}$. Then*

$$\|f^{=d}\|_2^2 \leq \left(\frac{10C}{d}\right)^d \alpha^2 \log^d(1/\alpha).$$

Proof. Let $q = \frac{\log(1/\alpha)}{d}$. Let q' be the Hölder conjugate of q , i.e. the number satisfying $\frac{1}{q} + \frac{1}{q'} = 1$. Then by (r, C) -hypercontractivity and Lemma 3.7 we have

$$\|f^{=d}\|_2^2 = \langle f, f^{=d} \rangle \leq \|f^{=d}\|_q \|f\|_{q'} \leq (Cq)^{d/2} \|f^{=d}\|_2 \cdot \alpha^{1-1/q}.$$

After rearranging we obtain

$$\|f^{=d}\|_2^2 \leq (e^2 Cq)^d \alpha^2.$$

The theorem follows by plugging in the value of q . \square

Theorem 4.5 together with Lemma 3.15 also shows that Theorem 3.10 implies Theorem 1.5.

Combining Lemma 4.4 with Theorem 4.5, we have the following.

Lemma 4.6. *For each $c, C > 0$ there exist $c', n_0 > 0$ such that the following holds. Let $n > n_0$ and let G be an n -graded (C, c) -good group. Let $A \subseteq G$ be measurable, and suppose that*

$$\alpha := \mu_G(A) \in (e^{-c'\sqrt{n}}, c').$$

Write $f = \frac{1_A}{\alpha}$ and $t = \frac{\log(1/\alpha)}{2}$. Then for all $1 \leq d \leq t$, we have

$$\|T_f\|_{V_{=d}} \leq \left(\frac{C' \log(1/\alpha)}{n} \right)^{d/2},$$

where $C' := \frac{10C}{c}$. Moreover,

$$\|T_f\|_{V_{>t}} \leq \left(\frac{\alpha}{n} \right)^{10}.$$

Proof. The lemma follows by applying Lemma 4.4 with the values of Q_d and Q which are promised by the goodness of G , and then bounding $\|f^{=d}\|_2$ using Theorem 4.5.

Let us begin with the range $d \leq t$. Since we know that G is c -strongly-quasirandom, he have that G is $((Q_d)_{d=1}^{\lceil n/2 \rceil - 1}, Q)$ graded where $Q_d \geq \left(\frac{cn}{d}\right)^d$ for all $d \leq t$. Hence, by Lemma 4.4, we have

$$\begin{aligned} \|T_f\|_{V_{=d}} &\leq \frac{\|f^{=d}\|_2}{\sqrt{Q_d}} \\ &\leq \|f^{=d}\|_2 \cdot \left(\frac{cn}{d}\right)^{-d/2} \\ &= \frac{\|\alpha \cdot f^{=d}\|_2}{\alpha} \cdot \left(\frac{cn}{d}\right)^{-d/2} \\ &\leq \left(\frac{d}{cn}\right)^{d/2} \cdot \left(\frac{10C}{d}\right)^{d/2} \log(1/\alpha)^{d/2} \end{aligned}$$

(using Lemma 4.5)

$$\leq \left(\frac{10C \log(1/\alpha)}{cn} \right)^{d/2},$$

which is the desired bound.

Next, consider the range $d \geq t$. For d in this range, and provided that n_0 is sufficiently large, we have from c -strong-quasirandomness that $Q_d \geq \left(\frac{cn}{t}\right)^t$. By Lemma 4.4 we therefore have, by a similar computation to before,

$$\|T_f\|_{V_{>t}} \leq \|f\|_2 \left(\frac{cn}{t}\right)^{-t/2} \leq n^{-t/4} e^{-21t} \alpha^{-1/2} \leq \left(\frac{\alpha}{n}\right)^{10},$$

where we used

$$\frac{cn}{t} \geq e^{42} n^{1/2},$$

which holds provided c' is sufficiently small. \square

Lemma 4.7. *Let $c, C, c' > 0$. Let G be an n -graded c -strongly-quasirandom group. Let $A \subseteq G$ be measurable, and suppose that*

$$\alpha := \mu_G(A) \geq c'.$$

Write $f = \frac{1_A}{\alpha}$. Then for $1 \leq d < cn/(1+c)$ we have

$$\|T_f\|_{V=d} \leq (c')^{-1/2} \left(\frac{d}{cn}\right)^{d/2},$$

and for $d \geq cn/(1+c)$ we have

$$\|T_f\|_{V=d} \leq (c')^{-1/2} (1+c)^{-cn/(2(1+c))}.$$

Proof. The lemma follows by applying Lemma 4.4 with the values of Q_d and Q that are guaranteed by the goodness of G , and then upper-bounding $\|f^{=d}\|_2$ using $\|f^{=d}\|_2 \leq \|f\|_2 = \alpha^{-1/2} \leq (c')^{-1/2}$. \square

Lemmas 4.6, 4.7 and 3.15 together show that Theorem 3.10 implies Theorem 1.9.

The following is a version of Theorem 4.5 that is perhaps easier to comprehend.

Theorem 4.8. *For each $c, C > 0$ there exist $c', C' > 0$ such that the following holds. Let G be an n -graded, (c, C) -good group, let $f: G \rightarrow \{0, 1\}$ be measurable, and suppose that $\alpha := \mathbb{E}[f] \geq e^{-c'\sqrt{n}}$. Then for all $d \in \mathbb{N}$, we have*

$$\|f^{=d}\|_2^2 \leq (C')^d \alpha^2 \log^d(e/\alpha). \quad (2)$$

Proof. Provided that C' is sufficiently large with respect to c' we may assume that $\alpha \leq c'$. Indeed, for $\alpha \geq c'$ the right hand side is greater than one, and we always have the trivial upper bound $\|f^{=d}\|_2^2 \leq \|f\|_2^2 = \alpha$. Let $r = c\sqrt{n}$, and note that in the case where $d > r$ the upper bound is trivial, as in this case the right-hand side of 2 is (again) greater than one.

Let $q = \log(1/\alpha)$. Let q' be the Hölder conjugate of q . For $d \leq r$, we have by (r, C) -hypercontractivity and Lemma 3.7 that

$$\|f^{=d}\|_2^2 = \langle f, f^{=d} \rangle \leq \|f^{=d}\|_q \|f\|_{q'} \leq (Cq)^{d/2} \|f^{=d}\|_2 \alpha^{1-1/q}.$$

After rearranging we obtain

$$\|f^{=d}\|_2^2 \leq e^2 (Cq)^d \alpha^2 = e^2 C^d \alpha^2 \log^d(1/\alpha).$$

This gives the right bound, provided C' is sufficiently large depending on C . \square

4.2 Upper bounds on the measures of product-free sets

Let us now show how Lemma 4.6 implies an upper bound on the measure of a product-free set in a good group.

Theorem 4.9. *There exist $c', n_0 > 0$ such that the following holds. Let $n > n_0$ and let G be a (c, C) -good n -graded group. Then every measurable product-free set in G has Haar measure at most $e^{-c'n^{1/3}}$.*

Before proving Theorem 4.9, let us note that together with Theorem 3.10 it implies Theorem 1.2. This follows since, by Lemma 3.15, for all $n > n_0$ every compact connected Lie group of min-rank n is a (c, C) -good n -graded group for some absolute constants $c, C > 0$.

Proof. Let $A \subseteq G$ be product-free and measurable; write $\alpha = \mathbb{E}[1_A]$ and $t = \frac{\log(1/\alpha)}{2}$. Assume w.l.o.g. that $\alpha < c'$, where c' is to be chosen later (if not, then replace A by a smaller product-free set). Let $f = \frac{1_A}{\alpha}$. Suppose for a contradiction that $\alpha \geq e^{-c'n^{1/3}}$. We have

$$0 = \langle T_f f, f \rangle = \mathbb{E}^3[f] + \sum_{d=1}^{\lfloor t \rfloor} \langle T_f f^{=d}, f^{=d} \rangle + \langle T_f f^{>t}, f^{>t} \rangle. \quad (3)$$

Provided c' is sufficiently small, we may now apply Lemma 4.6 and Theorem 4.5 with $1_A = f\alpha$ to obtain, for sufficiently large C' and all $1 \leq d \leq t$,

$$|\langle T_f f^{=d}, f^{=d} \rangle| \leq \|T_f\|_{V=d} \|f^{=d}\|_2^2 \leq \left(\frac{C' \log^3(1/\alpha)}{nd^2} \right)^{d/2} \leq 100^{-d},$$

provided c' is sufficiently small depending on C' . We may also apply Lemma 4.6 to obtain

$$|\langle T_f f^{>t}, f^{>t} \rangle| \leq \|T_f\|_{V>t} \|f\|_2^2 \leq \left(\frac{\alpha}{n} \right)^{10} \alpha^{-1}.$$

As $\mathbb{E}[f] = 1$, these two upper bounds contradict (3). □

4.3 Product mixing

The proof of Theorem 4.9 in fact gives the following stronger statement, which implies Theorem 1.11.

Theorem 4.10. *For any $\varepsilon > 0$, there exist c', n_0 , such that the following holds. Let $n > n_0$ and let G be a (c, C) -good n -graded group. Let A, B, C be measurable subsets of G , each with Haar measure at least $e^{-c'n^{1/3}}$. Let $f = \frac{1_A}{\mu(A)}, g = \frac{1_B}{\mu(B)}, h = \frac{1_C}{\mu(C)}$. Then*

$$|\langle f * g, h \rangle - 1| < \varepsilon.$$

Proof. Assume without loss of generality that B has the smallest measure of the three sets. (Note that, while the trilinear form $\mathcal{T}(f, g, h) := \langle f * g, h \rangle$ is not quite symmetric with respect to permuting f, g and h , we may swap the positions of f and g or of g and h if we replace some of A, B and C by their inverses, meaning $A^{-1} := \{x^{-1} : x \in A\}$ etc, which have the same measures. So there is indeed no loss of generality in assuming the above.) Write $\mu(A) = \alpha, \mu(B) = \beta, \mu(C) = \gamma$ and $C' = \frac{C}{10c}$.

Note that $\langle f * g, h \rangle = \langle T_g f, h \rangle$. First we quickly handle the case where $\beta \geq c'$. Here we may apply Lemma 4.7 to obtain that $\|T_g - I_0\|_{2 \rightarrow 2} < \varepsilon c'$, where I_0 denotes operator $F \mapsto \mathbb{E}[F]$ which sends a function to the constant function of the same expectation, provided that n_0 is sufficiently large. Using the fact that $\mathbb{E}[g] = 1$ we then have

$$|\langle T_g f, h \rangle - 1| = |\langle T_g f, h \rangle - \langle I_0 f, h \rangle| \leq \|T_g - I_0\|_{2 \rightarrow 2} \cdot \|f\|_2 \cdot \|h\|_2 < \varepsilon \frac{c'}{\alpha^{1/2} \gamma^{1/2}} \leq \varepsilon,$$

yielding the conclusion of the theorem.

The proof of the case $\min\{\alpha, \beta, \gamma\} = \beta < c'$ proceeds similarly to the proof of Theorem 4.9. Let $t = \frac{\log(1/\beta)}{2}$. We have

$$\langle T_g f, h \rangle - 1 = \langle T_g f, h \rangle - \langle T_g f^{=0}, h^{=0} \rangle = \sum_{d=1}^{\lfloor t \rfloor} \langle T_g f^{=d}, h^{=d} \rangle + \langle T_g f^{>t}, h^{>t} \rangle. \quad (4)$$

Using Lemma 4.6, we obtain the upper bound

$$|\langle T_g f^{>t}, h^{>t} \rangle| \leq \|T_g\|_{V^{>t}} \|f\|_2 \|h\|_2 \leq \frac{\beta^{10}}{\alpha^{1/2} \cdot \gamma^{1/2} \cdot n^{10}} < \varepsilon/2, \quad (5)$$

provided that n_0 is sufficiently large. For $1 \leq d \leq t$, we use the upper bound

$$|\langle T_g f^{=d}, h^{=d} \rangle| \leq \|T_g\|_{V^{=d}} \|f^{=d}\|_2 \|h^{=d}\|_2. \quad (6)$$

By Lemma 4.6, we have $\|T_g\|_{V^{=d}} \leq \left(\frac{C' \log(1/\alpha)}{n}\right)^{d/2}$. By Theorem 4.8, we have the upper bound

$$\|f^{=d}\|_2^2 \leq C'^d \log^d(e/\alpha),$$

and similarly for h ,

$$\|h^{=d}\|_2^2 \leq C'^d \log^d(e/\gamma).$$

Substituting the last three bounds into (6), we obtain

$$|\langle T_g f^{=d}, h^{=d} \rangle| \leq \left(\frac{C'^3 \log(e/\alpha) \log(e/\beta) \log(e/\gamma)}{n}\right)^{d/2} \leq \varepsilon 4^{-d}, \quad (7)$$

provided that c' is sufficiently small depending on C, c and ε .

The sum of the contribution from (5) and of those from (7) for $1 \leq d \leq t$, to the right-hand side of (4), is clearly less than ε , yielding $|\langle T_g f, h \rangle - 1| < \varepsilon$, as required. \square

4.4 Equidistribution of convolutions

Let $A \subseteq G$ be a positive-measure subset of a good Lie group, and suppose that X is a G -homogeneous topological space (equipped with its G -invariant Haar probability measure μ_X), and that $B \subseteq X$ is a positive-measure subset. The next theorem states that as long as the measures of A and of B are not too small, applying a uniformly random element of A to a uniformly random element of B yields an almost uniformly random element of X (meaning, a random element with respect to the Haar probability measure). Note that Theorem 4.11 below, together with Lemma 3.15, imply Theorems 1.10 and 1.12.

Theorem 4.11. *For each $C, c, \varepsilon > 0$ there exists $c', n_0 > 0$, such that the following holds. Let $n > n_0$, let G be an n -graded (C, c) -good compact connected Lie group, and let X be a G -homogeneous topological space (equipped with the G -action $(g, x) \mapsto gx$), and let μ_X denote the G -invariant Haar probability measure on X . Suppose that $A \subseteq G$ and $B \subseteq X$ are measurable sets of Haar probability measures $\geq e^{-c'\sqrt{n}}$. Let μ_A denote the Haar probability measure on G , conditioned on the event A , and let μ_B denote the Haar probability measure on X conditioned on the event B , i.e. $\mu_B(Y) = \mu_X(B \cap Y)/\mu_X(B)$ for a measurable set $Y \subseteq X$, and $\mu_A(Z) = \mu_G(A \cap Z)/\mu_G(A)$ for a measurable set $Z \subseteq G$. Then the total variation distance between μ_X and the distribution of ab where $a \sim \mu_A$ and $b \sim \mu_B$ independently, is less than ε .*

Proof. Consider first the case where $X = G$ and G acts on itself by left multiplication; in this case, since the distribution of ab is $\mu_A * \mu_B$, we need to show that $\|\mu_A * \mu_B - \mu_G\|_{\text{TV}} < \varepsilon$. We associate μ_A with the function $f_A = \frac{1_A}{\mu(A)} \in L^2(G)$ and similarly, we associate μ_B with the function $f_B = 1_B/\mu(B) \in L^2(G)$; it follows easily from the Cauchy-Schwarz inequality that $\|\mu_A * \mu_B - \mu_G\|_{\text{TV}} \leq \|f_A * f_B - 1\|_2$. So our aim is now to prove that

$$\|f_A * f_B - 1\|_2 < \varepsilon. \quad (8)$$

In proving (8), we argue that we may assume, without loss of generality, that $\mu(B) \leq c'$. Indeed, if this does not hold, then write $B = \cup_{i \in I} B_i$ as a finite, disjoint union of sets B_i such that $c'/2 \leq \mu(B_i) \leq c'$. Once we have proved (8) for sets of measure at most c' , we obtain the desired bound for B by convexity, noting that f_B is a convex combination of the functions f_{B_i} .

Set $t = \frac{\log(1/\mu(B))}{2}$. We now have

$$\|f_A * f_B - 1\|_2^2 = \sum_{d=1}^t \|T_{f_B} f_A^{-d}\|_2^2 + \|T_{f_B} f_A^{>t}\|_2^2.$$

Now

$$\|T_{f_B} f_A^{-d}\|_2 \leq \|T_{f_B}\|_{V=d} \|f_A^{-d}\|_2.$$

and

$$\|T_{f_B} f_A^{>t}\|_2 \leq \|T_{f_B}\|_{V>t} \|f_A^{>t}\|_2.$$

The bound (8) now easily follows from Theorem 4.8 and Lemma 4.6, similarly to in the proof of Theorem 4.10. Indeed, these yield

$$\|T_{f_B}\|_{V=d} \|f_A^{-d}\|_2 \leq \left(\frac{C'^2 \log(1/\mu(A)) \log(1/\mu(B))}{n} \right)^{d/2} \leq \varepsilon 4^{-d}$$

for all $1 \leq d \leq t$, and

$$\|T_{f_B}\|_{V>t} \|f_A^{>t}\|_2 \leq \left(\frac{\beta}{n} \right)^{10} \alpha^{-1/2} \leq \varepsilon/2,$$

provided that n_0 is sufficiently large and c' sufficiently small depending on c, C and ε .

To prove the general case, note that we may choose an arbitrary $x_0 \in X$ and set $\tilde{B} = \{b \in G : bx_0 \in B\}$. If $\tilde{b} \sim \mu_{\tilde{B}}$ then $\tilde{b}x_0 \sim \mu_B$, and if $g \sim \mu_G$ then $gx_0 \sim \mu_X$; it follows that $\|\mu_{ab} - \mu_X\|_{\text{TV}} \leq \|\mu_A * \mu_{\tilde{B}} - \mu_G\|_{\text{TV}}$, where μ_{ab} denotes the distribution of ab for $a \sim \mu_A$ and $b \sim \mu_B$ (independently). The result for the pair (A, \tilde{B}) therefore implies the result for (A, B) . \square

5 Growth in fine groups

In this section we show that Theorems 1.4, 1.7 and 1.8 follow from Theorem 3.14.

First, the theorem below is a variation on Theorem 4.5, with hypercontractivity replaced by weak hypercontractivity. Together with Lemma 3.16 it shows that Theorem 3.14 implies Theorem 1.4.

Theorem 5.1. *Let $c > 0$, $C > 1$ and let G be an (r, C) -weakly hypercontractive group. Let $f: G \rightarrow \{0, 1\}$ be measurable, and write $\alpha := \mathbb{E}[f]$. Let $d \in \mathbb{N}$ be such that $0 < d \leq \min\left(\frac{\log(1/\alpha)}{2}, r\right)$. Then*

$$\|f^{-d}\|_2^2 \leq \left(\frac{e \log(1/\alpha)}{d} \right)^{2Cd} \alpha^2.$$

Proof. Let $q = \log(1/\alpha)/d$, and let q' be the Hölder conjugate of q . Then by Hölder, weak hypercontractivity and Lemma 3.12, we have

$$\|f^{-d}\|_2^2 = \langle f, f^{-d} \rangle \leq \|f^{-d}\|_q \|f\|_{q'} \leq q^{Cd} \alpha^{1-1/q} \|f^{-d}\|_2.$$

The theorem follows by rearranging and substituting $\alpha^{-1/q} = e^d$. (Note that $C > 1$, by the definition of weak hypercontractivity.) \square

The following lemma is a variant of Lemma 4.6 for fine groups – the proof is the same as for Lemma 4.6, only using Theorem 5.1 instead of Theorem 4.5. We will make use of it when proving our diameter bounds for fine groups.

Lemma 5.2. *For each $c > 0$ and $C > 1$ there exist $c', C', n_0 > 0$ such that the following holds. Let $n > n_0$ and let G be an n -graded (C, c) -fine group. Let $A \subseteq G$ be measurable, and suppose that*

$$\alpha := \mu_G(A) \in (e^{-c'n}, c').$$

Write $f = \frac{1_A}{\alpha}$ and $t = \frac{\log(1/\alpha)}{2}$. Then for $1 \leq d \leq t$ we have

$$\|T_f\|_{V=d} \leq \left(\frac{e \log(1/\alpha)}{d} \right)^{Cd} \cdot \left(\frac{d}{cn} \right)^{d/2}.$$

We also have

$$\|T_f\|_{V>t} \leq \frac{\alpha^{10}}{n^{10}}.$$

We now show that if f has small expectation, then most of the Fourier mass of f lies on the high degrees.

Lemma 5.3. *For each $c > 0$ and $C > 1$ there exist $c', n_0 > 0$ such that the following holds. Let $n > n_0$, let G be a (c, C) -fine n -graded group, and let $f: G \rightarrow \{0, 1\}$ be measurable. Suppose that $\alpha := \mathbb{E}[f] \geq e^{-c'n}$, and let $0 \leq t \leq \frac{\log(1/\alpha)}{10^6 C^2}$. Then $\|f^{\leq t}\|_2^2 \leq 2 \cdot 2\alpha^{1.99}$.*

Proof. Inserting a factor of 2 into the bound from Theorem 5.1, we have for any $0 < d \leq t$ that

$$\|f^{=d}\|_2^2 \leq 2^{-d} \left(\frac{2e \log(1/\alpha)}{d} \right)^{2Cd} \alpha^2,$$

and for $d = 0$ we have $\|f^{=0}\|_2^2 = \alpha^2$. Therefore $\|f^{\leq t}\|_2^2$ is upper-bounded by $2\alpha^2$ multiplied by the maximum of $\left(\frac{2e \log(1/\alpha)}{d} \right)^{2Cd}$ in the range where $d \leq \frac{\log(1/\alpha)}{10^6 C^2}$. Taking logs and computing the derivative with respect to d , it is easy to show that the maximum is obtained at the end point when $d = \frac{\log(1/\alpha)}{10^6 C^2}$. This shows that

$$\|f^{\leq t}\|_2^2 \leq 2\alpha^2 (2 \cdot 10^6 \cdot eC^2)^{\frac{2 \log(1/\alpha)}{10^6 C}} \leq 2\alpha^{1.99},$$

as required. □

For smaller values of d , we have better bounds.

Lemma 5.4. *For each $\varepsilon, c > 0$ and $C > 1$, there exist $c', n_0 > 0$ such that the following holds. Let $n > n_0$, let G be a (C, c) -fine n -graded group, and let $f: G \rightarrow \{0, 1\}$ be measurable. Suppose that $\alpha := \mathbb{E}[f] \geq e^{-c'n}$ and let $d \in \mathbb{N}$ with $0 < d \leq \frac{\log(1/\alpha)}{n^\varepsilon}$. Then*

$$\|f^{\leq d}\|_2^2 \leq \alpha^{2-n^{-\varepsilon/2}}.$$

Proof. Let c' be sufficiently small and n_0 sufficiently large with respect to ε, c, C . Let $t = \frac{\log(1/\alpha)}{n^\varepsilon}$. Similarly to in the proof of Lemma 5.3, it is easy to see that

$$\|f^{=d}\|_2^2 \leq \left(\frac{e \log(1/\alpha)}{t} \right)^{2Ct} \alpha^2 = (e \cdot n^\varepsilon)^{2C \log(1/\alpha)/n^\varepsilon}.$$

Now

$$(e \cdot n^\varepsilon)^{2C \log(1/\alpha)/n^\varepsilon} = \alpha^{-n^{-\varepsilon} \cdot 2C \log(en^\varepsilon)} \leq \alpha^{-n^{-\varepsilon/2}},$$

provided that $n \geq n_0$. □

5.1 Product mixing in fine groups

The next lemma is a version of Theorem 4.10, except for fine instead of for good groups. The only difference in the proof is that we apply Lemma 5.2 in place of Lemma 4.6 and Theorem 5.1 in place of Theorem 4.5.

Lemma 5.5. *For each $\varepsilon, c > 0$ and $C > 1$, there exists $\delta > 0$ such that the following holds. Let G be a (C, c) -fine, n -graded group, let $A, B, C \subseteq G$ be measurable sets of measures at least e^{-n^δ} , and let $f = \frac{1_A}{\mu(A)}, g = \frac{1_B}{\mu(B)}, h = \frac{1_C}{\mu(C)}$. Then $|\langle f * g, h \rangle - 1| < \varepsilon$.*

We remark that Lemma 5.5 is only weaker than the corresponding Theorem 4.10 for good groups. We include it even though the groups of interest to us are both good and fine. We decided to include the lemma mainly for aesthetic reasons. Our diameter bounds rely on Lemma 5.5 and we preferred to show that our diameter bounds hold for fine groups rather than groups that are both good and fine.

Below we use a trick of Nikolov and Pyber [34] (who observed that product mixing implies an upper bound on the diameter), to bound the diameter in fine groups.

Corollary 5.6. *For each $c > 0$ and $C > 1$, there exists $\delta > 0$ such that if G is a (C, c) -fine group and $\mathcal{A} \subseteq G$ is a measurable set of measure at least e^{-n^δ} , then $\mu(\mathcal{A}^2) > 1 - e^{-n^\delta}$, and $\mathcal{A}^3 = G$.*

Proof. The claim about $\mu(\mathcal{A}^2)$ follows by applying Lemma 5.5 while taking $A = B = \mathcal{A}$, $C = G \setminus \mathcal{A}^2$ and $\varepsilon = 1/2$ (in fact, any value of ε less than one, will do). As for the claim about \mathcal{A}^3 , suppose for a contradiction that $\mathcal{A}^3 \neq G$. Let $x \in G \setminus \mathcal{A}^3$. Then $\mathcal{A}^2 \cap x\mathcal{A}^{-1} = \emptyset$. This contradicts Lemma 5.5 when setting $A = B = \mathcal{A}$ and $C = x\mathcal{A}^{-1}$. □

5.2 Non-Abelian Brunn-Minkowski type inequalities for fine groups

The following theorem is a restatement of Theorem 1.8.

Lemma 5.7. *There exists absolute constants $c', n_0 > 0$ such that the following holds. Let G be a compact connected Lie group with $n := n(G) > n_0$, and let $A \subseteq G$ be a measurable set of measure at least $e^{-c'n}$. Then $\mu(A)^2 \geq \mu(A)^{0.1}$.*

Proof. First note that $\mu(A) > \frac{1}{2}$ implies that $A^2 = G$, as if $x \in G \setminus A^2$, then A and xA^{-1} are disjoint sets each of measure greater than $1/2$, a contradiction. By Corollary 5.6, we may also assume that $\mu(A) \leq e^{-n^{c'}}$, provided c' is sufficiently small.

Let $f = \frac{1_A}{\mu(A)}$ and $g = 1_{A^2}$. Then we have $\langle f * f, g \rangle = 1$. On the other hand by Cauchy–Schwarz we have

$$|\langle f * f, g \rangle| \leq \|f * f\|_2 \|g\|_2 = \|f * f\|_2 \sqrt{\mu(A^2)}.$$

This yields $\mu(A^2) \geq \frac{1}{\|f * f\|_2^2}$. Let $t = \frac{\log(1/\mu(A))}{10^6 C^2}$. We have $\|f * f\|_2^2 = \|f^{\leq t} * f^{\leq t}\|_2^2 + \|f^{> t} * f^{> t}\|_2^2$. By applying lemma 5.3 with 1_A we obtain

$$\|f^{\leq t} * f^{\leq t}\|_2^2 \leq \|f^{\leq t}\|_2^4 \leq \alpha^{-0.02}.$$

By applying Lemma 5.2 we obtain

$$\|f^{> t} * f^{> t}\|_2^2 \leq \|T_f\|_{V_{>t}}^2 \|f\|_2^2 \leq \left(\frac{\alpha}{n}\right)^{20} \alpha^{-1} \leq 1.$$

Combining the bounds completes the proof. \square

Lemma 5.8. *For each $\varepsilon, c > 0$ and $C > 1$ there exist $\delta, n_0 > 0$ such that the following holds. Let $n > n_0$ and let G be a (C, c) -fine n -graded group. If*

$$\mu(A) := e^{-n^\zeta} \in \left(e^{-n^{1-\varepsilon}}, e^{-n^\varepsilon}\right),$$

then $\mu(A^2) \geq e^{-n^{\zeta-\delta}}$.

Proof. We may and shall assume, throughout the proof, that δ is sufficiently small depending on ε, C and c , and that n_0 is sufficiently large depending on δ . Let $f = \frac{1_A}{\mu(A)}$. As in the proof of the previous lemma, we have

$$\mu(A^2) \geq \frac{1}{\|f * f\|_2^2}. \quad (9)$$

Let $t_1 = \frac{\log(1/\mu(A))}{n^{4\delta}}$ and $t_2 = \frac{\log(1/\mu(A))}{10^6 C^2}$. We bound $\|f * f\|_2^2$ from above by decomposing it as follows:

$$\|f * f\|_2^2 = \|f^{<t_1} * f^{<t_1}\|_2^2 + \sum_{d=t_1}^{t_2} \|f^{=d} * f^{=d}\|_2^2 + \|f^{>t_2} * f^{>t_2}\|_2^2. \quad (10)$$

By applying Lemma 5.2, we obtain

$$\|f^{>t_2} * f^{>t_2}\|_2^2 \leq \|T_f\|_{V_{>t_2}}^2 \|f\|_2^2 \leq \frac{\alpha^{10}}{n^{10}} \alpha^{-1/2} \leq 1.$$

Applying Lemma 5.4 (with αf in place of f , and ε taken to be 4δ), we have

$$\|f^{<t_1} * f^{<t_1}\|_2^2 \leq \|f^{<t_1}\|_2^2 \leq \alpha^{-2} \cdot \alpha^{2-n-2\delta} \leq \frac{1}{4} \alpha^{-n-\delta},$$

provided that δ is sufficiently small and n is sufficiently large depending on δ .

Finally, for $t_1 < d < t_2$ we combine Lemma 5.2 with Theorem 5.1 to obtain the upper bound

$$\begin{aligned} \|f^{=d} * f^{=d}\|_2 &\leq \|Tf\|_{V=d} \|f^{=d}\|_2 \leq \alpha^2 \cdot \left(\frac{e \log(1/\alpha)}{d}\right)^{2Cd} \cdot \left(\frac{cn}{d}\right)^{-d/2} \\ &\leq 1 \cdot \left(\frac{e \log(1/\alpha)}{d}\right)^{2Cd} \cdot \left(\frac{c \log(1/\alpha) n^\varepsilon}{d}\right)^{-d/2}. \end{aligned}$$

We may now use the fact that

$$\left(\frac{e \log(1/\alpha)}{d}\right) \leq e \cdot n^{4\delta}$$

to obtain

$$\|f^{=d} * f^{=d}\|_2 \leq \left(\frac{e \log(1/\alpha)}{d}\right)^{2Cd} \cdot (cn^\varepsilon)^{-d/2} \leq n^{9\delta Cd} n^{-\varepsilon d/2} \leq \frac{1}{2n},$$

provided δ is sufficiently small. Substituting all of these upper bounds into (10) yields

$$\|f * f\|_2^2 \leq 2 + \frac{1}{4} \alpha^{-n^{-\delta}} \leq \alpha^{-n^{-\delta}}$$

provided n_0 is sufficiently large, and substituting this into (9) completes the proof. \square

5.3 Diameter bounds for fine groups

Theorem 5.9. *For each $\varepsilon, c > 0$ and $C > 1$, there exist $m, n_0 > 0$, such that the following holds. Let $n > n_0$ and suppose that G is a (C, c) -fine n -graded group. Let $A \subseteq G$ be a measurable set with $\mu(A) > e^{-n^{1-\varepsilon}}$. Then $A^m = G$.*

Proof. Note that we may assume ε is as small as we please (depending on c and C). First apply Lemma 5.8 repeatedly (N times, say), until A^{2^N} has measure $\geq e^{-n^\varepsilon}$. In other words, $\mu(A^{m_1}) \geq e^{-n^\varepsilon}$, where m_1 depends upon ε alone. We can now apply Corollary 5.6 to obtain $A^{3m_1} = G$, provided that ε is sufficiently small depending on c and C . \square

6 The strong quasirandomness of the simply connected compact Lie groups

In this section we describe the degree decomposition of the n -graded simply connected simple compact Lie groups in terms of their irreducible subrepresentations. We also show that all of them are c -strongly-quasirandom, for some absolute constant $c > 0$. In addition, we introduce the notion of *comfortable d -juntas*. These will be important in our proofs.

One of the goals of this section is to show that each Peter-Weyl ideal $W_\rho \subseteq V_{=d}$ contains a comfortable d -junta. This is useful because any linear operator on $L^2(G)$ that commutes with the action of G from both sides, has each W_ρ as an eigenspace. The operators we use in Section 8 will have this commuting property, and so when computing their eigenvalues we can simply consider the action of the relevant operator on a comfortable d -junta.

6.1 The Peter-Weyl theorem

We now recall some classical facts from the representation theory of compact groups. The Peter-Weyl theorem states that if G is a compact group, equipped with its Haar probability measure, then $L^2(G)$ has the following decomposition as an orthogonal direct sum:

$$L^2(G) = \bigoplus_{\rho \in \hat{G}} W_\rho,$$

where \hat{G} denotes a complete set of complex irreducible unitary representations of G (here, *complete* means having one irreducible representation from each equivalence class of irreducible representations), and W_ρ is the subspace of $L^2(G)$ spanned by functions of the form $g \mapsto u^t \rho(g)v$, for $u, v \in V$, where V is the vector space on which ρ acts. The latter functions are known as the *matrix coefficients* or the *matrix entries* of ρ . The subspaces W_ρ are two-sided ideals (meaning, they are closed under both left and right actions of G), and they are also topologically closed; in fact, they are precisely the minimal non-zero topologically closed two-sided ideals of $L^2(G)$, and they are therefore irreducible as $G \times G$ -modules (the $G \times G$ action being defined in the obvious way, with the first factor acting on $L^2(G)$ from the left and the second from the right). We call them the *Peter-Weyl ideals* of $L^2(G)$, though this terminology is non-standard. The space W_ρ can be decomposed as a direct sum of $\dim(\rho)$ irreducible left-representations.

Since the Peter-Weyl ideals W_ρ are precisely the minimal closed two-sided ideals of $L^2(G)$, every closed two-sided ideal of $L^2(G)$ can be decomposed as a direct sum of some of the W_ρ . Let $d \in \mathbb{N} \cup \{0\}$; since $V_{=d}$ is a closed, two-sided ideal of $L^2(G)$, there exists a set \mathcal{L}_d of irreducible representations of G such that

$$V_{=d} = \bigoplus_{\rho \in \mathcal{L}_d} W_\rho.$$

If $\rho \in \mathcal{L}_d$ for some integer $0 \leq d < n/2$, we say that the *level* of ρ is equal to d . (Note that, since the $V_{=d}$ are pairwise orthogonal, the sets \mathcal{L}_d are pairwise disjoint.)

6.2 Weyl's construction for $\mathrm{SO}(n)$, and its applications

Our goal is now to show that for the group $\mathrm{SO}(n)$ each $\rho \in \mathcal{L}_d$ has dimension at least $(\frac{cn}{d})^d$ for some absolute constant $c > 0$, for each $d < n/2$. We will also show that the other irreducible representations of $\mathrm{SO}(n)$ all have dimension at least exponential in n , i.e. at least $\exp(c'n)$ for some absolute constant $c' > 0$.

We briefly recall Weyl's construction of the irreducible representations of $\mathrm{SO}(n)$. For more detail on Weyl's construction, the reader is referred for example to the book [12] of Fulton and Harris. (We note that, though the description in [12] is of $\mathrm{SO}(n, \mathbb{C})$, the irreducible representations of $\mathrm{SO}(n) := \mathrm{SO}(n, \mathbb{R})$ are in a dimension-preserving one-to-one correspondence with those of its complexification $\mathrm{SO}(n, \mathbb{C})$.) We start by describing the irreducible representations of $\mathrm{O}(n) := \mathrm{O}(n, \mathbb{R})$. Let $V = \mathbb{R}^n$ denote the standard representation of $\mathrm{O}(n)$, defined by $\rho_V(g)(v) = g \cdot v$ — meaning, multiplication of the matrix g with the column-vector v . (We note, for later, that the restriction of this representation to $\mathrm{SO}(n)$ is known as ‘the standard representation of $\mathrm{SO}(n)$ ’.) For a partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ of some non-negative integer, let $d = \sum_{i=1}^{\ell} \lambda_i$. Consider the group algebra of the symmetric group on d elements, $\mathbb{R}[S_d]$, with the standard basis $\{e_g : g \in S_d\}$, and with multiplication defined by $e_g e_h = e_{gh}$ for $g, h \in S_d$. (Where there is no risk of confusion, we will sometimes write g in place of e_g , as an element of $\mathbb{R}[S_d]$, as is usual practice.) Let T be the standard Young tableau of shape λ with the numbers $1, 2, \dots, \lambda_1$ (in order) in the first row, the numbers $\lambda_1 + 1, \lambda_1 + 2, \dots, \lambda_1 + \lambda_2$ (in order) in the second row, and so on. Also, let P be the subgroup of S_d stabilising each of the rows of T (as sets), let Q be the subgroup of S_d stabilising each of the columns of T (as sets), and let

$$c_\lambda = \left(\sum_{g \in P} e_g \right) \left(\sum_{g \in Q} \mathrm{sign}(g) e_g \right)$$

be the *Young symmetrizer* of λ corresponding to T . The group S_d acts on $V^{\otimes d}$ from the right, permuting the factors:

$$(v_1 \otimes v_2 \otimes \dots \otimes v_d)g = v_{g(1)} \otimes v_{g(2)} \otimes \dots \otimes v_{g(d)},$$

and, extending linearly, so does $\mathbb{R}[S_d]$.

We define the *Weyl module* $\mathbb{S}_\lambda(V) := V^{\otimes d} c_\lambda$. Clearly, $\mathbb{S}_\lambda(V)$ is a left $\mathrm{O}(n)$ -submodule of $V^{\otimes d}$. It is reducible in general. However, we can obtain an irreducible left $\mathrm{O}(n)$ -module by considering $\mathbb{S}_{[\lambda]}(V) := V^{[d]} c_\lambda$, where $V^{[d]}$ is defined to be the intersection of the kernels of all $\binom{d}{2}$ linear maps on $V^{\otimes d}$ of the form

$$v_1 \otimes v_2 \otimes \dots \otimes v_d \mapsto \langle v_i, v_j \rangle v_1 \otimes v_2 \otimes \dots \otimes v_{i-1} \otimes v_{i+1} \otimes \dots \otimes v_{j-1} \otimes v_{j+1} \otimes \dots \otimes v_d.$$

Such linear maps are called *contractions*. It turns out that when the sum of the lengths of the first two columns of the Young diagram of λ is greater than n , we have $\mathbb{S}_{[\lambda]}(V) = \{0\}$. The other modules $\mathbb{S}_{[\lambda]}(V)$ (corresponding to those partitions λ such that the sum of the first two columns of the Young diagram of λ is at most n) form a complete set of pairwise inequivalent irreducible complex representations of $\mathrm{O}(n)$.

Weyl's construction for $\mathrm{SO}(n)$ requires only one additional ingredient. We say two partitions λ and μ are *associated* if the sum of the lengths of the first column of λ and the first column of μ is equal to n , and the i th column of λ has the same length as the i th column of μ for each $i > 1$. If λ and μ are a pair of distinct associated partitions, then $\mathbb{S}_{[\lambda]}(V)$

and $\mathbb{S}_{[\mu]}(V)$ restrict to isomorphic representations of $\mathrm{SO}(n)$. If λ is self-associated (which happens iff n is even and the first column of λ has length $n/2$), then $\mathbb{S}_{[\lambda]}(V)$ restricts to a direct sum of two isomorphic irreducible representations of $\mathrm{SO}(n)$; if λ is not self-associated, then $\mathbb{S}_{[\lambda]}(V)$ restricts to an irreducible representation of $\mathrm{SO}(n)$, and if λ' is the partition associated to λ , then $\mathbb{S}_{[\lambda']}(V)$ restricts to the same irreducible representation of $\mathrm{SO}(n)$. In the latter case, it is customary to choose (as the representative of its equivalence class), the partition with first column of length less than $n/2$. Note that, importantly for us, for any partition λ with $\sum_i \lambda_i < n/2$, $\mathbb{S}_{[\lambda]}(V)$ is irreducible as an $\mathrm{SO}(n)$ -representation, as well as being irreducible as an $\mathrm{O}(n)$ -representation, and moreover, as λ ranges over partitions of integers less than $n/2$, the $\mathbb{S}_{[\lambda]}(V)$ are pairwise inequivalent as $\mathrm{SO}(n)$ -representations, as well as being pairwise inequivalent as $\mathrm{O}(n)$ -representations.

An alternative definition of the level of a representation

The purpose of this section is to show that for a partition λ with $\sum_i \lambda_i < n/2$, the level of the irreducible representation $\mathbb{S}_{[\lambda]}(V)$ of $\mathrm{SO}(n)$ is equal to $\sum_i \lambda_i$.

For $0 \leq d < n/2$, define (as above) $\mathcal{L}_d := \{\rho \in \widehat{\mathrm{SO}(n)} : \rho \text{ has level } d\}$, and define $\tilde{\mathcal{L}}_d$ to be the set of irreducible representations of $\mathrm{SO}(n)$ (up to equivalence) that have the form $\mathbb{S}_{[\lambda]}(V)$, where $\sum_i \lambda_i = d$. We wish to show that $\mathcal{L}_d = \tilde{\mathcal{L}}_d$ for all $0 \leq d < n/2$.

Lemma 6.1. *Let $V = \mathbb{R}^n$ be the standard representation of $\mathrm{SO}(n)$ and let $0 \leq d < n/2$. Then all irreducible $\mathrm{SO}(n)$ -subrepresentations of $V^{\otimes d}$ are elements of $\cup_{i=0}^d \tilde{\mathcal{L}}_i$.*

Proof. We prove this lemma by induction on d . The case where $d = 0$ is trivial. Let $d \in \mathbb{N}$ and assume the statement of the lemma holds whenever d is replaced by some $d' < d$. Recall that, since $V^{\otimes d}$ can be expressed a sum of $V^{[d]}$ and some other modules all isomorphic to $V^{\otimes(d-2)}$, all the irreducible $\mathrm{SO}(n)$ -subrepresentations of $V^{\otimes d}$ appear either as $\mathrm{SO}(n)$ -subrepresentations of the module $V^{[d]}$ or as $\mathrm{SO}(n)$ -subrepresentations of $V^{\otimes(d-2)}$. By the induction hypothesis, it therefore suffices to show that each irreducible $\mathrm{SO}(n)$ -subrepresentation of $V^{[d]}$ is an element of $\tilde{\mathcal{L}}_i$ for some i . Let $c_{\lambda,T}$ be the Young symmetrizer corresponding to the Young tableau T (not necessarily the standard one) of shape λ . It is well-known that the $c_{\lambda,T}$ (as λ ranges over all partitions of d and T over all Young tableaux of shape λ) spans a subspace of $\mathbb{R}[S_d]$ containing the class functions; in particular, we may write $\mathrm{Id} \in \mathbb{R}[S_d]$ as a real linear combination of the $c_{\lambda,T}$'s. It follows that $V^{[d]}$ is a sum of left $\mathrm{O}(n)$ -modules of the form $V^{[d]}c_{\lambda,T}$, and clearly $V^{[d]}c_{\lambda,T}$ is isomorphic to $V^{[d]}c_{\lambda} = \mathbb{S}_{[\lambda]}(V)$, as either a left $\mathrm{O}(n)$ -module or a left $\mathrm{SO}(n)$ -module. This completes the proof of the lemma. \square

The following lemma implies that $\mathcal{L}_d = \tilde{\mathcal{L}}_d$ for all $0 \leq d < n/2$.

Lemma 6.2. *If $0 \leq d < n/2$, then*

$$V_{=d} = \bigoplus_{\rho \in \tilde{\mathcal{L}}_d} W_{\rho}.$$

Proof. We prove the lemma by induction on d . For $d = 0$, the statement is trivial. Suppose now that $d > 0$. Since the spaces W_ρ are pairwise orthogonal, the induction hypothesis reduces our task to showing that

$$V_{\leq d} = \bigoplus_{i=0}^d \bigoplus_{\rho \in \tilde{\mathcal{L}}_d} W_\rho.$$

Let us write $\tilde{V}_{\leq d} := \bigoplus_{i=0}^d \bigoplus_{\rho \in \tilde{\mathcal{L}}_d} W_\rho$. We now use Lemma 6.1, namely that all the irreducible subrepresentations of $V^{\otimes d}$ are elements of $\tilde{\mathcal{L}}_i$ for some $i \leq d$. The matrix coefficients of the representation $V^{\otimes d}$ include the entries of the matrix $X^{\otimes d}$, where $X \in \text{SO}(n)$ is the input matrix. These are exactly the degree- d monomials in the entries of X . Decomposing $V^{\otimes d}$ into irreducible representations, we see that all the homogeneous degree- d polynomials belong to $\tilde{V}_{\leq d}$; using the induction hypothesis again, all polynomials of degree at most $d-1$ belong to $\tilde{V}_{\leq d-1} \subset \tilde{V}_{\leq d}$, and therefore $V_{\leq d} \subseteq \tilde{V}_{\leq d}$. The reverse inclusion ($\tilde{V}_{\leq d} \subseteq V_{\leq d}$) follows immediately from the fact that if $\sum_i \lambda_i = d$, then the matrix coefficients of $\mathbb{S}_{[\lambda]}(V)$ are homogeneous degree- d polynomials in the entries of the input matrix. \square

We can now extend the notion of level to all the representations of $\text{SO}(n)$.

Definition 6.3. *Let ρ be an irreducible representation of $\text{SO}(n)$ and let λ be the corresponding partition, whose Young diagram has first column of length at most $n/2$. Then we define the level of ρ to be $\sum_i \lambda_i$.*

Comfortable d -juntas

We now digress a little and show that Weyl's construction implies that each Peter-Weyl ideal W_ρ contains a certain 'nice' function. This will be used later, in Section 8.

Our 'nice' functions are as follows.

Definition 6.4. *The comfortable d -juntas on $\text{SO}(n)$ are the functions on $\text{SO}(n)$ of the form*

$$X \mapsto \sum_{\sigma \in S_d} a_\sigma x_{1,\sigma(1)} \cdots x_{d,\sigma(d)}$$

for $a_\sigma \in \mathbb{R}$.

We remark that we use the term 'junta' here, by analogy with juntas in the theory of Boolean functions (on $\{0,1\}^n$), because functions of the above form depend only upon the upper $d \times d$ minor, though we stress that we will be interested in the case where d is polynomial in n (e.g. $d \sim \sqrt{n}$), rather than just the case of d fixed and n large.

Letting e_1, \dots, e_n be the standard orthonormal basis of \mathbb{R}^n , since $\langle e_i, e_j \rangle = 0$ for all $i, j \in [d]$ we have $e_1 \otimes \dots \otimes e_d \in V^{[d]}$. Therefore $(e_1 \otimes \dots \otimes e_d)c_\lambda \in S_{[\lambda]}(V)$, and thus the function P_λ in $L^2(\text{SO}(n))$ defined by

$$P_\lambda(X) := \langle X((e_1 \otimes \dots \otimes e_d)c_\lambda), e_1 \otimes \dots \otimes e_d \rangle$$

is a matrix coefficient of $\mathbb{S}_{[\lambda]}(V)$. Moreover, the function P_λ is clearly a comfortable d -junta: writing $c_\lambda = \sum_{\sigma \in S_d} \varepsilon_\sigma \sigma$, where $\varepsilon_\sigma \in \{-1, 0, 1\}$ for each $\sigma \in S_d$, we have

$$P_\lambda(X) = \sum_{\sigma \in S_d} \varepsilon_\sigma \prod_{i=1}^d x_{i, \sigma(i)}.$$

Moreover, we clearly have $P_\lambda(\text{Id}) = 1$, so P_λ is a non-zero element of $L^2(\text{SO}(n))$. We obtain the following conclusion, upon which we rely crucially in the sequel.

Fact 6.5. *Let $0 \leq d < n/2$. For each irreducible representation $\rho \in \mathcal{L}_d$ of $\text{SO}(n)$, the Peter-Weyl ideal W_ρ contains a nonzero comfortable d -junta.*

6.3 Obtaining strong quasirandomness for $\text{SO}(n)$.

The following lower bound on the dimension of an irreducible representation of $\text{SO}(n)$ follows immediately from the analysis in [39] of Weyl's original dimension formulae [42]. (We note that our comfortable d -junta machinery could be used to easily obtain a slightly weaker lower bound of $\binom{\lfloor n/2 \rfloor}{d}$. We use such an argument later, when showing strong quasirandomness for $\text{SU}(n)$.)

Lemma 6.6. *If ρ is an irreducible representation of $\text{SO}(n)$ of level $d \leq n$, then*

$$\dim(\rho) \geq \frac{(n-d)^d}{d!}.$$

We also need the following lower bound, whose proof is deferred to the Appendix.

Lemma 6.7. *Let $n \geq 5$. If ρ is an irreducible representation of $\text{SO}(n)$ of level $d \geq n/2$, then*

$$\dim(\rho) \geq \exp(n/32).$$

Lemmas 6.6 and 6.7 immediately give strong quasirandomness.

Theorem 6.8. *For each $n \geq 2$, the n -graded group $\text{SO}(n)$ is c -strongly-quasirandom, for some absolute constant $c > 0$.*

6.4 Obtaining strong quasirandomness for $\text{Spin}(n)$.

The strong quasirandomness of the group $\text{Spin}(n)$ follows from the fact that it is a double cover of $\text{SO}(n)$.

Theorem 6.9. *For each $n \geq 3$, the n -graded group $\text{Spin}(n)$ is c -strongly-quasirandom, for some absolute constant $c > 0$.*

Proof. Recall that the spin group $\text{Spin}(n)$ is the double-cover of $\text{SO}(n)$ for all $n \geq 2$. It is a simply-connected real Lie group for all $n \geq 3$, so its complex irreducible representations are in an explicit (and dimension-preserving) one-to-one correspondence with those of its Lie algebra. It has the same Lie algebra as $\text{SO}(n)$; this Lie algebra $\mathfrak{so}(n, \mathbb{R})$ is simple for all $n \geq 5$, and its complexification $\mathfrak{so}(n, \mathbb{C})$ is likewise simple (for all $n \geq 5$), so by e.g. [12] (26.14), for all $n \geq 5$ the complex irreducible representations of $\mathfrak{so}(n, \mathbb{R})$ are restrictions of unique complex irreducible representations of $\mathfrak{so}(n, \mathbb{C})$. The dimensions of the complex irreducible representations of $\text{Spin}(n)$ (for all $n \geq 5$) are therefore given by equations (24.29) and (24.41) in [12] (pages 408 and 410). For all $n = 2k + 1 \geq 5$ odd, we have

$$\dim(\rho_\lambda) = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j - i + j}{j - i} \prod_{1 \leq i \leq j \leq k} \frac{\lambda_i + \lambda_j + 2k + 1 - i - j}{2k + 1 - i - j},$$

where the k -tuple $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ ranges over all k -tuples defined by

$$\lambda_i = a_i + a_{i+1} + \dots + a_{k-1} + \frac{1}{2}a_k$$

for some $(a_i)_{i=1}^k \in (\mathbb{N} \cup \{0\})^k$. The case of a_k even corresponds to irreducible representations of $\text{Spin}(n)$ that are also irreducible representations of $\text{SO}(n)$; the dimensions of these were bounded previously. The case of a_k odd corresponds to ‘new’ irreducible representations of $\text{Spin}(n)$, but the above equation implies that any such has dimension at least $2^{\Omega(n)}$. This calculational check is deferred to the Appendix (see Section A.2). \square

6.5 Two descriptions of the compact symplectic group $\text{Sp}(n)$

We now turn to the case of the compact symplectic group. This group has two common descriptions, and both will be useful for us.

We start with its description as the unitary group over the field of quaternions, \mathbb{H} . A shorthand is useful at this point: a *quaternionic matrix* is a matrix with quaternion entries. The conjugate \bar{x} of a quaternion $x = a + ib + jc + kd$ is defined, as usual, by $\bar{x} = a - ib - jc - kd$. The conjugate \bar{M} of a quaternionic matrix M is defined by $(\bar{M})_{s,t} = \overline{M_{s,t}}$ for all s, t , and the Hermitian conjugate M^* is defined by $M^* = (\bar{M})^t$. We say an n by n quaternionic matrix is *unitary* if

$$M^*M = I = MM^*;$$

we note that the second equality above is equivalent to the first. For $n \in \mathbb{N}$, the *quaternionic unitary group* $U_n(\mathbb{H})$ is defined to be the group of n by n quaternionic unitary matrices. This is one way of viewing $\text{Sp}(n)$.

A second way of viewing $\text{Sp}(n)$ is as follows. Note that an n by n quaternionic matrix can be written in the form $A + jB$, with A and B being complex n by n matrices. If $A + jB$ and $C + jD$ are two quaternionic matrices, then their product satisfies

$$(A + jB)(C + jD) = AC - \bar{B}D + j(BC + \bar{A}D),$$

and the Hermitian conjugate of the quaternionic matrix $A + jB$ satisfies $(A + jB)^* = A^* - jB^t$. It follows that the map

$$\Phi : \mathbb{H}^{n \times n} \rightarrow \mathbb{C}^{2n \times 2n}$$

from n by n quaternionic matrices to $2n$ by $2n$ complex matrices defined by

$$\Phi(A + jB) = \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}$$

is an (injective) ring homomorphism that preserves Hermitian conjugates. It is easily checked that $M \in \mathbb{H}^{n \times n}$ is unitary if and only if $\Phi(M) \in \mathbb{C}^{2n \times 2n}$ is unitary (writing $M = A + jB$, either condition holds iff $A^*A + B^*B = I$ and $A^tB = B^tA$ both hold). Finally, defining

$$\Omega := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

we observe that an arbitrary unitary matrix

$$X = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \in \mathbb{C}^{2n \times 2n}$$

satisfies $X^t\Omega X = \Omega$ (or, equivalently, $X^t\Omega = \Omega X^*$) if and only if $C = -\bar{B}$ and $D = \bar{A}$, i.e. if and only if X lies in $\Phi(U_n(\mathbb{H}))$. It follows that

$$\Phi(U_n(\mathbb{H})) = \{X \in \mathbb{C}^{2n \times 2n} : X^t\Omega X = \Omega\} \cap U_{2n}(\mathbb{C}).$$

Hence, we can also view $\mathrm{Sp}(n)$ as

$$\{X \in \mathbb{C}^{2n \times 2n} : X^t\Omega X = \Omega\} \cap U_{2n}(\mathbb{C}),$$

i.e., as the intersection of the compact Lie group $U_{2n}(\mathbb{C})$ with

$$\mathrm{Sp}(2n, \mathbb{C}) := \{X \in \mathbb{C}^{2n \times 2n} : X^t\Omega X = \Omega\}.$$

(The group $\mathrm{Sp}(2n, \mathbb{C})$ is known as the *complex symplectic group*, and it is the complexification of $\mathrm{Sp}(n)$.) It can be checked (e.g. by taking the Pfaffian of the equation $X^t\Omega X = \Omega$) that any element of $\mathrm{Sp}(2n, \mathbb{C})$ has determinant one, so $U_{2n}(\mathbb{C})$ can be replaced by $\mathrm{SU}_{2n}(\mathbb{C}) = \mathrm{SU}(2n)$ in the above identification:

$$\mathrm{Sp}(n) = \{X \in \mathbb{C}^{2n \times 2n} : X^t\Omega X = \Omega\} \cap \mathrm{SU}(2n).$$

6.6 Weyl's construction in $\mathrm{Sp}(n)$, and its applications

Weyl's construction for $\mathrm{Sp}(n)$ is similar to his construction for $\mathrm{O}(n)$. Our exposition follows Fulton and Harris [12], as before.

The standard representation of $\mathrm{Sp}(n)$ is given by the second description above, regarding the elements of $\mathrm{Sp}(n)$ as $2n$ by $2n$ complex unitary matrices, and taking their natural (left) action on \mathbb{C}^{2n} .

Let $V = \mathbb{C}^{2n}$ denote the standard representation of $\mathrm{Sp}(n)$. Similarly to in the $\mathrm{O}(n)$ case, we have contraction maps $\psi_{i,j}: V^{\otimes d} \rightarrow V^{\otimes(d-2)}$, given by

$$\psi_{i,j}: v_1 \otimes \cdots \otimes v_d \mapsto Q(v_i, v_j) v_1 \otimes \cdots \hat{v}_i \otimes \cdots \hat{v}_j \otimes \cdots \otimes v_d,$$

where \hat{v} denotes that v is omitted from the tensor product, and $Q(v, w) := v^t \Omega w$, where Ω is the matrix above.

Let $V^{(d)}$ be the intersection of the kernels of all the contractions $\psi_{i,j}$; since the elements of $\mathrm{Sp}(n)$ preserve the skew-symmetric form Q , $V^{(d)}$ is a left $\mathrm{Sp}(n)$ -module. Moreover, $V^{(d)}$ is acted upon by S_d from the right, by permutation of the factors. (The fact that this action preserves $V^{(d)}$ follows from the skew-symmetry of Q .) For a partition $\lambda \vdash d$, we define $S_{\langle \lambda \rangle}(V)$ to be the representation (or left $\mathrm{Sp}(n)$ -module) $V^{(d)} c_\lambda$, where c_λ is the Young symmetrizer defined in Section 6.2. It turns out that $S_{\langle \lambda \rangle}(V)$ is nonzero precisely when (the Young diagram of) λ has at most n rows, and the nonzero left $\mathrm{Sp}(n)$ -modules of this form constitute a complete set of pairwise inequivalent complex irreducible representations of $\mathrm{Sp}(n)$. Note that the situation here is, if anything, even simpler than that for $\mathrm{SO}(n)$, where we have to worry, if only momentarily, about distinct $S_{\langle \lambda \rangle}(V)$ (for two distinct values of λ) restricting to the same irreducible representation of $\mathrm{SO}(n)$.

Recall that the *level* of an irreducible representation ρ of $\mathrm{Sp}(n)$ was defined to be the non-negative integer d such that $\rho \in \mathcal{L}_d$, where

$$V_{=d}^{\mathrm{Sp}(n)} = \bigoplus_{\rho \in \mathcal{L}_d} W_\rho.$$

Let λ be a partition such that $\sum_i \lambda_i < n/2$. Our next aim is to show that the level of the irreducible representation $S_{\langle \lambda \rangle}(V)$ is equal to $\sum_i \lambda_i$, as in the case of $\mathrm{SO}(n)$.

Earlier, in Section 3, we wrote our input quaternion matrix as $A + iB + jC + kD$, where A, B, C, D are n by n real matrices, and we defined $V_{\leq d}$ to consist of the degree $\leq d$ polynomials in the entries of the matrices A, B, C and D . Alternatively, we may write the input matrix as $A + jB$, where A and B are n by n complex matrices; then $V_{\leq d}$ may be viewed as the vector space of degree $\leq d$ polynomials in the entries of $A, B \in \mathbb{C}^{n \times n}$ and their complex conjugates. Finally, we may use the identification Φ above (in Section 6.5) of a quaternion matrix $A + jB$ with the $2n \times 2n$ complex matrix $\begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}$ to obtain that

$V_{\leq d}$ simply consists of polynomials of degree at most d in the entries of $\begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}$; this is precisely what we need to generalise our $\mathrm{SO}(n)$ proofs.

For each $0 \leq d < n/2$, let $\tilde{\mathcal{L}}_d$ denote the set of irreducible representations of $\mathrm{Sp}(n)$ (up to equivalence) of the form $S_{\langle \lambda \rangle}(V)$, where λ is a partition of d . Using the fact that, as with $\mathrm{SO}(n)$, any irreducible $\mathrm{Sp}(n)$ -subrepresentation of $V^{\otimes d}$ appears either as an $\mathrm{Sp}(n)$ -subrepresentation of the module $V^{(d)}$ or as an $\mathrm{Sp}(n)$ -subrepresentation of $V^{\otimes(d-2)}$, the proof of Lemma 6.1 generalizes straightforwardly to give:

Lemma 6.10. *Let $V = \mathbb{C}^{2n}$ be the standard representation of $\mathrm{Sp}(n)$ and let $0 \leq d < n/2$. Then all irreducible $\mathrm{Sp}(n)$ -subrepresentations of $V^{\otimes d}$ are elements of $\cup_{0 \leq i \leq d} \tilde{\mathcal{L}}_i$.*

Then, taking the input matrix X in the form $\begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}$, the proof of Lemma 6.2 generalises straightforwardly to give:

Lemma 6.11. *Let $0 \leq d < n/2$. Then $V_{=d}^{\mathrm{Sp}(n)} = \bigoplus_{\rho \in \tilde{\mathcal{L}}_d} W_\rho$.*

As in the $\mathrm{SO}(n)$ case, we may now extend the notion of level to all irreducible representations of $\mathrm{Sp}(n)$.

Definition 6.12. *Let ρ be an irreducible representation of $\mathrm{Sp}(n)$. The level of ρ is the integer d such that ρ is isomorphic to $S_{\langle \lambda \rangle}(V)$, where λ is a partition of d .*

Comfortable d -juntas

Viewing $\mathrm{Sp}(n)$ as $U_n(\mathbb{H})$, we say a function $f : \mathrm{Sp}(n) \rightarrow \mathbb{C}$ is a *comfortable d -junta* if it is a complex linear combination of monomials of the form

$$M \mapsto \prod_{\ell=1}^d (M_{\ell, \sigma(\ell)})_{q_{\ell\text{-part}}}$$

where $\sigma \in S_d$ and $q_1, \dots, q_d \in \{i, j, k, \mathbf{real}\}$.

An easy variant of our argument for $\mathrm{SO}(n)$, yields the following.

Lemma 6.13. *Let ρ be a representation of level $d \leq n$ of $\mathrm{Sp}(n)$. Then W_ρ contains a non-zero comfortable d -junta.*

Proof. We first adopt the second perspective on $\mathrm{Sp}(n)$, viewing it as

$$\{X \in \mathbb{C}^{2n \times 2n} : X^t \Omega X = \Omega\} \cap \mathrm{SU}(2n).$$

Letting e_1, \dots, e_{2n} be the standard orthonormal basis of \mathbb{C}^{2n} , since $Q(e_i, e_j) = 0$ for all $i, j \in [n]$ we have $e_1 \otimes \dots \otimes e_d \in V^{(d)}$. Therefore $(e_1 \otimes \dots \otimes e_d)c_\lambda \in S_{\langle \lambda \rangle}(V)$, and thus the function P_λ in $L^2(\mathrm{Sp}(n))$ defined by

$$P_\lambda(X) := \langle X((e_1 \otimes \dots \otimes e_d)c_\lambda), e_1 \otimes \dots \otimes e_d \rangle$$

is a matrix coefficient of $S_{\langle \lambda \rangle}(V)$. Note that this is exactly the same function as we exhibited for $\mathrm{SO}(n)$, except that its domain is

$$\{X \in \mathbb{C}^{2n \times 2n} : X^t \Omega X = \Omega\} \cap \mathrm{SU}(2n) \cong \mathrm{Sp}(n)$$

rather than $\mathrm{SO}(n)$. We claim that the function P_λ is a comfortable d -junta. Indeed, writing $c_\lambda = \sum_{\sigma \in S_d} \varepsilon_\sigma \sigma \in \mathbb{R}[S_n]$, where $\varepsilon_\sigma \in \{-1, 0, 1\}$ for each $\sigma \in S_d$, we have

$$P_\lambda(X) = \sum_{\sigma \in S_d} \varepsilon_\sigma \prod_{\ell=1}^d x_{\ell, \sigma(\ell)}.$$

Writing $X = \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}$, we see that each $x_{\ell, \sigma(\ell)}$ appearing above is actually $a_{\ell, \sigma(\ell)}$ (since $d \leq n$ and $\sigma \in S_d$); writing each $a_{\ell, \sigma(\ell)}$ as a sum of its real and imaginary parts and expanding, we see that P_λ is indeed a complex linear combination of monomials of the required form (in fact, with each q_ℓ being either **real** or i , and with the coefficient of each monomial being in $\{\pm 1, \pm i\}$).

The function P_λ is non-zero element of $L^2(\mathrm{Sp}(n))$ since (as with $\mathrm{SO}(n)$) we have $P_\lambda(\mathrm{Id}) = 1$. \square

6.7 Obtaining strong quasirandomness for $\mathrm{Sp}(n)$.

The dimensions of the complex irreducible representations of $\mathrm{Sp}(n)$ are given by equation (24.19) in [12]. These representations are in one-to-one correspondence with partitions of non-negative integers, whose Young diagrams have at most n rows; the dimension of the irreducible representation ρ_λ corresponding to the partition $\lambda = (\lambda_1, \dots, \lambda_n)$ (with $\lambda_1 \geq \dots \geq \lambda_n$) is given by

$$\dim(\rho_\lambda) = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j - i + j}{j - i} \prod_{1 \leq i \leq j \leq n} \frac{\lambda_i + \lambda_j + 2n + 2 - i - j}{2n + 2 - i - j}.$$

As with the (odd) special orthogonal groups, we define the *level* of the representation ρ_λ to be the number of cells in the Young diagram of λ (i.e., it is the non-negative integer of which λ is a partition). The proof of the following lemma is similar to that of Lemma 6.7. We defer it to the Appendix.

Lemma 6.14. *If ρ_λ is an irreducible representation of $\mathrm{Sp}(n)$ of level $d \geq n/2$, then $\dim(\rho_\lambda) \geq \exp(n/16)$.*

Again, as with the special orthogonal groups, the following lower bound is immediate from the analysis in [39] of Weyl's dimension formulae [42].

Lemma 6.15. *If ρ is an irreducible representation of $\mathrm{Sp}(n)$ of level $d \leq n$, then*

$$\dim(\rho) \geq \frac{(n-d)^d}{d!}.$$

These yield the following.

Theorem 6.16. *For each $n \geq 2$, the n -graded group $\mathrm{Sp}(n)$ is c -strongly-quasirandom, for some absolute constant $c > 0$.*

6.8 Weyl's construction for $\mathrm{SU}(n)$, and its applications.

The final group to consider is the special unitary group. We start by relating our degree decomposition of $L^2(\mathrm{SU}(n))$ to the decomposition of $L^2(\mathrm{SU}(n))$ into Peter-Weyl ideals.

Degree decomposition

Earlier, we defined $V_{\leq d}$ to consist of the polynomials of (total) degree at most d in the real parts and the imaginary parts of the entries of the input matrix $X \in \mathrm{SU}(n)$. Equivalently, $V_{\leq d}$ consists of the polynomials of (total) degree at most d in the entries of the input matrix and their complex conjugates.

Weyl's construction for $\mathrm{SU}(n)$

We now recall Weyl's construction of the irreducible representations of $\mathrm{SU}(n)$, and deduce from it the consequences we need. (As before, for more detail on Weyl's construction, the reader is referred to Fulton and Harris [12], noting that the complex irreducible representations of $\mathrm{SU}(n)$ are the same as those of $\mathrm{SL}(n, \mathbb{C})$, since $\mathrm{SU}(n)$ is a maximal compact subgroup of $\mathrm{SL}(n, \mathbb{C})$.) Let $V = \mathbb{C}^n$ denote the standard representation of $\mathrm{SU}(n)$, defined by $\rho_V(g)(v) = g \cdot v$. For a partition λ with at most $n - 1$ parts, let $d = d(\lambda) = \sum_i \lambda_i$. Let T be the standard Young tableau of shape λ (defined in Section 6.2) and let c_λ be the corresponding Young symmetrizer (also defined in Section 6.2). We define the corresponding *Weyl module* by $\mathbb{S}_\lambda(V) := V^{\otimes d} c_\lambda$. Clearly, $\mathbb{S}_\lambda(V)$ is a left $\mathrm{SU}(n)$ -submodule of $V^{\otimes d}$. The modules $\mathbb{S}_\lambda(V)$, as λ ranges over all partitions with at most $n - 1$ parts, constitute a complete set of pairwise inequivalent complex irreducible representations of $\mathrm{SU}(n)$. (Unlike in the cases of $\mathrm{SO}(n)$ and $\mathrm{Sp}(n)$, we do not need to pass to a subrepresentation of the Weyl module; the latter is already irreducible as a left $\mathrm{SU}(n)$ -module.)

Unlike in the case of $\mathrm{Sp}(n)$, however, the complex conjugates of the entries of the input matrix are no longer matrix coefficients of the standard representation. Instead, they are matrix coefficients of the dual of the standard representation, i.e. they are the entries of the matrix $(A^{-1})^t = \bar{A}$. (Recall that the dual of a representation ρ is the representation ρ^* defined by $\rho^*(g) = (\rho(g^{-1}))^t$.) We note that $(\mathbb{S}_\lambda(V))^* \cong \mathbb{S}_\lambda(V^*) = (V^*)^{\otimes d} c_\lambda$.

We have three notions of level for a representation ρ of $\mathrm{SU}(n)$, and our goal is to show that they agree when the level is $< n/2$.

Recall that we defined $V_{=0} := V_{\leq 0}$ (the space of constant functions), and $V_{=d} = V_{\leq d} \cap V_{\leq d-1}^\perp$ for each $d \in \mathbb{N}$. Since $V_{=d}$ is closed under both left and right actions of $\mathrm{SU}(n)$ for each $d \in \mathbb{N} \cup \{0\}$, there exists a set \mathcal{L}_d of irreducible representations of $\mathrm{SU}(n)$ such that

$$V_{=d}^{\mathrm{SU}(n)} = \bigoplus_{\rho \in \mathcal{L}_d} W_\rho.$$

If $\rho \in \mathcal{L}_d$ for some $0 \leq d < n/2$, we define the *level* of ρ to be d . (Note that, since the $V_{=d}$ are pairwise orthogonal, the sets \mathcal{L}_d are pairwise disjoint.)

In addition, we define the *tensor level* of an irreducible representation ρ of $\mathrm{SU}(n)$ to be the minimal non-negative integer d such that there exist $d_1, d_2 \in \mathbb{N} \cup \{0\}$ with $d_1 + d_2 = d$ and with ρ being isomorphic to a subrepresentation of $V^{\otimes d_1} \otimes V^{*\otimes d_2}$. We write $\tilde{\mathcal{L}}_d$ for the set of irreducible representations of $\mathrm{SU}(n)$ (up to equivalence) that have tensor level equal to d .

The third notion of level will soon be described in terms of the Young diagram/partition corresponding to the representation.

The following analogue of Lemma 6.1 is immediate, and implies that for $0 \leq d < n/2$, having level d is equivalent to having tensor level d .

Lemma 6.17. *If $0 \leq d < n/2$, then $V_{=d}^{\text{SU}(n)} = \bigoplus_{\rho \in \tilde{\mathcal{L}}_d} W_\rho$.*

Step vectors

For an irreducible representation $\rho = \mathbb{S}_\lambda(V)$ of $\text{SU}(n)$ with corresponding partition $\lambda = (\lambda_1, \dots, \lambda_{n-1})$, write $a_i := \lambda_i - \lambda_{i+1}$ for each $i \in [n-1]$. We call the vector (a_1, \dots, a_{n-1}) the *step vector* of the representation ρ (or, abusing terminology slightly, the *step vector* of the partition λ). We order such vectors with respect to the lexicographic ordering, i.e. $\lambda >_{\text{lex}} \lambda'$ iff $\lambda_i > \lambda'_i$ where $i = \min\{j : \lambda_j \neq \lambda'_j\}$. For a (not necessarily irreducible) representation ρ of $\text{SU}(n)$, its step vector is defined to be the lexicographically largest step vector of an irreducible subrepresentation of ρ .

The step vector is better-behaved than the corresponding partition, with respect to taking duals and tensors. The dual of ρ has the reversed step vector (a_{n-1}, \dots, a_1) , corresponding to the partition $(a_1 + \dots + a_{n-1}, \dots, a_1 + a_2, a_1)$. Moreover, if ρ_1 and ρ_2 are two representations whose step vectors are (a_1, \dots, a_{n-1}) and (b_1, \dots, b_{n-1}) , then the step vector of their tensor product is $(a_1 + b_1, \dots, a_{n-1} + b_{n-1})$.

We say that an irreducible representation $\rho = \mathbb{S}_\lambda(V)$ of $\text{SU}(n)$ is *efficient* if $\lambda_{\lfloor n/2 \rfloor} = 0$. Equivalently, ρ is efficient if its step vector $w = (a_1, \dots, a_{n-1})$ has the property that its *second half* $w'' := (a_{\lfloor n/2 \rfloor}, \dots, a_{n-1})$ consists of zeros. We say that ρ is *dually-efficient* if its *first half* $w' := (a_1, \dots, a_{\lfloor n/2 \rfloor - 1})$ consists of zeroes. Alternatively, in terms of the corresponding partition λ , the irreducible representation ρ is dually-efficient if $\lambda_1 = \lambda_2 = \dots = \lambda_{\lfloor n/2 \rfloor - 1}$. (Note that the dual to each efficient representation is dually-efficient, and if n is odd, the converse also holds. For n even, our definition leads to a somewhat arbitrary choice of how to handle the middle part of the step vector, but this does not matter when the level is smaller than $n/2$.) We call the partition α with step vector w' the *efficient truncation* of λ and the partition β with step vector w'' the *dually-efficient truncation* of λ .

Definition 6.18. *We define the total level of a representation $\rho = \mathbb{S}_\lambda(V)$ with step vector (a_1, \dots, a_{n-1}) to be*

$$\sum_{i=1}^{n-1} a_i \min\{i, n-i\}.$$

For each $d \in \mathbb{N} \cup \{0\}$, let $\tilde{\mathcal{L}}_d$ denote set of irreducible representations of $\text{SU}(n)$ with total level d . Our next aim is to show that $\mathcal{L}_d = \tilde{\mathcal{L}}_d$ for all $0 \leq d < n/2$. For this, we first need the following.

Lemma 6.19. *Let $\mathbb{S}_\lambda(V)$ be an irreducible representation of $\text{SU}(n)$ of total level d , with λ having α as its efficient truncation and β as its dually-efficient truncation. Then the*

representation $\mathbb{S}_\alpha(V) \otimes \mathbb{S}_\beta(V)$ can be decomposed as a direct sum of one copy of the representation $\mathbb{S}_\lambda(V)$ and some other irreducible representations, all of which have total level less than d .

Proof. This follows from the Littlewood-Richardson rule, e.g. as given in Fulton and Harris [12, Section A.8]. Indeed, by the Littlewood-Richardson rule, writing $\alpha = (\alpha_1, \dots, \alpha_\ell)$, the irreducible constituents of $\mathbb{S}_\beta(V) \otimes \mathbb{S}_\alpha(V)$ are exactly those $\mathbb{S}_\lambda(V)$ such that the Young diagram of λ can be produced by the following algorithm. Take the Young diagram of β , and first add α_1 new boxes to the rows (in such a way as to produce the Young diagram of another partition, but with no two of the α_1 added boxes being added to the same column), and place a ‘1’ in each of these α_1 new boxes. Then add a further α_2 boxes to the rows (again in such a way as to produce the Young diagram of another partition, but with no two of the α_2 added boxes being added to the same column), and place a ‘2’ in each of these α_2 new boxes. Continue in this way (so that at the last step, α_ℓ new boxes are added). Now consider the sequence of length $\alpha_1 + \dots + \alpha_\ell$ formed by concatenating the reversed rows of newly-added boxes, and check that if one looks at the first t entries in this sequence (for any t between 1 and $\alpha_1 + \dots + \alpha_\ell$), the integer p appears at least as many times as the integer $p + 1$ among these first t entries, for any $1 \leq p < \ell$. If this ‘concatenation’ condition holds, keep the Young diagram / partition; if not, reject it.

It is easy to check that the only way of performing this algorithm in such a way as to obtain a partition of level at least d , is to produce the (Young diagram of the) partition λ itself: the first α_1 new boxes must all be added to the first row of the Young diagram of β , the second α_2 new boxes must all be added to the second row, and so on. Indeed, if at the j th stage (when adding α_j new boxes containing the integer j), any box is added to a row above the j th row, then (inductively) one sees that the concatenation condition would be violated, and moreover if some new box is added to a column of depth greater than $n/2$, then clearly, at the end of the process, less than E cells will be in columns of depth at most $n/2$, and moreover less than F cells will be missing from columns of depth greater than $n/2$, so the irreducible constituent of $\mathbb{S}_\beta(V) \otimes \mathbb{S}_\alpha(V)$ which is obtained, will have level less than d . \square

We note the following consequence.

Lemma 6.20. *Let λ be a partition with $n - 1$ rows. Let α be its efficient truncation and β its dually-efficient truncation. Write $E = \sum_{i=1}^{\lceil n/2 \rceil} i \cdot a_i$ and $F = \sum_{i=1}^{\lceil n/2 \rceil - 1} i a_{n-i}$. Then the representation $\mathbb{S}_\alpha(V)$ is a subrepresentation of $V^{\otimes E}$, the representation $\mathbb{S}_\beta(V)$ is a subrepresentation of $(V^*)^{\otimes F}$, and the representation $\mathbb{S}_\lambda(V)$ is a subrepresentation of*

$$V^{\otimes E} \otimes (V^*)^{\otimes F}.$$

Proof. The first assertion, concerning $\mathbb{S}_\alpha(V)$, is immediate from the construction of the Weyl module, since α is a partition of the integer E . The second assertion, concerning $\mathbb{S}_\beta(V)$, follows from the first, together with the fact that taking duals reverses the step vector. The third assertion, concerning $\mathbb{S}_\lambda(V)$, now follows from the previous lemma. \square

Lemma 6.21. *For all $0 \leq d < n/2$, we have $\mathcal{L}_d = \tilde{\mathcal{L}}_d$.*

Proof. By Lemma 6.20, we have $\tilde{\mathcal{L}}_d \subseteq \mathcal{L}_d$ for all $0 \leq d < n/2$. It therefore suffices to show that for each $0 \leq d < n/2$,

$$\bigcup_{i=0}^d \mathcal{L}_i \subseteq \bigcup_{i=0}^d \tilde{\mathcal{L}}_i.$$

So suppose that ρ is an irreducible representation of $\mathrm{SU}(n)$ of level at most d . Our goal is to show that the total level of ρ is at most d . Let $E+F \leq d$ be such that ρ is a subrepresentation of $V^{\otimes E} \otimes (V^*)^{\otimes F}$. Just as in the proof of Lemma 6.1, we may decompose $V^{\otimes E}$ into a direct sum of submodules of the form $\mathbb{S}_\alpha(V)$ with $\alpha \vdash E$, and likewise we can decompose $(V^*)^{\otimes F}$ into a direct sum of submodules of the form $\mathbb{S}_\beta(V^*)$ with $\beta \vdash F$. Hence, we may assume that ρ is isomorphic to a subrepresentation of $\mathbb{S}_\alpha(V) \otimes \mathbb{S}_\beta(V^*) \cong \mathbb{S}_\alpha(V) \otimes (\mathbb{S}_\beta(V))^*$ for some $\alpha \vdash E$ and $\beta \vdash F$. Now as $d < n/2$, both $\mathbb{S}_\alpha(V)$ and $\mathbb{S}_\beta(V)$ are efficient, so $(\mathbb{S}_\beta(V))^*$ is dually efficient, and therefore we may apply Lemma 6.19 to deduce that the total level of ρ is at most the sum of the total levels of $\mathbb{S}_\alpha(V)$ and $(\mathbb{S}_\beta(V))^*$, which is $E+F$, as required. \square

Comfortable d -juntas

We say that a function $f : \mathrm{SU}(n) \rightarrow \mathbb{C}$ is a *comfortable d -junta* if it is a complex linear combination of monomials of the form

$$X \mapsto \prod_{j=1}^d (x_{j,\sigma(j)})_{q_j\text{-part}}$$

where $\sigma \in S_d$ and $q_1, \dots, q_d \in \{\mathbf{imaginary}, \mathbf{real}\}$.

We start by showing that the comfortable d -juntas are in $V_{=d}$, i.e. that they are degree d polynomials that are orthogonal to all polynomials of degree $\leq d-1$. (Here, the degree is in terms of the matrix entries and their complex conjugates, or equivalently, in terms of the real and the imaginary parts of the matrix entries.)

Lemma 6.22. *Every comfortable d -junta belongs to $V_{=d}$.*

Proof. Every comfortable d -junta clearly lies in $V_{\leq d}$. It suffices to show that any such is orthogonal to all polynomials (in the matrix entries and their complex conjugates) of degree $\leq d-1$. Let T be one of the degree d monomials that appear in our comfortable d -junta of degree $\leq d$. Let S be an arbitrary monomial of degree $\leq d-1$. It suffices to show that S and T are orthogonal. Let k_1, \dots, k_d be the rows of the matrix-entry variables appearing in the monomial T . Since S is a monomial of degree less than d , not all of these rows can appear amongst the variables in S ; without loss of generality, assume row k_1 does not. Let k_0 be a row that appears neither in the variables in S nor those in T (such exists, since $d \leq n/2$, so $d + (d-1) < n$). Let U_0 be the diagonal unitary matrix with ones everywhere on the diagonal except in the (k_0, k_0) and (k_1, k_1) entries, with a $-i$ in the (k_0, k_0) entry and an i in the (k_1, k_1) entry. If X is distributed according to the Haar

measure on $SU(n)$, then so is U_0X , but multiplying X by U_0 simply multiplies $\overline{S}T$ by i , so $\mathbb{E}_X[\overline{S}(X)T(X)] = \mathbb{E}_X[\overline{S}(U_0X)T(U_0X)] = i\mathbb{E}_X[\overline{S}(X)T(X)]$ and therefore $\mathbb{E}_X[\overline{S}T] = 0$ as required. \square

For the proof of the next lemma we need another definition. We say a polynomial $p : SU(n) \rightarrow \mathbb{C}$ in the matrix entries of $X \in SU(n)$ is *weakly-comfortable* if it is a complex linear combination of monomials of the form

$$X \mapsto \prod_{k=1}^{\ell} x_{i_k, j_k},$$

where i_1, \dots, i_{ℓ} are distinct and j_1, \dots, j_{ℓ} are distinct.

Lemma 6.23. *Let ρ be a representation of $SU(n)$ of total level d , where $0 \leq d \leq n$. Then W_{ρ} contains a comfortable d -junta.*

Proof. Let ρ be an irreducible representation of $SU(n)$ of total level d . Let λ be the corresponding integer partition (with at most $n - 1$ rows). Let α (respectively β) be its efficient part and dually efficient part respectively. Note that the Young diagram of β^* has F cells, and all are in columns of depth at most $n/2$. As in the $SO(n)$ case, there exists a weakly-comfortable, homogeneous polynomial P of total degree E , such that $P \in W_{\rho_{\alpha}}$, namely the polynomial

$$P_{\alpha}(X) := \langle X((e_1 \otimes \dots \otimes e_d)c_{\alpha}), e_1 \otimes \dots \otimes e_d \rangle,$$

where c_{α} is the Young symmetrizer corresponding to α ; writing $c_{\alpha} = \sum_{\sigma \in S_d} \varepsilon_{\sigma} \sigma$, where $\varepsilon_{\sigma} \in \{-1, 0, 1\}$ for each $\sigma \in S_d$, we have

$$P_{\lambda}(X) = \sum_{\sigma \in S_d} \varepsilon_{\sigma} \prod_{i=1}^d x_{i\sigma(i)}.$$

Similarly, there exists a non-zero, weakly-comfortable, homogeneous polynomial Q of total degree F , such that $Q \in W_{\rho_{\beta^*}}$. Since $E + F = d \leq n$, we may further take P to depend only upon matrix entries in the top E by E minor, and Q to depend only upon matrix entries in the minor $[E + 1, E + F] \times [E + 1, E + F]$.

Note that $P\overline{Q}$ is a comfortable d -junta and is spanned by the matrix coefficients of $\rho_{\alpha} \otimes \rho_{\beta}$. We claim that, in fact, $P\overline{Q}$ is spanned by the matrix coefficients of ρ_{λ} . This follows immediately from the fact that, firstly, by Lemma 6.19, $\rho_{\alpha} \otimes \rho_{\beta}$ can be decomposed into a direct sum of a copy of ρ_{λ} and some other irreducible representations of total level less than d , and that secondly, by Lemma 6.22, the comfortable d -junta $P\overline{Q}$ is orthogonal to $V_{\leq d-1}^{SU(n)}$. \square

6.9 Obtaining strong quasirandomness for $SU(n)$.

Lemma 6.24. *Let $k, n \in \mathbb{N}$ with $k \leq n/2$, and let*

$$P = P(X_{11}, X_{12}, \dots, X_{kk}) \in \mathbb{C}[X_{11}, X_{12}, \dots, X_{kk}] \setminus \{0\}$$

be a multivariate polynomial in the variables $(X_{i,j})_{i,j \in [k]}$ that is not the zero polynomial. Let $\pi : SU(n) \rightarrow \mathbb{C}^{k \times k}$ denote projection onto the top-left k by k minor of a matrix in $SU(n)$. Then $P \circ \pi$ cannot vanish on all of $SU(n)$.

Proof. The image of π is easily seen to have a nonempty interior inside $\mathbb{C}^{d \times d}$ (indeed, a suitably small open neighbourhood of 0 is contained in the image of π). Since P is a nontrivial polynomial in the variables $X_{11}, X_{12}, \dots, X_{kk}$, it cannot vanish on all of a nonempty open subset of $\mathbb{C}^{k \times k}$. \square

The following lemma is analogous to Lemma 6.6, and allows us to lower-bound the dimensions of irreducible representations of not-too-large degree. We prove it by considering comfortable d -juntas.

Lemma 6.25. *If ρ is an irreducible representation of $SU(n)$ of level d where $0 \leq d < n/2$, then*

$$\dim(\rho) \geq \binom{\lfloor n/2 \rfloor}{d}.$$

Proof. Let P be a non-zero comfortable d -junta contained in W_ρ . Recall that the left action of $SU(n)$ on $L^2(SU(n))$ is defined as follows: for $A \in SU(n)$ and $f \in L^2(SU(n))$, we define $L_A f \in L^2(SU(n))$ by $L_A f(X) = f(AX)$. Since $P \in W_\rho \setminus \{0\}$, the set $\{L_A P : A \in SU(n)\}$ is contained in a left submodule V of $L^2(SU(n))$ which is isomorphic to the representation ρ . In particular, for any even permutation σ , $L_{A(\sigma)} P \in V$, where $A(\sigma)$ is the permutation matrix corresponding to σ ; explicitly, $(A_\sigma)_{i,j} = 1_{\{\sigma(i)=j\}}$ for each $i, j \in [n]$. Note that $L_{A(\sigma)} P$ is the polynomial obtained from P by replacing the variable $X_{i,j}$ with the variable $X_{\sigma(i),j}$, for all $i, j \in [d]$; write $\sigma P := L_{A(\sigma)} P$, for brevity. For each subset $S \in \binom{[d]}{d}$, choose an even permutation σ_S sending $[d]$ to S . The polynomials $\sigma_S P$ are linearly independent as polynomials (as for any two distinct sets $S \neq S'$, the set of monomials appearing in $\sigma_S P$ is disjoint from the set of monomials appearing in $\sigma_{S'} P$); moreover, each polynomial $\sigma_S P$ depends only upon variables in the top left $\lfloor n/2 \rfloor$ by $\lfloor n/2 \rfloor$ minor. It follows from the previous lemma, applied with $k = \lfloor n/2 \rfloor$, that the polynomials $\sigma_S P$ are linearly independent as elements of $L^2(G)$, and therefore $\dim(V) \geq \binom{\lfloor n/2 \rfloor}{d}$, as required. \square

We remark that above method for lower-bounding the dimensions of the irreducible representations of $SU(n)$ works just as well for $SO(n)$ and $Sp(n)$, so we could have used it in place of Lemmas 6.6 and 6.15 to give an alternative, self-contained proof of the c -strong-quasirandomness of $SO(n)$ and of $Sp(n)$.

The following lemma is analogous to Lemma 6.7 and lower-bounds the dimensions of irreducible representations of $SU(n)$ with high levels. We defer the proof till the Appendix.

Lemma 6.26. *There exists an absolute constant $c > 0$ such that if $d \geq n/4$ and ρ is an irreducible representation of $SU(n)$ of level d , then $\dim(\rho) \geq 2^{cn}$.*

Lemmas 6.25 and 6.26 immediately yield the strong quasirandomness of $SU(n)$.

Lemma 6.27. *For each $n \geq 2$, the group $SU(n)$ is c -strongly-quasirandom as an n -graded group, where $c > 0$ is an absolute constant.*

7 Simply connected compact Lie groups are fine

In this section, we prove Theorem 3.14. The proof has two parts. In the first part, we identify a natural noise operator U_δ on $L^2(G)$ for the groups $G = SU(n), Sp(n), Spin(n)$ which is guaranteed to satisfy a certain hypercontractive inequality, thanks to the fact that the Ricci curvature of these groups is bounded from below. (We note that this noise operator U_δ is not quite the same as the Beckner operator $T_{\delta,r}$ that we defined earlier.) The second part consists of inferring the weak hypercontractivity of the operator $T_{\delta,r}$ from the hypercontractive inequality for U_δ . We accomplish that by analyzing the eigenvalues of U_δ and showing that they are all larger than the eigenvalues of the operator $T_{\delta^C,r}$ for some absolute constant $C > 1$. This will allow us to write $T_{\delta^C,r} = U_\delta S$ for a linear operator S on $L^2(G)$ satisfying $\|S\|_{2 \rightarrow 2} \leq 1$. We will thus have

$$\|T_{\delta^C,r}\|_{2 \rightarrow q} \leq \|S\|_{2 \rightarrow 2} \cdot \|U_\delta\|_{2 \rightarrow q} \leq 1,$$

as needed.

The hypercontractive inequality

Here we rely on concepts from differential geometry, such as a Riemannian metric, the Laplace–Beltrami operator, and the Ricci curvature/tensor. We use the notation of Anderson, Guionnet and Zeitouni [1, Sections E and F], and we refer the reader to that work for more details.

The compact Lie groups are equipped with the structure of a bi-invariant Riemannian manifold (M, g) ; this is unique up to normalization if the compact Lie group is simple. Once a normalization is set, and denoting by Δ the Laplace–Beltrami operator, it is also known that the Hilbert space $L^2(M)$ has an orthonormal basis of eigenvectors of Δ , that Δ is self-adjoint and negative semidefinite, and that 0 is a simple eigenvalue of Δ (with the constant functions as corresponding eigenvectors). For a given compact Lie group G with a Riemann manifold structure, we let u_0, u_1, u_2, \dots be such a basis, with $0 = \lambda_0 > \lambda_1 \geq \lambda_2 \geq \dots$ being the corresponding eigenvalues, so that $\lambda_i < 0$ for all $i \geq 1$, and with u_0 being the constant function with value 1. For any $f \in L^2(M)$, we may write f uniquely in the form

$$f = \sum_{i=0}^{\infty} c_i u_i,$$

where $c_i \in \mathbb{R}$ for each $i \geq 0$ (we have $c_i = \langle f, u_i \rangle$ for each $i \geq 0$). For $\delta > 0$, we define the *noise operator* U_δ by

$$U_\delta : L^2(M) \rightarrow L^2(M); \quad U_\delta(f) = \sum_{i=0}^{\infty} c_i \delta^{-\lambda_i} u_i,$$

for $f = \sum_{i=0}^{\infty} c_i u_i$. We note that $U_{e^{-t}}$ is, in fact, the heat kernel corresponding to Δ , which is the averaging operator with respect to the Brownian motion on the corresponding manifold.

For a Riemannian manifold (M, g) and $C > 0$, we say that (M, g) has *Ricci curvature bounded from below by C* if for all points $p \in M$, the Ricci tensor $\text{Ric}_p(\cdot, \cdot)$ at p satisfies

$$\text{Ric}_p(X, X) \geq C g_p(X, X)$$

for all tangent vectors X at p .

The hypercontractive inequality we need is the following.

Theorem 7.1. *Let $C > 0$ and let (M, g) be a compact, connected Riemann manifold whose Ricci curvature is bounded from below by C , let $2 \leq p \leq q$ and let $f \in L^p(M)$. Then*

$$\|U_\delta(f)\|_q \leq \|f\|_p \quad \forall 0 \leq \delta \leq \left(\frac{p-1}{q-1}\right)^{1/C}.$$

As explained in Klartag and Regev [38] the Bakry-Emery criterion yields a log-Sobolev inequality (as given for example in [1, Corollary 4.4.25] applied with $\Phi = 0$), which implies a hypercontractive inequality by a theorem of Gross [16, Theorem 6]).

Utilizing the Riemannian structure. As mentioned above, the compact simple Lie groups have a unique (up to normalization) bi-invariant Riemannian manifold structure. In order to set it up one needs to assign an inner-product on the tangent space at the identity Id of G , i.e., on the Lie algebra of G . (The inner product on all the other tangent spaces is then determined by using a push-forward with respect to left multiplication by the appropriate group element.)

The tangent space of $\text{Spin}(n)$ at the identity is the Lie algebra of $\text{SO}(n)$. As in [1], we equip it with the usual Euclidean / Hilbert-Schmidt norm, i.e. the norm of a matrix is the sum of the squares of its entries. This norm gives rise to a bi-invariant metric when applying push-forward maps to define the norm on all the other tangent spaces. The norm on the Lie algebras of $\text{SU}(n)$ and $\text{Sp}(n)$ is defined similarly by taking the sum of squares of the components of each entry.

It is well-known (see [1, 4.4.30]), that the Ricci curvature of the simply connected compact Lie groups $\text{Spin}(n)$, $\text{SU}(n)$, $\text{Sp}(n)$ is bounded from below by $(n-2)/4$, $\frac{n}{2}$, $n+1$ respectively.

Using Theorem 7.1, we can prove Theorem 3.14, assuming the following lemma.

Lemma 7.2. *There exists an absolute constant C , such that the following holds. Let G be either $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$ or $\mathrm{SO}(n)$. Let $d \in \mathbb{N}$ with $0 \leq d < n/2$ and let $\rho \in \mathcal{L}_d$ be a representation of level d . Then the corresponding eigenvalue of the Laplace–Beltrami operator for G satisfies $\lambda_\rho \geq -Cnd$.*

The proof of this lemma uses a formula for the eigenvalue λ_ρ in terms of the fundamental weight corresponding to ρ , which is well-known and appears e.g. in Berti and Procesi [7]. We defer the proof to the Appendix.

Proof of Theorem 3.14. We have already established the strong quasirandomness for all the groups G of the form $\mathrm{Spin}(n)$, $\mathrm{Sp}(n)$, and $\mathrm{SU}(n)$. It remains to establish their weak hypercontractivity. By Lemma 3.24, it suffices to prove that $\mathrm{SO}(n)$, $\mathrm{Sp}(n)$ and $\mathrm{SU}(n)$ are weakly hypercontractive.

By Theorem 7.1 there exists an absolute constant C_1 , such that setting $\delta = \left(\frac{1}{q-1}\right)^{\frac{1}{nC_1}}$, we have $\|U_\delta\|_{2 \rightarrow q} \leq 1$. Let $\rho \in \mathcal{L}_d$ be of level d . Then by Lemma 7.2 we obtain that the eigenvalues of U_δ given by $\delta^{-\lambda_\rho}$ are $\geq q^{-Cd}$ for some absolute constant C . This implies that we may write $T_{q^{-C}, r} = U_\delta \circ S$, where all the eigenvalues of S are ≤ 1 . Hence,

$$\|T_{q^{-C}, r}\|_{2 \rightarrow q} \leq \|U_\delta\|_{2 \rightarrow q} \|S\|_{2 \rightarrow 2} \leq 1.$$

This completes the proof of the theorem. □

8 Showing that $\mathrm{Sp}(n)$, $\mathrm{Spin}(n)$ and $\mathrm{SU}(n)$ are good

We now outline our coupling-based approach to showing that $\mathrm{SO}(n)$, $\mathrm{Sp}(n)$ and $\mathrm{SU}(n)$ are good. (By Lemma 3.24, the goodness of $\mathrm{Spin}(n)$ will follow from that of $\mathrm{SO}(n)$.) We focus on the case of $\mathrm{SO}(n)$, and discuss the necessary adaptations for $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$ in the Appendix.

Our coupling approach for proving hypercontractivity

Our approach is based on constructing a coupling between matrices X sampled according to the Haar measure on $\mathrm{SO}(n)$ and matrices $Y \in \mathbb{R}^{n \times n}$ whose entries are independent standard Gaussians, with the intuition that the distributions of $\sqrt{n}X$ and Y are ‘locally’ close to one another.

We use this coupling to define a noise operator on $L^2(\mathrm{SO}(n), \mu)$: first we use the coupling to move a function to Gaussian space, then we apply the well-known Gaussian noise operator (the Ornstein–Uhlenbeck operator), and then we go back to $\mathrm{SO}(n)$ via the coupling. It turns out that a noise operator defined this way inherits the hypercontractive property from the Gaussian noise operator (this is easy to see), so we get hypercontractivity ‘for free’. The real work of the proof is to show that the eigenvalues of our noise operator on $L^2(\mathrm{SO}(n), \mu)$ that correspond to functions in $V_{=d}$ are not too small. We show these eigenvalues are at least $2^{-O(d)}$, provided $d \leq \delta \cdot n^{1/2}$, for a sufficiently small absolute constant $\delta > 0$.

8.1 The Gaussian noise operator, a.k.a. the Ornstein–Uhlenbeck operator

In this section, we recall the definition of the Gaussian noise operator and several of its properties. For simplicity of notation, we present this theory for functions in $L^2(\mathbb{R}^n, \gamma)$, however everything applies more generally to functions in $L^2(\mathbb{R}^{n \times m}, \gamma)$. (Here and elsewhere, we abuse notation slightly and denote by γ a Gaussian distribution, where the domain is clear from context.)

Definition 8.1. For $\rho \in [0, 1]$, we define $U_\rho: L^2(\mathbb{R}^n, \gamma) \rightarrow L^2(\mathbb{R}^n, \gamma)$ by

$$U_\rho f(X) = \mathbb{E}_{Y \sim \gamma}[f(\rho X + \sqrt{1 - \rho^2} Y)].$$

It is a well known fact that U_ρ is hypercontractive [33]:

Theorem 8.2. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ and let $0 \leq \rho \leq \frac{1}{\sqrt{q-1}}$. Then $\|U_\rho f\|_{L^q(\gamma)} \leq \|f\|_{L^2(\gamma)}$.

Below we use U_ρ to construct an operator T_ρ over $L^2(\sqrt{n}\text{SO}(n))$ which is hypercontractive, and on which we have lower bounds on the eigenvalues corresponding to low-degrees, thereby showing that the group $\text{SO}(n)$ is good.

8.2 Constructing the noise operator T_ρ

In this section, we design our noise operator T_ρ on $L^2(\text{SO}(n))$. En route, we define auxiliary operators that act on both $L^2(\mathbb{R}^{n \times n}, \gamma^{n \times n})$ and $L^2(\sqrt{n}\text{SO}(n), \mu)$.

Left and right multiplication by matrices from $\text{SO}(n)$

Definition 8.3. For a matrix $U \in \text{SO}(n)$, we define the operator L_U acting both on $L^2(\mathbb{R}^{n \times n}, \gamma^{n \times n})$ and $L^2(\sqrt{n}\text{SO}(n), \mu)$, as follows. For a function $f: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$, the function $L_U f: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is defined by

$$L_U f(X) = f(UX).$$

For a function $f: \sqrt{n}\text{SO}(n) \rightarrow \mathbb{R}$, we similarly define $L_U f(X) = f(UX)$.

We similarly define the operator R_V corresponding to right multiplication.

Definition 8.4. For a matrix $V \in \text{SO}(n)$, we define the operator R_V acting both on $L^2(\mathbb{R}^{n \times n}, \gamma)$ and $L^2(\sqrt{n}\text{SO}(n), \mu)$, as follows. For a function $f: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$, the function $R_V f: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is defined by

$$R_V f(X) = f(XV).$$

For a function $f: \sqrt{n}\text{SO}(n) \rightarrow \mathbb{R}$, we similarly define $R_V f(X) = f(XV)$.

The Gram–Schmidt operators.

Next, we define the operators $T_{\text{row}}, T_{\text{col}}$ that capture our coupling and map $L^2(\text{SO}(n), \mu)$ to $L^2(\mathbb{R}^{n \times n}, \gamma)$, as well as their adjoint operators that go in the reverse direction. To do so, we use the Gram-Schmidt process.

Fix a matrix $X \in \mathbb{R}^{n \times n}$ and let c_1, \dots, c_n be its columns. Provided $\det(X) \neq 0$ (which for a Gaussian matrix happens with probability one), we may apply the Gram-Schmidt process on (c_1, \dots, c_n) to get an orthonormal set of vectors $\tilde{c}_1, \dots, \tilde{c}_n$. Abusing notation slightly, we define the matrix $\text{GS}_{\text{col}}(X) \in \sqrt{n}\text{SO}(n)$ as the matrix whose i^{th} column is $\sqrt{n}\tilde{c}_i$ for all $i < n$ and whose n^{th} column is either $\sqrt{n}\tilde{c}_n$ (if $\det(X) > 0$) or $-\sqrt{n}\tilde{c}_n$ (if $\det(X) < 0$); this is of course a (column-) dilation of the (column-) Gram-Schmidt matrix corresponding to X . Since the Gram-Schmidt process preserves the sign of the determinant, this matrix $\text{GS}_{\text{col}}(X)$ is indeed in $\sqrt{n}\text{SO}(n)$.

Similarly, letting r_1, \dots, r_n be the rows of X , we let $\tilde{r}_1, \dots, \tilde{r}_n$ be the resulting set of vectors by applying the Gram-Schmidt process on (r_1, \dots, r_n) and define the matrix $\text{GS}_{\text{row}}(X)$ as the matrix whose i^{th} row is $\sqrt{n}\tilde{r}_i$ for all $i < n$, and whose n^{th} row is either $\sqrt{n}\tilde{r}_n$ (if $\det(X) > 0$) or $-\sqrt{n}\tilde{r}_n$ (if $\det(X) < 0$).

The dilated Gram–Schmidt processes above define couplings $(X, \text{GS}_{\text{col}}(X))$ and $(X, \text{GS}_{\text{row}}(X))$ between γ and μ , and we use these to define the operators T_{row} and T_{col} :

Definition 8.5. We define $T_{\text{row}}: L^2(\sqrt{n}\text{SO}(n), \mu) \rightarrow L^2(\mathbb{R}^{n \times n}, \gamma)$ as follows. For a function $f: \sqrt{n}\text{SO}(n) \rightarrow \mathbb{R}$, we define $T_{\text{row}}f: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ by

$$T_{\text{row}}f(X) = f(\text{GS}_{\text{row}}(X)).$$

Definition 8.6. We define $T_{\text{col}}: L^2(\sqrt{n}\text{SO}(n), \mu) \rightarrow L^2(\mathbb{R}^{n \times n}, \gamma)$ as follows. For a function $f: \sqrt{n}\text{SO}(n) \rightarrow \mathbb{R}$, we define $T_{\text{col}}f: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ by

$$T_{\text{col}}f(X) = f(\text{GS}_{\text{col}}(X)).$$

The operator T_ρ .

The operators T_{col} and T_{row} allow us to move from $L^2(\mu)$ to $L^2(\gamma)$. We can also go in the reverse direction, using their adjoints. It is easy to see, using Jensen’s inequality, that $T_{\text{col}}^*U_\rho T_{\text{col}}$ has the same hypercontractive properties as U_ρ . Thus, it is natural to consider the operator $T_{\text{col}}^*U_\rho T_{\text{col}}$ as an analogue of the Gaussian noise operator, for $\sqrt{n}\text{SO}(n)$. We do not know, however, how to bound from below the eigenvalues of $T_{\text{col}}^*U_\rho T_{\text{col}}$ so as to deduce Theorem 1.13. The reason is that to bound its eigenvalues, we (naturally) need some information about the eigenvectors corresponding to them, and we only know how to obtain such information from classical representation facts about $\text{SO}(n)$. To use these facts (so as to ensure the eigenspaces are ‘nice’, and easy to analyse), it is necessary that our operator commutes with the action of $\text{SO}(n)$ from both sides. For the operator $T_{\text{col}}^*U_\rho T_{\text{col}}$ above, one can show that it commutes with multiplication from the left, i.e. with the operators L_U , but unfortunately, it does not commute with multiplication from the right.

To overcome this, we obtain commutation with the action of $\mathrm{SO}(n)$ from the right with an averaging trick, which is an analogue of the famous Weyl unitary trick.

Definition 8.7. We set $\mathbb{T}_\rho = \mathbb{E}_{V \sim \mathrm{SO}(n)} [R_V^* \mathbb{T}_{\mathrm{col}}^* U_\rho \mathbb{T}_{\mathrm{col}} R_V]$.

The following result asserts that \mathbb{T}_ρ is hypercontractive, and it also gives lower bounds on its eigenvalues (which are required for deducing our level d inequalities).

Theorem 8.8. For each $\rho \in (0, 1)$, the operator \mathbb{T}_ρ is self adjoint on $L^2(\sqrt{n}\mathrm{SO}(n), \mu)$ and has the following properties:

1. \mathbb{T}_ρ commutes with both left and right multiplication by matrices from $\mathrm{SO}(n)$.
2. If $\rho \leq \frac{1}{\sqrt{q-1}}$, and $f \in L^2(\sqrt{n}\mathrm{SO}(n), \mu)$, then $\|\mathbb{T}_\rho f\|_{L^q(\mu)} \leq \|f\|_{L^2(\mu)}$.
3. There exist absolute constants $\delta > 0, C > 1$, such that if $d \leq \delta n^{1/2}$ and $f \in V_d$, then

$$\|\mathbb{T}_\rho f\|_{L^2(\mu)} \geq C^{-d} \rho^d \|f\|_{L^2(\mu)}.$$

Let us show how Theorem 8.8 immediately implies that $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$ are good.

Theorem 8.9. There exist absolute constants $c, C > 0$ such that the n -graded groups $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$ are (C, c) -good.

Proof, given Theorem 8.8. The strong quasirandomness of $\mathrm{Spin}(n)$ and $\mathrm{SO}(n)$ was already established in Section 6. It remains to show that they are $(cn^{1/2}, C)$ -hypercontractive for absolute constants $C, c > 0$. Let C be the constant in the statement of Theorem 8.8 and let $c = \delta$ where δ is the constant in the statement of Theorem 8.8. By Lemma 3.24, it suffices to show that $\mathrm{SO}(n)$ is $(cn^{1/2}, C)$ -hypercontractive. Let $T = T_{1/(C\sqrt{q})}$, $cn^{1/2}$ be the Beckner operator. By the monotonicity (in δ) of $\|T_{\delta,r}\|_{2 \rightarrow q}$, it suffices to show that $\|T\|_{2 \rightarrow q} \leq 1$. Let us show (using Theorem 8.8) that we may write $T = T_{1/\sqrt{q-1}} \circ S$, where $\|S\|_{2 \rightarrow 2} \leq 1$. Indeed, as $T_{1/\sqrt{q-1}}$ is self-adjoint and commutes with the action of $\mathrm{SO}(n)$ from both sides, it has the Peter-Weyl ideals W_ρ as its eigenspaces. By Part 3 of Theorem 8.8, the eigenvalues of $T_{1/\sqrt{q-1}}$ corresponding to a representations ρ of level d are at least $(C\sqrt{q})^{-d}$, and thus are at least the corresponding eigenvalue of T on W_ρ . This shows that the desired operator S exists. We may now apply Part 2 of Theorem 8.8 to obtain:

$$\|T\|_{2 \rightarrow q} \leq \left\| T_{\frac{1}{\sqrt{q-1}}} \right\|_{2 \rightarrow q} \|S\|_{2 \rightarrow 2} \leq 1.$$

□

The noise operator \mathbb{T}_ρ commutes with the action of $\mathrm{SO}(n)$ from both sides

In this section, we establish Part 1 of Theorem 8.8. We phrase it as a lemma.

Lemma 8.10. The operator \mathbb{T}_ρ commutes with the action of $\mathrm{SO}(n)$ from both sides.

Proof. We show the commuting from the left and the right separately.

Commuting from the left. For this, it suffices to show that for each $U, V \in \text{SO}(n)$, the operators L_U and $R_V^* \text{T}_{\text{col}}^* \text{T}_\rho \text{T}_{\text{col}} R_V$ commute. It is easy to see that L_U and R_V commute. Hence, it suffices to show that L_U and $\text{T}_{\text{col}}^* \text{T}_\rho \text{T}_{\text{col}}$ commute.

First note that if $X \in \mathbb{R}^{n \times n}$ is a matrix, it holds that $\text{GS}_{\text{col}}(UX) = U \text{GS}_{\text{col}}(X)$. It follows that

$$L_U \text{T}_{\text{col}}^* f(X) = \text{T}_{\text{col}}^* f(UX) = \mathbb{E}_{Y: \text{GS}_{\text{col}}(Y)=UX} [f(Y)] = \mathbb{E}_{Z: \text{GS}_{\text{col}}(Z)=X} [f(UZ)] = \text{T}_{\text{col}}^* L_U f(X),$$

so L_U commutes with T_{col}^* . The adjointness immediately implies that L_U also commutes with the operator T_{col} . It is easy to see directly that L_U commutes with the operator U_ρ , and so it commutes with the composition $\text{T}_{\text{col}}^* \text{T}_\rho \text{T}_{\text{col}}$.

Commuting from the right. Fix $V' \in \text{SO}(n)$, then

$$R_{V'} \text{T}_\rho = \mathbb{E}_{V \sim \text{SO}(n)} [R_{V'} R_V^* \text{T}_{\text{col}}^* \text{T}_\rho \text{T}_{\text{col}} R_V] = \mathbb{E}_{V \sim \text{SO}(n)} [R_{V' V'^t}^* \text{T}_{\text{col}}^* \text{T}_\rho \text{T}_{\text{col}} R_V]$$

Making the change of variables $V \leftarrow V V'^t$, we obtain

$$R_{V'} \text{T}_\rho = \mathbb{E}_{V \sim \text{SO}(n)} [R_V^* \text{T}_{\text{col}}^* \text{T}_\rho \text{T}_{\text{col}} R_{V V'}] = \mathbb{E}_{V \sim \text{SO}(n)} [R_V^* \text{T}_{\text{col}}^* \text{T}_\rho \text{T}_{\text{col}} R_V] R_{V'} = \text{T}_\rho R_{V'}. \quad \square$$

8.3 The noise operator T_ρ is hypercontractive

In this section, we prove part 2 of Theorem 8.8. To do so, we first adopt a different point of view of the couplings defined by T_{row} and T_{col} that will often be easier for us to work with.

The ‘Gaussian maker distribution’

Rather than going from $Y \sim \gamma$ to $X \sim \mu$ by applying the Gram–Schmidt process on its columns and dilating by \sqrt{n} (and flipping the sign of the last column if necessary), we can go the other way and construct Y from X . This is accomplished as follows. We define a pair of independent random variables (X, G) such that XG is distributed according to γ and $X \sim \mu$. We call the distribution of G the *Gaussian maker distribution* and we abbreviate it to GMD.

Definition 8.11. *We define the Gaussian maker distribution to be the distribution of the upper-triangular matrix $G = (g_{ij})$ constructed as follows. First, independently choose one-dimensional Gaussians $g_{ij} \sim N(0, \frac{1}{n})$ of expectation zero and variance $1/n$, for each $i < j$. For each $i < n$, we independently choose g_{ii} to be $1/\sqrt{n}$ times the (Euclidean) length of an $(n - i + 1)$ -dimensional Gaussian $z \sim (\mathbb{R}^{n-i+1}, \gamma)$. We also independently choose g_{nn} to be a standard Gaussian random variable, $z \sim N(0, 1)$. Finally, we set $g_{ij} = 0$ for all $j < i$.*

It is clear that when we sample a matrix $Y \sim \gamma$ and apply the (dilated) Gram-Schmidt process in Section 8.2, we get a matrix X which is \sqrt{n} times a matrix sampled from the Haar

measure on $\text{SO}(n)$ (provided we condition on the probability-one event that $\det(Y) \neq 0$, of course). We would like to show that $X^{-1}Y \sim \text{GMD}$, independently of X . Indeed, this follows by choosing the columns of X and Y one after another. The first column of Y is uniformly distributed according to (\mathbb{R}^n, γ) . By rotational symmetry, its length and its normalization are independent, and therefore $XGe_1 = g_{11}Xe_1$ is indeed distributed as Ye_1 . Note that Xe_2 is independent of g_{11} , as it is a uniformly random unit vector orthogonal to X_1 . Thus, completing X_1 to a basis arbitrarily we obtain, by rotational invariance of the Gaussian distribution, that the correlation of Ye_2 with Xe_1 is normally distributed. After we present Ye_2 with respect to an extension of $\frac{1}{\sqrt{n}}Xe_1$ to an orthonormal basis, we see that the last $n - 1$ coordinates are distributed as a random $n - 1$ dimensional Gaussian. This shows that indeed Ye_2 is distributed as $g_{22}Xe_2 + g_{12}Xe_1$. Continuing in this fashion column by column (being a little careful with the last column), we see that indeed $X^{-1}Y$ is distributed according to GMD, independently of X . We thus have the following formulae for the adjoint operators $\text{T}_{\text{row}}^*, \text{T}_{\text{col}}^*$:

Lemma 8.12. *For each $f \in L^2(\mathbb{R}^{n \times n}, \gamma)$, we have*

$$\begin{aligned} \text{T}_{\text{col}}^* f(X) &= \mathbb{E}_{G \sim \text{GMD}} [f(XG)] & \forall X \in \text{SO}(n), \\ \text{T}_{\text{row}}^* f(X) &= \mathbb{E}_{G \sim \text{GMD}} [f(G^t X)] & \forall X \in \text{SO}(n). \end{aligned}$$

In addition to Lemma 8.12 being useful on its own, it allows us to extend the domain of $\text{T}_{\text{col}}^* f$ to $\mathbb{R}^{n \times n}$. Indeed, abusing notation, we shall often view $\text{T}_{\text{col}}^* f$ as a function from $\mathbb{R}^{n \times n}$ to \mathbb{R} , by using the above formula:

$$\text{T}_{\text{col}}^* f(X) = \mathbb{E}_{G \sim \text{GMD}} [f(XG)].$$

This allows us to view T_{col}^* as an operator on Gaussian space $L^2(\mathbb{R}^{n \times n}, \gamma)$, and we note that if $f: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ has degree at most d , then $\text{T}_{\text{col}}^* f$ also has degree at most d , and thus V_d is an invariant subspace of T_{col}^* .

Another consequence of Lemma 8.12 is that the operators $\text{T}_{\text{col}}^*, \text{T}_{\text{row}}^*$ do not increase q -norms:

Fact 8.13. *The following hold for all $q \geq 1$:*

1. *The operators T_{col} and T_{row} preserve q -norms.*
2. *The operators T_{col}^* and T_{row}^* (from $L^q(\mu)$ to $L^q(\gamma)$) cannot increase q -norms.*

Proof. We prove both items for T_{col} , as the proof for T_{row} is exactly analogous.

For the first item, we note that

$$\|\text{T}_{\text{col}} f\|_{L^q(\gamma)}^q = \mathbb{E}_{Y \sim \gamma} [|\text{T}_{\text{col}} f(Y)|^q] = \mathbb{E}_{Y \sim \gamma} [|f(GS_{\text{col}}(Y))|^q] = \mathbb{E}_{X \sim \mu} [|f(X)|^q] = \|f\|_{L^q(\mu)}^q.$$

For the second item, we use Jensen's inequality:

$$\|\text{T}_{\text{col}}^* f\|_{L^q(\mu)}^q = \mathbb{E}_{X \sim \mu} \left[\left| \mathbb{E}_{G \sim \text{GMD}} [f(XG)] \right|^q \right] \leq \mathbb{E}_{X, G} [|f(XG)|^q] = \mathbb{E}_{Y \sim \gamma} [|\text{T}_{\text{col}} f(Y)|^q] = \|f\|_{L^q(\gamma)}^q. \quad \square$$

Deducing hypercontractivity.

We now show that the operator T_ρ is hypercontractive, proving Part 3 of Theorem 8.8.

Lemma 8.14. *For all $0 \leq \rho \leq \frac{1}{\sqrt{q-1}}$ and $f: \text{SO}(n) \rightarrow \mathbb{R}$ we have $\|T_\rho f\|_{L^q(\mu)} \leq \|f\|_{L^2(\mu)}$.*

Proof. By the triangle inequality and the fact that R_V preserves the L^r norm for all r , it suffices to show that

$$\|T_{\text{col}}^* U_\rho T_{\text{col}} f\|_{L^q(\mu)} \leq \|f\|_{L^2(\mu)}.$$

To see that this holds, we apply Fact 8.13 and Theorem 8.2:

$$\|T_{\text{col}}^* U_\rho T_{\text{col}} f\|_{L^q(\gamma)} \leq \|U_\rho T_{\text{col}} f\|_{L^q(\gamma)} \leq \|T_{\text{col}} f\|_{L^2(\gamma)} = \|f\|_{L^2(\gamma)}. \quad \square$$

9 Comfortable polynomials on $\text{SO}(n)$, and their properties.

Recall that in Section 6 we defined the *comfortable d -juntas* on $\text{SO}(n)$ to be the multilinear polynomials of the form $X \mapsto \sum_{\sigma \in S_d} a_\sigma \prod_{i=1}^d x_{i,\sigma(i)}$, for $a_\sigma \in \mathbb{R}$. We also showed that for any irreducible representation ρ of level d , where $0 \leq d < n/2$, the Peter-Weyl ideal W_ρ contains a comfortable d -junta. In this section we define comfortable polynomials in general (the comfortable d -juntas are a special case). We then show that some of them are eigenfunctions of T_{col}^* (or of T_{row}^*).

We use comfortable d -juntas as they are both easy to work with, and each low degree eigenspace of T_ρ contains one; the latter is guaranteed by the following, since T_ρ commutes with the action of $\text{SO}(n)$ from both sides.

Claim 9.1. *Let T be a linear operator on $L^2(\text{SO}(n))$ that commutes with the action of $\text{SO}(n)$ from both sides, and let $0 \leq d < n/2$. Then the space $V_{=d}$ is T -invariant, and each eigenspace of T inside $V_{=d}$ contains a comfortable d -junta.*

Proof. A linear map from $L^2(\text{SO}(n))$ to itself that commutes with the action of $\text{SO}(n)$ from both sides is precisely an $\text{SO}(n) \times \text{SO}(n)$ -homomorphism. As each W_ρ is an irreducible $\text{SO}(n) \times \text{SO}(n)$ -module and the W_ρ are pairwise non-isomorphic, Schur's lemma implies that any $\text{SO}(n) \times \text{SO}(n)$ -homomorphism from $L^2(\text{SO}(n))$ to itself acts as a scalar multiple of the identity when restricted to W_ρ . Since the linear operator T commutes with the action of $\text{SO}(n)$ from both sides, each W_ρ is contained in an eigenspace of T . Since the Peter-Weyl ideals span $L^2(\text{SO}(n))$, each eigenspace of T is a direct sum of some of the W_ρ 's.

As $V_{=d}$ is a direct sum of finitely many Peter-Weyl ideals (each of which is T -invariant), $V_{=d}$ itself is T -invariant. The claim now follows from Fact 6.5, which implies that each of the Peter-Weyl constituents of $V_{=d}$ contains a comfortable d -junta. \square

9.1 Comfortable polynomials

Claim 9.1 is important for us as it says that if we want to understand the eigenvalues of an operator T that commutes with the action of $\text{SO}(n)$, it suffices to understand its

action on low-degree polynomials. One can already carry out some non-trivial analysis of our hypercontractive operator using this observation, however to push our analysis all the way to $d = \Theta(n^{1/2})$ we need to work with a restricted class of polynomials, which we call ‘comfortable polynomials’.

Definition 9.2 (Comfortable polynomial). *We say a multilinear monomial in the matrix entries of $X \in \text{SO}(n)$ is comfortable if it only contains variables from the top left $\lfloor \frac{n}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor$ minor of X , and it contains at most one variable from each row and at most one variable from each column. A polynomial in the matrix entries is said to be comfortable if it is a linear combination of (multilinear) comfortable monomials.*

We may naturally index monomials in $L^2(\mathbb{R}^{n \times n}, \gamma)$ by multisets of elements of $[n] \times [n]$. Given such a multiset $S = \{(i_1, j_1), \dots, (i_r, j_r)\}$, where each pair (i_k, j_k) may appear multiple times, the corresponding monomial is $H_S(X) := \prod_{r=1}^k x_{i_r, j_r}$.

We now define the notion of comfortable polynomial.

Definition 9.3. *Let $S = \{(i_1, j_1), \dots, (i_d, j_d)\}$ be a multiset of elements of $[n] \times [n]$; we define its transpose by $S^t := \{(j_1, i_1), \dots, (j_d, i_d)\}$.*

1. *We say H_S is row comfortable if i_1, \dots, i_d are distinct and all $\leq n/2$.*
2. *We say H_S is column comfortable if H_{S^t} is row comfortable, i.e. if j_1, \dots, j_d are distinct and all $\leq n/2$.*
3. *Finally, we say H_S is comfortable if it is both row comfortable and column comfortable.*

Definition 9.4. *We say a polynomial is row comfortable if it lies in the span of the row comfortable monomials; similarly, we say it is column comfortable if it lies in the span of the column comfortable monomials, and that it is comfortable if it is both row comfortable and column comfortable.*

We also need to define row- and column- comfortable d -juntas, generalizing the notion of comfortable d -juntas.

Definition 9.5. *A row- (respectively column-) comfortable d -junta is a row (respectively column) comfortable polynomial whose monomials contain variables only from the top left $d \times d$ minor.*

Finally, we need to define row comfortable d -row-juntas and column comfortable d -column-juntas, which are slight weakenings of the notion of a junta that will be useful.

Definition 9.6. *A row comfortable d -row-junta is a row comfortable polynomial whose monomials contain variables only from the first d rows. A column comfortable d -column-junta is a column comfortable polynomial whose monomials contain variables only from the first d columns.*

Comfortable monomials are eigenvectors of $\mathbb{T}_{\text{col}}^*/\mathbb{T}_{\text{row}}^*$

The following claim shows that the column/row comfortable monomials are eigenvectors of $\mathbb{T}_{\text{col}}^*/\mathbb{T}_{\text{row}}^*$.

Claim 9.7. *There exists an absolute constant $C \geq 1$ such that the following holds. For each multiset $S = \{(i_1, j_1), \dots, (i_d, j_d)\}$ of elements of $[n] \times [n]$, there exists $\lambda_S > 0$ such that:*

1. *If H_S is column comfortable (i.e. the j_k are distinct), then $\mathbb{T}_{\text{col}}^* H_S = \lambda_S H_S$.*
2. *If H_S is row comfortable, then $\mathbb{T}_{\text{row}}^* H_{S^t} = \lambda_S H_{S^t}$.*
3. *$C^{-d} \leq \lambda_S \leq 1$ for all S .*
4. *If S is supported on $[n] \times [d]$, then $|\lambda_S - 1| = O(d^2/n)$.*

Proof. It suffices to prove the first, third and fourth items, as the second follows from the first by taking transposes. By Lemma 8.12 we have $(\mathbb{T}_{\text{col}}^* H_S)(X) = \mathbb{E}_{G \sim \text{GMD}} [H_S(XG)]$. Using the fact that the entries of XG corresponding to different columns are independent and that each j_k appears at most once, we obtain

$$\begin{aligned} (\mathbb{T}_{\text{col}}^* H_S)(X) &= \mathbb{E}_{G \sim \text{GMD}} [H_S(XG)] \\ &= \mathbb{E}_{G \sim \text{GMD}} \left[\prod_{k=1}^d (XG)_{i_k, j_k} \right] \\ &= \prod_{k=1}^d \mathbb{E}_{G \sim \text{GMD}} [(XG)_{i_k, j_k}] \\ &= \prod_{k=1}^d \left(X \mathbb{E}_{G \sim \text{GMD}} [G] \right)_{i_k, j_k}. \end{aligned}$$

Observe that $\mathbb{E}_{G \sim \text{GMD}} [G]$ is a diagonal matrix with $(\mathbb{E}[G])_{j,j}$ being equal to $1/\sqrt{n}$ times the expectation of the length (= Euclidean norm) of an $(n-j+1)$ -dimensional standard Gaussian random vector, for each $j < n$. For $m \in \mathbb{N}$, let $N(0, I_m)$ denote an m -dimensional standard Gaussian random vector, and let $\|N(0, I_m)\|_{\ell^2}$ denote its Euclidean norm. We have

$$(\mathbb{T}_{\text{col}}^* H_S)(X) = \prod_{k=1}^d X_{i_k, j_k} \left(\mathbb{E}_{G \sim \text{GMD}} [G] \right)_{j_k, j_k} = H_S(X) \prod_{k=1}^d \left(\mathbb{E}_{G \sim \text{GMD}} [G] \right)_{j_k, j_k} = \lambda_S H_S(X),$$

where

$$\lambda_S = n^{-d/2} \prod_{k=1}^d \mathbb{E}[\|N(0, I_{n-j_k+1})\|_{\ell^2}].$$

To estimate the eigenvalues λ_S we need the following fact.

Fact 9.8. For any $m \in \mathbb{N}$, we have $\sqrt{m} - \frac{1}{2\sqrt{m}} \leq \mathbb{E} [\|N(0, I_m)\|_{\ell^2}] \leq \sqrt{m}$.

Proof. Let $Z_1, \dots, Z_m \sim N(0, 1)$ be independent and $Z = \sum_{i=1}^m Z_i^2$ so that $\|N(0, I_m)\|_{\ell^2} = \sqrt{Z}$. Then $\mathbb{E}[Z] = m$, so $\mathbb{E}[\sqrt{Z}] \leq \sqrt{\mathbb{E}[Z]} = \sqrt{m}$ by Cauchy-Schwarz, proving the upper bound. Secondly $\text{var}(Z) = \sum_{i=1}^m \text{var}(Z_i^2) = m$, which implies that

$$\mathbb{E} \left[(\sqrt{Z} - \sqrt{m})^2 \right] = \mathbb{E} \left[\frac{(Z - m)^2}{(\sqrt{Z} + \sqrt{m})^2} \right] \leq \frac{1}{m} \text{var}(Z) = 1.$$

Thus,

$$2m - 2\sqrt{m}\mathbb{E}[\sqrt{Z}] = \mathbb{E}[Z] + m - 2\sqrt{m}\mathbb{E}[\sqrt{Z}] \leq 1,$$

implying that

$$\mathbb{E}[\sqrt{Z}] \geq \sqrt{m} - \frac{1}{2\sqrt{m}},$$

proving the lower bound. \square

Continuing the proof of the claim, the above fact yields $C^{-d} \leq \lambda_S \leq 1$ for all S . If S is supported upon $[n] \times [d]$, then it yields

$$\lambda_S \geq n^{-d/2} \left(\sqrt{n-d+1} - \frac{1}{2\sqrt{n-d+1}} \right)^d \geq 1 - O(d^2/n),$$

completing the proof of Claim 9.7. \square

Projections onto comfortable subspaces

Define the operator $\Pi_{\text{comf}}: L^2(\mathbb{R}^{n \times n}, \gamma) \rightarrow L^2(\mathbb{R}^{n \times n}, \gamma)$ to be orthogonal projection onto the linear subspace of comfortable polynomials. This projection has a neat Fourier expression:

$$(\Pi_{\text{comf}} f)(X) = \sum_{H_\alpha \text{ a comfortable monomial}} \hat{f}(\alpha) H_\alpha(X),$$

where $\hat{f}(\alpha) := \langle f, H_\alpha \rangle_{L^2(\gamma)}$ for each Hermite polynomial H_α . We also define the operators $\Pi_{\text{comf},d}$, $\Pi_{\text{comf,col},d}$ and $\Pi_{\text{comf,row},d}$ to be the orthogonal projections onto the space of comfortable, row comfortable and column comfortable polynomials of degree at most d , respectively. Finally, if $S \subset [n]$ is a set of rows, we define $\Pi_{\text{comf},=S}$ to be the projection onto the subspace spanned by the comfortable homogeneous monomials of degree $|S|$ which depend only upon variables from the rows in S .

9.2 Reducing Theorem 8.8 to a statement about low-degree truncations of T_{col} .

In this section, we prove Theorem 8.8 modulo the following lemma, asserting that on the space of comfortable d -juntas, the operator $T_{\text{col}}R_V$ (for a typical V) preserves some of the mass of a function even after projection onto V_d .

Lemma 9.9. *There exist absolute constants $C > 0$ and $\delta > 0$ such that the following holds for all $d \leq \delta n^{1/2}$. For all comfortable d -juntas $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$, we have*

$$\mathbb{E}_{V \sim \text{SO}(n)} \left[\left\| (T_{\text{col}}R_V f)^{\leq d} \right\|_{L^2(\gamma)}^2 \right] \geq C^{-d} \|f\|_{L^2(\mu)}^2.$$

We now show how to complete the proof of Theorem 8.8, assuming Lemma 9.9.

Proof of Theorem 8.8 (assuming Lemma 9.9). Lemmas 8.10 and 8.14 give the first two items, so it remains to prove the third item. Namely, letting $f \in V_d$ we want to show that $\|T_\rho f\|_{L^2(\mu)} \geq (c\rho)^d \|f\|_{L^2(\mu)}$. By Claim 9.1 we may assume that f is a comfortable d -junta. By Cauchy–Schwarz, it is enough to show that $\langle T_\rho f, f \rangle_{L^2(\mu)} \geq (c\rho)^d \|f\|_{L^2(\mu)}^2$, and we next show that the last assertion follows by Lemma 9.9.

Using the fact that $U_\rho = U_{\sqrt{\rho}}^* U_{\sqrt{\rho}}$, we have

$$\begin{aligned} \langle T_\rho f, f \rangle_{L^2(\mu)} &= \mathbb{E}_{V \sim \text{SO}(n)} \left[\|U_{\sqrt{\rho}} T_{\text{col}} R_V f\|_{L^2(\gamma)}^2 \right] \geq \mathbb{E}_{V \sim \text{SO}(n)} \left[\|(U_{\sqrt{\rho}} T_{\text{col}} R_V f)^{\leq d}\|_{L^2(\gamma)}^2 \right] \\ &\geq \rho^d \mathbb{E}_{V \sim \text{SO}(n)} \left[\|(T_{\text{col}} R_V f)^{\leq d}\|_{L^2(\gamma)}^2 \right] \\ &\geq \rho^d C^{-d} \|f\|_{L^2(\mu)}^2, \end{aligned}$$

where we used Lemma 9.9 and the fact that for each function g of degree $\leq d$ it holds that

$$\|U_{\sqrt{\rho}} g\|_{L^2(\gamma)} \geq \rho^{d/2} \|g\|_{L^2(\gamma)}.$$

□

10 Low-degree truncations of T_{col} : Proof of Lemma 9.9.

In this section, we prove Lemma 9.9. Our main idea is to show that one can approximate the L^2 -norms of (row) comfortable d -juntas with respect to $(\sqrt{n}\text{SO}(n), \mu)$ by their L^2 -norms with respect to $(\mathbb{R}^{n \times n}, \gamma)$.

10.1 Comparing $L^2(\mu)$ and $L^2(\gamma)$.

The following lemmas assert that the 2-norm in $L^2(\sqrt{n}\text{SO}(n))$ of a row comfortable d -junta is roughly bounded by its 2-norm in Gaussian space. We defer the proofs of the lemmas to Sections 10.3 and 10.4.

Lemma 10.1. *For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $d \leq \delta n^{1/2}$, then for any function $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ that is either (i) a column comfortable, d -column-junta or (ii) a row comfortable, d -row-junta, we have*

$$\|f\|_{L^2(\mu)} \leq (1 + \varepsilon) \|f\|_{L^2(\gamma)}.$$

In the case where the function f is comfortable, we show that in fact the same inequality with the measures swapped.

Lemma 10.2. *For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $d \leq \delta n^{1/2}$ and $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is a comfortable d -junta, then*

$$\|f\|_{L^2(\gamma)} \leq (1 + \varepsilon) \|f\|_{L^2(\mu)}.$$

10.2 The main argument for proving Lemma 9.9.

We now give the proof of Lemma 9.9, assuming Lemmas 10.1 and 10.2.

Swapping T_{row} and T_{col} .

Our first step is to show that on the left-hand side of Lemma 9.9, we can replace T_{col} by T_{row} . The benefit of this exchange is that T_{row} and R_V commute.

Lemma 10.3. *There exist absolute constants $C > 0, \delta > 0$ such that the following holds. Let $d < \delta n$ and let $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ be a comfortable d -junta. Then*

$$\mathbb{E}_{V \sim \text{SO}(n)} \left[\|(T_{\text{col}} R_V f)^{\leq d}\|_{L^2(\gamma)}^2 \right] \geq C^{-d} \mathbb{E}_{V \sim \text{SO}(n)} \left[\|\Pi_{\text{comf}, d} T_{\text{row}} R_V f\|_{L^2(\gamma)}^2 \right].$$

Proof. Applying Claim 9.7, we have

$$\begin{aligned} \|(T_{\text{col}} R_V f)^{\leq d}\|_{L^2(\gamma)}^2 &\geq \sum_{H_S \text{ comfortable of degree } d} \langle T_{\text{col}} R_V f, H_S \rangle_{L^2(\gamma)}^2 \\ &= \sum_{H_S \text{ comfortable of degree } d} \langle R_V f, T_{\text{col}}^* H_S \rangle_{L^2(\mu)}^2 \\ &= \lambda_S^2 \sum_{H_S \text{ comfortable of degree } d} \langle R_V f, H_S \rangle_{L^2(\mu)}^2 \\ &= \frac{\lambda_S^2}{\lambda_{S^t}^2} \sum_{H_S \text{ comfortable of degree } d} \langle R_V f, T_{\text{row}}^* H_S \rangle_{L^2(\mu)}^2 \\ &\geq C^{-d} \sum_{H_S \text{ comfortable of degree } d} \langle T_{\text{row}} R_V f, H_S \rangle_{L^2(\gamma)}^2. \end{aligned}$$

The lemma follows by plugging in the definition of $\Pi_{\text{comf}, d}$. □

$T_{\text{row}}f$ is close to f .

We have now reduced our task to understanding the average of the square of the 2-norm of $\Pi_{\text{comf},d}T_{\text{row}}R_Vf = \Pi_{\text{comf},d}R_VT_{\text{row}}f$ (this equality holds because T_{row} and R_V commute). The following claim further simplifies our task and shows that $T_{\text{row}}f$ is close to f , thereby effectively reducing our task to estimating the 2-norm of $\Pi_{\text{comf},d}R_Vf$. (Though some care is required to make this precise, as we are applying a projection operator on top, which may decrease norms considerably.)

Claim 10.4. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. Let $d \leq \delta n^{1/2}$, and let $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ be a comfortable d -junta. Then*

$$\|T_{\text{row}}f - f\|_{L^2(\gamma)}^2 \leq \varepsilon \|f\|_{L^2(\gamma)}^2.$$

Proof. Let g be a comfortable d -junta satisfying $T_{\text{row}}^*g = f$, i.e. writing $f = \sum_S \alpha_S H_S$, we take $g = \sum \lambda_S^{-1} \alpha_S H_S$, where λ_S is as in Claim 9.7. Then by Parseval, we have

$$\|g - f\|_{L^2(\gamma)} \leq \varepsilon \|f\|_{L^2(\gamma)},$$

provided δ is sufficiently small. Hence, using Cauchy–Schwarz, we have

$$\begin{aligned} \|f\|_{L^2(\mu)}^2 &= \langle f, T_{\text{row}}^*g \rangle = \langle T_{\text{row}}f, g \rangle = \langle T_{\text{row}}f, f \rangle + \langle T_{\text{row}}f, g - f \rangle \\ &\leq \langle T_{\text{row}}f, f \rangle + \|T_{\text{row}}f\|_{L^2(\gamma)} \|g - f\|_{L^2(\gamma)}, \end{aligned}$$

implying that

$$\langle T_{\text{row}}f, f \rangle \geq \|f\|_{L^2(\mu)}^2 - \varepsilon \|T_{\text{row}}f\|_{L^2(\gamma)} \|f\|_{L^2(\gamma)} = \|f\|_{L^2(\mu)}^2 - \varepsilon \|f\|_{L^2(\mu)} \|f\|_{L^2(\gamma)}.$$

Thus, we obtain

$$\begin{aligned} \|T_{\text{row}}f - f\|_{L^2(\gamma)}^2 &= \|f\|_{L^2(\gamma)}^2 + \|T_{\text{row}}f\|_{L^2(\gamma)}^2 - 2\langle T_{\text{row}}f, f \rangle \\ &\leq \|f\|_{L^2(\gamma)}^2 - \|f\|_{L^2(\mu)}^2 + 2\varepsilon \|f\|_{L^2(\mu)} \|f\|_{L^2(\gamma)} \\ &\leq 3\varepsilon \|f\|_{L^2(\gamma)}^2, \end{aligned}$$

using Lemmas 10.1 and 10.2 (again provided δ is sufficiently small depending on ε). This completes the proof. \square

The projection of $R_V T_{\text{row}}f$ onto the subspace of comfortable polynomials

At this point, it would seem that to finish the proof of Lemma 9.9 it suffices to estimate the typical 2-norm of $\Pi_{\text{comf},d}R_Vf$. While this is indeed the case, some care is needed, as one cannot switch particularly smoothly from $T_{\text{row}}f$ to f in the previous statement. To address this, we must be able to estimate the 2-norm of $\Pi_{\text{comf},d}R_Vf$ under the weaker hypothesis that f is a *row-comfortable* d -junta. This is the content of the following claim.

Claim 10.5. *Let $d \leq n/2$ and let $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ be a row-comfortable d -row-junta. Then*

$$\mathbb{E}_{V \sim \text{SO}(n)} \|\Pi_{\text{comf},d} R_V f\|_{L^2(\gamma)}^2 \geq \frac{(n/2)!}{2^d n^d ((n/2) - d)!} \|f\|_{L^2(\mu)}^2.$$

Proof. We first note that if H_α and H_β are row-comfortable monomials such that the set of rows appearing in α is different from the set of rows appearing in β , then we have $\langle H_\alpha, H_\beta \rangle_{L^2(\gamma)} = \langle H_\alpha, H_\beta \rangle_{L^2(\mu)} = 0$. Indeed, the first equality is immediate from the pairwise orthogonality of the Hermite polynomials with respect to $L^2(\gamma)$, and the second equality follows from the fact that, if H_α contains a variable from the i th row and H_β does not, then flipping the sign of both the i th row and the $(\lfloor n/2 \rfloor + 1)$ th row of a matrix $X \in \text{SO}(n)$ yields a (Haar-)measure-preserving map on $\text{SO}(n)$ that sends $H_\alpha(X)H_\beta(X)$ to its negative, and therefore $\langle H_\alpha, H_\beta \rangle_{L^2(\mu)} = \mathbb{E}_{X \sim \mu}[H_\alpha(X)H_\beta(X)] = 0$.

For $S \subset [n]$, let W_S denote the linear span of the row-comfortable homogeneous monomials of degree $|S|$ which depend only upon variables from the rows in S (and therefore depend upon precisely one variable from each of the rows in S). It is clear that the W_S are pairwise orthogonal with respect to $L^2(\gamma)$, and by the above observation they are also pairwise orthogonal with respect to $L^2(\mu)$.

By the above remarks, for any row-comfortable d -row junta $g : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$, we may write g as an orthogonal direct sum,

$$g = \sum_{S \subset [d]} g_{(=S)},$$

where $g_{(=S)}$ denotes the orthogonal projection of g onto W_S (with respect to $L^2(\gamma)$). It is clear that $(R_V g)_{(=S)} = R_V(g_{(=S)})$ for any $V \in \text{SO}(n)$ and any row-comfortable d -row-junta g .

Now let $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ be a row-comfortable d -row-junta. Since the $f_{(=S)}$ are pairwise orthogonal with respect to $L^2(\mu)$ as well as with respect to $L^2(\gamma)$, we have

$$\|f\|_{L^2(\mu)}^2 = \sum_{S \subset [d]} \|f_{(=S)}\|_{L^2(\mu)}^2,$$

and therefore by averaging, there exists $S \subset [d]$ such that

$$\|f_{(=S)}\|_{L^2(\mu)}^2 \geq \frac{1}{2^d} \|f\|_{L^2(\mu)}^2. \quad (11)$$

For brevity, write $h = f_{(=S)}$, $d' := |S|$ and $S = \{i_1, \dots, i_{d'}\}$. Our next aim is to show that

$$\mathbb{E}_{V \sim \text{SO}(n)} \|\Pi_{\text{comf},=S} R_V h\|_{L^2(\gamma)}^2 \geq \frac{(n/2)!}{n^{d'} ((n/2) - d')!} \|h\|_{L^2(\mu)}^2. \quad (12)$$

In proving (15), we may and shall assume (without loss of generality) that $S = \{1, 2, \dots, d'\}$. We first assert that, in this case,

$$\mathbb{E}_{V \sim \text{SO}(n)} \|\Pi_{\text{comf},=S} R_V h\|_{L^2(\gamma)}^2 = \frac{(n/2)!}{((n/2) - d')!} \mathbb{E}_V \langle R_V h, H_{\{(1,1), \dots, (d', d')\}} \rangle_{L^2(\gamma)}^2.$$

Indeed, this follows from the fact that we may write V as the product of a random $\text{SO}(n)$ matrix V' and a random permutation matrix P_σ , $V = P_\sigma V'$ say. Now the S_n -orbit $\{R_{P_\sigma} m : \sigma \in S_n\}$ of a monomial $m = m(X)$ of the form $\prod_{i=1}^{d'} X_{i,v_i}$ with the v_i all distinct, consists of all the monomials of the form $\prod_{i=1}^{d'} X_{i,w_i}$, with the w_i all distinct.

Now write $h = \sum_{\alpha=((1,i_1),\dots,(d',i_{d'}))} \hat{h}(\alpha) H_\alpha$. Then for each α of the form $\{(1, j_1), \dots, (d', j_{d'})\}$ (for $j_1, \dots, j_{d'}$ all distinct), we have

$$R_V H_\alpha(X) = H_\alpha(XV) = \prod_{k=1}^d [XV]_{k,j_k} = \sum_{r_1, \dots, r_{d'}} \prod_{k=1}^{d'} X_{k,r_k} V_{r_k,j_k}.$$

Therefore,

$$\langle R_V H_\alpha(X), H_{\{(1,1),\dots,(d',d')\}} \rangle_{L^2(\gamma)} = H_\alpha(V) = n^{-d'/2} H_\alpha(\sqrt{n}V).$$

Expanding $h = \sum_{\alpha} \hat{h}(\alpha) H_\alpha$ and taking 2-norms with respect to $\sqrt{n}V \sim \mu$, we see that indeed,

$$\mathbb{E}_{V \sim \text{SO}(n)} \|\Pi_{\text{comf},=S} R_V h\|_{L^2(\gamma)}^2 \geq \frac{(n/2)!}{n^{d'}((n/2) - d')!} \|h\|_{L^2(\mu)}^2,$$

proving (15).

Combining this with (11) and using the fact that $d' \leq d$, yields

$$\mathbb{E}_{V \sim \text{SO}(n)} \|\Pi_{\text{comf},=S} R_V f_{(=S)}\|_{L^2(\gamma)}^2 \geq \frac{(n/2)!}{2^d n^d ((n/2) - d)!} \|f\|_{L^2(\mu)}^2.$$

Finally, since

$$\begin{aligned} \Pi_{\text{comf},d} R_V f &= \Pi_{\text{comf},d} R_V \left(\sum_{S \subset [d]} f_{(=S)} \right) \\ &= \Pi_{\text{comf},d} \left(\sum_{S \subset [d]} (R_V f)_{(=S)} \right) \\ &= \sum_{S \subset [d]} \Pi_{\text{comf},=S} ((R_V f)_{(=S)}), \end{aligned}$$

and since the last sum is an orthogonal direct sum (again using the pairwise orthogonality of the W_S 's with respect to $L^2(\gamma)$), we have

$$\begin{aligned} \mathbb{E}_{V \sim \text{SO}(n)} \|\Pi_{\text{comf},d} R_V f\|_{L^2(\gamma)}^2 &\geq \mathbb{E}_{V \sim \text{SO}(n)} \|\Pi_{\text{comf},=S} ((R_V f)_{(=S)})\|_{L^2(\gamma)}^2 \\ &= \mathbb{E}_{V \sim \text{SO}(n)} \|\Pi_{\text{comf},=S} R_V (f_{(=S)})\|_{L^2(\gamma)}^2 \\ &\geq \frac{(n/2)!}{2^d n^d ((n/2) - d)!} \|f\|_{L^2(\mu)}^2, \end{aligned}$$

proving the claim. \square

We now show how to lower-bound the right-hand side of the expression in the statement of Lemma 10.3.

Lemma 10.6. *There exist $\delta > 0$ and $C > 0$ such that the following holds. Let $d \leq \delta n^{1/2}$. Then for all comfortable d -juntas $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$, we have*

$$\mathbb{E}_{V \sim \text{SO}(n)} \left[\|\Pi_{\text{comf},d} R_V \text{T}_{\text{row}} f\|_{L^2(\gamma)}^2 \right] \geq C^{-d} \|f\|_{L^2(\mu)}^2.$$

Proof. We write $\Pi_{\text{comf},d} = \Pi_{\text{comf},d} \Pi_{\text{comf},\text{row},d}$, and set $g = \Pi_{\text{comf},\text{row},d} \text{T}_{\text{row}} f$. Note that g is a row-comfortable d -row-junta. We have

$$\Pi_{\text{comf},d} R_V \text{T}_{\text{row}} f = \Pi_{\text{comf},d} R_V g,$$

where we used the fact that $\Pi_{\text{comf},\text{row},d}$ commutes with R_V . Taking the squares of the 2-norms and expectations over V we may apply Claim 10.5 to g , obtaining

$$\mathbb{E}_V \left[\|\Pi_{\text{comf},d} \text{T}_{\text{row}} R_V f\|_{L^2(\gamma)}^2 \right] = \frac{(n/2)!}{n^d ((n/2) - d)!} \|g\|_{L^2(\mu)}^2 \geq C^{-d} \|g\|_{L^2(\mu)}^2. \quad (13)$$

By Lemma 10.1, as $g - f$ is a row comfortable, d -row-junta, we have

$$\|g\|_{L^2(\mu)} \geq \|f\|_{L^2(\mu)} - \|g - f\|_{L^2(\mu)} \geq \|f\|_{L^2(\mu)} - 2\|g - f\|_{L^2(\gamma)}$$

As f is comfortable, we can apply Claim 10.4 to it to obtain

$$\|g - f\|_{L^2(\gamma)} = \|\Pi_{\text{comf},\text{row},d} (\text{T}_{\text{row}} f - f)\|_{L^2(\gamma)} \leq \|\text{T}_{\text{row}} f - f\|_{L^2(\gamma)} \leq \varepsilon \|f\|_{L^2(\gamma)}.$$

On the other hand Lemma 10.2 shows that $\|f\|_{L^2(\gamma)} \leq 2\|f\|_{L^2(\mu)}$. Putting these three facts together, we obtain $\|g\|_{L^2(\mu)} > (1 - 2\varepsilon)\|f\|_{L^2(\mu)}$. Plugging this into (13), and adjusting the value of C , completes the proof. \square

Finishing the proof of Lemma 9.9.

Using Lemma 10.3, the left-hand side of Lemma 9.9 is at least

$$C^{-d} \mathbb{E}_{V \sim \text{SO}(n)} \left[\|\Pi_{\text{comf},d} \text{T}_{\text{row}} R_V f\|_{L^2(\gamma)}^2 \right] = C^{-d} \mathbb{E}_{V \sim \text{SO}(n)} \left[\|\Pi_{\text{comf},d} R_V \text{T}_{\text{row}} f\|_{L^2(\gamma)}^2 \right],$$

and using Lemma 10.6 the last quantity is at least $C'^{-d} \|f\|_{L^2(\mu)}^2$, as required. \square

10.3 $L^2(\mu)$ is dominated by $L^2(\gamma)$ on column or row comfortable juntas: Proof of Lemma 10.1.

Our remaining tasks are to prove Lemmas 10.1 and 10.2. First, we show that the operator T_{col}^* is close to the identity on column comfortable d -column-juntas.

Claim 10.7. *For all $\varepsilon > 0$ there exists $\delta > 0$ such that if $d \leq \delta n^{1/2}$ and f is a column comfortable d -column-junta, then*

$$\|\mathbb{T}_{\text{col}}^* f - f\|_{L^2(\gamma)} \leq \varepsilon \|f\|_{L^2(\gamma)}.$$

Proof. The lemma follows immediately from Parseval and Claim 9.7 as $|\lambda_S - 1| < \varepsilon$ for each S as above, provided δ is sufficiently small (similarly to in the proof of Claim 10.4). \square

Claim 10.7 is particularly useful as it implies that $\mathbb{T}_{\text{col}}^*$ is invertible on the space spanned by column-comfortable d -column-juntas, and thus gives us a natural way of going from $L^2(\mu)$ to $L^2(\gamma)$. We are now ready to prove Lemma 10.1, restated below.

Lemma 10.1 (Restated). *For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $d \leq \delta n^{1/2}$, then for any function $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ that is either (i) a column comfortable, d -column-junta or (ii) a row comfortable, d -row-junta, we have*

$$\|f\|_{L^2(\mu)} \leq (1 + \varepsilon) \|f\|_{L^2(\gamma)}.$$

Proof. Let $\delta = \delta(\varepsilon) > 0$ be as in the statement of Claim 10.7. Without loss of generality, we assume that f is a column comfortable d -column-junta, i.e., is contained in the subspace

$$W := \text{Span}\{H_S : H_S \text{ is column comfortable and } S \text{ is supported on } [n] \times [d]\},$$

otherwise we may consider $f'(X) = f(X^t)$. Let $g : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ be a function such that $\mathbb{T}_{\text{col}}^* g = f$. Then as $\mathbb{T}_{\text{col}}^*$ is a contraction, we have $\|f\|_{L^2(\mu)} \leq \|g\|_{L^2(\gamma)}$. In order to complete the proof, we note (using Claim 10.7) that $\mathbb{T}_{\text{col}}^*(W) = W$, $\mathbb{T}_{\text{col}}^*$ is invertible on the subspace W , and that

$$\|(\mathbb{T}_{\text{col}}^*|_W)^{-1}\|_{L^2(\gamma) \rightarrow L^2(\gamma)} \leq 1 + 2\varepsilon. \quad (14)$$

Indeed, the facts that $\mathbb{T}_{\text{col}}^*(W) = W$ and that $\mathbb{T}_{\text{col}}^*$ is invertible on the subspace W , follow immediately from Claim 9.7. Moreover, if (14) fails, then there exists $h \in W$ with $\|h\|_{L^2(\gamma)} < 1/(1 + 2\varepsilon)$ and $\|(\mathbb{T}_{\text{col}}^*|_W)^{-1}h\|_{L^2(\gamma)} = 1$, and then for $h' := (\mathbb{T}_{\text{col}}^*|_W)^{-1}h$ (which lies in W), we obtain

$$\begin{aligned} \|\mathbb{T}_{\text{col}}^* h' - h'\|_{L^2(\gamma)} &= \|h - (\mathbb{T}_{\text{col}}^*|_W)^{-1}h\|_{L^2(\gamma)} \\ &\geq \|(\mathbb{T}_{\text{col}}^*|_W)^{-1}h\|_{L^2(\gamma)} - \|h\|_{L^2(\gamma)} \\ &> 1 - 1/(1 + 2\varepsilon) \\ &\geq \varepsilon \\ &= \varepsilon \|h'\|_{L^2(\gamma)} \end{aligned}$$

provided $\varepsilon \leq 1/2$, contradicting Claim 10.7.

We have $g = (\mathbb{T}_{\text{col}}^*|_W)^{-1}f$, and so

$$\|f\|_{L^2(\mu)} \leq \|g\|_{L^2(\gamma)} = \|(\mathbb{T}_{\text{col}}^*|_W)^{-1}f\|_{L^2(\gamma)} \leq (1 + 2\varepsilon) \|f\|_{L^2(\gamma)},$$

completing the proof. \square

10.4 $L^2(\gamma)$ is dominated by $L^2(\mu)$ on comfortable juntas: Proof of Lemma 10.2.

To prove Lemma 10.2, we define an auxiliary distribution on $\mathbb{R}^{n \times n}$, which we refer to as the ‘over-Gaussian’ distribution.

Definition 10.8. *Let $G \sim \text{GMD}$, and choose $Y \sim \gamma$ independently. We define the distribution ν to be the distribution of YG , and call it the over-Gaussian distribution.*

We refer to ν by this name since it can be produced taking $X \sim \mu$ (i.e., at random according to the Haar measure on $\text{SO}(n)$), and then multiplying X by two independent copies of GMD, thereby ‘overshooting’ the Gaussian distribution.

In the following section, we show that the distribution ν is close to γ in the sense that the expectation of a certain kind of test function is roughly the same under both measures. In fact, the relevant test functions are the squares of the comfortable d -juntas, and their expectations are roughly the same even if we allow the degree d to be as large as $\Theta(\sqrt{n})$. The following lemma asserts that if f is a comfortable d -junta, then its over-Gaussian 2-norm cannot be much larger than its Gaussian 2-norm.

Lemma 10.9. *For all $\varepsilon > 0$ there exists $\delta > 0$ such that if $d \leq \delta n^{1/2}$, then for all comfortable d -juntas $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$, we have*

$$\|f\|_{L^2(\nu)} \leq (1 + \varepsilon) \|f\|_{L^2(\gamma)}.$$

To prove Lemma 10.9, we need some more notation, and two technical claims. For a permutation $\sigma \in S_d$, we write $x_\sigma := x_{1,\sigma(1)} \cdots x_{d,\sigma(d)}$; this is a function on $\mathbb{R}^{d \times d}$. Fix a comfortable d -junta $f : \mathbb{R}^{d \times d} \rightarrow \mathbb{R}$ and write $f = \sum_{I=(i_1, \dots, i_d)} a_I x_I$ where $x_I := x_{1,i_1} x_{2,i_2} \cdots x_{d,i_d}$, the sum ranging over all I such that $i_1, \dots, i_d \in [d]$ are distinct. (As usual, we write $(S)_d$ for the set of ordered d -tuples of distinct elements of the set S ; using this notation we may write $I \in ([d])_d$.) We have

$$\|f\|_{L^2(\nu)}^2 = \sum_I a_I^2 \|x_I\|_{L^2(\nu)}^2 + \sum_{I \neq J} a_I a_J \langle x_I, x_J \rangle_\nu.$$

We now need two technical claims, which handle respectively the diagonal terms and the off-diagonal terms in the above expansion of $\|f\|_{L^2(\nu)}^2$. To handle the diagonal terms, we will use the following claim.

Claim 10.10. *If $i_1, \dots, i_d \in [n]$ are distinct, then $x_I := x_{1,i_1} x_{2,i_2} \cdots x_{d,i_d}$ satisfies*

$$\|x_I\|_{L^2(\nu)}^2 = 1.$$

For $I, J \in [n]^d$ we let $d(I, J) := |\{r \mid i_r \neq j_r\}|$ denote the Hamming distance from I to J . To handle the off-diagonal terms, we need the following claim.

Claim 10.11. *For any I, J such that $d(I, J) = \ell$, we have $|\langle x_I, x_J \rangle_{L^2(\nu)}| \leq \varepsilon_\ell$, where*

$$\varepsilon_\ell := 2^{\ell+4} n^{-\ell/2} 2^{d\ell/\sqrt{n}}.$$

Claims 10.10 and 10.11 imply Lemma 10.9

We first show how to deduce Lemma 10.9 from Claims 10.10 and 10.11. Writing $f = \sum_I a_I x_I$, we have

$$\begin{aligned}
\|f\|_{L^2(\nu)}^2 &= \sum_I a_I^2 \|x_I\|_{L^2(\nu)}^2 + \sum_{I \neq J} a_I a_J \langle x_I, x_J \rangle_\nu \\
&\leq \|f\|_{L^2(\gamma)}^2 + \sum_{I \neq J} \frac{a_I^2 + a_J^2}{2} |\langle x_I, x_J \rangle_\nu| \\
&\leq \|f\|_{L^2(\gamma)}^2 + \sum_{\ell=1}^d \sum_I a_I^2 |\{J : d(J, I) = \ell\}| \cdot \varepsilon_\ell \\
&\leq \|f\|_{L^2(\gamma)}^2 + \|f\|_{L^2(\gamma)}^2 \sum_{\ell=1}^d \varepsilon_\ell \binom{d}{\ell} \ell! \\
&= \|f\|_{L^2(\gamma)}^2 \left(1 + \sum_{\ell=1}^d \varepsilon_\ell d^\ell \right).
\end{aligned}$$

Using the upper bound on ε_ℓ , we obtain

$$\sum_{\ell=1}^d \varepsilon_\ell d^\ell \leq \sum_{\ell=1}^d 2^{\ell+4} n^{-\ell/2} 2^{d\ell/\sqrt{n}} d^\ell \leq 16 \sum_{\ell=1}^{\infty} \left(\frac{2d \cdot 2^{d/\sqrt{n}}}{\sqrt{n}} \right)^\ell \leq \frac{\varepsilon}{2},$$

where we used $d \leq \delta n^{1/2}$ and the fact that δ is sufficiently small compared to ε .

Proof of Claims 10.10 and 10.11.

To prove the two claims, we need the following simple fact about the Gaussian maker distribution.

Claim 10.12. *Let $I = (i_1, \dots, i_d) \in [n]^d$ and $J = (j_1, \dots, j_d) \in [n]^d$ be such that $d(I, J) = \ell$, and such that in the product*

$$G_{i_1 j_1} G_{i_2 j_2} \cdots G_{i_d j_d},$$

no matrix entry of G appears more than twice. Then

$$|\mathbb{E}_{G \sim GMD} [G_{i_1 j_1} G_{i_2 j_2} \cdots G_{j_d i_d}]| \leq \left(\frac{1}{n} \right)^{\ell/2}.$$

Proof. If, in the product

$$G_{i_1 j_1} G_{i_2 j_2} \cdots G_{i_d j_d},$$

some off-diagonal matrix entry of G appears exactly once, then the expectation of the product is zero. We may therefore assume that every off-diagonal matrix entry of G appears

either exactly twice, or not at all, in the above product. If there are exactly ℓ values of r such that $i_r \neq j_r$, then the above expectation factorises into a product of the expectations of the squares of $\ell/2$ off-diagonal and of the squares of $(d - \ell)/2$ diagonal entries:

$$\prod_{k \in \mathcal{D}} \mathbb{E}[G_{k,k}^2] \prod_{(i,j) \in \mathcal{E}} \mathbb{E}[G_{i,j}^2],$$

where $\mathcal{E} \subset [n]^2 \setminus \{(k,k) : k \in [n]\}$, $|\mathcal{D}| = (d - \ell)/2$ and $|\mathcal{E}| = \ell/2$. We have $\mathbb{E}[G_{i,j}^2] = 1/n$ for all $(i,j) \in \mathcal{E}$ and $\mathbb{E}[G_{k,k}^2] = (n - k + 1)/n \leq 1$ for all $k \in \mathcal{D}$, proving the claim. \square

We are now ready to prove Claim 10.10.

Proof of Claim 10.10. Let $x_I = x_{1,i_1} x_{2,i_2} \cdots x_{d,i_d}$, where $i_1, \dots, i_d \in [n]$ are distinct. We have

$$\|x_I\|_{L^2(\nu)}^2 = \mathbb{E}_{G \sim \text{GMD}} \mathbb{E}_{Y \sim \gamma} [(YG)_{1,i_1}^2 (YG)_{2,i_2}^2 \cdots (YG)_{d,i_d}^2].$$

Since for each $h \in [d]$, $(YG)_{h,i_h} = \sum_{k=1}^{i_h} Y_{h,k} G_{k,i_h}$ involves only entries of Y in row h and entries of G in column i_h (and the i_h are distinct), the random variables $((YG)_{h,i_h}^2 : h \in [d])$ form a system of independent random variables, and therefore

$$\|x_I\|_{L^2(\nu)}^2 = \mathbb{E}_{G \sim \text{GMD}} \mathbb{E}_{Y \sim \gamma} [(YG)_{1,i_1}^2] \mathbb{E}_{G \sim \text{GMD}} \mathbb{E}_{Y \sim \gamma} [(YG)_{2,i_2}^2] \cdots \mathbb{E}_{G \sim \text{GMD}} \mathbb{E}_{Y \sim \gamma} [(YG)_{d,i_d}^2].$$

For each $h \in [d]$, we have

$$\begin{aligned} \mathbb{E}_{G \sim \text{GMD}} \mathbb{E}_{Y \sim \gamma} [(YG)_{h,i_h}^2] &= \mathbb{E}_{G \sim \text{GMD}} \mathbb{E}_{Y \sim \gamma} \left[\left(\sum_{k=1}^{i_h} Y_{h,k} G_{k,i_h} \right)^2 \right] \\ &= 2 \sum_{1 \leq k < k' \leq i_h} \mathbb{E}_{G \sim \text{GMD}} \mathbb{E}_{Y \sim \gamma} [Y_{h,k} Y_{h,k'} G_{k,i_h} G_{k',i_h}] \\ &\quad + \sum_{k=1}^{i_h} \mathbb{E}_{G \sim \text{GMD}} \mathbb{E}_{Y \sim \gamma} [Y_{h,k}^2 G_{k,i_h}^2] \\ &= 0 + \sum_{k=1}^{i_h} \mathbb{E}_{G \sim \text{GMD}} [G_{k,i_h}^2] \mathbb{E}_{Y \sim \gamma} [Y_{h,k}^2] \\ &= \sum_{k=1}^{i_h} \mathbb{E}_{G \sim \text{GMD}} [G_{k,i_h}^2] \cdot 1 \\ &= (i_h - 1)(1/n) + (n - i_h + 1)/n \\ &= 1. \end{aligned}$$

(Here, for the third equality we use the independence of $Y_{h,k}, Y_{h,k'}, G_{k,i_h}, G_{k',i_h}$ and the fact that $Y_{h,k}$ and $Y_{h,k'}$ both have zero expectation.) Hence, $\|x_I\|_{L^2(\nu)}^2 = 1$, as required. \square

We now move on to the proof of Claim 10.11.

Proof of Claim 10.11. Let $\ell \geq 1$ and fix $I, J \in ([d])_d$ such that $d(I, J) = \ell \geq 1$. Since G is upper-triangular and $i_h, j_h \leq d$ for all $h \in [d]$, we have

$$(YG)_{h,i_h} = \sum_{k=1}^{i_h} Y_{h,k} G_{k,i_h} = \sum_{k=1}^d Y_{h,k} G_{k,i_h}$$

and

$$(YG)_{h,j_h} = \sum_{k=1}^{j_h} Y_{h,k} G_{k,j_h} = \sum_{k=1}^d Y_{h,k} G_{k,j_h}$$

for all $h \in [d]$. Hence,

$$x_I(YG) = \prod_{h=1}^d (YG)_{h,i_h} = \sum_{K=(k_1, \dots, k_d) \in [d]^d} Y_{1,k_1} \cdots Y_{d,k_d} G_{k_1,i_1} \cdots G_{k_d,i_d}$$

and

$$x_J(YG) = \sum_{K=(k_1, \dots, k_d) \in [d]^d} Y_{1,k_1} \cdots Y_{d,k_d} G_{k_1,j_1} \cdots G_{k_d,j_d},$$

so, using the fact that, under ν , the $(Y_{i,j} : i, j \in [n])$ are independent and of expectation zero (and are independent of the $G_{i,j}$), we obtain

$$\langle x_I, x_J \rangle_\nu = \sum_{K \in [d]^d} \mathbb{E}_{G \sim \text{GMD}} [G_{k_1 i_1} G_{k_1 j_1} \cdots G_{k_d i_d} G_{k_d j_d}].$$

For a d -tuple $K = (k_1, \dots, k_d) \in [d]^d$ we write $m_1 = m_1(K) := |\{r : j_r = i_r, k_r \neq i_r\}|$, $m_2 = m_2(K) := |\{r : j_r \neq i_r, k_r \notin \{i_r, j_r\}\}|$ and $m_3 = m_3(K) := |\{r : j_r \neq i_r, k_r \in \{i_r, j_r\}\}|$, and we let $\mathcal{K}(m_1, m_2, m_3)$ denote the set of d -tuples K with parameters m_1, m_2 and m_3 . For $K \in \mathcal{K}(K_1, K_2, K_3)$, by Claim 10.12 we have

$$|\mathbb{E}_{G \sim \text{GMD}} [G_{k_1 i_1} G_{k_1 j_1} \cdots G_{k_d i_d} G_{k_d j_d}]| \leq n^{-\frac{2m_1 + 2m_2 + m_3}{2}}.$$

Summing over all K , we see that $|\langle x_I, x_J \rangle|$ is at most

$$\begin{aligned} & \sum_{m_1, m_2, m_3} \sum_{K \in \mathcal{K}(m_1, m_2, m_3)} n^{-\frac{2m_1 + 2m_2 + m_3}{2}} \\ & \leq \sum_{m_1, m_2, m_3} n^{-\frac{2m_1 + 2m_2 + m_3}{2}} |\mathcal{K}(m_1, m_2, m_3)|. \end{aligned}$$

Now

$$|\mathcal{K}(m_1, m_2, m_3)| \leq \binom{d}{m_1} d^{m_1} \binom{\ell}{m_2} d^{m_2} 2^{m_3} \leq \frac{d^{2m_1 + m_2} \ell^{m_2} 2^{m_3}}{m_1! m_2!}.$$

Summing over all m_1, m_2, m_3 with $m_2 + m_3 = \ell$ completes the proof. Indeed,

$$\begin{aligned}
\sum_{m_1, m_2, m_3} n^{-\frac{2m_1+2m_2+m_3}{2}} |\mathcal{K}(m_1, m_2, m_3)| &\leq \sum_{m_1, m_2, m_3} n^{-\frac{2m_1+2m_2+m_3}{2}} \frac{d^{2m_1+m_2} \ell^{m_2} 2^{m_3}}{m_1! m_2!} \\
&= \sum_{m_2+m_3=\ell} n^{-\frac{2m_2+m_3}{2}} \frac{d^{m_2} \ell^{m_2} 2^{m_3}}{m_2!} \sum_{m_1=0}^{d-\ell} \frac{1}{m_1!} (d^2/n)^{m_1} \\
&\leq 2 \sum_{m_2+m_3=\ell} n^{-\frac{2m_2+m_3}{2}} \frac{d^{m_2} \ell^{m_2} 2^{m_3}}{m_2!} \\
&= 2^{\ell+1} n^{-\ell/2} \sum_{m_2=0}^{\ell} n^{-m_2/2} \frac{d^{m_2} \ell^{m_2}}{2^{m_2} m_2!} \\
&\leq 2^{\ell+1} n^{-\ell/2} \left(1 + \sum_{m_2=1}^{\ell} \left(\frac{ed\ell}{2\sqrt{nm_2}} \right)^{m_2} \right) \\
&\leq 2^{\ell+4} n^{-\ell/2} 2^{d\ell/\sqrt{n}},
\end{aligned}$$

as required. \square

With Lemma 10.9 in hand, the proof of Lemma 10.2 (restated below) is easy.

Lemma 10.2 (Restated). *For all $\varepsilon > 0$, there exists $\delta > 0$ such that if $d \leq \delta n^{1/2}$ and $f : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is a comfortable d -junta, then $\|f\|_{L^2(\gamma)} \leq (1 + \varepsilon) \|f\|_{L^2(\mu)}$.*

Proof. We may assume that $\varepsilon \leq 1/2$. By the triangle inequality, it suffices to show that $\|f(X) - f(XG)\|_{L^2(\mu, \text{GMD})} < \frac{\varepsilon}{2} \|f\|_{L^2(\gamma)}$, where $X \sim \mu$ and $G \sim \text{GMD}$. Since for any fixed, upper triangular $G \in \mathbb{R}^{n \times n}$, the function g_G defined by

$$g_G : X \mapsto f(X) - f(XG)$$

is a row-comfortable d -row-junta, by Lemma 10.1 we have

$$\begin{aligned}
\|f(X) - f(XG)\|_{L^2(\mu, \text{GMD})}^2 &= \mathbb{E}_{G \sim \text{GMD}} \mathbb{E}_{X \sim \mu} [(f(X) - f(XG))^2] \\
&= \mathbb{E}_{G \sim \text{GMD}} \mathbb{E}_{X \sim \mu} [g_G(X)^2] \\
&= \mathbb{E}_{G \sim \text{GMD}} [\|g_G\|_{L^2(\mu)}^2] \\
&\leq (1 + \varepsilon)^2 \mathbb{E}_{G \sim \text{GMD}} \|g_G\|_{L^2(\gamma)}^2 \\
&= (1 + \varepsilon)^2 \mathbb{E}_{G \sim \text{GMD}} \mathbb{E}_{Y \sim \gamma} [(f(Y) - f(YG))^2] \\
&= (1 + \varepsilon)^2 \|f(Y) - f(YG)\|_{L^2(\gamma, \text{GMD})}^2,
\end{aligned}$$

and therefore it suffices to show that

$$\|f(Y) - f(YG)\|_{L^2(\gamma, \text{GMD})} < \frac{\varepsilon}{4} \|f\|_{L^2(\gamma)},$$

where $Y \sim \gamma$ and $G \sim \text{GMD}$ is independent of Y . Expanding, we note that

$$\|f(Y) - f(YG)\|_{L^2(\gamma, \text{GMD})}^2 = \|f\|_{L^2(\gamma)}^2 + \|f\|_{L^2(\nu)}^2 - 2\langle f, T_{\text{col}}^* f \rangle_{L^2(\gamma)}.$$

To handle the cross term, we note that by Cauchy-Schwarz and Claim 10.7, we have

$$|\langle f, T_{\text{col}}^* f - f \rangle_{L^2(\gamma)}| \leq \|f\|_{L^2(\gamma)} \|T_{\text{col}}^* f - f\|_{L^2(\gamma)} \leq \frac{\varepsilon}{10} \|f\|_{L^2(\gamma)}^2,$$

so $\langle f, T_{\text{col}}^* f \rangle_{L^2(\gamma)} \geq (1 - \varepsilon/10) \|f\|_{L^2(\gamma)}^2$. Using Lemma 10.9, we have

$$\|f\|_{L^2(\nu)}^2 \leq (1 + \varepsilon/20) \|f\|_{L^2(\gamma)}^2$$

provided δ is sufficiently small, and so

$$\|f(Y) - f(YG)\|_{L^2(\gamma, \text{GMD})}^2 \leq \|f\|_{L^2(\gamma)}^2 + \|f\|_{L^2(\nu)}^2 - 2(1 - \varepsilon/10) \|f\|_{L^2(\gamma)}^2 \leq (\varepsilon/4) \|f\|_{L^2(\gamma)}^2,$$

completing the proof. \square

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A Bounding the dimensions of high level representations

A.1 The special orthogonal group $\mathrm{SO}(n)$

Lemma 6.7 (Restated). *Let $n \geq 5$. If ρ is an irreducible representation of $\mathrm{SO}(n)$ of level $d \geq n/2$, then*

$$\dim(\rho) \geq \exp(n/32).$$

Proof. This follows from Weyl’s original dimension formulae, together with a short computation. First suppose that $n = 2k + 1$ is odd. As mentioned above, the equivalence classes of irreducible representations of $\mathrm{SO}(2k + 1, \mathbb{R})$ are in an explicit one-to-one correspondence with the partitions λ (of non-negative integers) whose Young diagrams have at most k rows. Weyl’s dimension formula states that for any such partition λ , the corresponding irreducible representation ρ_λ of $\mathrm{SO}(2k + 1, \mathbb{R})$ has

$$\dim(\rho_\lambda) = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{1 \leq i \leq j \leq k} \frac{\lambda_i + \lambda_j + 2k + 1 - i - j}{2k + 1 - i - j}.$$

Fix λ a partition of degree $d \in \mathbb{N}$; trivially,

$$\dim(\rho_\lambda) \geq \prod_{1 \leq i \leq j \leq k} \frac{\lambda_i + 2k + 1 - i - j}{2k + 1 - i - j} \geq \prod_{\substack{1 \leq i \leq k/2 \\ k/2 \leq j \leq k}} \left(1 + \frac{\lambda_i}{2k}\right) = \prod_{1 \leq i \leq k/2} \left(1 + \frac{\lambda_i}{2k}\right)^{k/2}.$$

If $\lambda_1 \geq k$, then the previous product is at least $(1 + 1/2)^{k/2} \geq \exp(n/16)$ and we are done, so assume that $\lambda_1 < k$ and hence all λ_i ’s are smaller than k . For all $0 \leq \delta < 1/2$ we have $1 + \delta \geq e^{\delta/2}$, so the previous product is at least

$$\prod_{1 \leq i \leq k/2} e^{\frac{\lambda_i}{4k} \cdot \frac{k}{2}} = e^{\frac{1}{8} \sum_{i=1}^{k/2} \lambda_i}.$$

Since $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$, we get that $\sum_{i=1}^{k/2} \lambda_i \geq \frac{1}{2} \sum_{i=1}^k \lambda_i = \frac{d}{2}$, so the last expression is at least $e^{d/16}$ and we are done.

The case of even n is very similar. Let $n = 2k$; then the equivalence classes of irreducible representations of $\mathrm{SO}(2k, \mathbb{R})$ are in an explicit correspondence with the partitions λ (of non-negative integers) whose Young diagrams have at most k rows: this correspondence is one-to-one when the number of rows is less than k , but when the number of rows is equal to k , the correspondence is two-to-one (each partition λ with k rows corresponds to two irreducible representations ρ_λ and $\tilde{\rho}_\lambda$ of the same dimension). Writing $c = c(\lambda) = 1$ if λ

has less than k rows and $c = c(\lambda) = 1/2$ if λ has exactly k rows, the dimensions of the corresponding irreducible representations are given by the formula

$$\dim(\rho_\lambda) = c \prod_{1 \leq i < j \leq k} \left(\frac{\lambda_i - \lambda_j + j - i}{j - i} \right) \left(\frac{\lambda_i + \lambda_j + 2k - i - j}{2k - i - j} \right).$$

From now on we can apply exactly the same argument as in the case of n odd; we omit the details. \square

A.2 The spin group $\text{Spin}(n)$

Lemma A.1. *Let ρ_λ be as in Theorem 6.9. Then*

$$\dim(\rho_\lambda) = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j - i + j}{j - i} \prod_{1 \leq i \leq j \leq k} \frac{\lambda_i + \lambda_j + 2k + 1 - i - j}{2k + 1 - i - j} \geq 2^{-\Omega(n)}.$$

Proof. Indeed, if a_k is odd then we must have $a_k \geq 1$ and therefore $\lambda_i \geq 1/2$ for all $i \in [k]$. Hence, rather crudely, we have

$$\begin{aligned} \dim(\rho_\lambda) &\geq \prod_{1 \leq i < j \leq k} \frac{\lambda_i + \lambda_j + 2k + 1 - i - j}{2k + 1 - i - j} \\ &= \prod_{1 \leq i < j \leq k} \left(1 + \frac{\lambda_i + \lambda_j}{2k + 1 - i - j} \right) \\ &\geq \prod_{\substack{1 \leq i \leq k/2, \\ k/2 \leq j \leq k}} \left(1 + \frac{\lambda_i}{2k} \right) \\ &\geq \prod_{1 \leq i \leq k/2} \left(1 + \frac{\lambda_i}{2k} \right)^{k/2} \\ &\geq \prod_{1 \leq i \leq k/2} \left(1 + \frac{\min\{\lambda_i, 2k\}}{2k} \right)^{k/2} \\ &\geq \prod_{1 \leq i \leq k/2} \exp(\min\{\lambda_i, 2k\}/8) \\ &= \exp \left(\frac{1}{8} \sum_{1 \leq i \leq k/2} \min\{\lambda_i, 2k\} \right) \\ &\geq \exp(k/32) \\ &= \exp((n-1)/64), \end{aligned}$$

as required, using the fact that $1 + x \geq e^{x/2}$ for all $x \leq 1$.

For all $n = 2k \geq 6$ even, we have

$$\dim(\rho_\lambda) = \prod_{1 \leq i < j \leq k} \frac{\lambda_i - \lambda_j - i + j}{j - i} \frac{\lambda_i + \lambda_j + 2k - i - j}{2k - i - j},$$

where the k -tuple $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ ranges over all k -tuples defined by

$$\lambda_i = a_i + a_{i+1} + \dots + a_{k-2} + \frac{1}{2}(a_{k-1} + a_k) \quad \forall i \leq k-2, \quad \lambda_{k-1} = \frac{1}{2}(a_{k-1} + a_k), \quad \lambda_k = \frac{1}{2}(a_k - a_{k-1}),$$

for some $(a_i)_{i=1}^k \in (\mathbb{N} \cup \{0\})^k$. The case of $a_{k-1} + a_k$ even corresponds to irreducible representations of $\text{Spin}(n)$ that are also irreducible representations of $\text{SO}(n, \mathbb{R})$; the dimensions of these were bounded in the previous subsection. The case of $a_{k-1} + a_k$ odd corresponds to ‘new’ irreducible representations of $\text{Spin}(n)$, but the above equation quickly implies that any such has dimension at least $2^{\Omega(n)}$. Indeed, if $a_{k-1} + a_k$ is odd then we must have $a_{k-1} + a_k \geq 1$ and therefore $\lambda_i \geq 1/2$ for all $i \leq k-1$. Hence, again rather crudely, we have

$$\begin{aligned} \dim(\rho_\lambda) &\geq \prod_{1 \leq i < j \leq k} \frac{\lambda_i + \lambda_j + 2k - i - j}{2k - i - j} \\ &= \prod_{1 \leq i < j \leq k} \left(1 + \frac{\lambda_i + \lambda_j}{2k - i - j} \right) \\ &\geq \prod_{\substack{1 \leq i \leq k/2, \\ k/2 < j \leq k}} \left(1 + \frac{\lambda_i}{2k} \right) \\ &\geq \prod_{1 \leq i \leq k/2} \left(1 + \frac{\lambda_i}{2k} \right)^{(k-1)/2} \\ &\geq \prod_{1 \leq i \leq k/2} \left(1 + \frac{\min\{\lambda_i, 2k\}}{2k} \right)^{k/4} \\ &\geq \prod_{1 \leq i \leq k/2} \exp(\min\{\lambda_i, 2k\}/16) \\ &= \exp \left(\frac{1}{16} \sum_{1 \leq i \leq k/2} \min\{\lambda_i, 2k\} \right) \\ &\geq \exp(k/64) \\ &= \exp(n/128), \end{aligned}$$

as required, again using the fact that $1 + x \geq e^{x/2}$ for all $x \leq 1$. □

A.3 The special unitary group $\text{SU}(n)$

Lemma 6.26 (Restated). *If ρ is an irreducible representation of $\text{SU}(n)$ of level D , then $\dim(\rho) \geq 2^{\Omega(\min(D, n))}$.*

Proof. Weyl's dimension formula states that the dimension of the irreducible representation of $SU(n)$ corresponding to the partition λ (with at most $n - 1$ parts) is

$$\prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

Assume first that $\lambda_1 \geq n$. If $\lambda_{\lfloor n/2 \rfloor + 1} \leq n/2$ then, considering all the terms in the above product corresponding to $i = 1$ and $j \geq \lfloor n/2 \rfloor + 1$, we see that the above product satisfies

$$\begin{aligned} \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} &\geq \prod_{j \geq \lfloor n/2 \rfloor + 1} \frac{\lambda_1 - \lambda_j + j - 1}{j - 1} \\ &\geq \prod_{j \geq \lfloor n/2 \rfloor + 1} \frac{\lambda_1 - \lambda_j + n - 1}{n - 1} \\ &\geq \prod_{j \geq \lfloor n/2 \rfloor + 1} \frac{n - n/2 + n - 1}{n - 1} \\ &= \exp(\Theta(n)). \end{aligned}$$

If, on the other hand, we have $\lambda_{\lfloor n/2 \rfloor + 1} > n/2$, then considering all the terms in the above product corresponding to $j = n$ and $i \leq \lfloor n/2 \rfloor + 1$, we obtain

$$\begin{aligned} \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} &\geq \prod_{i \leq \lfloor n/2 \rfloor + 1} \frac{\lambda_i - \lambda_n + n - i}{n - i} \\ &\geq \prod_{i \leq \lfloor n/2 \rfloor + 1} \frac{\lambda_i - \lambda_n + n - 1}{n - 1} \\ &\geq \prod_{i \leq \lfloor n/2 \rfloor + 1} \frac{n/2 - 0 + n - 1}{n - 1} \\ &= \exp(\Theta(n)). \end{aligned}$$

We may henceforth assume that $\lambda_1 < n$. In this case we have $\lambda_i - \lambda_j < n$ for all i, j . Hence, the above product satisfies

$$\begin{aligned}
\prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} &\geq \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + n - 1}{n - 1} \\
&\geq \prod_{1 \leq i < j \leq n} \left(1 + \frac{\lambda_i - \lambda_j}{n - 1} \right) \\
&\geq \prod_{1 \leq i < j \leq n} \exp\left(\frac{\lambda_i - \lambda_j}{2(n - 1)}\right) \\
&= \exp\left(\frac{1}{2(n - 1)} \sum_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)\right) \\
&= \exp\left(\frac{1}{2(n - 1)} \sum_{1 \leq i < j \leq n} (a_i + \dots + a_{j-1})\right) \\
&= \exp\left(\frac{(n - 1)a_1 + 2(n - 2)a_2 + \dots + 2(n - 2)a_{n-2} + (n - 1)a_{n-1}}{2(n - 1)}\right) \\
&\geq \exp(\Theta(D)).
\end{aligned}$$

□

B The required adaptations for showing that $\mathrm{Sp}(n)$ and $\mathrm{SU}(n)$ are good

Here we complete the proof that $\mathrm{Sp}(n)$ and $\mathrm{SU}(n)$ are good. In fact, we will only explain how to adapt the proof for $\mathrm{Sp}(n)$, as $\mathrm{SU}(n)$ is only simpler (one just works with complex matrices rather than quaternionic ones).

To construct our noise operator T_ρ on $\mathrm{Sp}(n)$ we use the identification

$$\mathrm{Sp}(n) = \{X \in \mathbb{H}^{n \times n} : XX^h = I\} = \{X \in \mathbb{H}^{n \times n} : X^h X = I\}$$

of $\mathrm{Sp}(n)$ with the group of unitary quaternionic matrices. (Here, X^h denotes the quaternionic conjugate of X , i.e. $(X^h)_{p,q} = \overline{X_{q,p}}$, where $\overline{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$ for $a, b, c, d \in \mathbb{R}$.) We couple $\mathrm{Sp}(n)$ with the space $(\mathbb{H}^{n \times n}, \gamma)$ of quaternionic normal random matrices, where the real part, the \mathbf{i} -part (= coefficient of \mathbf{i}), the \mathbf{j} -part and the \mathbf{k} -part of each entry are independent normal (real-valued) random variables with mean zero and variance $1/4$, and all the entries are independent. (The following terminology will be useful in the sequel. Recall that a *standard normal quaternionic* or \mathcal{QN} random variable is a quaternion-valued random variable where the real part, the \mathbf{i} -part, the \mathbf{j} -part and the \mathbf{k} -part of the value of the random variable are independent normal (real-valued) random variables with mean zero and variance $1/4$; so a quaternionic normal random matrix $\sim (\mathbb{H}^{n \times n}, \gamma)$ is

simply a matrix where each entry is an independent \mathcal{QN} random variable. A \mathcal{QN} random vector in n dimensions is a vector of n independent \mathcal{QN} random variables.)

To make this coupling work, we need to define a ‘Gram-Schmidt’ type process (on the columns, and also on the rows) which, when applied to a matrix in $(\mathbb{H}^{n \times n}, \gamma)$, yields an element of $\text{Sp}(n)$ with probability one. We define two inner products on \mathbb{H}^n :

$$\langle x, y \rangle := \sum_{i=1}^n \bar{x}_i y_i, \quad \langle x, y \rangle' := \sum_{i=1}^n x_i \bar{y}_i.$$

It is easy to check that, for $H \in \mathbb{H}^{n \times n}$, we have $\langle Hx, Hy \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{H}^n$ if and only if $H \in \text{Sp}(n)$.

Our Gram-Schmidt process on the columns, $\text{GS}_{\text{col}}(X)$ for $X \in \mathbb{H}^{n \times n}$, is defined as follows. If c_1, c_2, \dots, c_n denote the columns of X , then we (inductively) define

$$\gamma_k := c_k - \sum_{\ell < k} \tilde{c}_\ell \langle \tilde{c}_\ell, c_k \rangle$$

and (if $\gamma_k \neq 0$)

$$\tilde{c}_k := \gamma_k / \sqrt{\langle \gamma_k, \gamma_k \rangle},$$

for $1 \leq k \leq n$. If any $\gamma_k = 0$ then $\text{GS}_{\text{col}}(X)$ is undefined; otherwise we define $\text{GS}_{\text{col}}(X)$ to be \sqrt{n} times the matrix \tilde{X} in $\mathbb{H}^{n \times n}$ with columns $\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_n$; it is easy to see that $\tilde{X} \in \text{Sp}(n)$. (One checks, by induction on k , that $\langle \tilde{c}_\ell, \tilde{c}_k \rangle = \delta_{j,k}$ for all $1 \leq j \leq k \leq n$; taking conjugates this implies that $\langle \tilde{c}_k, \tilde{c}_\ell \rangle = \delta_{j,k}$ for all $1 \leq j \leq k \leq n$, and these two statements together imply that $\tilde{X} \in \text{Sp}(n)$). It is clear that, if X is sampled according to γ , then all the γ_k are non-zero with probability one, so $\text{GS}_{\text{col}}(X)$ is defined outside a set of zero probability measure.

We define the Gaussian noise operator $U_\rho : L^2(\mathbb{H}^{n \times n}, \gamma) \rightarrow L^2(\mathbb{H}^{n \times n}, \gamma)$ in the obvious way: for $f \in L^2(\mathbb{H}^{n \times n}, \gamma)$ we define

$$U_\rho f(X) = \mathbb{E}_{Y \sim \gamma} [f(\rho X + \sqrt{1 - \rho^2} Y)].$$

It is easy to check that U_ρ is self-adjoint; indeed,

$$\mathbb{E}_{X \sim \gamma} [\overline{f(X)} U_\rho g(X)] = \mathbb{E}_{X, Z \sim \gamma, \rho\text{-correlated}} \overline{f(X)} g(Z) = \mathbb{E}_{X \sim \gamma} [\overline{U_\rho f(X)} g(X)].$$

As in the case of $\text{SO}(n)$, we let μ denote the measure on $\sqrt{n}\text{Sp}(n)$ induced (under dilation by \sqrt{n}) from the Haar probability measure on $\text{Sp}(n)$. For $f \in L^2(\text{Sp}(n))$ we define $\text{T}_{\text{col}} : L^2(\sqrt{n}\text{Sp}(n), \mu) \rightarrow L^2(\mathbb{H}^{n \times n}, \gamma)$ by

$$\text{T}_{\text{col}} f(X) = f(\text{GS}_{\text{col}}(X)).$$

We similarly let T_{col}^* denote its (Hilbert-space) adjoint.

Again, as in the case of $\text{SO}(n)$, we define

$$\text{T}_\rho = \mathbb{E}_{V \sim \text{Sp}(n)} [R_V^* \text{T}_{\text{col}}^* U_\rho \text{T}_{\text{col}} R_V].$$

Since U_ρ is self-adjoint, so is T_ρ . The fact that T_{col} commutes with L_V for all $V \in \text{Sp}(n)$ follows from the fact that $\text{GS}_{\text{col}}(VX) = V\text{GS}_{\text{col}}(X)$ for all $X \in \mathbb{H}^{n \times n}$ and all $V \in \text{Sp}(n)$, which in turn follows from the fact that $\langle Vx, Vy \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{H}^n$ and all $V \in \text{Sp}(n)$. The fact that L_V and R_V both commute with U_ρ follows from the fact that, if $X \sim (\mathbb{H}^{n \times n}, \gamma)$ and $V \in \text{Sp}(n)$, then $VX \sim (\mathbb{H}^{n \times n}, \gamma)$ and $XV \sim (\mathbb{H}^{n \times n}, \gamma)$, as in the case of $\text{SO}(n)$ and (\mathbb{R}^n, γ) (the proof is very similar). For any $V \in \text{Sp}(n)$, we have $L_V^* = L_{V^{-1}}$ and $R_V^* = R_{V^{-1}}$, and therefore, taking the adjoints of

$$L_V T_{\text{col}} = T_{\text{col}} L_V, \quad R_V T_{\text{col}} = T_{\text{col}} R_V$$

yields

$$T_{\text{col}}^* L_{V^{-1}} = L_{V^{-1}} T_{\text{col}}^*, \quad T_{\text{col}}^* R_{V^{-1}} = R_{V^{-1}} T_{\text{col}}^*,$$

and therefore L_V and R_V commute with T_{col}^* , for all $V \in \text{Sp}(n)$. It follows that L_V commutes with T_ρ . The fact that for any $W \in \text{Sp}(n)$, R_W commutes with T_ρ , follows from a simple change of variables:

$$\begin{aligned} R_W T_\rho &= R_W \mathbb{E}_{V \sim \text{Sp}(n)} [R_V^* T_{\text{col}}^* U_\rho T_{\text{col}} R_V] \\ &= \mathbb{E}_{V \sim \text{Sp}(n)} [R_{W^{-1}}^* R_V^* T_{\text{col}}^* U_\rho T_{\text{col}} R_V] \\ &= \mathbb{E}_{V \sim \text{Sp}(n)} [(R_V R_{W^{-1}})^* T_{\text{col}}^* U_\rho T_{\text{col}} R_V] \\ &= \mathbb{E}_{V \sim \text{Sp}(n)} [R_{V W^{-1}} T_{\text{col}}^* U_\rho T_{\text{col}} R_V] \\ &= \mathbb{E}_{V \sim \text{Sp}(n)} [R_V T_{\text{col}}^* U_\rho T_{\text{col}} R_{VW}] \\ &= \mathbb{E}_{V \sim \text{Sp}(n)} [R_V T_{\text{col}}^* U_\rho T_{\text{col}} R_V R_W] \\ &= T_\rho R_W. \end{aligned}$$

Hence, T_ρ commutes with the action of $\text{Sp}(n)$ from both left and right, as in the $\text{SO}(n)$ case.

We also need to define an analogue of T_{row} . However, this is a little different to in the $\text{SO}(n)$ case. For $V \in \mathbb{H}^{n \times n}$, $X \in \text{Sp}(n)$ does not imply that $\langle e_i X, e_j X \rangle = \delta_{i,j}$ for each $i, j \in [n]$ (the latter would be the analogue of ‘orthonormal rows’). The condition $X \in \text{Sp}(n)$ is, however, equivalent to the condition $\langle e_i X, e_j X \rangle' = \delta_{i,j}$ for each $i, j \in [n]$ (which is in turn equivalent to the condition $\langle e_i \bar{X}, e_j \bar{X} \rangle = \delta_{i,j}$ for each $i, j \in [n]$). We therefore define our Gram-Schmidt process on the rows, $\text{GS}_{\text{row}}(X)$ for $X \in \mathbb{H}^{n \times n}$, as follows. If r_1, r_2, \dots, r_n denote the rows of X , then we (inductively) define

$$\delta_k := r_k - \sum_{\ell < k} \overline{\langle \tilde{r}_\ell, r_k \rangle'} \tilde{r}_\ell$$

and (if $\delta_k \neq 0$)

$$\tilde{r}_k := \delta_k / \sqrt{\langle \delta_k, \delta_k \rangle'},$$

for $1 \leq k \leq n$. If any $\delta_k = 0$, then $\text{GS}_{\text{row}}(X)$ is undefined; otherwise we define $\text{GS}_{\text{row}}(X)$ to be \sqrt{n} times the matrix $\tilde{X} \in \mathbb{H}^{n \times n}$ with rows $\tilde{r}_1, \tilde{r}_2, \dots, \tilde{r}_n$; it is easy to see that $\tilde{X} \in \text{Sp}(n)$,

similarly to in the case of $\text{GS}_{\text{col}}(X)$. Again, as with $\text{GS}_{\text{col}}(X)$, if X is sampled according to γ , then all the δ_k are non-zero with probability one, so $\text{GS}_{\text{row}}(X)$ is defined outside a set of zero probability measure.

For $f \in L^2(\text{Sp}(n))$, we define $T_{\text{row}} : L^2(\sqrt{n}\text{Sp}(n), \mu) \rightarrow L^2(\mathbb{H}^{n \times n}, \gamma)$ by

$$T_{\text{row}}f(X) = f(\text{GS}_{\text{row}}(X)),$$

and of course, we let T_{row}^* denote its (Hilbert-space) adjoint.

As in the $\text{SO}(n)$ case, we observe that T_{row} commutes with R_V for all $V \in \text{Sp}(n)$; this follows from the fact that $\text{GS}_{\text{row}}(XV) = \text{GS}_{\text{row}}(X)V$ for all $X \in \mathbb{H}^{n \times n}$ and all $V \in \text{Sp}(n)$, which in turn follows from the fact that $\langle xV, yV \rangle' = \langle x, y \rangle'$ for all $x, y \in \mathbb{H}^n$ and all $V \in \text{Sp}(n)$.

Similarly to as in the $\text{SO}(n)$ case, if $Y \sim (\mathbb{H}^{n \times n}, \gamma)$, and $X = \text{GS}_{\text{col}}(Y)$, we obtain $Y = XG$, where $g_{i,i}$ is $1/\sqrt{n}$ times the (Euclidean) length of a \mathcal{QN} random vector in $n - i + 1$ dimensions, $g_{i,j} = 0$ for all $i > j$, $g_{i,j}$ is $1/\sqrt{n}$ times a \mathcal{QN} random variable, the entries of G are independent, and independent of all the entries of X . This (distribution over) quaternionic upper-triangular matrices G is our Gaussian Maker Distribution (or GMD, for short) in the $\text{Sp}(n)$ case. To generalise the $\text{SO}(n)$ proof, the only (important) facts we need are the fact that $\text{Sp}(n)$ acts transitively (from either the left or the right) on the set $\mathcal{S} = \{v \in \mathbb{H}^n : \langle v, v \rangle = 1\}$ of quaternionic vectors of unit norm, and that $(\mathbb{H}^{n \times n}, \gamma)$ is invariant under both left and right actions of $\text{Sp}(n)$.

We can therefore write

$$T_{\text{col}}^*f(X) = \mathbb{E}_{G \sim \text{GMD}} f(XG) \quad \forall X \in \sqrt{n}\text{Sp}(n).$$

Similarly, we can write

$$T_{\text{row}}^*f(X) = \mathbb{E}_{G \sim \text{GMD}} f(G^T X) \quad \forall X \in \sqrt{n}\text{Sp}(n).$$

Our ‘nice’ functions on $L^2(\mathbb{H}^{n \times n}, \gamma)$ are polynomials (with complex coefficients) where each variable is a real-part, an **i**-part, a **j**-part or a **k**-part of one of the matrix entries. We say a monomial in these variables is *comfortable* if no two of its variables come from the same row, no two of its variables come from the same column, and all variables come from the first $\lfloor n/2 \rfloor$ rows and the first $\lfloor n/2 \rfloor$ columns. Similarly, we say it is *row comfortable* (respectively *column comfortable*) if no two of its variables come from the same row (respectively column), and all of its variables come from the first $\lfloor n/2 \rfloor$ rows (respectively columns). A comfortable polynomial (respectively row-comfortable polynomial or column-comfortable polynomial) is a complex linear combination of comfortable (respectively row-comfortable or column-comfortable) monomials. Such a polynomial is said to be a *d-junta* if it is homogeneous of degree d and depends only upon the top-left d by d minor, it is said to be a *d-row-junta* if it depends only upon the first d rows, and it is said to be a *d-column-junta* if it depends only upon the first d columns.

As in the $\text{SO}(n)$ -case, we define $\Pi_{\text{comf}} : L^2(\mathbb{H}^{n \times n}, \gamma) \rightarrow L^2(\mathbb{H}^{n \times n}, \gamma)$ to be orthogonal projection onto the linear subspace of comfortable polynomials. Similarly, we define the

operators $\Pi_{\text{comf},d}$, $\Pi_{\text{comf,col},d}$ and $\Pi_{\text{comf,row},d}$ to be the orthogonal projections onto the space of comfortable, row comfortable and column comfortable polynomials of degree at most d , respectively. Finally, if $S \subset [n]$ is a set of rows, we define $\Pi_{\text{comf},=S}$ to be the projection onto the subspace spanned by the comfortable homogeneous monomials of degree $|S|$ which depend only upon variables from the rows in S .

Since the real-part, the \mathbf{i} -part, the \mathbf{j} -part and the \mathbf{k} -part of each matrix entry of $X \sim (\mathbb{H}^{n \times n}, \gamma)$ is $N(0, 1/4)$ -distributed rather than $N(0, 1)$ -distributed, to guarantee orthonormality we must multiply by a factor of 2^d , so a ‘generic’ row-comfortable monomial of degree d is of the form

$$2^d (X_{i_1, j_1})_{q_1\text{-part}} \cdot (X_{i_2, j_2})_{q_2\text{-part}} \cdot \dots \cdot (X_{i_d, j_d})_{q_d\text{-part}},$$

where $q_k \in \{\text{real}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ for all $k \in [d]$ and i_1, \dots, i_d are distinct integers between 1 and $n/2$; for brevity we denote this by H_α where $\alpha = \{(i_1, j_1; q_1), (i_2, j_2; q_2), \dots, (i_d, j_d; q_d)\}$.

The proof (and statement) of Claim 9.7 is readily adapted. If

$$S = \{(i_1, j_1; q_1), (i_2, j_2; q_2), \dots, (i_d, j_d; q_d)\}$$

is such that the j_k are all distinct, then we have

$$\begin{aligned} T_{\text{col}}^* f(X) &= \mathbb{E}_{G \sim \text{GMD}} H_S(XG) \\ &= 2^d \mathbb{E}_{G \sim \text{GMD}} \left[\prod_{k=1}^d ((XG)_{i_k, j_k})_{q_k\text{-part}} \right] \\ &= 2^d \mathbb{E}_{G \sim \text{GMD}} \left[\prod_{k=1}^d \left(\sum_{\ell=1}^{j_k} (X_{i_k, \ell} G_{\ell, j_k})_{q_k\text{-part}} \right) \right] \\ &= 2^d \prod_{k=1}^d \mathbb{E}_{G \sim \text{GMD}} \left[\sum_{\ell=1}^{j_k} (X_{i_k, \ell} G_{\ell, j_k})_{q_k\text{-part}} \right] \\ &= 2^d \prod_{k=1}^d \left(\sum_{\ell=1}^{j_k} (X_{i_k, \ell} \mathbb{E}_G[G_{\ell, j_k}])_{q_k\text{-part}} \right) \\ &= 2^d \prod_{k=1}^d (X_{i_k, j_k} \mathbb{E}_G[G_{j_k, j_k}])_{q_k\text{-part}} \\ &= 2^d \prod_{k=1}^d (X_{i_k, j_k})_{q_k\text{-part}} \mathbb{E}_G[G_{j_k, j_k}] \\ &= \lambda_S H_S(X), \end{aligned}$$

where

$$\lambda_S := \prod_{k=1}^d \mathbb{E}_G[G_{j_k, j_k}].$$

(Note that we used the fact that $\mathbb{E}_G[G_{\ell, j_k}] = 0$ for all $\ell \neq j_k$, and that $\mathbb{E}_G[G_{j_k, j_k}] \in \mathbb{R}$.) The rest of the proof is almost exactly the same as before.

The analogue of Claim 10.5 is that, for any row-comfortable d -row-junta $f : \mathbb{H}^{n \times n} \rightarrow \mathbb{C}$, we have

$$\mathbb{E}_{V \sim \text{Sp}(n)} \|\Pi_{\text{comf}, d} R_V f\|_{L^2(\gamma)}^2 \geq \frac{(n/2)!}{2^{d n^d} ((n/2) - d)!} \|f\|_{L^2(\mu)}^2.$$

The proof of Claim 10.5 is readily adapted. We first note that, as before, if H_α and H_β are row-comfortable monomials such that the set of rows appearing in α is different from the set of rows appearing in β , then we have $\langle H_\alpha, H_\beta \rangle_{L^2(\gamma)} = \langle H_\alpha, H_\beta \rangle_{L^2(\mu)} = 0$; the same proof works as before.

For $S \subset [n]$, let W_S denote the (complex) linear span of the row-comfortable monomials of degree exactly $|S|$ which depend only upon variables from the rows in S . As before, the W_S are pairwise orthogonal with respect to both $L^2(\gamma)$ and $L^2(\mu)$.

Again as before, for any row-comfortable d -row junta $g : \mathbb{H}^{n \times n} \rightarrow \mathbb{C}$, we may write g as an orthogonal direct sum,

$$g = \sum_{S \subset [d]} g_{(=S)},$$

where $g_{(=S)}$ denotes the orthogonal projection of g onto W_S (with respect to $L^2(\gamma)$). It is clear that $(R_V g)_{(=S)} = R_V(g_{(=S)})$ for any $V \in \text{Sp}(n)$ and any row-comfortable d -row-junta g .

Now let $f : \mathbb{H}^{n \times n} \rightarrow \mathbb{C}$ be a row-comfortable d -row-junta. Since the $f_{(=S)}$ are pairwise orthogonal with respect to $L^2(\mu)$ as well as with respect to $L^2(\gamma)$, we have

$$\|f\|_{L^2(\mu)}^2 = \sum_{S \subset [d]} \|f_{(=S)}\|_{L^2(\mu)}^2,$$

so there exists $S \subset [d]$ such that $\|f_{(=S)}\|_{L^2(\mu)}^2 \geq \frac{1}{2^d} \|f\|_{L^2(\mu)}^2$.

For brevity, write $h = f_{(=S)}$, $d' := |S|$ and $S = \{i_1, \dots, i_{d'}\}$. We need to show that

$$\mathbb{E}_{V \sim \text{Sp}(n)} \|\Pi_{\text{comf}, =S} R_V h\|_{L^2(\gamma)}^2 \geq \frac{(n/2)!}{n^{d'} ((n/2) - d')!} \|h\|_{L^2(\mu)}^2. \quad (15)$$

In proving this, we may assume without loss of generality that $S = \{1, 2, \dots, d'\}$. In this case, we may write

$$h = \sum_{\alpha = \{(1, j_1; q_1), \dots, (d', j_{d'}; q_{d'})\}} \hat{f}(\alpha) H_\alpha. \quad (16)$$

For brevity, write $H_{\alpha_0} := H_{\{(1, 1; \mathbb{R}), (2, 2; \mathbb{R}), \dots, (d', d'; \mathbb{R})\}}$, i.e., H_{α_0} denotes the degree- d' monomial

$$X \mapsto 2^{d'} (X_{1,1})_{\text{real-part}} \cdot (X_{2,2})_{\text{real-part}} \cdot \dots \cdot (X_{d',d'})_{\text{real-part}}.$$

If $\alpha = \{(1, j_1; q_1), (2, j_2; q_2), \dots, (d', j_{d'}; q_{d'})\}$ is such that the j_k are all distinct, then $H_\alpha = R_{V_\alpha} H_{\alpha_0}$, where $V_\alpha = \Sigma D$, Σ is the permutation matrix corresponding to some permutation $\sigma \in S_n$ satisfying $\sigma^{-1}(i) = j_i$ for all $i \in [d']$ ($\Sigma_{i,j} := \delta_{\{j=\sigma(i)\}}$ for all $i, j \in [n]$),

and D is a diagonal matrix with $D_{i,i} = \bar{q}_i$ for all $i \in [d']$. It follows that $H_{\alpha_0} = R_{V_\alpha^{-1}} H_\alpha$. Since $V_\alpha \in \text{Sp}(n)$ for any such V_α , and since $V_\alpha V \sim \text{Sp}(n)$ for $V \sim \text{Sp}(n)$, we have

$$\begin{aligned}
\mathbb{E}_{V \sim \text{Sp}(n)} \|\Pi_{\text{comf},d} R_V f\|_{L^2(\gamma)}^2 &= \sum_{\alpha: H_\alpha \text{ comfortable}} \mathbb{E}_{V \sim \text{Sp}(n)} |\langle R_V f, H_\alpha \rangle|^2 \\
&= \sum_{\alpha: H_\alpha \text{ comfortable}} \mathbb{E}_{V \sim \text{Sp}(n)} |\langle R_{V_\alpha V} f, H_\alpha \rangle|^2 \\
&= \sum_{\alpha: H_\alpha \text{ comfortable}} \mathbb{E}_{V \sim \text{Sp}(n)} |\langle R_{V_\alpha} R_V f, H_\alpha \rangle|^2 \\
&= \sum_{\alpha: H_\alpha \text{ comfortable}} \mathbb{E}_{V_\alpha V \sim \text{Sp}(n)} |\langle R_V f, R_{V_\alpha^{-1}} H_\alpha \rangle|^2 \\
&= 4^{d'} \frac{(n/2)!}{((n/2) - d')!} \mathbb{E}_{V \sim \text{Sp}(n)} |\langle R_V f, H_{\alpha_0} \rangle|^2.
\end{aligned}$$

Now observe that

$$R_V H_\alpha(X) = H_\alpha(XV) = 2^{d'} \prod_{k=1}^{d'} ((XV)_{k,j_k})_{q_k\text{-part}} = 2^{d'} \prod_{k=1}^{d'} \left(\sum_{i=1}^n X_{k,i} V_{i,j_k} \right)_{q_k\text{-part}},$$

and therefore

$$\langle R_V H_\alpha(X), H_{\alpha_0}(X) \rangle = \prod_{k=1}^{d'} (V_{k,j_k})_{q_k\text{-part}} = 2^{-d'} H_\alpha(V).$$

Using the expansion (16) of h , it follows that

$$\langle R_V h, H_{\alpha_0} \rangle = 2^{-d'} \sum_{\alpha} \hat{f}(\alpha) H_\alpha(V) = 2^{-d'} f(V),$$

and therefore

$$\begin{aligned}
\mathbb{E}_{V \sim \text{Sp}(n)} \|\Pi_{\text{comf},=S} R_V h\|_{L^2(\gamma)}^2 &= 4^{d'} \frac{(n/2)!}{((n/2) - d')!} \mathbb{E}_{V \sim \text{Sp}(n)} 4^{-d'} |h(V)|^2 \\
&= \frac{(n/2)!}{n^{d'} ((n/2) - d')!} \|h\|_{L^2(\mu)}^2.
\end{aligned}$$

The rest of the proof is exactly as before.

We now need analogues of some of the results of Section 10. (We omit those whose statements and proofs are trivial to adapt.) As in the $\text{SO}(n)$ case, we define the ‘over-Gaussian’ distribution ν to be the distribution of YG , where $Y \sim \gamma$ and $G \sim \text{GMD}$. Fix a comfortable d -junta f and write

$$f = \sum_S a_I (2^d x_S),$$

where for $S = \{(1, i_1; q_1), \dots, (d, i_d; q_d)\}$ (with $i_1, \dots, i_d \in [n]$ distinct, and $q_1, \dots, q_d \in \{\text{real}, \mathbf{i}, \mathbf{j}, \mathbf{k}\} := \mathcal{R}$), we write

$$x_S := \prod_{h=1}^d (x_{h, i_h})_{q_h\text{-part}}.$$

Note that $\{2^d x_S\}_S$ forms an orthonormal set of vectors in $L^2(\gamma)$, where S ranges over tuples of the above form.

We need the following analogue of Claim 10.10.

Claim B.1. *Let $S = \{(1, i_1; q_1), \dots, (d, i_d; q_d)\}$ where $i_1, \dots, i_d \in [n]$ are distinct and $q_1, \dots, q_d \in \{\text{real}, \mathbf{i}, \mathbf{j}, \mathbf{k}\} := \mathcal{R}$, and let $x_S := \prod_{h=1}^d (x_{h, i_h})_{q_h\text{-part}}$. Then*

$$\|2^d x_S\|_{L^2(\nu)}^2 = 1.$$

Proof. In what follows, for $q \in \{\text{real}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and $h \in \mathbb{H}$, we define $(h)_{-q\text{-part}} := -(h)_{q\text{-part}}$, for notational convenience. Observe that

$$\|2^d x_S\|_{L^2(\nu)}^2 = 4^d \mathbb{E}_{G \sim \text{GMD}} \mathbb{E}_{Y \sim \gamma} [(((Y G)_{1, i_1})_{q_1\text{-part}})^2 ((Y G)_{2, i_2})_{q_2\text{-part}}^2 \cdots ((Y G)_{d, i_d})_{q_d\text{-part}}^2].$$

Since for each $h \in [d]$, $((Y G)_{h, i_h})_{q_h\text{-part}} = \sum_{k=1}^{i_h} \sum_{r \in \mathcal{R}} (Y_{h, k})_{r\text{-part}} (G_{k, i_h})_{r^{-1} q_h\text{-part}}$ involves only entries of Y in row h and entries of G in column i_h (and the i_h are distinct), the random variables $\{(((Y G)_{h, i_h})_{q_h\text{-part}})^2 : h \in [d]\}$ form a system of independent random variables, and therefore

$$\|2^d x_S\|_{L^2(\nu)}^2 = 4^d \prod_{h=1}^d \mathbb{E}_{G \sim \text{GMD}} \mathbb{E}_{Y \sim \gamma} [(((Y G)_{h, i_h})_{q_h\text{-part}})^2].$$

For each $h \in [d]$, we have

$$\begin{aligned} & \mathbb{E}_{G \sim \text{GMD}} \mathbb{E}_{Y \sim \gamma} [(((Y G)_{h, i_h})_{q_h\text{-part}})^2] \\ &= \mathbb{E}_G \mathbb{E}_Y \left[\left(\sum_{k=1}^{i_h} \sum_{r \in \mathcal{R}} (Y_{h, k})_{r\text{-part}} (G_{k, i_h})_{r^{-1} q_h\text{-part}} \right)^2 \right] \\ &= \sum_{(k, r) \neq (k', r')} \mathbb{E}_G \mathbb{E}_Y [(Y_{h, k})_{r\text{-part}} (Y_{h, k'})_{r'\text{-part}} (G_{k, i_h})_{r^{-1} q_h\text{-part}} (G_{k', i_h})_{(r')^{-1} q_h\text{-part}}] \\ &+ \sum_{k=1}^{i_h} \mathbb{E}_{G \sim \text{GMD}} \mathbb{E}_{Y \sim \gamma} [Y_{h, k}^2 G_{k, i_h}^2] \\ &= 0 + \sum_{k=1}^{i_h} \sum_{r \in \mathcal{R}} \mathbb{E}_{G \sim \text{GMD}} [((G_{k, i_h})_{r^{-1} q_h\text{-part}})^2] \mathbb{E}_{Y \sim \gamma} [((Y_{h, k})_{r\text{-part}})^2] \\ &= \sum_{k=1}^{i_h} \sum_{r \in \mathcal{R}} \mathbb{E}_{G \sim \text{GMD}} [((G_{k, i_h})_{r^{-1} q_h\text{-part}})^2] \cdot \frac{1}{4} \\ &= \frac{1}{4} (4(i_h - 1)(1/(4n)) + (n - i_h + 1)/n) \\ &= 1/4. \end{aligned}$$

(Here, for the third equality we use the independence of

$$(Y_{h,k})_{r\text{-part}}, (Y_{h,k'})_{r'\text{-part}}, (G_{k,i_h})_{r^{-1}q_h\text{-part}}, (G_{k',i_h})_{(r')^{-1}q_h\text{-part}}$$

and the fact that $(Y_{h,k})_{r\text{-part}}$ and $(Y_{h,k'})_{r'\text{-part}}$ both have zero expectation.) Hence, $\|2^d x_S\|_{L^2(\nu)}^2 = 1$, as required. \square

Similarly, we need the following analogue of Claim 10.11. For $S = \{(1, i_1; q_1), \dots, (d, i_d; q_d)\}$ and $T = \{(1, j_1; p_1), \dots, (d, j_d; p_d)\}$ we set

$$d(S, T) := |\{h \in [d] : i_h \neq j_h \text{ or } q_h \neq p_h\}|.$$

Claim B.2. *For any S, T such that $d(S, T) = \ell$, we have $|\langle 2^d x_S, 2^d x_T \rangle|_{L^2(\nu)} \leq \varepsilon_\ell$, where*

$$\varepsilon_\ell := 2^{\ell+4} n^{-\ell/2} 2^{d\ell/\sqrt{n}}.$$

To prove this we first need the following simple analogue of Claim 10.12.

Claim B.3. *Let $(i_1, \dots, i_d) \in [n]^d$ and $(j_1, \dots, j_d) \in [n]^d$ be such that $|\{h \in [d] : i_h \neq j_h\}| = \ell$ and such that in the product*

$$(G_{i_1 j_1})_{r_1\text{-part}} (G_{i_2 j_2})_{r_2\text{-part}} \cdots (G_{i_d j_d})_{r_d\text{-part}},$$

no matrix entry of G appears more than twice. Then

$$|\mathbb{E}_{G \sim \text{GMD}} [(G_{i_1 j_1})_{r_1\text{-part}} (G_{i_2 j_2})_{r_2\text{-part}} \cdots (G_{i_d j_d})_{r_d\text{-part}}]| \leq \left(\frac{1}{4n}\right)^{\ell/2}.$$

Proof. If, in the product

$$(G_{i_1 j_1})_{r_1\text{-part}} (G_{i_2 j_2})_{r_2\text{-part}} \cdots (G_{i_d j_d})_{r_d\text{-part}},$$

some off-diagonal matrix entry of G appears exactly once, then the expectation of the product is zero. We may therefore assume that every matrix entry of G appears either exactly twice, or not at all, in the above product. If there are exactly ℓ values of h such that $i_h \neq j_h$, then the above expectation factorises into a product of the expectations of the squares of $\ell/2$ off-diagonal and of the squares of $(d - \ell)/2$ diagonal entries:

$$\prod_{k \in \mathcal{D}} \mathbb{E}[(G_{k,k})_{q_k\text{-part}}^2] \prod_{(i,j) \in \mathcal{E}} \mathbb{E}[(G_{i,j})_{r_i\text{-part}}^2],$$

where $\mathcal{E} \subset [n]^2 \setminus \{(k, k) : k \in [n]\}$, $|\mathcal{D}| = (d - \ell)/2$, $|\mathcal{E}| = \ell/2$ and $q_k, r_i \in \mathcal{R}$ for all i and k . We have $\mathbb{E}[(G_{i,j})_{r_i\text{-part}}^2] = 1/(4n)$ for all $(i, j) \in \mathcal{E}$ and $\mathbb{E}[(G_{k,k})_{q_k\text{-part}}^2] \leq 1$ for all $k \in \mathcal{D}$, proving the claim. \square

Proof. Let $\ell \geq 1$ and fix $S = \{(1, i_1; q_1), \dots, (d, i_d; q_d)\}$ and $T = \{(1, j_1; p_1), \dots, (d, j_d; p_d)\}$ such that $d(S, T) = \ell \geq 1$. Since G is upper-triangular and $i_h, j_h \leq d$ for all $h \in [d]$, we have

$$((YG)_{h, i_h})_{q_h\text{-part}} = \sum_{k=1}^{i_h} \sum_{r \in \mathcal{R}} (Y_{h,k})_{r\text{-part}} (G_{k, i_h})_{r^{-1}q_h\text{-part}} = \sum_{k=1}^d \sum_{r \in \mathcal{R}} (Y_{h,k})_{r\text{-part}} (G_{k, i_h})_{r^{-1}q_h\text{-part}}$$

and

$$((YG)_{h, j_h})_{p_h\text{-part}} = \sum_{k=1}^{j_h} \sum_{r \in \mathcal{R}} (Y_{h,k})_{r\text{-part}} (G_{k, j_h})_{r^{-1}p_h\text{-part}} = \sum_{k=1}^d \sum_{r \in \mathcal{R}} (Y_{h,k})_{r\text{-part}} (G_{k, j_h})_{r^{-1}p_h\text{-part}}$$

for all $h \in [d]$. Hence,

$$\begin{aligned} x_S(YG) &= \prod_{h=1}^d ((YG)_{h, i_h})_{q_h\text{-part}} \\ &= \sum_{\substack{K=(k_1, \dots, k_d) \in [d]^d, \\ R=(r_1, \dots, r_d) \in \mathcal{R}^d}} (Y_{1, k_1})_{r_1\text{-part}} \cdots (Y_{d, k_d})_{r_d\text{-part}} (G_{k_1, i_1})_{r_1^{-1}q_1\text{-part}} \cdots (G_{k_d, i_d})_{r_d^{-1}q_d\text{-part}} \end{aligned}$$

and

$$x_T(YG) = \sum_{\substack{K=(k_1, \dots, k_d) \in [d]^d, \\ R=(r_1, \dots, r_d) \in \mathcal{R}^d}} (Y_{1, k_1})_{r_1\text{-part}} \cdots (Y_{d, k_d})_{r_d\text{-part}} (G_{k_1, j_1})_{r_1^{-1}p_1\text{-part}} \cdots (G_{k_d, j_d})_{r_d^{-1}p_d\text{-part}},$$

so, using the fact that, under ν , the $((Y_{i,j})_{r\text{-part}} : i, j \in [n], r \in \mathcal{R})$ are independent and of expectation zero (and are independent of the $G_{i,j}$), we obtain

$$\begin{aligned} &\left\langle 2^d x_S, 2^d x_T \right\rangle_\nu \\ &= \sum_{\substack{K \in [d]^d, \\ R \in \mathcal{R}^d}} \mathbb{E}_{G \sim \text{GMD}} \left[(G_{k_1 i_1})_{r_1^{-1}q_1\text{-part}} (G_{k_1 j_1})_{r_1^{-1}p_1\text{-part}} \cdots (G_{k_d i_d})_{r_d^{-1}q_d\text{-part}} (G_{k_d j_d})_{r_d^{-1}p_d\text{-part}} \right]. \end{aligned} \tag{17}$$

For a d -tuple $(K; R) = (k_1; r_1 \dots, k_d; r_d) \in ([d] \times \mathcal{R})^d$, we write $m_1 = m_1(K) = m_1(K; R) := |\{h \in [d] : j_h = i_h, k_h \neq i_h\}|$, $m_2 = m_2(K) = m_2(K; R) := |\{h \in [d] : j_h \neq i_h, k_h \notin \{i_h, j_h\}\}|$ and $m_3 = m_3(K; R) := |\{h \in [d] : j_h \neq i_h, k_h \in \{i_h, j_h\}\}|$; note that these quantities depend only on K and not on R . We let $\mathcal{K}(m_1, m_2, m_3)$ denote the set of d -tuples $(K; R)$ with parameters m_1, m_2 and m_3 . For $(K; R) \in \mathcal{K}(m_1, m_2, m_3)$, by Claim 10.12 we have

$$\left| \mathbb{E}_G \left[(G_{k_1 i_1})_{r_1^{-1}q_1\text{-part}} (G_{k_1 j_1})_{r_1^{-1}p_1\text{-part}} \cdots (G_{k_d i_d})_{r_d^{-1}q_d\text{-part}} (G_{k_d j_d})_{r_d^{-1}p_d\text{-part}} \right] \right| \leq (4n)^{-\frac{2m_1+2m_2+m_3}{2}}.$$

We further note that, for $(K; R) \in \mathcal{K}(m_1, m_2, m_3)$, the expectation in the above inequality is zero unless the following four conditions hold:

- Whenever $i_h = j_h$ and $k_h \neq i_h$, we have $p_h = q_h$.
- Whenever $i_h = j_h = k_h$ we have $p_h = q_h$ and $r_h^{-1}p_h = \text{real}$.
- Whenever $i_h \neq j_h$ and $k_h = i_h$ we have $r_h^{-1}q_h = \text{real}$.
- Whenever $i_h \neq j_h$ and $k_h = j_h$ we have $r_h^{-1}p_h = \text{real}$.

In view of this we let $\mathcal{K}^*(m_1, m_2, m_3)$ be the set of all d -tuples $(K; R) \in \mathcal{K}(m_1, m_2, m_3)$ such that the above four conditions hold. For $(K; R) \in \mathcal{K}^*(m_1, m_2, m_3)$ we have $m_2 + m_3 = \ell$. Summing over all K , we see that $|\langle x_S, x_T \rangle|$ is at most

$$\begin{aligned} & \sum_{m_1, m_2, m_3} \sum_{K \in \mathcal{K}^*(m_1, m_2, m_3)} (4n)^{-\frac{2m_1 + 2m_2 + m_3}{2}} \\ & \leq \sum_{m_1, m_2, m_3} (4n)^{-\frac{2m_1 + 2m_2 + m_3}{2}} |\mathcal{K}^*(m_1, m_2, m_3)|. \end{aligned}$$

Now

$$|\mathcal{K}^*(m_1, m_2, m_3)| \leq \binom{d}{m_1} d^{m_1} \binom{\ell}{m_2} d^{m_2} 2^{m_3} \cdot 4^{m_1 + m_2} \leq \frac{d^{2m_1 + m_2} \ell^{m_2} 2^{m_3}}{m_1! m_2!} \cdot 4^{m_1 + m_2};$$

note that the only difference with the corresponding expression in the proof of Claim 10.12 is the extra factor of $4^{m_1 + m_2}$, which comes from the fact that r_h can vary freely over \mathcal{R} (and still satisfy the above conditions) when $i_h \neq j_h$ and $k_h \notin \{i_h, j_h\}$, or when $i_h = j_h$ and $k_h \neq i_h$, but in no other cases.

Summing over all m_1, m_2, m_3 with $m_2 + m_3 = \ell$ completes the proof, just as in the proof of Claim 10.11; the extra factor of $4^{-\frac{2m_1 + 2m_2 + m_3}{2}}$ cancels out (or more than cancels out) the extra factor of $4^{m_1 + m_2}$. \square

The analogue of Lemma 10.9 is proven from the above claims in a very similar way. Writing

$$f = \sum_S \alpha_S (2^d x_S),$$

we obtain

$$\begin{aligned}
\|f\|_{L^2(\nu)}^2 &\leq \sum_S |\alpha_S|^2 \|2^d x_S\|_{L^2(\nu)}^2 + \sum_{S \neq T} |\alpha_S| |\alpha_T| \left| \langle 2^d x_S, 2^d x_T \rangle_\nu \right| \\
&\leq \|f\|_{L^2(\gamma)}^2 + \sum_{S \neq T} \frac{|a_S|^2 + |a_T|^2}{2} \left| \langle 2^d x_S, 2^d x_T \rangle_\nu \right| \\
&\leq \|f\|_{L^2(\gamma)}^2 + \sum_{\ell=1}^d \sum_S |a_S|^2 |\{T : d(T, S) = \ell\}| \cdot \varepsilon_\ell \\
&\leq \|f\|_{L^2(\gamma)}^2 + \|f\|_{L^2(\gamma)}^2 \sum_{\ell=1}^d \varepsilon_\ell \binom{d}{\ell} \ell! 4^\ell \\
&= \|f\|_{L^2(\gamma)}^2 \left(1 + \sum_{\ell=1}^d \varepsilon_\ell (4d)^\ell \right),
\end{aligned}$$

and the rest of the proof is essentially unchanged, up to reducing the value of δ by a constant factor.

C Bounding the eigenvalues of the Laplace-Beltrami operator

C.1 Bounding the eigenvalues of the Laplace-Beltrami operator in $\mathrm{SO}(n)$

In this section, we obtain our desired bound on the eigenvalues of the Laplace-Beltrami operator in SO_n , which we need in order to prove fineness.

Lemma C.1. *Let $\rho \in \hat{\mathrm{SO}}(n)$ be of level D , then the corresponding eigenvalue λ_ρ of the Laplacian Δ satisfies*

$$\lambda_\rho \geq -C(nD + D^2).$$

To prove this, we will use [7, Theorems 2.3, 2.4]. It is well-known that the matrix coefficients of the irreducible representations of $\mathrm{SO}(n)$ are eigenvectors of the Laplace-Beltrami operator. The aforementioned theorems in [7] give a formula for the eigenvalues, in terms of the fundamental weights of the irreducible representations. (The reader is referred to [17, p. 219] for a relatively concise definition of the fundamental weights.) The following statement (which suffices for our purposes) combines Theorem 2.3 and 2.4 from [7].

Theorem C.2. *Let G be a simply connected Lie group of rank k . Then the eigenvalues of the Laplace-Beltrami operator $\Delta : L^2(G) \rightarrow L^2(G)$ are in one-to-one correspondence with the equivalence classes of complex irreducible representations of G . Furthermore, there is a set of vectors $\{w_1, \dots, w_k\}$ in a finite-dimensional inner-product space, known as a system of fundamental weights for G , such that there is an explicit one-to-one correspondence between*

the equivalence classes of irreducible representations of G , and the elements of the discrete cone

$$\left\{ \sum_{i=1}^k r_i w_i \mid r_1, \dots, r_k \in \mathbb{N} \right\}.$$

Furthermore, defining $\rho := \sum_{i=1}^k w_i$, the eigenvalue λ_v of Δ corresponding to $v = \sum_{i=1}^k r_i w_i$ satisfies

$$\lambda_v = -\|v + \rho\|_2^2 + \|\rho\|_2^2 = -2\langle v, \rho \rangle - \|v\|_2^2.$$

We will choose a known system of fundamental weights for $\mathrm{SO}(n)$ as in [35, Section 5.1]. We denote by $E_{i,j} \in \mathbb{R}^{n \times n}$ the n by n matrix with a one in the (i, j) -th entry and zeros elsewhere. (We omit n from notation, as it will always be clear from context.) The system of fundamental weights depends on whether n is even or odd, and we deal with the two cases separately.

The case of odd n

Let $n = 2k + 1$. In this case, the rank of $\mathrm{SO}(n)$ is k , and a system of fundamental weights $\{w_1, w_2, \dots, w_k\}$ is given by $w_i = \sum_{j=1}^i u_j$ for $1 \leq i \leq k - 1$ and $w_k = \frac{1}{2} \sum_{j=1}^k u_j$, where $u_j := E_{j, k+j} - E_{k+j, j}$ for each $1 \leq j \leq k$.

The equivalence class of representations corresponding to $v = \sum_{i=1}^k r_i w_i$ is indexed by Young diagrams $\lambda = (\lambda_1, \dots, \lambda_k)$, where $r_i = \lambda_i - \lambda_{i+1}$ for each $1 \leq i \leq k - 1$ and $r_k = 2\lambda_k$. The corresponding degree is therefore

$$D = \sum_{i=1}^k \lambda_i = \sum_{i=1}^{k-1} \left(\sum_{i \leq j \leq k-1} r_j + \frac{r_k}{2} \right) + \frac{r_k}{2} = \sum_{1 \leq j \leq k-1} j r_j + \frac{k}{2} r_k. \quad (18)$$

We will use this equality to estimate the eigenvalue given in Theorem C.2. Using the formula therein, the corresponding eigenvalue is

$$\lambda_v = -\|v + \rho\|_2^2 + \|\rho\|_2^2,$$

where here and henceforth we view the matrices v and ρ as vectors of length n^2 , using the the natural flattening, and we use the unnormalized inner product on \mathbb{R}^{n^2} (equivalently, $\langle A, B \rangle = \mathrm{Tr}(A^t B)$). Then

$$\lambda_v = -2\langle v, \rho \rangle - \|v\|_2^2.$$

It remains to estimate the norm of v and the inner product of v and ρ . To estimate the norm of v , we write

$$v = \sum_{i=1}^k w_i = \sum_{i=1}^{k-1} r_i \sum_{j=1}^i u_j + \frac{r_k}{2} \sum_{j=1}^k u_j = \sum_{j=1}^k \left(\frac{r_k}{2} + \sum_{i=j}^{k-1} r_i \right) u_j,$$

and since the u_j 's are mutually orthogonal and each has $\|u_j\|_2^2 = 2$, we obtain

$$\|v\|_2^2 = 2 \sum_{j=1}^k \left(\frac{r_k}{2} + \sum_{i=j}^{k-1} r_i \right)^2 \leq 2 \left(\sum_{j=1}^k \left(\frac{r_k}{2} + \sum_{i=j}^{k-1} r_i \right) \right)^2 = 2 \left(\sum_{j=1}^k \frac{kr_k}{2} + \sum_{i=1}^{k-1} ir_i \right)^2,$$

which is at most $2D^2$ by (18).

To bound the inner product of v and ρ , note that ρ is the vector v in which we take all the r_i 's to be 1, hence by the computation above, we have

$$\rho = \sum_{i=1}^k w_i = \sum_{j=1}^k \left(k - j + \frac{1}{2} \right) u_j,$$

and so

$$\langle \rho, v \rangle = 2 \sum_{j=1}^k \left(k - j + \frac{1}{2} \right) \left(\frac{r_k}{2} + \sum_{i=j}^{k-1} r_i \right) \leq 2k \sum_{j=1}^k \left(\frac{r_k}{2} + \sum_{i=j}^{k-1} r_i \right) \leq 2kD.$$

where in the last inequality we used (18). Overall, we see that the eigenvalue λ_v satisfies $\lambda_v \geq -2D^2 - nD$, as required.

The case of even n

Let $n = 2k$. In this case, the rank of $\text{SO}(n)$ is k , and a system of fundamental weights $\{w_1, \dots, w_k\}$ is given by $w_i = \sum_{j=1}^i u_j$ for $1 \leq i \leq k-2$, $w_{k-1} = \frac{1}{2} \sum_{j=1}^k u_j$ and $w_k = w_{k-1} - u_k$, where again we define $u_j := E_{j,k+j} - E_{k+j,j}$ for $1 \leq j \leq k$.

Consider the the equivalence class of representations corresponding to $v = \sum_{i=1}^k r_i w_i$. We now need to inspect the corresponding Young diagram to relate the r_i 's to the degree of the representation, and we recall that the corresponding Young diagram $\lambda = (\lambda_1, \dots, \lambda_k)$ maybe either have k non-zero rows or at most $k-1$ non-zero rows.

Young diagrams with at most $k-1$ rows. In this case we have $r_i = \lambda_i - \lambda_{i+1}$ for each $1 \leq i \leq k$, and so the degree is

$$D = \sum_{i=1}^k \lambda_i = \sum_{i=1}^k \sum_{j=i}^{k-1} r_j = \sum_{j=1}^{k-1} jr_j.$$

Using the same estimates as before, we obtain

$$v = \sum_{i=1}^{k-2} r_i \sum_{j=1}^i u_j + \sum_{j=1}^k \frac{r_{k-1}}{2} u_j = \sum_{j=1}^k \left(\sum_{i=j}^{k-2} r_i + \frac{r_{k-1}}{2} \right) u_j,$$

so

$$\|v\|_2^2 \leq 2 \sum_{j=1}^k \left(\sum_{i=j}^{k-2} r_i + \frac{r_{k-1}}{2} \right)^2 \leq 2 \left(\sum_{i=j}^{k-2} i r_i + \frac{k r_{k-1}}{2} \right)^2 \leq 2D^2.$$

We also obtain $\rho = \sum_{j=1}^{k-2} (k - \frac{1}{2} - j) u_j + \frac{1}{2} u_{k-1} + \frac{1}{2} u_k$, and so

$$\langle v, \rho \rangle = 2 \sum_{j=1}^{k-2} \left(k - \frac{1}{2} - j \right) \left(\sum_{i=j}^{k-2} r_i + \frac{r_{k-1}}{2} \right) + r_{k-1} \leq 2k \sum_{j=1}^{k-2} \left(\sum_{i=j}^{k-2} r_i + \frac{r_{k-1}}{2} \right) + r_{k-1} \leq 2kD + D,$$

which is at most $2nD$, as desired.

Young diagrams with k rows. In this case, we have $r_i = \lambda_i - \lambda_{i+1}$ for each $1 \leq i \leq k-2$, and (r_{k-1}, r_k) is either $(\lambda_{k-1} - \lambda_k, \lambda_{k-1} + \lambda_k)$ or $(\lambda_{k-1} + \lambda_k, \lambda_{k-1} - \lambda_k)$. The computation in both cases is similar and goes along the same lines as the computations so far, hence we deal only with the case where $(r_{k-1}, r_k) = (\lambda_{k-1} - \lambda_k, \lambda_{k-1} + \lambda_k)$.

The degree here is

$$D = \sum_{i=1}^k \lambda_i = \sum_{i=1}^{k-2} \left(\sum_{j=i}^{k-2} r_j + \frac{1}{2}(r_{k-1} + r_k) \right) = \sum_{j=1}^{k-2} j r_j + \frac{k-2}{2} r_{k-1} + \frac{k-2}{2} r_k.$$

We have the same formulae for v and ρ as before, and so

$$\|v\|_2^2 \leq 2 \sum_{j=1}^k \left(\sum_{i=j}^{k-2} r_i + \frac{r_{k-1}}{2} \right)^2 \leq 2 \left(\sum_{i=j}^{k-2} i r_i + \frac{k r_{k-1}}{2} \right)^2 \leq 2D^2,$$

and

$$\langle v, \rho \rangle = 2 \sum_{j=1}^{k-2} \left(k - \frac{1}{2} - j \right) \left(\sum_{i=j}^{k-2} r_i + \frac{r_{k-1}}{2} \right) + r_{k-1} \leq 2k \sum_{j=1}^{k-2} \left(\sum_{i=j}^{k-2} r_i + \frac{r_{k-1}}{2} \right) + r_{k-1} \leq 2kD + D,$$

which is at most $2nD$, as desired.

C.2 Bounding the eigenvalues of the Laplace–Beltrami operator in $SU(n)$

The following lemma, analogous to Lemma C.1, gives our desired bound on the eigenvalues of the Laplace–Beltrami operator in $SU(n)$.

Lemma C.3. *For $\rho \in \widehat{SU(n)}$ of level D , the corresponding eigenvalue λ_ρ of Δ satisfies*

$$\lambda_\rho \geq -C(nD + D^2),$$

where C is an absolute constant.

Proof. The proof proceeds by a similar computation to the proof of Lemma C.1. We apply Theorem C.2 in the case of $G = \text{SU}(n)$, which has rank $k = n - 1$, and use a system of fundamental weights from [7]. The system of fundamental weights $\{w_1, \dots, w_{n-1}\}$ is defined by

$$w_i = \sum_{j=1}^i e_j - \frac{i}{n} \sum_{j=1}^n e_j \quad (1 \leq i \leq n-1),$$

where $\{e_i\}_{i=1}^n$ is the standard basis of \mathbb{R}^n ; here, e_i corresponds to

$$\mathbf{i}E_{i,i}, \tag{19}$$

where $\mathbf{i} = \sqrt{-1}$ and $E_{i,j}$ is the matrix with a one in the (i, j) th-entry and zeros elsewhere, as before. For each $1 \leq k \leq l \leq n - 1$, we have

$$\langle w_k, w_l \rangle = k(n - l)/n.$$

Set $\sigma := \sum_{i=1}^{n-1} w_i$. For a partition λ whose Young diagram has less than n rows, the corresponding weight vector is

$$v = \sum_{i=1}^{n-1} a_i w_i,$$

where $a_i = \lambda_i - \lambda_{i+1}$ for all $i \in [n - 1]$ and $\lambda_n := 0$; the level D of the corresponding representation is given by

$$D = \sum_{i=1}^{n-1} a_i \min\{i, n - i\}.$$

It follows that, if $v = \sum_{k=1}^{n-1} a_k w_k$, then

$$\begin{aligned} \langle v, \sigma \rangle &= \left\langle \sum_{k=1}^{n-1} a_k w_k, \sum_{k=1}^{n-1} w_k \right\rangle \\ &= \sum_{1 \leq k \leq l \leq n-1} a_k k(n - l)/n + \sum_{1 \leq l < k \leq n-1} a_k l(n - k)/n \\ &= \sum_{k=1}^{n-1} (n - k)(n - k + 1)k a_k / (2n) + \sum_{k=1}^{n-1} k(k + 1)l(n - k)a_k / (2n) \\ &\leq nD, \end{aligned}$$

and

$$\begin{aligned} \langle v, v \rangle &= \left\langle \sum_{k=1}^{n-1} a_k w_k, \sum_{k=1}^{n-1} a_k w_k \right\rangle \\ &= \sum_{1 \leq k \leq l \leq n-1} a_k a_l k(n - l)/n + \sum_{1 \leq l < k \leq n-1} a_k a_l l(n - k)/n \\ &\leq 2D^2. \end{aligned}$$

Hence, by Theorem , the corresponding eigenvalue λ_v of Δ satisfies

$$\lambda_v = -2\langle v, \sigma \rangle - \langle \sigma, \sigma \rangle \geq -2nD - 2D^2.$$

□

C.3 Bounding the eigenvalues of the Laplace-Beltrami operator in $\mathrm{Sp}(n)$

For bounds on the eigenvalues of the Laplace-Beltrami operator of $\mathrm{Sp}(n)$, we use the following system of fundamental weights (which are a scalar multiple of those in [12]). Setting $u_j = \mathbf{i}(E_{j,j} - E_{n+j,n+j})$ for each $j \in [n]$, we let

$$w_i = \sum_{j=1}^i u_j$$

be the i th fundamental weight, for each $i \in [n]$. The irreducible representation corresponding to $v = \sum_{i=1}^n r_i w_i$ also corresponds to the Young diagram $\lambda = (\lambda_1, \dots, \lambda_n)$, where $r_i = \lambda_i - \lambda_{i+1}$ for all $1 \leq i \leq n$ and $\lambda_{n+1} := 0$; the corresponding degree is

$$D = \sum_{i=1}^n \lambda_i = \sum_{j=1}^n j r_j. \quad (20)$$

From here on, the analysis is very similar to that for $\mathrm{SO}(n)$ (for n odd). Using the formula in Theorem C.2, the eigenvalue of Δ corresponding to $v = \sum_{i=1}^n r_i w_i$ is

$$\lambda_v = -\|v + \rho\|_2^2 + \|\rho\|_2^2,$$

where here we view the matrices v and ρ as vectors of length $(2n)^2$, using the natural flattening, and we use the unnormalized inner product on $\mathbb{R}^{(2n)^2}$ (equivalently, $\langle A, B \rangle = \mathrm{Tr}(A^t B)$). Then

$$\lambda_v = -2\langle v, \rho \rangle - \|v\|_2^2.$$

It remains to estimate the norm of v and the inner product of v and ρ . To compute the norm of v , we write

$$v = \sum_{i=1}^n w_i = \sum_{i=1}^n r_i \sum_{j=1}^i u_j = \sum_{j=1}^n \left(\sum_{i=j}^n r_i \right) u_j,$$

and since the u_j 's are mutually orthogonal and each has 2-norm-squared equal to 2, we get that

$$\|v\|_2^2 = 2 \sum_{j=1}^n \left(\sum_{i=j}^n r_i \right)^2 \leq 2 \left(\sum_{j=1}^n \left(\sum_{i=j}^n r_i \right) \right)^2 = 2 \left(\sum_{j=1}^n \sum_{i=1}^n i r_i \right)^2,$$

which is at most $2D^2$ by (20).

To bound the inner product of v and ρ , note that

$$\rho = \sum_{i=1}^n w_i = \sum_{j=1}^n (n-j+1) u_j,$$

and so

$$\langle \rho, v \rangle = 2 \sum_{j=1}^n (n-j+1) \left(\sum_{i=j}^n r_i \right) \leq 2n \sum_{j=1}^n \left(\sum_{i=j}^n r_i \right) \leq 2nD.$$

where in the last inequality we used (20). Overall, we see that the eigenvalue λ_v satisfies $\lambda_v \geq -2D^2 - nD$, as desired.