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Degree 2 lower bound for Permanent in arbitrary characteristic

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Abstract

An elementary proof of quadratic lower bound for determinantal complexity of the permanent in positive characteristic is stated. This is achieved by constructing a sequence of matrices with zero permanent, but the rank of Hessian is bounded below by a degree two polynomial.

1 Introduction

The objective of this paper is to prove that $\mathtt{dc}(\mathrm{Perm}_d) \geq d(d-1)/2$ over a field of arbitrary characteristic. The stated objective is achieved by constructing a Hessian whose rank is bounded below by a quadratic polynomial. In the first section an overview of the paper is given and the results are proved in the second section.

1.1 Literature Review and Result

The determinantal complexity $\mathtt{dc}$ of a polynomial was first defined in [1, Def 1.2]. It can also be found in [2, Def 1.4],[4, p. 15, Chap. 1] or [6, Def 1.2].

1.1.1 Definition. A polynomial $f \in k[\mathrm{X}_1, \ldots, \mathrm{X}_n]$ has determinantal complexity $\mathtt{dc}(f) = m$, if there are affine linear forms $\varphi_{ij} \in k[\mathrm{X}_1, \ldots, \mathrm{X}_n], 1 \leq i, j \leq m$ such that $f = \det_m[\varphi_{ij}]$ and m is the smallest such integer.

The best known lower bound for determinantal complexity of the permanent is quadratic, as shown in [6]. It is conjectured that the determinantal complexity of the permanent grows faster than any polynomial. The reader can find the latest on this in [5].

The main result in [6] is the following.

Theorem A. *If* $\operatorname{char} k = 0$, *then* $\mathtt{dc}(\mathrm{Perm}_d) \geq d^2/2$.

The computations of Hessian in [6] cannot be carried to positive characteristic. The case for $\operatorname{char} p > 0$ is studied in [2] with the main result as the following theorem.

Theorem B. *Let p be an odd prime, then*

1. *If $p \neq 23$, then for every $d > 2$ that satisfies $p|(d+1)$, there exists a $(d+1) \times (d+1)$ matrix X_0 over finite field $\mathbb{F}_p$ such that*

$$\mathrm{Perm}(\mathrm{X}_0) \equiv 0 \mod p \text{ and } \operatorname{rank}(\mathbf{H}(\mathrm{Perm}(\mathrm{X}_0))) > (d-2)(d-3);$$

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2. *If $p \neq 3, 5$ then for every $d > 1$ that satisfies $p|(d+2)$, there exists a $(d+1) \times (d+1)$ matrix X_0 over finite field $\mathbb{F}_p$ such that*

$$\mathrm{Perm}(\mathrm{X}_0) \equiv 0 \mod p \text{ and } \mathrm{rank}(\mathbf{H}(\mathrm{Perm}(\mathrm{X}_0))) > (d-2)(d-3).$$

The above theorem implies a quadratic lower bound for $\mathtt{dc}(\mathrm{Perm}_d)$ over the field $\mathbb{F}_p$, as shown in next section.

Theorem B is reproved in this paper using elementary techniques of row and column reduction. The main result of the paper is the following.

Theorem 2.2.17. *Let k be a field of $\mathrm{char}\, 0$ or $\mathrm{char}\, p$ where p is an odd prime. Then there exists a $d \times d$ matrix A over k such that $\mathrm{Perm}(\mathrm{A}) = 0$ and the rank of Hessian of $\mathrm{Perm}(\mathrm{A})$ is at least $d^2 - d$.*

In the next section we show the strategy to link rank of hessian of the permanent with determinantal complexity.

1.2 Determinantal complexity via Hessian

Let us now give a short recap of the proof of Theorem A which relates determinantal complexity to the rank of the Hessian.

Let $\mathrm{Perm}(\mathrm{X}) = \det_m[\varphi_{ij}]$ for φ_{ij} affine linear (with m a smallest possible integer) and X a $d \times d$ matrix, that is $\mathtt{dc}(\mathrm{Perm}_d) = m$. Expanding affine linear functions around the matrix A as a sum of a linear homogeneous functions L_{ij} and a constant matrix say $[\mathrm{Y}_{ij}]$ (as in the definition of affine linear form) gives

$$\mathrm{Perm}(\mathrm{X}) = \det(\mathrm{L}_{ij}(\mathrm{X} - \mathrm{A}) + \mathrm{Y}_{ij}).$$

Taking the second derivative, that is the Hessian (denoted by $\mathbf{H}$), and multiplying with suitable matrices furnishes $\mathrm{rank}\ \mathbf{H}(\mathrm{Perm}(\mathrm{A})) \leq \mathrm{rank}\ \mathbf{H}(\det([\mathrm{Y}_{ij}]))$. The infinite sequence of matrices A is chosen such that $\mathrm{Perm}(\mathrm{A}) = 0$ but $\mathrm{rank}\ \mathbf{H}(\mathrm{Perm}(\mathrm{A})) = d^2$. On the other hand, it is shown that $\mathrm{rank}\ \mathbf{H}(\det([\mathrm{Y}_{ij}])) \leq 2m$, implying the result $d^2 \leq 2m$. Thus, we have $m \geq d^2/2$ or $\mathtt{dc}(\mathrm{Perm}_d) \geq \mathrm{rank}\ \mathbf{H}(\mathrm{Perm}(\mathrm{A}))/2$.

All the work in [6],[2] and this paper is in finding a suitable matrix A such that $\mathrm{Perm}(\mathrm{A}) = 0$ and $\mathrm{rank}\ \mathbf{H}(\mathrm{Perm}(\mathrm{A}))$ is bounded below by a quadratic polynomial. These matrices are given in the next section.

1.3 Suitable Matrices

The matrix A used in [6] is

$$\begin{pmatrix} 1-d & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}_{d \times d}.$$

The Hessian of the permanent of A has rank d^2. The entries in the Hessian of the permanent of A consist of factorials, and thus become zero modulo p. Hence, the approach of [6] fails for $\mathrm{char}\, k = p$. This defect was rectified in [2] by considering a different sequence of matrix X_0 given as

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 1 \\ 0 & 1 & 0 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & v \end{pmatrix}_{(d+1)\times(d+1)} \qquad \text{and } v \in \{1, 2\}.$$

The permanent of X_0 is a positive non zero integer. Thus, additional assumptions such as $p|d+1$ or $p|d+2$ are needed so that the permanent becomes zero modulo p. The rank of the Hessian of the permanent of X_0 is a quadratic polynomial in d as stated in Theorem B.

The matrix used in this paper is

$$\begin{pmatrix} 1-d & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 1 \end{pmatrix}_{d\times d}.$$

The Hessian of the permanent of the above matrix contains a sub matrix of size $(d^2-d)\times(d^2-d)$ of full rank. Thus, implying that the rank of the Hessian of the permanent is at least d^2-d. The sub matrix is called the truncated Hessian and the following proposition shows it is full rank.

Proposition 2.2.14. *The rank of the truncated Hessian is d^2-d.*

Although d is a positive integer, it is shown that the entries of the permanent of the truncated Hessian lie in the set $\{0, 1, -2\}$. Furthermore, row and column reduction leads to entries in the set $\{0, 1, -4\}$. Thus, the non-zero entries remain non-zero when reduced modulo p for all odd primes. Therefore, all the work is done over $\mathbb{Z}$ and the results carry over to $\mathbb{F}_p$.

The relationship between the permanent and the determinant is explained in great detail in [7] or [3]. The origins of permanent versus determinant problem can be found in [9].

1.4 Hessian

Given a polynomial f in n variables $X_1, \dots, X_n$, there is a corresponding $n\times n$ Hessian matrix $\mathbf{H}(f) := (\partial^2 f/\partial X_i \partial X_j)_{ij}$ where $1 \le i, j \le n$. For example if f is a function of two variables X, Y then the Hessian is given by the 2×2 matrix

$$\begin{pmatrix} \dfrac{\partial^2 f}{\partial X^2} & \dfrac{\partial^2 f}{\partial X \partial Y} \\ \\ \dfrac{\partial^2 f}{\partial Y \partial X} & \dfrac{\partial^2 f}{\partial Y^2} \end{pmatrix} \tag{1.4.0.1}$$

Consider the matrix below with d^2 indeterminates $\mathrm{X}_{11}, \mathrm{X}_{12}, \ldots, \mathrm{X}_{dd}$

$$\mathrm{G} := \begin{pmatrix} \mathrm{X}_{11} & \ldots & \mathrm{X}_{1d} \\ \vdots & \ddots & \vdots \\ \mathrm{X}_{d1} & \ldots & \mathrm{X}_{dd} \end{pmatrix}, \tag{1.4.0.2}$$

then Perm(G) and det(G) each have d^2 variables and thus the corresponding Hessians are each a $d^2 \times d^2$ matrix. This Hessian matrix is expressed in terms of variables $\mathrm{X}_{11}, \ldots, \mathrm{X}_{dd}$, and thus can be evaluated at these variables.

1.4.1 Notation. 1. The set $\{1, \ldots, \hat{j}, \ldots n\}$ means that the set has all elements from $1, \ldots, n$ except j, or more succinctly

$$\{1, \ldots, \hat{j}, \ldots n\} := \{1, \ldots, n\} \backslash \{j\}.$$

2. If M is a $d \times d$ matrix, then M_{ij} will denote the matrix obtained by removing row i and column j from M. Additionally, $\mathrm{M}_{ij,i'j'}$ (or $\mathrm{M}_{\{ij,i'j'\}}$) will denote the matrix obtained from M by removing two rows i, i' and two columns j, j'. Thus, M_{ij} is a $(d-1) \times (d-1)$ matrix and $\mathrm{M}_{ij,i'j'}$ is a $(d-2) \times (d-2)$ matrix.

1.4.2 Remark. $\mathrm{G}_{ij,i'j'}$ in this paper corresponds to $\mathrm{G}_{\{i,i'\},\{j,j'\}}$ in [6].

1.4.3 Notation. The Hessian in (1.4.0.1) is written as

$$\begin{array}{ccc} & \mathrm{X} & \mathrm{Y} \\ \mathrm{X} & * & * \\ \mathrm{Y} & * & * \end{array}$$

where $*$ means take the partial first with respect to variable in the row and then take partial with respect to variable in the column.

1.4.4 Hessian of the Permanent Let G be the matrix as given in (1.4.0.2), then the Hessian is a $d^2 \times d^2$ matrix.

$$\begin{array}{cccccccc} & \mathrm{X}_{11} & \mathrm{X}_{12} & \mathrm{X}_{13} & \ldots & \mathrm{X}_{i'j'} & \ldots & \mathrm{X}_{dd} \\ \mathrm{X}_{11} & \mathbf{H}_{11,11} & \mathbf{H}_{11,12} & \mathbf{H}_{11,13} & \ldots & \mathbf{H}_{11,i'j'} & \ldots & \mathbf{H}_{11,dd} \\ \vdots & \vdots & \vdots & & & & & \\ \mathrm{X}_{ij} & \mathbf{H}_{ij,11} & \mathbf{H}_{ij,12} & \mathbf{H}_{ij,13} & \ldots & \mathbf{H}_{ij,i'j'} & \ldots & \mathbf{H}_{ij,dd} \\ \vdots & \vdots & \vdots & & & & & \\ \mathrm{X}_{dd} & \mathbf{H}_{dd,11} & \mathbf{H}_{dd,12} & \mathbf{H}_{dd,13} & \ldots & \mathbf{H}_{dd,i'j'} & \ldots & \mathbf{H}_{dd,dd} \end{array} \tag{1.4.4.1}$$

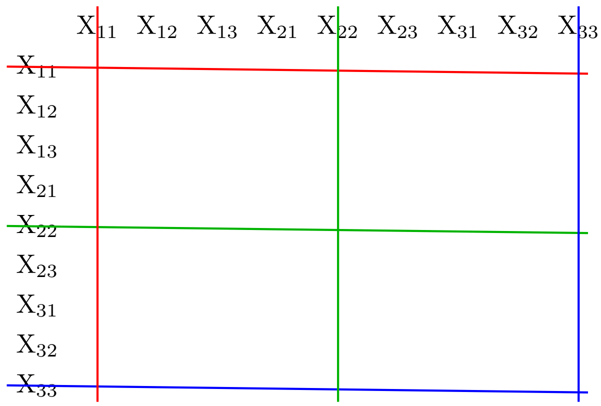
where

$$\mathbf{H}_{ij,i'j'} = \frac{\partial^2 \mathrm{Perm}(\mathrm{G})}{\partial \mathrm{X}_{ij} \partial \mathrm{X}_{i'j'}}$$

$\mathbf{H}_{ij,i'j'}$ can be computed for G via the following rules given in [6] or [2].

1. If $i \neq i'$ and $j \neq j'$ then $\mathbf{H}_{ij,i'j'} = \mathrm{Perm}(\mathrm{G}_{ij,i'j'})$.
2. If $i = i'$ or $j = j'$ then $\mathbf{H}_{ij,i'j'} = 0$.

1.4.5 Truncated Hessian of Permanent The Hessian given in (1.4.4.1) is a $d^2 \times d^2$ matrix. Removing all rows and columns indexed by the variables X_{ii} gives the *truncated Hessian*, which is a $(d^2 - d) \times (d^2 - d)$ matrix. For example if $d = 3$, then the rows and columns indexed by X_{ii} are deleted (as shown below) resulting in $3^2 - 3 = 6$ rows and columns in the truncated Hessian.



The symmetry of the deletions ensures that the truncated Hessian is still symmetric.

2 Results

2.1 Matrix A with zero Permanent

2.1.1 Definition. Let $A := [a_{ij}]_{d\times d}$ where

$$a_{ij} = \begin{cases} 1-d & \text{for } i = 1 = j \\ 1 & \text{for } i = 1 \text{ and } 2 \le j \le d \\ 1 & \text{for } j = 1 \text{ and } 2 \le i \le d. \\ 1 & \text{for } i = j \text{ and } i,j > 1 \\ 0 & \text{otherwise.} \end{cases}$$

In other words, the matrix A can be expressed as

$$\begin{pmatrix} 1-d & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Notice that deleting the first row and first column of A leads to an identity matrix A_{11} (following notation 1.4.1). The *identity submatrix* of A will always refer to A_{11}.

2.1.2 Example. If $d = 6$ the matrix A in Definition 2.1.1 is given as

$$\begin{pmatrix} -5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

2.1.3 Remark. The following two observations will be used repeatedly throughout the paper.

1. Consider the matrix below where we have added a column of all 1's to the left of an identity matrix.

$$\mathrm{R} := \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Let R_i denote the matrix obtained from R by deleting the ith column.

   (a) The matrix R_1 is just the identity matrix, thus $\mathrm{Perm}(\mathrm{R}_1) = 1$.

   (b) If $i > 1$, then a column in the identity submatrix has been deleted. Thus, R_i has a row which has all zeros except the first position which is 1.

$$\mathrm{R}_i = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 1 \end{pmatrix}, \qquad i > 1.$$

Expanding from this row gives,

$$\mathrm{Perm}(\mathrm{R}_i) = 1 \cdot \mathrm{Perm}(\text{identity matrix}) = 1.$$

2. Similarly, consider the matrix where we add a row of all 1's on top of the identity matrix. This is a transpose of the matrix R.

$$\mathrm{R}^t := \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Deleting row i is denoted by R_i^t and its permanent is 1 just as in the previous case.

2.1.4 Lemma. *The permanent of matrix* A *as defined in 2.1.1 is zero.*

Proof. The permanent of A when expanded from the first row gives

$$(1-d)\cdot \mathrm{Perm}(\mathrm{R}_1) + 1 \cdot \sum_{i\geq 2} \mathrm{Perm}(\mathrm{R}_i)$$

where R_i is defined Remark 2.1.3, and its permanent is shown to be one. Thus,

$$\begin{aligned} &(1-d)\cdot \mathrm{Perm}(\mathrm{R}_1) + 1\cdot \mathrm{Perm}(\mathrm{R}_2) + \cdots + 1\cdot \mathrm{Perm}(\mathrm{R}_d) \\ &(1-d) + \underbrace{1+\cdots+1}_{d-1 \text{ times}} = 1-d+d-1 = 0. \end{aligned} \tag{2.1.4.1}$$

□

2.2 Computations of Hessian

In this section we explore the implications of our choice of A. The Lemma below is illustrated in the Examples 2.2.5.

2.2.1 Lemma. *1. Deletion of the jth row and jth column from A_{11} gives an identity matrix.*

2. The permanent of the matrix $\mathrm{A}_{\{ij,ji\}}, i \neq j$ lies in the set $\{1,-2\}$.

1. Deleting the jth row and jth column from A_{11} gives an identity matrix of size $(d-2)\times(d-2)$ whose permanent is trivially 1.
2. (a) If $i=1$, then we are deleting the first row and the first column. The symmetry for deletion of the jth row and jth column will preserve the identity matrix giving 1 for the value of the permanent.

   (b) If $i \neq 1$, then we are keeping the first row and the first column. The symmetry for deletion of the jth row and jth column will preserve the identity matrix within A_{11}. Now expanding from the first row will give

   $$(1-d) + \underbrace{1+\cdots+1}_{1-d-2 \text{ times}} = -2.$$

   Note that deleting distinct i and j columns (and rows) has decreased the number of 1's by 2 which is reflected in the above sum.

   - The possible choices for i, j are in the set $\{2,3,\ldots,d\}$ or $\binom{d-1}{2}$ for the entries above the diagonal. For the matrix the total number of values are $2\cdot\binom{d-1}{2} = (d-1)(d-2)$. A row or column will contain only one entry of -2 which lies at the intersection of row indexed X_{ij} and column indexed X_{ji}.

2.2.2 Preserving Symmetry for only one row/column If A is a $d\times d$ matrix, then deleting row j and column j where $j>1$ again gives A but now it is $(d-1)\times(d-1)$ matrix. Next, delete row i and column k to obtain $\mathrm{A}_{\{jk,ij\}}$ (or $\mathrm{A}_{\{ij,jk\}}$) such that $i \neq k$.

1. If $i=1$, then $\mathrm{A}_{\{jk,1j\}} = \mathrm{R}_k$ which has permanent 1 by Remark 2.1.3
2. If $k=1$, then $\mathrm{A}_{\{j1,ij\}} = \mathrm{R}_i^t$ which has permanent 1 by Remark 2.1.3
3. If $i>1, k>1, i\neq k$, then deletion of row i and column k will lead to a row of all zeros in the identity submatrix with 1 on the left and a column of all zeros in the identity submatrix with 1 on the top (see

Example 2.2.3). Expanding first by the row of zeros with 1 on the left followed by expanding from column with 1 on the top and zeros below it will lead to an identity matrix giving 1 as the permanent.

$$\begin{pmatrix} 1-d & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & \dots & 0 & 1 & 0 \\ 1 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

2.2.3 Example. For example deleting row 4 and column 5 will lead to a row of zeros and a column of zeros in the identity submatrix, also highlighted in color below. First expand via the row which contains all zeros for the identity submatrix, then expand from the column which has all zeros in the identity submatrix. This yields the identity matrix which has permanent 1.

$$\begin{pmatrix} -5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \to 1 \cdot \mathrm{Perm} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to 1 \cdot \mathrm{Perm} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1$$

2.2.4 Breaking Symmetry leads to zero First recall that many of the zero entries in the Hessian come from the second rule of computation of Hessian given in Section 1.4.4. We now consider other cases which lead to a zero entry. This section is complementary to the section 2.2.2.

(2.2.4.a) **Case 1** The first row and first column are deleted, followed by deletion of the ith row and the jth column ($i \neq j$). In other words we have the matrix $\mathrm{A}_{\{11,ij\}}$ which has a row of zeros. Expanding from the row of zeros gives a zero permanent. Since, the order of deletion of the rows or columns does not matter the permanent of $\mathrm{A}_{\{1j,i1\}}$ (lying at the intersection of row X_{1j} and column X_{i1}) is also zero. Similarly for $\mathrm{A}_{\{j1,1i\}}$.

(2.2.4.b) **Case 2** All indices i, i', j, j' are distinct.

1. The first row and first column are not deleted. Two distinct rows and columns are deleted from A_{11}. After deletion there are two rows (and two columns) of all zeros coming from the identity sub matrix A_{11}. These two rows of zeros are padded with non-zero entries on the left from column 1 of A. Expanding from one such row gives

$$1 \times \mathrm{Perm}(\text{sub matrix of } \mathrm{A}_{11}) + 0 \ldots + 0.$$

Since the sub matrix of A_{11} has a row of zeros the above value is zero.

2. Either the first row or first column is deleted. Suppose, first column is deleted now deleting two distinct row and column will lead to a row of zeros which gives zero permanent. The same holds if we start by deleting a row.

2.2.5 Example. The above results can be illustrated via some examples.

1. The permanent of the matrix below is $-5+1+1+1=-2$, when expanded from the first row.

$$\begin{pmatrix} -5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

2. Deleting the 2nd row and the 3rd column from A_{11} leads to a row and a column of zeros in the sub matrix A_{11}.

   Case 1 If row 1 and column 1 are deleted, then expanding from the row of zeros gives zeros.

$$\begin{pmatrix} -5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

   Case 2 If row 1 and column 1 are not deleted, then expand from the row of zeros in A_{11}. There are two rows (and columns) consisting of all zeros in a submatrix of A_{11}. Expanding from the third row gives

$$\begin{pmatrix} -5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \to 1 \cdot \mathrm{Perm} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

   If column 1 is deleted, then we can expand from the row of zeros in sub matrix A_{11} to get a zero permanent.

$$\begin{pmatrix} \cancel{-5} & 1 & 1 & 1 & 1 & \cancel{1} \\ \cancel{1} & \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{1} & 0 & 1 & 0 & 0 & \cancel{0} \\ \cancel{1} & 0 & 0 & 1 & 0 & \cancel{0} \\ \cancel{1} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{1} & \cancel{0} \\ \cancel{1} & \color{red}{0} & \color{red}{0} & \color{red}{0} & \color{red}{0} & \cancel{1} \end{pmatrix}$$

2.2.6 Example. Let us now illustrate the results obtained until now by filling up the entries in the Hessian for $d = 4$. The non-zero entries lie at the cross section of row X_{ij} and column X_{ji}.

1. The entries shaded in red come from the second rule of computing the Hessian.
2. The entries shaded in gray come from Case 1 of Section 2.2.4.
3. The entries shaded in blue come from Case 2 of Section 2.2.4, i.e. all indices are distinct.

(2.2.6.1)

	X_{12}	X_{13}	X_{14}	X_{21}	X_{23}	X_{24}	X_{31}	X_{32}	X_{34}	X_{41}	X_{42}	X_{43}
X_{12}	0	0	0	1			0	0	0	0	0	0
X_{13}	0	0	0	0	0	0	1			0	0	0
X_{14}	0	0	0	0	0	0	0	0	0	1		
X_{21}	1	0	0	0	0	0	0		0	0		0
X_{23}		0	0	0	0	0		-2		0		0
X_{24}		0	0	0	0	0	0		0		-2	
X_{31}	0	1	0	0		0	0	0	0	0	0	
X_{32}	0		0		-2		0	0	0	0	0	
X_{34}	0		0	0		0	0	0	0			-2
X_{41}	0	0	1	0	0		0	0		0	0	0
X_{42}	0	0				-2	0	0		0	0	0
X_{43}	0	0		0	0				-2	0	0	0

The missing entries in the truncated Hessian above are all 1 as we now show.

2.2.7 Lemma. *1. The permanent of* $A_{\{1j,jk\}}$ *is* 1 *if* $j > 1, k \in \{1, \ldots, \hat{j}, \ldots, d\}$.

2. The permanent of $A_{\{j1,kj\}}$ *is* 1 *if* $j > 1, k \in \{1, \ldots, \hat{j}, \ldots, d\}$.

Proof. The condition $j > 1$ ensures that $j \neq 1$.

1. Removing row one and a column j and row j leads to the first case of Remark 2.1.3. Now removing any column gives then value of permanent as 1.
2. The symmetry of the Hessian gives the result. Alternatively, removing column one and a column j and row j leads to the second case of Remark 2.1.3. Now removing any row gives the permanent as 1.

□

2.2.8 Remark. The Lemma 2.2.7 gives the non-zero entries in rows X_{1j} or columns X_{j1}. The row corresponding to X_{1j} has $d-1$ consecutive 1's corresponding to consecutive columns X_{jk}. Since, $j \in \{1, \ldots, \hat{j}, \ldots, d\}$ there are a total of $d-1$ such rows. The same story is repeated for the first $d-1$ columns since the truncated Hessian is symmetric. All other entries in rows X_{1j} or columns X_{j1} are zero (shown in Section 2.2.4).

(2.2.8.2)
$$\begin{array}{cccccc} & X_{j1} & \ldots & X_{jk} & \ldots & X_{jd} \\ X_{1j} & 1 & \ldots & 1 & \ldots & 1 \end{array} \qquad \text{where } k \in \{1, \ldots, \hat{j}, \ldots d\}.$$

Interchanging the indices also gives other rows with $j > 1$ but with second index 1.

(2.2.8.3)
$$\begin{array}{cccccc} & X_{1j} & \ldots & X_{kj} & \ldots & X_{dj} \\ X_{j1} & 1 & \ldots & 1 & \ldots & 1 \end{array} \qquad \text{where } k \in \{1, \ldots, \hat{j}, \ldots d\}.$$

Note that the 1's in (2.2.8.2) are consecutive but it is not so in (2.2.8.3). Compare the rows labelled X_{12} and X_{21} in (2.2.13.1).

Transposing the above also gives columns.

$$\begin{array}{cc} & X_{1j} \\ X_{j1} & 1 \\ \vdots & \vdots \\ X_{jk} & 1 \\ \vdots & \vdots \\ X_{jd} & 1 \end{array} \qquad\qquad \begin{array}{cc} & X_{j1} \\ X_{1j} & 1 \\ \vdots & \vdots \\ X_{kj} & 1 \\ \vdots & \vdots \\ X_{dj} & 1 \end{array}$$

Compare the columns labelled X_{12} and X_{21} in (2.2.13.1).

The above observation is extremely important since it will allow us to carry out elementary row and column operations later.

2.2.9 Proposition. *The entries of the truncated Hessian lie in the set* $\{0, 1, -2\}$.

Proof The truncated Hessian is symmetric, thus the arguments for row vectors carry over to column vectors.
Entry -2 This comes from the permanent of $A_{\{ij,ji\}}, j \neq i$ and $j, i \in \{2, \ldots, d\}$ as proved in lemma 2.2.1.
Entry 1 There are two separate cases.

(2.2.9.c) If the initial matrix is A_{1j}, (that is the first row and column j is deleted) then the permanent of $A_{\{1j,jk\}}$ is 1 if $k \in \{1, \ldots, \hat{j}, \ldots, d\}$. This is shown in Lemma 2.2.7 and Remark 2.2.8. The symmetry ensures that the above is true for $A_{\{j1,kj\}}$. This finishes the case for the first $d-1$ rows and $d-1$ columns.

(2.2.9.d) Now Consider $A_{ij}, i > 1, j > 1$, in other words we are skipping $d-1$ rows and $d-1$ columns which have been covered above. Then, Section 2.2.2 implies the following.

1. By the symmetry for j the permanent $A_{\{ij,jk\}}$ is 1 for $k \in \{1, \ldots, \hat{i}, \hat{j}, \ldots, d\}$.
2. By the symmetry for i the permanent $A_{\{ij,\ell i\}}$ is 1 for $\ell \in \{1, \ldots, \hat{i}, \hat{j}, \ldots, d\}$.

Entry 0: All other entries will be zero in the truncated Hessian. This follows from the following

1. The second rule for computation of the Hessian (as in section 1.4.4).

2. The breaking of symmetry as explained in Section 2.2.4.

$\square$

2.2.10 Table The above result can be packaged into a table given below. The first column gives the row and column intersection.

Row/Column	Value	Reference
X_{i*}/X_{i*} or X_{*j}/X_{*j}	0	End of Section 1.4.4.
$X_{ij}/X_{ji}, i, j > 1, i \neq j$	-2	Lemma 2.2.1.
X_{jk}/X_{1j} or X_{kj}/X_{j1} or X_{j1}/X_{kj} or $X_{1j}/X_{jk}, j \neq k$	1	Lemma 2.2.7.
X_{jk}/X_{ij} or $X_{ij}/X_{jk}, i > 1, k > 1, i \neq k$	1	Remark 2.2.2.
$X_{ij}/X_{k\ell}$ all indices distinct	0	Section 2.2.4.
X_{1j}/X_{k1} or $X_{j1}/X_{1k}, j \neq k$	0	Section 2.2.4.

2.2.11 Example. Let us illustrate (2.2.9.d) for $d = 6$ for a 5×5 block. Fix $i = 2, j = 6$ and thus $k \in \{1, 3, 4, 5\} = \{1, \hat{2}, 3, 4, 5, \hat{6}\}$. The shaded row gives the entries corresponding to $A_{\{ij,jk\}}$ (except -2).

(2.2.11.1)

	X_{61}	X_{62}	X_{63}	X_{64}	X_{65}
X_{21}	0	1	0	0	0
X_{23}	0	1	0	0	0
X_{24}	0	1	0	0	0
X_{25}	0	1	0	0	0
X_{26}	1	-2	1	1	1

The entries corresponding to $A_{\{ij,\ell i\}}$ for $i = 2, j = 6$ and thus $\ell \in \{1, 3, 4, 5\}$ are given as

	X_{12}	X_{32}	X_{42}	X_{52}
X_{26}	1	1	1	1

Interchanging row and column indices above gives the non-zero column in (2.2.11.1).

	X_{62}
X_{21}	1
X_{23}	1
X_{24}	1
X_{25}	1

2.2.12 Vicinity of -2 In the proof of Lemma 2.2.1 it is shown that -2 lies at the intersection of row X_{ij} and column X_{ji} for $i, j > 1$. The row given below will be called the *row vicinity of* -2.

It is obtained from the permanent $A_{\{ij,jk\}}$ in (2.2.9.d).

	X_{j1}	$\dots$	X_{ji}	$\dots$	X_{jd}
X_{ij}	1	$\dots$	-2	$\dots$	1

Similarly, the *column vicinity of* -2 is given by

$$\begin{array}{cc} & X_{ij} \\ X_{j1} & 1 \\ \vdots & \vdots \\ X_{ji} & -2 \\ \vdots & \vdots \\ X_{jd} & 1 \end{array}$$

2.2.13 Entries of the truncated Hessian In this section non-zero entries for (2.2.6.1) are filled based upon the results obtained above.

1. The Lemma 2.2.7 shows the sequence of 1's in the first $d-1$ rows and $d-1$ columns, shaded in gray.
2. If $i \neq j$ and $i, j > 1$, the non-zero entries of the truncated Hessian below come from the vicinity of -2 which is shaded in blue.

(2.2.13.1)

	X_{12}	X_{13}	X_{14}	X_{21}	X_{23}	X_{24}	X_{31}	X_{32}	X_{34}	X_{41}	X_{42}	X_{43}
X_{12}	0	0	0	1	1	1	0	0	0	0	0	0
X_{13}	0	0	0	0	0	0	1	1	1	0	0	0
X_{14}	0	0	0	0	0	0	0	0	0	1	1	1
X_{21}	1	0	0	0	0	0	0	1	0	0	1	0
X_{23}	1	0	0	0	0	0	1	-2	1	0	1	0
X_{24}	1	0	0	0	0	0	0	1	0	1	-2	1
X_{31}	0	1	0	0	1	0	0	0	0	0	0	1
X_{32}	0	1	0	1	-2	1	0	0	0	0	0	1
X_{34}	0	1	0	0	1	0	0	0	0	1	1	-2
X_{41}	0	0	1	0	0	1	0	0	1	0	0	0
X_{42}	0	0	1	1	1	-2	0	0	1	0	0	0
X_{43}	0	0	1	0	0	1	1	1	-2	0	0	0

2.2.14 Proposition. *The rank of the truncated Hessian is* $d^2 - d$.

Proof:

1. Fix $i \geq 2$ and subtract column X_{i1} from columns $X_{ij}, j > 1$ in the truncated Hessian.
   (a) Then (2.2.9.c) ensures that the row $i-1$ will have an elementary basis vector $(0, 0, \ldots, 1, 0 \ldots, 0)$ with 1 in position X_{i1}. Thus, we have an elementary basis for the first $d-1$ rows.
   (b) The above operation ensures that all 1's in the row vicinity of -2 get converted to zero (except of course the corresponding entry in X_{i1}) and -2 becomes $-2-1=-3$.
   (c) Since the basis vector $(0, 0, \ldots, 1, 0 \ldots, 0)$ with 1 in position X_{i1} can be subtracted from each row, set the entries for the column corresponding to $X_{i1}, i > d-1$ to be zero.
2. Fix $i \geq 2$ and subtract row X_{i1} from rows $X_{ij}, j > 1$ and repeat the same procedure as above. The column vicinity will make -3 above as $-3-1=-4$.

The above operations give the span of the row space of the truncated Hessian with elementary basis vectors of the form $(0, \dots, 0, c, 0, \dots, 0), c \neq 0$.

For the row indexed by X_{ij} the unique non-zero entry in the row will lie in the column indexed X_{ji}.

1. If $i = 1$ or $j = 1$ the non-zero entry is 1.
2. If $i \neq 1$ or $j \neq 1$ the non-zero entry is -4.

Since, 1 and -4 are non zero modulo p for all odd primes, we have obtained a linearly independent basis implying full rank.

$\square$

2.2.15 Example. For $d = 4$ we obtain the following truncated Hessian. Notice that the basis elements lie at the cross-section of row X_{ij} and column X_{ji}.

	X_{12}	X_{13}	X_{14}	X_{21}	X_{23}	X_{24}	X_{31}	X_{32}	X_{34}	X_{41}	X_{42}	X_{43}
X_{12}	0	0	0	1	0	0	0	0	0	0	0	0
X_{13}	0	0	0	0	0	0	1	0	0	0	0	0
X_{14}	0	0	0	0	0	0	0	0	0	1	0	0
X_{21}	1	0	0	0	0	0	0	0	0	0	0	0
X_{23}	0	0	0	0	0	0	0	-4	0	0	0	0
X_{24}	0	0	0	0	0	0	0	0	0	0	-4	0
X_{31}	0	1	0	0	0	0	0	0	0	0	0	0
X_{32}	0	0	0	0	-4	0	0	0	0	0	0	0
X_{34}	0	0	0	0	0	0	0	0	0	0	0	-4
X_{41}	0	0	1	0	0	0	0	0	0	0	0	0
X_{42}	0	0	0	0	0	-4	0	0	0	0	0	0
X_{43}	0	0	0	0	0	0	0	0	-4	0	0	0

2.2.16 Proof Overview The first $d-1$ rows and $d-1$ columns give rise to a basis of $2(d-1)$ rows. The non-zero entries of these rows and columns are shown in Lemma 2.2.7.

In the proof of Lemma 2.2.1 it is shown that there are a total of $(d-1)(d-2)$ number of -2's. These -2's become -4's after the elementary row column operations and give the basis for $(d-1)(d-2)$ rows. Therefore, the total of non-zero entries is

$$2(d-1) + (d-1)(d-2) = (d-1)(d-2+2) = d^2 - d.$$

Python code to reproduce truncated Hessian for any d, with a visualization of reduction to basis elements is given in [8].

2.2.17 Theorem. *Let k be a field of* $\operatorname{char} 0$ *or* $\operatorname{char} p$ *where p is an odd prime. Then there exists a $d \times d$ matrix* A *over k such that* $\operatorname{Perm}(\mathrm{A}) = 0$ *and the rank of the Hessian of* $\operatorname{Perm}(\mathrm{A})$ *is at least $d^2 - d$.*

Proof. The matrix A is defined in Section 2.1 and in Lemma 2.1.4 it is shown that $\operatorname{Perm}(\mathrm{A}) = 0$. Proposition 2.2.14 gives the rank of the truncated Hessian. The theorem follows from the observation that the rank of the Hessian $\geq$ rank of the truncated Hessian. $\square$

2.2.18 Remark. Numerical analysis of the rank of the full Hessian comes out to be $d^2 - d + 2$ for all tested $3 \leq d \leq 25$.

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