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Constant Degree Direct Product Testers with Small Soundness

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Abstract

Let X be a d-dimensional simplicial complex. A function $F\colon X(k) \to \{0,1\}^k$ is said to be a direct product function if there exists a function $f\colon X(1) \to \{0,1\}$ such that $F(\sigma) = (f(\sigma_1), \dots, f(\sigma_k))$ for each k-face σ. In an effort to simplify components of the PCP theorem, Goldreich and Safra [GS00] introduced the problem of direct product testing, which asks whether one can test if $F\colon X(k) \to \{0,1\}^k$ is correlated with a direct product function by querying F on only 2 inputs. Dinur and Kaufman [DK17] conjectured that there exist bounded degree complexes with a direct product test in the small soundness regime. We resolve their conjecture by showing that for all $\delta > 0$, there exists a family of high-dimensional expanders with degree $O_\delta(1)$ and a 2-query direct product tester with soundness δ.

We use the characterization given by [BM23a] and independently by [DD23a], who showed that some form of non-Abelian coboundary expansion (which they called "Unique-Games coboundary expansion") is a necessary and sufficient condition for a complex to admit such direct product testers. Our main technical contribution is a general technique for showing coboundary expansion of complexes with coefficients in a non-Abelian group. This allows us to prove that the high dimensional expanders constructed by [CL23] satisfy the conditions of [BM23a], thus admitting a 2-query direct product tester with small soundness.

1 Introduction

The primary goal of this paper is to construct direct product testers with constant degree. Earlier works by a subset of the authors [BM23a] and by Dinur and Dikstein [DD23a] have shown that there are coboundary-type properties that are sufficient for a complex to have in order to admit such a direct product tester. The main contribution of the current paper is to establish that the simplicial complex of [CL23] has the sufficient condition presented in [BM23a], implying that it admits direct product testers with small soundness.

1.1 Direct Product Testers

The goal in direct product testing is to encode the values of a function $f\colon [n] \to \{0,1\}$ via a table F which is testable. By that, we mean that there is an efficient, randomized tester that makes queries to F, performs a test on the received values, and decides to accept or reject. The tester should always accept if F is indeed a valid encoding of a function f; this property is often referred to as the completeness of the test. In the converse direction, if the tester accepts F with probability at least $s > 0$, then F must be close (in the case s is close to 1) or correlated (in the case s is close to 0) to a valid encoding of some function f. This property is often referred to as the soundness of the tester. The smallest s for which the last property holds is called

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the soundness parameter of the encoding, and the number of queries the algorithm makes is called the query complexity of the algorithm. In this paper, we will restrict our attention to direct product testers with 2 queries, which is the smallest query complexity one can hope for.

The study of direct product testing and the importance of its parameters such as completeness, soundness and query complexity, originally comes from the work of Goldreich and Safra [GS00]. Their motivation was to simplify components in the proof of the PCP theorem [FGL91, AS98, ALM+98], and the parameters of the direct product tester correspond exactly to the parameters of the PCP verifier. Another motivation for the study of direct product testing comes from hardness amplification, particularly in the style of the parallel repetition theorem [Raz98, Hol09, Rao11, DS14a, BG15]. Indeed, direct product testers with small soundness can be viewed as a combinatorial analog of parallel repetition theorems, and these can sometimes be turned into proper parallel repetition theorems [IKW09, DM11]. We refer the reader to [DK17, BM23a] for further discussion on the role of direct product testing in theoretical computer science.

The most natural direct product encoding of a function $f\colon [n] \to \{0,1\}$ is given by the Johnson scheme. Here, for $k \in \mathbb{N}$ thought of as a constant, one may encode f via the table $F\colon \binom{[n]}{k} \to \{0,1\}^k$ as $F[A] = f|_A$.[1] The Johnson encoding scheme also admits a corresponding natural direct product tester:

1. Sample $I \subseteq [n]$ of size t.
2. Independently sample $A, A' \subseteq [n]$ of size k containing I.
3. Query $F[A]$ and $F[A']$ and check that they agree on the coordinates of I.

The above direct product tester has received significant attention, as well as its variations. By now, all of these are well understood in the entire range of parameters t, see for example [DG08, IKW09, DS14b, BKM23]. In particular, it is well known that the tester has vanishing soundness in the setting that $t \approx \sqrt{k}$ and for sufficiently large k.[2]

The primary disadvantage with the Johnson encoding scheme and the corresponding direct product tester is its size. Indeed, the size of the encoding F of a function f is n^k, which is polynomially large in the size of f. Additionally, if one wishes the soundness of the Johnson direct product tester to be small, say δ, one must take k to be sufficiently large, hence getting a large blow-up in the encoding size. For some applications, this size blow-up is too costly. For instance, in applications in PCPs, the blow-up introduced by the use of parallel repetition theorem is often what dominates the blow-up in the instance size reductions produce.

1.2 Direct Product Testers via High Dimensional Expanders

In the quest for more efficient ways to amplify hardness (often called derandomized hardness amplification), Dinur and Kaufman [DK17] suggested high dimensional expanders as a sparse object that may facilitate direct product testers. To describe their result, we first take a quick detour to present several basic notions from the field of high dimensional expansion, abbreviated HDX henceforth.

A d-dimensional simplicial complex $X = (X(0), \ldots, X(d))$ with vertex set $X(1) = [n]$ is a downwards closed collection of subsets of $[n]$. We follow the convention that $X(0) = \{\emptyset\}$, and for each $i > 1$ the set of i-faces $X(i)$ is a collection of subsets of $X(1)$ of size i. The size of X is the total number of faces in X. The degree of a vertex $v \in X(1)$ is the number of faces containing it, and the degree of X is the maximum degree over all $v \in X(1)$.

[1] We think of $[n]$ as being order in a canonical way, thus for a set A of size k containing $i_1 < \ldots < i_k$ we define $F[A] = (f(i_1), \ldots, f(i_k))$.

[2] In fact, in that case it is known that the soundness is exponentially small in t.

Definition 1.1. *For a d-dimensional simplicial complex $X = (X(0), X(1), \dots, X(d))$, $0 \leqslant i \leqslant d-2$ and $I \in X(i)$, the link of I is the $(d-i)$-dimensional complex X_I whose faces are given as*

$$X_I(j-i) = \{J \setminus I \mid J \in X(j), J \supseteq I\}.$$

For a d-dimensional complex $X = (X(0), X(1), \dots, X(d))$ and $I \in X$ of size at most $d-2$, the graph underlying the link of I is the graph whose vertices are $X_I(1)$ and whose edges are $X_I(2)$. We associate with X a collection of distributions over faces. The distribution μ_d is the uniform distribution over $X(d)$, and for each $i < d$ the distribution μ_i is a distribution over $X(i)$ which results by picking $D \sim \mu_d$, and then taking $I \subseteq D$ of size i uniformly. For convenience, we encourage the reader to think of μ_i as the uniform distribution over $X(i)$ (though often times this is not the case, the fact that μ_i is not actually the uniform distribution is rarely an issue).

Definition 1.2. *We say a d-dimensional simplicial complex X is a γ one-sided local spectral expander if for every $I \in X$ of size at most $d-2$, the second eigenvalue of the normalized adjacency matrix of the graph $(X_I(1), X_I(2))$ is at most γ.*

With this definition, the result of Dinur and Kaufman [DK17] asserts that if X is a γ one-sided local spectral expander for a sufficiently small γ, then X admits a 2-query direct product tester with soundness $s = 1-\varepsilon$. Their direct product tester is a direct analog of the Johnson direct product tester. It is parameterized by $k \in \mathbb{N}$ which is much smaller than d, and proceeds as follows. Given an assignment $F\colon X(k) \to \{0,1\}^d$, the tester proceeds as follows:

1. Sample $D \sim \mu_d$.
2. Sample $I \subseteq D$ of size $\sqrt{k}$ uniformly.
3. Sample $I \subseteq A, A' \subseteq D$ independently.
4. Query $F[A]$ and $F[A']$ and check that they agree on the coordinates of I.

The result of Dinur and Kaufman [DK17] asserts that provided that γ is small enough, if F passes the above tester with probability $1-\varepsilon$, then there exists $f\colon X(1) \to \{0,1\}$ such that with probability $1 - O(\varepsilon)$ over the choice of $A \sim \mu_k$ we have that $F[A] = f|_A$. We refer to the above tester as the canonical direct product tester of X.

The main open problem left from the work [DK17] is whether there are high-dimensional expanders that facilitate direct product testers in the low soundness regime. With regards to this, the works [BM23a, DD23a] both show that spectral expansion is insufficient. Namely, there are high-dimensional expanders with arbitrarily good local spectral expansion for which the above natural direct product tester fails in the low-soundness regime. These two works then went on to seek additional properties of high-dimensional expanders that will imply that the canonical direct product tester has small soundness.

1.3 Main Result

The main result of this paper is that there exist families of high-dimensional expanders for which the canonical direct product tester has small soundness. In fact, the complexes we prove this for are variants of the complexes constructed by Chapman and Lubotzky [CL23, Section 5] for sufficiently large parameters $p, n \in \mathbb{N}$, which are explicit and efficiently computable. The relevance of variants of the complexes of [CL23] was communicated to us in [DDL23].

Theorem 1.3. *For all $\delta, \gamma > 0$, for sufficiently large d, there is $\Delta \in \mathbb{N}$ such that the following holds. There are infinitely many n for which there is a simplicial complex X such that:*

1. *The complex X has n vertices and is a γ two-sided spectral expander.*
2. *The canonical direct product tester for X has soundness δ.*
3. *The complex X has degree at most Δ.*

It has been brought to our attention that Dikstein, Dinur, and Lubotzky have concurrently and independently established Theorem 1.3.

1.4 UG Coboundary Expansion

The starting point of the proof of Theorem 1.3 is the characterization of [BM23a] of high-dimensional expanders for which the canonical tester has small soundness. Towards this end, we begin by defining non-Abelian affine unique games over graphs.

Definition 1.4. *An instance of Affine Unique Games $\Psi = (G, \Pi)$ over the symmetric group S_m consists of a graph $G = (V, E)$ and a collection of permutations, one for each ordered edge, $\Pi = \{\pi_{u,v}\}_{(u,v)\in E}$, where $\pi_{u,v} \in S_m$ and $\pi_{u,v} = \pi_{v,u}^{-1}$. An assignment to Ψ is a function $A : V \to S_m$, and we denote by $\mathrm{val}(A)$ the fraction of constraints satisfied by A, that is,*

$$\mathrm{val}(A) = \Pr_{(u,v)\sim E}[A(u) = \pi_{u,v}A(v)].$$

We denote by viol(A) the fraction of constraints violated by A, that is, viol$(A) = 1 - \mathrm{val}(A)$. The value of the instance Ψ is defined as $\mathrm{val}(\Psi) = \max_{A:V\to S_m} \mathrm{val}(A)$.

We refer to the above instances as affine Unique Games because they can be thought of as systems of equations of the form $A(u)A(v)^{-1} = \pi_{u,v}$ over S_m, wherein the goal is to find an S_m-labeling of the vertices that satisfies as many of the equations as possible.

Definition 1.5. *Let $G = (V, E)$ be a graph, equipped with a distribution $\mathcal{D}$ over triangles in G. We say that a UG instance $\Phi = (G, \{\pi_{u,v}\}_{(u,v)\in E(G)})$ is $(1-\delta)$-triangle consistent if:*

$$\Pr_{(u,v,w)\sim\mathcal{D}}[\pi_{u,v}\pi_{v,w}\pi_{w,u} = id] \geqslant 1 - \delta.$$

We let incons(Φ) denote the fraction of triangles that are inconsistent.

Note that if we have a graph $G = (V, E)$ and an assignment $A\colon V \to S_m$, then defining $\pi_{u,v} = A(u)A(v)^{-1}$ gives a fully triangle consistent instance of affine Unique-Games. The property of UG coboundary expansion asserts, roughly speaking, that the only instances that are highly triangle consistent over certain graphs associated with a complex X arise in this way.

Definition 1.6. *Let X be a d-dimensional simplicial complex, and let $r \leqslant d/3$. We define the graph $G_r[X]$ whose vertex set is $X(r)$, and whose set of edges $E_r[X]$ consists of pairs of vertices (u, v) such that $u \cup v \in X(2r)$. The distribution over triangles associated with this graph is the distribution where we pick a face from $X(3r)$ according to the measure μ_{3r}, and then split it randomly as $u \cup v \cup w$ where $u, v, w \in X(r)$.*

With the definition of the graph $G_r[X]$, we may now define the notion of UG coboundary expansion.

Definition 1.7. *We say that a d-dimensional simplicial complex X is an (m,r,ξ,c) UG coboundary expander if for all $t \leqslant r$ and for all functions $f\colon E_t[X] \to S_m$ that are $(1-\xi)$-consistent on triangles, there is $g\colon X(t) \to S_m$ such that*

$$\Pr_{u\cup v\sim\mu_{2t}}\left[\pi(u,v) = g(u)g(v)^{-1}\right] \geqslant 1-c.$$

The notion of UG coboundary expansion can be seen as a non-Abelian version of the usual notion of coboundary expansion. For $r=1$ this definition appeared in earlier works of [DM19, GK23]; for larger r it is not clear what is the correct way to generalize the notion of coboundary expansion, as it is often stated for Abelian groups. Our notion is motivated by looking at the graph as a constraint satisfaction instance. In this paper, this will be the only notion of coboundary expansion discussed, and as such we will often abbreviate it as refer to it simply as coboundary expansion.

The main result of [BM23a] asserts that a high-dimensional expander X with sufficiently strong UG coboundary parameters has a canonical direct product tester with small soundness; see [BM23a, Theorem 1.9, Theorem B.1]. For future reference, we give here the version we use in our application:

Theorem 1.8. *There exists $c>0$ such that for all $\varepsilon,\delta>0$ there is η and sufficiently large $m,r\in\mathbb{N}$, such that for sufficiently large k, sufficiently large d and sufficiently small $\gamma>0$ the following holds. Suppose that X is a d-dimensional simplicial complex such that:*

1. *A γ one-sided local spectral expander.*
2. *An $(m, r, 2^{-r/\log\log\log r}, c)$ UG coboundary expander.*

If $F\colon X(k)\to\{0,1\}^k$ passes the canonical direct product tester on X with probability at least δ, then there exists $f\colon X(1)\to\{0,1\}$ such that

$$\Pr_{A\sim\mu_k}\left[\Delta(F[A], f|_A) \leqslant \varepsilon k\right] \geqslant \eta.$$

Using Theorem 1.8, it suffices to construct a complex which is simultaneously a strong spectral expander, as well as a UG coboundary expander for sufficiently good parameters. With regard to this, we mention the work of [DD23b] that presents a technique that allows one to prove coboundary-type properties with bounds that are independent of the dimension of the complex. We also mention the recent work [DD23c] which shows coboundary-type properties for general buildings with for up to $\delta = 2^{-O(r)}$ inconsistent triangles, and an improved bound of $\delta = 2^{-O(\sqrt{r})}$ for the spherical building of type A.

Studying spherical buildings and proving coboundary expansion results for them is a central part of our proof as well, however the result of [DD23c] is insufficient as far as we know. The spherical buildings we have to study are spherical buildings of type C_n, and for them we have to be able to handle Unique-Games instances with $\delta = 2^{-o(r)}$ fraction of inconsistent triangles. Indeed, the main technical contribution is a fairly general technique for proving that a simplicial complex is a UG coboundary expander.

1.5 Our Techniques

The rest of this introductory section is devoted to an overview of the proof of Theorem 1.3, and we start with the following definition.

Definition 1.9. *We say that a graph is a $(C(G), \beta(G))$-coboundary expander over S_m if for all $\delta \in [0,1]$ and all UG instances Ψ over S_m, with incons$(\Psi) \leqslant \delta$, there exists an assignment $A : V \to S_m$ with viol$_\Psi(A) \leqslant C(G)\delta + \beta(G)$. When $\beta(G)$ is 0, we say that a graph is a $C(G)$-coboundary expander over S_m.*

The additive error $\beta(G)$ is necessary in our proofs, but we encourage the reader to ignore it and think of it as 0 for the purposes of this section. Fix a simplicial complex X; we will eventually take X to be Chapman Lubotzky complex with an appropriate choice of parameteres [CL23]. The main components of our proof proceed as follows:

1. **Local to global for cosystolic expansion:** we use an idea of [EK16, DD23b] who show that to prove that a simplicial complex is a cosystolic expander, it suffices to show that its links are coboundary expanders. Here and throughout, we say that a graph $G = (V, E)$ is a C-cosystolic expander if for every Unique-Games instances over G with at most δ fraction of inconsistent triangles, we may modify the constraints of G in at most $C\delta$ fraction of the edges and get that all triangles are fully consistent. We use this idea to show that if the links of X are coboundary expanders, then X itself is a cosystolic expander with similar parameters.

2. **Vanishing Cohomology of X:** inspired by the first item, one is motivated to ask the question of what is the constraint structure of fully triangle consistent Unique Games instances on the graphs associated with X. Here, we use the fact, communicated to us by Dikstein, Dinur and Lubotzky [DDL23], that the parameters of the Chapman–Lubotzky [CL23] can be chosen appropriately so that the complex has vanishing cohomology over S_m.

3. **Coboundary expansion of the links of X:** Together, the first two items imply that to prove that X is a coboundary expander, it suffices to prove that the links of X are coboundary expanders. Indeed, in that case, if Ψ is a $(1-\delta)$-consistent Unique-Games instance over S_m, by the first item we can modify the constraints of Ψ on at most $C\delta$ fraction of edges for $C = 2^{o(r)}$ to get an instance Ψ' which is fully triangle consistent. Invoking the second item, we conclude a structural result about the constraints of Ψ', which automatically implies a similar result for Ψ.

   Proving that the links of X have sufficiently good coboundary expansion consists of the bulk of our effort in this paper. We do so by an inductive argument on the parameter r.

In the rest of this section, we elaborate on the third item above as we consider it to be the main contribution of this paper. We begin by noting that for our complex X, the links are either isomorphic to the spherical buildings of type C_n or to product of two such spherical buildings. Given a prime p and a dimension d, the spherical building of type A refers to the complex whose vertex set is the set of all nontrivial subspaces of $\mathbb{F}_p^d$, and whose k-faces are k-flags of subspaces, namely $\{A_1, \ldots, A_k\}$ where $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_k$.

The spherical buildings with type C are similar, except that the vertices only consist of isotropic subspaces with respect to a symplectic form. Here, the dimension d is an even number $2n$, and one defines a symplectic form $\omega \colon \mathbb{F}_p^{2n} \times \mathbb{F}_p^{2n} \to \mathbb{F}_p$ by

$$\omega(x, y) = \sum_{i=1}^{n} x_i y_{i+n} - x_{i+n} y_i.$$

A subspace $V \subseteq \mathbb{F}_p^{2n}$ is then called *isotropic* if $\omega(x, y) = 0$ for all $x, y \in V$. With this in mind, the spherical building of type C certainly has a nice structure. However, as we are aiming for an inductive approach which will necessitate for us to look at various "restrictions" of this complex, it turns out to be convenient to think about this complex more abstractly as a measure μ over $\prod_{i=1}^{n} X_i$, where X_i is the collection of all isotropic subspaces of $\mathbb{F}_p^d$ of dimension i. Namely, μ is just the uniform distribution over top dimensional faces of X.

The most crucial feature of μ that makes it easier to work with is the fact that it is an ε-product measure in the sense of [GLL22], where $\varepsilon = O(1/\sqrt{p})$. Informally, an ε-product measure is one in which one can perform the usual type of discrete Fourier analysis as in product domains such as the Boolean hypercube $\{0,1\}^n$, or product-like domains as the Johnson scheme $\binom{[n]}{k}$ with constant k. Additionally, ε-productness is a property that is preserved under conditioning, making it friendly for inductive processes. We defer a precise definition of ε-product measures to Section 2. Below, we will focus on the measure μ itself, but we remark that our arguments have to also work with restrictions of μ (which they do; the notations become somewhat more complicated, and hence we omit this discussion).

1.5.1 Moving to Tripartite Instances and ε-product Distributions

Set-up: moving to the tripartite problem: consider the n-dimensional spherical building of type C complex X as above, and let Ψ be a Unique-Games instance over its r faces which is $(1-\delta)$-triangle consistent. Sample $R_1, R_2, R_3 \subseteq [n]$ independently of size r, and consider the Unique-Games instance Ψ' induced by Ψ on the tripartite graph $T(R_1, R_2, R_3)$. Here and throughout, the tripartite weighted graph $T(R_1, R_2, R_3)$ is the graph whose faces are r-faces that are subsets of R_1, R_2 and R_3, and whose edges and constraints are induced by Ψ. It can be shown via standard arguments that the fraction of inconsistent triangles in Ψ' is $(1 + o(1))\delta$. Furthermore, using an idea from [DD23b], we show that it suffices to prove that the instance Ψ' has a good cosystolic coefficient for a good fraction of choices of R_1, R_2, R_3. Roughly speaking, if we show that with probability at least p, the induced instance Ψ' has cosystolic coefficient C, then the instance Ψ will have a cosystolic coefficient at most $O(C/p)$.

Our argument will not show a strong enough cosystolic coefficient for every choice of R_1, R_2, R_3, and it depends on two features of them:

1. Separatedness: we would like the elements in $R_1 \cup R_2 \cup R_3$ to be as far apart as possible form each other. Note that the expected magnitude of an element in the union is $\Theta(n)$, and as we are choosing $3r$ of them uniformly, we expect them to be roughly $\Theta(n/r)$ apart. Ideally, we would have liked any two distinct elements from $R_1 \cup R_2 \cup R_3$ to be $\Theta(n/r)$ apart, however the probability for that (over the choice of R_1, R_2, R_3) is $2^{-\Theta(r)}$, which is too small for our purposes as we are shooting for a $2^{o(r)}$ coboundary constant. We thus settle for weaker separatedness, and for our argument it suffices to have R_1, R_2, R_3 to be $\Theta(n/r^2)$-separated (which happens with probability $\Theta(1)$).

2. Well spread: consider an interval $I \subseteq [d]$, say $I = \{s, s+1, \ldots, s+L-1\}$, and think of L as being of the order $n/r^{0.99}$. Note that in expectation, each one of R_1, R_2, R_3 contains $L\frac{r}{n}$ element from the interval. We say that R_1 is well spread if for every interval I of length L we have that $R_1 \cap I$ has size roughly $L\frac{r}{n}$, and say that R_1, R_2, R_3 is well spread if each one of R_1, R_2 and R_3 is well spread. We would like R_1, R_2, R_3 to be well spread, and a standard application of concentration bounds show that the probability a randomly chosen R_1, R_2, R_3 is well spread with probability $1 - o(1)$.

For the rest of this section we fix R_1, R_2, R_3 that are well spread and $\Theta(n/r^2)$ separated.

Moving to the language of ε-product distributions: now that we are considering tripartite instances Ψ' over R_1, R_2, R_3, it will be useful to think of the measure μ over $\prod_{i \in R_1 \cup R_2 \cup R_3} X_i$ that underlies this complex;

namely, this is the distribution μ_{3r} restricted to triangles in Ψ'. This measure can be proved to be an ε-product measure for $\varepsilon = \varepsilon(p)$ which is a vanishing function of the field size p; for our intents this means that ε can be guaranteed to be as small as we wish compared to the dimension of the complex and r. We denote by $C_{r,r,r}(\mu)$ the coboundary constant of the tripartite graph underlying Ψ'. With these notations, our primary objective now is to prove that $C_{r,r,r}(\mu) = 2^{o(r)}$. More generally, for $1 \leqslant t \leqslant r$, we define

$$C_{t,t,t}(\mu) = \max_{\substack{R_1' \subseteq R_1, |R_1'|=t \\ R_2' \subseteq R_2, |R_2'|=t \\ R_3' \subseteq R_3, |R_3'|=t}} \max_{a} \text{Coboundary constant of } T(R_1', R_2', R_3' \mid x_{(R_1 \cup R_2 \cup R_3) \setminus (R_1' \cup R_2' \cup R_3')} = a),$$

where the graph $T(R_1', R_2', R_3' \mid x_{(R_1 \cup R_2 \cup R_3) \setminus (R_1' \cup R_2' \cup R_3')} = a)$ is the induced subgraph of T on the vertices that agree with the restriction $x_{(R_1 \cup R_2 \cup R_3) \setminus (R_1' \cup R_2' \cup R_3')} = a$. The main benefit of ε-product measures is that it provides a clean abstraction of he spectral properties of our tripartite graphs which is suitable for induction.

1.5.2 An Inductive Approach to Coboundary Expansion: the Base Case

To get some intuition, we first consider the case that $r = 1$. In that case, the underlying graph of Ψ' is a tripartite inclusion graph induced by 3 distinct dimensions, say $i < j < \ell$, which we know to be $\Theta(n/r^2)$ separated. Inspecting this graph, one can show that the diameter of this graph is at most

$$O\left(\max\left(\frac{j}{|i-j|}, \frac{\ell}{|\ell - j|}\right)\right) = O(r^2),$$

which suggests that the graph is very well connected. In this case, we use the cones method [Gro10, LMM16, KM22, KM19, KO19]. The cones method is a standard combinatorial "local correction" technique allowing one to deduce global assignments from good local consistency. The method requires us to construct a set of canonical paths between vertices which will be used for "propagation", as well as triangulations of cycles that are formed by an edge (V, W) in the graph and the two canonical constructed paths from some vertex U, using only a small number of triangles.[3] To make this possible, we have to construct the set of canonical paths in a rather careful manner. With these paths, we are able to break each formed cycle between U, V, W into a small collection of 8-cycles (as well as some triangles), which we then triangulate via careful case analysis.

In our context, using the cones method we can show that the coboundary constant of the graph underlying G is polynomial in its diameter. In particular, this implies that $C_{1,1,1}(\mu) \leqslant r^{O(1)}$. We remark that working with well-separated indices is a useful idea if one wishes to prove results that are independent of the dimension of the complex, first introduced by [DD23b]. For us, it will be important that the above applies not only to the measure μ itself, but rather to any restriction of μ leaving at least 3 coordinates alive.

1.5.3 An Inductive Approach to Coboundary Expansion: the Extended Base Case

The inductive base case above for $C_{1,1,1}(\mu)$ is insufficient for our purposes; indeed, using it and our inductive step yields a bound of $2^{\tilde{O}(r)}$ on the coboundary expansion of $T(R_1, R_2, R_3)$.[4] To go beyond this bound we

[3] The cones method also requires sufficient good transitive symmetry from the complex, which holds for free in the case of the spherical buildings we study.

[4] We remark that with additional work, this bound may be improved to $2^{O(r)}$, but we do not know how to break this barrier using only the base case for $r = 1$.

identify another base case, that we explain next, for which we can also directly bound the coboundary constant.

Let k be a parameter to be thought of as a small power of r, and consider subsets $S_1, S_2, S_3 \subseteq R_1 \cup R_2 \cup R_3$ with the following properties:

1. Equal sizes: we have that $|S_1| = |S_2| = |S_3| = 3k$ and $|S_i \cap R_j| = k$ for any $i, j \in \{1, 2, 3\}$.
2. Well ordered: we have that $\max_{s_1 \in S_1} s_1 < \min_{s_2 \in S_2} s_2 \leqslant \max_{s_2' \in S_2} s_2' < \min_{s_3 \in S_3} s_3$. In words, all elements of S_1 are smaller than all elements of S_2, and all elements of S_2 are smaller than all elements of S_3.

Our extended base case gives good bounds for the coboundary constant of the graph $T(S_1, S_2, S_3; \mu)$. To get some intuition, note that when we attempt to construct paths and triangulations in the graph $T(S_1, S_2, S_3)$, incidences between vertices depend only on partial information on them. For example, if $U \in \mathsf{supp}(\mu^{S_1})$ and $V \in \mathsf{supp}(\mu^{S_2})$ are vertices, then the fact of whether (U, V) is an edge or not depends only on the subspace in U corresponding to dimension $\max_{s_1 \in S_1} s_1$ and the subspace in V corresponding to dimension $\min_{s_2 \in S_2} s_2$. Thus, incidences in this graph depend only on the 4 parameters in the second condition above, and it thus stands to reason one can extend the triangulation from the case $r = 1$ to the current case.

Luckily, it turns out there is a surprisingly clean way to go about proving a statement along these lines that only uses the base case $r = 1$ in a black box manner. Indeed, letting $i = \max_{s_1 \in S_1} s_1$, $j = \min_{s_2 \in S_2} s_2$, $j' = \max_{s_2 \in S_2} s_2$ and $\ell = \min_{s_3 \in S_3} s_3$, we show, using an argument similar to the inductive step explained below, that

$$C(T(S_1, S_2, S_3; \mu)) \leqslant C(T(\{i\}, \{j\}, \{\ell\}; \mu) \cdot \max(C_1, C_2, C_3), \tag{1}$$

where

$$C_1 = \max_a C(T(S_1 \setminus \{i\}, S_2, S_3; \mu \mid X_i = a)), \qquad C_2 = \max_b C(T(S_1, S_2 \setminus \{j\}, S_3; \mu \mid X_j = b)),$$
$$C_3 = \max_c C(T(S_1, S_2, S_3 \setminus \{\ell\}; \mu \mid X_\ell = c)).$$

The first term on the right hand side of (1) can be bounded by the base case $r = 1$ above. The second term, namely $\max(C_1, C_2, C_3)$, is more interesting, and we focus on C_1 for concreteness. Thinking of μ as the uniform distribution over top faces of the type C spherical building, one observes that once we condition on the vertex of dimension i in a face, the vertices of dimension strictly less than i and the vertices of dimension more than i form a product structure. In particular, since all elements of $S_1 \setminus \{i\}$ are smaller than i and all elements of S_2, S_3 are bigger than it, we have that for $\nu = \mu \mid X_i = a$ it holds that

$$\nu^{(S_1 \setminus \{i\}) \cup S_2} = \nu^{S_1 \setminus \{i\}} \times \nu^{S_2}, \qquad \nu^{(S_1 \setminus \{i\}) \cup S_3} = \nu^{S_1 \setminus \{i\}} \times \nu^{S_3}.$$

Thus, the tripartite graph $T(S_1 \setminus \{i\}, S_2, S_3; \nu)$ is composed of the bipartite graph between S_2 and S_3, and the graph between $S_1 \setminus \{i\}$ and them is complete. This trivializes the task of constructing triangulations, and indeed we show that the coboundary constant C_1 is dominated by the diameter of the bipartite graph between S_2 and S_3, which is easily seen to be $O(r^2)$ using the separatedness.

Overall, the extended base case allows us to argue that $C(S_1, S_2, S_3; \mu) \leqslant r^{O(1)}$ even when the sets S_1, S_2 and S_3 are large, so long as they are "well ordered". The ability to find well ordered subsets is precisely the reason we require R_1, R_2, R_3 to be well spread.

Remark 1.10. *We remark that the additional feature of μ we use here is that restricting a coordinate of it breaks μ into a proper product distribution of 2 distributions. While this property is easy to see directly for* *type A and C spherical buildings discussed herein, it is in fact more general for any spherical building. This is best seen by looking at the Coxeter diagrams that correspond to a given spherical building, and inspecting that removing a vertex from them (which corresponds to the operation of restriction/ taking a link) either turns a type into an easier-to-handle type, or else it disconnects the diagram, in which case the product structure as discussed above emerges.*

1.5.4 The Inductive Approach: the Inductive Step

Armed with the base case, we are now ready to describe an inductive approach to bound $C_t(\mu) = C_{t,t,t}(\mu)$.

Restrictions and finding good local solutions: Suppose that we have $R_1' \subseteq R_1$, $R_2' \subseteq R_2$ and $R_3' \subseteq R_3$ of size t during our induction, and choose disjoint well ordered subsets S_1, S_2, S_3 of $R_1' \cup R_2' \cup R_3'$ that each contains k elements from each one of R_1, R_2 and R_3. An easy argument shows that for appropriate choice of parameters, so long as $t \geqslant r^{0.99}$ we are able to find such S_1, S_2, S_3, and we fix such sets henceforth. We later explain how to handle the case $t < r^{0.99}$.

Choose restrictions $(a_1, a_2, a_3) \sim \mu^{S_1 \cup S_2 \cup S_3}$. We may consider 3 distinct induced graphs on $T = T(R_1', R_2', R_3')$ corresponding to these restrictions, which are $T_1 = T(R_1' \setminus S_1, R_2' \setminus S_1, R_3' \setminus S_1 \mid \mu \mid x_{S_1} = a_1)$, $T_2 = T(R_1' \setminus S_2, R_2' \setminus S_2, R_3' \setminus S_2 \mid \mu \mid x_{S_2} = a_2)$ and $T_3 = T(R_1' \setminus S_3, R_2' \setminus S_3, R_3' \setminus S_3 \mid \mu \mid x_{S_3} = a_3)$. Each one of these graphs refers to the induced graph on vertices that agree with the restriction a_1, a_2 and a_3 respectively, and we may consider the induced affine Unique-Games instance Ψ'_{a_1} Ψ'_{a_2} and Ψ'_{a_3} on them. Let ε_{a_1}, ε_{a_2} and ε_{a_3} denote the fraction of inconsistent triangles in Ψ'_{a_1}, Ψ'_{a_2} and Ψ'_{a_3} respectively, and choose X_{a_1} X_{a_2} and X_{a_3} to assignments that satisfy the maximum fraction of constraints. Thus, by definition we get that

$$\text{viol}(X_{a_1}; T_1) \leqslant C_{t-k}\varepsilon_{a_1}, \qquad \text{viol}(X_{a_2}; T_2) \leqslant C_{t-k}\varepsilon_{a_2}, \qquad \text{viol}(X_{a_3}; T_3) \leqslant C_{t-k}\varepsilon_{a_3}.$$

Since we are working with affine instances of Unique-Games, once we have one good solution we may apply affine shifts to it to get a collection of good solutions; in our case, for each $\pi \in S_m$ we may define $L_{a_1}[\pi] = X_{a_1}\pi$, and have that L_{a_1} consists of solutions to Ψ_{a_1} each satisfying all but $C_{t-k}\varepsilon_{a_1}$ of the constraints. Similarly, we may define L_{a_2} and L_{a_3}.

Relating good local solutions: consider now the instance induced by Ψ on the graph

$$G_{a_1,a_2} = T(R_1', R_2', R_3' \mid x_{S_1 \cup S_2} = (a_1, a_2)).$$

Denote by $\text{viol}(X_{a_1}; G_{a_1,a_2})$, $\text{viol}(X_{a_2}; G_{a_1,a_2})$ the fraction of constraints violated by X_{a_1}, X_{a_2} in G_{a_1,a_2} respectively. From the point of view of the restriction S_1, a_1, this is a randomly chosen induced sub-instance of T_1, and hence we expect that $\text{viol}(X_{a_1}; G_{a_1,a_2}) \lesssim \text{viol}(X_{a_1})$, and analogously $\text{viol}(X_{a_2}; G_{a_1,a_2}) \lesssim \text{viol}(X_{a_2})$. Using the ε-productness of μ, the graph G_{a_1,a_2} has second singular value bounded away from 1, and therefore any two good solutions to an Affine Unique-Games over it must be the same up to an affine shift (we remark that this is an idea whose origin is algorithms for solving affine Unique-Games over some special classes of graphs [BBK+21, BM23b]). Thus, we may find a shift $\pi_{a_1,a_2} \in S_m$ that nearly forms a matching between the lists L_{a_1}, L_{a_2}. More precisely, we are able to show that

$$\mathop{\mathbb{E}}_{a_1,a_2}\left[\Pr_{u \in G_{a_1,a_2}}[X_{a_1}(u) \neq \pi_{a_1,a_2}X_{a_2}(u)]\right] \lesssim \mathop{\mathbb{E}}_{a_1,a_2}[\text{viol}(X_{a_1}; G_{a_1,a_2}) + \text{viol}(X_{a_2}; G_{a_1,a_2})] \lesssim C_{t-k}\delta.$$

Using our observations so far, we may consider the tripartite graph over restrictions, whose triangles correspond to a triplet of restrictions a_1, a_2, a_3 that are valid under μ. We consider an Affine Unique-Games instance Ψ_{restrict} on it, that has the constraint $\pi^{-1}_{a_1,a_2}$ on the edge (a_1, a_2), and the goal is to choose labels from S_m to each vertex so as to satisfy as many of the constraints as possible. Using our arguments so far, we can argue that the fraction of inconsistent triangles is at most $\eta = O(C_{t-k}\delta)$, and as the underlying graph of Ψ_{restrict} is $T(S_1, S_2, S_3)$ we are able to conclude that we may find a solution to Ψ_{restrict} that satisfies all but $O(C_k C_{t-k}\delta)$ of constraints. Going in this route, one can conclude the recursion $C_t \lesssim C_k C_{t-k}$, which ultimately gives $C_t \leqslant 2^{O(t\log r)}$. This is, of course, insufficient, and the reason for why we require the extended base case. Using it, we have that $C(T(S_1, S_2, S_3)) \leqslant r^{O(1)}$ giving us the recursion $C_t \leqslant r^{O(1)}C_{t-k}$ so long as $t \geqslant r^{0.99}$. Iterating, this gives

$$C_r \leqslant r^{r/k} C_{r^{0.99}} \leqslant r^{r/k} 2^{O(r^{0.99}\log r)} \leqslant 2^{O(r^{1-c})}$$

for some absolute constant $c > 0$.

To complete the overall picture, we now explain how we lift good solutions to Ψ_{restrict} to a good solution of Ψ'. Suppose that A is an assignment to Ψ_{restrict} satisfying $1-\eta$ fraction of constraints, and let U be a vertex in Ψ', say $U \in \mathsf{supp}(\mu^{R'_1})$ without loss of generality. The idea is to ask the opinion of a random vertex in S_1 that is consistent with U, and assign U accordingly. More precisely, we sample $a_1 \sim \mu \mid x_{R'_1} = U$, and assign $B[U] = A[a_1](U)$. Indeed, using standard spectral arguments, we show that for typical U, the value of $A[a_1](U)$ for a_1 chosen in this way is almost fixed, and furthermore that the assignment B satisfies all but η fraction of constraints in Ψ'.

2 Preliminaries

Notations: We use standard big-O notations: we denote $A = O(B)$ or $A \lesssim B$ if $A \leqslant c \cdot B$ for some absolute constant $c > 0$. Similarly, we denote $A = \Omega(B)$ or $A \gtrsim B$ if $A \geqslant cB$ for some absolute constant $c > 0$. We also denote $k \ll d$ to denote the fact that d is taken to be sufficiently large compared to any function of k. For a distribution μ over $X_1 \times \ldots \times X_r$ and a subset $S \subseteq [r]$, we denote by μ^S the marginal distribution of μ on the coordinates of S. We denote by $\mathsf{supp}(\mu)$ the support of μ, and for a subset $S \subseteq [r]$ and an assignment $V \in \prod_{i\in S} X_i$ we denote by $\mu \mid X_S = V$ the distribution of $X \sim \mu$ conditioned on $X_S = V$. If A is a finite set and $i \leqslant |A|$, the notation $B \subseteq_i A$ means that we sample a subset of size i of A uniformly.

2.1 Graphs Associated with Distributions and ε-product Distributions

Let μ over $X_1 \times \ldots \times X_r$. The following definition describes bipartite graphs that can be associated with μ:

Definition 2.1. *Let μ be a distribution on $X_1 \times \ldots \times X_r$. Let $L, R \subseteq [r]$ be two disjoint non-empty sets. Let $A(L, R; \mu)$ be the bipartite graph produced as follows: the vertices are $\mathsf{supp}(\mu^L)$ and $\mathsf{supp}(\mu^R)$ and to sample an edge we choose a sample X from μ, and output (X_L, X_R). We will let $A_{L,R}$ denote the corresponding operator $A_{L,R} : L^2(X_L, \mu^L) \to L^2(X_R, \mu^R)$, with $A_{L,R}f(v) = \mathbb{E}_{w\sim\mu^L|X_R=v}[f(w)]$.*

Similarly, one can create tripartite graphs from μ:

Definition 2.2. *Let μ be a distribution on $X_1 \times \ldots \times X_r$. Let $S_1, S_2, S_3 \subseteq [r]$ be three disjoint non-empty sets. Let $T(S_1, S_2, S_3; \mu)$ be the tripartite graph produced as follows: the vertices are $\mathsf{supp}(\mu^{S_1})$, $\mathsf{supp}(\mu^{S_2})$*

and $\mathsf{supp}(\mu^{S_3})$*, and to sample an edge we choose a sample* X *from* μ*, and with probability* $1/3$ *each output* (X_{S_i}, X_{S_j}) *for* $i \neq j, i, j \in [3]$*.*

Note that when one of the S_i*'s is* $\emptyset$*, say* S_1*, the graph* $T(\emptyset, S_2, S_3; \mu)$ *denotes the tripartite graph with one vertex* $\emptyset$ *in its first part,* $\mathsf{supp}(\mu^{S_2}|a), \mathsf{supp}(\mu^{S_3}|a)$ *in its second and third parts, and we sample an edge by sampling* $X \sim \mu$ *and and outputting the pairs* $(\emptyset, X_{S_2})$*,* $(\emptyset, X_{S_3})$ *or* (X_{S_2}, X_{S_3}) *with equal probability. When more of the* S_i*'s are* $\emptyset$ *we can similarly create these tripartite graphs.*

We now discuss the definition of ε-product distributions from [GLL22]. We begin by defining ε-pseudorandom distributions.

Definition 2.3. *We say that a distribution* $\mathcal{D}$ *over* $X_1 \times X_2$ *is* ε*-pseudorandom if the second largest singular value of* $A_{\{1\},\{2\}}$ *is at most* ε*.*

Next, we define the notion of having ε-pseudorandom skeletons.

Definition 2.4. *We say that a distribution* $\mathcal{D}$ *over* $Y_1 \times \ldots \times Y_t$ *has* ε*-pseudorandom skeletons if for all* $i \neq j \in [t]$*, the marginal distribution* $\mathcal{D}^{\{i,j\}}$ *is* ε*-pseudorandom.*

We are now ready to define the notion of ε-product distributions.

Definition 2.5. *We say that* μ *is an* ε*-product distribution over* $X_1 \times \ldots \times X_r$ *if for all* $S \subseteq [r]$ *of size at most* $r-2$ *and all* $V \in \mathsf{supp}(\mu^S)$*, the conditional distribution* $\mu | X_S = V$ *has* ε*-pseudorandom skeletons.*

Many properties of ε-product distributions were established in [GLL22], and we will require a few of them. In particular, we will need [GLL22, Lemma 3.3], which asserts that if $L, R \subseteq [r]$ are disjoint and μ is an ε-product distribution, then the second singular values of the bipartite graphs $A(L, R; \mu)$ are small.

Lemma 2.6. *Let* μ *be an* ε*-product distribution over* $X_1 \times \ldots \times X_r$ *and let* $L, R \subseteq [r]$ *be two disjoint sets. Then the second largest singular value of* $A_{L,R}$ *and* $A_{R,L}$ *is at most* $\mathrm{poly}(r)\varepsilon$*.*

2.2 Properties of Expanders

We need the following well known version of the expander mixing lemma for bipartite graphs.

Lemma 2.7. *Let* $G = (U, V, E)$ *be a bipartite graph in which the second singular value of the normalized adjacency matrix is at most* λ*. Then for all* $A \subset U$ *and* $B \subset V$ *we have that*

$$\left| \Pr_{(u,v)\in E}[u \in A, v \in B] - \mu(A)\mu(B) \right| \leqslant \lambda \sqrt{\mu(A)(1-\mu(A))\mu(B)(1-\mu(B))}.$$

We also use the following standard sampling property of bipartite expanders.

Lemma 2.8. *Let* $G = (U, V, E)$ *be a weighted bipartite graph with second singular value at most* λ*. Let* $B \subset U$ *and set* $T = \{v \in V \mid \Pr_{u \sim v}[u \in B] > \varepsilon + \Pr[B]\}$*. Then* $\Pr[T] \leqslant \lambda^2 \delta / \varepsilon^2$*.*

2.3 Properties of Local Spectral Expanders

Recall that we associated with each d-dimensional simplicial complex X a sequence of measures $\{\mu_k\}_{1 \leqslant k \leqslant d}$, where μ_k is a probability measure over $X(k)$. Note that for all $0 \leqslant t \leqslant r \leqslant d$, a sample according to μ_t can be drawn by first sampling $R \sim \mu_r$, and then sampling $T \subseteq_t R$ uniformly. The converse is also true: a sample from μ_r can be drawn by first sampling $T \sim \mu_t$, and then sampling R from μ_r conditioned on containing T. These observations give rise to the standard "up" and "down" operators, which we present next. We only mention a few of their properties that are necessary for our arguments, and refer the reader to [DDFH18] for a more comprehensive exposition.

Definition 2.9. *The operator U_i^{i+1} is a map from $L_2(X(i);\mu_i)$ to $L_2(X(i+1);\mu_{i+1})$ defined as*

$$U_i^{i+1} f(u) = \mathop{\mathbb{E}}_{v \subset_i u} \left[f(v) \right]$$

for all $u \in X(i+1)$. For $j \geqslant k+1$, we define U_k^j via composition of up operators: $U_k^j = U_{j-1}^j \circ \ldots \circ U_k^{k+1}$.

Definition 2.10. *The operator D_i^{i+1} is a map from $L_2(X(i+1);\mu_{i+1})$ to $L_2(X(i);\mu_i)$ defined as*

$$D_i^{i+1} f(u) = \mathop{\mathbb{E}}_{v \supseteq_{i+1} u} \left[f(v) \right]$$

for all $u \in X(i)$. For $j \geqslant k+1$, we define D_k^j via composition of down operators: $D_k^j = D_k^{k+1} \circ \ldots \circ D_{j-1}^j$.

Abusing notations, we use the notations U_k^j, D_k^j to denote the operators, as well as the real valued matrices associated with them. A key property of the down and up operators is that they are adjoint:

Claim 2.11. *For all $k \leqslant j \leqslant d$, U_k^j and D_k^j are adjoint operators: for all functions $f\colon X(k) \to \mathbb{R}$ and $g\colon X(j) \to \mathbb{R}$ it holds that $\langle U_k^j f, g \rangle = \langle f, D_k^j g \rangle$.*

We need the following result due to [Opp18] known as the trickling-down theorem that uses the eigenvalues of links at $X(d-2)$ to show that X is a one-sided local spectral expander.

Theorem 2.12. *Let X be a d-dimensional simplicial complex such that the 1-skeleton of every link (including the empty one) is connected and for all $I \in X(d-2)$, the 1-skeleton of I has second eigenvalue at most λ. Then X is a $\frac{\lambda}{1-(d-1)\lambda}$-one-sided local spectral expander.*

We need the following lemma regarding the second eigenvalue of the down-up walks $U_k^j D_k^j$ on $X(j)$ ($j \geqslant k$), that can be found in [AL20].

Lemma 2.13. *Let (X,μ) be a d-dimensional γ one-sided local spectral expander. For all $i \leqslant d$ and $\alpha \in (1/i, 1)$, the largest singular value of $U_{\alpha i}^i$ and $D_{\alpha i}^i$ is at most $\sqrt{\alpha} + \mathrm{poly}(i)\gamma$. Thus the down-up random walk $U_{\alpha i}^i D_{\alpha i}^i$ on $X(i)$ has second largest singular value at most $\alpha + \mathrm{poly}(i)\gamma$.*

2.4 Spherical Buildings of Type A

Analyzing the Chapman-Lubotzky complex will require us to study its links. In this section and in the next one, we present the spherical buildings of type A and of type C, which are morally the graphs we will end up needing to study.

Definition 2.14. *The spherical building of type A over $\mathbb{F}_q^{d+1}$ is a d-dimensional complex denoted by $SB_d^A(\mathbb{F}_q)$ with the set of maximal faces:*

$$\{(V_1, \ldots, V_d) : V_1 \subset \ldots \subset V_d, V_i \subset_i \mathbb{F}_q^{d+1}\}.$$

The d-faces are equipped with the uniform distribution which we also denote by $SB_d^A(\mathbb{F}_q)$.

The following is a well-known fact, but we give the proof here for completeness.

Lemma 2.15. *The distribution $SB_d^A(\mathbb{F}_q)$ is a $O(1/\sqrt{q})$-product distribution.*

Proof. Let $\mu = SB_d^A(\mathbb{F}_q)$. Take any set $S \subseteq [d], |S| \leqslant r-2$ and a valid restriction of it, say V. We need to show that $\mu|X_S = V$ has $1/q$-pseudorandom skeletons. Consider two coordinates $i \neq j \in [d] \setminus S$ and let $A_{i,j}$ be the bipartite graph/normalized adjacency operator corresponding to $\mu^{\{i,j\}}|X_S = V$. If there exists $k \in S$ such that $i < k < j$, then $A_{i,j}$ has second largest singular value 0. So let us consider the case where there is no such coordinate k between i and j. Let i' be the largest coordinate in S that is less than i and j' be the smallest coordinate in S that is greater than j ($i' < i < j < j'$). Then the bipartite graph $A_{i,j}$ is isomorphic to the weighted inclusion graph between $i-i'$ and $j-i'$-dimensional subspaces of $\mathbb{F}_q^{j'-i'}$. It is well-known (see for example [BCN12, GM16]) that this graph has second largest singular value $\lesssim 1/\sqrt{q^{j-i}} \leqslant 1/\sqrt{q}$. This shows that $\mu|X_S = V$ has $1/\sqrt{q}$-pseudorandom skeletons for all S and V, thus showing that μ is a $1/\sqrt{q}$-product distribution. □

2.5 Spherical Buildings of Type C

Next, we present the spherical buildings of type C. Like the spherical buildings of type A, type C spherical buildings are too defined using subspaces. However, we only consider subspaces that are isotropic with respect to a *symplectic form*.

Definition 2.16. *A symplectic bilinear form is a mapping $\omega : \mathbb{F}^{2n} \times \mathbb{F}^{2n} \to \mathbb{F}$ is a map which bi-linear, anti-symmetric – $\omega(v,w) = -\omega(w,v), \forall v,w \in \mathbb{F}^{2n}$ and non-degenerate – $\omega(u,v) = 0$ for all v implies $u = 0$.*

In this paper we fix the symplectic form:

$$\omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

which gives the bi-linear form $\omega(v,w) = \sum_{i=1}^n v_i w_{n+i} - w_i v_{n+i}$. One can check that $\omega(\cdot,\cdot)$ is a valid symplectic bilinear form.

Definition 2.17. *A subspace $V \subseteq \mathbb{F}_q^{2d}$ is called isotropic if $\omega(v,w) = 0$ for all $v,w \in V$.*

Definition 2.18. *The symplectic group Sp$(2d, F)$ is defined as the set of $2d \times 2d$ matrices M over a field F that preserve the symplectic form ω. This group is characterized by matrices in GL$_{2d}(\mathbb{F})$ that satisfy $M^T \omega M = \omega$, where ω is the matrix representation of the symplectic form.*

Properties of SP$_{2d}(\mathbb{F})$: For an invertible matrix M and subspace $V \subseteq \mathbb{F}^{2d}$, let $M \circ V$ denote the subspace span$(Mv)_{v \in V}$. Then the group SP$_{2d}(\mathbb{F})$ has the following properties:

1. If V is a t-dimensional isotropic subspace of $\mathbb{F}_q^{2d}$, then $M \circ V$ is also a t-dimensional isotropic subspace of $\mathbb{F}_q^{2d}$.
2. Furthermore, SP$_{2d}(\mathbb{F})$ acts transitively on the set of t-dimensional isotropic subspaces for all $t \leqslant d$.

Definition 2.19. *The spherical buildings of type C over $\mathbb{F}_q^{2d}$ is a d-dimensional complex denoted by $SB_d^C(\mathbb{F}_q)$ with the set of maximal faces:*

$$\{(V_1, \ldots, V_d) : V_1 \subset \ldots \subset V_d, V_i \text{ is an isotropic subspace of dimension } i\}.$$

The d-faces are equipped with the uniform distribution which we also denote by $SB_d^C(\mathbb{F}_q)$.

The following lemma asserts that the natural distribution associated with the spherical building of type C is ε-product for a small ε.

Lemma 2.20. *Let $d, q \in \mathbb{N}$ with $q \geqslant \mathrm{poly}(d)$. Then $\mu = SB_d^C(\mathbb{F}_q)$ is a $O(1/\sqrt{q})$-product distribution.*

Proof. Fix $\varepsilon = O(1/\sqrt{q})$. One can check that proving μ is a λ-product distribution is equivalent to showing that $SB_d^C(\mathbb{F}_q)$ denoted by X is a λ-one-sided local spectral expander, and we focus on the latter task. To do so, it suffices to show that the 1-skeleton of all links $I \in X(d-2)$ have second eigenvalue at most ε. Once we show that, the trickling-down theorem, Theorem 2.12, implies that X is a $\frac{\varepsilon}{1-d\varepsilon} \lesssim \varepsilon$ one-sided local spectral expander.

Fix a $(d-2)$-sized link I, and say that it fixes the set of coordinates $S \subset [d]$ of size $d-2$ to the isotropic subspaces $V_a, a \in S$, that form a valid inclusion chain. Let $i < j \in [d]$ be the two unfixed coordinates and let $\mathcal{D}$ be the resulting conditional distribution on $X_{\{i,j\}}$. The second largest eigenvalue of the 1-skeleton of I is equal to the second largest singular value of $A(\{i\}, \{j\}; \mathcal{D})$, since the 1-skeleton is a bipartite graph. We bound the latter using case analysis.

Coordinates i, j are not consecutive: In this case there exists $k \in S$ such that $i < k < j$. So we know that $\mathcal{D}$ is a product distribution and $A(\{i\}, \{j\}; \mathcal{D})$ is the complete bipartite graph, hence the corresponding second largest singular value is 0.

Coordinates i, j are consecutive but not equal to $(d-1, d)$: In this case, $j = i + 1$ and the coordinates $i+2$ and $i-1$ belong to S. Then the resulting bipartite graph is over the set of isotropic subspaces contained within V_{i+1} and containing V_{i-1}. Since V_{i+1} is an isotropic subspace and every subspace within an isotropic subspace is isotropic, we get that $A(\{i\}, \{j\}; \mathcal{D})$ is isomorphic to the bipartite weighted inclusion graph between 1 and 2-dimensional subspaces of $\mathbb{F}_q^3$. This has largest singular value $\leqslant \varepsilon$ as we saw in Lemma 2.15.

Coordinates $(i, j) = (d-1, d)$: In this case the coordinate $(d-2)$ belongs to S. The set of $d-1$ and d-dimensional isotropic subspaces containing V_{d-2} is in one-to-one correspondence with the set of 1 and 2-dimensional subspaces respectively, that are symplectically orthogonal to V_{d-2} and are not contained in V_{d-2}. Taking a quotient by V_{d-2} we get that this is the set of 1 and 2-dimensional isotropic subspaces of $\mathbb{F}_q^4$. Therefore $A(\{i\}, \{j\}; \mathcal{D})$ is isomorphic to the bipartite inclusion graph between 1 and 2-dimensional isotropic subspaces of $\mathbb{F}_q^4$. Using [BCN12, Theorem 9.4.3], we get that this graph has second largest singular value $\lesssim \varepsilon$ as required. □

3 A Local to Global Theorem for Coboundary Expansion

The primary goal of this section is to present an inductive approach to prove upper bounds on the coboundary constants corresponding to the level r faces of a simplicial complex X. Roughly speaking, starting with an initial assumption regarding the coboundary constant of X on constant levels, we show how to lift it to a reasonable bound on higher levels.

3.1 Tools

3.1.1 Basic Notions and Properties of the Tripartite Graph $T(R_1, R_2, R_3)$

In this section, we present some of the tools and notions that are necessary for our proof. Throughout this section, we fix a set of indices $I = \{i_1, \ldots, i_{3r}\}$ and an ε-product distribution μ over $\prod_{i \in I} X_i$. Here and throughout, ε should be thought of as very small compared to all other parameters, and we encourage the reader to think of $\varepsilon = 0$ at first reading.

We begin by formally defining the notion of coboundary expansion (with additive error) for measures.

Definition 3.1. *Let μ be a measure over $\prod_{i\in I} X_i$ and let $0 \leqslant r_1, r_2, r_3 \leqslant r \in \mathbb{N}$ be integers. We say that μ is a $(C_{r_1,r_2,r_3}, \beta_{r_1,r_2,r_3})$-coboundary expander over S_m if for all sets S of size at most $3r - (r_1 + r_2 + r_3)$, for all restrictions $a \in \mathsf{supp}(\mu^S)$, and for all disjoint $R_1, R_2, R_3 \subseteq I \setminus S$ of sizes r_1, r_2, r_3 respectively, the tripartite graph $T(R_1, R_2, R_3; \mu|X_S = a)$ with respect to the distribution $\mu^{\cup R_i}|(X_S = a)$ over triangles, is a $(C_{r_1,r_2,r_3}, \beta_{r_1,r_2,r_3})$-coboundary expander over S_m as per Definition 1.9.*

When $\beta_{r_1,r_2,r_3} = 0$, we simply say that μ is a C_{r_1,r_2,r_3}-coboundary expander over S_m.

Definition 3.1 will be of central interest to us, and to use it we must develop some tools to investigate coboundary expansion in the tripartite graphs $T(R_1, R_2, R_3; \mu)$. We begin with the following claim, asserting that this graph always has second singular value bounded away from 1.

Claim 3.2. *Let μ be an ε-product distribution over $X_1 \times \ldots \times X_r$, and let $S_1, S_2, S_3 \subseteq [r]$ be three disjoint non-empty sets. Let T be the normalized adjacency operator of the graph $T(S_1, S_2, S_3; \mu)$. Then the second largest singular value of $T$is at most $1/2 + \mathrm{poly}(r)\varepsilon$.*

Proof. Let T be the normalized adjacency operator of $T(S_1, S_2, S_3; \mu)$ and let f be the second eigenvector of T with $\mathbb{E}[f] = 0$ and $\|f\|_2 = 1$. We will now bound $\|Tf\|_2$.

For any $i = 1, 2, 3$, let f_i be the function f restricted to X_{S_i} thought of as an element in $L_2(X_{S_i}; \mu^{S_i})$; we also denote $a_i = \mathbb{E}[f_i] = |\,\mathbb{E}_{x\sim\mu^{S_i}}[f(x)]|$. We have that for all $x \in S_1$,

$$Tf(x) = \frac{1}{2}A_{S_2,S_1}f(x) + \frac{1}{2}A_{S_3,S_1}f(x),$$

and similarly for $x \in S_2$ or S_3. Similarly, we denote by $(Tf)_i$ the restriction of Tf to X_{S_i}. With these notations, we have that

$$\|(Tf)_1\|_2 \leqslant \frac{1}{2}\|A_{S_2,S_1}f_2\|_2 + \frac{1}{2}\|A_{S_3,S_1}f_3\|_2 \leqslant \frac{1}{2}(a_2 + \mathrm{poly}(r)\varepsilon) + \frac{1}{2}(a_3 + \mathrm{poly}(r)\varepsilon),$$

where in the last transition we used Claim 2.6 to bound the second singular value of A_{S_2,S_1} and A_{S_3,S_1} to bound $\|A_{S_2,S_1}f_2 - a_2\|_2 = \|A_{S_2,S_1}(f_2 - a_2)\|_2 \leqslant \mathsf{poly}(r)\varepsilon$. Squaring and simplifying gives us that

$$\|(Tf)_1\|_2^2 \leqslant \frac{1}{4}(a_2 + a_3)^2 + \mathrm{poly}(r)\varepsilon.$$

As $a_1 + a_2 + a_3 = 3\,\mathbb{E}[f] = 0$, we conclude that $\|(Tf)_1\|_2^2 \leqslant \frac{1}{4}a_1^2 + \mathsf{poly}(r)\varepsilon$, and similarly $\|(Tf)_2\|_2^2 \leqslant \frac{1}{4}a_2^2 + \mathsf{poly}(r)\varepsilon$ and $\|(Tf)_3\|_2^2 \leqslant \frac{1}{4}a_3^2 + \mathsf{poly}(r)\varepsilon$. Multiplying by $1/3$ and summing up we get

$$\begin{aligned}\|Tf\|_2^2 = \mathop{\mathbb{E}}_{i\in[3]}[\|(Tf)_i\|_2^2] &\leqslant \frac{1}{12}(a_1^2 + a_2^2 + a_3^2) + \mathrm{poly}(r)\varepsilon \\ &= \frac{1}{12}(\mathbb{E}[f_1]^2 + \mathbb{E}[f_2]^2 + \mathbb{E}[f_3]^2) + \mathrm{poly}(r)\varepsilon \\ &\leqslant \frac{1}{12}(\mathbb{E}[f_1^2] + \mathbb{E}[f_2^2]^2 + \mathbb{E}[f_3^2]) + \mathrm{poly}(r)\varepsilon \\ &= \frac{1}{4}\|f\|_2^2 + \mathrm{poly}(r)\varepsilon,\end{aligned}$$

which is at most $\frac{1}{4} + \mathsf{poly}(r)\varepsilon$ as $\|f\|_2 = 1$. Taking a square root gives that $\|Tf\|_2 \leqslant 1/2 + \mathrm{poly}(r)\varepsilon$. □

3.1.2 Almost Uniqueness of Good Solutions to Affine UG Instances

We begin by defining shifts of assignments to Affine Unique-Games instances G.

Definition 3.3. *Given an affine Unique-Games instance Φ with alphabet S_m, an assignment A to it and $\pi \in S_m$, we denote by $A \circ \pi$ denote the assignment that $(A \circ \pi)(v) = X(v)\pi$.*

Suppose Φ is an instance of affine Unique-Games as in Definition 1.4, and suppose that A is an assignment to it. It can easily be seen that $\mathrm{val}(A) = \mathrm{val}(A \circ \pi)$ for every $\pi \in S_m$. Thus, if A satisfies many of the constraints of Ψ, then so does $A \circ \pi$. The following claim says that if the underlying constraint graph is an expander, then all good assumptions are essentially shifts of one good assignment X. This idea appeared in [BBK+21] in the context of the Abelian version of affine Unique-Games, but the proof in the non-Abelian case is essentially the same and we give it for completeness.

Claim 3.4. *Let $\Phi = (G, \Pi, S_m)$ be a UG instance on G which is an expander graph with second eigenvalue λ, and let $X, X' \in S_m^{V(G)}$ be two solutions to Φ. Then there exists a permutation $\pi \in S_m$ such that*

$$\Pr_{v \in G}[X(v) \neq X'(v)\pi] \leqslant \frac{\mathit{viol}(X) + \mathit{viol}(X')}{1-\lambda}.$$

Proof. Fix X and X' as in the statement of the claim, and partition $V(G) = \cup_{\pi \in S_m} C_\pi$ where for each $\pi \in S_m$ we define $C_\pi = \{u \in G : X(v) = X'(v)\pi\}$. Note that if an edge $(u, v) \in G$ is satisfied by both X and X', then its endpoints lie in the same part C_π. Indeed, suppose that $X(v) = X'(v)\pi$, then

$$X(u) = \pi_{u,v} X(v) = \pi_{u,v} X'(v)\pi = X'(u)\pi.$$

Thus, if (u, v) goes across distinct parts of the partition $\{C_\pi\}_{\pi \in S_m}$, then it must be violated either by X or by X'.

For a set of vertices S let $\mu(S)$ denote the measure of S in $V(G)$ and for a set of edges T let $\mu(T)$ denote the measure of T in $E(G)$. Let $E(S, \overline{S})$ denote the set of edges that cross between S and its complement. By (the easy direction of) Cheeger's inequality, for each $\pi \in S_m$ we have that

$$\frac{1}{2}\mu(E(C_\pi, \overline{C_\pi})) \geqslant (1-\lambda)\mu(C_\pi)(1-\mu(C_\pi)).$$

Summing this over $\pi \in S_m$ we get,

$$\frac{1}{2}\sum_{\pi \in S_m} \mu(E(C_\pi, \overline{C_\pi})) \geqslant (1-\lambda)(1-\sum_{\pi}\mu(C_\pi)^2).$$

Note that each edge that crosses between distinct parts of $\{C_\pi\}_{\pi \in S_m}$ is counted twice on the left side, and by our earlier observation it must be violated either by X or by X', and so $\frac{1}{2}\mu(E(C_\pi, \overline{C_\pi})) \leqslant \mathrm{viol}(X) + \mathrm{viol}(X')$. Plugging this and simplifying gives that

$$\sum_{\pi} \mu(C_\pi)^2 \geqslant 1 - \frac{\mathrm{viol}(X) + \mathrm{viol}(X')}{1-\lambda},$$

and as $\sum_{\pi \in S_m} \mu(C_\pi)) = 1$, it follows that there exists $\pi \in S_m$ such that $\mu(C_\pi) \geqslant 1 - \frac{\mathrm{viol}(X)+\mathrm{viol}(X')}{1-\lambda}$. □

3.2 Exponential Bound on the Coboundary Constant via Lopsided Induction

Our bounds on the coboundary constants C_{r_1,r_2,r_3} will always follow an inductive strategy. All of our inductive arguments will be of similar spirit, though they differ in some technical aspects that tailor them for different uses. This section is devoted for the most simplistic of these inductive approaches, for which we have two utilities. First, it will be useful for us to handle r_1, r_2, r_3 that are relatively small (e.g. $r^{0.99}$). Secondly, it will be useful for us when we extend the base case into the extended base case as described the introduction. In this case, we have the following result:

Lemma 3.5. *There exists an absolute constant $K > 0$ such that the following holds. Suppose that μ is an ε-product measure with $\varepsilon \leqslant r^{-K}$ which is a $(C_{1,1,1}(\mu), \beta_{1,1,1}(\mu))$-coboundary expander over S_m. Then for all $k_1, k_2, k_3 \in \mathbb{N}$ such that $k_1 + k_2 + k_3 \leqslant r$, μ is a $(C_{k_1,k_2,k_3}(\mu), \beta_{k_1,k_2,k_3}(\mu))$-coboundary expander over S_m with*

$$C_{k_1,k_2,k_3}(\mu) \leqslant O(C_{1,1,1}(\mu))^{k_1+k_2+k_3}, \quad \beta_{k_1,k_2,k_3}(\mu) \leqslant O(C_{1,1,1}(\mu))^{k_1+k_2+k_3}\beta_{1,1,1}(\mu).$$

Proof. Fix any three pairwise disjoint sets R_1, R_2, R_3 with sizes $0 < r_1, r_2, r_3 \in \mathbb{N}$ such that $\sum r_i \leqslant r$, indices $j_1 \in R_1, j_2 \in R_2$ and $j_3 \in R_3$, a set $S \subset [d] \setminus \bigcup R_i$, and a restriction $A_0 \in \mathsf{supp}(\mu^S)$. Let $\mathcal{D}$ denote $\mu \mid X_S = A_0$. For any restriction $a_1 \in \mathsf{supp}(\mathcal{D}^{j_1})$, let G_{a_1} denote the induced subgraph $T(R_1 \setminus j_1, R_2, R_3; \mathcal{D}|X_{j_1} = a_1)$ and similarly define the graphs G_{a_2}, G_{a_3} for every $a_2 \in \mathsf{supp}(\mathcal{D}^{j_2}), a_3 \in \mathsf{supp}(\mathcal{D}^{j_3})$. Then we will show that,

$$C(T(R_1, R_2, R_3; \mathcal{D})) \leqslant C(T(\{j_1\}, \{j_2\}, \{j_3\}; \mathcal{D})) \cdot \max_{\substack{i \in [3] \\ a_i \in \mathsf{supp}(\mathcal{D}^{j_i})}} (C(G_{a_i})). \tag{2}$$

$$\beta(T(R_1, R_2, R_3; \mathcal{D})) \leqslant C(T(\{j_1\}, \{j_2\}, \{j_3\}; \mathcal{D})) \cdot \max_{\substack{i \in [3] \\ a_i \in \mathsf{supp}(\mathcal{D}^{j_i})}} (\beta(G_{a_i})) + \beta(T(\{j_1\}, \{j_2\}, \{j_3\}; \mathcal{D})). \tag{3}$$

Additionally, when either of the r_i's are 0 we show that $C(T(R_1, R_2, R_3; \mathcal{D})) \leqslant 1$, as well as that $\beta(T(R_1, R_2, R_3; \mathcal{D})) = 0$. This along with the equations above is enough to conclude the lemma as we show in the end of the proof.

To prove (2) and (3) fix any affine UG instance Φ on $T(R_1, R_2, R_3; \mathcal{D})$ which has δ-fraction of inconsistent triangles.

Setting up lists on restrictions: For every $a_i \in \mathsf{supp}(\mathcal{D}^{j_i})$ let $X_{a_i} \in S_m^{V(G_{a_i})}$ be an assignment on $V(G_{a_i})$ with maximum value, and let $L_{j_i \to a_i}$ be an ordered list of solutions indexed by permutations in S_m, where $L_{S_i \to a_i}[\pi] = X_{a_i} \circ \pi$.

Setting up permutations between restrictions: For every two restrictions of different parts, say a of j_1 and b of j_2 where (a, b) is a valid restriction (that is, where $(a, b) \in \mathsf{supp}(\mathcal{D}^{\{j_1,j_2\}})$), let $G_{a,b}$ be the induced subgraph $T(R_1, R_2, R_3; \mathcal{D}|(X_{j_1} = a, X_{j_2} = b))$. Then by Claim 3.4 there exists a permutation $\pi_{a,b}$ that satisfies the following:

$$\Pr_{v \sim G_{a,b}}[X_a(v) \neq X_b(v)\pi_{a,b}] \leqslant \frac{\text{viol}(X_a; G_{a,b}) + \text{viol}(X_b; G_{a,b})}{1 - \lambda(G_{a,b})} \lesssim \text{viol}(X_a; G_{a,b}) + \text{viol}(X_b; G_{a,b}), \tag{4}$$

where we used Claim 3.2 to bound $\lambda(G_{a,b}) \leqslant 1/2 + \text{poly}(r)\varepsilon \leqslant 0.51$. Let

$$\text{Bad}(a,b) = \{v \in G_{a,b} \mid X_a(v) \neq X_b(v)\pi_{a,b}\}.$$

For any restriction b as above, let $L_b \circ \pi$ denote the list where $L_b \circ \pi[\pi'] = L_b[\pi\pi'] = X_b \circ \pi\pi'$. With these notations and the definition of $\text{Bad}(a,b)$, we get that $L_a|_{G_{a,b}\cap\overline{\text{Bad}(a,b)}} = L_b \circ \pi_{a,b}|_{G_{a,b}\cap\overline{\text{Bad}(a,b)}}$.

Counting bad triangles: For any restriction $(a_1,a_2,a_3) \in \mathsf{supp}(\mathcal{D}^{\{j_1,j_2,j_3\}})$ let G_{a_1,a_2,a_3} denote the graph $T(R_1 \setminus j_1, R_2 \setminus j_2, R_3 \setminus j_3; \mathcal{D}|(a_1,a_2,a_3))$. Clearly, we have that,

$$\mathop{\mathbb{E}}_{a_3\sim\mathcal{D}^{j_3}|(a_1,a_2)}[\mu_{G_{a_1,a_2,a_3}}(\text{Bad}(a_1,a_2))] = \mu_{G_{a_1,a_2}}(\text{Bad}(a_1,a_2)).$$

Fix a_1, a_2. Using Markov's inequality and (4) it follows that

$$\Pr_{a_3\sim\mathcal{D}^{j_3}|(a_1,a_2)}\left[\mu_{G_{a_1,a_2,a_3}}(\text{Bad}(a_1,a_2)) > \frac{1}{100}\right] \lesssim \text{viol}(X_{a_1}; G_{a_1,a_2}) + \text{viol}(X_{a_2}; G_{a_1,a_2}). \tag{5}$$

We say a restriction (a_1,a_2,a_3) is bad if $\mu_{G_{a_1,a_2,a_3}}(\text{Bad}(a_i,a_k)) > 1/100$ for some $i \neq k$. By (5) we get

$$\begin{aligned}
&\mathop{\mathbb{E}}_{(a_1,a_2,a_3)\sim\mathcal{D}^{\cup j_i}}[\mathbb{1}((a_1,a_2,a_3)\text{ is bad})] \\
&\leqslant \sum_{i\in[3]} \mathop{\mathbb{E}}_{\substack{(a_\ell,a_k)\sim\mathcal{D}^{j_\ell\cup j_k}\\ \ell,k\neq i}} \mathop{\mathbb{E}}_{a_i\sim\mathcal{D}^{j_i}|(a_\ell,a_k)}[\mathbb{1}(\mu_{G_{a_1,a_2,a_3}}(\text{Bad}(a_\ell,a_k)) > \frac{1}{100})] \\
&\lesssim \sum_{i\in[3]} \mathop{\mathbb{E}}_{\substack{(a_\ell,a_k)\sim\mathcal{D}^{j_\ell\cup j_k}\\ \ell,k\neq i}}[\text{viol}(X_{a_\ell}; G_{a_\ell,a_k}) + \text{viol}(X_{a_k}; G_{a_\ell,a_k})] \\
&\lesssim \sum_{i\in[3]} \mathop{\mathbb{E}}_{a_i\sim\mathcal{D}^{j_i}}[\text{viol}(X_{a_i}; G_{a_i})],
\end{aligned} \tag{6}$$

where in the first inequality we used the union bound, the second inequality we used (5). By definition we have that $\text{viol}(X_{a_1}; G_{a_1}) \leqslant C(G_{a_1})\varepsilon_{a_1} + \beta(G_{a_1})$, where ε_{a_1} is the fraction of violated triangles in G_{a_1}. Therefore,

$$(6) \lesssim \sum_{i\in[3]} \mathop{\mathbb{E}}_{a_i\sim\mathcal{D}^{j_i}}[C(G_{a_i})\varepsilon_{a_i} + \beta(G_{a_i})] \lesssim \max_{\substack{i\in[3]\\ a_i\in\mathsf{supp}(\mathcal{D}^{j_i})}}(C(G_{a_i}))\delta + \max_{\substack{i\in[3]\\ a_i\in\mathsf{supp}(\mathcal{D}^{j_i})}}(\beta(G_{a_i})) := \delta'.$$

Creating a UG instance on graph over restrictions: Let Ψ be the following UG instance on $H = T(\{j_1\},\{j_2\},\{j_3\};\mathcal{D})$: each edge (a_i,a_k) has the constraint $\pi^{-1}_{a_i,a_k}$. That is, we want to find a solution A that maximizes the fraction of edges satisfying $A(a_i) = \pi^{-1}_{a_i,a_k}A(a_k)$. Note that a triangle (a_1,a_2,a_3) is consistent if $\pi^{-1}_{a_3,a_1}\pi^{-1}_{a_1,a_2}\pi^{-1}_{a_2,a_3} = \text{id}$ or equivalently if $\pi_{a_2,a_3}\pi_{a_1,a_2}\pi_{a_3,a_1} = \text{id}$.

First note that if a triangle (a_1,a_2,a_3) is not bad, then it is consistent. To see this, note that in this case, by the union bound we have $\mu_{G_{a_1,a_2,a_3}}(\text{Bad}(a_1,a_2) \cup \text{Bad}(a_2,a_3) \cup \text{Bad}(a_1,a_3)) \leqslant 0.03$, which gives that $\mu_{G_{a_1,a_2,a_3}}(\text{Good}(a_1,a_2,a_3)) \geqslant 0.97 > 0$, where the latter set is defined as those vertices in G_{a_1,a_2,a_3} that don't belong to any of $\text{Bad}(a_i,a_k)$. We have that:

$$L_{a_1}|_{\text{Good}(a_1,a_2,a_3)} = L_{a_2} \circ \pi_{a_1,a_2}|_{\text{Good}(a_1,a_2,a_3)}$$

$$L_{a_2}|_{\text{Good}(a_1,a_2,a_3)} = L_{a_3} \circ \pi_{a_2,a_3}|_{\text{Good}(a_1,a_2,a_3)}$$
$$L_{a_3}|_{\text{Good}(a_1,a_2,a_3)} = L_{a_1} \circ \pi_{a_3,a_1}|_{\text{Good}(a_1,a_2,a_3)},$$

which implies that:

$$L_{a_3}|_{\text{Good}(a_1,a_2,a_3)} = L_{a_3} \circ \pi_{a_2,a_3} \circ \pi_{a_1,a_2} \circ \pi_{a_3,a_1}|_{\text{Good}(a_1,a_2,a_3)} = L_{a_3} \circ \pi_{a_2,a_3}\pi_{a_1,a_2}\pi_{a_3,a_1}|_{\text{Good}(a_1,a_2,a_3)}.$$

Taking any $v \in \text{Good}(a_1, a_2, a_3)$ we get that, $L_{a_3}[\text{id}][v] = L_{a_3} \circ \pi_{a_2,a_3}\pi_{a_1,a_2}\pi_{a_3,a_1}[\text{id}][v]$, where the former is $X_{a_3}(v)$ and the latter is $X_{a_3}(v)\pi_{a_2,a_3}\pi_{a_1,a_2}\pi_{a_3,a_1}$, which implies $\pi_{a_2,a_3}\pi_{a_1,a_2}\pi_{a_3,a_1} = \text{id}$.

Using the coboundary expansion of the graph $T(\{j_1\}, \{j_2\}, \{j_3\}; \mathcal{D})$ we get that there exists a UG solution to the vertices violating at most $(C(H)\delta' + \beta(H))$-fraction of the edges. Call this solution A.

Lifting the solution: We will now use A to create a highly satisfying solution B to $G = T(R_1, R_2, R_3; \mathcal{D})$. To each vertex $u \in R_i$, define $B(u) = L_{u|_{j_i}}[A(u|_{j_i})][u]$. We will now upper bound the fraction of edges that B violates for Φ.

Consider an edge $(u_i, u_k) \in G$ between parts R_i, R_k, and let (a_i, a_k) denote the edge $(u_i|_{j_i}, u_k|_{j_k})$ in the restriction graph H. Let $\text{Good}(G)$ be the set of edges where the following three events hold: (1) A satisfies the edge (a_i, a_k), i.e. $A(a_i) = \pi^{-1}_{a_i,a_k} A(a_k)$, (2) X_{a_i} satisfies the edge (u_i, u_k), i.e. $X_{a_i}(u_i) = \pi_{u_i,u_k} X_{a_i}(u_k)$ and (3) u_k does not belong to $\text{Bad}(a_i, a_k)$, i.e. $X_{a_i}(u_k) = X_{a_k}(u_k)\pi_{a_i,a_k}$. It is easy to see that B satisfies every good edge, that is, $B(u_i) = \pi_{u_i,u_k} B(u_k)$. Indeed, using the events (1), (2) and (3), we have that,

$$\begin{aligned}
B(u_i) &= L_{a_i}[A(a_i)][u_i] = X_{a_i}(u_i)A(a_i) \\
&= X_{a_i}(u_i)\pi^{-1}_{a_i,a_k} A(a_k) && \text{by (1)} \\
&= \pi_{u_i,u_k} X_{a_i}(u_k)\pi^{-1}_{a_i,a_k} A(a_k) && \text{by (2)} \\
&= \pi_{u_i,u_k} X_{a_k}(u_k)\pi_{a_i,a_k}\pi^{-1}_{a_i,a_k} A(a_k) && \text{by (3)} \\
&= \pi_{u_i,u_k} B(u_k).
\end{aligned}$$

So it suffices to upper bound the fraction of edges that are not in $\text{Good}(G)$, which we denote by $\text{Bad}(G)$. Sampling an edge, we can upper bound the probability it doesn't satisfy at least one of the events. For (1),

$$\Pr_{(u_i,u_k)\in G}[A(a_i) \neq \pi^{-1}_{a_i,a_k} A(a_k)] = \text{viol}(A) \lesssim C(H)\delta' + \beta(H).$$

For (2),

$$\begin{aligned}
\Pr_{(u_i,u_k)\in G}[X_{a_i}(u_i) \neq \pi_{u_i,u_k} X_{a_i}(u_k)] &= \mathop{\mathbb{E}}_{\substack{i\sim[3] \\ a_i\sim\mathcal{D}^{j_i}}} \mathop{\mathbb{E}}_{\substack{(u_1,u_2,u_3)\sim\mathcal{D}|(X_{j_i}=a_i) \\ k\sim[3]\setminus i}} [\mathbb{1}(X_{a_i}(u_i) \neq \pi_{u_i,u_k} X_{a_i}(u_k))] \\
&\lesssim \mathop{\mathbb{E}}_{\substack{i\sim[3] \\ a_i\sim\mathcal{D}^{j_i}}} [\text{viol}(X_{a_i}; G_{a_i})] \lesssim \delta'.
\end{aligned}$$

For (3), using (4) we have

$$\begin{aligned}
\Pr_{(u_i,u_k)\in G}[u_k \notin \text{Bad}(a_i, a_k)] &\lesssim \mathop{\mathbb{E}}_{(a_i,a_k)\sim\mathcal{D}^{j_i\cup j_k}} \mathop{\mathbb{E}}_{\substack{(u_i,u_k,u_\ell)\sim\mathcal{D}|(a_i,a_k) \\ r\sim\{i,k\}}} [\mathbb{1}(u_r \notin \text{Bad}(a_i, a_k))] \\
&\lesssim \mathop{\mathbb{E}}_{(a_i,a_k)\sim\mathcal{D}^{j_i\cup j_k}} [\text{viol}(X_{a_i}; G_{a_i,a_k}) + \text{viol}(X_{a_k}; G_{a_i,a_k})]
\end{aligned}$$

$$\lesssim \mathop{\mathbb{E}}_{a_i \sim \mathcal{D}^{j_i}} [\text{viol}(X_{a_i}; G_{a_i})]$$
$$\lesssim \delta'.$$

Adding up these probabilities we get,

$$\Pr_{(u_i,u_k)\in G}[(u_i, u_k) \in \text{Bad}(G)] \lesssim \delta' + C(H)\delta' + \beta(H)$$
$$\lesssim \left(C(H) \max_{\substack{i\in[3] \\ a_i \in \mathsf{supp}(\mathcal{D}^{j_i})}} (C(G_{a_i})) \right) \delta + \left(C(H) \max_{\substack{i\in[3] \\ a_i \in \mathsf{supp}(\mathcal{D}^{j_i})}} (\beta(G_{a_i})) + \beta(H) \right),$$

which completes the proof of (2) and (3).

Proving base cases when r_i's are 0: Without loss of generality assume that $|R_1| = 0$. Recall that the graph $G = T(R_1, R_2, R_3; \mathcal{D})$ has one vertex $\emptyset$ in its first part that is connected to all the vertices $\mathsf{supp}(\mu^{R_2})$ and $\mathsf{supp}(\mu^{R_3})$ in the second and third parts named P_2, P_3. We can show that $C(G) \leqslant 1$ via the following algorithm to get a satisfying solution to Φ. We assign the identity permutation to $\emptyset$, and then propagate this solution to all the vertices in parts P_2 and P_3. Now note that by definition all the edges between $\emptyset$ and P_2 or P_3 are satisfied. For an edge (u, v) between P_2, P_3, it is violated only if the triangle $(\emptyset, u, v)$ is violated which happens only with probability δ. Therefore we get a solution violating at most $\delta/3$-fraction of the edges of G, which shows that $C(G) \leqslant 1$. The same argument holds when more of the R_i's have size 0.

Concluding the induction: Firstly note that (2), (3) imply that:

$$C_{r_1,r_2,r_3}(\mu) \lesssim C_{1,1,1}(\mu)C', \quad \beta_{r_1,r_2,r_3}(\mu) \lesssim C_{1,1,1}(\mu)\beta' + \beta_{1,1,1}(\mu) \tag{7}$$

where we take $C' = \max(C_{r_1-1,r_2,r_3}(\mu), C_{r_1,r_2-1,r_3}(\mu), C_{r_1,r_2,r_3-1}(\mu))$ and analogously pick β' to be $\beta' = \max(\beta_{r_1-1,r_2,r_3}(\mu), \beta_{r_1,r_2-1,r_3}(\mu), \beta_{r_1,r_2,r_3-1}(\mu))$.

To prove the final statement we can then use induction on $\sum_{i=1}^3 r_i$. The base case of induction is when either $\sum r_i = 3$ and each $r_i = 1$, or one of the r_i's is 0. In the first case $C_{r_1,r_2,r_3}(\mu) \leqslant C_{1,1,1}(\mu) \leqslant O(C_{1,1,1}(\mu))^3$, $\beta_{r_1,r_2,r_3}(\mu) \leqslant \beta_{1,1,1}(\mu) \leqslant O(C_{1,1,1}(\mu))^3\beta_{1,1,1}(\mu)$ and in the second case $C_{r_1,r_2,r_3} \leqslant 1 \leqslant O(C_{1,1,1}(\mu))^{\sum r_i}, \beta_{r_1,r_2,r_3} = 0 \leqslant C_{1,1,1}(\mu)^{\sum r_i}\beta_{1,1,1}(\mu)$ too. Therefore now let us assume the lemma statement holds for all (r_1, r_2, r_3) with $\sum r_i \leqslant t$ and let us prove it for (r'_1, r'_2, r'_3) with $\sum r'_i = t + 1$ and each $r'_i > 0$. Applying (7) on $C_{r'_1,r'_2,r'_3}(\mu)$ we get that,

$$C_{r'_1,r'_2,r'_3}(\mu) \leqslant O(C_{1,1,1}(\mu))C' \leqslant O(C_{1,1,1}(\mu))O(C_{1,1,1}(\mu))^{r'_1+r'_2+r'_3-1} = O(C_{1,1,1}(\mu))^{r'_1+r'_2+r'_3},$$

where in the second inequality we used the inductive bound on $C_{r'_1-1,r'_2,r'_3}(\mu), C_{r'_1,r'_2-1,r'_3}(\mu), C_{r'_1,r'_2,r'_3-1}(\mu)$. A similar argument for $\beta_{r'_1,r'_2,r'_3}(\mu)$ completes the induction thus proving the lemma. □

3.3 Subexponential Bounds via Non-Lopsided Induction

Throughout this section we fix $k = r^{0.01}$, three r-sized pairwise disjoint sets $R_1, R_2, R_3 \subset [d]$, $I = \bigcup_{i\in[3]} R_i$ and μ over $\prod_{i\in I} X_i$ that satisfy the following assumptions:

Assumption 1. *The following conditions hold:*

1. *The measure μ is an ε-product distribution with $\varepsilon \leqslant 2^{-r^{12}}$.*

2. *For all $k \leqslant r$, $S \subseteq I$ of size at most $3r - 3k$ and restrictions $a_0 \in \mathsf{supp}(\mu^S)$, for all k-sized pairwise disjoint sets A, B, C of $I \setminus S$, such that*

$$\max_{a \in A} a < \min_{b \in B} b \leqslant \max_{b \in B} b < \min_{c \in C} c,$$

the graph $T(A, B, C; \mu | X_S = a_0)$ is a $(\text{poly}(r), 2^{-r^{12}})$-coboundary expander over S_m.

3. *For each interval $I_j = \left\{\frac{jdk^{10}}{r}, \ldots, \frac{(j+1)dk^{10}}{r}\right\}$ and for each $i \in [3]$ we have that*

$$k^{10} - k^6 \leqslant |R_i \cap I_j| \leqslant k^{10} + k^6.$$

In words, the number of elements in R_i in the interval I_j is roughly k^{10} (which is the number of points a typical interval of that length has).

The main result of this section is the following lemma, asserting that if μ satisfies Assumption 1, then the corresponding tripartite graph is a coboundary expander with good parameters. More precisely:

Lemma 3.6. *Let $R_1, R_2, R_3 \subseteq$ and let μ be a probability measure over $\prod_{i \in I} X_i$ satisfying Assumption 1. Then the graph $T(R_1, R_2, R_3; \mu)$ is a $(\text{poly}(r)^{r/k}, 2^{-\Omega(r^{12})})$-coboundary expander over S_m.*

Proof. Let $I_j \subset [d]$ denote the interval $\left\{\frac{jdk^{10}}{r}, \ldots, \frac{(j+1)dk^{10}}{r}\right\}$, and write $|R_i \cap I_j| = k^{10} + c_{i,j}$ where $c_{i,j} \in [-k^6, k^6]$; note that $\sum_j c_{i,j} = 0$. We will use induction to prove the lemma. Throughout the induction, we will have three sets R_1', R_2', R_3', satisfying:

1. $R_i' \subseteq R_i$.

2. For all i, j, $|R_i' \cap I_j| = r_j + c_{i,j}$ for some $r_j \leqslant k^{10}$.

3. There exist at least three intervals with $r_j \geqslant k^9$, say $I_{j_1}, I_{j_2}, I_{j_3}$, with $j_1 < j_2 < j_3$.

Initially, we will take $R_1' = R_1$, $R_2' = R_2$, $R_3' = R_3$, and later steps will take subsets of these. Note that in particular (2) above implies that all R_i' are equal in size. Henceforth, we fix such R_1', R_2', R_3'. Note that by the second and third items, we may find three $3k$-sized disjoint sets S_1, S_2, S_3 satisfying that $S_i \subset I_{j_i}$ for $i = 1, 2, 3$ and furthermore for all $j \in [3]$, $|S_i \cap (R_j' \cap I_{j_i})| = k$, and we fix such S_1, S_2, S_3 henceforth.

Consider a distribution $\mathcal{D} = \mu | X_B = B_0$, where $B \subseteq I \setminus \bigcup_{i \in [3]} R_i'$ and $B_0 \in \mathsf{supp}(\mu^B)$. The bulk of the argument will be devoted to proving that

$$C(T(R_1', R_2', R_3'; \mathcal{D})) \leqslant \text{poly}(r) \cdot \max_{\substack{i \in [3], \\ a_i \in \mathsf{supp}(\mathcal{D}^{S_i})}} (C(T(R_1' \setminus S_i, R_2' \setminus S_i, R_3' \setminus S_i; \mathcal{D} | X_{S_i} = a_i))), \quad (8)$$

$$\beta(T(R_1', R_2', R_3'; \mathcal{D})) \leqslant \text{poly}(r) \cdot \max_{\substack{i \in [3], \\ a_i \in \mathsf{supp}(\mathcal{D}^{S_i})}} (\beta(T(R_1' \setminus S_i, R_2' \setminus S_i, R_3' \setminus S_i; \mathcal{D} | X_{S_i} = a_i))). \quad (9)$$

Once we establish these two inequalities, it is straightforward to conclude the lemma, and we do so in the end of the proof. Towards proving (8) and (9), fix any UG instance Φ on $G = T(R_1', R_2', R_3'; \mathcal{D})$ with δ-fraction of inconsistent triangles.

Setting up lists on restrictions: For every S_i and every restriction $a \in \mathsf{supp}(\mathcal{D}^{S_i})$, let G_a be the subgraph $T(R_1' \setminus S_i, R_2' \setminus S_i, R_3' \setminus S_i; \mathcal{D}|X_{S_i} = a)$. Let $X_a \in S_m^{V(G_a)}$ be an assignment on $V(G_a)$ with maximum value, and let L_a be an ordered list of solutions indexed by permutations in S_m, where $L_a[\pi] = X_a \circ \pi$.

Setting up permutations between restrictions: For every $i \neq j$, and every restriction a of S_i and b of S_j where $(a, b) \in \mathsf{supp}(\mathcal{D}^{S_i \cup S_j})$, let $G_{a,b}$ be the induced subgraph $T(R_1', R_2', R_3'; \mathcal{D}|(X_{S_i} = a, X_{S_j} = b))$. Let $\mathrm{viol}(X_a; G_{a,b})$ denote the fraction of edges in $G_{a,b}$ that are violated by the assignment X_a. Then by Claim 3.4 there exists a permutation $\pi_{a,b}$ satisfying that

$$\Pr_{v \sim G_{a,b}}[X_a(v) \neq X_b(v)\pi_{a,b}] \leqslant \frac{\mathrm{viol}(X_a; G_{a,b}) + \mathrm{viol}(X_b; G_{a,b})}{1 - \lambda(G_{a,b})} \lesssim \mathrm{viol}(X_a; G_{a,b}) + \mathrm{viol}(X_b; G_{a,b}), \quad (10)$$

where we used Claim 3.2 to bound $\lambda(G_{a,b})$ by $1/2 + \mathrm{poly}(r)\varepsilon < 0.51$. Let

$$\mathrm{Bad}(a, b) = \{v \in G_{a,b} \mid X_a(v) \neq X_b(v)\pi_{a,b}\}.$$

For any restriction b as above, let $L_b \circ \pi$ denote the list where $L_b \circ \pi[\pi'] = L_b[\pi\pi'] = X_b \circ \pi\pi'$. With these notations and the definition of $\mathrm{Bad}(a, b)$, we get that $L_a|_{G_{a,b} \cap \overline{\mathrm{Bad}(a,b)}} = L_b \circ \pi_{a,b}|_{G_{a,b} \cap \overline{\mathrm{Bad}(a,b)}}$.

Counting bad triangles: Let $S = \bigcup_i S_i$. For any restrictions a_1, a_2, a_3 of S_1, S_2, S_3 let G_{a_1,a_2,a_3} denote the graph $T(R_1' \setminus S, R_2' \setminus S, R_3' \setminus S; \mathcal{D}|(a_1, a_2, a_3))$ when (a_1, a_2, a_3) is a valid restriction. Clearly, we have thats

$$\mathop{\mathbb{E}}_{a_3 \sim \mathcal{D}^{S_3}|(a_1,a_2)}[\mu_{G_{a_1,a_2,a_3}}(\mathrm{Bad}(a_1, a_2))] = \mu_{G_{a_1,a_2}}(\mathrm{Bad}(a_1, a_2)).$$

Fix a_1, a_2. Using Markov's inequality and (10) we get

$$\Pr_{a_3 \sim \mathcal{D}^{S_3}|(a_1,a_2)}\left[\mu_{G_{a_1,a_2,a_3}}(\mathrm{Bad}(a_1, a_2)) > \frac{1}{100}\right] \lesssim \mathrm{viol}(X_{a_1}; G_{a_1,a_2}) + \mathrm{viol}(X_{a_2}; G_{a_1,a_2}). \quad (11)$$

We say a restriction (a_1, a_2, a_3) is bad if $\mu_{G_{a_1,a_2,a_3}}(\mathrm{Bad}(a_i, a_j)) > 1/100$ for some $i \neq j$. By (11) we get that,

$$\begin{aligned}
&\mathop{\mathbb{E}}_{(a_1,a_2,a_3) \sim \mathcal{D}^{\cup S_i}}[\mathbb{I}((a_1, a_2, a_3) \text{ is bad})] \\
&\leqslant \sum_{i \in [3]} \mathop{\mathbb{E}}_{\substack{(a_j,a_k) \sim \mathcal{D}^{S_j \cup S_k} \\ j,k \neq i}} \mathop{\mathbb{E}}_{a_i \sim \mathcal{D}^{S_i}|(a_j,a_k)}[\mathbb{I}(\mu_{G_{a_1,a_2,a_3}}(\mathrm{Bad}(a_j, a_k)) > \frac{1}{100})] \\
&\lesssim \sum_{i \in [3]} \mathop{\mathbb{E}}_{\substack{(a_j,a_k) \sim \mu^{S_j \cup S_k} \\ j,k \neq i}} [\mathrm{viol}(X_{a_j}; G_{a_j,a_k}) + \mathrm{viol}(X_{a_k}; G_{a_j,a_k})] \\
&= \sum_{i \in [3]} \mathop{\mathbb{E}}_{a_i \sim \mathcal{D}^{S_i}}[\mathrm{viol}(X_{a_i}; G_{a_i})], \quad (12)
\end{aligned}$$

where in the first inequality we used the union bound and in the second inequality we used (11). By definition we have that $\mathrm{viol}(X_{a_i}; G_{a_i}) \leqslant C(G_{a_i})\varepsilon_{a_i} + \beta(G_{a_i})$, where ε_{a_i} is the fraction of violated triangles in G_{a_i}. Therefore,

$$(12) \leqslant \sum_{i \in [3]} \mathop{\mathbb{E}}_{a_i \sim \mathcal{D}^{S_i}}[C(G_{a_i})\varepsilon_{a_i} + \beta(G_{a_i})] \lesssim \max_{\substack{i \in [3] \\ a_i \in \mathsf{supp}(\mathcal{D}^{S_i})}} (C(G_{a_i})\delta + \beta(G_{a_i})) := \delta''.$$

Creating a UG instance on graph over restrictions: Let Ψ be the following UG instance on $H = T(S_1, S_2, S_3; \mathcal{D})$: each edge (a_i, a_j) has the constraint $\pi^{-1}_{a_i,a_j}$. That is, we want to find a solution A that maximizes the fraction of edges satisfying $A(a_i) = \pi^{-1}_{a_i,a_j} A(a_j)$. Note that a triangle (a_1, a_2, a_3) is consistent if $\pi^{-1}_{a_3,a_1}\pi^{-1}_{a_1,a_2}\pi^{-1}_{a_2,a_3} = \mathrm{id}$ or equivalently if $\pi_{a_2,a_3}\pi_{a_1,a_2}\pi_{a_3,a_1} = \mathrm{id}$.

First note that if a triangle (a_1, a_2, a_3) is not bad, then it is consistent. The proof is the same as that in Lemma 3.5 hence we omit it here. Using the coboundary expansion of H, we get that there exists a UG solution A to the vertices violating at most $C(H)\delta'' + \beta(H)$-fraction of the edges.

Since $S_i \subset I_{j_i}$ for all i, we get that $\max_{a\in S_1} a < \min_{b\in S_2} b$ and $\max_{b\in S_2} b < \min_{c\in S_3} c$. Therefore by Assumption 1, $C(H) \leqslant \mathrm{poly}(r)$ and $\beta(H) \leqslant 2^{-r^{12}}$. Therefore $\mathrm{viol}(A) \leqslant \mathrm{poly}(r)\delta'' + 2^{-r^{12}} := \delta'$.

Lifting the solution: We will now use A to create a highly satisfying solution B to $G = T(R'_1, R'_2, R'_3; \mathcal{D})$. For every vertex $v \in \mathsf{supp}(\mu^{R'_i})$ and restriction $a \in \mathsf{supp}(\mathcal{D}^{S_1}|(X_{R'_i} = v))$, let $g_a(v)$ denote the permutation $L_a[A(a)][v]$. We will choose a randomized assignment as follows: to every vertex $u \in \mathsf{supp}(\mu^{R'_i})$, choose a random $s \sim \mathcal{D}^{S_1}|(X_{R'_i} = u)$ and assign $B(u) = g_s(u)$. We will now upper bound the expected fraction of edges that B violates for Φ.

Consider an edge $(u_i, u_j) \in G$ between parts R'_i, R'_j. This edge is satisfied if there exists $s' \in \mathsf{supp}(\mathcal{D}^{S_1}|(X_{R'_i} = u_i, X_{R'_j} = u_j))$ such that, (1) $B(u_i) = g_{s'}(u_i)$, (2) $B(u_j) = g_{s'}(u_j)$ and (3) The assignment $(g_{s'}(u_i), g_{s'}(u_j))$ satisfies the edge (u_i, u_j) or equivalently (u_i, u_j) is satisfied by the assignment $X_{s'}$. To evaluate the probability there is such s', we sample $s' \sim \mathcal{D}^{S_1}|(X_{R'_i} = u_i, X_{R'_j} = u_j)$ and consider each one of the events, starting with event (1). For it, the probability it doesn't hold is at most

$$\mathop{\mathbb{E}}_{B}\ \mathop{\mathbb{E}}_{(u_i,u_j)\sim G}\ \mathop{\mathbb{E}}_{s'\sim\mathcal{D}^{S_1}|(X_{R'_i}=u_i, X_{R'_j}=u_j)}[\mathbb{1}(B(u_i) \neq g_{s'}(u_i))] = \mathop{\mathbb{E}}_{u\in G}\ \mathop{\mathbb{E}}_{s,s'\sim\mathcal{D}^{S_1}|u}[\mathbb{1}(g_{s'}(u) \neq g_s(u))].$$

To calculate this, let us first calculate a bound on:

$$\mathop{\mathbb{E}}_{u\in G}\ \mathop{\mathbb{E}}_{\substack{i\neq j\in[3]\\ (s,s')\sim\mathcal{D}^{S_i\cup S_j}|u}}[\mathbb{1}(g_{s'}(u_i) \neq g_s(u_i))].$$

Fix a vertex $u \in R'_i$ for some i. It is easy to check that an if an edge $(s, s') \in T(S_1, S_2, S_3; \mathcal{D}|u)$ is satisfied by A and $u \notin \mathrm{Bad}(s, s')$, then $g_s(u) = g_{s'}(u)$. Thus,

$$\begin{aligned}
&\mathop{\mathbb{E}}_{u\in G}\ \mathop{\mathbb{E}}_{\substack{i\neq j\in[3]\\ (s,s')\sim\mathcal{D}^{S_i\cup S_j}|u}}[\mathbb{1}(g_{s'}(u) \neq g_s(u))]\\
&\leqslant \mathop{\mathbb{E}}_{u\in G}\ \mathop{\mathbb{E}}_{\substack{i\neq j\in[3]\\ (s,s')\sim\mathcal{D}^{S_i\cup S_j}|u}}[\mathbb{1}((s,s') \text{ not satisfied by } A)] + \mathop{\mathbb{E}}_{u\in G}\ \mathop{\mathbb{E}}_{\substack{i\neq j\in[3]\\ (s,s')\sim\mathcal{D}^{S_i\cup S_j}|u}}[\mathbb{1}(u \in \mathrm{Bad}(s,s'))]\\
&= \mathrm{viol}(A) + \mathop{\mathbb{E}}_{\substack{(s,s')\sim E(T(S_1,S_2,S_3;\mathcal{D}))\\ u\in G_{s,s'}}}[\mathbb{1}(u \in \mathrm{Bad}(s,s'))]\\
&= \delta' + 2\mathop{\mathbb{E}}_{u\in T(S_1,S_2,S_3;\mathcal{D})}[\mathrm{viol}(X_u)]\\
&\lesssim \delta'.
\end{aligned}$$

We are ready to bound the probability that event (1) does not happen. Towards this end, for each vertex u define $p_u := \Pr_{(s,s')\sim\mathcal{D}^{S_1\cup S_2}|u}[g_{s'}(u) \neq g_s(u)]$, so that the above inequality translates to $\mathbb{E}_{u\in G}[p_u] \lesssim \delta'$.

Let $p'_u = \Pr_{s,s' \sim \mathcal{D}^{S_1}|u}[g_{s'}(u) \neq g_s(u)]$. By Lemma 2.6, the second largest singular value of the bipartite graph $A(S_1, S_2; \mathcal{D}|u)$ is at most $\varepsilon \cdot \mathrm{poly}(r) \leqslant 0.01$ for all u, so by the easy direction of Cheeger's inequality we get that, $p'_u \leqslant O(p_u)$, which gives us that,

$$\mathop{\mathbb{E}}_{u \in G} \mathop{\mathbb{E}}_{s,s' \sim \mathcal{D}^{S_1}|u}[\mathbb{1}(g_{s'}(u) \neq g_s(u))] \lesssim \delta',$$

thus bounding the probability of event (1). One can check that the probability of event (2) is the same as event (1), hence let us proceed to event (3). For that we get,

$$\mathop{\mathbb{E}}_{(u_i,u_j) \sim G} \mathop{\mathbb{E}}_{s \sim \mathcal{D}^{S_1}|(X_{R'_i}=u_i, X_{R'_j}=u_j)}[\mathbb{1}((u_i,u_j) \in \mathrm{viol}(X_s))] \lesssim \delta'.$$

We see that sampling an edge (u_i, u_j), s and s' fails to satisfy at least one of the events (1), (2) and (3) with probability at most $O(\delta')$. Thus, with probability at most $O(\delta')$ over the choice of (u_i, u_j) and s, there is no s' like that and otherwise we get that s' satisfies all of (1), (2) and (3). This shows that in expectation the assignment B that we get violates at most $O(\delta') \leqslant \mathrm{poly}(r) \max_{i \in [3] a_i \in \mathsf{supp}(\mathcal{D}^{S_i})}(C(G_{a_i})\delta + \beta(G_{a_i}))$ fraction of the edges of G, which completes the proof of (8), (9).

Concluding the induction: Let us now use (8), (9) to prove the lemma. For any sets R'_1, R'_2, R'_3 satisfying the first two items in our assumption about R'_i's above with $|R'_i| = \ell k$ and distribution $\mathcal{D}$ as above we will show by induction that $C(T(R'_1, R'_2, R'_3; \mathcal{D})) \leqslant \mathrm{poly}(r)^{\ell} \cdot \mathrm{poly}(r)^{r/k}$ and $\beta(T(R'_1, R'_2, R'_3; \mathcal{D})) \leqslant 2^{-\Omega(r^{12})}$. Let us prove the base case first. When $\ell = 1$, we can use Lemma 3.5 to bound $C(T(R'_1, R'_2, R'_3; \mathcal{D}))$ by $O(C_{1,1,1}(\mu))^{3k} \leqslant \mathrm{poly}(r)^k \leqslant \mathrm{poly}(r)^{r/k}$, since $C_{1,1,1}(\mu) \leqslant \mathrm{poly}(r)$ by Assumption 1. Our next base case is when the sets R'_i's do not satisfy the third item in the assumptions about R'_i's. That is, for all but at most two intervals I_j, $|R'_i \cap I_j| \leqslant k^9 + c_{i,j} \leqslant k^9 + k^6$. This implies that $|R'_i| \leqslant (r/k^{10}) \cdot (k^9 + k^6) + 2 \cdot (k^{10} + k^6) \lesssim r/k$. In this case, again using Lemma 3.5 we get that $C(T(R'_1, R'_2, R'_3; \mathcal{D})) \leqslant \mathrm{poly}(r)^{r/k}$ as required.

Let us now show the inductive step. Assume that we have proved the statement for ℓ and let us prove it for sets R'_i's satisfying assumptions (1), (2) and (3) with $|R'_i| = (\ell+1)k$. Applying (8) we get that there are $3k$-sized S_i's so that,

$$C(T(R'_1, R'_2, R'_3; \mathcal{D})) \leqslant \mathrm{poly}(r) \cdot \max_{\substack{i \in [3], \\ a_i \in \mathsf{supp}(\mathcal{D}^{S_i})}} (C(T(R'_1 \setminus S_i, R'_2 \setminus S_i, R'_3 \setminus S_i; \mathcal{D}|X_{S_i} = a_i))).$$

One can now check that the sets $(R'_1 \setminus S_1, R'_2 \setminus S_1, R'_3 \setminus S_1)$ satisfy assumptions (1) and (2), therefore $C(T(R'_1 \setminus S_1, R'_2 \setminus S_1, R'_3 \setminus S_1; \mathcal{D}|a_1)) \leqslant \mathrm{poly}(r)^{\ell} \cdot \mathrm{poly}(r)^{r/k}$ by the induction hypothesis. The same holds for S_2 and S_3, thus giving us that, $C(T(R'_1, R'_2, R'_3; \mathcal{D})) \leqslant \mathrm{poly}(r)^{\ell+1}\mathrm{poly}(r)^{r/k}$ as required. This implies that for $\ell = r/k$, we get that $C(T(R_1, R_2, R_3; \mu)) \leqslant \mathrm{poly}(r)^{r/k}$.

Applying a similar argument on β gives that $\beta(T(R_1, R_2, R_3; \mu)) \leqslant 2^{-\Omega(r^{12})}$. □

4 Base Case and Extended Base Case for Spherical Buildings

In this section we establish bounds on the coboundary constants that serve as the base case presented in this section. The discussion will be specialized to spherical buildings of type A, type C (for technical reasons, in the end we will also have to discuss tensors of two types, but this will be straightforward).

To establish the base case of our induction we will use the cones method [Gro10, LMM16, KM22, KM19, KO19]. The cones method allows one to prove coboundary expansion properties for a graph given an efficient way *triangulate* certain cycles in our graphs of interest.

Definition 4.1. *A triangulation of a cycle C in a graph G is given by a set of vertices $S_C \subseteq V(G)$ such that the induced subgraph on $S_C \cup V(C)$ breaks into a union of triangles, where any two distinct triangles are either disjoint or share an edge.*

Notations: throughout this section, for subspaces A, B, we will often use the notation (A, B) to denote the subspace $A + B$. Similarly, for a vector v and a subspace A, we will denote by (v, A) the subspace $\text{span}(\{v\}) + A$.

4.1 The Base Case for Spherical Buildings of Type A

Let $\text{Gr}_d(k_1, k_2, k_3)$ denote the tripartite graph whose vertices are k_1, k_2, k_3-dimensional subspaces of $\mathbb{F}_q^d$ and an edge is sampled by sampling a random chain $V_1 \subset V_2 \subset V_3$ and choosing $(V_i, V_j), i \neq j \sim [3]$. We drop the subscript d when clear from context.

Lemma 4.2. *For all $m \in \mathbb{N}$, $k_1 < k_2 < k_3$ integers, let $K = \max(\lceil \frac{k_2}{k_3-k_2} \rceil, \lceil \frac{k_2}{k_2-k_1} \rceil)$. Then $Gr_d(k_1, k_2, k_3)$ is an $(O(K^2), \text{poly}(K)/q)$-coboundary expander over S_m.*

We break the proof into two cases according to which term larger, $k_2 - k_1$ or $k_3 - k_2$. Let us first discuss the case when $k_3 - k_2 \leqslant k_2 - k_1$ and then later we discuss how to modify the proof in the other case. For simplicity of notation we will assume that k_1, k_2, k_3 are all multiples of $k_3 - k_2$, so set $k = k_3 - k_2$, $t = k_3/k$ and $t' = k_1/k$. The same proof with some slight modifications works when this is not the case, hence we omit those details.

Let G denote the graph $\text{Gr}_d(k_1, k_2, k_3)$ and G_i denote the vertices of dimension k_i. First fix an arbitrary vertex $U \in G_3$ and an arbitrary basis for it: $u_1, \ldots, u_{k_3}$. Let $U_{(1)}$ denote the set of first k vectors, $\{u_1, \ldots, u_k\}$, $U_{(2)}$ the second set of k vectors and so on upto $U_{(t)}$. We now fix a set of paths from U to V for most $V \in G$. Let $\mathcal{B}$ be a set of "block decompositions" for each subspace $V \in G$, i.e. $\mathcal{B}$ assigns $V \in G_i$ the blocks, $\mathcal{B}(V) = (V_{(1)}, \ldots, V_{(t)})$ with $V_{(i)} = U_{(i)}$ for all $i \leqslant t - k_i/k$ and $V = (V_{>t-k_i/k})$.

4.1.1 Set of Good Vertices and Edges with respect to $\mathcal{B}$ when $k_3 - k_2 \leqslant k_2 - k_1$

1. Let $\text{Good}_i(\mathcal{B}) \subseteq G_i$ be the set of $V \in G_i$ that satisfy for all $i \in [t], \dim(V_{(1)}, \ldots, V_{(i)}, U_{>i}) = k_3$. In particular taking $i = t$ this implies that $\dim(U_{\leqslant t-k_i/k}, V) = k_3$.
2. For all $b < a \in [3]$ let $\text{Good}_{ab}(\mathcal{B}) \subseteq E(G_a, G_b)$ be the set of edges (V, W) with $W \subset V$ that satisfy
   (a) Both $V, W \in \bigcup_{a' \in [3]} \text{Good}_{a'}(\mathcal{B})$.
   (b) For all $j < i \in [t], \dim(V_{\leqslant j}, W_{(j+1)}, \ldots, W_{(i)}, U_{>i}) = k_3$.

Claim 4.3. *Let $V_1, V_2 \subseteq \mathbb{F}_q^d$ be d_1, d_2 dimensional subspaces respectively, and suppose that $d_3 + d_1 \leqslant d_2$. Then*

$$\Pr_{V_3 \subseteq_{d_3} V_2}[\dim(V_3 + V_1) < d_3 + d_1] \leqslant \frac{1}{q-1}.$$

Proof. Let $U = V_1 \cap V_2$ and write $V = U + V_1'$ with $V_1' \cap V_2 = \{0\}$. Then:

$$\Pr_{V_3 \subset_{d_3} V_2}[V_3 \cap U = \{0\}] \geqslant \prod_{i=0}^{d_3-1} \left(\frac{q^{d_2} - q^{d_1+i}}{q^{d_2}} \right) \geqslant 1 - \frac{1}{q-1}.$$

If $V_3 \cap U = \{0\}$ then $V_3 \cap V_1 = \{0\}$ too, hence $\dim(V_3 + V_1) = d_3 + d_1$ thus completing the proof. □

Lemma 4.4. *There exists a block decomposition $\mathcal{B}$ such that for all $b < a \in [3]$:*

$$\Pr_{V\sim G_a}[V \in \mathit{Good}_a(\mathcal{B})] \geqslant 1 - O\left(\frac{t}{q}\right)$$

$$\Pr_{\substack{V\sim G_a \\ W\subset_b V}}[(V,W) \in \mathit{Good}_{ab}(\mathcal{B})] \geqslant 1 - \frac{\mathrm{poly}(t)}{q}.$$

Proof. Let us consider a random block decomposition $\mathcal{B}$, that for each subspace $V \in G_a$, picks a sequence of random subspaces:

$$V_{(t-k_a/k+1)} \subset_k (V_{(t-k_a/k+1)}, V_{(t-k_a/k+1)}) \subset_{2k} \ldots \subset_{k_a-k} V,$$

and sets $V_{(i)} = U_{(i)}$ for all $i \leqslant t - k_a/k$. Let us prove that with high probability $V \in \mathrm{Good}_a$. Fix some $i \in [t]$ and let $F \subset [t]$ be the set $\{1, \ldots, t - k_a/k\} \cup \{i+1, \ldots, t\}$. For a set $S \subset [t]$, let U_S denote the subspace $\mathrm{span}(U_{(i)} \mid i \in S)$. Then using Claim 4.3 we get,

$$\Pr_{V\sim G_a,\mathcal{B}}[\dim(V_{\leqslant i}, U_{>i}) = k_3] = \Pr_{V\sim G_a,\mathcal{B}}[\dim(V_{\overline{F}}, U_F) = k_3] \geqslant 1 - \frac{1}{q-1},$$

where we used that $V_{\overline{F}}$ is distributed as a uniformly random $k(t-|F|)$-dimensional subspace. Taking a union bound over $i \in [t]$ we get that,

$$\Pr_{V\sim G_a,\mathcal{B}}[V \in \mathrm{Good}_a(\mathcal{B})] \geqslant 1 - \frac{t}{q-1}, \tag{13}$$

establishing the first item. For the second item, fix $b < a \in [3]$, $j < i \in [t]$, and let

$$F = [t - k_a/k] \cup \{i+1, \ldots, t\} \cup ([t - k_b/k] \cap \{j+1, \ldots, i\}), \quad R_1 = [j] \setminus F, \quad R_2 = \{j+1, \ldots, i\} \setminus F.$$

We have:

$$\begin{aligned}
&\Pr_{W\subset_{k_1} V,\mathcal{B}}[\dim(V_{\leqslant j}, W_{(j+1)}, \ldots, W_{(i)}, U_{>i}) = k_3] \\
&= \Pr_{W\subset V,\mathcal{B}}[\dim(V_{R_1}, W_{R_2}, U_F) = k_3] \\
&= \Pr_{V,\mathcal{B}}[\dim(V_{R_1}, U_F) = (|R_1| + |F|)k] \\
&\cdot \Pr_{W\subset_{k_2} V,\mathcal{B}}[\dim(V_{R_1}, W_{R_2}, U_F) = k_3 \mid \dim(V_{R_1}, U_F) = (|R_1| + |F|)k].
\end{aligned} \tag{14}$$

The first term is at least $1 - \frac{1}{q-1}$ by Claim 4.3, and we next lower bound the second term. By symmetry, it suffices to bound it for a fixed V_{R_1} which satisfies $\dim(V_{R_1}, U_F) = (|R_1| + |F|)k$. We will first show that the distribution over W_{R_2} conditioned on V_{R_1} is $O(1/q)$-close to uniform. Indeed, by Claim 4.3 with probability $1 - O(1/q)$ we have that $W \cap T = \{0\}$. Conditioned on this, W_{R_2} is a uniformly random $|R_2|k$-dimensional subspace amongst $|R_2|k$-dimensional subspaces intersecting V_{R_1} at $\{0\}$. The latter distribution is $O(1/q)$-close to a uniformly random $|R_2|k$-dimensional subspace, as by Claim 4.3 the probability a random $|R_2|k$-dimensional subspace intersects V_{R_1} at $\{0\}$ is $1-O(1/q)$. When W_{R_2} is chosen to be a uniformly random $|R_2|k$-dimensional subspace we can easily bound the probability that $\dim(V_{R_1}, W_{R_2}, U_F) = k_3$ using Claim 4.3. Therefore we get,

$$\Pr_{W\subset_{k_2} V,\mathcal{B}}[\dim(V_{R_1}, W_{R_2}, U_F) = k_3 \mid \dim(V_{R_1}, U_F) = (|R_1| + |F|)k] \geqslant 1 - O(1/q).$$

Plugging this back into (14) and taking a union bound over all $j < i \in [t]$ we get,

$$\Pr_{(V,W)\sim E(G_a,G_b),\mathcal{B}}[(V,W) \in \text{Good}_{ab}(\mathcal{B})] \geqslant 1 - \frac{\text{poly}(t)}{q}. \tag{15}$$

An averaging argument on (13) and (15) now gives us that there is a $\mathcal{B}$ for which most vertices and edges are good as required. ☐

Henceforth fix a block decomposition $\mathcal{B}$ satisfying the conclusions of Lemma 4.4.

4.1.2 Constructing the Paths when $k_3 - k_2 \leqslant k_2 - k_1$

In this section, fixing U, we construct collections of canonical paths between U and good vertices in our graph. For each $V \in \bigcup_{i\in[3]} \text{Good}_i(\mathcal{B})$, this is achieved using the block decomposition of V as follows:

$$\begin{aligned} P(U,V) =& (U_{(1)},\dots,U_{(t)}) \to (U_{(2)},\dots,U_{(t)}) \to (V_{(1)},U_{(2)},\dots,U_{(t)}) \\ &\to (V_{(1)},U_{(3)},\dots,U_{(t)}) \to (V_{(1)},V_{(2)},U_{(3)},\dots,U_{(t)}) \to \dots \\ &\to (V_{(1)},\dots,V_{(t-1)},U_{(t)}) \to (V_{(1)},\dots,V_{(t-1)}) \to (V_{(1)},\dots,V_{(t)}) \to V, \end{aligned}$$

where we omit the last step if $V \in G_3$ since $V = (V_{(1)},\dots,V_{(t)})$. Note that since $V \in \text{Good}_i$, this path alternates between vertices from S_3 and S_2. When $V \in G_1$ or G_2, the subspace U appears more than once on the path, and the path from U to V could be shortened. We use this longer path instead to keep things notationally simpler, since now all paths are of length either $2t$ or $2t+1$.

4.1.3 Triangulating Cycles when $k_3 - k_2 \leqslant k_2 - k_1$

Having constructed paths from U to all good vertices, we notice that if (W,V) is an edge in the graph, then together with the paths from U we have a cycle $C(U,V,W)$. In the following lemma, we show how to triangulate this cycle using a small number of triangles.

Lemma 4.5. *When $k_3-k_2 \leqslant k_2-k_1$, for every edge $(V,W) \in \bigcup_{b<a\in[3]} \text{Good}_{ab}(\mathcal{B})$, the cycle $C(U,V,W) = U \xrightarrow{P(U,V)} V \to W \xrightarrow{P(W,U)} U$ has a triangulation of size $O((\frac{k_3}{k_3-k_2})^2)$.*

Proof. To make notation simpler we will give a triangulation $T(U,V,W)$ of the cycle with some repeating vertices. We put in edges between two vertices that are the same, and label them with the identity permutation. These are called "equality edges". This introduces triangles that might have at least two identical vertices, but it is easy to see that such a triangle is consistent, hence we can use these triangles essentially for free.

Tiling by 8-cycles when $V \in G_3$ and $W \in G_2$: Let $W' = (W_{(1)},\dots,W_{(t)})$ with $W_{(1)} = U_{(1)}$, $W = (W_{\geqslant 2})$, and $V = (V_{(1)},\dots,V_{(t)})$. To get a triangulation we first create paths between the $(2i)^{th}$ vertex on the path $P(U,V)$ and the $(2i)^{th}$ vertex on $P(U,W)$ for all $i \in \{1,\dots,t\}$. Recall that $P_{2i}(U,V) = (V_{\leqslant i}, U_{>i})$ and $P_{2i}(U,W) = (W_{\leqslant i}, U_{>i})$. For $i \in [t]$ we take the obvious path $R^i(U,V,W)$ between $P_{2i}(U,W)$ and $P_{2i}(U,V)$ that flips a block of W to a block of V one at a time:

$$R^i(U,V,W) := P_{2i}(U,W) \to (W_{(2)},\dots,W_{(i)},U_{>i}) \to (V_{(1)},W_{(2)},\dots,W_{(i)},U_{>i}) \to \dots \to P_{2i}(U,V),$$

that alternates between k_3 and k_2 dimensional vertices. Here we used the fact that $(V, W) \in \text{Good}_{32}(\mathcal{B})$ which implies that the intermediate odd vertices – $(V_{\leqslant j}, W_{(j+1)}, \ldots, W_{(i)}, U_{>i})$, on the path R^i above are of dimension k_3.

Let us henceforth drop the notation U, V, W in $R^i(U, V, W)$ since U, V, W are fixed. First note that by creating the paths R^i, we have broken the original cycle $C(U, V, W)$ into $O(t)$ cycles of the form:

$$C^i = P_{2i}(U, W) \xrightarrow{R^i} P_{2i}(U, V) \to P_{2i+1}(U, V) \to P_{2i+2}(U, V) \xrightarrow{R^{i+1}} P_{2i+2}(U, W) \to P_{2i+1}(U, W) \\ \to P_{2i}(U, W),$$

for $i \in [t-1]$ and

$$C^t = W' = (W_{(1)}, \ldots, W_{(t)}) \xrightarrow{R^t} V = (V_{(1)}, \ldots, V_{(t)}) \to W \to V.$$

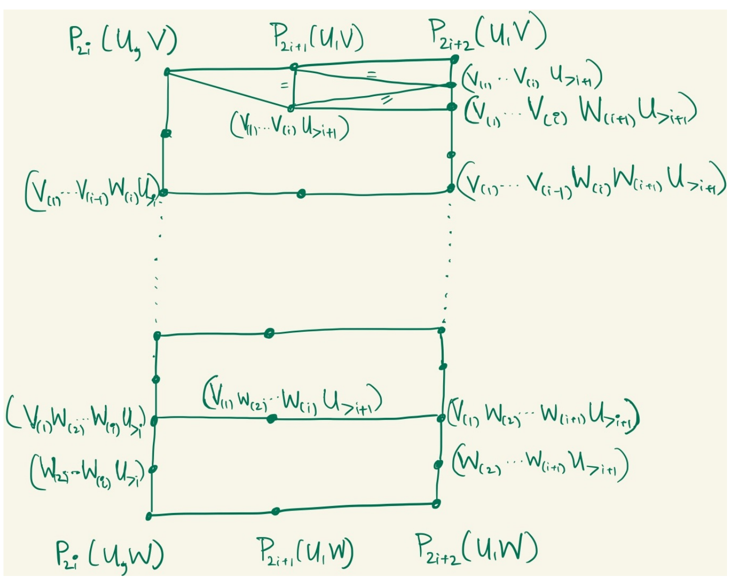
We will tile each C^i by 8-cycles and triangles, starting with $i \in [t-1]$. For all $j \in [1, i+1]$, let R^i_j denote the $(2j-1)^{th}$ vertex on the path R^i, that is,

$$R^i_j = (V_{<j}, W_{(j)}, \ldots, W_{(i)}, U_{>i}).$$

It is easy to see that for all $j \in [1, i+1]$, the vertices R^i_j and R^{i+1}_j are connected via a path of length two:

$$R^i_j \to (V_{<j}, W_{(j)}, \ldots, W_{(i)}, U_{\geqslant i+2}, \ldots, U_{(t)}) \to R^{i+1}_j.$$

As for the last part of the cycle, the vertex $P_{2i+1}(U, V)$, occurs as the middle vertex in all the following length 2 paths: 1) $R^i_{i+1} = P_{2i}(U, V) \to R^{i+1}_{i+1}$, 2)$P_{2i}(U, V) \to P_{2i+2}(U, V)$ and $R^{i+1}_{i+1} \to R^{i+1}_{i+2}$. Thus using equality edges between these vertices we can check that the whole cycle has been broken into $O(t)$ 8-cycles and $O(1)$ triangles (see figure below).



Let us now tile the cycle C^t. We have that the first three vertices of R^t are W', $(W_{\geqslant 2})$ and $(V_{(1)}, W_{\geqslant 2})$ which are all connected to W. Additionally including $(V_{(1)}, W_{\geqslant 2})$, every other odd vertex on R^t, i.e. for all $j \in [2, t+1]$, $R^t_j = (V_{<j}, W_{(j)}, \ldots, W_{(t)})$ is contained (as a subspace) in V since both W and V' are in V.

Since the dimensions match it means that these are all equal to V and therefore we can put in equality edges between them. Similarly every intermediate vertex (k_2-dimensional) on R^t is contained in V. This means that C^t is broken into $O(t)$ triangles. So overall $C(U, V, W)$ has been broken into $O(t^2)$ 8-cycles and $O(t)$ triangles.

Tiling by 8-cycles when $V \in G_3$ and $W \in G_1$: Let $W' = (W_{(1)}, \dots, W_{(t)})$, where $W = (W_{>t-k_1/k+1}) \subset V$. Now we create the same paths R^i of length $2i + 1$ between $P_{2i}(U, W) \to P_{2i}(U, V)$ for all $i \in [t]$, and each vertex on these paths is in S_3 or S_2 because $(V, W) \in \text{Good}_{31}(\mathcal{B})$. This breaks the cycle into the cycles C^i, for $i \in [t]$. The tiling of $C^i, i \in [t-1]$ proceeds identical to the first case, therefore let us discuss the tiling of the last cycle:

$$C^t = W' = (W_{(1)}, \dots, W_{(t)}) \xrightarrow{R^t} V = (V_{(1)}, \dots, V_{(t)}) \to W \to W'.$$

As in the case above, we have that the first $2(t - k_1/k) + 1$ vertices of R^t are all connected to W, since they are of the form $(V_{<j}, U_{(j)}, \dots, U_{(t-k_1/k)}, W)$. After that all subsequent odd vertices on R^t, i.e. for all $j \in [t - k_1/k + 1, t]$, $R^t_j = (V_{<j}, W_{(j)}, \dots, W_{(t)})$ are connected to V. This means that C^t is broken into $O(t)$ triangles. So overall $C(U, V, W)$ has been broken into $O(t^2)$ 8-cycles and $O(t)$ triangles.

(V, W) is an edge between G_2 and G_1: We can show that $C(U, V, W)$ in this case too can be broken into $O(t^2)$ 8-cycles and $O(t)$ triangles. The proof for the cycles C^i for $i \in [t-1]$ is the same, so we only discuss the tiling of C^t. We have that,

$$C^t = W' = (W_{(1)}, \dots, W_{(t)}) \xrightarrow{R^t} V' = (V_{(1)}, \dots, V_{(t)}) \to V \to W \to W',$$

where $W = (W_{>t-k_1/k}), V = (V_{>t-k_2/k})$. Again we have that the first $2(t - k_1/k) + 1$ vertices of R^t are all connected to W, since they are of the form $(V_{<j}, U_{(j)}, \dots, U_{(t-k_1/k)}, W)$. Then the vertex $(V_{\leqslant t-k_1/k}, W)$ is additionally connected to V', V. After that all subsequent vertices on R^t, i.e. for all $j \in [t - k_1/k + 1, t - 1]$, $R^t_j = (V_{\leqslant j}, W_{(j)}, \dots, W_{(t)})$ and the intermediate vertices in G_1 are all connected to V', by the same reasoning as the above two cases. This means that C^t is broken into $O(t)$ triangles.

Triangulating the 8-cycles: Having shown that the cycle $C(U, V, W)$ can always be tiled by at most $O(t^2)$ 8-cycles and $O(t)$ triangles, it suffices to show that each one of the resulting 8-cycles can be triangulated individually. Towards this end, we first notice that the 8-cycles we formed consist of edges between subspaces of dimension k_3 and k_2. Thus, to triangulate them we will have to use auxiliary vertices of dimension k_1.

Note that each 8-cycle in the above tiling is of the following form for some $1 \leqslant j < i \in [t]$:

$$\begin{aligned} &(W_{(j)}, U_{(i)}, X) \to (W_{(j)}, X) \to (W_{(j)}, W_{(i)}, X) \to (W_{(i)}, X) \\ \to (V_{(j)}, W_{(i)}, X) \to (V_{(j)}, X) \to &(V_{(j)}, U_{(i)}, X) \to (U_{(i)}, X) \to (W_{(j)}, U_{(i)}, X), \end{aligned}$$

with $X = (V_{\leqslant j-1}, W_{(j+1)}, \dots, W_{(i-1)}, U_{(>i)})$. We know that $\dim(X) = k_2 - k = 2k_2 - k_3$ and since $k_2 - k_1 \geqslant k_3 - k_2$, $\dim(X) \geqslant k_1$. This implies that there is some k_1-dimensional subspace $Y \subseteq X$ that is contained within all the vertices of this 8-cycle. Putting this vertex in the middle of the 8-cycle completes the triangulation of this cycle.

Size of Triangulation: In total we used $O((k_3/k_3 - k_2)^2)$ 8-cycles, each of which used $O(1)$ triangles thus completing the proof. □

4.1.4 The Case that $k_3 - k_2 > k_2 - k_1$

We now move on to discuss the case that $k_2 - k_1 \leqslant k_3 - k_2$ and let $k = k_2 - k_1, t = k_2/k$ henceforth. We will need to use a different set of paths, and towards this end we fix an arbitrary vertex $U \in G_2$ and an arbitrary basis for it: $u_1, \dots, u_{k_2}$. Let $U_{(1)}$ denote the set of first k vectors, $\{u_1, \dots, u_k\}$, $U_{(2)}$ the second set of k vectors and so on up to $U_{(t)}$. We now fix a set of paths from U to V for most $V \in G$. We will choose a block decomposition $\mathcal{B}$ for each subspace $V \in G$, namely:

1. For $V \in G_2$, $\mathcal{B}(V) = (V_{(1)}, \dots, V_{(t)}) = V$.
2. For every $V \in G_1$, let $V' = (V_{(1)}, \dots, V_{(t)})$ for $V_{(1)} = U_{(1)}$ and $V = (V_{\geqslant 2})$.
3. For $V \in G_3$, the decomposition $\mathcal{B}$ first associates with V a k_2-dimensional subspace $V' \subseteq V$ along with its block decomposition $V' = (V_{(1)}, \dots, V_{(t)})$.

As before, we will need the block decomposition $\mathcal{B}$ to satisfy several genericness properties, that we explain next.

Set of Good Vertices and Edges with respect to $\mathcal{B}$ when $k_3 - k_2 > k_2 - k_1$:

1. Let $\mathrm{Good}_i(\mathcal{B}) \subseteq G_i$ be the set of $V \in G_i$ that satisfy for all $i \in [t], \dim(V_{(1)}, \dots, V_{(i)}, U_{>i}) = k_2$.
2. For all $b < a \in [3]$ let $\mathrm{Good}_{ab}(\mathcal{B}) \subseteq E(G_a, G_b)$ be the set of edges (V, W) with $W \subset_b V$ that satisfy,
   (a) Both $V, W \in \bigcup_{a \in [3]} \mathrm{Good}_a(\mathcal{B})$.
   (b) For all $j < i \in [t], \dim(V_{\leqslant j}, W_{(j+1)}, \dots, W_{(i)}, U_{>i}) = k_2$.

Analogously to Lemma 4.4 one can show that there is a $\mathcal{B}$ for which $1 - \mathrm{poly}(t)/q$-fraction of vertices and edges are good, and we fix such $\mathcal{B}$ henceforth.

4.1.5 Constructions the Paths when $k_3 - k_2 > k_2 - k_1$

For a vertex $V \in \bigcup_{i \in [3]} \mathrm{Good}_i(\mathcal{B})$, we first construct a path $P(U, V')$, where $V' = (V_{(1)}, \dots, V_{(t)})$, by flipping a block of U to a block of V' one at a time. One can check that this path alternates between k_2 and k_1-dimensional vertices since V is good. If $V \in G_2$, then $V' = V$ and we are done, else in the last step we go from $V' \to V$.

4.1.6 Triangulating Cycles when $k_3 - k_2 > k_2 - k_1$

Having formed the paths $P(U, V)$, we now note that if (V, W) is an edge, then $P(U, V), (V, W), P(U, W)$ form a cycle. We call this cycle $C(U, V, W)$ as before, and show that it can be triangulated using a small number of triangles.

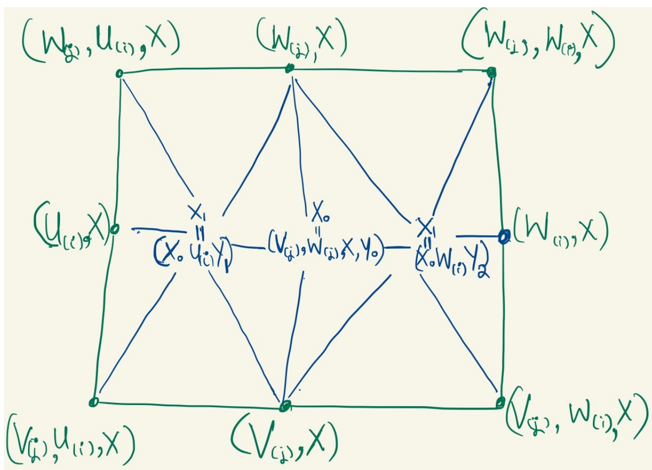
Lemma 4.6. *Suppose that $k_2 - k_1 \leqslant k_3 - k_2$, and let $(V, W) \in \bigcup_{b<a \in [3]} \mathrm{Good}_{ab}(\mathcal{B})$ be an edge. Then cycle $C(U, V, W) = U \xrightarrow{P(U,V)} V \to W \xrightarrow{P(W,U)} U$ has a triangulation of size $O((\frac{k_2}{k_2 - k_1})^2)$.*

Proof. We only give a proof sketch here since the details are exactly the same as Lemma 4.5. To create the paths R^i we take the obvious path between $P_{2i}(U, W)$ and $P_{2i}(U, V)$ by flipping one block at a time, alternating between k_2 and k_1-dimensional vertices (instead of k_2, k_3-dimensional ones). Then putting in the length two paths between R_j^i and R_j^{i+1} (that are two k_2-dimensional vertices, that can be connected using one k_1-dimensional vertex) we break the cycle into $O(t^2)$ 8-cycles and $O(t)$ triangles.

Note that each 8-cycle in the above tiling is of the following form for some $1 \leqslant j < i \in [t]$:

$$\begin{aligned}&(W_{(j)}, U_{(i)}, X) \to (W_{(j)}, X) \to (W_{(j)}, W_{(i)}, X) \to (W_{(i)}, X)\\ &\to (V_{(j)}, W_{(i)}, X) \to (V_{(j)}, X) \to (V_{(j)}, U_{(i)}, X) \to (U_{(i)}, X) \to (W_{(j)}, U_{(i)}, X),\end{aligned}$$

with $X = (V_{\leqslant j-1}, W_{(j+1)}, \ldots, W_{(i-1)}, U_{(>i)})$, with $\dim(X) = k_1 - k = 2k_1 - k_2$. Since $k_3 - k_2 \geqslant k_2 - k_1$, we get that $k_3 \geqslant k_2 + k$. Therefore to tile this cycle we can use the vertices: $X_0 = (V_{(j)}, W_{(j)}, X, Y_0)$, where Y_0 is chosen so that $\dim(X_0) = k_2$, $X_1 = (X_0, U_{(i)}, Y_1)$, and $X_2 = (X_0, W_{(i)}, Y_2)$ where Y_1, Y_2 are chosen so that $\dim(X_1) = \dim(X_2) = k_3$. This is possible since $\dim(X_0, W_{(i)}), \dim(X_0, U_{(i)}) \leqslant k_2 + k \leqslant k_3$. The figure below shows that after adding these in the cycle breaks into triangles.



As for the size of the triangulation: we had $O(t^2)$ 8-cycles, each tiled by $O(1)$ triangles, which gives an overall triangulation size of $O((k_2/k_2 - k_1)^2)$ as required. □

4.1.7 Proof of Lemma 4.2

Given the paths $P(U, V)$ and triangulations $T(U, V, W)$ for every good edge (V, W), it is easy to complete the proof of Lemma 4.2 using the fact that $\mathrm{GL}_d(\mathbb{F}_q)$ acts transitively on the triangles of $\mathrm{Gr}(k_1, k_2, k_3)$.

For each invertible linear transformation $L \in \mathrm{GL}_d(\mathbb{F}_q)$, and a subspace V let $L(V)$ denote the subspace $\mathrm{span}(Lv \mid v \in V)$ and let $L^{-1}(V)$ denote the subspace W such that $L(W) = V$. For every $V \in \bigcup_{i \in [3]} \mathrm{Good}_i(\mathcal{B})$ let $P_L(L(U), L(V))$ denote the path from $L(U) \to L(V)$ where at the i^{th}-step we have the vertex $L(P_i(U, V))$. It is easy to see that this is a valid path from $L(U)$ to $L(V)$. For a triangle Δ let $L(\Delta)$ denote the triangle whose vertices are $L(U_i), \forall U_i \in \Delta$. In fact for every good edge (V, W) we can let $T_L(L(U), L(V), L(W))$ be the triangulation of the cycle $L(U) \xrightarrow{P_L(L(U),L(V))} L(V) \to L(W) \xrightarrow{P_L(L(W),L(U))} L(U)$ where a triangle in this triangulation is given by $L(\Delta)$ for $\Delta \in T(U, V, W)$. Again it is easy to see that this is a valid triangulation of the cycle for every edge $(V, W) \in \bigcup_{b<a} \mathrm{Good}_{ab}(\mathcal{B})$ with the same size as $T(U, V, W)$.

We have the following randomized algorithm to get a highly satisfying UG solution to an arbitrary UG instance Φ.

Algorithm 1 ($\Phi = (\text{Gr}(k_1, k_2, k_3), \Pi)$).
Input: UG instance Φ on $\text{Gr}(k_1, k_2, k_3)$.
Output: A function $f : V(\text{Gr}(k_1, k_2, k_3)) \to S_m$.

1. Choose a random linear transformation $L \in \text{GL}_d(F_q)$ and set $f(L(U)) = \text{id}$.
2. For each subspace $V \in \cup_{i\in[3]}\text{Good}_i(\mathcal{B})$, assign $f_L(L(V))$ the label obtained by propagating the label of $L(U)$ to $L(V)$ via the path $P_L(L(U), L(V))$, chosen appropriately according to whether $k_3 - k_2$ or $k_2 - k_1$ is larger.
3. For every subspace $V \notin \cup_{i\in[3]}\text{Good}_i(\mathcal{B})$ choose an arbitrary label for $L(V)$.

We now complete the proof of Lemma 4.2 via the following lemma:

Lemma 4.7. *Let Φ be any UG instance over S_m with incons$(\Phi) = \delta$. Then in expectation over $L \sim$ $GL_d(\mathbb{F}_q)$, the algorithm violates at most $O(K^2)\delta + \text{poly}(K)/q$-fraction of edges, where*

$$K = \max\left(\left\lceil \frac{k_2}{k_3 - k_2}\right\rceil, \left\lceil \frac{k_2}{k_2 - k_1}\right\rceil\right).$$

Proof. Suppose the propagation algorithm chooses a linear transformation L. Let $f_L : V(\text{Gr}(k_1, k_2, k_3)) \to S_m$ denote the assignment outputted by Algorithm 2 in this case. Let E denote $E(\text{Gr}(k_1, k_2, k_3))$ and let $\text{Good}(E) = \bigcup_{b<a} \text{Good}_{ab}(\mathcal{B})$. For every $(V, W) \in \text{Good}(E)$, the edge $(L(V), L(W))$ is satisfied by f_L if the cycle $L(U) \xrightarrow{P_L(L(U),L(V))} L(V) \to L(W) \xrightarrow{P_L(L(W),L(U))} L(U)$ is consistent. Furthermore this is true if every triangle in $T_L(L(U), L(V), L(W))$ is consistent. Recall that this is the set of triangles $L(\Delta), \Delta \in T(U, V, W)$. So we get that,

$$\begin{aligned}
\text{viol}(f_L) &\leqslant \Pr_{(V,W)\sim E}[(V, W) \notin \text{Good}(E)] + \mathop{\mathbb{E}}_{(V,W)\sim\text{Good}(E)}[\mathbb{1}(\exists\Delta \in T(L(U), L(V), L(W)) \cap \text{incons}(\Phi)] \\
&\leqslant \frac{\text{poly}(K)}{q} + \max_{(V,W)\in\text{Good}(E)}(|T(U, V, W)|) \mathop{\mathbb{E}}_{(V,W)\sim\text{Good}(E)} \mathop{\mathbb{E}}_{\Delta\in T(U,V,W)}[\mathbb{1}(L(\Delta) \in \text{incons}(\Phi)] \\
&\leqslant \frac{\text{poly}(K)}{q} + O(K^2) \mathop{\mathbb{E}}_{(V,W)\sim\text{Good}(E)} \mathop{\mathbb{E}}_{\Delta\in T(U,V,W))}[\mathbb{1}(L(\Delta) \in \text{incons}(\Phi)],
\end{aligned}$$

where in the second inequality we used Lemma 4.4 and the last one we used Lemmas 4.5 and 4.6 to bound the size of the triangulation. Now taking an expectation over $L \sim \text{GL}_d(\mathbb{F}_q)$ we get:

$$\begin{aligned}
\mathop{\mathbb{E}}_{L}[\text{viol}(f_L)] &\leqslant \frac{\text{poly}(K)}{q} + O(K^2) \mathop{\mathbb{E}}_{(V,W)\sim\text{Good}(E)} \mathop{\mathbb{E}}_{\Delta\in T(U,V,W))} \mathop{\mathbb{E}}_{L}[\mathbb{1}(L(\Delta) \in \text{incons}(\Phi)] \\
&\leqslant \frac{\text{poly}(K)}{q} + O(K^2)\delta,
\end{aligned}$$

which completes the proof. □

4.2 The Base Case for Spherical Buildings of Type C

Let μ be the uniform distribution over chains of isotropic subspaces $(V_1 \subset \ldots V_d)$ of $\mathbb{F}_q^{2d}$, with $\dim(V_i) = i$. For $k_1, k_2, k_3 \leqslant d$ let $S_d(k_1, k_2, k_3)$ denote the tripartite graph $T(\{k_1\}, \{k_2\}, \{k_3\}; \mu)$, where we drop the subscript d when clear from context. We will prove that:

Lemma 4.8. *For all $m \in \mathbb{N}$, $0 < k_1, k_2, k_3 \leqslant d$, let $K = \max(\lceil \frac{k_2}{k_3 - k_2} \rceil, \lceil \frac{k_2}{k_2 - k_1} \rceil)$. Then $S(k_1, k_2, k_3)$ is an $(O(K^2), \mathrm{poly}(K)/q)$-coboundary expander over S_m.*

The proof of this statement will follow closely along the lines of Lemma 4.2.

4.2.1 Auxiliary Claims

Claim 4.9. *For $t < k < d \in \mathbb{N}$, given isotropic subspaces $W_1, W_2 \subseteq \mathbb{F}_q^{2d}$ with $\dim(W_1) = d_0 - k$ and $\dim(W_2) = d_0$, there exists an isotropic subspace $W \subseteq W_2$ satisfying $\dim(W) = k$, and $W \cap (W_1 \cap W_2) = \{0\}$ such that $W + W_1$ is a d_0-dimensional isotropic subspace.*

Proof. Define $U = W_1 \cap W_2$ and denote $d_1 = \dim(U)$. We can write $W_1 = U + W_1'$ and $W_2 = U + W_2'$ with $W_1' \cap W_2' = \{0\}$. Let V be the subspace of vectors v satisfying $\omega(v, w') = 0$ for all $w' \in W_1'$. Note that V has dimension $2d - \dim(W_1') = 2d - d_0 + k + d_1$. Also

$$\dim(V \cap W_2') = \dim(V) + \dim(W_2') - \dim(V + W_2') \geqslant (2d - d_0 + k + d_1) + (d_0 - d_1) - 2d = k.$$

Take $W \subseteq V \cap W_2'$ of dimension k. Since $W \subseteq W_2'$ we get that W is isotropic, and as $W \subseteq V$ we get that $W + W_1'$ is isotropic. Next, note that $W \cap W_1 \cap W_2 \subseteq W \subseteq W_2' \cap U = \{0\}$. Finally, now that

$$\dim(W + W_1) = k + (d_0 - k) - \dim(W \cap W_1) = d_0,$$

as $W \cap W_1 \subseteq W \cap W_1 \cap W_2' \subseteq W \cap W_1 \cap W_2 = \{0\}$. □

For an isotropic subspace U, let $\mathrm{Symp}(U)$ denote the subspace that is symplectically orthogonal to U, i.e. for all $w \in \mathrm{Symp}(U), u \in U, \omega(w, u) = 0$. We note the standard facts that $\dim(\mathrm{Symp}(U)) = 2d - \dim(U)$ and $\mathrm{Symp}(U \oplus V) = \mathrm{Symp}(U) \cap \mathrm{Symp}(V)$. We also have that $\dim(U \cap \mathrm{Symp}(W)) = \dim(U) - \dim(W) + \dim(\mathrm{Symp}(U) \cap W)$. In particular, when $\dim(U) = \dim(W)$ we get that $\dim(U \cap \mathrm{Symp}(W)) = \dim(\mathrm{Symp}(U) \cap W)$.

Claim 4.10. *The following facts are true for sufficiently large q:*

1. *Fix a subspace U, let $1 \leqslant i \leqslant d$ and choose an isotropic subspace W of dimension i uniformly at random. Then:*
   (a) *With probability at least $1 - \frac{4}{q}$ we have that*
   $$\dim(U \cap \mathit{Symp}(W)) = \max(0, \dim(U) - \dim(W)).$$
   (b) *With probability at least $1 - \frac{4}{q}$ we have that*
   $$\dim(\mathit{Symp}(U) \cap W) = \max(0, \dim(W) - \dim(U)).$$
2. *Choosing maximal isotropic subspaces U, W of $\mathbb{F}_q^{2d}$ randomly, we have that $U \cap W = \{0\}$ with probability at least $1 - \frac{4}{q}$.*

Proof. We prove each item separately, and we begin with the first item. We start with (a), and assume that $\dim(U) \geqslant i$; otherwise we replace U by a subspace of dimension i that contains it. Choose $w_1, \ldots, w_i$ uniformly conditioned on w_j being symplectically orthogonal to $w_1, \ldots, w_{j-1}$ for all j. Define the vectors $w_1', \ldots, w_i'$ by $w_j(s) = -w_j(s + d)$ for $s \leqslant d$, and $w_j(s) = w_j(s - d)$ for $s > d$. Thus, the space

$U \cap \text{Symp}(\{w_1, \dots, w_i\})$ corresponds to all vectors in U that are orthogonal to $w'_1, \dots, w'_i$. Note that for each $1 \leqslant j \leqslant i$ and $\alpha_1, \dots, \alpha_{j-1} \in \mathbb{F}_q$, the marginal distribution of $w'_j + \sum_{\ell=1}^{j-1} \alpha_\ell w'_\ell$ is uniform, and hence it is orthogonal to U with probability at most $q^{(2d-\dim(U))-2d} = q^{-\dim(U)}$. It follows that no vector in $\text{span}(\{w'_1, \dots, w'_i\})$ is orthogonal to U with probability at least

$$1 - \sum_{j=1}^{i} q^{j-1} q^{-\dim(U)} \geqslant 1 - \frac{2}{q^{(i-1)-\dim(U)}} \geqslant 1 - \frac{2}{q},$$

where we used the fact that $\dim(U) \geqslant i$. Next, we note that the probability that $w_1, \dots, w_i$ is linearly independent is at least

$$1 - \sum_{j=1}^{i} \frac{q^{2j}}{q^{2d}} \geqslant 1 - 2\frac{q^{2i}}{q^{2d}},$$

in which case $w'_1, \dots, w'_i$ are linearly independent too. It follows from the union bound that with probability at least $1 - \frac{3}{q}$, the set $w_1, \dots, w'_i$ is linearly independent and no vector in it is orthogonal to U, in which case the subspace in U of vectors orthogonal to $w_1, \dots, w'_i$ has dimension $\dim(U) - i$, and (a) follows.

We move on to proving (b). Note that

$$\begin{aligned} \dim(\text{Symp}(U) \cap W) &= \dim(\text{Symp}(U)) + \dim(W) - \dim(\text{Symp}(U) \oplus \dim(W)) \\ &= \dim(\text{Symp}(U)) + \dim(W) - 2d + \dim(U \cap \text{Symp}(W)) \\ &= i - \dim(U) + \dim(U \cap \text{Symp}(W)), \end{aligned}$$

and the result follows from (a).

The second item follows immediately from the first item with $i = d$. □

Claim 4.11. *Let A, B be such that $\dim(A) = \dim(B) = k$. Then there exists a randomized algorithm to choose $A' \subseteq_{k-1} A, b \in B$ such that $A' + \text{span}(\{b\})$ is an isotropic subspace and for all d_0-dimensional C where $(A + C)$ and $(B + C)$ are two $d_0 + k$-dimensional isotropic subspaces,*

$$\Pr_{A',b}[\dim(A' + \text{span}(\{b\}) + C) = d_0 + k] \geqslant 1 - O\left(\frac{1}{q}\right).$$

Proof. We know that $A \cap C = B \cap C = \{0\}$, and there are two cases:

1. If there is no $b \in B$ that is isotropic to all of A, then choose $A' \subset_{k-1} A$ arbitrarily, and let $b \in B$ so that $A' + \text{span}(\{b\})$ is a k-dimensional isotropic subspace by Claim 4.9. In this case, we know that $B \cap (A + C) = \{0\}$, because every vector in $A + C$ is in $\text{Symp}(A)$. Therefore $b \notin A' + C$, which gives that,

   $$\dim(A' + \text{span}(\{b\}) + C) = \dim(A' + C) + \dim(\text{span}(\{b\})) = \dim(A') + \dim(C) + 1 = d_0 + k,$$

   as required.

2. If there is some non-zero $b_0 \in B \cap \text{Symp}(A)$, then choose $b = b_0$ and choose a uniformly random $A' \subset_{k-1} A$. Note that $A' + \text{span}(\{b\})$ is an isotropic subspace. Furthermore $\dim((C + \text{span}(\{b_0\})) \cap A) \leqslant 1$ since $\dim(C \cap A) = 0$. If $\dim((C + \text{span}(\{b_0\})) \cap A) = 0$, then $\dim(C + \text{span}(\{b_0\}) + A') = d_0 + k$, as required. Else, $(C + \text{span}(\{b_0\})) \cap A = \text{span}(\{v\})$ for some $v \neq 0$; by Claim 4.3 we know that $A' \cap \text{span}(\{v\}) = \{0\}$ with probability $1 - O(1/q)$, in which case $(C + \text{span}(\{b_0\})) \cap A' = \emptyset$ and $\dim(C + \text{span}(\{b_0\}) + A) = d_0 + k$ as required. □

4.2.2 Constructing Paths

In this section, we set up paths between vertices in the spherical building that will be convenient for us to triangulate. Let S_i denote the vertices of dimension k_i, and let $k = k_2 - k_1$. For simplicity of notation we will assume that k_2 is a multiple of k and that $k_1 \geqslant k$ (and so k_1 is also a multiple of k). The same proof with some slight modifications works when this is not the case, and in Section 4.2.6 we explain the necessary modifications.

Fix an arbitrary vertex $U \in S_2$, and pick an arbitrary basis for U: $\{u_1, \ldots, u_{k_2}\}$. Let $U_{(1)}$ denote the set of first k vectors, $\{u_1, \ldots, u_k\}$, $U_{(2)}$ the second set of vectors and so on up to $U_{(t)}$ for $t = k_2/k$. Note that $k_1 = k(t-1)$. Define:

1. $\text{Good}_1 \subseteq S_1$ to be the set of subspaces V that satisfy the following: $V \cap U = \{0\}$, and for all $i \in [t-1]$, $\dim(V \cap \text{Symp}(U_{>i})) = (i-1)k$ and $\dim(\text{Symp}(V) \cap \text{Symp}(U_{>i})) = 2d - k_1 - k_2 + ik$.

2. $\text{Good}_2 \subseteq S_2$ to be the set of subspaces V that satisfy the following: $V \cap U = \{0\}$, and for all $i \in [t-1]$, $\dim(V \cap \text{Symp}(U_{>i})) = ik$.

3. $\text{Good}_3 \subseteq S_3$ to be the set of subspaces V that satisfy the following: $V \cap U = \{0\}$ and $\dim(V \cap \text{Symp}(U)) = k_3 - k_2$.

Lemma 4.12. *For all $i \in [3]$:*

$$\Pr_{V \sim S_i}[V \in \mathit{Good}_i] \geqslant 1 - O\left(\frac{t}{q}\right).$$

Proof. Immediate from Claim 4.10 and the union bound. □

Setting up Paths from U: We will now be interested in constructing paths from U to other vertices V in the graph. Towards this end, for fixed U and V we will associate with V a vertex V'. In the case that $\dim(V) = k_2$, we will take $V' = V$, and otherwise V' will be an appropriately chosen subspace or superspace of V.

1. For a vertex V of dimension k_2, using Claim 4.9 on V and $U_{\geqslant 2}$ we find $V_{(1)} \subseteq_k V$ such that $(V_{(1)}, U_{(2)}, \ldots, U_{(t)}) \in S_2$. Applying this claim iteratively, we find $V_{(2)}$ such that

$$(V_{(1)}, V_{(2)}, U_{(3)}, \ldots, U_{(t)}) \in S_2,$$

and so on. Then consider the following path from $U \to V$ which flips a block of U to a block of V one at a time:

$$\begin{aligned} P(U,V) =& (U_{(1)}, \ldots, U_{(t)}) \to U_{\geqslant 2} \to (V_{(1)}, U_{\geqslant 2}) \to (V_{(1)}, U_{\geqslant 3}) \to (V_{(1)}, V_{(2)}, U_{\geqslant 3}) \to \ldots \\ & \to (V_{<t}, U_{(t)}) \to V_{<t} \to V. \end{aligned}$$

Note that this path alternates between vertices of S_2 and S_1. We set $V' = V$.

2. For a vertex V of dimension k_3, we know that $\dim(V \cap \text{Symp}(U)) = k_3 - k_2$, and thus we may choose $V' \subset_{k_2} V$ such that $V' \cap \text{Symp}(U) = \{0\}$. Consider its block decomposition $V' = (V_{(1)}, \ldots, V_{(t)})$ such that for all $i \leqslant t$, $(V_{(1)}, \ldots, V_{(i)}, U_{(i+1)}, \ldots, U_{(t)}) \in S_2$. As shown above, we know that such a decomposition is possible using Claim 4.9 iteratively. Then consider the following path from $U \to V$ which flips a block of U to a block of V one at a time similar to the above:

$$\begin{aligned} P(U,V) =& (U_{(1)}, \ldots, U_{(t)}) \to U_{\geqslant 2} \to (V_{(1)}, U_{\geqslant 2}) \to (V_{(1)}, U_{\geqslant 3}) \to (V_{(1)}, V_{(2)}, U_{\geqslant 3}) \to \ldots \\ & \to V' \to V. \end{aligned}$$

3. We only give a path from U to $V \in \text{Good}_1$. For such a V, we first find $V_{(1)}$ of dimension k such that $(V_{(1)}, V)$ and $(V_{(1)}, U_{\geqslant 2})$ are in S_2 and $V_{(1)} \cap U_{(1)} = 0$. Such a subspace exists because

$$\dim(\text{Symp}(U_{\geqslant 2}) \cap \text{Symp}(V)) = 2d - k_1 - k_2 + k \geqslant 2(d - k_2) + 2k \geqslant 2k.$$

   By Claim 4.10 picking a random k-dimensional isotropic subspace $V_{(1)}$ from the intersection will satisfy that $V_{(1)} \cap \text{Symp}(U_{(1)}) = \{0\}$, since $\text{Symp}(V) \cap \text{Symp}(U)$ is a co-dimension k subspace of $\text{Symp}(U_{\geqslant 2}) \cap \text{Symp}(V)$. Additionally both $(V_{(1)}, V)$ and $(V_{(1)}, U_{\geqslant 2})$ are isotropic. Since $V \in \text{Good}_1$, $\text{Symp}(V) \cap U_{\geqslant 2} = V \cap \text{Symp}(U_{\geqslant 2}) = \{0\}$, which gives us that $V_{(1)} \cap V = V_{(1)} \cap U_{\geqslant 2} = \{0\}$ implying that $(V_{(1)}, V)$ and $(V_{(1)}, U_{\geqslant 2})$ are in S_2.

   Now set $V' = (V_{(1)}, V)$ and consider the a path from $U \to V$. This path is the result of the above process for k_2 dimensional vertices applied on V', where we already picked $V_{(1)}$, followed by a final step from V' to V.

$$\begin{aligned} P(U, V) = & U \to U_{\geqslant 2} \to (V_{(1)}, U_{\geqslant 2}) \to (V_{(1)}, U_{\geqslant 3}) \to (V_{(1)}, V_{(2)}, U_{\geqslant 3}) \to \dots \\ & \to (V_{(1)}, \dots, V_{(t)}) \to V. \end{aligned}$$

   Note that this path too alternates between vertices from S_2 and S_1.

The block decomposition above satisfies the following properties:

Claim 4.13. *Fix U and take $V \in \bigcup_{i \in [3]} \text{Good}_i$, let V' be the associated vertex with V and write $V' = (V_{(1)}, \dots, V_{(t)})$ the block decomposition chosen by the path from U as above. Then for all $i \in [t]$, $V_{(i)} \cap \text{Symp}(U_i) = \{0\}$.*

Proof. We split the proof to cases.

The case that $\dim(V) = k_2$**:** in that case, by construction and definition of Good_2 we have that for $i \geqslant 2$ it holds that $V \cap \text{Symp}(U_{\geqslant i}) = V_{\leqslant i-1}$. By this equality for $i+1$ instead of i, it follows that each $v \in V_{(i)}$ is symplectic to $U_{\leqslant i+1}$, and we conclude that

$$V_{(i)} \cap \text{Symp}(U_i) = V_{(i)} \cap \text{Symp}(U_{\geqslant i}) = V_{(i)} \cap V \cap \text{Symp}(U_{\geqslant i}) = V_{(i)} \cap V_{\leqslant i-1} = \{0\}.$$

The case that $\dim(V) = k_3$**.** In that case, we picked $V' \subseteq V$ of dimension k_2 such that $V' \cap \text{Symp}(U) = \{0\}$, and the proof is exactly as in the previous case.

The case that $\dim(V) = k_1$**:** by construction $V_{(1)} \cap \text{Symp}(U_1) = \{0\}$. For $i \geqslant 2$, the fact that $V_{(i)} \cap \text{Symp}(U_i) = \{0\}$ follows from the same argument above, as the path from U to V' is constructed in the same way. □

4.2.3 Constructing the Triangulations: Handling Special 8-cycles

With the paths $P(U, V)$, we would like to construct triangulations of cycles that they form. Towards this end, we will have to triangulate 8-cycles of a special form as in the following lemma:

Lemma 4.14. *Let $k \leqslant k_1 < k_2 < k_3 \leqslant d$ with $k_3 \geqslant k_2 + k$, let A, B, C, D be k-dimensional isotropic subspaces, let X be a $k_2 - 2k$-dimensional isotropic subspace and let $X_1, X_2 \subseteq X$ be of dimension $k_1 - k$. Suppose that $C \cap \text{Symp}(D) = \text{Symp}(C) \cap D = \{0\}$. Then the following 8-cycle, whose odd vertices are in S_2 and even ones are in S_1,*

$$C_8 = (A, C, X) \to (A, X_1) \to (A, D, X) \to (D, X_2) \to (B, D, X)$$
$$\to (B, X_1) \to (B, C, X) \to (C, X_2) \to (A, C, X),$$

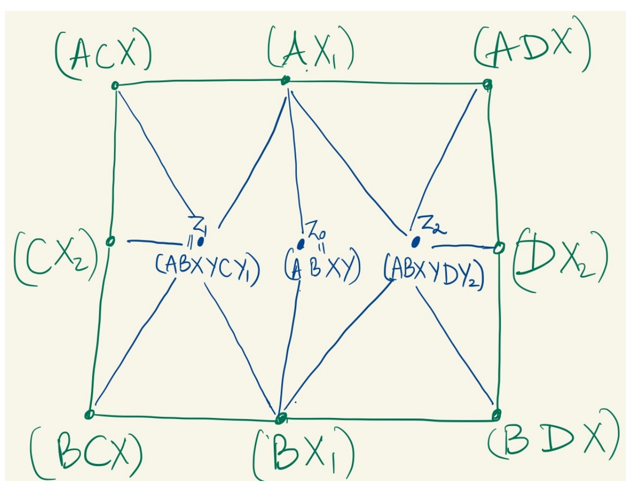
has a triangulation of size $O(1)$.

Proof. We break the proof into two cases. We will use the fact that $d \geqslant k_3 \geqslant k_2 + k$.

Case 1–$A \subseteq$ Symp(B): This implies that $B \subseteq \text{Symp}(A)$ too. Let $Z = (A, B, X)$ and $\dim(Z) = k_2 - \ell$ for some $\ell \in [0, k]$. Let V be a subspace such that: $V \subseteq \text{Symp}(Z)$ and $V \cap Z = \{0\}$. We get that $\dim(V) \geqslant 2d - 2(k_2 - \ell) \geqslant 2k + 2\ell$. Let $G_1 \subseteq V$ be such that $G_1 \subseteq \text{Symp}(C) \cap V$ and $G_2 \subseteq V \cap \text{Symp}(D)$. We get that $\dim(G_1), \dim(G_2) \geqslant \dim(V) - k$, therefore:

$$\dim(G_1 \cap G_2) \geqslant \dim(V) - 2k \geqslant 2\ell.$$

Pick an arbitrary ℓ-dimensional isotropic subspace $Y \subset G_1 \cap G_2$ and add in the isotropic subspace $Z_0 = (Z, Y)$. As $Y \subset V$ it follows that $Y \cap Z = \{0\}$, and therefore $\dim(Z_0) = k_2$. Then add in the vertices $Z_1 = (Z, Y, C, Y_1)$ and $Z_2 = (Z, Y, D, Y_2)$, where Y_1, Y_2 are chosen such that Z_1, Z_2 are k_3-dimensional isotropic subspaces. This is possible since $(Z, Y, C), (Z, Y, D)$ are subspaces of dimension $\leqslant \dim(Z, Y) + k \leqslant k_3$ and they are isotropic since $Y \subset G_1 \cap G_2$. The figure below shows that adding in Z_0, Z_1, Z_2 breaks the 8-cycle into triangles.



Case 2 – A may not be in Symp(B): Write $A = A_1 + A_2$, where $A_1 \subset \text{Symp}(B)$ and $A_2 \cap \text{Symp}(B) = 0$. Similarly write $B = B_1 + B_2$, where $B_1 \subset \text{Symp}(A)$ and $B_2 \cap \text{Symp}(A) = 0$. We know that $\dim(A_1) = \dim(B_1)$, and say they are equal to $k - \ell$ for some $\ell \in [0, k]$; thus $\dim(A_2) = \dim(B_2) = \ell$. We know that $\dim(A_1, B_1, X) \leqslant 2(k - \ell) + k_2 - 2k = k_2 - 2\ell$, and so we write $\dim(A_1, B_1, X) = k_2 - 2\ell - z$ for some $z \geqslant 0$. Let us find a subspace V' that satisfies:

$$V' \subseteq \text{Symp}(A_2) \cap \text{Symp}(B_2) \cap \text{Symp}(C) \cap \text{Symp}(D) \cap \text{Symp}(A_1, B_1, X), \quad V' \cap (A_1, B_1, X) = \{0\},$$

By dimension counting we get,

$$\begin{aligned}\dim(V') &\geqslant 2d - \dim(A_2) - \dim(B_2) - \dim(C) - \dim(D) - 2\dim(A_1, B_1, X) \\ &= 2d - 2\ell - 2k - 2(k_2 - 2\ell - z) \\ &= 2(d - k_2 - k) + 2(\ell + z) \\ &\geqslant 2(\ell + z).\end{aligned}$$

We can now pick any $(\ell+z)$-dimensional isotropic subspace V inside V'. Also fix an arbitrary ℓ-dimensional subspace $\widetilde{V}$ inside V.

Below we elaborate on the vertices we add inside the cycle. It is easy to check that by construction each of these vertices are isotropic subspaces, therefore we will only check that they are of the correct dimensions (k_1, k_2 or k_3). Let $A' = (B_1, \widetilde{V})$.

1. (A', X_1): By construction $V \cap (B_1, X_1) = \{0\}$ and $\widetilde{V} \subseteq V$, hence $(B_1, X_1) \cap \widetilde{V} = \{0\}$ and by the dimension formula we get

   $$\dim(A', X_1) = \dim(B_1, X_1) + \dim(\widetilde{V}) = \dim(B_1) + \dim(X_1) + \dim(\widetilde{V}) = k - \ell + k_1 - k + \ell = k_1.$$

2. (A, B_1, V, X): This is equal to (A_2, A_1, B_1, X, V). We know that $A_2 \cap (A_1, B_1, V, X) = \{0\}$, since the latter is symplectically orthogonal to B and $A_2 \cap \mathrm{Symp}(B) = \{0\}$. Also, by construction $V \cap (A_1, B_1, X) = \{0\}$, and applying the dimension formula twice we get that:

   $$\dim(A, B_1, V, X) = \dim(A_2) + \dim(A_1, B_1, X) + \dim(V) = \ell + k_2 - 2\ell - z + \ell + z = k_2.$$

3. (A, B_1, V, C, X, Y_1) where Y_1 is chosen so that the whole subspace is k_3-dimensional. This is possible since:

   $$\dim(A, B_1, V, C, X) = \dim(A_2) + \dim(C) + \dim(A_1, B_1, X) + \dim(V) = k_2 + k \leqslant k_3,$$

   where we used that $A_2 \cap (A_1, B_1, V, C) = \{0\}$ since the latter is symplectically orthogonal to B and $A_2 \cap \mathrm{Symp}(B) = \{0\}$, and further $C \cap (A_1, B_1, V, X)$, since the latter is symplectically orthogonal to D and $C \cap \mathrm{Symp}(D) = \{0\}$ by assumption.

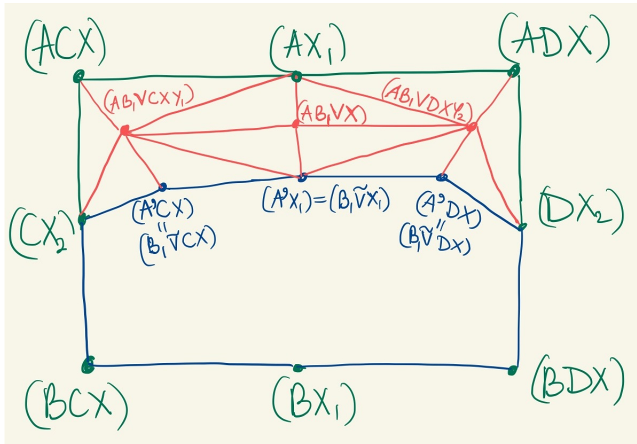
4. (A', C, X): This equals $(B_1, \widetilde{V}, C, X)$ and like the above, we get

   $$\dim(A', C, X) = \dim(C) + \dim(B_1) + \dim(X) + \dim(\widetilde{V}) = k_2.$$

5. (A, B_1, V, D, X, Y_2) where Y_2 is chosen so that the whole subspace is k_3-dimensional, where the proof that this is possible is the same as the third item above.

6. (A', D, X): This is a k_2-dimensional isotropic subspace like the fourth item above.

The figure below shows that this breaks the cycle into triangles and another 8-cycle corresponding to the subspaces A', B, C, D, X, X_1, X_2.



Now note that $A' = (B_1, \widetilde{V})$ satisfies $A' \subseteq \text{Symp}(B)$. Therefore we can triangulate the remaining 8-cycle (in blue above) using Case 1, which gives a triangulation of the original cycle using $O(1)$ triangles. $\square$

4.2.4 Triangulating General Cycles

Lemma 4.15. *For every edge $(V, W) \in S(k_1, k_2, k_3)$, where $V, W \in \bigcup_{i=1}^{3} Good_i$, the cycle $C(U, V, W) = U \xrightarrow{P(U,V)} V \to W \xrightarrow{P(W,U)} U$ has a triangulation of size $O(K^2)$, where $K = \max\left(\frac{k_2}{k_2 - k_1}, \frac{k_2}{k_3 - k_2}\right)$.*

Proof. Without loss of generality we assume that $\dim(V) > \dim(W)$, meaning that $W \subset V$. First, we break the cycle $C(U, V, W)$ into $O(t^2)$ cycles of length 8 and $O(t)$ triangles. This step is slightly different depending on which type of edge we have, and we proceed by case analysis.

Tiling by 8-cycles when $V \in S_3, W \in S_2$: Let $V' \subset V$ be the vertex chosen by the path from U, with $V' = (V_{(1)}, \dots, V_{(t)})$ being the corresponding block decomposition. Let $(W_{(1)}, \dots, W_{(t)})$ be the block decomposition of W. For all $0 \leqslant i < j \leqslant t$ we can check that:

$$(V_{(1)}, \dots, V_{(i)}, W_{(i+1)}, \dots, W_{(j)}, U_{(j+1)}, \dots, U_{(t)}) \in S_2. \tag{16}$$

Indeed, fix i, j. First note that this vertex is an isotropic subspace because (1) we know that $(V_{\leqslant i}, U_{>i}) \in S_2$ so $V_{\leqslant i}$ is symplectically orthogonal to $U_{>j}$, (2) $(W_{\leqslant j}, U_{>j}) \in S_2$ implying $(W_{(i+1)}, \dots, W_{(j)})$ is symplectically orthogonal to $U_{>j}$, and (3) $W, V' \subset V$ implying that $(W_{(i+1)}, \dots, W_{(j)}), V_{\leqslant i}$ are symplectically orthogonal. Therefore all vectors in the subspace in (16) are symplectically orthogonal to each other.

Second, we check that the dimension of the space in (16) is tk. Since $W \in \text{Good}_2$, $\dim(W \cap \text{Symp}(U_{>i})) = \dim(W) - \dim(U_{>i}) = ik$. Since $\dim(W_{\leqslant i}) = ik$, and by construction $W_{\leqslant i}$ is symplectic to $U_{>i}$, it follows that $W \cap \text{Symp}(U_{>i}) = W_{\leqslant i}$, and so nothing in $W_{>i}$ is symplectically orthogonal to $U_{>i}$. Thus, as $V_{\leqslant i} \subset \text{Symp}(U_{>i})$, we conclude that $V_{\leqslant i} \cap W_{>i} = \{0\}$. We also argue that

$$(V_{\leqslant i}, W_{(i+1)}, \dots, W_{(j)}) \cap (U_{>j}) = \{0\}.$$

Indeed, if $v + w \in (V_{\leqslant i}, W_{(i+1)})$ and $w \neq 0$, then v is symplectic to $U_{>j}$ and w is not, so $v + w$ is not symplectic to $U_{>j}$ and hence $v + w \notin U_{>j}$. If $w = 0$, then the claim follows as $V_{\leqslant i} \cap U_{>j} = \{0\}$ for $i \leqslant j$. Combining everything, we get from the dimension formula that

$$\dim(V_{\leqslant i}, W_{(i+1)}, \dots, W_{(j)}, U_{>j}) = \dim(V_{\leqslant i}, W_{(i+1)}, \dots, W_{(j)}) + \dim(U_{>j})$$

$$= \dim(V_{\leqslant i}) + \dim(W_{(i+1)}, \ldots, W_{(j)}) + \dim(U_{>j})$$
$$= tk.$$

Recall that $P_{2i}(U,V) = (V_{\leqslant i}, U_{>i})$ and same for W. We will now create paths R^i of length $2i+1$ between $P_{2i}(U,W)$ and $P_{2i}(U,V)$ for all $i \in [t]$ using the intermediate vertices from (16). These paths are constructed as:

$$R^i := P_{2i}(U,W) \to (W_{(2)}, \ldots, W_{(i)}, U_{>i}) \to (V_{(1)}, W_{(2)}, \ldots, W_{(i)}, U_{>i}) \to \ldots \quad \to P_{2i}(U,V).$$

Note that this is a valid path because the odd vertices (starting from $P_{2i}(U,W)$) are in S_2 by (16), which also implies that the even vertices are in S_1. By creating the paths R^i, we have broken the original cycle $C(U,V,W)$ into $O(t)$ cycles of the form:

$$C^i = P_{2i}(U,W) \xrightarrow{R^i} P_{2i}(U,V) \to P_{2i+1}(U,V) \to P_{2i+2}(U,V) \xrightarrow{R^{i+1}} P_{2i+2}(U,W) \to P_{2i+1}(U,W) \to P_{2i}(U,W),$$

for $i \in [1, t-1]$ and

$$C^t = W \xrightarrow{R^t} V' = (V_{(1)}, \ldots, V_{(t)}) \to V \to W.$$

We will tile each C^i by 8-cycles and triangles, starting with $i \in [t-1]$. For all $j \in [1, i+1]$, let R^i_j denote the $(2j-1)^{th}$ vertex on the path R^i, that is,

$$R^i_j = (V_{<j}, W_{(j)}, \ldots, W_{(i)}, U_{>i}).$$

It is easy to see that for all $j \in [1, i+1]$, the vertices R^i_j and R^{i+1}_j are connected via a path of length two:

$$R^i_j \to (V_{<j}, W_{(j)}, \ldots, W_{(i)}, U_{\geqslant i+2}, \ldots, U_{(t)}) \to R^{i+1}_j.$$

Additionally the middle vertex in the length 2 path is equal to $P_{2i+1}(U,V)$ in the path from $R^i_{i+1} = P_{2i}(U,V) \to R^{i+1}_{i+1}$, $P_{2i}(U,V) \to P_{2i+2}(U,V)$ and $R^{i+1}_{i+1} \to R^{i+1}_{i+2}$, thus showing that the whole cycle has been broken into $O(t)$ 8-cycles and $O(1)$ triangles.

Let us now tile the cycle C^t. We have that for all $j \in [1, t+1]$, $R^t_j = (V_{<j}, W_{(j)}, \ldots, W_{(t)})$ is contained in V since both W and V' are in V. This means that C^t is broken into $O(t)$ triangles. So overall $C(U,V,W)$ has been broken into $O(t^2)$ 8-cycles and $O(t)$ triangles.

Tiling by 8-cycles when $V \in S_3, W \in S_1$: Let $V' \subset V$ be the vertex chosen by the path from U, with $V' = (V_{(1)}, \ldots, V_{(t)})$ being the corresponding block decomposition. Let $W_{(1)}$ be the additional vertex used in the path from U to W, with the block decomposition $W' = (W_{(1)}, \ldots, W_{(t)})$, where $W = (W_{(2)}, \ldots, W_{(t)}) \subset V$. For all $0 \leqslant i < j \in [t]$, using the same argument as in (16) we have that

$$(V_{(1)}, \ldots, V_{(i)}, W_{(i+1)}, \ldots, W_{(j)}, U_{(j+1)}, \ldots, U_{(t)}) \in S_2. \tag{17}$$

Now we create the same paths R^i of length $2i+1$ between $P_{2i}(U,W) \to P_{2i}(U,V)$ for all $i \in [t]$, and each vertex on these paths is in S_2 or S_1 because of (17). This breaks the cycle into the cycles C^i, for $i \in [t]$. The tiling of $C^i, i \in [t-1]$ proceeds identical to the first case, therefore let us discuss the tiling of the last cycle:

$$C^t = W' = (W_{(1)}, \ldots, W_{(t)}) \xrightarrow{R^t} V' = (V_{(1)}, \ldots, V_{(t)}) \to V \to W \to W'.$$

We have that the first three vertices of R^t are W', $(W_{\geqslant 2})$ and $(V_{(1)}, W_{\geqslant 2})$ which are connected to W. Additionally including $(V_{(1)}, W_{\geqslant 2})$, every other vertex on R^t, i.e. for all $j \in [2, t{+}1]$, $R^t_j = (V_{<j}, W_{(j)}, \ldots, W_{(t)})$ and the intermediate vertices in S_1, is contained in V since both W and V' are in V. This means that C^t is broken into $O(t)$ triangles. So overall $C(U,V,W)$ has been broken into $O(t^2)$ 8-cycles and $O(t)$ triangles.

Tiling by 8**-cycles when** $V \in S_2, W \in S_1$**:** We can show that $C(U,V,W)$ in this case too can be broken into $O(t^2)$ 8-cycles and $O(t)$ triangles. The proof for the cycles C^i for $i \in [t-1]$ is the same, so we only discuss the tiling of C^t. We have that,

$$C^t = W' = (W_{(1)}, \ldots, W_{(t)}) \xrightarrow{R^t} V = (V_{(1)}, \ldots, V_{(t)}) \to W \to W',$$

where $W = (W_{(2)}, \ldots, W_{(t)})$. The first and second vertex on R^t, W', W are connected to W, and then one can check that every vertex on R^t except for the first one, is connected to V. This breaks C^t into $O(t)$ triangles.

Triangulating the 8**-cycles:** Having shown that the cycle $C(U,V,W)$ can always be tiled by at most $O(t^2)$ 8-cycles and $O(t)$ triangles, it suffices to show that each one of the resulting 8-cycles can be triangulated individually. Towards this end, we first notice that the 8-cycles we formed consist of edges between isotropic subspaces of dimension k_1 and isotropic subspaces of dimension k_2. Thus, to triangulate them we will have to use auxiliary vertices of dimension k_3. Intuitively, the difference $k_2 - k_1$ measures how "different" adjacent vertices in the cycle are, and it stands to reason that the closer these vertices are, the easier time we will have triangulating it. In the proof below we handle two cases separately: the case $k_2 - k_1 \leqslant k_3 - k_2$, and the case that $k_2 - k_1 > k_3 - k_2$, and we begin with the former easier case.

Triangulating the 8-cycles when $k_3 - k_2 \geqslant k_2 - k_1$**:** Note that each 8-cycle in the above tiling is of the following form for some $1 \leqslant j < i \in [t]$:

$$\begin{aligned} &(W_{(j)}, U_{(i)}, X) \to (W_{(j)}, X) \to (W_{(j)}, W_{(i)}, X) \to (W_{(i)}, X) \\ &\to (V_{(j)}, W_{(i)}, X) \to (V_{(j)}, X) \to (V_{(j)}, U_{(i)}, X) \to (U_{(i)}, X) \to (W_{(j)}, U_{(i)}, X), \end{aligned} \tag{18}$$

with $X = (V_{\leqslant j-1}, W_{(j+1)}, \ldots, W_{(i-1)}, U_{(>i)})$. Let $A = W_{(j)}, B = V_{(j)}, C = U_{(i)}, D = W_{(i)}$. By Claim 4.13, $C \cap \mathrm{Symp}(D) = \{0\}$. Furthermore each of these blocks are of size $k = k_2 - k_1$ and our assumption reads that $k_3 \geqslant k_2 + k$. This cycle therefore satisfies the assumptions of Lemma 4.14 and can thus be triangulated with $O(1)$ triangles.

Triangulating the 8-cycles when $k_3 - k_2 \leqslant k_2 - k_1$**:** This case is more difficult, and we first start with an 8-cycle of the form we created, and transform it into an 8-cycle that can be triangulated using Lemma 4.14. Fix some $i_0 < j_0 \in [t]$ consider an 8-cycle as in (18), namely:

$$\begin{aligned} C_8 = (A, C, X) &\to (A, X) \to (A, D, X) \to (D, X) \to (B, D, X) \\ &\to (B, X) \to (B, C, X) \to (C, X) \to (A, C, X), \end{aligned}$$

where A, B, C, D, X are defined appropriately as in the above paragraph. We have the property that $C \cap \mathrm{Symp}(D) = D \cap \mathrm{Symp}(C) = \{0\}$.

Picture the 8-cycle as a square, with the k_2-dimensional vertices at the 4 corners. We will construct two "horizontal" paths: $(A, C, X) \to (A, C, D)$ and $(B, C, X) \to (B, D, X)$ and two vertical paths:

$(A, C, X) \to (B, C, X)$ and $(A, C, X) \to (B, D, X)$. To do so first we apply the randomized algorithm from Claim 4.11 on the subspaces $C_1 := C, D := D_1$ to get $C_2 \subset_{k-1} C_1, d_1 \in D$ such that (d_1, C_2) is an isotropic subspace. We can then write $D = D_2 + d_1$, such that $D_2 \cap \text{span}(\{d_1\}) = \emptyset$, and apply Claim 4.11 again to get $d_2 \in D_2, C_3 \subset_{k-2} C_2$ such that (d_2, C_3) is an isotropic subspace and in fact (d_1, d_2, C_3) is also isotropic. Similarly applying the claim k times we get a sequence of subspaces:

$$C, \quad (d_1, C_2), \quad (d_1, d_2, C_3), \ldots, (d_1, \ldots, d_{k-1}, C_k), \quad D.$$

We know that $C_k \subset_1 C_{k-1} \subset_2 \ldots C_2 \subset_1 C$. Similarly we create the following sequence between A and B:

$$A, \quad (b_1, A_2), \quad (b_1, b_2, A_2), \ldots, (b_1, \ldots, b_{k-1}, A_k), \quad B.$$

We will show that for all $i, j \in [k]$:

$$\Pr_{a,b,c,d}[\dim(b_{\leqslant i}, A_i, d_{\leqslant j}, C_j, X) = 2k] \geqslant 1 - \frac{\text{poly}(k)}{q}. \tag{19}$$

This is because:

$$\begin{aligned}\Pr_{a,b}[\dim(b_{\leqslant i}, A_i, X) = k] &= \prod_{i'=1}^{i} \Pr_{a,b}[\dim(b_{\leqslant i'-1}, b_{i'}, A_{i'}, X) = k \mid \dim(b_{\leqslant i'-1}, A_{i'-1}, X) = k] \\ &\geqslant \left(1 - O\left(\frac{1}{q}\right)\right)^i \geqslant 1 - O\left(\frac{k}{q}\right),\end{aligned}$$

where for each term in the product we used Claim 4.11 with the subspaces $C = (b_{\leqslant i'-1}, X)$, $A = A_{i'-1}$ and $B = B_{i'-1}$. To prove (19) we use the same trick again:

$$\begin{aligned}&\Pr_{a,b,c,d}[\dim(b_{\leqslant i}, A_i, d_{\leqslant j}, C_j, X) = 2k] \\ &= \Pr_{a,b}[\dim(b_{\leqslant i}, A_i, X) = k] \Pr_{a,b,c,d}[\dim(b_{\leqslant i}, A_i, d_{\leqslant j}, C_j, X) = 2k \mid \dim(b_{\leqslant i}, A_i, X) = k] \\ &\geqslant \left(1 - O\left(\frac{k}{q}\right)\right) \prod_{j'=1}^{j} \Pr_{a,b,c,d}[\dim(d_{\leqslant j'-1}, b_{\leqslant i}, A_i, d_{j'}, C_{j'}, X) = 2k \mid \dim(d_{\leqslant j'-1}, C_{j'-1}, b_{\leqslant i}, A_i, X) = 2k] \\ &\geqslant 1 - O\left(\frac{\text{poly}(k)}{q}\right),\end{aligned}$$

where for each term in the product we used Claim 4.11 with the subspaces $C = (d_{\leqslant j'-1}, b_{\leqslant i}, A_i, X)$, $A = C_{j'-1}$ and $B = D_{j'-1}$, which proves (19). Since $q > \text{poly}(k)$ we can now union bound over all choices of $i, j \in [k]$ to get that there exists a choice of a, b, c, d's such that for all $i, j \in [k]$:

$$\dim(b_{\leqslant i}, A_i, d_{\leqslant j}, C_j, X) = 2k. \tag{20}$$

Henceforth fix such a choice of a, b, c, d's.

Denote $k' = k_3 - k_2$ and write $k = t'k' + r'$, with $r' < k'$ and $t' \geqslant 1$. For all $i \in [0, t']$, set $P_i(C, D) = (d_{\leqslant ik'}, C_{ik'+1})$ and $P_{t'+1}(C, D) = D$. Then we have the path

$$\begin{aligned}R^1 = (A, C, X) &\to (A, X) \to (A, P_1(C, D), X) \to \ldots \\ &\to (A, X) \to (A, P_{t'}(C, D), X) \to (A, X) \to (A, D, X)\end{aligned}$$

that alternates between S_2 and S_1 by (20). Similarly for all $i \in [0, t']$, let $P_i(A, B)$ denote the subspace, $(b_{\leqslant ik'}, A_{ik'+1})$ and $P_{t'+1}(A, B) = B$. We get the following parallel path between (B, C, X) and (B, D, X):

$$\begin{aligned} R^2 = (B, C, X) &\to (B, X) \to (B, P_1(C, D), X) \to \dots \\ &\to (B, X) \to (B, P_{t'}(C, D), X) \to (B, X) \to (B, D, X). \end{aligned}$$

We now create horizontal paths between $(A, X, P_i(C, D))$ and $(B, X, P_i(C, D))$ for all $0 \leqslant i \leqslant t' + 1$:

$$\begin{aligned} H^i = (A, P_i(C, D), X) &\to (P_i(C, D), X) \to (P_1(A, B), P_i(C, D), X) \to \dots \\ &\to (P_i(C, D), X) \to (P_{t'}(A, B), P_i(C, D), X) \to (P_i(C, D), X) \to (B, P_i(C, D), X), \end{aligned}$$

where every vertex is either in S_2 or S_1 by (20). Let H^i_j denote the vertex $(P_j(A, B), P_i(C, D), X)$ – this is connected to H^{i+1}_j via the vertex $(P_j(A, B), X) \in S_1$. Therefore we have broken the whole cycle into multiple triangles on the periphery and a grid of 8-cycles of the form:

$$\begin{aligned} &(P_j(A, B), P_i(C, D), X) \to (P_j(A, B), X) \to (P_j(A, B), P_{i+1}(C, D), X) \\ &\to (P_{i+1}(C, D), X) \to (P_{j+1}(A, B), P_{i+1}(C, D), X) \to (P_{j+1}(A, B), X) \\ &\to (P_{j+1}(A, B), P_i(C, D), X) \to (P_i(C, D), X) \to (P_j(A, B), P_i(C, D), X), \end{aligned}$$

for $i, j \in [t']$. To rewrite this cycle in a more convenient form, let $A_{jk'+1} = A_{(j+1)k'+1} + A'$, for some k'-dimensional A' satisfying $A' \cap A_{(j+1)k'+1} = \{0\}$. Let $B' = (b_{jk'+1}, \dots, b_{(j+1)k'})$, and $Y_{AB} = (b_{\leqslant jk'}, A_{(j+1)k'+1})$. In these notations we have that

$$P_j(A, B) = (R_A, Y_{AB}) \quad P_{j+1}(A, B) = (b_{jk'+1}, \dots, b_{(j+1)k'}, Y_{AB}).$$

Let $D' = (d_{ik'+1}, \dots, d_{(i+1)k'})$, so that $D_{ik'+1} = D_{(i+1)k'+1} + D'$. Let $C' = \text{Symp}(D_{(i+1)k'+1}) \cap C_{ik'+1}$, and $Y_{CD} = (d_{\leqslant ik'}, C_{(i+1)k'+1})$. We will now show that $C' \cap \text{Symp}(D') = 0$ and $\dim(C') = k'$. First note that for all $x \in [t']$, $C_x \cap \text{Symp}(D_x) = \text{Symp}(C_x) \cap D_x = \{0\}$. This is because $D = \text{span}(d_{\leqslant x-1}) + D_x$ and $C_x \subset \text{Symp}(d_{\leqslant x-1})$ therefore if any vector $v \in C_x$ was symplectically orthogonal to D_x, then v would be symplectically orthogonal to D and we would get a contradiction to $C \cap \text{Symp}(D) = \{0\}$; thus $C_x \cap \text{Symp}(D_x) = \{0\}$, and the proof for $\text{Symp}(C_x) \cap D_x = \{0\}$ is analogous. Since $C_{(i+1)k'+1} \cap \text{Symp}(D_{(i+1)k'+1}) = \{0\}$, we get that $C_{(i+1)k'+1} \cap C' = 0$. By dimension counting we know that $\dim(C') \geqslant k'$, which gives us the decomposition:

$$C_{ik'+1} = C_{(i+1)k'+1} + C',$$

and $\dim(C') = k'$. Since $C_{ik'+1} \cap \text{Symp}(D_{ik'+1}) = \{0\}$ and $C_{(i+1)k'+1} \subset \text{Symp}(D')$, we conclude that $C' \cap \text{Symp}(D') = \{0\}$. Finally we get that,

$$P_i(C, D) = (C', Y_{CD}) \quad P_{i+1}(C, D) = (d_{ik'+1}, \dots, d_{(i+1)k'}, Y_{CD}).$$

Letting $Y = (Y_{AB}, Y_{CD})$, we can rewrite the cycle as:

$$\begin{aligned} &(A', C', Y) \to (A', Y_{AB}) \to (A', D', Y) \to (D', Y_{CD}) \to (B', D', Y) \\ &\to (B', Y_{AB}) \to (B', C', Y_{AB}) \to (C', Y_{CD}) \to (A', C', Y). \end{aligned}$$

We have shown that each of A', B', C', D' is of size $k' = k_3 - k_2$, with $C' \cap \text{Symp}(D') = 0$ and $k_3 \geqslant k_2 + k'$. Thus the cycle satisfies the assumptions of Lemma 4.14 and can be triangulated in $O(1)$ triangles.

Size of triangulation: Since we showed a tiling of $C(U,V,W)$ by $O(t^2)$ 8-cycles, and a tiling of each 8-cycle by $O(\lceil k_2-k_1/k_3-k_2\rceil^2)$ triangles, $C(U,V,W)$ has a triangulation of size $O(K^2)$ as required. □

4.2.5 Using Triangulations for Coboundary Expansion

Given the triangulations from Lemma 4.15 and as is standard in applications of the cones method, we will use the property that $\mathrm{SP}_{2d}(\mathbb{F}_q)$ acts transitively on the triangles of $S(k_1,k_2,k_3)$ to complete the proof of Lemma 4.8.

For a matrix $M \in \mathrm{SP}_{2d}(\mathbb{F})$, and a subspace V recall that $M(V)$ denotes the subspace $\mathrm{span}(Mv \mid v \in V)$. Let $M^{-1}(V)$ denote the subspace W such that $M(W) = V$. Given a path $P(U,V)$ from U to V, let $P_M(M(U),M(V))$ denote the path from $M(U)$ to $M(V)$ where at the i^{th}-step we have the vertex $M(P_i(U,V))$. It is easy to see that this is a valid path from $M(U)$ to $M(V)$ if $V \in \bigcup_{i\in[3]}\mathrm{Good}_i$. For a triangle Δ let $M(\Delta)$ denote the triangle whose vertices are $M(U_i), \forall U_i \in \Delta$. In fact we can let $T_M(M(U),M(V),M(W))$ be the triangulation of the cycle

$$M(U) \xrightarrow{P_M(M(U),M(V))} M(V) \to M(W) \xrightarrow{P_M(M(W),M(U))} M(U)$$

where a triangle in this triangulation is given by $M(\Delta)$ for $\Delta \in T(U,V,W)$. Again it is easy to see that this is a valid triangulation of the cycle.

We have the following randomized algorithm to get a good solution to an arbitrary UG instance Φ.

Algorithm 2 ($\Phi = (S(k_1,k_2,k_3),\Pi)$)**.**
Input: UG instance Φ on $S(k_1,k_2,k_3)$.
Output: A function $f : V(S(k_1,k_2,k_3)) \to S_m$.

1. Choose a random linear transformation $M \sim \mathrm{SP}_{2d}(\mathbb{F}_q)$ and set $f(M(U)) = \mathrm{id}$.
2. For each subspace $V \in \bigcup_i \mathrm{Good}_i$, assign $M(V)$ the label obtained by propagating the label of $M(U)$ to $M(V)$ via the path $P_M(M(U),M(V))$. For $V \notin \bigcup_i \mathrm{Good}_i$, assign an arbitrary label to $M(V)$.

We now complete the proof of Lemma 4.8 via the following lemma:

Lemma 4.16. *Let Φ be any UG instance over S_m with incons$(\Phi) = \delta$. Then in expectation over $M \sim$ SP$_{2d}(\mathbb{F}_q)$, the algorithm violates at most $O(K^2)\delta + \mathrm{poly}(K)/q$-fraction of edges where*

$$K = \max\left(\left\lceil \frac{k_2}{k_3-k_2}\right\rceil, \left\lceil \frac{k_2}{k_2-k_1}\right\rceil\right).$$

Proof. Suppose the propagation algorithm chooses a linear transformation M. Consider the assignment $f_M : V(S(k_1,k_2,k_3)) \to S_m$ that Algorithm 2 outputs in this case. For $V,W \in \bigcup_i \mathrm{Good}_i$, an edge $(M(V),M(W))$ is satisfied if the cycle $M(U) \xrightarrow{P_M(M(U),M(V))} M(V) \to M(W) \xrightarrow{P_M(M(W),M(U))} M(U)$ is consistent, i.e. the permutations on the edges product to id. Furthermore this is true if every triangle in $T_M(M(U),M(V),M(W))$ is consistent. Recall that this is the set $M(\Delta)$ as Δ ranges over $T(U,V,W)$. Let E denote $E(S(k_1,k_2,k_3))$ and Good(E) denote the set of edges (V,W) for $V,W \in \bigcup_i \mathrm{Good}_i$. We get that,

$$\mathrm{viol}(f_M) \leqslant \Pr_{(V,W)\sim E}[(V,W) \notin \mathrm{Good}(E)] + \mathop{\mathbb{E}}_{(V,W)\sim \mathrm{Good}(E)}[\mathbb{1}(\exists \Delta \in T_M(M(U),M(V),M(W)) \cap \mathrm{incons}(\Phi)]$$

$$\leqslant O\left(\frac{K}{q}\right) + \max_{(V,W)\in\text{Good}(E)}(|T(U,V,W)|) \mathop{\mathbb{E}}_{(V,W)\sim\text{Good}(E)} \mathop{\mathbb{E}}_{\Delta\in T(U,V,W)}[\mathbb{I}(M(\Delta)\in\text{incons}(\Phi)]$$

$$\leqslant O\left(\frac{K}{q}\right) + O(K^2) \mathop{\mathbb{E}}_{(V,W)\sim\text{Good}(E)} \mathop{\mathbb{E}}_{\Delta\in T(U,V,W))}[\mathbb{I}(M(\Delta)\in\text{incons}(\Phi)],$$

where in the second inequality we used Lemma 4.12 and the last one we used Lemmas 4.15 to bound the size of the triangulation.

Recall that M acts transitively on the set of t-dimensional isotropic subspaces, therefore the distribution $M(\Delta)$ over $M \sim \text{SP}_{2d}(\mathbb{F}_q)$ is uniform over the triangles in $S(k_1, k_2, k_3)$. Using this fact, we can now take an expectation over $M \sim \text{SP}_{2d}(\mathbb{F}_q)$ for the above equation to get:

$$\mathbb{E}[\text{viol}(f_M)] \leqslant O\left(\frac{K}{q}\right) + O(K^2) \mathop{\mathbb{E}}_{(V,W)\sim\text{Good}(E)} \mathop{\mathbb{E}}_{\Delta\in T(U,V,W))} \mathop{\mathbb{E}}_{M}[\mathbb{I}(M(\Delta)\in\text{incons}(\Phi)]$$

$$\leqslant O\left(\frac{K}{q}\right) + O(K^2)\delta,$$

which completes the proof. □

4.2.6 Modifications in Triangulations in Edge Cases

In this section, we explain the necessary modifications to the triangulation argument in the case that k_2 and k_1 are not multiples of $k = k_2 - k_1$. Let $k_2 = (t-1)k + a$ for some $2 \leqslant t \in \mathbb{N}$ and $a \in \{0, \dots, k-1\}$. Then $k_1 = (t-2)k + a$. Fix a vertex $U \in S_2$ and let $U = (U_{(1)}, \dots, U_{(t)})$ where each block has dimension k, except for the last one which has dimension a. We can define the set $\bigcup_i \text{Good}_i$ accordingly which gives the following set of paths from U.

Set of Paths from U: The paths from U are the same for all but the second to last step. For every V we will associate with V a vertex V'. In the case that $\dim(V) = k_2$, we will take $V' = V$, and otherwise V' will be an appropriately chosen subspace or superspace of V. We will also associate with V an a-dimensional subspace $V'_{(t-1)}$ which is a subset of $V_{(t-1)}$ that is obtained while creating a path from U to V.

1. For a vertex V of dimension k_2, using Claim 4.9 on V and $U_{\geqslant 2}$ we find $V_{(1)} \subseteq_k V$ such that $(V_{(1)}, U_{(2)}, \dots, U_{(t)}) \in S_2$. Applying this claim iteratively, we find $V_{(2)}$ such that

$$(V_{(1)}, V_{(2)}, U_{(3)}, \dots, U_{(t)}) \in S_2,$$

   and so on. Let $V'_{(t-1)}$ be a random a-dimensional subspace of $V_{(t-1)}$. Then consider the following path from $U \to V$ which flips a block of U to a block of V one at a time:

$$P(U,V) = (U_{(1)}, \dots, U_{(t)}) \to U_{\geqslant 2} \to (V_{(1)}, U_{\geqslant 2}) \to (V_{(1)}, U_{\geqslant 3}) \to (V_{(1)}, V_{(2)}, U_{\geqslant 3}) \to \dots$$
$$\to (V_{<t}, U_{(t)}) \to (V_{<t-1}, V'_{(t-1)}) \to V.$$

   Note that this path alternates between vertices of S_2 and S_1. We set $V' = V$.

2. We do the same for vertices in $\text{Good}_1, \text{Good}_3$. The paths remain the same for all but the step where we put in the vertex $(V_{<t-1}, V'_{(t-1)})$ and one can show that these are valid paths alternating between S_2 and S_1.

Triangulating the cycles $C(U,V,W)$**:** Let us focus on the case where (V,W) is an edge for $V \in \text{Good}_2, W \in \text{Good}_1$. The other two cases follow analogously. We create the same paths R^i from $P_{2i}(U,W)$ to $P_{2i}(U,V)$. The only path that is slightly different is the path R^t:

$$R^t := P_{2i}(U,W) \to (W_{(2)},\dots,W_{(t)}) \to (V_{(1)},W_{(2)},\dots,W_{(t)}) \to \dots \to (V_{<t},W_{(t)}) \to (V_{<t-1},V'_{(t-1)}) \\ \to P_{2i}(U,V).$$

One can now tile each cycle C^i created using 8-cycles and triangles in an identical manner when $i \in [1,t-2] \cup \{t\}$. Let us discuss the tiling of C^{t-1}. Putting in horizontal length two paths between R_j^{t-1} and R_j^t we can break C^{t-1} into 8-cycles of the form:

$$(W_{(1)},U_{(t)},X,Y) \to (W_{(1)},X) \to (W_{(1)},W_{(t)},X,Y) \to (W_{(t)},X,Y) \\ \to (V_{(1)},W_{(t)},X,Y) \to (V_{(1)},X) \to (V_{(1)},U_{(t)},X,Y) \to (U_{(t)},X,Y) \to (W_{(1)},U_{(t)},X,Y), \quad (21)$$

with $X = (W_{(2)},\dots,W_{(t-2)},W'_{(t-1)})$ and $Y = W''_{(t-1)})$, where $W''_{(t-1)} + W'_{(t-1)} = W_{(t-1)}$ and $W''_{(t-1)} \cap W'_{(t-1)} = \{0\}$. The only difference in this 8-cycle from the one in (18) is that the vertices $(W_{(1)},U_{(t)},X,Y)$ and $(W_{(1)},W_{(t)},X,Y)$ have an intersection $(W_{(1)},X,Y)$ which is larger than the k_1-dimensional subspace $(W_{(1)},X)$, and the same holds for the vertices $(V_{(1)},U_{(t)},X,Y)$ and $(V_{(1)},W_{(t)},X,Y)$. Intuitively this is only easier to tile than (18) where every two adjacent points on the square intersect in a k_1-dimensional subspace. Indeed this holds, and in both the cases $k_3-k_2 \geqslant k_2-k_1$ and $k_3-k_2 \leqslant k_2-k_1$ we can use the same strategy to tile this 8-cycle with triangles.

Given this tiling of $C(U,V,W)$, the rest of the proof remains the same and we omit the details.

4.3 The Extended Base Cases

In this section we use Lemmas 4.2 and 4.8 to show that the spherical buildings of type A and C, and their tensor products, satisfy the Assumptions 1. That is, we will prove a bound on the coboundary constant of tripartite graphs $T(A,B,C;\mu|X_S = A_0)$, where $\max_{a\in A} a < \min_{b\in B} b$ and $\max_{b\in B} b < \min_{c\in C} c$, and μ is the uniform distribution over the maximal faces of one of these complexes.

4.3.1 The Extended Base Case for Restrictions of Type A and Type C

To establish Assumption 1 we proceed in two steps. First, we prove an auxiliary lemma handling the case that μ has a product structure in the sense that $\mu^{S_1\cup S_2} = \mu^{S_1} \times \mu^{S_2}$ and $\mu^{S_1\cup S_3} = \mu^{S_1} \times \mu^{S_3}$. Let $\text{diam}(G)$ denote the diameter of a graph G.

Lemma 4.17. *Let μ be a distribution over $\prod_{i\in[d]} X_i$, and let G be a group acting on $\prod X_i$ such that for all $g \in G$ it holds that $g(X_i) = X_i$ and $g(\mathsf{supp}(\mu)) = \mathsf{supp}(\mu)$, as well as for all $X \in \mathsf{supp}(\mu)$, the distribution over $g(X)$ for a uniformly chosen $g \in G$ is μ. Then for all pairwise disjoint sets $S_1,S_2,S_3 \subset [d]$ that satisfy $\mu^{S_1\cup S_2} = \mu^{S_1} \times \mu^{S_2}$ and $\mu^{S_1\cup S_3} = \mu^{S_1} \times \mu^{S_3}$, we get that*

$$C(T(S_1,S_2,S_3;\mu)) \lesssim \textit{diam}(A(S_2,S_3;\mu)).$$

Proof. We first construct a set of paths and triangulations for propagation. Let P_i denote the set of vertices corresponding to the set S_i in $H = T(S_1,S_2,S_3;\mu)$. Since μ^{S_1,S_2} and μ^{S_1,S_3} are both product distributions, we have an edge between U,W for all $U \in P_1$ and $W \in P_3$, as well as between any $U \in P_1$ and $V \in P_2$. Fix two vertices $U_0 \in P_1, V_0 \in P_2$. For every $V \in P_2$ fix the path $P(U_0,V) = U_0 \to V$ and similarly for

$W \in P_3$ fix the path $P(U_0, W) = U_0 \to W$. For any $U \in P_1$ fix the path $P(U_0, U) = U_0 \to V_0 \to U$. Now for an edge (V, W) between P_2 and P_3 the cycle (U_0, V, W) is already a triangle, therefore has a triangulation of size 1. For an edge (U, V) we need to tile the cycle $U_0 \to V_0 \to U \to V \to U_0$. To do so we use a path R of length $\mathrm{diam}(A(S_2, S_3; \mu))$ from $V_0 \to V$ that alternates between vertices of P_2 and P_3. Every vertex in this path is connected to both U and U_0 therefore this cycle has a triangulation of size $\lesssim \mathrm{diam}(A(S_2, S_3; \mu))$. We can follow the same strategy for triangulating the cycle $U_0 \to W \to U \to V_0 \to U_0$ corresponding to the edge (V, W) – we build a path R of length $\mathrm{diam}(A(S_2, S_3; \mu))$ from V_0 to W, and given that every vertex in the path is connected to both U_0 and U this gives us a triangulation of size $\lesssim \mathrm{diam}(A(S_2, S_3; \mu))$. We can now use the fact that the triangles in H are transitive under the action of G to get that the coboundary constant of H is $\lesssim \mathrm{diam}(A(S_2, S_3; \mu))$. The proof of this fact is the same as that of Lemma 4.7, wherein the group G was $\mathrm{GL}_d(\mathbb{F}_q)$, and therefore we omit the details. □

Armed with Lemma 4.17, we now prove that the restrictions of type A spherical building give tripartite graphs that are coboundary expanders.

Lemma 4.18. *Let $\mu = SB_d^A(\mathbb{F}_q)$. For all $t \leqslant d/3$, $S \subseteq I$ of size at most $d - 3t$ and restrictions $A_0 \in \mathsf{supp}(\mu^S)$, for all t-sized pairwise disjoint sets A, B, C of $I \setminus S$, with $i = \max_{a \in A} a, j = \min_{b \in B} b, j' = \max_{b \in B} b$ and $k = \min_{c \in C} c$ satisfying $i < j \leqslant j' < k$, the graph $T(A, B, C; \mu | X_S = A_0)$ is a $(\mathrm{poly}(K), \mathrm{poly}(K)/q)$-coboundary expander over S_m, for $K = \max\left(\frac{k}{j-i}, \frac{k}{k-j'}\right)$.*

Proof. In this proof we will apply Lemma 4.17 multiple times. To do so, we will use the fact that for all $S \subseteq [d], A_0 \in \mathsf{supp}(\mu^S)$, the subgroup H of $\mathrm{GL}_d(\mathbb{F}_q)$ that fixes A_0, has the property that for all $X \in \mathsf{supp}(\mathcal{D})$, the distribution $h(X), h \sim H$ is equal to $\mathcal{D}$ for $\mathcal{D} = \mu | X_S = A_0$.

We first consider the case of $t = 1$. Fix a set $S \subset [d]$ of size at most $d - 3$, $A_0 \in \mathsf{supp}(\mu^S)$ and consider coordinates $i, j, k \in [d] \setminus S$ with $i < j < k$. Let $\mathcal{D}$ denote the distribution $\mu | X_S = A_0$. We divide the proof into two cases:

There exists $\ell \in S$ such that $\ell \in (i, j)$ or $\ell \in (j, k)$: Let us assume that there is $\ell \in S$ with $\ell \in (i, j)$. Then note that $\mathcal{D}^{\{i\},\{j\}} = \mathcal{D}^{\{i\}} \times \mathcal{D}^{\{j\}}$ and $\mathcal{D}^{\{i\},\{k\}} = \mathcal{D}^{\{i\}} \times \mathcal{D}^{\{k\}}$. Therefore we can apply Lemma 4.17 to get that $C(T(\{i\}, \{j\}, \{k\}; \mathcal{D})) \lesssim \mathrm{diam}(A(\{j\}, \{k\}; \mathcal{D}))$, which we know is at most $O(k/(k-j))$.

The same proof works when $\ell \in (j, k)$ to give a coboundary constant $O(j/j - i)$.

There is no $\ell \in S$ with $\ell \in (i, k)$: In this case, let $d_1 \in S$ be the largest index smaller than i and let $d_2 \in S$ be the smallest index larger than k. We know that G is isomorphic to $\mathrm{Gr}(i - d_1, j - d_1, k - d_1)$ over the ambient space $\mathbb{F}_q^{d_2 - d_1}$. Therefore we can use Lemma 4.2 to conclude that our graph is a $(\mathrm{poly}(K'), \mathrm{poly}(K')/q)$-coboundary expander for

$$K' = \max\left(\frac{j - d_1}{(j - d_1) - (i - d_1)}, \frac{j - d_1}{(k - d_1) - (j - d_1)}\right) \leqslant \max\left(\frac{k}{k - j}, \frac{k}{j - i}\right) \leqslant K,$$

as required.

We now move on to the case that $t > 1$. Namely fix a set $S \subset [d]$ of size at most $d - 3t$, $A_0 \in \mu^S$ and consider three pairwise disjoint t-sized sets $A, B, C \subset [d] \setminus S$ with $i < j < j' < k$. Let $\mathcal{D}$ denote the distribution $\mu | X_S = A_0$. Using the argument in Lemma 3.5, and more specifically (2), we get that

$$C(T(A, B, C; \mathcal{D})) \leqslant C(T(\{i\}, \{j\}, \{k\}; \mathcal{D})) \cdot$$

$$\max_{\substack{a\in\mathsf{supp}(\mathcal{D}^i)\\ b\in\mathsf{supp}(\mathcal{D}^j)\\ c\in\mathsf{supp}(\mathcal{D}^k)}} (C(T(A\setminus\{i\},B,C;\mathcal{D}|a)), C(T(A,B\setminus\{j\},C;\mathcal{D}|b)), C(T(A,B,C\setminus\{k\};\mathcal{D}|c)))$$

The first term above is at most K using the case $t=1$ from above. As for the second term, consider for instance $C(T(A\setminus\{i\},B,C;\mathcal{D}|a)$, and consider $\mathcal{D}'=\mathcal{D}|a$ and $A'=A\setminus\{i\}$. Note that $(\mathcal{D}')^{A'\cup B}=(\mathcal{D}')^{A'}\times(\mathcal{D}')^{B}$ and $(\mathcal{D}')^{A'\cup C}=(\mathcal{D}')^{A'}\times(\mathcal{D}')^{C}$, so applying Lemma 4.17 we get that the second term above is at most $\mathrm{diam}(A(S_2,S_3;\mathcal{D}'))$. Note that the diameter of the graph $A(S_2,S_3;\mathcal{D}')$ is at most a constant times the diameter of $A(\{j'\},\{k\};\mathcal{D}')$, which is easily seen to be at most $O\left(\frac{k}{k-j'}\right)\leqslant O(K)$. Combining the two bounds, we conclude that $C(T(A,B,C;\mathcal{D}))\lesssim K^2$. Similarly using (3) It is easy to check that the additive error β is at most $\mathrm{poly}(K)/q$. ☐

Next, we handle restrictions of type C spherical buildings.

Lemma 4.19. *Let $\mu=SB_d^C(\mathbb{F}_q)$. For all $t\leqslant d/3$, $S\subseteq I$ of size at most $d-3t$ and restrictions $A_0\in\mathsf{supp}(\mu^S)$, for all t-sized pairwise disjoint sets A,B,C of $I\setminus S$, with $i=\max_{a\in A}a, j=\min_{b\in B}b, j'=\max_{b\in B}b$ and $k=\min_{c\in C}c$ satisfying $i<j\leqslant j'<k$, the graph $T(A,B,C;\mu|X_S=A_0)$ is a $(\mathrm{poly}(K),\mathrm{poly}(K)/q)$-coboundary expander over S_m, for $K=\max\left(\frac{k}{j-i},\frac{k}{k-j'}\right)$.*

Proof. The proof is very similar to the proof of Lemma 4.18, and we will need to use the following facts (which were already used in Lemma 2.20):

1. Let $V_{i'}\subset V_{k'}$ be two i' and k'-dimensional isotropic subspaces. For $i'<i<j<k<k'$, the tripartite graph over i,j and k-dimensional isotropic subspaces that are contained in $V_{k'}$ and contain $V_{i'}$ is isomorphic to $\mathrm{Gr}_{k'-i'}(i-i',j-i',k-i')$ as defined in Section 4.1.
2. Let $V_{i'}$ be some i'-dimensional isotropic subspace. For $i'<i<j<k$, the tripartite graph over i,j and k-dimensional isotropic subspaces that contain $V_{i'}$, is isomorphic to $S_{d-i'}(i-i',j-i',k-i')$ as defined in Section 4.2.

We will again apply Lemma 4.17 and to do so, we will use the fact that for all $S\subseteq[d]$, $A_0\in\mathsf{supp}(\mu^S)$, the subgroup H of $\mathrm{SP}_{2d}(\mathbb{F}_q)$ that fixes A_0, has the property that for all $X\in\mathsf{supp}(\mathcal{D})$, the distribution $h(X), h\sim H$ is equal to $\mathcal{D}$ for $\mathcal{D}=\mu|X_S=A_0$.

With these facts in mind, we begin with the proof in the case that $t=1$. Let $\mathcal{D}=\mu|X_S=A_0$. We now prove that $G=T(\{i\},\{j\},\{k\};\mathcal{D})$ is a $(\mathrm{poly}(K),\mathrm{poly}(K)/q)$-coboundary expander, and there are a few cases to consider depending on the set of coordinates S that we restricted.

There exists $\ell\in S$ such that $\ell\in(i,j)$ or $\ell\in(j,k)$: Let us assume that there is $\ell\in S$ with $\ell\in(i,j)$. Then note that $\mathcal{D}^{\{i\},\{j\}}=\mathcal{D}^{\{i\}}\times\mathcal{D}^{\{j\}}$ and $\mathcal{D}^{\{i\},\{k\}}=\mathcal{D}^{\{i\}}\times\mathcal{D}^{\{k\}}$. Therefore we can apply Lemma 4.17 to get that $C(T(\{i\},\{j\},\{k\};\mathcal{D}))\lesssim\mathrm{diam}(A(\{j\},\{k\};\mathcal{D}))$, which we can check is at most $O(k/(k-j))$. The same proof works when $\ell\in(j,k)$ to give a coboundary constant $O(j/j-i)$.

There is no $\ell\in S$ with $\ell\in(i,k)$ and there is $k'\in S, k'>k$: In this case, let $k'\in S$ be the smallest index larger than k and let $i'\in S$ be the largest index smaller than i (set it to 0 if no such index). By the above, G is isomorphic to $\mathrm{Gr}_{k'-i'}(i-i',j-i',k-i')$ which is a $(\mathrm{poly}(K),\mathrm{poly}(K)/q)$-coboundary expander by Lemma 4.2.

There is no $\ell \in S$ with $\ell \in (i,k)$ and no $k' \in S, k' > k$: In this case, again let $k' \in S$ be the smallest index larger than k and let $i' \in S$ be the largest index smaller than i ($i' = 0$ if no such index). By Fact 2 above, we know that G is isomorphic to $S_{d-i'}(i-i', j-i', k-i')$ which is a $(\mathrm{poly}(K), \mathrm{poly}(K)/q)$-coboundary expander by Lemma 4.8.

We now move on to handle $t > 1$. Using the argument in Lemma 3.5, and more specifically (2), we get that

$$C(T(A,B,C;\mathcal{D})) \leqslant C(T(\{i\},\{j\},\{k\};\mathcal{D}))\cdot \max_{\substack{a\in\mathsf{supp}(\mathcal{D}^i)\\ b\in\mathsf{supp}(\mathcal{D}^j)\\ c\in\mathsf{supp}(\mathcal{D}^k)}} (C(T(A\setminus\{i\},B,C;\mathcal{D}|a)), C(T(A,B\setminus\{j\},C;\mathcal{D}|b)), C(T(A,B,C\setminus\{k\};\mathcal{D}|c)))$$

The first term above is at most K using the case $t = 1$ from above. As for the second term, consider for instance $C(T(A\backslash\{i\},B,C;\mathcal{D}|a)$, and consider $\mathcal{D}' = \mathcal{D}|a$ and $A' = A\backslash\{i\}$. Note that $(\mathcal{D}')^{A'\cup B} = (\mathcal{D}')^{A'} \times (\mathcal{D}')^B$ and $(\mathcal{D}')^{A'\cup C} = (\mathcal{D}')^{A'} \times (\mathcal{D}')^C$, so applying Lemma 4.17 we get that the second term above is at most $\mathrm{diam}(A(S_2,S_3;\mathcal{D}')) \lesssim K$. Combining the two bounds, we conclude that $C(T(A,B,C;\mathcal{D})) \lesssim K^2$. Similarly using (3) It is easy to check that the additive error β is at most $\mathrm{poly}(K)/q$. □

4.3.2 The Extended Base Case for Tensors of Type A and C

We now extend the base case from the previous section to a base case regarding *tensors* of type A and type C complexes.

Definition 4.20. *Let $X = (X(0), X(1), \ldots, X(d_1))$ and $Y = (Y(0), Y(1), \ldots, Y(d_2))$ be simplicial complexes. We define the tensored simplicial complex $X \otimes Y$ as the complex $(Z(0), \ldots, Z(d_1+d_2))$ where for each $1 \leqslant \ell \leqslant d_1 + d_2$ we have*

$$Z(\ell) = \{A \cup B \,|A \in X, B \in Y, |A| + |B| = \ell\,\}.$$

We note that if we let μ, ν be the uniform distributions on top dimensional faces of X, Y respectively, then the distribution $\mu \otimes \nu$ is uniform over the top dimensional faces of $X \otimes Y$. Thus, it will often be convenient for us to discuss tensors of complexes using the language of associated probability distributions.

Lemma 4.21. *Let μ_1, μ_2 be two distributions over the domains $\prod_{i\in[d_1]} X_i$ and $\prod_{i=d_1+1}^{d} X_i$, with μ_1 being either $SB^A_{d_1}(\mathbb{F}_q)$ or $SB^C_{d_1}(\mathbb{F}_q)$ and μ_2 being either $SB^A_{d-d_1}(\mathbb{F}_q)$ or $SB^C_{d-d_1}(\mathbb{F}_q)$. Let $\mu = \mu_1 \times \mu_2$, $t \leqslant d/3$, $S \subseteq [d]$ be of size at most $d - 3t$, $A_0 \in \mathsf{supp}(\mu^S)$ a restriction of S, and let $A, B, C \subseteq [d] \setminus S$ be of size t such that*

$$i = \max_{a\in A} a < j = \min_{b\in B} b \leqslant j' = \max_{b\in B} b < k = \min_{c\in C} c.$$

Then the graph $G = T(A,B,C;\mu|X_S = A_0)$ is a $(\mathrm{poly}(K), \mathrm{poly}(K)/q)$-coboundary expander over S_m for $K = \max\left(\frac{k}{k-j'}, \frac{k}{j-i}\right)$.

Proof. Let $G_1 = \mathrm{GL}_{d_1}(\mathbb{F}_q)$ or $\mathrm{SP}_{2d_1}(\mathbb{F}_q)$ depending on whether μ_1 is $SB^A_{d_1}(\mathbb{F}_q)$ or $SB^C_{d_1}(\mathbb{F}_q)$, and similarly define G_2 as $\mathrm{GL}_{d-d_1}(\mathbb{F}_q)$ or $\mathrm{SP}_{2(d-d_1)}(\mathbb{F}_q)$. The set $\mathsf{supp}(\mu)$ is transitive under the action of $G_1 \times G_2$ and moreover for all $S \subseteq [d]$, $A_0\mathsf{supp}(\mu^S)$, the subgroup H of $G_1 \times G_2$ that fixes A_0, has the property that for all $X \in \mathsf{supp}(\mathcal{D})$, the distribution $h(X), h \sim H$ is equal to $\mathcal{D}$ for $\mathcal{D} = \mu|X_S = A_0$.

Given this fact, we begin with the case that $t = 1$. Fix a set $S \subset [d]$ of size at most $d - 3$, $A_0 \in \mathsf{supp}(\mu^S)$ and consider coordinates $i, j, k \in [d] \setminus S$ with $i < j < k$. Let $\mathcal{D} = \mu|X_S = A_0$, $I_1 = [d_1]$ and $I_2 = \{d_1 + 1, \ldots, d\}$. We divide the proof into 4 cases:

The case that $i \in I_1$ and $j, k \in I_2$: First note that $\mathcal{D}^{\{i,j\}}$ and $\mathcal{D}^{\{i,k\}}$ are both product distributions. Hence we can apply Lemma 4.17 to get that the coboundary constant of G is $\lesssim \text{diam}(A(\{j\}, \{k\}; \mathcal{D})) \lesssim K$.

The case that $i, j \in I_1$ and $k \in I_2$: This case is similar to the above and we get a bound of $O(j/(j-i)) \lesssim K$ on the coboundary constant.

All three coordinates in I_2: Depending on whether μ_1 is $SB^A_{d_1}$ or $SB^C_{d_1}$ we can use Lemma 4.18 or 4.19 to get that G is a $(\text{poly}(K), 2^{-\Omega(r^{12})})$-coboundary expander.

All three coordinates in I_1: This case follows analogously to the third case above.

We now move on to the case that $t > 1$, and the argument here is the same as in Lemma 4.18 and 4.19. Using the argument in Lemma 3.5, and more specifically (2), we get that

$$C(T(A, B, C; \mathcal{D})) \leqslant C(T(\{i\}, \{j\}, \{k\}; \mathcal{D}))\cdot \max_{\substack{a \in \mathsf{supp}(\mathcal{D}^i) \\ b \in \mathsf{supp}(\mathcal{D}^j) \\ c \in \mathsf{supp}(\mathcal{D}^k)}} (C(T(A \setminus \{i\}, B, C; \mathcal{D}|a)), C(T(A, B \setminus \{j\}, C; \mathcal{D}|b)), C(T(A, B, C \setminus \{k\}; \mathcal{D}|c)))$$

The first term above is at most K using the case $t = 1$ from above. As for the second term, consider for instance $C(T(A \setminus \{i\}, B, C; \mathcal{D}|a)$, and consider $\mathcal{D}' = \mathcal{D}|a$ and $A' = A \setminus \{i\}$. Note that $(\mathcal{D}')^{A' \cup B} = (\mathcal{D}')^{A'} \times (\mathcal{D}')^B$ and $(\mathcal{D}')^{A' \cup C} = (\mathcal{D}')^{A'} \times (\mathcal{D}')^C$, so applying Lemma 4.17 we get that the second term above is at most $\text{diam}(A(S_2, S_3; \mathcal{D}')) \lesssim K$. Combining the two bounds, we conclude that $C(T(A, B, C; \mathcal{D})) \lesssim K^2$. Similarly using (3) It is easy to check that the additive error β is at most $\text{poly}(K)/q$. $\square$

5 UG coboundary expansion of Spherical Buildings

We can now use the base case of induction along with the local to global lemma to get that the spherical buildings of type A and C are UG coboundary expander.

Claim 5.1. *There is a constant $C > 0$ such that if $\alpha(r)$ is a function of r such that $\alpha(r) \geqslant 6$, and suppose that $d \geqslant 10r\alpha(r)$. Then*

$$\Pr_{i_1, \ldots, i_r \sim [d]} \left[\min_{j \neq k \in [r]} |i_j - i_k| \geqslant \frac{d}{r\alpha(r)} \right] \geqslant 2^{-r/\alpha(r)C}.$$

Proof. Denote $r' = r\alpha(r)$ for simplicity. We will show that the event holds even if we pick elements from $[d]$ with repetitions, which is clearly stronger. To do so, we first choose a_1, then calculate the probability that a_2 lies outside the d/r' interval around a_1, then further calculate the probability that a_3 lies outside the d/r' interval around a_1 and a_2 and so on. Formally it suffices to lower bound:

$$\Pr_{a_i \sim [d]} \left[\min_{a \in \{a_1, \ldots, a_{i-1}\}} |a - a_i| \geqslant \frac{d}{r'} \;\middle|\; a_1, \ldots a_{i-1} \right],$$

for all $i \in [k]$.

Let us start by bounding the first expression, and let us fix an i. There are a total of at most $(i-1)\frac{d}{r'}$ elements that are $\frac{d}{r'}$-close to some $a_{i'}$ for $1 \leqslant i' \leqslant i-1$. Thus, the probability a_i is chosen among them is at most $\frac{i-1}{r'}$, so the first expression is at least $1 - \frac{i-1}{r'}$.

As the probability of the event we are interested in the product of the expressions over $i = 1, \ldots, r$, we get that

$$\Pr_{a_1,\ldots,a_r \sim [d]}\left[\min_{a \neq b \in \cup a_i} |a-b| \geqslant \frac{d}{r'}\right] \geqslant \prod_{i \in [r]} \left(1 - \frac{i-1}{r'}\right) \geqslant \left(1 - \frac{r}{r'}\right)^r \geqslant 2^{-O(r/\alpha(r))}. \qquad \square$$

Definition 5.2. *Let $r \leqslant 3d$ and $\mathcal{D}$ be a distribution over $\prod_{i \in [d]} X_i$. We define a $\binom{d}{r}$-partite graph $G_r(\mathcal{D})$ as follows: for every $S \subset_r [d]$, let the part P_S contain the vertices $\mathsf{supp}(\mathcal{D}^S)$ and to sample an edge we sample $S, S' \subset_r [d]$ with $S \cap S' = \emptyset$, $X \sim \mathcal{D}$ and output $(X_S, X_{S'})$. We associate $G_r(\mathcal{D})$ with the distribution on triangles that samples disjoint sets $S_1, S_2, S_3 \subset_r [d]$, $X \sim \mathcal{D}$ and outputs $(X_{S_1}, X_{S_2}, X_{S_3})$.*

The next lemma shows that the graph $G_r(\mathcal{D})$ is a coboundary expander over S_m with strong parameters if $\mathcal{D}$ is any restriction of a spherical building of type A, of type C or a tensor of them.

Lemma 5.3. *Let $m, r, d, q \in \mathbb{N}$ with $r \ll d \ll q$ and let μ be the distribution $SB_d^A(\mathbb{F}_q)$, $SB_d^C(\mathbb{F}_q)$ or one of their tensor products. Then for all $S \subset_{\leqslant r} [d]$ and $A_0 \in \mathsf{supp}(\mu^S)$, $G_r(\mu | X_S = A_0)$ is a $(2^{O(r^{0.99}\log r)}, 2^{-\Omega(r^{12})})$-coboundary expander over S_m.*

Proof. Let us fix μ to be the distribution corresponding to the spherical building of type C. The proof when μ equals $SB_d^A(\mathbb{F}_q)$ is identical except that we use Lemmas 2.15 and 4.18 for type A instead of Lemmas 2.20 and 4.19 for type C that are used below. Similarly, in the case that μ is one of the various possible tensor products, we use the fact that μ is an $O(1/\sqrt{q})$-product distribution (being the product of two $O(1/\sqrt{q})$-product distributions), and Lemma 4.21 instead of Lemma 4.19.

Consider $\mathcal{D} = \mu | (X_S = A_0)$ for $S \subset [d]$ of size at most r and $A_0 \in \mathsf{supp}(\mu^S)$. Let $\Phi = (G_r(\mathcal{D}), \Pi)$ be a UG instance over S_m with $\mathrm{incons}(\Phi) = \delta$. Let $\mathcal{S}$ be the set of tuples (S_1, S_2, S_3) of subsets of $[d] \setminus S$ of size r that are pairwise disjoint, and let $\mathcal{S}' \subseteq \mathcal{S}$ be the set of tuples $(S_1, S_2, S_3) \in \mathcal{S}$ where $\cup S_i$ is $\Omega(d/r^2)$-separated and satisfies the third item in Assumption 1. Using Claim 5.1 we have

$$\Pr_{(S_1,S_2,S_3) \in \mathcal{S}}[(S_1, S_2, S_3) \in \mathcal{S}'] \geqslant \Omega(1) - \Pr_{(S_1,S_2,S_3) \in \mathcal{S}}[\exists j \in [r^{0.9}], i \in [3] \text{ such that } ||S_i \cap I_j| - r^{0.1}| \geqslant r^{0.06}]. \tag{22}$$

By the union bound, symmetry and the fact that $r \leqslant \sqrt{d}$ we have that

$$\Pr_{(S_1,S_2,S_3) \in \mathcal{S}}[\exists j \in [r^{0.9}], i \in [3] \text{ such that } ||S_i \cap I_j| - r^{0.1}| \geqslant r^{0.06}] \lesssim r \Pr_{S_1}[||S_1 \cap I_1| - r^{0.1}| \geqslant r^{0.06}].$$

We bound the latter probability using the multiplicative Chernoff bound (that holds for sampling without replacement too):

$$\Pr_{S_1 \subset_r [d]}[||S_1 \cap I_1| - r^{0.1}| \geqslant r^{0.06}] \leqslant 2^{-\Omega(r^{0.1}(r^{0.06}/r)^2)} = 2^{-\Omega(r^{0.02})}.$$

Overall, plugging in these estimates into (22) we get that

$$\Pr_{(S_1,S_2,S_3) \in \mathcal{S}}[(S_1, S_2, S_3) \in \mathcal{S}'] \gtrsim 1.$$

Let $\mathcal{T}$ be the set of $S_4 \subset_r [d] \setminus S$ with $S_4 \cap (\cup_{i\in[3]} S_i) = \emptyset$, and note that

$$\mathop{\mathbb{E}}_{(S_1,S_2,S_3)\sim\mathcal{S}'} \mathop{\mathbb{E}}_{(a_1,a_2,a_3)\sim\mu^{\cup S_i}} [(a_1,a_2,a_3) \in \mathrm{incons}(\Phi)] \lesssim \delta, \tag{23}$$

$$\mathop{\mathbb{E}}_{(S_1,S_2,S_3)\sim\mathcal{S}'} \mathop{\mathbb{E}}_{S_4\sim\mathcal{T}} \mathop{\mathbb{E}}_{(a_1,a_2,a_4)\sim\mu^{S_1\cup S_2\cup S_4}} [(a_1,a_2,a_4) \in \mathrm{incons}(\Phi)] \lesssim \delta, \tag{24}$$

and

$$\mathop{\mathbb{E}}_{(S_1,S_2,S_3)\sim\mathcal{S}'} \mathop{\mathbb{E}}_{\substack{S_4,S_5\sim\mathcal{T}:\\ S_4\cap S_5=\emptyset}} \mathop{\mathbb{E}}_{(a_1,a_4,a_5)\sim\mu^{S_1\cup S_4\cup S_5}} [(a_1,a_4,a_5) \in \mathrm{incons}(\Phi)] \lesssim \delta, \tag{25}$$

where we used the fact that the expectations of the same event under $(S_1,S_2,S_3) \sim \mathcal{S}$ is equal to δ. Using Markov's inequality we can henceforth fix a tuple $(S_1,S_2,S_3) \in \mathcal{S}'$ where both the above expectations are bounded by $O(\delta)$. Let $I = \cup_{i\in[3]} S_i$. We verify that $\mathcal{D}^I$ and I satisfy the Assumptions 1. Indeed,

1. By Lemma 2.20, the measure $\mathcal{D}$ is an ε-product distribution with $\varepsilon \lesssim 1/\sqrt{q}$ which can be made at most $2^{-r^{12}}$ by taking q large.
2. The second item in Assumption 1 follows from Lemma 4.19.
3. The third item holds by the definition of $\mathcal{S}'$.

This means that we may apply Lemma 3.6, and indeed we do so.

Solving Φ on $H = T(S_1,S_2,S_3;\mathcal{D}^I)$: Since $\mathcal{D}^I$ and I satisfy Assumption 1 we can apply Lemma 3.6 to get that H is an $(2^{O(r^{0.99}\log r)}, 2^{-\Omega(r^{12})})$-coboundary expander over S_m. In particular if $\Phi|_H$ is the restricted UG instance on H then there exists a solution A to the vertices of H with:

$$\mathrm{viol}(A) \leqslant 2^{O(r^{0.99}\log r)} \mathrm{incons}(\Phi|_H) + 2^{-\Omega(r^{12})}) \leqslant 2^{O(r^{0.99}\log r)}\delta + 2^{-\Omega(r^{12})}.$$

Lifting the solution: We will now use A to create a highly satisfying solution B to $G = G_r(\mathcal{D})$. This proof is similar to the lifting proof from Lemma 3.6, but we give the full proof here for completeness. For every $S_4 \in \mathcal{T}$, every vertex $u \in \mathsf{supp}(\mathcal{D}^{S_4})$ and every restriction $s \in \mathsf{supp}(\mathcal{D}^{S_i}|X_{S_4} = u), i \in [3]$, let $g_s(v)$ denote the permutation $\pi(v,s)A(s)$. We will choose a randomized assignment as follows: for every set $S_4 \in \mathcal{T}$, every vertex $u \in \mathsf{supp}(\mathcal{D}^{S_4})$, choose a random $s \sim \mathcal{D}^{S_1}|(X_{S_4} = u)$ and assign $B(u) = g_s(u)$. We will now upper bound the expected fraction of edges that B violates for Φ.

Consider an edge $(u,v) \in G$ between two parts $S_4, S_5 \in \mathcal{T}$ (with $S_4 \cap S_4 = \emptyset$). This edge is satisfied if there exists $s' \in \mathsf{supp}(\mathcal{D}^{S_1}|(X_{S_4} = u, X_{S_5} = v))$ such that, (1) $B(u) = g_{s'}(u)$, (2) $B(v) = g_{s'}(v)$ and (3) The triangle (s',u,v) is consistent.

To evaluate the probability there is such s', we sample $s' \sim \mathcal{D}^{S_1}|(X_{S_4} = u, X_{S_5} = v)$ and consider each one of the events, starting with event (1). For it, the probability it doesn't hold is at most

$$\mathop{\mathbb{E}}_{B} \mathop{\mathbb{E}}_{(u,v)\sim\mathcal{D}^{S_4\cup S_5}} \mathop{\mathbb{E}}_{s'\sim\mathcal{D}^{S_1}|(X_{S_4}=u,X_{S_5}=v)} [\mathbb{1}(B(u) \neq g_{s'}(u))] = \mathop{\mathbb{E}}_{u\sim\mathcal{D}^{S_4}} \mathop{\mathbb{E}}_{s,s'\sim\mathcal{D}^{S_1}|u} [\mathbb{1}(g_{s'}(u) \neq g_s(u))].$$

To calculate this, let us first calculate a bound on:

$$\mathop{\mathbb{E}}_{u\sim\mathcal{D}^{S_4}} \mathop{\mathbb{E}}_{(s,s')\sim\mathcal{D}^{S_1\cup S_2}|u} [\mathbb{1}(g_s(u) \neq g_{s'}(u))].$$

It is easy to check that an if an edge $(s,s') \in T(S_1,S_2,S_3;\mathcal{D}|u)$ is satisfied by A, and the triangle (s,s',u) is consistent then $g_s(u) = g_{s'}(u)$. Thus,

$$\begin{aligned}
&\mathop{\mathbb{E}}_{u\in\mathcal{D}^{S_4}}\ \mathop{\mathbb{E}}_{(s,s')\sim\mathcal{D}^{S_1\cup S_2}|u}[\mathbb{1}(g_{s'}(u)\neq g_s(u))]\\
&\leqslant \mathop{\mathbb{E}}_{u\in G}\ \mathop{\mathbb{E}}_{(s,s')\sim\mathcal{D}^{S_1\cup S_2}|u}[\mathbb{1}((s,s')\text{ not satisfied by }A)] + \mathop{\mathbb{E}}_{u\in\mathcal{D}^{S_4}}\ \mathop{\mathbb{E}}_{(s,s')\sim\mathcal{D}^{S_1\cup S_2}|u}[\mathbb{1}((s,s',u)\in\text{incons}(\Phi))]\\
&\lesssim \text{viol}(A) + \mathop{\mathbb{E}}_{(s,s',u)\sim\mathcal{D}^{S_1\cup S_2\cup S_4}}[\mathbb{1}((s,s'u)\in\text{incons}(\Phi))] := \delta'(S_4).
\end{aligned}$$

We are ready to bound the probability that event (1) does not happen. Towards this end, for each vertex $u \in \mathsf{supp}(\mathcal{D}^{S_4})$ define $p_u := \Pr_{(s,s')\sim\mathcal{D}^{S_1\cup S_2}|u}[g_{s'}(u)\neq g_s(u)]$, so that the above inequality translates to $\mathbb{E}_{u\sim\mathcal{D}^{S_4}}[p_u] = \delta'(S_4)$. Let $p'_u = \Pr_{s,s'\sim\mathcal{D}^{S_1}|u}[g_{s'}(u)\neq g_s(u)]$. By Lemma 2.6, the second largest singular value of the bipartite graph $A(S_1,S_2;\mathcal{D}|u)$ is at most $\varepsilon\cdot\text{poly}(r)\leqslant 0.01$ for all u, so by the easy direction of Cheeger's inequality we get that, $p'_u \leqslant O(p_u)$, which gives us that,

$$\mathop{\mathbb{E}}_{u\sim\mathcal{D}^{S_4}}\ \mathop{\mathbb{E}}_{s,s'\sim\mathcal{D}^{S_1}|u}[\mathbb{1}(g_{s'}(u)\neq g_s(u))] \lesssim \delta'(S_4),$$

thus bounding the probability of event (1). One can check that the probability of event (2) is the same as event (1), hence let us proceed to event (3). For that we get,

$$\mathop{\mathbb{E}}_{(u,v)\sim\mathcal{D}^{S_4\cup S_5}}\ \mathop{\mathbb{E}}_{s'\sim\mathcal{D}^{S_1}|(X_{S_1}=u,X_{S_2}=v)}[\mathbb{1}((s',u,v)\in\text{incons}(\Phi))] = \mathop{\mathbb{E}}_{(s',u,v)\sim\mathcal{D}^{S_1\cup S_4\cup S_5}}[\mathbb{1}((s',u,v)\in\text{incons}(\Phi))].$$

Adding up the probabilities that events (1),(2) or (3) do not happen and taking an expectation over $S_4, S_5 \sim \mathcal{T}$ with $S_4\cap S_5 = \emptyset$ we get that,

$$\mathop{\mathbb{E}}_{B}\ \mathop{\mathbb{E}}_{\substack{S_4,S_5\sim\mathcal{T}:\\ S_4\cap S_5}}\ \mathop{\mathbb{E}}_{(u,v)\sim\mathcal{D}^{S_4\cup S_5}}\ \mathop{\mathbb{E}}_{s'\sim\mathcal{D}^{S_1}|(u,v)}[\mathbb{1}[\text{events (1),(2), or (3) don't hold}]] \lesssim 2^{O(r^{0.99}\log r)}\delta + 2^{-\Omega(r^{12})} := \delta',$$

where we used (24) and (25).

We see that sampling sets $S_4, S_5 \sim \mathcal{T}$ and an edge $(u,v)\sim\mathcal{D}^{S_4\cup S_5}$, the restrictions s_u, s_v chosen by the randomized assignment B and s' fails to satisfy at least one of the events (1), (2) and (3) with probability $\lesssim \delta'$. Thus, with probability at most $O(\delta')$ over the choice of $S_4, S_5, (u,v), s_u, s_v$, there is no s' that satisfies all events and otherwise we get that s' satisfies all of (1), (2) and (3). This shows that in expectation the assignment B that we get violates $\lesssim \delta'$-fraction of the edges of G that are between disjoint $S_4, S_5 \in \mathcal{T}$. Noting that this is a $1-o(1)$-fraction of all the edges of G (since $r \ll d$) completes the proof. □

6 Construction of Sparse UG Coboundary Expander

We show that the complex from [CL23] is a sufficiently strong UG coboundary expander. As an immediate corollary of [BM23a] we get that the Chapman-Lubotzky complex admits direct product testers.

6.1 Vanishing Cohomology for $G_1[X]$ over S_m

The main goal of this section is to present the Chapman-Lubotzky complex. In particular, we need the statement that for all $m \in \mathbb{N}$, for an appropriate choice of parameters, this complex has vanishing 1-cohomology. This statement was communicated to us by Dikstein, Dinur and Lubotzky [DDL23], and below we give an alternative proof.

Definition 6.1. *We say a graph G has vanishing cohomology with respect to S_m if the following holds. Let Φ be a Unique-Games instance on G over S_m in which every triangle in G is consistent. Then there exists a solution $S \in S_m^{V(G)}$ that satisfies all the constraints of Φ. A complex X has vanishing 1-cohomology over S_m if $G_1[X]$ has vanishing cohomology over S_m.*

We note that the above definition is equivalent to the standard topological definition of vanishing 1-cohomology. Therein, given a function $f\colon E \to H$ defined on the edges where H is some finite group, one defines the coboundary map ∂f on triangles via

$$\partial f(u, v, w) = f(u, v)f(v, w)f(w, u).$$

With these notations, we care about functions f such that $\partial f \equiv \text{id}$. If $f(u, v) = g(u)g(v)^{-1}$ for all edges $(u, v) \in E$ for some $g\colon V \to H$, then $\partial f \equiv \text{id}$ clearly. The first cohomology group of G with coefficients in H is defined via

$$H^1(G, H) = \{f \mid \partial f \equiv \text{id}\} \backslash \{f \mid \exists g\colon V \to H, f(u, v) = g(u)g(v)^{-1}\},$$

and note that the fact that G has vanishing cohomology over S_m with respect to the definition above is equivalent to $H^1(G, H) = 0$. We will use this language henceforth in this section.

First, we need the following general lemma that relates the cohomology of a complex to the homomorphisms of groups.

Lemma 6.2. *Let G be a group acting transitively on a contractible topological space X, let Γ be a discrete subgroup of G that acts simply on X, and let H be a finite group. Then $H^1(\Gamma\backslash X, H) \cong \text{Hom}(\Gamma, H)$.*

Proof. We may identify functions on $\Gamma\backslash X$ with functions on X that are invariant under Γ. For a cycle x let us write $[x]$ for its equivalence class modulo the boundaries. Given $[f] \in H^1(\Gamma\backslash X, H)$ we obtain that $\partial(f) = \text{id}$. Since X is contractible, we may find a function g on X with $\partial(g) = f$. Without loss of generality $g(x) = \text{id}$ for some $x \in X$. We now set $\varphi(\gamma) = g(\gamma(x))$ for each $\gamma \in \Gamma$. We assert that φ is a homomorphism. Indeed, for each γ in Γ choose a path from γx to x. By hypothesis, $f = \partial(g)$ is invariant under left-multiplication by Γ. Therefore, the value of $\partial(g)$ on the path is $\varphi(\gamma) = g(\gamma x)g(x)^{-1} = g(\gamma\tau x)g(\tau(x))^{-1} = \varphi(\gamma\tau)\varphi(\tau)^{-1}$. This yields that φ is indeed a homomorphism. To show that the homomorphism is well defined we need to show that if g is γ-invariant, then $\varphi = 1$, which of course holds. To construct the homomorphism in the reverse direction we choose a representative x for each coset Γx and define a function g on X by setting its value on γx to be $\varphi(\gamma)$. We then obtain back an element $[\partial(g)] \in H^1(\Gamma\backslash X, H)$. It is easy to verify that the maps that we defined are inverses of each other. □

Recall that a quarternion algebra over $\mathbb{F}$ is a field extension $\mathbb{F}[i, j, k]$ with $i^2 = a, j^2 = b, k^2 = c$ and $k = ij = -ji$ for $a, b, c \in \mathbb{F}$. Let $\mathbb{Q}_p$ be the p-adic rationals and $\mathbb{Q}_\infty$ be $\mathbb{R}$. Let ν be either a prime p or infinity. A quarternion algebra D over $\mathbb{Q}$ is said to be split at ν if $D \otimes \mathbb{Q}_p$ is isomorphic to the algebra of 2×2 matrices over $\mathbb{Q}_\nu$. Otherwise, it is a division ring and it is said to be *ramified* or unsplit. Given a quarternion algebra D there exists a natural involution τ sending each of i, j, k to its negation. For a matrix A with coordinates in D let us write $A^* = (\tau(a_{ji}))$. Then the group $SU(n, D)$ consists of all the matrices with coordinates in D, such that $A^*A = I$. The first result from number theory used by Chapman and Lubotzky is the following.

Fact 6.3. *For every p_0 there exits a quarternion algebra that is ramified (non-split) only over p_0 and ∞.*

The Chapman-Lubotzky high dimensional expanders consist of $\Gamma\backslash B$, where Γ is a lattice and B is the Bruhat-Tits building of type $\tilde{C}_n$ over $\mathbb{Q}_p$. The lattice Γ is given by taking a lattice inside the set $SU(n, D)$, where D is a quarternion algebra splitting over p_0, ∞ for some p_0, ∞. In our case we choose the prime p_0 to be larger than m. The lattice is obtained as follows: we first embed $SU(n, D)$ diagonally inside the product $\prod_{\nu\neq p,\infty} SU(n, D \otimes Q_\nu)$. We then intersect it with a product of the compact groups K_ν, where each K_ν can be an arbitrary compact open subgroup, which we choose to be $SP_{2n}(\mathbb{Z}_\nu)$ for each $\nu \neq p_0$, and when $\nu = p_0$, K_ν can be taken to be a pro p_0-group, which means that all its finite quotients have order which is a power of p_0:

Fact 6.4. *The subgroup $SU(n, D) \otimes Q_p$ contains a compact open pro-p group for each p.*

Following Chapman-Lubotzky we set $K = \prod_{\nu\neq p,\infty} K_\nu$ and $\Gamma = K \cap SU(n, D)$.

Finally, using the congruence subgroup property and the strong approximation theorem, Chapman–Lubotzky showed that K is the profinite completion of Γ, which means that every homomorphism from γ to a finite group can be extended uniquely into K. The final number theoretic facts that we need is the following. Recall that a discrete subgroup $L \leqslant G$ is said to be a *lattice* if there exists a G-invariant probability measure on $L\backslash G$.

Fact 6.5. Γ *is a lattice inside $SP_{2n}(\mathbb{Q}_p)$.*

We are now ready to state the main lemma of this section.

Lemma 6.6. *For all $m \in \mathbb{N}$, choosing n and p to be sufficiently lage, the Chapman-Lubotzky $\Gamma\backslash B$ has vanishing cohomology. Namely, $H^1(\Gamma/B, S_m) = 0$.*

The rest of this section is devoted to the proof of Lemma 6.6. First, we note that Lemma 6.2, to show that $H^1(\Gamma\backslash B, S_m) = 0$ it suffices to show that $\mathrm{Hom}(\Gamma, S_m) = 0$. By the above, it is sufficient to establish that $\mathrm{Hom}(K_\nu, S_m) = 0$ for each $\nu \neq p, \infty$.

When $\nu = p_0$, this follows from the fact that K_ν is a pro p_0-group and $p_0 > m$. This implies that its quotients are p-groups and therefore cannot be embedded inside S_m when $m < p$.

For $\nu \neq p_0$ this follows from the fact that the minimal dimension of a sub-representation of $\mathrm{SP}_{2n}(\mathbb{Z}_p)$ goes to infinity with n, and so if n is sufficiently large $\mathrm{Hom}(\mathrm{SP}_{2n}(\mathbb{Z}_p), S_m) = 0$ as each such permutation representation would give rise to a complex representation of dimension m, which would therefore have to be trivial. The lower bound on the dimension can be easily deduced by adapting the method due to Howe and Gurevich of U-rank, and we give the details below. [5]

Lemma 6.7. *The minimal dimension of a non-trivial representation of $SP_{2n}(\mathbb{Z}_p)$ tends to infinity as a function of n uniformly over all p.*

Proof. Let $\chi = \mathrm{tr} \circ \rho$ be the character of a nontrivial representation. Let $\mathcal{B}$ be the Abelian group of symmetric matrices over $\mathbb{Z}_p$ and let $G = \mathrm{GL}_n(\mathbb{Z}_{\mathbb{p}})$, which acts on the group $\mathcal{B}$ via $A^B = B^t AB$. The group $\mathrm{SP}_{2n}(\mathbb{Z}_{\mathbb{p}})$. Recall that the group $SP_{2n}(\mathbb{Z}_{\mathbb{p}})$ conists of the matrices that preserves a symplectic form. They are generated by the matrices of the form $\begin{pmatrix} I & A \\ O & I \end{pmatrix}$, where A is symmetric matrix with entries in $\mathbb{Z}_p$, those matrices of the form $\begin{pmatrix} C & O \\ O & (C^t)^{-1} \end{pmatrix}$, where C is in $\mathrm{GL}_n(\mathbb{Z}_p)$ together with the element $w = \begin{pmatrix} O & I \\ -I & O \end{pmatrix}$.

[5] A stronger form of the lemma below will appear in a future paper of Evra, Kindler, Lifshitz, and Pirani that will extend the theory of U-rank over p-adic groups.

Let us denote the first group of elements by B and the second group of elements G. Then the product BG is a semi direct product $B \rtimes G$ and it known as the *Siegel parabolic* subgroup.

Now the normal subgroup that B generates can easily be seen to be all of $\mathrm{SP}_{2n}(\mathbb{Z}_p)$. This shows that the restriction of ρ to B is also nontrivial. Indeed, if the restriction of ρ to B is trivial this would imply that ρ on all the conjugates of B and therefore also on the normal subgroup that it generates.

Therefore, the restriction of χ to B is a a non-constant function. Let us denote the restriction by f. Then since f is invariant under the conjugation action of G on B we have $f(B^tAB) = f(A)$ for all $B \in G$. This implies that for each character χ_X in the Pontryagin dual of B all its equivalence classes χ_{BXB^t} also lies in the support of f. Now the dimension of χ is equal to $f(1)$ and standard representation theory implies that the Fourier coefficients of f are nonnegative integers.

We may therefore lower bound the dimension $\chi(1)$ by the minimal size of an orbit of a nontrivial character in the Pontryagin dual of B. Each character in the Pontryagin dual corresponds to a symmetric matrix X over $\mathbb{Z}/p^k$ and $\chi_X(A) = \omega_p^{\mathrm{tr}(XA)}$. Now the orbit of X under the action of G can be lower bounded by the orbit of P^iX for each i. We may therefore multiply X by powers of p until all its entries are multiples of p^{k-1} and lower bound the orbit of the resulting character. However, the resulting matrix has entries in $\mathbb{F}_{\mathbb{p}}$ and the lower bound on the orbit of X was established for that case by Gurevich and Howe [GH17]. □

6.2 Cosystolic Expansion for $G_r[X]$

In this section we show that $G_r[X]$ has sufficiently strong cosystolic expansion. To prove this we use a local to global theorem of [DD23b] that shows that when the links are coboundary expanders then the complex is a cosystolic expander. Let us first show that the vertex links of the Chapman-Lubotzky complex are spherical buildings of type C, as well as tensor products of type A and type C buildings.

Lemma 6.8. *The Chapman-Lubotzky complex X discussed in Section 6.1 has vertex links that are spherical buildings of either type C_{n-1}, tensor of type A_1 and type C_{n-2}, or tensor of type C_k and type C_{n-k-1}. Furthermore, X is an $O(1/\sqrt{p})$-one-sided local spectral expander.*

Proof. Consider the complex X, and note that by construction the links of X are the same as the links of Γ. The affine building Γ has a Coxeter diagram of type $\tilde{C}_n$. By [AB08, Proposition 3.16], the links of Γ correspond to spherical buildings whose Coxeter diagram is the result of deleting one vertex from the diagram $\tilde{C}_n$ (see also [Ess10, Lemma 3.1.13]). By inspection, it follows that all links are spherical buildings that are either type C_{n-1}, tensor of type A_1 and type C_{n-2}, or tensor of type C_k and type C_{n-k-1}. These complexes correspond to subspaces over the field $\mathbb{F}_p$ (see [Nel23] for a description of the $\tilde{C}_n$ building).

Using Lemmas 2.20 and 2.15 we know that the spherical buildings of type C and type A are $O(1/\sqrt{p})$-product distributions or equivalently $O(1/\sqrt{p})$-one-sided local spectral expanders. This implies that the same also holds for their tensor products. Using the trickling-down theorem, Theorem 2.12 we then get that X is also an $O(1/\sqrt{p})$-one-sided local spectral expander. □

Below we state the formal definition of cosystolic expansion for graphs and complexes that will be useful for us.

Definition 6.9. *We say a graph G is a C-cosystolic expander with respect to S_m if the following holds. Let Φ be a Unique-Games instance on G over S_m with incons$(\Phi) = \delta$. Then one can change the constraints of Φ on $C\delta$-fraction of the edges to get Φ' such that incons$(\Phi') = 0$. We say that a complex X is a C-cosystolic expander over S_m if the graph $G_1[X]$ is a C-cosystolic expander over S_m.*

In literature the above definition is known as the 1-cosystolic expansion of X over S_m, and has various generalizations to higher levels of X (when S_m is replaced by an Abelian group) but we refrain from stating those definitions.

Let us now state the local to global theorem. For a complex X let $Y = X^{\leqslant R}$ denote the "cutoff" of X, i.e. $Y = (X(1), \dots, X(R))$, equipped with the same distributions.

Theorem 6.10 ([DD23b] Theorem 1.2 modified)**.** *There exists a constant R such that for all $\beta, \lambda > 0$ and $m, d, C \in \mathbb{N}$ with $\beta, \lambda \leqslant \mathrm{poly}(1/C)$ the following holds. Let X be a d-dimensional complex such that $X^{\leqslant R}$ is a λ-one-sided local spectral expander and for all $v \in X(1)$, $G_1[X_v]$ is a (C, β)-coboundary expander over S_m. Then X is a $\mathrm{poly}(C)$-cosystolic expander over S_m.*

Proof. First, [DD23b, Theorem 1.2] asserts that if the one-skeletons of all links of X are λ-one-sided expanders and for all $v \in X(1)$ the complex X_v is a C-coboundary expander over S_m, then X is a $\mathrm{poly}(C)$ cosystolic expander over S_m.

We discuss how to change their argument to get the stronger theorem. Firstly their argument only uses the link expansion to derive spectral gaps of the up-down walks on X on $R' < R$ levels. Therefore we can look at $Y = X^{\leqslant R}$ instead and using the local spectral expansion of Y, derive that these up-down walks on Y are sufficiently expanding. But these are the same as the walks on X therefore hence the latter also have the required expansion properties. Hence we only require the local spectral expansion assumption on $X^{\leqslant R}$.

Now let us discuss the requirements on the coboundary expansion of links – we only have that the 1-links are (C, β)-coboundary expanders over S_m, instead of C-coboundary expanders. Their argument works as is, except in the step that they apply coboundary expansion. They use the fact that given any UG instance: $\mathrm{incons}(\Phi) \geqslant \frac{1}{C}\mathrm{viol}(\Phi)$, but we can instead get that, $\mathrm{incons}(\Phi) \geqslant \frac{1}{C}\mathrm{viol}(\Phi) - \frac{\beta}{C}$. This error of β/C gets absorbed into the other additive errors that depend on "η" (check the proof overview) and λ as long as $\beta \leqslant \mathrm{poly}(1/C)$. Hence this error does not affect the rest of the argument and we get the same conclusion. $\square$

Lemma 6.11. *For all $r \ll d \ll q$ the following holds. Let X be the d-dimensional Chapman-Lubotzky complex over $\mathbb{F}_q$. Then $G_r[X]$ is a $2^{O(r^{0.99}\log r)}$-cosystolic expander over S_m for all $m \in \mathbb{N}$.*

Proof. Consider the complex X^r, with

$$X^r(1) = \{v \mid v \in X(r)\}, \qquad X^r(2) = \{(u, v) \mid u, v \in X(r) \text{ such that } u \cup v \in X(2r)\}$$

and more generally

$$X^r(\ell) = \{(u_1, \dots, u_\ell) : u_1, \dots, u_\ell \in X(r) \text{ such that } u_1 \cup \ldots \cup u_\ell \in X(\ell r)\}.$$

Note that that $G_r[X] = G_1[X^r]$. We intend to use Theorem 6.10, and for that we verify that the constant-sized links have one-sided expansion on the 1-skeletons, and that the 1-links of X^r are coboundary expanders over S_m.

One-sided local spectral expansion of $(X^r)^{\leqslant R}$: We will show that $(X^r)^{\leqslant R}$ is an $\exp(-r^{10})$-one-sided local spectral expander. Let us fix an i-face F of X^r for $i \leqslant R-2$ and upper bound the second eigenvalue of the 1-skeleton of $(X^r)_F$. Let Y be the complex X_F with $d' := \dim(Y) = d - ir$. Then bounding the second eigenvalue of the 1-skeleton corresponds to bounding the second eigenvalue of the random walk W that picks a random r-face $A \in Y(r)$, goes up to a random $2r$-face $(A, B) \in Y(2r)$ and then outputs the r-face $B \in Y(r)$. We can check that W is $1 - O(r^2/d')$-close to the walk W' that picks a random d'-face I

in Y and two uniformly random r-faces inside I. This is because W' conditioned on outputting two disjoint r-faces is the same as W. The probability that W' outputs disjoint faces is at least $1 - O(r^2/d')$, therefore giving us that the second eigenvalues of W and W' differ by at most $O(r^2/d')$. But W' is the same as the up-down walk from $X(r) \to X(d') \to X(r)$, which by Lemma 2.13 has a second eigenvalue of at most $O(r/d')$ since X_F is a $O(1/\sqrt{q})$-one-sided local spectral expander. This gives us that every i-link of X^r has a second eigenvalue of at most $O(r/(d - ir))$ which is at most $2^{-r^{12}}$ by taking d to be a large enough function of r, R, as required.

Coboundary expansion of 1-links of X^r: We will show that for all links $I \in X^r(1)$, $G_1[(X^r)_I]$ are $(2^{O(r^{0.99}\log r)}, \exp(-r^{10}))$-coboundary expanders. Fix some $I \in X^r(1)$. By definition of X^r, I corresponds to an r-face in X, which we also denote by I. Let $v \in X(1)$ be some vertex that belongs to I. By Lemma 6.8 we know that X_v is a spherical building either of type C_{n-1}, tensor of type A_1 and type C_{n-2}, or tensor of type C_k and type C_{n-k-1}. Let $Y = X_v$, associated with the distribution μ over its maximal faces. We get that the complex X_I is some $r-1$-link of Y denoted by $Y_{S\to A_0}$ and associated with the distribution $\mu | Y_S = A_0$. One can check that the complex $(X^r)_I$ is then equal to $(X_I)^r$ which in turn equals $(Y_{S\to A_0})^r$. So we get that, $G_1[(X^r)_I] = G_1[(Y_{S\to A_0})^r] = G_r(\mu | X_S = A_0)$ which is a $(2^{O(r^{0.99}\log r)}, 2^{-\Omega(r^{12})})$-coboundary expander by Lemma 5.3. So we get that all the vertex links of X^r are coboundary expanders.

In the two paragraphs above, we have shown that X^r satisfies the hypotheses of Lemma 6.10. Therefore applying the lemma on X^r we get that X^r or equivalently the graph $G_1[X^r] = G_r[X]$ is a $2^{O(r^{0.99}\log r)}$-cosystolic expander over S_m. □

6.3 Chapman-Lubotzky complex is a UG Coboundary Expander

Theorem 6.12. *For all $m, r \ll d \ll q \in \mathbb{N}$ the following holds. Let X be the d-dimensional Chapman-Lubotzky complex over $\mathbb{F}_q$. Then X is an $(m, r, \alpha(r), \alpha(r))$-UG coboundary expander for*

$$\alpha(r) = 2^{O(r^{0.99}\log r)}.$$

Proof. From Lemma 6.6 we know that X has vanishing 1-cohomology over S_m. We also know that X is a well-connected clique complex [CL23]. We now use [DD23a, Lemma 3.7], which asserts that if a complex X is a well-connected clique complex with $G_1[X]$ having vanishing cohomology then $G_r[X]$ also has vanishing cohomology. We also know from Lemma 6.11 that $G_r[X]$ is an $2^{O(r^{0.99}\log r)}$-cosystolic expander over S_m. Combining the two facts proves that $G_r[X]$ is an $2^{O(r^{0.99}\log r)}$-coboundary expander over S_m, therefore an $(m, r, \alpha(r), \alpha(r))$-UG coboundary expander. □

6.4 Proof of Theorem 1.3

Fix $\varepsilon, \delta > 0$ and take r and m sufficiently large. Take the Chapman Lubotzky complex X for sufficiently large n and prime p, which is a one-sided local spectral expander [CL23]. Combining this with Theorem 6.12 we get that X satisfies the conditions of Theorem 1.8, and therefore we conclude that the canonical direct product tester of X has soundness at most δ. □

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