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Low Acceptance Agreement Tests via Bounded-Degree Symplectic HDXs

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Abstract

We solve the derandomized direct product testing question in the low acceptance regime, by constructing new high dimensional expanders that have no small connected covers. We show that our complexes have swap cocycle expansion, which allows us to deduce the agreement theorem by relying on previous work.

Derandomized direct product testing, also known as agreement testing, is the following problem. Let X be a family of k-element subsets of $[n]$ and let $\{f_s : s \to \Sigma \mid s \in X\}$ be an ensemble of local functions, each defined over a subset $s \subset [n]$. Suppose that we run the following so-called agreement test: choose a random pair of sets $s_1, s_2 \in X$ that intersect on $\sqrt{k}$ elements, and accept if f_{s_1}, f_{s_2} agree on the elements in $s_1 \cap s_2$. We denote the success probability of this test by $\mathrm{Agree}(\{f_s\})$. Given that $\mathrm{Agree}(\{f_s\}) = \varepsilon > 0$, is there a global function $G : [n] \to \Sigma$ such that $f_s = G|_s$ for a non-negligible fraction of $s \in X$?

We construct a family X of k-subsets of $[n]$ such that $|X| = O(n)$ and such that it satisfies the low acceptance agreement theorem. Namely,

$$\mathrm{Agree}(\{f_s\}) > \varepsilon \quad \Longrightarrow \quad \exists G : [n] \to \Sigma, \quad \mathop{\mathbb{P}}_s[f_s \overset{0.99}{\approx} G|_s] \geqslant \mathrm{poly}(\varepsilon).$$

A key idea is to replace the well-studied LSV complexes by symplectic high dimensional expanders (HDXs). The family X is just the k-faces of the new symplectic HDXs. The later serve our needs better since their fundamental group satisfies the congruence subgroup property, which implies that they lack small covers.

1 Introduction

Let X be a family of k-element subsets of $[n]$ and let $\{f_s : s \to \Sigma \mid s \in X\}$ be an ensemble of local functions, each defined over a subset $s \subset [n]$. Is there a global function $G : [n] \to \Sigma$ such that $f_s = G|_s$ for all $s \in X$? An agreement test is a randomized property tester for this question. The natural two-query test, called the V-test, is as follows: choose a random pair of sets $s_1, s_2 \in X$ with prescribed intersection size ($\sqrt{k}$ in our case) and accept if f_{s_1}, f_{s_2} agree on the elements in $s_1 \cap s_2$.

Where agreement tests are concerned, there are two important parameters: the number of subsets in X and the soundness parameter ε. In this paper we are concerned with the most efficient families X, that have linearly many subsets. This is also called the "derandomized direct product testing question" (since the first works considered families X consisting of all possible $\binom{[n]}{k}$ subsets). As for the soundness parameter, the "99%" or high acceptance regime has been resolved in [DK17] using high dimensional expanders (and improved in [DD19]). In this paper we resolve the problem in the "1%" or low acceptance regime. Namely, we construct linear size complexes X for which, given any ensemble $\{f_s\}_{s \in X}$,

$$\mathrm{Agree}(\{f_s\}) > \varepsilon \quad \Longrightarrow \quad \exists G : [n] \to \Sigma, \quad \mathop{\mathbb{P}}_s[f_s \overset{0.99}{\approx} G|_s] \geqslant \mathrm{poly}(\varepsilon). \tag{1.1}$$

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In words, we show that ε agreement implies global structure. This has appeared as an open question in [DK17] and as Conjecture 1.1 in [DD23a].

For a more detailed history of the problem we refer the reader to [DD23a] and the references therein. Recent works [DD23a; BM23] have realized that high dimensional expansion alone does not suffice for a low acceptance agreement theorem. In [DD23a] it was shown that any high dimensional expander with no small covers satisfies (1.1), as long as it is also a swap cocycle expander. A similar result was shown in [BM23]. Then, in [DD23c] swap coboundary expansion was shown for the LSV complexes [LSV05]. This implied that agreement can be achieved replacing X by a small cover Y of it (see Theorem 1.4 below).

In this work we construct new high dimensional expanders with no small covers, and show that they are swap cocycle expanders, adapting techniques from [DD23c].

1.1 Results

Our main theorem is as follows,

Theorem 1.1. *For every $\varepsilon > 0$ there exists large enough $k < g$ and prime p such that the following holds. There exists an infinite family of constant degree connected g-dimensional simplicial complexes X that are finite quotients of the Bruhat-Tits building associated with $Sp(2g, \mathbb{Q}_p)$, such that the following holds. Let Σ be a finite alphabet, and let $\{f_s : s \to \Sigma \mid s \in X(k)\}$ be an ensemble of local functions on $X(k)$.*

$$Agree(\{f_s\}) > \varepsilon \quad \Longrightarrow \quad \exists G : X(0) \to \Sigma, \quad \mathbb{P}_s[f_s \approx G|_s] \geqslant \mathrm{poly}(\varepsilon). \tag{1.2}$$

In a previous work we showed [DD23a] a similar conclusion under the assumption that X has no small covers. The novel idea of the current paper is the existence of such complexes without small covers when one replaces the family of high dimensional expanders coming from buildings associated with $SL_n(\mathbb{Q}_p)$ by those associated with $Sp(2g, \mathbb{Q}_p)$.

Theorem 1.2. *Let $m \geqslant 2$ and $g \geqslant 100\sqrt{m \log(m)}$. Then for every prime p, the following holds. There exists an infinite family of connected simplicial complexes X that are finite quotients of the Bruhat-Tits building associated with $Sp(2g, \mathbb{Q}_p)$, such that every X has no connected m'-covers for any $1 < m' \leqslant m$.*

The case of $m = 2$ follows from [CL23]. The rest of the work is to adapt techniques from [DD23c] showing swap coboundary expansion (a key requirement for the agreement theorem to hold, see Definition 2.11) for buildings of symplectic type. This was previously shown for buildings associated with SL_n.

Theorem 1.3. *Let d be an integer. There is some $p_0 = p_0(d)$ such that for all primes $p > p_0$ the following holds. Let $\mathcal{S}$ be a quotient of the affine symplectic building associated with $Sp(2g, \mathbb{Q}_p)$ for $g \geqslant d^5$. Then $\mathcal{S}$ is a $(d, \exp(-O(\sqrt{d})))$-swap cocycle expander.*

We rely on the following low soundness agreement theorem from [DD23a]:

Theorem 1.4. *Let $k \in \mathbb{N}$, and let $\varepsilon > \Omega(1/\log k)$. Let $d > k$ be sufficiently large and let X be a d-dimensional high dimensional expander with sufficiently good swap-cosystolic-expansion. Let $\{f_s : s \to \Sigma \mid s \in X(k)\}$ be an ensemble of local functions on $X(k)$.*

$$Agree(\{f_s\}) > \varepsilon \quad \Longrightarrow \quad \exists Y \overset{\rho}{\twoheadrightarrow} X, \exists G : Y(0) \to \Sigma, \quad \mathbb{P}_s[f_s \text{ is explained by } G] \geqslant \mathrm{poly}(\varepsilon).$$

where $\rho : Y \to X$ is a $\ell = \mathrm{poly}(1/\varepsilon)$ covering map.

Since we construct in Theorem 1.2 complexes X with no connected m-covers, for $m \leqslant \mathrm{poly}(1/\varepsilon)$, we deduce that Y must be a collection of disjoint copies of X and this proves our main result.

We recently learned that an upcoming manuscript by Mitali Bafna, Noam Lifschitz, and Dor Minzer contains a similar result.

2 Preliminaries

Most of this section follows definitions used in previous works [DD23a], [DD23b] and [DD23c].

2.1 Local spectral expanders

A pure d-dimensional simplicial complex X is a hypergraph that consists of an arbitrary collection of sets of size $(d+1)$ together with all their subsets. The sets of size $i+1$ in X are denoted by $X(i)$. The vertices of X are denoted by $X(0)$ (we identify between a vertex v and its singleton $\{v\}$). We will sometimes omit set brackets and write for example $uvw \in X(2)$ instead of $\{u,v,w\} \in X(2)$. As a convention $X(-1) = \{\emptyset\}$. Let X be a d-dimensional simplicial complex. Let $k \leqslant d$. We denote the set of oriented k-faces in X by $\overrightarrow{X}(k) = \{(v_0, v_1, ..., v_k) \mid \{v_0, v_1, ..., v_k\} \in X(k)\}$.

For $k \leqslant d$ we denote by $X^{\leqslant k} = \bigcup_{j=-1}^{k} X(j)$ the k-skeleton of X. When $k=1$ we call this complex *the underlying graph of* X, since it consists of the vertices and edges in X (as well as the empty face).

A *clique complex* is a simplicial complex such that if $s \subseteq X(0)$ has that if s is a clique, that is, for every two vertices $v, u \in s$ the edge $vu \in X(1)$, then $s \in X$.

For a simplicial complex X we denote by $diam(X)$ the diameter of the underlying graph.

Partite Complexes

A $(d+1)$-partite d-dimensional simplicial complex is a generalization of a bipartite graph. It is a complex X such that one can decompose $X(0) = A_0 \mathbin{\dot\cup} A_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} A_d$ such that for every $s \in X(d)$ and $i \in [d]$ it holds that $|s \cap A_i| = 1$. The *color* of a vertex $col(v) = i$ such that $v \in A_i$. More generally, the color of a face s is $c = col(s) = \{col(v) \mid v \in s\}$. We denote by $X[c]$ the set of faces of color c in X, and for a singleton $\{i\}$ we sometimes write $X[i]$ instead of $X[\{i\}]$.

We also denote by X^c, for $c \subset [d+1]$, the complex induced on vertices whose colors are in c.

A join of complexes

Definition 2.1 (Join of complexes)**.** Let $S_1, S_2, \ldots, S_k$ be $\ell_1, \ell_2, \ldots, \ell_k$ dimensional complexes respectively. Let $n = \sum_{i=1}^{k} \ell_i - (k-1)$. The join $\mathcal{S} = \bigvee S_i$ is the n-dimensional complex whose faces are all the $s_1 \mathbin{\dot\cup} s_2 \mathbin{\dot\cup} \ldots s_k$ so that $s_i \in S_i$. The distribution over top-level faces is to (independently) choose $s_i \sim S_i(\ell_i)$ and output $s_1 \mathbin{\dot\cup} s_2 \mathbin{\dot\cup} \ldots s_k$.

Observe that if the S_i's are partite, then so is S. Moreover, if we restrict S to colors I so that every two colors $j_1, j_2 \in I$ come from different complexes S_i, then S^I is a complete partite complex.

Probability over simplicial complexes

Let X be a simplicial complex and let $\mathbb{P}_d : X(d) \to (0,1]$ be a density function on $X(d)$ (that is, $\sum_{s \in X(d)} \mathbb{P}_d(s) = 1$). This density function induces densities on lower level faces $\mathbb{P}_k : X(k) \to (0,1]$ by $\mathbb{P}_k(t) = \frac{1}{\binom{d+1}{k+1}} \sum_{s \in X(d), s \supset t} \mathbb{P}_d(s)$. We can also define a probability over directed faces, where we choose an ordering uniformly at random. Namely, for $s \in \overrightarrow{X}(k)$, $\mathbb{P}_k(s) = \frac{1}{(k+1)!} \mathbb{P}_k(set(s))$ (where $set(s)$ is the set of vertices participating in s). When clear from the context, we omit the level of the faces, and just write $\mathbb{P}[T]$ or $\mathbb{P}_{t \in X(k)}[T]$ for a set $T \subseteq X(k)$.

Links and local spectral expansion

Let X be a d-dimensional simplicial complex and let $s \in X$ be a face. The link of s is the $d' = d - |s|$-dimensional complex

$$X_s = \{t \setminus s \mid t \in X, t \supseteq s\}.$$

For a simplicial complex X with a measure $\mathbb{P}_d : X(d) \to (0,1]$, the induced measure on $\mathbb{P}_{d',X_s} : X_s(d-|s|) \to (0,1]$ is

$$\mathop{\mathbb{P}}_{d',X_s}(t \setminus s) = \frac{\mathbb{P}_d(t)}{\sum_{t' \supseteq s} \mathbb{P}_d(t')}.$$

We denote by $\lambda(X_s)$ to be the (normalized) second largest eigenvalue of the adjacency operator of the graph $X_s^{\leqslant 1}$. We denote by $|\lambda|(X_s)$ to be the (normalized) second largest eigenvalue of the adjacency operator of the graph $X_s^{\leqslant 1}$ *in absolute norm*.

Definition 2.2 (local spectral expander)**.** Let X be a d-dimensional simplicial complex and let $\lambda \in (0,1)$. We say that X is a λ-*one sided local spectral expander* if for every $s \in X^{\leqslant d-2}$ it holds that $\lambda(X_s) \leqslant \lambda$. We say that X is a λ-*two sided local spectral expander* if for every $s \in X^{\leqslant d-2}$ it holds that $|\lambda|(X_s) \leqslant \lambda$.

We stress that this definition includes $s = \emptyset$, which also implies that the graph $X^{\leqslant 1}$ should have a small second largest eigenvalue.

Walks on local spectral expanders

Let X be a d-dimensional simplicial complex. Let $\ell \leqslant k \leqslant d$. The (k,ℓ)-containment graph $G_{k,\ell} = G_{k,\ell}(X)$ is the bipartite graph whose vertices are $L = X(k), R = X(\ell)$ and whose edges are all (t,s) such that $t \supseteq s$. The probability of choosing such an edge is as in the complex X.

Theorem 2.3 ([KO20])**.** *Let X be a d-dimensional λ-one sided local spectral expander. Let $\ell \leqslant k \leqslant d$. Then the second largest eigenvalue of $G_{k,\ell}(X)$ is upper bounded by $\lambda(G_{k,\ell}(X)) \leqslant \frac{\ell+1}{k+1} + O(k\lambda)$.*

A related walk is the *swap walk*. Let k, ℓ, d be integers such that $\ell + k \leqslant d - 1$. The k,ℓ-swap walk $S_{k,\ell} = S_{k,\ell}(X)$ is the bipartite graph whose vertices are $L = X(k), R = X(\ell)$ and whose edges are all (t,s) such that $t \mathbin{\dot\cup} s \in X$. The probability of choosing such an edge is the probability of choosing $u \in X(k+\ell+1)$ and then uniformly at random partitioning it to $u = t \mathbin{\dot\cup} s$. This walk has been defined and studied independently by [DD19] and by [AJT19], who bounded its spectral expansion.

Theorem 2.4 ([DD19; AJT19])**.** *Let X be a λ-two sided local spectral expander. Then the second largest eigenvalue of $S_{k,\ell}(X)$ is upper bounded by $\lambda(S_{k,\ell}(X)) \leqslant (k+1)(\ell+1)\lambda$.*

For a d-partite complex and two disjoint set of colors $J_1, J_2 \subseteq [d]$ one can also define the *colored swap walk* S_{J_1,J_2} as the bipartite graph whose vertices are $LX[J_1], R = X[J_2]$. and whose edges are all (s,t) such that $t \mathbin{\dot\cup} s \in X[J_1 \mathbin{\dot\cup} J_2]$. The probability of choosing this edge is $\mathbb{P}_{X[J_1 \dot\cup J_2]}[t \mathbin{\dot\cup} s]$.

Theorem 2.5 ([DD19])**.** *Let X be a d-partite λ-one sided local spectral expander. Then the second largest eigenvalue of $S_{J_1,J_2}(X)$ is upper bounded by $\lambda(S_{J_1,J_2}(X)) \leqslant |J_1| \cdot |J_2| \cdot \lambda$.*

We note that this theorem also make sense even when $J_1 = \{i\}, J_2 = \{i'\}$, and the walk is between $X[i]$ and $X[i']$ that are subsets of the vertices.

2.2 Coboundary and Cosystolic Expansion

In this paper we focus on coboundary and cosystolic expansion on 1-cochains, with respect to non-abelian coefficients. For a more thorough introduction, we refer the reader to [DD23b].

Let X be a d-dimensional simplicial complex for $d \geqslant 2$ and let Γ be any group. For $i = -1, 0$ let $C^i(X,\Gamma) = \{f : X(i) \to \Gamma\}$. We sometimes identify $C^{-1}(X,\Gamma) \cong \Gamma$. For $i = 1, 2$ let

$$C^1(X,\Gamma) = \left\{ f : \vec{X}(1) \to \Gamma \;\middle|\; f(u,v) = f(v,u)^{-1} \right\}$$

and

$$C^2(X,\Gamma) = \left\{ f : \vec{X}(i) \to \Gamma \;\middle|\; \forall \pi \in Sym(3), (v_0,v_1,v_2) \in \vec{X}(2)\ f(v_{\pi(0)}, v_{\pi(1)}, v_{\pi(2)}) = f(v_0,v_1,v_2)^{\mathrm{sign}(\pi)} \right\}.$$

be the spaces of so-called *asymmetric* functions on edges and triangles. For $i = -1, 0, 1$ we define functions $\delta_i : C^i(X,\Gamma) \to C^{i+1}(X,\Gamma)$ by

1. $\delta_{-1} : C^{-1}(X,\Gamma) \to C^0(X,\Gamma)$ is $\delta_{-1}h(v) = h(\emptyset)$.
2. $\delta_0 : C^0(X,\Gamma) \to C^1(X,\Gamma)$ is $\delta_0 h(v,u) = h(v)h(u)^{-1}$.
3. $\delta_1 : C^1(X,\Gamma) \to C^2(X,\Gamma)$ is $\delta_1 h(v,u,w) = h(v,u)h(u,w)h(w,v)$.

Let $Id = Id_i \in C^i(X,\Gamma)$ be the function that always outputs the identity element. It is easy to check that $\delta_{i+1} \circ \delta_i h \equiv Id_{i+2}$ for all $i = -1, 0$ and $h \in C^i(X,\Gamma)$. Thus we denote by

$$Z^i(X,\Gamma) = \ker \delta_i \subseteq C^i(X,\Gamma),$$
$$B^i(X,\Gamma) = \mathrm{Im}\delta_{i-1} \subseteq C^i(X,\Gamma),$$

and have that $B^i(X,\Gamma) \subseteq Z^i(X,\Gamma)$.

Henceforth, when the dimension i of the cochain f is clear from the context we denote $\delta_i f$ by δf.

Coboundary and cosystolic expansion is a property testing notion so for this we need a notion of distance. Let $f,g \in C^i(X,\Gamma)$. Then

$$\mathrm{dist}(f,g) = \mathop{\mathbb{P}}_{s \in \vec{X}(i)} \left[f(s) \neq g(s) \right]. \tag{2.1}$$

We also denote the weight of the function $\mathrm{wt}(f) = \mathrm{dist}(f, Id)$.

We are ready to define coboundary and cosystolic expansion.

Definition 2.6 (Cosystolic expansion)**.** Let X be a d-dimensional simplicial complex for $d \geqslant 2$. Let $\beta > 0$. We say that X is a β-cosystolic expander if for every group Γ, and every $f \in C^1(X,\Gamma)$ there exists some $g \in Z^1(X,\Gamma)$ such that

$$\beta\, \mathrm{dist}(f,g) \leqslant \mathrm{wt}(\delta f). \tag{2.2}$$

In this case we denote $h^1(X) \geqslant \beta$.

Definition 2.7 (Coboundary expansion)**.** Let X be a d-dimensional simplicial complex for $d \geqslant 2$. Let $\beta > 0$. We say that X is a β-coboundary expander if it is a β-cosystolic expander and in addition $Z^1(X,\Gamma) = B^1(X,\Gamma)$ for every group Γ.

Another way of phrasing coboundary expansion is the following. If X is a β-coboundary expander, then it holds that for every $f \in C^1(X,\Gamma)$ there exists a function $h \in C^0(X,\Gamma)$ such that

$$\beta\, \mathrm{dist}(f, \delta h) \leqslant \mathrm{wt}(\delta f).$$

Although this definition of cosystolic and coboundary expansion related to such expansion over *every* group Γ, one can also consider cosystolic expansion with respect to a specific group Γ. All the results in this paper apply to all groups simultaneously, so we do not make this distinction.

Dinur and Meshulam already observed that cosystolic expansion (and coboundary expansion) is closely in fact equivalent testability of covers, which they call *cover stability* [DM22].

2.3 Covering maps

In this subsection we give a short introduction to covers and their connection to 1-cohomology. We stress that everything we state in this subsection is well known. For a more in depth discussion, see [Sur84].

Definition 2.8 (Covering map)**.** Let Y, X be simplicial complexes. We say that a map $\rho : Y(0) \to X(0)$ is a covering map if the following holds.

1. ρ is a surjective homomorphism.
2. For every $v \in X(0)$, and $(v,i) \in \rho^{-1}(\{v\})$ it holds that $\rho|_{Y_{(v,i)}} : Y_{(v,i)}(0) \to X_v(0)$ is an isomorphism.

We often denote $\rho : Y \to X$. We say that ρ is an ℓ-cover if for every $v \in X(0)$ it holds that $\left|\rho^{-1}(\{v\})\right| = \ell$. If there exists such a covering map $\rho : Y \to X$ we say that Y covers X.

Covers are intimately connected to the fundamental group. We will not define it as we will only be using the following facts about it. For a thorough definition and discussion see [Sur84].

Fact 2.9. *Let X be a connected simplicial complex and locally finite simplicial complex. Let $\pi_1(X, v_0)$ be the fundamental group of X (with $v_0 \in X(0)$ an arbitrary vertex). Then X has a connected ℓ-cover if and only if $\boldsymbol{P}_1(X, v_0)$ has a subgroup of index ℓ.*

2.4 The faces complex

Definition 2.10. Let X be a d-dimensional simplicial complex. Let $r \leqslant d$. We denote by $\mathrm{F}X$ the simplicial complex whose vertices are $\mathrm{F}X(0) = X(r)$ and whose faces are all $\left\{\{s_0, s_1, ..., s_j\} \mid s_0 \uplus s_1 \uplus \cdots \uplus s_j \in X((j+1)(r+1)-1)\right\}$.

It is easy to verify that this complex is $\left(\lfloor \frac{d+1}{r+1} \rfloor - 1\right)$-dimensional and that if X is a clique complex then so is $\mathrm{F}X$.

Let X be a d-dimensional simplicial complex, and let $r < d$. The distribution on the top-level faces of $\mathrm{F}X$ is given by the following. Let $m = \left(\lfloor \frac{d+1}{r+1} \rfloor - 1\right)$

1. Sample a d-face $t = \{v_0, v_1, \ldots, v_d\} \in X(d)$.
2. Sample $s_0, s_1, \ldots, s_m \subseteq t$ such that $|s_i| = r+1$, $s_i \cap s_j = \emptyset$ and output $\{s_0, s_1, \ldots, s_m\}$.

Definition 2.11. A simplicial complex X is said to have (β, r)-*swap coboundary (cosystolic) expansion* if $\mathrm{F}^r X$ is a β coboundary (cosystolic) expander for 1-cochains.

It is convenient to view the faces complex as a subcomplex of the following complex.

Definition 2.12 (Generalized faces complex)**.** Let X be a simplicial complex. The generalized faces complex, denoted FX, has a vertex for every $w \in X$, and a face $s = \{w_0, \ldots, w_i\} \in FX$ iff $\uplus s := w_0 \uplus w_1 \uplus \cdots \uplus w_i \in X$.

This complex is not pure so we do not define a measure over it. One can readily verify that links of the faces complex correspond to faces complexes of links in the original complex. That is,

Claim 2.13. Let $s \in FX$. Then $FX_s = F(X_{\cup s})$ where $\cup s = \bigcup_{t \in s} t$. The same holds for $\mathrm{F}^r X_s = F^r(X_{\cup s})$. □

We are therefore justified to look at generalized links of the form $FX_{\cup s}$,

Definition 2.14 (Generalized Links)**.** Let $w \in X$. We denote by $FX_w = F(X_w)$. We also denote by $\mathrm{F}X_w = \mathrm{F}X \cap FX_w$. Note that this is not necessarily a proper link of $\mathrm{F}X$.

2.4.1 Colors of a faces complex

Definition 2.15 (Simplicial homomorphism)**.** Let X, Y be two simplicial complexes. A map $\varphi : X \to Y$ is called a simplicial homomorphism if $\varphi : X(0) \to Y(0)$ is onto and for every $s = \{v_0, \ldots, v_i\} \in X(i)$, $\varphi(s) = \{\varphi(v_0), \ldots, \varphi(v_i)\} \in Y(i)$.

Claim 2.16. Let $\varphi : X \to Y$ be a simplicial homomorphism. Then there is a natural homomorphism $\varphi : FX \to FY$ given by $\varphi(\{s_0, \ldots, s_i\}) = \{\varphi(s_0), \ldots, \varphi(s_i)\}$.

Proof. Suppose $s = \{s_0, \ldots, s_i\} \in FX(i)$. By definition this means that $\uplus s \in X$ so $\varphi(\uplus s) \in Y$. But $\varphi(\uplus s) = \varphi(s_0 \uplus \cdots \uplus s_i) = \varphi(s_0) \uplus \cdots \uplus \varphi(s_i)$ (because for a simplicial homomorphism $\varphi : X \to Y$ whenever $a \uplus b \in X$, $\varphi(a \uplus b) = \varphi(a) \uplus \varphi(b) \in Y$). Thus $\{\varphi(s_0), \ldots, \varphi(s_i)\} \in Y$. □

Let $Y = \Delta_n$ be the complete complex on n vertices. Recall the definition of a partite complex and observe that X is n-partite if and only if there is a homomorphism $col : X \to \Delta_n$.

We say that a complex is n colorable if its underlying graph is n colorable, namely one can partition the vertices into n color sets such that every edge crosses between colors.

Claim 2.17. Let X be an n-colorable complex. Then $\mathrm{F}X$ is $\binom{n}{r+1}$-colorable.

We denote the set of colors of $\mathrm{F}X$ by $\mathbb{C} = \mathrm{F}\Delta(0)$ (supressing n from the notation). This is the set of all subsets of $[n]$ of size $r+1$.

Fix a set $J \subset \Delta_n$, namely $J = \{c_1, \ldots, c_m\}$ and $c_j \subset [n]$ are pairwise disjoint. Let $\mathrm{F}^J X = \{s \in FX \mid col(s) \subseteq J\}$ be the sub-complex of FX whose vertex colors are in J, so $\mathrm{F}^J X(0) = \bigcup_{j=1}^m X[c_j]$. We will be particularly interested in the case where $J \in \mathrm{F}\Delta$, namely, J consists of pairwise disjoint subsets. In this case $\mathrm{F}^J X$ is $|J|$-partite and $|J|-1$ dimensional. We abuse notation in this section allowing multiple c_j's to be empty sets. In this case $X[c_j]$ are copies of $\{\emptyset\}$, and every empty set set is in all top level faces of $\mathrm{F}^J X$.

The measure induced on the top level faces of $\mathrm{F}^J X$ is the one obtained by sampling $t \in X[\cup J]$ and partitioning it to $t = s_1 \cup s_2 \cup \cdots \cup s_m$ such that $s_i \in X[c_i]$.

Finally, throughout the paper we use the following notation. Let $J', J \subseteq \mathrm{F}\Delta$ We write $J' \leqslant J$, if $J = \{c_1, c_2, \ldots, c_m\}$ and $J' = \{c'_1, \ldots, c'_m\}$ where $c'_j \subseteq c_j$.

2.5 The SL_n-spherical and affine building

We denote by $A = A_g(\mathbb{F}_p)$ the spherical building associated with $SL_n(\mathbb{F}_p)$. The vertices of this complex are all non-trivial subspaces of $\mathbb{F}_p^g$. A face in this complex is all flags, i.e. all $\{v_0, v_1, v_{g-2}\}$ such that there exists an ordering so that $v_0 \subseteq v_1 \subseteq \cdots \subseteq v_{g-2}$.

Previous work has shown that this complex is both a local spectral expander, and a (swap) coboundary expander.

Claim 2.18 ([EK16], [DD19] for the color restriction)*.* The spherical building A_n is a $O(\frac{1}{\sqrt{q}})$-one sided local spectral expander. Moreover, $A_n^{\leqslant k}$ is a $\max\{O(\frac{1}{\sqrt{q}}), \frac{1}{d-k}\}$-two sided local spectral expander. The same holds for A_n^J for all subsets $J \subseteq [d]$.

We give some additional background on the coboundary expansion of this building and its color restrictions later in the paper.

2.6 The Symplectic Spherical Building

Let $g > 0$ be an integer, and let $V = \mathbb{F}^{2g}$. For $x, y \in \mathbb{F}_p^g$ we denote $(x, y) \in V$ the vector whose first g-coordinates are x and the last g coordinates are y. Let $\langle\rangle : V \to \mathbb{F}_p$ be the following skew-symmetric bilnear form.

$$\langle (x, y), (z, w) \rangle = x \cdot w - y \cdot z \tag{2.3}$$

where $a \cdot b = \sum_{i=1}^g a_i b_i$ is the usual inner product over $\mathbb{F}_p^g$.

A subspace $v \subseteq V$ is called *isotropic* if for every $x_1, x_2 \in v$, $\langle x_1, x_2 \rangle = 0$. By Witt's theorem (cf. [Art57, Theorem 3.10]) all maximal isotropic spaces have the same dimension. For this form, the dimension is g, and a maximal isotropic subspace is $Span(\{(e_i, 0) : i = 1, 2, \ldots g\})$ where $e_i \in \mathbb{F}_p^g$ are the standard basis vectors.

Definition 2.19. The symplectic spherical building of dimension g over $\mathbb{F}_p$ denoted $C = C_g(\mathbb{F}_p)$ is the following simplicial complex. Its vertices are all non-trivial isotropic subspaces. Its faces are all *flags* of isotropic subspaces. That is $\{v_0, v_1, \ldots, v_k\} \in C(k)$ if $v_0 \subseteq v_1 \subseteq \ldots v_k$ and every v_i is isotropic. By the fact that all maximal isotropic subspaces have the same dimension, it follows that C is a pure $g-1$-dimensional simplicial complex.

We note that C is g-partite, where $C[i] = \{v \in C \mid \; dim(v) = i\}$.

2.7 Links of the Symplectic Spherical Building

We note that the form above is non-degenerate, that is, that the function $x \mapsto \langle x, \cdot \rangle$ is an isomorphism between V and V^*, linear forms on V. We note the following property on non-degenerate linear forms which is easy to verify so we omit its proof.

Claim 2.20. Let $\langle \cdot, \cdot \rangle$ be a non-degenerate bilinear form. Let $v^\perp = \{x \in V \mid \; \forall y \in v, \langle x, y \rangle = 0\}$. Then $dim(v^\perp) = dim(V) - dim(v)$. $\square$

Let v be an isotropic subspace. The fact that v is isotropic is the same as saying that $v \subseteq v^\perp$. Let $v' = v^\perp / v$ be the quotient space. Let us define the following skew-symmetric bilinear form $\langle\rangle_{v'} : v' \to \mathbb{F}_p$ by

$$\langle [x], [x'] \rangle_{v'} = \langle x, x' \rangle$$

for any two $[x], [x'] \in v'$.

Claim 2.21. The for $\langle\rangle_{v'}$ is a well defined skew-symmetric bilinear form. Moreover, it is non-degenerate.

Proof of Claim 2.21. We need to show that the definition does not depend on choice of representatives. Namely, for every $x_1, x_2 \in [u_1]$ and $[u']$ it holds that $\langle x_1, x' \rangle = \langle x_2, x' \rangle$ (we need to show it also for representatives on the right but this just holds from skew-symmetry). Indeed, note that $x_1 - x_2 \in v$ hence $\langle x_1 - x_2, x' \rangle = 0$ as $x' \in v^{\perp}$.

Bilinearity just follows from the fact that the quotient map is linear.

Finally, let us show that this form is non-degenerate. Let $[x] \in v'$ be so that $\forall [x'] \in v'$ it holds that $\langle [x], [x'] \rangle_{v'} = 0$. By definition this implies that $\forall x' \in v^{\perp}$, $\langle x, x' \rangle = 0$. Thus $x \in (v^{\perp})^{\perp}$. As $(W^{\perp})^{\perp} = v$ it holds that $x \in v$ or equivalently $[x] = 0$[1]. $\square$

With Claim 2.21 we can understand the structure of the subset of isotropic subspaces that contains a fixed subspace $v \in C(0)$. Below we use the notation of $C_g = C_g(\mathbb{F}_p)$ supressing the field.

Proposition 2.22. *Let $t \leqslant g-1$. Let $v \in C_g[t]$. Let $\rho : V \to v'$ be the quotient. Then $\rho|_{v^{\perp}}$ induces an isomorphism between isotropic subspaces that contain v with respect to $\langle \rangle$, to isotropic subspaces in v' with respect to $\langle \rangle_{v'}$. This isomorphism takes subspaces of dimension $t+i$ to subspaces of dimension i.*

As a corollary we get a concrete description of the link of v.

Corollary 2.23.

1. *Let $t \leqslant g$ and let $v \in C_g[t]$. Then the link of v is isomorphic to $A_t \vee C_{g-t-1}$ where C_{g-t-1} is as above and A_t is the poset of* all *non-trivial subspaces of $\mathbb{F}_p^t$.*
2. *Let $w = \{v_0 \leqslant v_1 \leqslant \ldots \leqslant v_i\} \in C(i)$. Then the link of w is isomorphic to $A_{j_0} \vee A_{j_1} \vee \cdots \vee A_{j_t} \vee C_{j_{t+1}}$. Here $j_0 = dim(v_0)$, $j_{t+1} = g - dim(v_i) - 1$ and for all $i = 1, 2, \ldots, t$, $j_i = dim(v_i) - dim(v_{i-1})$.*

Proof of Proposition 2.22. The second item follows from the inducting over the first item, so we focus on proving the first item, using the fact that $(C_g)_w = ((\ldots(C_g)_{v_0})_{v_1})\ldots)_{v_i}$.

It is well known by the correspondence theorem that the poset of subspaces of V containing v and the subspaces in V/v are isomorphic and that this isomorphism sends subspaces of dimension $t+i$ to subspaces of dimension t. Thus we need to show that a subspace $u \subseteq V$ that contains v is isotropic if and only if $\rho(u)$ is isotropic (with respect to the respective inner products).

Indeed $u \supseteq v$ is isotropic if and only if for every $x, y \in u$, $\langle x, y \rangle = 0$. In particular this also holds for every $x \in v, y \in u$, so $u \subseteq v^{\perp}$. So all isotropic subspaces containing v are in $v^{\perp}$. Moreover, by definition $\rho(u) = \{[x] \in v' \mid x \in u\}$ and $\langle [x], [y] \rangle_{v'} = \langle x, y \rangle$ so u is isotropic if and only if $\rho(u)$ is. $\square$

Proof of Corollary 2.23. Let $I_1 = \{0, 1, \ldots, t-1\}, I_2 = \{t+1, t+2, \ldots, g\}$. We first note that $(C_g)_v = (C_g)_v^{I_1} \vee (C_g)_v^{I_2}$ since choosing a top level face in the link corresponds to choosing a flag contained in v and (independently) a flag that contains v and taking the union of the two flags. Clearly any subspace contained in v is itself isotropic so clearly $(C_g)_v^{I_1} \cong A_t$. Moreover, by Proposition 2.22, $(C_g)_v^{I_2} \cong C_{g-t-1}$. $\square$

2.8 The affine symplectic building

In this section we give the following facts regarding the affine symplectic building. For proofs and a more in depth discussion see e.g. [AB08].

We denote the affine symplectic building associated with the group $Sp(2g, \mathbb{Q}_p)$ by $\tilde{C} = \tilde{C}_g(\mathbb{Q}_p)$. This is an infinite simplicial complex which has the following properties.

Fact 2.24.

1. *It is pure and g dimensional.*
2. *It is $g+1$-partite.*
3. *It is connected and simply connected. In fact it is contractible.*
4. *The group $Sp(2g, \mathbb{Q}_p)$ acts simplicially and transitively on the top level faces of the building. The action preserves the colors of the vertices.*

Fact 2.25. *One can label the parts of the building $\tilde{C}[0], \tilde{C}[1], \ldots, \tilde{C}[g]$ such that the link of a vertex $v \in \tilde{C}[i]$ is isomorphic to $C_{i-1}(\mathbb{F}_p) \vee C_{g-i-1}(\mathbb{F}_p)$ i.e. the join of two symplectic spherical buildings of dimensions $i, g-i-1$.*[2]

[1] We note that $(v^{\perp})^{\perp} = v$ follows since $v \subseteq (v^{\perp})^{\perp}$ and $dim((v^{\perp})^{\perp}) = dim(V) - dim(v^{\perp}) = dim(V) - (dim(V) - dim(v)) = dim(v)$.

[2] For $i = 0$ or $i = g$ one of the complexes is empty, i.e. this is a single symplectic spherical building of dimension $g-1$.

Together with Corollary 2.23 this gives a complete description of the links of the affine symplectic building.

Corollary 2.26. *Let $w \in \tilde{C}_g(\mathbb{Q}_p)(i)$. Then the link of w is a join of at most $i+2$ complexes that are either S_nL-spherical buildings or symplectic spherical buildings.*

The following fact is also important.

Fact 2.27. *Let $\Gamma \leqslant Sp(2g, \mathbb{Q}_p)$ be a discrete and torsion free subgroup. Then the action of Γ on the vertices of $\tilde{C}_g(\mathbb{Q}_p)$ is free.*

2.9 Covers as quotients of simplicial complexes

In this subsection we give a description (and proof sketch) of a general technique of constructing topological spaces with a certain fundamental group and certain local properties using deck transformations and quotient maps. For a general and more formal setup the reader can read [Hat02, Section 1.3].

Recall that for an action of a group Γ on a set B we denote by $\Gamma \setminus B$ the set of orbits of elements in B. Below we show that in certain cases we can embed this set with a simplicial complex structure that is covered by B. We denote by $[v]$ the orbit of a vertex $v \in B(0)$.

Let B be a locally finite, connected and simply connected simplicial complex. Let Γ be a group that acts simplicially on B. We say that the action is *proper* if for every $v \in B(0)$ and $\gamma \in \Gamma \setminus \{Id\}$, $\mathrm{dist}(v, \gamma.v) \geqslant 3$.[3]

The quotient of B by Γ is the following simplicial complex $X = \Gamma \setminus B$.

$$X = \{\{[v_0], [v_1], \ldots, [v_i]\} \mid \ \{v_0, v_1, \ldots, v_i\} \in B\}\,.$$

We denote the quotient map $\rho : B \to X$ by $\rho(\{v_0, v_1, \ldots, v_i\}) = \{[v_0], [v_1], \ldots, [v_i]\}$. By definition every $\tilde{s} \in B$ maps to a face $s \in X$.

The properties of the action promise that dimension is maintained, that is, that every $\tilde{s} \in B(i)$ maps to a face $s \in X(i)$. Indeed, this follows because every $v, u \in s$ have distance one so they must be in different orbits.

Claim 2.28. Let B be as above. Then $\rho : B(0) \to X(0)$ is a covering map.

Proof. Fix $[v_0] \in B(0)$. We need to show that for every $v_0 \in [v_0]$, the restriction of ρ to the link of v_0 is a simplicial isomorphism between B_{v_0} and $X_{[v_0]}$. Fix $v_0 \in [v_0]$ as well. First, we note that indeed $\rho(B_{v_0}) \subseteq X_{[v_0]}$: for every $v_1 \in B_{v_0}(0)$, $\{v_0, v_1\} \in B(1)$ so $\{[v_0], [v_1]\} \in X(1)$ or equivalently $[v_1] \in X_{[v_0]}(0)$.

Next we show that this is a bijection. Surjectivity is because if $[v_1] \in X_{[v_0]}(0)$ then v_1 is a neighbor of some $\gamma.v_0$, and in particular, $\gamma^{-1}.v_1 \in [v_1] \cap B_{v_0}(0)$. Injectivity is due to the distance assumption. Two neighbors v_1, v_2 of v_0 have distance 2 and therefore they belong to different orbits.

Finally we claim that this is a simplicial isomorphism. For this we claim that for every $v_0 \in [v_0]$ and every $\{[v_0], [v_1], \ldots, [v_i]\} \in X(i)$ containing $[v_0]$, there exists a set $\{v_0, v_1, \ldots, v_i\} \in B(i)$ such that $\rho(\{v_0, v_1, \ldots, v_i\}) = \{[v_0], [v_1], \ldots, [v_i]\}$. Indeed, if $\{[v_0], [v_1], \ldots, [v_i]\} \in X(i)$ there is some $v_0' \in [v_0]$ and face $\{v_0', v_1', \ldots, v_i'\} \in B(i)$. There is also an element γ sending v_0' to v_0. Thus by setting $v_j = \gamma v_j'$ we have that $\{v_0, v_1, \ldots, v_i\} \in B(i)$ has $\rho(\{v_0, v_1, \ldots, v_i\}) = \{[v_0], [v_1], \ldots, [v_i]\}$.

Hence if $\{v_1, v_2, \ldots, v_i\} \in B_{v_0}(i-1)$ if and only if $\{v_0, v_1, \ldots, v_i\} \in B(i)$, which implies that $\{[v_0], [v_1], \ldots, [v_i]\} \in X(i)$ if and only if $\{[v_1], \ldots, [v_i]\} \in X_{[v_0]}(i)$. On the other direction $\{[v_1], \ldots, [v_i]\} \in X_{[v_0]}(i)$ if and only if $\{[v_0], [v_1], \ldots, [v_i]\} \in X(i)$ which by the above implies that there exists $\{v_0, v_1, \ldots, v_i\} \in B(i)$ if and only if $\{v_1, v_2, \ldots, v_i\} \in B_{v_0}(i-1)$. ◻

2.10 On the freeness of actions on complexes and groups

In the paper we construct a family of subgroups that act freely on the affine symplectic building. However in the above, we assumed something a priori stricter. That is, that the distance between every two elements in the same orbit is at least 3. In this subsection, we show that this is easily obtained when given a family of subgroups acting freely on the building.

[3] To define covers, one can weaken this requirement. For the exact details see [Hat02].

Claim 2.29. Let B be a simplicial complex, Γ a group acting simplicially on B such that the action is free. Then for every $v \in B(0)$ and $S \subseteq B(0)$, there is at most $|S|$ elements in γ satisfying $\gamma.v \in S$.

Proof. Assume that there are more elements. This implies that there exists $\gamma_1 \neq \gamma_2$ sending v to $u = \gamma_1.v = \gamma_2.v$. This implies that $\gamma_1^{-1}\gamma_2.v = v$ which contradicts the freeness of the action. □

Recall that a fundamental domain of an action is a set of representations of orbits of elements of $B(0)$.

Claim 2.30. Let B be a simplicial complex, Γ a group acting simplicially on B. Let $F \subseteq B(0)$. Then if for every $v \in F, \gamma \in \Gamma \setminus \{Id\}$, $\text{dist}(v, \gamma.v) > r$, then for every $v \in B(0)$ and $\gamma \in \Gamma \setminus \{Id\}$, $\text{dist}(v, \gamma.v) > r$.

Proof. We prove the contrapositive. Assume there exists some $v \in B(0)$ and $\gamma \in \Gamma$ such that $\text{dist}(v, \gamma.v) \leqslant r$. Let $\gamma' \in \Gamma$ such that $\gamma'.v \in F$. As γ' preserves distances, $\text{dist}(\gamma'.v, \gamma'.\gamma.v) \leqslant r$. We observe that $\gamma'.\gamma = \gamma'.\gamma\gamma'^{-1}\gamma'$ so with $u = \gamma'.v$ we have that $u \in F$ and for $\gamma'' = \gamma'.\gamma\gamma'^{-1} \neq Id$, $\text{dist}(u, \gamma''.u) \leqslant r$. □

Combining the two claims above we get the following.

Claim 2.31. Let B be a locally finite simplicial complex. Let Γ be a group acting simplicially and freely on B such that $\Gamma \setminus B$ is finite. Let $\Gamma = \Gamma_1 \geqslant \Gamma_2 \geqslant \Gamma_3 \dots$ be a chain of subgroups such that $\bigcap_{i=1}^{\infty} \Gamma_i = \{Id\}$. Then for every $r > 0$ there exists i_0 such that for every $i > i_0$, every $\gamma \in \Gamma_i \setminus \{Id\}$ and every $v \in B(0)$ it holds that $\text{dist}(v, \gamma.v) > r$.

Proof. Let F be a fundamental domain of Γ, which is finite by assumption. Thus by Claim 2.30 there is a finite set $S \subseteq \Gamma$ such that for all $v \in F$ and $\gamma \notin S$ $dist(v, \gamma.v) > r$. Let i_0 be such that for all $i > i_0$, $\Gamma_i \cap S = \{I_d\}$. Claim 2.30 implies the claim. □

3 Complexes with no small covers

In this section we will prove the Theorem 1.2.

The Bruhat-Tits building is contractible and Γ acts freely on the building. Thus by [Hat02, Proposition 1.40] the simplicial complex $X = \Gamma \setminus B$ has fundamental group Γ.

It is well known that every m-cover of X corresponds to an m-index subgroup in the fundamental group of X [Sur84]. Thus Theorem 1.2 follows directly from this proposition.

Proposition 3.1 (Main)**.** *Let $m \geqslant 2$ be an integer and let $g \geqslant 100\sqrt{m \log m}$. Then for every prime p the p-adic group $G = Sp(2g, \mathbb{Q}_p)$ has infinitely many cocompact and torsion free lattices $\{\Gamma_i \leqslant G\}_{i=1}^{\infty}$ satisfying that for every i, Γ_i has no proper subgroup of index $\leqslant m$.*

The case for $m = 2$ was proven in [CL23]. The proof here is based on the same lattices and on similar arguments.

3.1 Background

3.1.1 Profinite groups

Before we prove Theorem 3.1, let us give some necessary background.

Definition 3.2 (Profinite topology)**.** Let Γ be a finitely generated group. Its profinite topology is defined as the topology generated by the basis of open sets $\{\gamma H \mid \ \gamma \in \Gamma, H \leqslant \Gamma, [\Gamma : H] < \infty\}$.

One can verify that in this topology the multiplication and inverse operations are continuous.

Definition 3.3 (Profinite completion)**.** Let Γ be a finitely generated group. Its profinite completion is the group $\widehat{\Gamma}$, which is the topological completion of Γ with respect to the profinite topology.

An equivalent definition is to say that

$$\widehat{\Gamma} = \varprojlim \{\Gamma/N \mid \ N \trianglelefteq \Gamma, [\Gamma : N] < \infty\}.$$

This means that

$$\widehat{\Gamma} \subseteq \prod_{N \trianglelefteq \Gamma, [\Gamma:N]<\infty} \Gamma/N$$

where $(\gamma_N)_N \in \widehat{\Gamma}$ if for all $N_1 \leqslant N_2$, $\pi_{N_1,N_2}(\gamma_{N_1}) = \gamma_{N_2}$, where $\pi_{N_1,N_2} : \Gamma/N_1 \to \Gamma/N_2$ is the natural projection (see e.g. [RZ00]).

We say that a group K is profinite if it is a profinite completion of some group Γ.

There is a homomorphism $p : \Gamma \to \widehat{\Gamma}$ where $p(\gamma) = (\gamma N)_N$. This homomorphism is injective exactly when Γ is residually finite (because then for every $\gamma \neq \gamma'$ there is a normal subgroup N such that $\gamma N \neq \gamma' N$). We will only work with residually finite groups so we hence assume that $\Gamma \subseteq \widehat{\Gamma}$ (and the inclusion is via this homomorphism).

Proposition 3.4 ([LD81])**.** *Let Γ be a finitely generated, residually finite group. Then there is a bijection between the finite index subgroups of Γ, and the open subgroups in $\widehat{\Gamma}$. This bijection preserves indexes. It is*

$$H \leqslant \Gamma \mapsto \overline{H}$$

and in the inverse direction by

$$H' \leqslant \widehat{\Gamma} \mapsto H' \cap \Gamma.$$

Moreover, we note that in this case for every finite index subgroup $H \leqslant \Gamma$, $\overline{H} = \widehat{H}$.

3.1.2 Preliminary observations

The following observations are elementary but we prove them here for concreteness.

Claim 3.5.

1. A group Γ has no proper subgroup of index $\leqslant m$ if and only if the only homomorphism from Γ to $Sym(m)$ is the trivial one.
2. A profinite group K has no proper *open* subgroup of index $\leqslant m$ if and only if the only *continuous* homomorphism from K to $Sym(m)$. (equipped with the discreet topology) is the trivial one.

Proof. Let us prove the contrapositive. I.e., that there is a proper subgroup of index $\leqslant m$ if and only if there is a non-trivial homomorphism $\phi : \Gamma \to Sym(m)$. For the first item, observe that if $H \leqslant \Gamma$ is of index $\leqslant m$, then Γ acts transitively on the cosets of H. This action gives rise to a non-trivial homomorphism to $Sym(\Gamma/H)$ which is (isomorphic to) a subgroup of $Sym(m)$. In the other direction, suppose there is a non-trivial homomorphism to $Sym(m)$. Then there is an element $i \in [m]$ such that $Orb(i) = \{\phi(\gamma).i \mid \ \gamma \in \Gamma\} \neq \{i\}$. It is easy to see that the stabilizer of i, i.e. $H = \{\gamma \in \Gamma \mid \ \phi(\gamma).i = i\}$ is indeed a subgroup, and its index is the size of the orbit. In particular, this is a proper subgroup of index $\leqslant m$.

Let us move on to the second item. Note that if $H \leqslant K$ is an open subgroup, then the homomorphism above is a continuous one. To show continuity we need to show that for any $\sigma \in Sym(m)$, $\phi^{-1}(\sigma)$ is closed. $\phi^{-1}(\sigma)$ is a coset of the kernel, hence it is equivalent to show that $ker(\phi) = \phi^{-1}(Id)$ is closed. It can be verified that $ker(\phi) = \bigcap_{g \in G} g^{-1}Hg$. Let us explain why this is closed. As this is an intersection, it is enough to show that every $g^{-1}Hg$ is closed. Multiplication is continuous, if H is closed then every $g^{-1}Hg$ is also closed, so it suffices to show that H is closed. But indeed, if H is open, then its complement is a union of cosets, which are also open - thus H is closed.

For the other direction, let ϕ be the homomorphism. The $\leqslant m$-index subgroup H constructed above contains $ker(\phi)$, which is open (from continuity of ϕ). Thus H is a union of cosets of an open subgroup, which implies it is open. $\square$

Claim 3.6. Let $\{K_i\}_{i \in I}$ be profinite groups and let $K = \prod_{i \in I} K_i$. The group K has a non-trivial continuous homomorphism to $Sym(m)$ if and only if there exists $j \in I$ such that K_j has a non-trivial continuous homomorphism to $Sym(m)$.

Proof. For the first direction, observe that we can embed every K_j in K where every $k \in K_j$ corresponds to $\tilde{k} \in K$ where

$$\tilde{k}_i = \begin{cases} k & i = j \\ Id & i \neq j \end{cases}.$$

Let $T = \langle K_j \rangle_{j \in I} \subseteq K$. It is easy to see that T consists of all elements that are not the identity on a finite number of components. We note that T is dense inside K. Thus every continuous homomorphism of $Sym(m)$ that is non-trivial, must also be non-trivial on T. This implies that it must be non-trivial on one of the sets generating T, i.e. that there exists a j such that $\phi|_{K_j}$ is non-trivial.

The other direction is simple. If $\phi : K_j \to Sym(m)$ is a non-trivial homomorphism, then $\phi \circ p_j : K \to Sym(m)$ is also a continuous and non trivial homomorphism, where p_j is the projection to the j-th coordinate. □

An immediate corollary from the two claims is:

Corollary 3.7. *Let $\{K_i\}_{i \in I}$ be profinite groups and let $K = \prod_{i \in I} K_i$. Then K has a proper open subgroup of index $\leqslant m$ if and only if there exists K_j that has a proper open subgroup of index $\leqslant m$.*

Finally, we also need the notion of Frattini subgroups.

Definition 3.8 (Frattini subgroup)**.** Let K be a profinite group. Its Frattini subgroup $\Phi(K)$, is the intersection of all maximal open subgroups $M \leqslant K$.

The subgroup $\Phi(K)$ is a normal subgroup since every conjugate of a maximal open group is also a maximal open subgroup. It is also closed, since it is an intersection of closed sets (recall that every open subgroup is also closed since its complement is a union of cosets, which are themselves open).

This is the main observation we need about Frattini subgroups.

Observation 3.9. Let K be a profinite group. Then K has a proper subgroup of index $\leqslant m$ if and only if $K/\Phi(K)$ has a proper subgroup of index $\leqslant m$.

Proof. Let $L \leqslant K$ be a proper subgroup of index $\leqslant m$. It is contained in a maximal proper subgroup M with index $\leqslant m$. By definition $\Phi(K) \leqslant M \leqslant K$ and by the correspondence theorem $M/\Phi(K)$ has the same index in $K/\Phi(K)$. The other direction follows from the same argument, reversed. □

3.1.3 Quaternion Algebras

In this subsection we present without proof some classical material from the theory of quaternion algebras and arithmetic groups. For more on this and complete references see [PRR23] or [Mor01].

Definition 3.10 (Quaternion Algebra)**.** Let $\mathbb{F}$ be a field and let $a, b \in \mathbb{F}^*$. The Quaternion algebra is the $\mathbb{F}$-algebra

$$H_{a,b}(\mathbb{F}) = \left\langle 1, i, j, k \middle| i^2 = a, j^2 = b, ij = -ji = k \right\rangle.$$

When $a = b = -1$ and $\mathbb{F} = \mathbb{R}$ this is the Hamilton's Quaternion algebra. If $\mathbb{F} \subseteq L$ are two fields and $a, b \in \mathbb{F}^*$ then $H_{a,b}(\mathbb{F}) \leqslant H_{a,b}(L)$.

We will use the following facts about the Quaternion algebras.

Fact 3.11. *Let ℓ be a prime. There exists $a, b \in \mathbb{Q}^*$ such that:*

1. *For every $p \neq \ell$, $H_{a,b}(\mathbb{Q}_p) \cong M_2(\mathbb{Q}_p)$ (in which case we say the algebra* splits *over $\mathbb{Q}_p$).*
2. *$H_{a,b}(\mathbb{Q}_\ell), H_{a,b}(\mathbb{R})$ are division algebras (in which case we say the algebra* ramifies *in $\mathbb{Q}_\ell, \mathbb{R}$).*

We denote this algebra by H^ℓ.

For $\alpha = w + xi + yj + zk$ we define the involution $\overline{\alpha} = \overline{w + xi + yj + zk} = w - xi - yj - zk$. For every $\mathbb{F} \supseteq \mathbb{Q}$ we also denote the sesquilinear[4] form

$$\langle , \rangle : H^\ell(\mathbb{F})^g \times H^\ell(\mathbb{F})^g \to H^\ell(\mathbb{F});\ \langle \alpha, \beta \rangle = \sum_{t=1}^{g} \alpha_t \overline{\beta}_t.$$

With this form in mind we denote by $SU_g(H_\ell(\mathbb{F}))$, i.e.

$$SU_g(H_\ell(\mathbb{F})) = \left\{ A \in M_{g\times g}(H^\ell(\mathbb{F})) \;\middle|\; det(A) = 1 \text{ and } \forall x, y \in H^\ell(\mathbb{F})^g,\ \langle Ax, Ay\rangle = \langle x, y\rangle \right\}.$$

When ℓ and g are clear from context we just write $\mathbb{G}(\mathbb{F})$.

Remark 3.12. We note that H^ℓ is not commutative so it is not a priori clear how the determinant in the definition of $SU_g(H_\ell(\mathbb{F}))$ should be defined. For a general definition one should use the reduced norm, see [PRR23]. In the cases that are relevant for us, $H^\ell \cong M_2(\mathbb{F})$ and thus $M_{g\times g}(H^\ell) \cong M_{2g\times 2g}(\mathbb{F})$. Therefore, the definition of the reduced norm coincides with the definition of the determinant of the $2g \times 2g$ matrix.

Fact 3.13. *Let ℓ be a prime. Then*

1. *For every prime $p \neq \ell$, $SU_g(H_\ell(\mathbb{Q}_p)) \cong Sp(2g, \mathbb{Q}_p)$.*
2. *The group $SU_g(H_\ell(\mathbb{Q}_\ell))$ (respectfully $SU_g(H_\ell(\mathbb{R}))$) is a compact $\mathbb{Q}_\ell$-Lie group (respectfully $\mathbb{R}$-Lie group).*
3. *The group $SU_g(H_\ell(\mathbb{Q}_p))$ is ℓ-adic Lie group, is virtually pro-ℓ, i.e. it has an open subgroup H_1 such that every index of an open subgroup in H_1 is a power of ℓ.*
4. *There is an infinite sequence of open (and finite index) subgroups inside $H_0 = SU_g(\ell, \mathbb{Q}_\ell)$, which we denote by $H_1 \geqslant H_2 \geqslant \dots$ such that $[H_i : H_{i+1}] = \ell$. Moreover, the intersection $\bigcap_{i=1}^{\infty} H_i = \{Id\}$.*

Let us fix $g \geqslant 1$ and two primes $p \neq \ell$. Let $\Gamma_0 = \mathbb{G}_g(\ell, \mathbb{Q}_p) \cap M_{2g\times 2g}(\mathbb{Z}[\frac{1}{p}])$, then the following is known:

Fact 3.14.

1. *The group Γ_0 is a discrete cocompact lattice of $SU_g(\ell, \mathbb{Q}_p) \cong Sp(2g, \mathbb{Q}_p)$.*
2. *The profinite completion $\widehat{\Gamma_0} \cong H_1 \times \prod_{q\neq \ell,p} Sp(2g, \mathbb{Z}_q)$. Here $\mathbb{Z}_q$ are the q-adic integers.*

The second item follows from the strong approximation theorem and the congruent subgroup property of Γ_0.

3.2 Proof of Theorem 3.1

Proof. Fix some two primes p, ℓ such that $\ell > m$. Let Γ_0 be as above.

Let $\tilde{H}_i \cong H_i \times \prod_{q\neq \ell,p} Sp(2g, \mathbb{Z}_q)$ be open subgroups $\widehat{\Gamma_0}$, where H_i are the subgroups from Fact 3.13. Let $\Gamma_i = \Gamma \cap \tilde{H}_i$. By Proposition 3.4, $\Gamma_i \subseteq \Gamma_0$ is a finite index subgroup of Γ_0. In particular, by Fact 3.14 it is a discrete cocompact lattice of $Sp(2g, \mathbb{Q}_p)$. In addition, by Proposition 3.4, $\tilde{H}_i \cong \widehat{\Gamma}_i$, so instead of showing that Γ_i has no subgroups of index $\leqslant m$, we will show that $\tilde{H}_i$ has no *open* subgroups of index $\leqslant m$.

Let us fix i. As above, $\widehat{\Gamma}_i$ is an infinite product of profinite groups. By the congruence subgroup property these subgroups are torsion free. By Corollary 3.7 showing that $\tilde{H}_i$ has no open subgroup of index $\leqslant m$ is equivalent to showing that none of the groups in the product have open subgroups of index $\leqslant m$.

The group H_i is a pro ℓ-group so all subgroups must have index at least $\ell > m$. Let us consider $Sp(2g, \mathbb{Z}_q)$. Fix some $q \neq p, \ell$. Assume towards contradiction that $K = Sp(2g, \mathbb{Z}_q)$ has a subgroup of index at most m. By Observation 3.9, $K/\Phi(K)$ also has a subgroup of index at most m. By [Wei96]

$$K/\Phi(K) \cong PSp(2g, \mathbb{F}_q).$$

It is well known that $PSp(2g, \mathbb{F}_q)$ is a simple group. By Claim 3.5, if it has a non-trivial homomorphism to $Sym(m)$. The kernel of this homomorphism is a proper *normal* subgroup of index at most $m!$. But the only proper normal subgroup in $PSp(2g, \mathbb{F}_q)$ is the trivial subgroup, which has index larger than $m!$, since the order of $PSp(2g, \mathbb{F}_q)$ is $|PSp(2g, \mathbb{F}_q)| > |PSp(2g, \mathbb{F}_2)| \geqslant 2^{g^2-g-1} > m!$ for $g \geqslant 100\sqrt{m \log m}$. $\square$

[4]Hermitian form with respect to the involution $\overline{\alpha}$.

4 Expansion Properties of the Symplectic Building

We say that a complex is *symplectic like* if it is isomorphic to a link of a vertex in the affine building associated with $Sp(2g,\mathbb{Q}_p)$. In this section we show that symplectic like complexes have local spectral expansion and swap cocycle expansion (proving Theorem 1.3).

4.1 Local spectral expansion

In this subsection we fix g and denote by $\mathcal{S}$ the spherical building associated with $Sp(2g,\mathbb{F}_p)$. We prove that this building is a local spectral expander.

Theorem 4.1. *There exists $C>0$ such that the following holds for every integer g and prime power p. Let $\mathcal{S}$ be the spherical building $Sp(2g,\mathbb{F}_p)$. Let $w\in\mathcal{S}(i)$ for $i\leqslant g-2$. Then the graph between every two parts $\mathcal{S}_w[j],\mathcal{S}_w[k]$ is an $O(\frac{1}{\sqrt{p}})$-one sided spectral expander. In particular, $\mathcal{S}$ is an $O(\frac{1}{\sqrt{p}})$-one sided local spectral expander.*

One observes that by the trickle down theorem [Opp18], this implies that quotients of the affine building are also $O(\frac{1}{\sqrt{p}})$-one sided local spectral expanders, since for every vertex, its link is either a spherical building associated with one of $Sp(2g,\mathbb{F}_p)$,$SL_g(\mathbb{F}_p)$, or a join of two such buildings.

The main proposition we need is the following one.

Proposition 4.2. *Let $j<\ell$, then $\mathcal{S}[j],\mathcal{S}[\ell]$ is a $\frac{C^{\ell-j}}{\sqrt{p}^{\ell-j}}$)-spectral expander where $C>1$ is some universal constant independent of p. In particular this is a $O(\frac{1}{\sqrt{p}})$-spectral expander.*

The theorem follows quite easy from Proposition 4.2.

Proof of Theorem 4.1. Let $w\in\mathcal{S}$ and we consider $\mathcal{S}_w[j],\mathcal{S}_w[\ell]$. As we saw in Section 2.9, $\mathcal{S}_w$ is a join of lower dimensional spherical buildings. If $\mathcal{S}_w[j],\mathcal{S}_w[\ell]$ belong to different complexes with respect to the join, then the graph between the two sides is a complete bipartite graph which is a 0-one sided spectral expander. If $\mathcal{S}_w[j],\mathcal{S}_w[\ell]$ belong to a lower dimensional spherical building associated with $SL_m(\mathbb{F}_p)$, then the vertices of $\mathcal{S}_w[j]$ are (isomorphic to) subspaces of dimension j' in $\mathbb{F}_p^m$, the vertices of $\mathcal{S}_w[\ell]$ are subspaces of dimension ℓ' in $\mathbb{F}_p^m$. There is an edge between u_1 and u_2 if and only if $u_1\subseteq u_2$. It was shown by e.g. [Dik+18] that this graph is an $O(\frac{1}{\sqrt{p}})$-expander (where the constant is independent of p).

The remaining case is when $\mathcal{S}_w[j],\mathcal{S}_w[\ell]$ belong to a part in the join which is itself isomorphic to a spherical building associated with $Sp(2m,\mathbb{F}_p)$ for some $m\leqslant g$. In this case the graph is a $O(\frac{1}{\sqrt{p}})$-one sided spectral expander by Proposition 4.2.

The in particular part follows from the following standard claim. For a proof see e.g. [Dik22].

Claim 4.3. Let G be a weighted multipartite graph between parts $V_1,V_2,\dots,V_m$. Assume that the induced subgraph between every two parts is a λ-one sided spectral expander, and that for every $i\neq j$, $\mathbb{P}_{e\in E}\left[e\in E(V_i,V_j)\right]=\frac{1}{\binom{m}{2}}$. Then G is a λ-one sided spectral expander. ◻

◻

The proof of Proposition 4.2 follows from the theory developed in [Dik+18] regarding expanding posets. We give a brief discussion of the parts of the theory we need.

4.1.1 Sub posets of the Grassmann

The (n,p,d)-Grassmann poset is the poset

$$Gr(n,p,d)=\left\{u\subseteq p^n \mid \ dim(u)\leqslant d\right\},$$

where the order is by containment.

A simplicial sub-poset of $Gr(n,p,d)$ is a subset $P\subseteq Gr(n,p,d)$ such that for every $v\in P$ and $u\subseteq v$. We denote the i-dimensional subspaces in P by $P(i)$. A simplicial sub-poset is *pure* if for every $u\in P$ there exists some $v\in P(d)$ such that $u\subseteq v$.

The measure on flags in P is via sampling a uniform $v_d \in P(d)$ and then a uniform flag $\{v_0, v_1, \dots, v_d\}$ where $v_i \in P(i)$.

For every $i < j$ we consider the containment graph $C(P, i, j)$ between $P(i)$ and $P(j)$ where the probability of an edge $\{v_i, v_j\}$ is the probability of sampling a flag containing $\{v_i, v_j\}$.

Fix i. We denote the bipartite graph operator of the containment graph between $P(i)$ and $P(i+1)$ by U_i. That is, for every $f : P(i) \to \mathbb{R}$, $U_i f : P(i+1) \to \mathbb{R}$ is given by $U_i f(v) = \mathbb{E}_{u \in P(i), u \subseteq v}[f(u)]$. Denote its adjoint by D_{i+1}.

The bipartite graph operator of the containment graph between $P(i)$ and $P(j)$ is the composition $U_{j-1} \circ \dots U_{i+1} \circ U_i$. Therefore

$$\lambda_2(C(P, i, j)) \leqslant \prod_{t=i}^{j-1} \lambda_2(C(P, t, t+1)). \tag{4.1}$$

There are two natural two-step walks on $P(i)$ using these containment graphs.

1. The upper walk that chooses a pair $v, v' \in P(i)$ by choosing $u \in P(i+1)$ and then two $v, v' \subseteq u$. The graph operator for this walk is $D_{i+1}U_i$. We also denote its non-lazy version by M_i (i.e. the walk that samples v, v' conditioned on $v \neq v'$). It holds that $D_{i+1}U_i = \frac{p-1}{p^{i+1}-1} I + \left(1 - \frac{p-1}{p^{i+1}-1}\right) M_i$.

2. The lower walk is the one that chooses a pair $v, v' \in P(i)$ by choosing $u \in P(i-1)$ and then two $v, v' \supseteq u$. The graph operator for this walk is $U_i D_i$.

The following notion generalizes graph expansion to posets.

Definition 4.4 (eposet)**.** Let $\overline{\gamma} = (\gamma_0, \gamma_1, \dots)$ be a vector of non-negative numbers. A sub-poset of the Grassmann is a $\overline{\gamma}$-eposet if for every $i = 1, \dots, d-1$

$$\|M_i - U_i D_i\| \leqslant \gamma_i.$$

The following theorem is by [Dik+18].

Theorem 4.5 (Theorem 8.23 in [Dik+18])**.** *Let P be a pure d-dimensional sub-poset of the (n, p, d)-Grassmann. Then if P is a $\overline{\gamma}$-eposet then.*

$$\lambda(D_{i+1}U_i) \leqslant \sum_{t=1}^{i} \frac{1}{p^t} + \sum_{t=0}^{i} \gamma_t.$$

Work in [Dik+18] also proposes a criterion for showing γ-eposetness.

Let $w \in P(i-1)$. Its link graph P_w is the graph whose vertices are all $P_w(0) = \{w' \in P(i) \mid \ w' \supseteq w\}$. The edges are sampled by sampling some $u \in P(i+1), u \supseteq w$ and then sampling $w \subseteq w', w'' \subseteq u$ conditioned on $w' \neq w''$. We say that a poset P is a $\overline{\gamma}$-expander if for every $i = 0, 1, \dots, d$ and every $w \in P(i)$ $\lambda(P_w) \leqslant \gamma_i$.

Theorem 4.6 (Theorem 8.21 in [Dik+18])**.** *Let P be a $\overline{\gamma}$-link expander. Then P is a $\overline{\gamma}$-eposet.*

4.1.2 Proof of Proposition 4.2

Recall that the graph between the two parts is the containment graph between istropic spaces of dimension j, ℓ respectively, inside some $2g$-dimensional space V. Hence the poset P we consider is the poset of isotropic subspaces, with respect to some non-degenerate skew symmetric bilinear form.

We observe that P is a pure n-dimensional sub-poset of the $(2n, q, n)$-Grassmann poset.

Proof of Proposition 4.2. With the notation U_ℓ in Section 4.1.1. To prove the proposition it suffices to show that there exists a universal constant $C > 1$ such that $\lambda(U_\ell) \leqslant \frac{C}{\sqrt{p}}$. If we show that P is a $\bar{\gamma}$-eposet for $\gamma_\ell = \frac{C'}{p^{n-\ell}}$ then this follows from Theorem 4.5. To prove this we will show that X is a $\bar{\gamma}$-link expander and invoke Theorem 4.6.

Fix $w \in X(\ell - 1)$. By Proposition 2.22 the link graph is isomorphic to the link of $\{0\}$ in the spherical building associated with $Sp(2(n - \ell + 1), \mathbb{F}_p)$. That is, the vertices are all 1-dimensional subspaces inside a

$2(n-\ell+1)$-space (which are all subspaces since the bilinear form is skew-symmetric), and we connect two subspaces via traversing through a 2-dimensional isotropic subspace - i.e. two subspaces are connected if and only if their sub is an isotropic subspace. If $v \oplus u$ is isotropic we write $v \perp u$. Recall we denote the adjacency operator of this graph by M_0. It will be more convenient to analyze this graph once we add a self loop to every vertex, i.e. add laziness. This corresponds the graph whose matrix is $M_0' = \frac{1}{D}I + \frac{D-1}{D}M_0$ where D is the regularity of the graph. As we will see shortly, in this case $D = \Omega(p^{n-\ell})$, so $\|M_0' - M_0\| = O(\frac{1}{p^{n-\ell}})$ and we can analyze M_0' instead of M_0. We note that this is not D_0U_0, since the amount of laziness we add is much smaller. It corresponds to the number of neighbors a one-dimensional space has, and not the number of one-dimensional spaces are contained in a two-dimensional space.

Let us consider the double cover of M_0', i.e. the bipartite graph whose vertices are all $V \times \{0,1\}$ and there is an edge between (v,i) and (u,j) if $v \perp u$ and $i \neq j$. This graph is isomorphic to the containment graph of the Grassmann between subspaces of dimension 1 and $2(n-i+1)-1$, and $v \sim u$ if and only if $v \subseteq u^{\perp}$. The isomorphism is given by $(v,0) \mapsto v$ and $(u,1) \mapsto u^{\perp} = \{x \in V \mid \forall y \in u, \langle x,y \rangle = 0\}$. It is well known that this graph is an $O(1/\sqrt{p}^{2(n-\ell)}) = O(1/p^{n-\ell}) = \gamma_i$ one-sided spectral expander (see e.g. [Dik+18]).

As for the degree of every vertex, one observes from the double cover that the degree D of a subspace v (in either M_0' or the double cover), is the number of co-dimension 1 subspaces that contain v, i.e. $D = \Omega(p^{n-\ell})$. □

4.2 Coboundary and Swap coboundary expansion

In this section we prove that the symplectic spherical building is a $\Omega(1)$-coboundary expander and use this to show that quotients of the affine symplectic building are $(d, \exp(-O(\sqrt{d})))$-swap cocycle expanders.

4.2.1 Coboundary expansion machinery

We start by mentioning some machinery we need to prove coboundary expansion. Most of which is from previous works. For showing coboundary expansion of the symplectic building, we follow a similar strategy as in [DD23b]: we first prove coboundary expansion of color restrictions of the symplectic building, and then "patch them up" to show coboundary expansion of the whole complex using the following theorem.

Theorem 4.7 ([DD23b, Theorem 1.3])**.** *Let ℓ, d be integers so that $3 \leqslant \ell \leqslant d$ and let $\beta, p, \lambda \in (0,1]$. Let Γ be some group. Let X be a d-partite simplicial complex so that*

$$\mathop{\mathbb{P}}_{F \in \binom{[d]}{\ell}} \left[X^F \text{ is a } \beta\text{-coboundary expander and } \forall s \in X(0)\ X^F_s \text{ is a } \lambda\text{-spectral expander} \right] \geqslant p.$$

Then X is a coboundary expander with $h^1(X) \geqslant \frac{p(1-\lambda)\beta}{6e}$. Here $e \approx 2.71$ is Euler's number.

We note that Theorem 4.7 is proven in [DD23b] assuming that the spectral expansion of the graph is $1-\beta$. This assumption is not needed in the proof; following the same steps with a separate parameter λ gives us a bound of $h^1(X) \geqslant \frac{p(1-\lambda)\beta}{6e}$.

To show coboundary expansion of the color restrictions, our workhorse is the cone method. This method was first observed by [Gro10], later formalized by [LMM16] and further developed by [KM19; KO21] . Its generalization to non-abelian coefficients is done in [DD23c]:

Lemma 4.8 (Cones)**.** *Let X be a simplicial complex such that $Aut(X)$ is transitive on k-faces. Suppose that there exists a cone C with diameter R. Then X is a $\frac{1}{\binom{k+1}{3} \cdot R}$-coboundary expander.*

The tools mentioned above are enough for proving constant coboundary expansion of the symplectic spherical building. However, we will need some additional tools to show swap cocycle expansion.

The following theorem is a local-to-global theorem that deduces cocycle expansion of a complex from coboundary expansion of the links. The first to show such a theorem were [KKL14] (for 1-cochains) and [EK16] (for arbitrary i-cochains). The following version by [DD23b] gives a quantitatively better bound.

Theorem 4.9 ([DD23b, Theorem 1.2])**.** *Let $\beta, \lambda > 0$ and let $k > 0$ be an integer. Let X be a d-dimensional simplicial complex for $d \geqslant k+2$ and assume that X is a λ-one-sided local spectral expander. Let Γ be any group. Assume that for every vertex $v \in X(0)$, X_v is a coboundary expander and that $h^1(X_v) \geqslant \beta$. Then*

$$h^1(X) \geqslant \frac{(1-\lambda)\beta}{24} - e\lambda.$$

Here $e \approx 2.71$ is Euler's number.

The following lemma follows from the trickle-down theorem.

Lemma 4.10 ([DD23c])**.** *Let X be a $\exp(-O(i))$-high dimensional expander so that every link in X is simply connected. Then*

$$h^1(X^J) \geqslant exp(-O(i)) \cdot \min_{s \in X(i)} h^1(X^J_s).$$

We also use the following three general claims on coboundary expansion from [DD23c].

Claim 4.11. Let X be a k-partite simplicial complex, for $k \geqslant 5$. Assume that for every $s \in X[\{0,1,\dots,k-2\}]$ and every $v \in X[k-1]$, $s \cup \{v\} \in X(k-1)$. Then $h^1(X) = \Omega(1)$.

Claim 4.12 (Color Swap). For every $\ell \geqslant 4$ there is a universal constant $c_\ell > 0$ so that the following holds. Let $I, I \in \mathrm{F}\Delta(\ell-1)$ be two sets of size ℓ such that their symmetric difference $I \Delta I' = \{i, i'\}$ where $i \in I, i' \in I'$. Let X be a n-partite λ-local spectral expander for $\lambda < \frac{1}{100}$. $h^1(X^I) \geqslant c_\ell h^1(X^{I'}) \min_{v \in X[i']} h^1(X^I_v)$.

We denote by $K_{n_1,n_2,\dots,n_m}$ the complete partite complex with n_i vertices on every side.

Claim 4.13. Let $m \geqslant 5$. Let X be a m-partite simplicial complex, such that $h^1(X) \geqslant \beta$. Assume that the colored swap walk between vertices to triangles is an η-spectral expander. Then $Y = X \otimes K_{n_1,n_2,\dots,n_m}$ is a coboundary expander and $h^1(Y) \geqslant (1-O(\eta))\exp(-O(\ell))\beta$ where $\ell = |\{i \in [m] \mid \ n_i > 1\}|$.

The following claim is proven in the end of this subsection.

Claim 4.14. Let $Z = A_1 \vee A_2$ be a join of two complexes A_1, A_2 of dimensions d_1, d_2 respectively. Assume that there is a group that act on Z so that the action on $Z(d_1+d_2+1)$ is transitive. Then there exists a constant $\beta = \beta(d_1, d_2)$ such that $h^1(Z) \geqslant \frac{\beta}{diam(A_1)}$.

Finally, we need these two claims on the Sl spherical building.

Claim 4.15. Let $\mathcal{S}$ be the Sl-spherical building. Let $I \subseteq [n], |I| \geqslant 2$. Then $diam(\mathcal{S}^I_w) = O(\frac{\max I}{\max I - \min I})$.

Lemma 4.16. *Let $I = \{i_0 < i_1 < i_2 < i_3\}$ such that $i_3 > 21$ and such that $i_j - i_{j-1} \geqslant 3$. Let $\mathcal{S}$ be an Sl-spherical building. Then*

$$h^1(\mathcal{S}^I) \geqslant \exp\left(-O\left(\log\left(\frac{i_3}{i_1 - i_0}\right) \cdot \log\left(\frac{i_3}{i_1}\right)\right)\right).$$

Moving on to swap coboundary expansion, there is a theorem similar to Theorem 4.7 in faces complexes by [DD23c].

Lemma 4.17 (Color restriction for faces complex)**.** *Let X be an n-partite complex for $n \geqslant d_1^5$. Let $m \in [3, n^{0.5}/d_1]$ and let $\mathcal{J} \subseteq \mathrm{F}\Delta(m)$ be set of relative size p. Assume that for every $J \in \mathcal{J}$, $h^1(\mathrm{F}XJ) \geqslant \beta$ then $h^1(\mathrm{F}^{d_1}X) \geqslant \omega(\beta p^2)$.*

The following two propositions show give a direct bound on swap cocycle expansion in terms of coboundary expansion in links. These bounds are not strong enough for using directly but are still used as a main component in showing swap cocycle expansion.

Proposition 4.18. *Let X be a n-partite complex that is a λ-local spectral expander for $\lambda \leqslant \frac{1}{2r^2}$. Let $\ell \geqslant 5$ and let $J = \{c_1, c_2, \dots, c_\ell\}$ be a set of mutually disjoint colors $c_j \subseteq [n]$, $|c_j| \leqslant r$. Denote by $R = \sum_{j=1}^{\ell} |c_j|$. Let $\beta > 0$ and assume that for every $I = \{i_1, i_2, \dots, i_\ell\}$ such that $i_j \in c_j$ and every $w \in X^{\cup J \setminus I}$, $h^1(X^I_w) \geqslant \beta$. Then $h^1(\mathrm{F}^J X) \geqslant \beta_1^R$ for $\beta_1 = \Omega_\ell(\beta)$.*

Let $q \leqslant R$ be an integer. $\mathcal{J}_q = \mathcal{J}_q(J)$ be all the $J' = \{c'_1, c'_2, \ldots, c'_\ell\} \leqslant J$ such that $d(J') = q$. Let

$$T_q(X, J) = \min_{(J', X_w), J' \in \mathcal{J}_q, w \in X[\cup J \setminus \cup J']} \left(\max_{i_1, i_2, \ldots, i_\ell \text{ s.t. } i_j \in c'_j} \left(h^1(X_w^{\{i_1, i_2, \ldots, i_\ell\}}) \right) \right).$$

To state this explicitly, this is the largest T_q such that Player 1 is guaranteed to get when at a node (J', X_w) where $J' \in \mathcal{J}_q(J)$.

Proposition 4.19. *Let X be a partite λ-one sided local spectral expander for $\lambda \leqslant \frac{1}{2r^2}$. Let $J = \{c_1, c_2, \ldots, c_\ell\}$ and let $R = \sum_{j=1}^{\ell} |c_j|$. Then $h^1(\mathrm{F}^J X) \geqslant \prod_{q=1}^{R} \Omega_\ell(T_q(X, J))$.*

4.2.2 Well Spread colors

Let I be an ordered set. Let $c = \{i_1 < i_2 < \cdots < i_m\} \subseteq I$ be any subset. A c-bin is one of the following sets

$$B_0 = \{i \in I \mid \ i < i_1\}, B_m = \{i \in I \mid \ i > i_m\}$$

or

$$\forall j = 1, 2, \ldots, m-1 \ B_j = \left\{i \in I \,\middle|\, \ i_j < i < i_{j+1}\right\}.$$

Fix c^* as above and let $J = \{c_1, c_2, \ldots, c_m\}$ be mutually disjoint and disjoint from c^*. We say that a c-bin B_j is J-crowded if there are two distinct $c_{\ell_1}, c_{\ell_2} \in J$ such that $B_j \cap c_{\ell_1}, B_J \cap c_{\ell_2} \neq \emptyset$. If there is only a single $c_\ell \in J$ such that $B_j \cap c_\ell \neq \emptyset$ we say that B_j is J-lonely. Otherwise, if for all $c_\ell \in J$, $B_j \cap c_\ell = \emptyset$ we say that B_j is J-empty.

We define a well-spread color to have good pseudo-random properties, that is, all indices are roughly equally spaced, and interlaced with one another so that many colors will be isolated. This will facilitate the lower bounds in the next sections.

Let $J \subseteq \mathcal{C}$. Recall that $\uplus J = \bigcup_{c \in J} c$.

Definition 4.20 (Well-spread subsets of colors)**.** Let J be a set of m colors in $\mathcal{C}$. We say that J is *well-spread* if the following properties hold.

1. Every $c_1, c_2 \in J$ are disjoint.
2. Renaming the colors $0, 1, \ldots, n$ (with the usual order), for every $\ell_1, \ell_2 \in (\cup J) \cup \{0, n\}$ it holds that $|\ell_1 - \ell_2| \geqslant \frac{n}{(m(d_1+1))^3}$.
3. For every $J' \subseteq J$ of size $|J'| = 5$ and $\overline{J'} = J \setminus J'$:
   (a) Every $\cup\overline{J'}$ bin has size at most $\frac{100 n \log(d_1+1)}{(d_1+1)m}$.
   (b) For every $c \in J'$, the number of colors $i \in c$ that are in J'-crowded $\cup\overline{J'}$-bins is at most $\frac{100(d_1+1)\log(d_1+1)}{m \log m}$.
   (c) For every $c \in J'$ and every $\overline{J'}$-bin B, it holds that $|B \cap c| \leqslant 20 \frac{\log(d_1+1)}{\log m}$.

We denote by $\mathcal{J} \subset \mathrm{F}^{d_1}\Delta$ the set of well-spread color sets.

Proposition 4.21. *Let d be an integer. Let $6 \leqslant m \leqslant (d_1+1)$. The probability that m uniformly chosen colors out of n colors are* well-spread *tends to 1 as $d, n \to \infty$ so long as $d^5 \leqslant n$.*

4.2.3 Coboundary expansion of a join

Proof of Claim 4.14. By Lemma 4.8 it is enough to show that there is a cone whose diameter in $O(diam(A_1))$. We construct the cone as follows. Fix $v_1^* \in A_1$ and $v_2^* \in A_2$. Our base of the cone is v_1^*. For every $u \in A_2$ we take the path $P_u = (v_1, u)$, for every $u \in A_1$ we take the path $P_u = (v_1, v_2, u)$.

Now we consider an edge $u_1u_2 \in Z$. If $u_1u_2 \in A_2$ then the cycle $C_0 = C_{u_1u_2} = P_{u_1} \circ (u_1, u_2) \circ P_{u_2}^{-1} = (v_1, u_1, u_2, v_1)$ is a triangle in Z so we can contract it in one step to $C_1 = (v_1, u_1, v_1)$ which contracts to

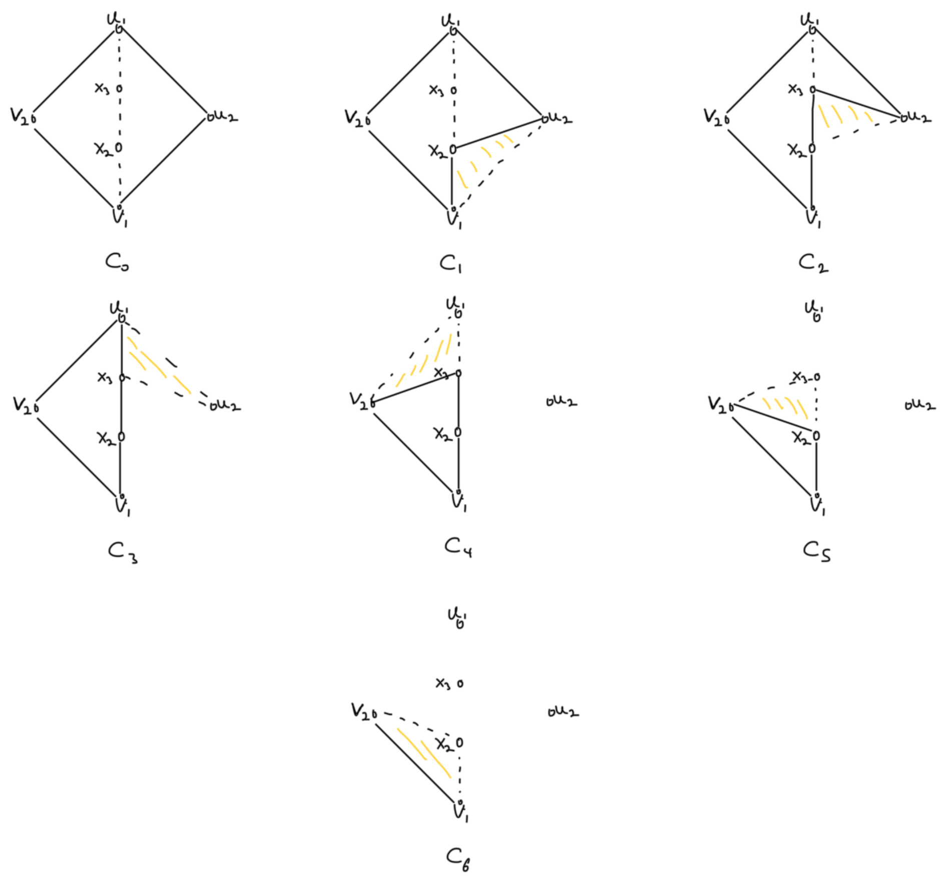

Figure 1: Contraction of the interesting case

the trivial loop using only backtrack relations. Similarly, if $u_1u_2 \in A_1$ then the loop we need to contract is $C_0 = C_{u_1u_2} = (v_1, v_2, u_1, u_2, v_2, v_1)$ which can also be contracted using a single triangle $v_2u_1u_2 \in Z$ only.

The interesting case is if (say) $u_1 \in A_1$ and $u_2 \in A_2$ so $C_0 = (v_1, u_2, u_1, v_2, v_1)$. In this case we take some shortest path in A_1 from v_1 to u_1, which we denote $Q = (v_1 = x_1, x_2, x_3, \dots, x_m = u_1)$, where $m \leqslant diam(A_1)$. From here we recommend to view Figure 1 that illustrates the contraction.We define for $i = 1, 2, \dots m-1$ $C_i = (v_1, x_2, x_3, \dots, x_i, u_2, u_1, v_2, v_1)$, where we go from C_i to C_{i+1} via the triangle $x_ix_{i+1}u_2 \in Z(2)$. And similarly we define $C_m = (v_1, x_2, \dots, x_m, v_2, v_1)$ (where again we use $x_{m-1}u_2x_m \in Z(2)$). Similarly, we define $C_{m+1}, C_{m+2}, \dots, C_{2m}$ where $C_{m+i} = (v_1, x_2, x_3, \dots, x_{m-i}, v_2, v_1)$ where we use the triangle $x_{m-i+1}v_2x_{m_i}$ to go from C_{m+i} to C_{m+i+1}. We end up with $C_{2m} = (v_1, v_2, v_1)$ which contracts to the trivial loop with a backtracking relation. Finally, note that we use $2m \leqslant 2diam(A_1)$ so the claim follows. ◻

4.3 Coboundary expansion of the symplectic spherical building

In this subsection we show that the spherical symplectic building is a $\Omega(1)$-coboundary expander for 1-cochains. As in the previous subsection, in this subsection we write $\mathcal{S}$ for the spherical building associated with $Sp(2g, \mathbb{F}_q)$.

Theorem 4.22. $h^1(\mathcal{S}) = \Omega(1)$.

We note that [LMM16] already gave a bound that depends on the rank g (for coefficients in $\mathbb{F}_2$, but the same technique applies for all coefficients). So without loss of generality we may assume that g is sufficiently large.

Proof of Theorem 4.22. We use Theorem 4.7. The colors we restrict ourselves are

$$C = \{\{i_0, i_1, i_2\} \mid \ i_1 \geqslant 2i_0, i_2 \geqslant 3i_1, 17i_1 \leqslant 2g\}.$$

We will prove below the following lemma.

Lemma 4.23. *Let $I \in C$. Then $h^1(\mathcal{S}^I) \geqslant \Omega(1)$.*

Ordering the colors of C by their size, we have that

$$C \supseteq \{\{i_0, i_1, i_2\} \mid \ i_0 \in [0, 0.01g], i_1 \in [0.02n, 0.1g], i_2 \in [0.3g, g]\}$$

so $|C| = \Omega\left(\binom{n}{3}\right)$, or equivalently $p = \frac{|C|}{\binom{n}{3}} = \Omega(1)$. Thus by Theorem 4.7 $h^1(S) = \Omega(1)$. ◻

Our main effort will be proving Lemma 4.23. We will do so using non-abelian cones. For constructing them we will need the following claim, that will imply that the diameter of $\mathcal{S}^I$ is constant.

Claim 4.24. Let $i < j$, and let G be the bipartite containment graph between $\mathcal{S}[i]$ and $\mathcal{S}[j]$. Let H be the bipartite containment graph between $\mathcal{S}'[i]$ and $\mathcal{S}'[j]$ where $\mathcal{S}'[x] = \{u \subseteq V \mid \ dim(u) = x\}$ (i.e. all subspaces, not just isotropic subspaces). Then for every $v_1, v_2 \in S[i]$ it holds that $\mathrm{dist}_G(v_1, v_2) \leqslant 2\,\mathrm{dist}_H(v_1, v_2)$. In particular, $diam(G) \leqslant 2diam(H)$.

Proof of Claim 4.24. The claim will follow if we show that for every $v_1, v_2 \in \mathcal{S}[i]$ that are of distance 2 in H have distance at most 4 in G. Indeed, let $A = \{a_1, a_2, \dots, a_\ell\}$ be a basis for $v_1 \cap v_2$. Let B_1, B_2 be such that $A \mathbin{\dot\cup} B_i$ is a basis for v_i. It follows that $A \mathbin{\dot\cup} B_1 \mathbin{\dot\cup} B_2$ is a basis for $v_1 + v_2$ that is of dimension $\leqslant j$.

Let $u_1 \in S[j]$ be such that $u_1 \supseteq v_1$, and let C be such that $(A \cup B_1) \mathbin{\dot\cup} C$ is a basis for u_1. Finally let $w = span(A \cup B_1 \cup B_2 \cup C)$.

Claim 4.25. There exists a j-dimensional isotropic subspace $u_2 \subseteq w$ such that $u_2 \supseteq v_2$.

This claim is an easy corollary from Witt's theorem, but we prove it in the end of this subsection.

Given this u_2 one can write a basis to it by $A \mathbin{\dot\cup} B_2 \mathbin{\dot\cup} D$ where every element $d_i \in D$ is a linear combination of elements in $B_1 \cup C$. We note that $A \mathbin{\dot\cup} D$ have more than i elements, since. In particular we can find some $v_3 \in \mathcal{S}[i]$ whose basis is $A \mathbin{\dot\cup} D'$ (for some $D' \subseteq D$. As we can see, v_3 is a neighbor of v_1 (through u_1) and a neighbor of v_2 (through u_2). Thus $\mathrm{dist}_H(v_1, v_2) \leqslant 4$. ◻

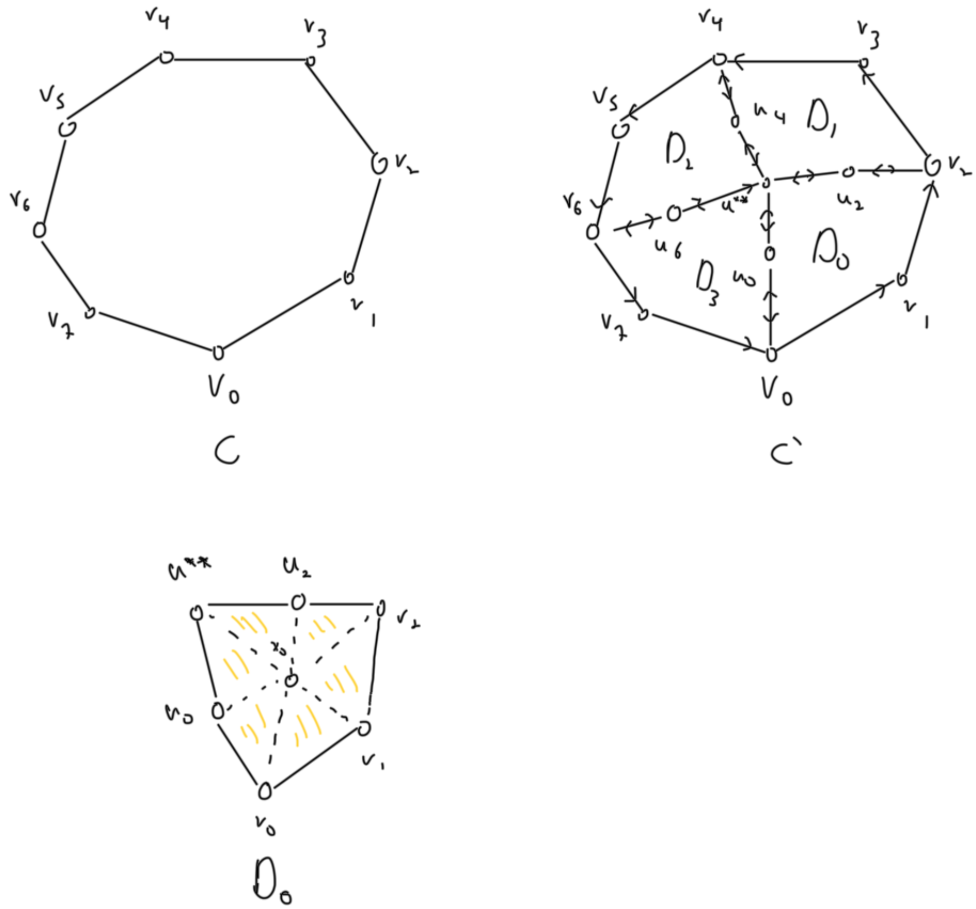

Figure 2: The contraction

Proof of Lemma 4.23. The symplectic group induces a transitive action on the triangles of $\mathcal{S}^I$ (for every $I \in C$), therefore by Lemma 4.8, if we find a constant sized cone for $\mathcal{S}^I$ we conclude that $h^1(\mathcal{S}^I) = \Omega(1)$.

We define the following cones. Fix $v_0 \in \mathcal{S}[i_0]$ as a base point.

The distance from v_0 to any other $u \in \mathcal{S}[i_0]$ is 4 from Claim 4.24. Moreover, this also implies that there exists a 5-path from v_0 to any $u \in \mathcal{S}^I$ so that for any $u' \neq u$ in this path it holds that $u' \notin \mathcal{S}[i_2]$ (we do so by going from u to some i_0-dimensional subspace of u' and then traversing from that subspace to v_0 in four steps).

Hence, for every $u \neq v_0$ we fix an arbitrary shortest path P_u from v_0 to u so that for every vertex $u' \neq u$ in the path, $u' \in \mathcal{S}[i_0] \cup \mathcal{S}[i_1]$.

Now for every $\{w_1, w_2\} \in \mathcal{S}^I(1)$ we need to define a contraction of $C = P_{w_1} \circ (w_1, w_2) \circ P_{w_2}^{-1}$ to the trivial loop around v_0.

We begin by showing how to contract C assuming that $w_1, w_2 \in \mathcal{S}[i_0] \cup \mathcal{S}[i_1]$ and then we show how to reduce from the general case to this case.

Indeed, C is a cycle of length $\leqslant 11$ with at most 5 subspaces of dimension i_1 (and these contain all subspaces of dimension i_0).

Claim 4.26. There exists an isotropic subspace $u_\perp$ of dimension $6i_1$ such that for every $x \in \bigoplus_{v \in C} v$ and $y \in u^*$, $\langle x, y \rangle = 0$.

In particular, we can find an isotropic subspace $u^* \subseteq u_\perp$ of dimension i_1 that intersects all subspaces in C trivially. Choose some (arbitrary) i_0-subspace $u^{**} \subseteq u^*$. Let us relabel $C = (v_0, v_1, v_2, v_3, \ldots, v_m, v_0)$ for $m \leqslant 11$, where $v_{2j} \in \mathcal{S}[i_0]$ and $v_{2j+1} \in \mathcal{S}[i_1]$. We note that for any v_{2j}, there exists some isotropic space $u_{2j} \in \mathcal{S}[i_1]$ so that $v_{2j}, u^{**} \subseteq u_{2j}$: we start from $u^{**} \oplus v_{2j}$ which is isotropic since u^{**}, v_{2j} are both isotropic and perpendicular to one another, and then we add independent vectors to it from u^* until getting an i_1-dimensional subspace u_{2j}.

We denote by

$$C' := (v_0, u_0, u^{**}, u_0, v_0, v_1, \tag{4.2}$$
$$v_2, u_2, u^{**}, u_2, v_2, v_3 \tag{4.3}$$
$$v_4, u_4, u^{**}, u_4, v_4, v_5, \tag{4.4}$$
$$\dots \tag{4.5}$$
$$v_m, v_0), \tag{4.6}$$

i.e. from every $v_{2j} \in C$ we add $(v_{2j}, u_{2j}, u^{**}, u_{2j}, v_{2j})$ before going to v_{2j+1}. and note that $C \overset{(BT)}{\sim} C'$ so we can contract C' instead of C. However, we note that C' (shifted to start and end at w_0) is composed from a constant number of loops of the form $D_j = (u^{**}, u_{2j}, v_{2j}, v_{2j+1}, v_{2j+2}, u_{2j+2}, u^{**})$. Thus if we can contract any such loop to the trivial loop with a constant number of steps, then we can find a contraction of all C with a constant number of steps.

Indeed, fix D_j, and note that by construction $v_{2j+1} \oplus u^* \supset \bigoplus_{u \in D_j} u$. This is because $v_{2j}, v_{2j+2} \subseteq v_{2j+1}$ and by construction, the vectors in u_{2j}, u_{2j+2} and u^{**} all lie in $v_{2j+1} \oplus u^*$ as well. We note that $u^* \oplus v_{2j+1}$ is isotropic since both subspaces are isotropic and perpendicular to one another. Thus there is an i_2-dimensional subspace x_j that contains all subspaces in D_j (there is an n-dimensional maximal isotropic space that contains $v_{2j+1} \oplus u^*$ so we take an i_2-subspace of it containing this subspace as well). Hence for every edge (a, b) in D_j the triangle $\{a, b, x_j\} \in S^I(2)$.We have shown that with a constant number of triangles D_j could be contracted to

$$(w_0, x_j, u_{2j}, x_j, v_{2j}, x_j, v_{2j+1}, x_j, v_{2j+2}, x_j, u_{2j+2}, x_j, w_0)$$

which is equivalent to the trivial loop around w_0 by a sequence of (BT) relations.

For the general case we can do the following contraction to a path that contains only subspaces from i_0, i_1 as above (we also recommend looking at Figure 3). For the case where $\{w_1, w_2\}$ is such that $w_2 \in \mathcal{S}[i_2]$ and $w_1 \in \mathcal{S}[i_0]$ we denote by w' the other neighbor of w_2 and note that $w' \in \mathcal{S}[i_0]$. Thus we can find an i_1-dimensional (isotropic) subspace $w_2' \subseteq w_2$ such that $w_2' \supseteq w_1 + w'$ (by assumption that $i_1 \geqslant 2i_0$). Thus using the triangles $\{w_1, w_2, w_2'\}, \{w', w_2, w_2'\} \in \mathcal{S}^I(2)$ we can contract (w_1, w_2, w') to (w_1, w_2', w') removing the subspace w_2 resulting in the previous case (using 2 triangles). Moreover, if $w_1 \in \mathcal{S}[i_1], w_2 \in \mathcal{S}[i_2]$ then we denote by $w'' \in \mathcal{S}[i_0]$ the other neighbor of w_1. By containment we have that $\{w'', w_1, w_2\} \in \mathcal{S}^I(2)$ so we can contract (w'', w_1, w_2) to (w'', w_2) using a single triangle. We then do the same contraction as above (using 2 triangles) to remove w_2 and get to a cycle of the same length with vertices only from i_0, i_1. □

Proof of Claim 4.25. A corollary from Witt's theorem [Art57, Theorem 3.10] says that every maximal isotropic subspace has the same dimension. We will use this corollary on the bilinear for restricted to w. In particular, $u_1 \subseteq w$ is an isotropic subspace in w of dimension j, so a maximal isotropic subspace inside w has dimension $\geqslant j$. Thus there is also an isotropic subspace $v_2 \subseteq u_2 \subseteq w$ of dimension j (that in contained in a maximal isotropic subspace that contains v_2. □

Proof of Claim 4.26. We find a basis for $u_\perp$ one vector at a time as follows. Let $B_0 = \emptyset$, and for $j = 1, 2, \dots 6i_1$ the set B_j will denote the basis vectors that we found thus far and $t_j = span(B_j) \oplus \bigoplus_{v \in C} v$. We note that $dim(t_j) \leqslant j + dim(t_0)$. Moreover, $t_0 = \bigoplus_{v \in C} v = \bigoplus_{v \in C : v \in S[i_1]} v$ since every subspace in this path is contained in an i_1-dimensional subspace. As we saw, there are at most 5 such spaces, therefore $dim(t_j) \leqslant j + 5i_1 \leqslant 11i_1$.

In particular, the subspace perpendicular to t_j always contains at least $2n - 11i_1$ independent vectors. This is greater or equal $6i_1$ from the assumption that $17i_1 \leqslant 2n$. Thus given B_j we take $B_{j+1} = B_j \cup \{x_{j+1}\}$ where x_{j+1} is perpendicular to t_j and independent from B_j. We note that by construction the vectors of B_i are perpendicular to one another, so $u_\perp = span(B_{6i_1})$ is indeed an isotropic subspace of dimension i_1. □

Figure 3: The case where $w_1 \in \mathcal{S}[i_0]$ is on the left. The case where $w_1 \in \mathcal{S}[i_1]$ is on the right.

4.3.1 Other color restrictions

Towards proving swap cocycle expansion we show that other color restrictions also have coboundary expansion. Our goal is to prove the following lemma which we will use later on in Section 4.4. Here we conitinue with the convention that $\mathcal{S}$ is the spherical symplectic building associated with $Sp(2g, \mathbb{F}_q)$.

Lemma 4.27. *Let $I = \{i_0, i_1, i_2, i_3\}$ such that $i_0 < i_1 - 79$ and $i_3 \geqslant 80$. Then*

$$h^1(\mathcal{S}^I) \geqslant \exp\left(-O\left(\log\left(\frac{i_3}{i_1 - i_0}\right)\min\left(\log(\frac{i_3 - i_0}{i_1 - i_0})\log(\frac{i_1 - i_0}{i_0})\right)\right)\right).$$

Our starting point is triples of colors as in Lemma 4.23, i.e. colors such that $i_1 \geqslant 2i_0, i2 \geqslant 3i_1$ and $17i_1 \leqslant 2g$. Then we will gradually relax the requirements from the the colors until we reach Lemma 4.27. For technical reasons, we will need to introduce a fourth color but Theorem 4.7 implies that this can be done with an expense of a constant multiple decrease only.

We start by removing the requirements from i_1.

Claim 4.28. Let $I = \{i_0, i_1, i_2\}$ be such that $80i_0 \leqslant i_2$ then

$$h^1(\mathcal{S}^I) \geqslant \Omega\left(\min\{\frac{i_1 - i_0}{i_0}, \frac{i_1}{i_2}\}\right).$$

Proof of Claim 4.28. Let $I' = \{i_0, i_1' = 2i_0, , i_2\}$. By Claim 4.12, $h^1(S^I) = \Omega(h^1(S^{I'})) \cdot \min_{v \in S[i_1']} h^1(S^I_v)$. By Lemma 4.23, $h^1(S^{I'}) = \Omega(1)$ (if $i_1' = 2i_0 \leqslant 40i_2$ then in particular $17i_1' \leqslant 2g$ so the lemma applies in this case). As for S^I_v, note that this is join of the complex whose vertices are subspaces that are contained in v, and isotropic subspaces that contain v. If $i_1' < i_1$ then by Claim 4.24 and Claim 4.15 the complex that has isotropic spaces that contain v has diameter $O(\frac{i_2 - i_1'}{i_2 - i_1}) = O(\frac{i_2}{i_1})$. Otherwise, the complex that has (all) subspaces contained in v has diameter $O(\frac{i_0}{i_1 - i_0})$: Indeed, this complex contains all subspaces of dimensions i_0, i_1 inside a space of dimension $i_1' = 2i_0$. This is isomorphic to the graph of subspaces of dimensions $2i_0 - i_1, 2i_0 - i_0$ inside a space of dimension $2i_0$, the isomorphism goes from a subspace of dimension i_j to its annahilator with respect to the standard bilinear form. Thus its diameter is $O(\frac{i_0}{i_1 - i_0})$.

In both cases the diameter is at most the maximum between the two expressions. By Claim 4.14, $h^1(\mathcal{S}^I_v) = \Omega(\min\{\frac{i_0}{i_1 - i_0}, \frac{i_1}{i_2}\})$. The claim follows. □

Using Theorem 4.7 the following corollary is immediate.

Corollary 4.29. *Let $I = \{i_0, i_1, i_2, i_3\}$ be such that $80i_0 \leqslant i_3$ then*

$$h^1(S^I) \geqslant \Omega\left(\min\{\frac{i_1 - i_0}{i_0}, \frac{i_1}{i_3}\}\right).$$

Our main effort in going from Corollary 4.29 to Lemma 4.23 is to relax the inequality on i_0. To do so we will apply Claim 4.12 multiple times to go from i_0 to i_0', to i_0'' etc. until finally we reach a complex that we can use Corollary 4.29 on. Given i_0, we would like the i_0' we choose to use Claim 4.12 on to be as small as possible. It will be apparent in the proof of Lemma 4.23 that the following function $T(x)$ will give us the smallest possible index.

Let $T(x) = T_{i_3}(x) = \max\{1, \left\lceil \frac{80x-i_3}{79} \right\rceil\}$. We denote by $T^m(x)$ the composition of T m times (and $T^0(x) = x$).

We record the following claim that bounds the number T's needed to apply on i_0 so that it reaches 1. We prove it after proving the lemma.

Claim 4.30. Let $n = n(I) = \log\left(\frac{i_3}{i_3-i_0}\right)$. Then $T^n_{i_3}(i_0) = 1$.

Proof of Lemma 4.27. Let $n(I) = \log\left(\frac{i_3}{i_3-i_0}\right)$. By Claim 4.30, $T^{n(I)}_{i_3}(i_0) = 1 \leqslant 80i_3$. In every iteration we will show the following guarantee below.

Proposition 4.31. *Let $c = \Omega(\min\{\frac{i_1-i_0}{i_0}, \frac{i_1-i_0}{i_3-i_0}\})$. Then for every $m \geqslant 0$,*

$$h^1(\mathcal{S}^{\{T^m(i_0),i_1,i_2,i_3\}}) \geqslant c \cdot h^1(\mathcal{S}^{\{T^{m+1}(i_0),i_1,i_2,i_3\}}).$$

By using Proposition 4.31 $n(I)$ times we have that

$$h^1(\mathcal{S}^I) \geqslant c^{n(I)} h^1(S^{1,i_1,i_2,i_3}) \geqslant \frac{i_1}{i_3} \cdot c^{n(I)} \geqslant \left(-O\left(\log\left(\frac{i_3}{i_1-i_0}\right)\min\left(\log(\frac{i_3-i_0}{i_1-i_0})\log(\frac{i_1-i_0}{i_0})\right)\right)\right).$$

Proof of Proposition 4.31. Fix $m \geqslant 0$. and let $I_m = \{T^m(i_0), i_1, i_2, i_3\}$, $I_{m+1} = \{T^{m+1}(i_0), i_1, i_2, i_3\}$. The complex S^{I_m} is a high dimensional expander so by Claim 4.12

$$h^1(\mathcal{S}^{I_m}) \geqslant h^1(\mathcal{S}^{I_{m+1}})\Omega(\min_{v\in\mathcal{S}[T^{m+1}(i_0)]} h^1(\mathcal{S}^{I_m}_v))$$

so it remains to show that for every $v \in \mathcal{S}[T^{m+1}(i_0)]$, $h^1(\mathcal{S}^{I_m}_v) = \Omega(\min\{\frac{i_1-i_0}{i_0}, \frac{i_1-i_0}{i_3-i_0}\})$.

Fix $v \in \mathcal{S}[T^{m+1}(i_0)]$ and denote by $J = \{j_0, j_1, j_2, j_3\}$ where $j_0 = T^m(I_0) - T^{m+1}(i_0)$ and for $t = 1, 2, 3$, $j_t = i_t - T^{m+1}(i_0)$. We have seen in Proposition 2.22 that $\mathcal{S}^{I_{m+1}}_v \cong (\mathcal{S}')^J$ where $\mathcal{S}'$ is the spherical building associated with $Sp(2(n - T^{m+1}(i_0)), \mathbb{F}_q)$. By definition of T,

$$79T^{m+1}(i_0) = 79T(T^m(i_0)) \geqslant 80T^m(i_0) - i_3$$

or equivalently,

$$80j_0 = 80\left(T^m(i_0) - T^{m+1}(i_0)\right) \leqslant i_3 - T^{m+1}(i_0) = j_3.$$

Thus $80j_0 \leqslant j_3$ and by Corollary 4.29 $h^1(\mathcal{S}^I_v) = \Omega(\min\{\frac{j_1-j_0}{j_0}, \frac{j_1}{j_3}\}) = \Omega(\min\{\frac{i_1-i_0}{i_0}, \frac{i_1-i_0}{i_3-i_0}\})$. □

□

It remains to prove Claim 4.30.

Proof of Claim 4.30. Let $\tilde{T}(x) = \frac{80x-i_3}{79} + 1$. We note that both T and $\tilde{T}$ are monotone and that for every n, if $T(x) > 1$ then $\tilde{T}^n(x) \geqslant T^m(x)$. Hence it is enough to show that after $n = \log\frac{i_3}{i_3-i_0}$ steps $\tilde{T}^n(x) \leqslant 1$. Indeed, solving a recursion relation yields $\tilde{T}^m(i_0) = (i_0 + 79 - i_3)\left(\frac{80}{79}\right)^m + (i_3 - 79)$. The term $i_0 + 79 - i_3 < 0$ by the assumption on i_0. Thus we have that

$$\tilde{T}^m(i_0) \leqslant 1 \Leftrightarrow m \geqslant \log_{\frac{80}{79}} \frac{i_3 - 80}{i_3 - i_0 + 79}.$$

The constant $n \geqslant \log_{\frac{80}{79}} \frac{i_3-80}{i_3-i_0+79}$ so we are done. □

4.4 Coboundary expansion of the of the links of the affine symplectic building

In this section we modify the proof in [DD23c] to show that the finite quotients of the affine symplectic building's faces complex is a $(\exp(-O(\sqrt{r})), r)$-swap coboundary expander. The proof follows the same lines as [DD23c], where the main difference is that in some cases we need to use Lemma 4.27 instead of Lemma 4.16.

As we saw in Section Section 2, the link of every j-face in a quotient of an affine building is a join of $j' \leqslant j+2$ spherical buildings (as in the definition of a join in Section 2). If the building is symplectic then these buildings are either symplectic or special linear. The following theorem deals with swap coboundary expansion of such complexes.

Theorem 4.32. *Let d be an integer. There is some $p_0 = p_0(d)$. Let $p > p_0$ be any prime power. Let $k \geqslant 1$ and let $\{\mathcal{S}_i\}_{i=1}^{k}$ be so that for every $i = 1, 2, \ldots, k$, $\mathcal{S}_i$ are either $SL_{\ell_i}(\mathbb{F}_p)$ or $Sp(2\ell_i, \mathbb{F}_p)$ spherical buildings. Assume that $\sum_{i=1}^{k} \ell_i = n \geqslant d^5$. $\mathcal{S} = \bigvee_{i=1}^{k} S_i$. Let $X = \mathrm{F}^d\mathcal{S}$ be its faces complex. Then X is a coboundary expander and $h^1(X) \geqslant \exp(-O(\sqrt{d}))$.*

From this theorem we immediately derive swap cocycle expansion of the quotients of the affine symplectic building.

Theorem (Restatement of Theorem 1.3)**.** *Let d be an integer. There is some $p_0 = p_0(d)$ such that for all primes $p > p_0$ the following holds. Let $\mathcal{S}$ be a quotient of the affine symplectic building associated with $Sp(2g, \mathbb{Q}_p)$ for $g \geqslant d^5$. Then $\mathcal{S}$ is a $(d, \exp(-O(\sqrt{d})))$-swap cocycle expander.*

Proof of Theorem 1.3 from Theorem 4.32. Let $X = \mathrm{F}^d\mathcal{S}$. For every $s \in X(0)$, X_s, being itself a faces complex of $\mathcal{S}_s$, is a faces complex of a complex that satisfies the conditions of Theorem 4.32 so it is a $\exp(-O(\sqrt{d}))$-coboundary expander. In addition, for p_0 large enough X is a sufficiently good spectral high dimensional expander, so by Theorem 4.9, it holds that $h^1(X) \geqslant \exp(-\Omega(\sqrt{d}))$ (as a cosystolic expander). □

4.4.1 Notation for this section

We denote by $X = \mathrm{F}^d\mathcal{S}$ and $\tilde{X} = \mathrm{F}\mathcal{S}$.

As all S_i's are partite, they come with colors which are associated with the dimension of the subspace.

For a vertex $v \in \mathcal{S}(0)$, we denote its color by $col(v) = col_{\mathcal{S}}(v)$, which is a tuple (i, j) such that $v \in S_i$ is a subspace of dimension j (i.e. $col_{\mathcal{S}_i}(v)$). We order the colors lexicographically, that is $(i, j) \leqslant (i', j)$ if $i \leqslant i'$ or $i = i'$ and $j \leqslant j'$.

Proof of the main theorem

Fix $d, n \in \mathbb{N}$ so that $d^5 \leqslant n$. Fix $m = \sqrt{d+1}$.

We let $\mathcal{S}$ be as in the theorem statement. We denote $X = \mathrm{F}^d\mathcal{S}$ and $\tilde{X} = \mathrm{F}\mathcal{S} \supset X$. We let $\mathcal{C} = \binom{C_0}{d+1}$ be the set of possible colors of vertices of X where $C_0 = \{(i, j) \mid \ i \in [k], j \in [\ell_i]\}$ are the possible colors of $\mathcal{S}$.

We use u, v to denote vertices of $\mathcal{S}$, and w to denote vertices of X, which are faces of $\mathcal{S}$. Faces of X are denoted by s. We denote subsets of colors of $\mathrm{F}\mathcal{S}$ that are mutually disjoint by the letters J, I (so $J, I \in \mathrm{F}\Delta$).

Proof of Theorem 4.32(weaker version). To bound $h^1(X)$ we follow the steps of the decomposition. Let $\mathcal{J}$ be the set of well-spread J's per Definition 4.20. By Proposition 4.21, at least half of the sets J are in $\mathcal{J}$. Therefore, by Lemma 4.17,

$$h^1(X) \geqslant \Omega(1) \cdot \min_{J \in \mathcal{J}} h^1(X^J) \tag{4.7}$$

Fix $J \in \mathcal{J}$. We note that every link of the faces complex is simply connected by Claim 4.35. By Lemma 4.10

$$h^1(X^J) \geqslant \exp(-O(m)) \cdot \min_{s \in X^J(m-6)} h^1(X_s^J) \tag{4.8}$$

Fix any $s \in X^J(m-6)$. By Corollary 4.33

$$h^1(X_s^J) \geqslant const \cdot h^1(\tilde{X}_s^{\tilde{J}}). \tag{4.9}$$

Recall that $\tilde{X}_s^{\tilde{J}} \cong \mathrm{F}^{\tilde{J}}\mathcal{S}_{\cup s}$ and that $\tilde{J}$ are the indexes that are in $J \setminus col(s)$-crowded bins as in the definition in Section 4.2.2.

Next, denote by $\beta = \min_{w,I} h^1(\mathcal{S}^I_{\cup s \cup w})$ where the minimum is taken over sets I consisting of five singletons such that $I \cup col(s) \leqslant J$, and $w \in \mathcal{S}_{\cup s}$ such that $col(w) \subseteq \cup J$ and $col(w) \cap I = \emptyset$.

By Proposition 4.18,

$$h^1(\tilde{X}_s^{\tilde{J}}) \geqslant const \cdot (\beta_1)^R \tag{4.10}$$

where $\beta_1 = \Omega(\beta)$ and $R = \sum_j |\tilde{c}_j|$. By item 3(c) of Definition 4.20, for every $\tilde{c}_j$, the number of indices in crowded bins is at most $O\left(\frac{d \log d}{m \log m}\right)$, so in total $R = O\left(\frac{d \log d}{m \log m}\right)$.

Finally, by Lemma 4.34,

$$\beta = \min_{w,I} h^1(\mathcal{S}^I_{\cup s \cup w}) \geqslant \exp(-O(\log^2 d)) \tag{4.11}$$

We now plug in each equation into the previous one, to get the desired bound,

$$h^1(X) \geqslant const \cdot \exp\left(-O\left(m + \frac{d \operatorname{poly}\log d}{m \log m}\right)\right) = \exp(-O(\sqrt{d} \log^2 d)).$$

$\square$

Proof of Theorem 4.32(full version). Our starting point is (4.9) in the proof above, namely that

$$h^1(X) \geqslant \ldots \geqslant \exp(-O(m)) \min_{J \in \mathcal{J}, s \in X^J(m-6)} h^1(\tilde{X}_s^{\tilde{J}}).$$

Fixing $J \in \mathcal{J}$ and s, we bound $h^1(\tilde{X}_s^{\tilde{J}})$. As we have seen $\tilde{X}_s^{\tilde{J}} \cong \mathrm{F}^{\tilde{J}}\mathcal{S}_{\cup s}$. We wish to use Proposition 4.19. Towards this, recall the definition of $T_q(\mathcal{S}_{\cup s}, \tilde{J})$ as in Section 4.2. $T_q(\mathcal{S}_{\cup s}, \tilde{J})$ is the largest constant such that for every choice of $J' = \{c'_1, c'_2, \ldots, c'_5\}$ such that $c'_j \subseteq \tilde{c}_j$ and $\sum_{j=1}^5 |c'_j| = q$ there are indexes $i_j \in c'_j$, such that the coboundary expansion of $h^1(\mathcal{S}^{i_1,i_2,\ldots,i_5}_{\cup s \cup w}) \geqslant T_q(\mathcal{S}_{\cup s}, \tilde{J})$ for every face $w \in \mathcal{S}_{\cup s}[\tilde{J} \setminus \cup J']$. By Proposition 4.19

$$h^1(\mathrm{F}^{\tilde{J}}\mathcal{S}_{\cup s}) \geqslant \exp(-O(R)) \cdot \prod_{q=1}^{R} T_q(\mathcal{S}_{\cup s}, \tilde{J})$$

where $R = \sum_{j=1}^5 |\tilde{c}_j|$. As we saw in the weaker version's proof, $R = O(\frac{d \log d}{m \log m})$. By the definition,

$$T_q(\mathcal{S}_{\cup s}, \tilde{J}) \geqslant \min_{w,I} h^1(S^I_{\cup s \cup w})$$

so by Lemma 4.34, $T_q(\mathcal{S}_{\cup s}, \tilde{J}) \geqslant \exp(-O(\log^2 d))$.

However, by Claim 4.36 we can obtain a tighter bound on $T_q(\mathcal{S}_{\cup s}, \tilde{J})$. Let

$$q_0 = \max_{B,c} |c \cap B|$$

where B is a $col(\cup s)$-bin and $c \in \tilde{J}$. By Definition 4.20 $q_0 = O\left(\frac{\log d}{\log m}\right)$ and by Claim 4.36 for every $q > 10q_0$, $T_q(\mathcal{S}_{\cup s}, \tilde{J}) = \Omega(1)$. Thus

$$\mathrm{F}^{\tilde{J}}\mathcal{S}_{\cup s} \geqslant \exp(-O(R)) \cdot \exp(-O(\log^2 d))^{10q_0} \cdot \exp(-O(R - 10q_0)).$$

Plugging in $m = \sqrt{d}$ we have that $q_0 = O(1)$ so this is at least $\exp(-O(R + \log^2 d)) = \exp(-O(\sqrt{d}))$. In conclusion, we have that $h^1(X) = \exp(-O(\sqrt{d}))$. $\square$

4.4.2 Links of the complex

We can write the link of v as

$$\mathcal{S}_v = \bigvee_{t=1, t\neq i}^{k} S_t \vee (\mathcal{S}_i)_v[[j-1]] \vee (\mathcal{S}_i)_v[[\ell_i] \setminus [j]]).$$

The reason is that $(\mathcal{S}_i)_v[[j-1]]$ is the subspaces contained in v, and $(\mathcal{S}_i)_v[[\ell_i] \setminus [j]]$ are the (possibly isotropic) subspaces that contain v - and containment is transitive.

Let us understand how this generalizes to links of arbitrary faces. Let $c = \{(i_0, j_0), (i_1, j_1), \dots, (i_r, j_r)\}$ be a set of colors. A c-bin is one of the following sets

$$B_0 = \{(i,j) \mid \ (i,j) < (i_0, j_0)\},$$

$$\forall \ell = 1, 2, \dots, r \ B_\ell = \{(i,j) \mid \ (i_{\ell-1}, j_{\ell-1}) < (i,j) < (i_\ell, j_\ell)\}$$

and

$$B_{r+1} = \{(i,j) \mid \ (i,j) > (i_r, j_r)\}.$$

For a general face $w = \{v_0, v_1, \cdots, v_r\} \in \mathcal{S}$ so that $col(v_0) < col(v_1) < \dots col(v_r)$, we can write the link $\mathcal{S}_w$ as

$$\mathcal{S}_w = \bigvee_{\ell=0}^{r} \mathcal{S}_w^{B_\ell} \tag{4.12}$$

where the B_ℓ's are $col(w)$-bins as above (and it is possible that $\mathcal{S}_w^{B_\ell}$ is itself also a join of complexes).

4.4.3 A tensor decomposition of the faces complex of a join

Let $w \in \mathcal{S}(r)$, and let $\mathcal{S}_w^{B_\ell}$ be the components of the decomposition of $\mathcal{S}_w$ as in (4.12). Let $J = \{c_1, c_2, \dots, c_m\}$ be subsets of mutually disjoint colors in X so that are disjoint from $col(w)$. We denote by $c_\ell^t = c_\ell \cap B_t$, and let $J_t = \{c_1^t, c_2^t, \dots, c_m^t\}$ be the corresponding subsets of mutually disjoint colors in $\mathcal{S}_w^{B_t}$ (technically this should be a multiset but only the empty set can appear more than once). Then

$$X_w^J = \bigotimes_{t=0}^{r} \mathcal{S}_w^{J_t}. \tag{4.13}$$

We can refine this decomposition using crowded bins. Recall that a $col(w)$-bin B_t is J-crowded if there are two distinct $c_{\ell_1}, c_{\ell_2} \in J$ such that $c_{\ell_1}^t, c_{\ell_2}^t \neq \emptyset$. If there is exactly one $c_\ell^t \neq \emptyset$ then we say that B_t is J-lonely and if all $c_\ell^t = \emptyset$ we say that B_t is J-empty. Let

$$I_1 = \{t \in [r] \mid \ B_t \text{ is } J\text{-crowded }\}$$

and let $I_2 = [r] \setminus I_1$. We can write (4.13) as

$$X_w^J = \left(\bigotimes_{t \in I_1} \mathcal{S}_w^{J_t} \right) \otimes \left(\bigotimes_{i \in I_2} \mathcal{S}_w^{J_t} \right).$$

For every $t \in I_2$, J_t has at most one non-empty set of colors, so $\mathcal{S}_w^{J_t}$ is a complete partite complex. Therefore, $\bigotimes_{i \in I_2} \mathcal{S}_w^{J_i}$ is also a complete partite complex. By Claim 4.13, we have that

Corollary 4.33. *There is some constant $\beta = \beta_m > 0$ such that*

$$h^1(\tilde{X}_w^J) \geqslant \beta \cdot (h^1(\tilde{X}_w^{\tilde{J}}))$$

Where $\tilde{J} = \{\tilde{c}_1, \tilde{c}_2', \dots, \tilde{c}_\ell\}$ and $\tilde{c}_j = \{i \in c_j \mid \ i \text{ is not in a lonely or empty bin}\}$. □

Proof. By (4.13)

$$X_w^J = \left(\bigotimes_{t\in I_1} \mathcal{S}_w^{J_t}\right) \otimes \left(\bigotimes_{i\in I_2} \mathcal{S}_w^{J_t}\right).$$

The second component is a complete partite complex so by Claim 4.13 there exists $\beta = \beta_m$ so that $h^1(X_w^J) \geqslant \beta\left(\bigotimes_{i\in I_1} \mathcal{S}_w^{J_t}\right)$. As $\left(\bigotimes_{i\in I_1} \mathcal{S}_w^{J_t}\right) = \tilde{X}^{\tilde{J}}$ we are done. □

4.4.4 Expansion of colored spherical buildings

Recall Definition 4.20 of the set $\mathcal{J}$ of well-spread color sets, and let $J = \{c_1, \dots, c_m\} \in \mathcal{J}$ and recall that we write $J' \leqslant J$, if $J' = \{c_1', \dots, c_m'\}$ where $c_j' \subseteq c_j$. In this subsection we prove the following lemma.

Lemma 4.34. *Let $I = \{\{(i_1, j_1)\}, \dots, \{(i_1, j_1)\}\}$, and let $s \in X$ be such that $|s| = m - 5$ and there is a well-spread set of colors $J \in \mathcal{J}$ such that $I \cup col(s) \leqslant J$. Let $w' \in \mathcal{S}_{\cup s}^{\cup J}$ be such that $col(w') \cap I = \emptyset$. Then $h^1(\mathcal{S}_{\cup s \cup w'}^I) \geqslant exp(-O(\log^2(d)))$.*

Proof of Lemma 4.34. Let us denote by $w = \cup s \cup w'$. Fix some I as above, and let $I' = \{(i_1, j_1) < (i_2, j_2) < (i_3, j_3) < (i_4, j_4)\}$ be any four indexes inside I. By Theorem 4.7, if we show that $\mathcal{S}_w^{I'}$ is a β-coboundary expander, then it holds that $h^1(\mathcal{S}_w^{I'}) \geqslant \Omega(\beta)$. So from now on we let $I = \{i_0 < i_1 < i_2 < i_3\}$. The coboundary expansion $h^1(\mathcal{S}_w^I)$ depends on I and w. We address first the easier "direct" cases, and then move to the general case which is gradually reduced to the easier cases, via decomposition steps.

If $\mathcal{S}_w^I$ is a join of three complexes (or more), i.e. we can write $\mathcal{S}_w^I = A_1 \vee (A_2 \vee A_3)$ then the diameter of $(A_2 \vee A_3)$ so by Claim 4.14 $h^1(\mathcal{S}_w^I) \geqslant \Omega(\frac{1}{diam(A_2 \vee A_2)}) = \Omega(1)$. If $\mathcal{S}_w^I$ is a join of exactly two complexes then one of the complexes is a color restriction of a spherical building. Without loss of generality let us assume that $(i_1, j_1), (i_2, j_2)$ belong to the same complex (i.e. $i_1 = i_2$) and let us assume that the complex has dimension t. By Claim 4.15 and Claim 4.24 the diameter of this complex is $O(\frac{j_2}{j_2 - j_1})$. By Definition 4.20 the dimension of the complex t is bounded from above by the maximal distance of two consecutive colors of $\cup s$ have length $\frac{100n\log d}{dm}$ (and the link of w can only split to more complexes of lower dimension). Therefore $j_2 \leqslant t \leqslant \frac{100n\log d}{dm}$. On the other hand by item 2 in Definition 4.20 $j_2 - j_1 \geqslant \frac{n}{(md)^3}$ so the diameter is bounded by $O(poly(d))$. By Claim 4.14 in this case $h^1(\mathcal{S}_w^I) = \exp(-O(\log(d)))$.

If $\mathcal{S}_w^I$ is a color restriction of an SL_n spherical building then by Lemma 4.16

$$h^1(\mathcal{S}^I) \geqslant \exp\left(-O\left(\log\left(\frac{j_3}{j_1 - j_0}\right) \cdot \log\left(\frac{j_3}{j_1}\right)\right)\right).$$

Similarly, if $\mathcal{S}_w^I$ is a color restriction of a spherical building then by Lemma 4.27

$$h^1(\mathcal{S}_w^I) \geqslant \exp\left(-O\left(\log\left(\frac{j_3}{j_1 - j_0}\right) \min\left(\log(\frac{j_3 - j_0}{j_1 - j_0}) \log(\frac{j_1 - j_0}{j_0})\right)\right)\right).$$

As before, the quantities $\frac{j_3}{j_1}, \frac{j_3}{j_1 - j_0}, \frac{j_3 - j_0}{j_1 - j_0}$ and $\frac{j_1 - j_0}{j_0}$ are poly(d) from well spreadness, so in both cases $h^1(\mathcal{S}_w^I) \geqslant \exp(-O(\log^2(d)))$. □

4.4.5 Simple connectivity of the links

Claim 4.35. Let J be a set of well spread colors. For every $i \leqslant d$ and $w \in X^J(i)$, $X_w^J(i)$ is simply connected.

Proof of Claim 4.35. Showing simple connectivity is equivalent to showing that the complex is a coboundary expander with some positive constant. We do so using Proposition 4.18 on $X_w^J = F^J(X_{\cup w})$. By Lemma 4.34, for every $I = \{i_1, i_2, i_3, i_4, i_5\}$ such that $i_j \in c_j$ and $w' \in X_{\cup w}^{\cup J \setminus I}$, we have $h^1(X_{w' \cup (\cup w)}) > \beta$. In addition, for large enough q, $X_{\cup w}$ is a $\frac{1}{2d^2}$-local spectral expander and hence $F^J X_{\cup w}$ is a coboundary expander with some $h^1(F^J(X_{\cup w})) > 0$. □

4.4.6 An improved bound for large sets of colors

The following claim is proven exactly as in [DD23c]. We prove it here as well for staying self contained.

Claim 4.36. Let $J \subseteq \mathrm{F}\Delta(4)$. Let $w \in \mathcal{S}$ be such that $col(w) \cap (\cup J) = \emptyset$. Let

$$q_0 = \max_{B,c} |c \cap B|$$

where B is a $col(w)$-bin and $c \in J$. Then for all $q > 10q_0$, $T_q(\mathcal{S}_w, J) = \Omega(1)$.

Note that when J is well spread, and $w = \cup s$ for $s \in \mathbb{Z}^J(m-6)$, then $q_0 = O(1)$.

Proof. To bound $T_q(\mathcal{S}_w, J)$ we need to show that there for every $J' = \{c'_1, c'_2, \dots, c'_5\} \in \mathcal{J}_q$ and for all $w' \in \mathcal{S}_w$ whose color is disjoint from I we can find 5 colors $I = \{i_1, i_2, i_3, i_4, i_5\}$ such that $i_j \in c'_j$ and such that $h^1(\mathcal{S}^I_{w \cup w'}) = \Omega(1)$.

Fix $q > 10q_0$ and $J' = \{c'_1, c'_2, \dots, c'_5\} \in \mathcal{J}_q$. Let us prove that there exists some set $I = \{i_1, i_2, \dots, i_5\}$ as above with the property that there exists a $col(s)$-bin B with $|B \cap I| = 1$. The reason we use such a set is because in this case $\mathcal{S}^I_{w \cup w'}$ is a complex as in Claim 4.11. That is, if $B \cap I = \{i_5\}$, then for any $w'' \in \mathcal{S}_{w \cup w'}[\{i_1, i_2, i_3, i_4\}]$ and any $v \in \mathcal{S}_{\cup s \cup w}[i_5]$, $w'' \cup \{v\} \in \mathcal{S}^I_{w \cup w'}$ (i_5 sits in a different bin so the space v which is compatible with $w \cup w'$ should also be compatible with the flag w'' that is contained in different bins). By Claim 4.11 this implies that $h^1(\mathcal{S}^I_{w \cup w'}) = \Omega(1)$.

By assumption $q > 10q_0$ so there exists a color c'_j such that $|c'_j| > 2q_0$. Without loss of generality let us assume this is c'_5. We begin by choosing $I' = \{i_1, i_2, i_3, i_4\}$ arbitrarily. If there are three bins B_1, B_2, B_3 so that $B_i \cap I' \neq \emptyset$, then at least two bins have that $|B_i \cap I'| = 1$. In this case no matter how we complete I' to $I = I' \cup \{i_5\}$, there will still be a bin B_i such that $|B_i \cap I'| = 1$. There are at most two bins B_1, B_2 with $B_i \cap I' \neq \emptyset$. By the definition of q_0 $|c_5 \cap B_i| \leqslant q_0$ but $|c_5| > 2q_0$ so there exists $i_5 \in c_5 \setminus (B_1 \cup B_2)$ which we can choose. ◻

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