

Polynomial Calculus sizes over the Boolean and Fourier bases are incomparable

Sasank Mouli

Mahindra University, Hyderabad

sasankm.ucsd@gmail.com

Abstract

For every $n > 0$, we show the existence of a CNF tautology over $O(n^2)$ variables of width $O(\log n)$ such that it has a Polynomial Calculus Resolution refutation over $\{0, 1\}$ variables of size $O(n^3 \text{polylog}(n))$ but any Polynomial Calculus refutation over $\{+1, -1\}$ variables requires size $2^{\Omega(n)}$. This shows that Polynomial Calculus sizes over the $\{0, 1\}$ and $\{+1, -1\}$ bases are incomparable (since Tseitin tautologies show a separation in the other direction) and answers an open problem posed by Sokolov [Sok20] and Razborov [Raz].

1 Introduction

Polynomial Calculus (PC), introduced in [CEI96], is a well-studied proof system with lower bounds known through a series of works [Raz98], [IPS99], [BGIP01], [AR01], [GL10], [MN15]. In spite of progress, these lower bounds have not shed light on limitations of systems such as $\text{AC}^0[p]$ -Frege, which was the original motive of studying Polynomial Calculus. A natural subsystem of $\text{AC}^0[p]$ -Frege for which lower bounds remained open is PC over $\{+1, -1\}$ variables or the “Fourier” basis (instead of the usual $\{0, 1\}$ or Boolean basis considered in prior works). After this was highlighted in [IMP20], Sokolov [Sok20] proved the first exponential lower bounds for PC over the Fourier basis, which were generalized in subsequent works [IMP23], [DMM23].

Sokolov [Sok20] posed the natural problem of separating PC sizes over the Fourier and Boolean bases, which was also highlighted by Razborov [Raz]. Since Tseitin tautologies (or unsatisfiable \mathbb{F}_2 -linear equations) have small Fourier proofs but require large Boolean proofs, he asked whether this is always the case, i.e. if any PCR proof of size s of a CNF formula F over the Boolean basis can always be converted to a proof of F over the Fourier basis of size $s^{O(1)}$. In this work, we answer this question in the negative. We show a family of CNF formulas over $O(n^2)$ variables that has a PCR refutation of size $O(n^3 \text{polylog}(n))$, but any PC refutation over ± 1 requires size $2^{\Omega(n)}$.

The formulae we use are a variant of the Linear Ordering Principle (LOP_n), introduced by Krishnamurthy [Kri85], who conjectured that they require long Resolution proofs. Stålmarck [Stå96] refuted this by demonstrating short Resolution proofs of these tautologies. They

were then used by Bonet-Galesi [BG01] and Galesi-Lauria [GL10] to show optimality of size-width/degree tradeoffs for Resolution and Polynomial Calculus respectively. [dRLNS21] used an *OR* lifted version of the graph variant of LOP_n to show a separation between PC and PCR sizes. We use an *OR* lifted version of BOP_n , which is obtained by shortening the clauses of LOP_n by using binary extension variables for our purposes. We state our main result below.

Theorem 1. *For any $n > 0$, there exists a CNF tautology over $O(n^2)$ variables of width $O(\log n)$ which has PCR proofs of size $O(n^3 \text{polylog}(n))$ over $\{0, 1\}$ but requires PC proofs of size $2^{\Omega(n)}$ over $\{+1, -1\}$.*

2 Preliminaries

Definition 1 (Polynomial Calculus/Polynomial Calculus Resolution). *Let $\Gamma = \{P_1 \dots P_m\}$ be an unsolvable system of polynomials in variables $\{x_1 \dots x_n\}$ over \mathbb{F} . A PC (Polynomial Calculus) refutation of Γ is a sequence of polynomials $\{R_1 \dots R_s\}$ where $R_s = 1$, and for every $\ell \in [s]$, R_ℓ is either a polynomial from Γ , or is obtained from two previous polynomials R_j, R_k , $j, k < \ell$, by one of the following derivation rules:*

$$R_\ell = \alpha R_j + \beta R_k \text{ for } \alpha, \beta \in \mathbb{F}$$

$$R_\ell = x_i R_k \text{ for some } i \in [n]$$

The size of the refutation is $\sum_{\ell=1}^s |R_\ell|$, where $|R_\ell|$ is the number of monomials in the polynomial R_ℓ . The degree of the refutation is $\max_\ell \deg(R_\ell)$.

A PCR (Polynomial Calculus Resolution) refutation is a PC refutation over the set of Boolean variables $\{x_1 \dots x_n, \bar{x}_1 \dots \bar{x}_n\}$ where $\{\bar{x}_1 \dots \bar{x}_n\}$ are twin variables of $\{x_1 \dots x_n\}$. That is, over the $\{0, 1\}$ encoding, the equations $x_i^2 - x_i = 0$, $\bar{x}_i^2 - \bar{x}_i = 0$ and $x_i + \bar{x}_i - 1 = 0$ are treated as axioms. Similarly, over the ± 1 encoding, the equations $x_i^2 - 1 = 0$, $\bar{x}_i^2 - 1 = 0$ and $x_i \bar{x}_i + 1 = 0$ are treated as axioms.

Definition 2 (Quadratic set, Quadratic degree, Quadratic terms over ± 1 ; taken from [Sok20], Section 3.2). *Given a proof Π over ± 1 variables, the Quadratic set of Π , denoted $\mathcal{Q}(\Pi)$, is the set of pairs of terms $\mathcal{Q}(\Pi) = \{(t_1, t_2) \mid t_1, t_2 \in P \text{ for some line } P \in \Pi\}$.*

Denote by $\mathcal{QT}(\Pi)$ the set of quadratic terms $\{t_1 t_2 \mid (t_1, t_2) \in \mathcal{Q}(\Pi)\}$, where the product is modulo the axioms $x_i^2 = 1$.

The Quadratic degree of Π is the max degree of a term in $\mathcal{QT}(\Pi)$.

Informally, Quadratic degree is the max degree of the square of each line (before cancellations).

Definition 3 (Split operation over x [Sok20], Section 5.4). *Given a proof $\Pi = (P_1, P_2, \dots, P_t)$ and a variable $x \in \{\pm 1\}$, each line P_i of Π is of the form $P_{i,1}x + P_{i,0}$, where $P_{i,1}, P_{i,0}$ do not contain x . The Split operation at x , denoted by $\text{Split}_x(\Pi)$, is the sequence Π' with the lines $\{P_{1,1}, P_{1,0}, P_{2,1}, P_{2,0}, \dots, P_{t,1}, P_{t,0}\}$.*

The following lemmas show that applying Split_x on a proof Π gives a valid proof when x does not appear in the axioms of Π , and it removes from the proof all Quadratic terms

containing x . They are adapted from [Sok20] and [IMP23], with the below versions from [DMM23]

Lemma 1 ([Sok20]). *Suppose that Π is a proof and x is a variable that does not appear in any axioms of Π except $x^2 = 1$. Then $\text{Split}_x(\Pi)$ outputs a valid proof of the axioms of Π , with no line containing x .*

Proof. Let Π be the sequence P_1, \dots, P_t . We show by induction on the line number j that both $P_{j,1}$ and $P_{j,0}$ are derivable and x -free.

If P_j is an axiom, then it is free of x . So the Split version is $P_{j,1} = 0$, $P_{j,0} = P_j$, and both these polynomials are derivable.

If $P_j = \alpha P_i + \beta P_k$ for some $i, k < j$, then $P_{j,b} = \alpha P_{i,b} + \beta P_{k,b}$ for $b = 0, 1$.

If $P_j = y P_i$ for some $i < j$ and some variable $y \neq x$, then $P_{j,b} = y P_{i,b}$ for $b = 0, 1$.

If $P_j = x P_i$ for some $i < j$, then since $x^2 = 1$ we obtain $P_{j,1} = P_{i,0}$ and $P_{j,0} = P_{i,1}$.

Thus all the lines $P_{j,b}$ are derivable and do not contain x .

Since the last line of the proof is $P_t = 1$, we have $P_{t,1} = 0$ and $P_{t,0} = P_t = 1$. Thus $\text{Split}_x(\Pi)$ derives 1 and is a valid proof from the axioms of Π . \square

Lemma 2 ([Sok20]). *Let $\mathcal{Q}_x(\Pi)$ be the set of pairs $(t_1, t_2) \in \mathcal{Q}(\Pi)$ such that $x \in t_1 t_2$, and let $\mathcal{QT}_x(\Pi)$ be the corresponding set of quadratic terms.*

If $(t_1, t_2) \in \mathcal{Q}(\text{Split}_x(\Pi))$, then t_1 and t_2 are both x -free, and at least one of (t_1, t_2) , $(t_1 x, t_2 x)$, is in $\mathcal{Q}(\Pi)$. Thus $\mathcal{QT}(\text{Split}_x(\Pi)) \subseteq \mathcal{QT}(\Pi) \setminus \mathcal{QT}_x(\Pi)$.

Proof. Consider a pair $(t_1, t_2) \in \mathcal{Q}(\text{Split}_x(\Pi))$. That t_1, t_2 are x -free follows from lemma 1. The pair (t_1, t_2) is contributed to $\mathcal{Q}(\text{Split}_x(\Pi))$ by P_b for some line $P = x P_1 + P_0$ of Π and some $b \in \{0, 1\}$. If P_0 contributes the pair, then P also contributes the pair to $\mathcal{Q}(\Pi)$. If P_1 contributes the pair, then P contributes the pair $(t_1 x, t_2 x)$ to $\mathcal{Q}(\Pi)$. \square

The following lemma from [Sok20] shows how to convert from low Quadratic degree to low degree. For completeness we include a proof of it from [DMM23] in the Appendix.

Lemma 3 ([Sok20], Lemma 3.6). *Let Π be a refutation of a set of axioms F of degree d_0 with Quadratic degree at most d . Then there exists a refutation Π' of F with (usual) degree at most $2 \max(d, d_0)$.*

3 Proof of the main theorem

3.1 The tautology

The tautology we use must be easy for PCR over $\{0, 1\}$, but must allow us to prove a lower bound for PC over $\{+1, -1\}$. For this we use a short-width, *OR*-lifted version of the tautology *Linear Ordering Principle* from [BG01], [GL10], [dRLNS21] which states that in any total ordering of n elements, there is a minimum element. We describe it formally below.

Definition 4 (Linear Ordering Principle, LOP_n). *Suppose that we have a set \mathcal{A} of n elements. Let x_{ij} , $i, j \in [n], i \neq j$ denote Boolean variables encoding a total order on the elements in \mathcal{A} where $x_{ij} = 1$ if and only if element i is ordered before element j . Then the*

unsatisfiable formula LOP_n is the following set of $O(n^3)$ clauses over $O(n^2)$ variables, which states that there exists an ordering with no minimum element.

$$\begin{aligned} & \bigvee_{i \neq j} x_{ij} \quad \forall j \\ & \bar{x}_{ij} \vee \bar{x}_{jk} \vee x_{ik} \quad \forall i, j, k \\ & \bar{x}_{ij} \vee \bar{x}_{ji} \quad \forall i, j \end{aligned}$$

The first set of clauses asserts that every element j must have some other element ordered before it. We call these the *vertex* axioms, and denote the j^{th} vertex axiom by V_j . The next two sets of clauses enforce the total order: transitivity and asymmetry must hold. We call these the *ordering* axioms T .

The lemma below states that LOP_n has small sized Resolution refutations. This is originally from [Stå96], [BG01], but we will use the version from [dRLNS21]

Lemma 4 ([dRLNS21], Proposition 7). *The formula LOP_n has resolution refutations of size $O(n^3)$, where each clause in the refutation has at most two negative literals.*

For our purposes we need that the above formula be transformed into one with short initial clauses.

Definition 5 (Ordering Principle with binary pointers, BOP_n). *The tautology BOP_n over variables x_{ij} , $i \in [n]$, $j \in [n]$ and variables y_{ja} , $j \in [n]$, $a \in [\log n]$, is the following set of clauses, where the expression $(y_i \neq j)$ denotes the clause $\bigvee_{b_a=0} y_{ia} \bigvee_{b_a=1} \bar{y}_{ia}$, with $b_{\log n} \dots b_1$ being the binary representation of j .*

$$\begin{aligned} & (y_j \neq i) \vee x_{ij} \quad \forall j \in [n] \\ & \bar{x}_{ij} \vee \bar{x}_{jk} \vee x_{ik} \quad \forall i, j, k \in [n] \\ & \bar{x}_{ij} \vee \bar{x}_{ji} \quad \forall i, j \in [n] \end{aligned}$$

BOP_n is obtained from LOP_n by using extension variables y_j to reduce the clause width of LOP_n . $y_j = i$ forces x_{ij} to be true. Similar to LOP_n , we denote by BV_j the set of all axioms $\{(y_j \neq i) \vee x_{ij}, i \in [n]\}$, and by T the set of all ordering axioms. BOP_n is of width $O(\log n)$ and has a similar number of variables and clauses as LOP_n . Below we define the *OR* lifted version $BOP_{\ell,n}^\vee$ of BOP_n .

Definition 6. *For a constant $\ell > 0$, let $BOP_{\ell,n}^\vee$ denote the tautology obtained from BOP_n by replacing each x_{ij} by an *OR* of ℓ new variables $x_{ij1} \dots x_{ij\ell}$. The following are clauses of $BOP_{\ell,n}^\vee$:*

$$\begin{aligned} & (y_j \neq i) \bigvee (\bigvee_l x_{ijl}) \quad \forall j \in [n] \\ & \bar{x}_{ijl} \bigvee \bar{x}_{jkl} \bigvee (\bigvee_l x_{ikl}) \quad \forall i, j, k \in [n], l \in [\ell] \\ & \bar{x}_{ijl} \bigvee \bar{x}_{jil} \quad \forall i, j \in [n], l \in [\ell] \end{aligned}$$

Let BV_j^\vee and T^\vee denote the j^{th} vertex axioms and the ordering axioms respectively of $BOP_{\ell,n}^\vee$. One of our contributions is a *special degree* lower bound for PC proofs of $BOP_{\ell,n}^\vee$, where we define this notion carefully. This is along the lines of a similar bound proved in [GL10] for the ordering principle based on expander graphs. We state the lower bound here and defer the proof to the next section. To state the bound, we need the following definitions of special degree.

Definition 7. *Let t be a term in the variables of $BOP_{\ell,n}^\vee$. We say that a vertex j is strongly touched by t if $x_{ijl} \in t$ for some i, l or $y_{ja} \in t$ for some a . We say that a vertex i is lightly touched (through j) if for some j and for all l , $x_{ijl} \in t$. The set of all touched vertices, denoted by $\tau(t)$, is the set of all vertices that are both strongly and lightly touched by t . We sometimes refer to $|\tau(t)|$ as special degree.*

Theorem 2 (Special PC degree lower bound for $BOP_{\ell,n}^\vee$). *In any PC refutation of the tautology $BOP_{\ell,n}^\vee$, there must exist a term t such that $|\tau(t)| = n$.*

On the other hand, since $\forall i (y_j \neq i) \vee x_{ij}$ implies $\bigvee_i x_{ij}$ we can derive LOP_n from BOP_n in size $O(n)$ (since the number of variables in y_j is logarithmic). [dRLNS21] show that the OR lifted version of LOP_n also has small Resolution refutations.

Lemma 5 ([dRLNS21], Lemma 9). *The formula LOP_n^\vee , obtained by lifting each variable of LOP_n by an OR of ℓ variables, has a Resolution refutation of size $O(n^3 \text{poly}(\ell))$.*

Lemma 6 (Short Resolution refutations of $BOP_{\ell,n}^\vee$). *$BOP_{\ell,n}^\vee$ has Resolution refutations of size $O(n^3 \text{poly}(\ell))$*

Proof. As stated above, LOP_n can be derived from BOP_n in Resolution size $O(n)$. Substituting OR gadget variables for the vertex variables of BOP_n , we obtain a derivation of LOP_n^\vee from $BOP_{\ell,n}^\vee$. We now use the previous lemma. \square

Part of our lower bound proof uses a random clustering argument on the gadget variables, where we pair variables $x_{ij_1} \dots x_{ij_\ell}$ randomly and assign each pair to a new variable z_{ijl} to obtain a smaller set of gadget variables $z_{ij_1} \dots z_{ij_{\ell/2}}$. The following lemma shows that any term t of degree at least $\ell/2$ in variables x_{ijl} in the ± 1 basis reduces to a term that misses at least one variable z_{ijl} with high probability after random clustering.

Lemma 7. *Suppose that t is a term over $\{+1, -1\}$ in variables $x_{ij_1} \dots x_{ij_\ell}$ of degree at least $\ell/2$. We iteratively pick a pair of variables $x_{ij_{l_1}}$ and $x_{ij_{l_2}}$ at random and assign them to a new variable z_{ijl} , so that t is now a term over variables $z_{ij_1} \dots z_{ij_{\ell/2}}$. Then with high probability for a large enough ℓ , t does not contain at least one variable from $z_{ij_1} \dots z_{ij_{\ell/2}}$.*

Proof. We observe that if the random process pairs any two variables $x_{ij_{l_1}}, x_{ij_{l_2}} \in t$, then the corresponding new variable z_{ijl} does not appear in t after substitution. Therefore, t contains all the variables z_{ijl} only if every variable in t is paired with a variable outside of t . The probability that this happens is at most $(3/4)^{\ell/2}$. \square

We are ready to prove our size lower bound for PC refutations of $BOP_{\ell,n}^\vee$ over ± 1 .

Theorem 3 (Main theorem). *For any $n > 0$ and $\ell > 10 \log n$, any refutation of $BOP_{\ell,n}^\vee$ in PC over ± 1 requires size $2^{\Omega(n)}$*

Proof. Let Π be a PC refutation of $BOP_{\ell,n}^\vee$ over ± 1 of size $2^{\gamma n}$ for a small enough constant $\gamma > 0$. We will reduce Π to a refutation Π' of $BOP_{\ell/2,n/2}^\vee$ of special degree less than $n/2$, contradicting Theorem 2. We consider two parts of special degree of any term $t \in \Pi$: the number of strongly touched vertices and the number of lightly touched vertices. We lower both of these separately to arrive at our lower bound. For the former, we consider Quadratic terms of the proof that strongly touch at least $n/8$ vertices. By iteratively picking a vertex j that is strongly touched by many Quadratic terms and applying Split on its variables, we remove all Quadratic terms with many strongly touched vertices. We then use Lemma 3 to get a refutation where each term of the proof strongly touches less than $n/4$ vertices. For the latter, we randomly cluster our gadget variables as in Lemma 7, and argue that after clustering no term lightly touches more than $n/4$ vertices with high probability. This gives us a refutation of special degree less than $n/2$.

Let H be the set of all Quadratic terms in Π which strongly touch at least $n/8$ vertices. Pick a vertex j that by averaging is strongly touched by at least $1/8^{\text{th}}$ of Quadratic terms in H . This means that an eighth of the terms in H contain x_{ijl} for some i, l or contain y_{ja} for some a . We pick the gadget indices l_i such that x_{ijl_i} occurs the most in terms containing x_{ijl} , $l \in [\ell]$. For every i and $l \neq l_i$, we set $x_{ijl} = 1$. We also set $x_{jkl} = 0$ for all k, l . Note that this eliminates all axioms in BV_j^\vee plus any axiom in T^\vee that mentions j and we recover copy of $BOP_{\ell,n-1}^\vee$ where each BV_k^\vee contains an additional axiom of the form $(y_k \neq j)$. Functionally this has no effect and the special degree lower bound still holds for this copy. We now apply Split on all variables y_{ja} and on all x_{ijl_i} . Note that since none of these variables are in the axioms, this gives a valid refutation by Lemma 1. Moreover, this eliminates from the refutation at least a $1/8\ell$ fraction of terms in H by Lemma 2. By repeating this for $8\ell\gamma n$ iterations, we remove all Quadratic terms in H . By using Lemma 3, we obtain a refutation of $BOP_{\ell,n/2}^\vee$ where each term strongly touches less than $n/4$ vertices (for a small enough γ).

We now cluster gadget variables $x_{ijl}, l \in [\ell]$ into variables $z_{ijl}, l \in [\ell/2]$ as in Lemma 7. We compute the probability that a term t lightly touches more than $n/4$ vertices after clustering. For any index i and some j , if t contained less than $\ell/2$ variables x_{ijl} prior to clustering, then it still contains less than $\ell/2$ variables after clustering, and thus does not lightly touch i through j . On the other hand, if t contained at least $\ell/2$ variables x_{ijl} , then by Lemma 7 it now misses some variable z_{ijl} with probability all but $(3/4)^{\ell/2} < 1/n^2$. Therefore, by a union bound, the probability that i is lightly touched by t through some j is at most $1/n$ and thus the probability that t lightly touches more than $n/4$ vertices is at most $\binom{n/2}{n/4} (1/n)^{n/4} < 2^{-O(n \log n)}$. Since the size of the proof is at most $2^{\gamma n}$, again by a union bound we have that there exists a clustering where each term lightly touches at most $n/4$ vertices.

Therefore, we have a refutation Π' of $BOP_{\ell/2,n/2}^\vee$ where the number of touched vertices by any term is less than $n/2$. This contradicts Lemma 2. □

4 Proof of the special degree lower bound

In this section we prove the lower bound for PC stated in Theorem 2. For this we largely follow the framework laid out in [GL10], which in turn was motivated by [AR01].

4.1 Preliminaries

We work with a field \mathbb{F} and polynomials in the ring $\mathbb{F}[x_1, \dots, x_n]$.

Definition 8 (Ideal generated by a set of polynomials F). *Given a set of polynomials $F = \{f_1 \dots f_k\}$, the ideal generated by F , denoted by $\text{Span}(F)$ is the set of all polynomials of the form $\{\sum_i f_i g_i \mid g_i \in \mathbb{F}[x_1, \dots, x_n]\}$*

Definition 9 (Semantic implication). *We say that a set of polynomials f_1, \dots, f_n semantically implies another polynomial g , if for every assignment to the underlying variables, $g = 0$ whenever each $f_i = 0$. We denote this by $f_1, \dots, f_n \models g$.*

Definition 10 (Graded lexicographic ordering). *Let \prec denote the following ordering on multilinear monomials in $\mathbb{F}[x_1, \dots, x_n]$. We have $1 \prec x_1 \prec x_2 \prec \dots \prec x_n$. For any two monomials M, M' , if $\deg(M) < \deg(M')$, then $M \prec M'$. Also, if $M \preceq M'$ then for any variable x , we have $xM \preceq xM'$ (where the product is modulo $x^2 = x$). Note that this is a total ordering on multilinear monomials.*

For a polynomial P , the leading term, denoted by $LT(P)$ is the greatest monomial in P according to \prec . This ordering is extended lexicographically to polynomials, i.e. for two polynomials $P = a_1 M_1 + a_2 M_2 + \dots + a_k M_k$ and $P' = a'_1 M'_1 + a'_2 M'_2 + \dots + a'_k M'_k$, we write $P \prec P'$ if there exists an $i \in [k]$ such that $M_j = M'_j$ for $j < i$ and $M_i \prec M'_i$. If no such i exists then P and P' are incomparable w.r.t. \prec and therefore \prec is not a total order on polynomials. However, in such a case we have that a linear combination of P and P' is less than both P and P' according to \prec . Therefore, in any set of polynomials there always exists a minimum element w.r.t. \prec .

Definition 11 (Residue of a polynomial P w.r.t. a set of polynomials F). *Given a set of polynomials F , the residue $R_F(P)$ of a polynomial P modulo the ideal generated by F is defined as the minimum element in the set $\{P - Q \mid Q \in \text{Span}(F)\}$ according to \prec (which exists by the previous definition).*

Definition 12 (Properties of residue, cf. [GL10], Property 1). *For a set of polynomials F and any two polynomials P, Q , we have:*

1. $R_F(P) \preceq P$
2. If $P - Q \in \text{Span}(F)$, $R_F(P) = R_F(Q)$
3. R_F is a linear operator
4. $R_F(PQ) = R_F(P \cdot R_F(Q))$

4.2 Proof of the special degree lower bound

Our lower bound proof relies on building a linear operator R with the following properties, which is sufficient for proving the desired lower bound from Theorem 2.

Lemma 8 ([GL10], [AR01]). *Suppose that there exists a linear operator R such that*

1. $R(A) = 0$ for every axiom A of BOP_n
2. For any polynomial P and any variable w such that $|\tau(wt)| < n$ for any $t \in P$,
 $R(wP) = R(w \cdot R(P))$
3. $R(1) = 1$

Then BOP_n requires proofs of where there exists a term t with $|\tau(t)| = n$.

We conclude by showing the existence of such a linear operator R . Again, we follow the framework of [GL10], [AR01] and choose $R(t)$ for a term t to be the residue $R_{\tau(t), T^\vee}(t)$, where we abuse notation to indicate by $\tau(t)$ the set of all axioms BV_j^\vee for $j \in \tau(t)$. We drop the T^\vee and write $R_{\tau(t)}(t)$ to mean $R_{\tau(t), T^\vee}(t)$.

We would like to show that for any term t with $|\tau(t)| < n - 1$ and a variable w , $\tau(wt)$ cannot do any better in reducing t than $\tau(t)$. This is made formal below.

Lemma 9. *Let t be term and let w be any variable such that $|\tau(wt)| < n$. Then we have $R_{\tau(wt)}(t) = R_{\tau(t)}(t)$.*

Proof. We have by definition of $R_{\tau(wt)}(t)$ that $t - R_{\tau(wt)}(t)$ belongs to $Span(\tau(wt))$, or in other words we have

$$T^\vee, \tau(wt) \models t - R_{\tau(wt)}(t)$$

Since we have $|\tau(wt)| \leq n - 1$ there exists a vertex axiom $BV_j^\vee \notin \tau(wt)$. This means that for every i, l $x_{ijl} \notin t$, and for every k there exists some l_k such that $x_{jkl_k} \notin t$. Consider the following partial assignment ρ to the variables of $BOP_{\ell, n}^\vee$ (called a j -cta in [GL10]): set $x_{ijl} = 0$ for all i, l and set $x_{jkl_k} = 1$ for every k . Also set $y_k = j$ for $BV_k^\vee \in \tau(wt) \setminus \tau(t)$ and leave the other variables unset. In other words, ρ sets the element j as the minimum, sets the pointers y_k of the vertices k touched by w and not touched by t (if they exist) to point to the vertex j and leaves everything else unset. We note here that ρ is chosen such that none of the variables in t are set, and each axiom in $\tau(wt) \setminus \tau(t)$ is killed.

Let us see the effect ρ has on the above semantic implication. Firstly, we claim that $T_{|\rho}^\vee \subseteq T^\vee$. This is easy to see: ρ kills all the ordering axioms involving j and does not touch the other ordering axioms. Similarly, we claim that the set of axioms in $\tau(wt)_{|\rho}$ is a subset of those in $\tau(t)$. This is because as mentioned above, every axiom in $\tau(wt) \setminus \tau(t)$ is killed by ρ . Moreover, for any $BV_k \in \tau(t)$, the axiom pointing to j is killed and others are left untouched. Therefore we have the semantic implication

$$T^\vee, \tau(t) \models t - R_{\tau(wt)}(t)_{|\rho}$$

This is because t does not contain any variable that is set by ρ . We now have by minimality of $R_{\tau(t)}(t)$ that $R_{\tau(t)}(t) \preceq R_{\tau(wt)}(t)_{|\rho}$. However, since a restriction can only

decrease the degree of any term, we have that $R_{\tau(wt)}(t)|_{\rho} \preceq R_{\tau(wt)}(t)$. Finally since $\tau(wt) \supseteq \tau(t)$, we have $R_{\tau(wt)}(t) \preceq R_{\tau(t)}(t)$. We therefore have $R_{\tau(t)}(t) \preceq R_{\tau(wt)}(t) \preceq R_{\tau(t)}(t)$ and thus $R_{\tau(t)}(t) = R_{\tau(wt)}(t)$. \square

We note that the above lemma also works when we replace $\tau(wt)$ by any set of the form BV_I^\vee , where $BV_I^\vee = \{BV_i^\vee \mid i \in I\}$, such that $|I| < n$ and I contains $\tau(t)$.

Lemma 10. *Let I be any set of vertices of size less than n and let t be a term such that $\tau(t) \subseteq BV_I$. Then we have $R_{BV_I, T^\vee}(t) = R_{\tau(t)}(t)$.*

The following technical lemma is needed to make the proof of Lemma 8 go through.

Lemma 11. *Let t a term and let t' be any term appearing in $R_{\tau(t)}(t)$. Then we have $\tau(t') \subseteq \tau(t)$*

Proof. The proof of this lemma is similar to the previous one. Suppose that there exists some term $t' \in R_{\tau(t)}(t)$ with $j \in \tau(t') \setminus \tau(t)$. Then we have

$$T^\vee, \tau(t) \models t - R_{\tau(t)}(t)$$

Let ρ be the partial assignment defined in the previous lemma where we additionally set y_j to an arbitrary value. Substituting this in the above semantic implication, we have similar to the previous lemma

$$T^\vee, \tau(t) \models t - R_{\tau(t)}(t)|_{\rho}$$

We now have $R_{\tau(t)}(t)|_{\rho} \prec R_{\tau(t)}(t)$ since $t' \in R_{\tau(t)}(t)$ is hit by ρ and its degree is now lesser. However from the above semantic implication we have $R_{\tau(t)}(t) \preceq R_{\tau(t)}(t)|_{\rho}$, which is a contradiction. \square

We conclude this section by showing the existence of a linear operator R from Lemma 8. As mentioned earlier, we define $R(t)$ to be $R_{\tau(t)}(t)$.

Lemma 12. *$R(t) = R_{\tau(t)}(t)$ satisfies the properties of Lemma 8 and therefore $BOP_{\ell, n}^\vee$ requires PC proofs where $|\tau(t)| = n$ for some term t*

Proof. We verify that our choice of R satisfies each property listed in Lemma 8.

1. Let A be an axiom of $BOP_{\ell, n}^\vee$. If $A \in T^\vee$ is one of the ordering axioms, $R(A) = 0$ since we reduce by T^\vee by default. On the other hand, if A is an axiom in BV_j^\vee , then for any term $t \in A$, $\tau(t)$ contains BV_j^\vee and thus $R(A)$ is again equal to zero.

2. Let $P = \sum_i a_i t_i$ be a polynomial and let w be any variable with $|\tau(wt_i)| < n$. Then we have $R(wP) = \sum_i a_i R(wt_i) = \sum_i a_i R_{\tau(wt_i)}(wt_i)$. Now, by Definition 12, pt. 4 we have $R_{\tau(wt_i)}(wt_i) = R_{\tau(wt_i)}(wR_{\tau(wt_i)}(t_i))$. Since $|\tau(wt_i)| < n$, by Lemma 9 we have $R_{\tau(wt_i)}(t_i) = R_{\tau(t_i)}(t_i) = R(t_i)$. Therefore we have $R(wP) = R_{\tau(wt_i)}(wR(t_i))$. We argue that the latter expression is in fact equal to $R(wR(t_i))$. To see this, consider any term $t \in wR(t_i)$. By Lemma 11, we have $\tau(t) \subseteq \tau(wt_i)$. Moreover, by Lemma 10, we have $R(t) = R_{\tau(t)}(t) = R_{\tau(wt_i)}(t)$ (since $|\tau(wt_i)| < n$). Putting everything together, we get that $R(wP) = R_{\tau(wt_i)}(wR(t_i)) = R(wR(t_i))$.

3. We have that $\tau(1) = \emptyset$ and since T^\vee is satisfiable, 1 is irreducible modulo T^\vee and thus $R_{\tau(1)}(1) = 1$. □

Acknowledgments

The author would like to thank Dmitry Sokolov and Alexander Razborov for posing this open problem, Russell Impagliazzo for pointing to the random clustering technique, Arkadev Chattopadhyay and Yogesh Dahiya for helpful discussions.

A Proof of Sokolov's low Quadratic degree to low degree lemma

We include here a proof from [DMM23] of Lemma 3 . We restate it below.

Lemma 13. *Let Π be a refutation of a set of axioms F of degree d_0 with Quadratic degree at most d . Then there exists a refutation Π' of F with (usual) degree at most $2 \max(d, d_0)$.*

Proof. Let $\Pi = \{P_j\}_j$. Now, consider $\Pi' = \{P'_j\}_j$ where $P'_j = t_j P_j$, with each $t_j \in P_j$ carefully selected. Since the degree of $t_j P_j$ is bounded by the Quadratic degree of P_j , every line in Π' is of degree at most d . However, Π' is not an immediate valid refutation of F , but it can be transformed into one. We will show that each line of Π' can be derived from previous lines and axioms of F in degree at most $2 \max(d, d_0)$, completing the proof. We proceed by induction on line number j .

If P_j is an axiom, then we set t_j to be an arbitrary term in P_j and derive $P'_j = t_j P_j$ in degree $2d_0$ starting from P_j .

(Note that in [Sok20], it is claimed that this step can be derived in degree d_0 . But this is not always so. For instance, if $p = x_1 x_2 x_3 + x_2 x_3 x_4 + x_3 x_4 x_1 + x_4 x_1 x_2$ has degree $d_0 = 3$, and $d = 2$, then for any term $t \in p$, tp has degree 2 but needs degree $4 > \max\{d, d_0\}$ for the derivation.)

If $P_i = x P_j$ for some $j < i$, then we select $t_i = x t_j$, and consequently, $P'_i = t_i P_i = t_j P_j = P'_j$ is derived without raising the degree.

Finally, if $P_i = P_{j_1} + P_{j_2}$, we choose t_i to be an arbitrary term in P_i and derive $P'_i = t_i P_i = t_i t_{j_1} P'_{j_1} + t_i t_{j_2} P'_{j_2}$. We argue that the degree of both $t_i t_{j_1}$ and $t_i t_{j_2}$ is at most d , and as a result, P'_i can be derived from P'_{j_1} and P'_{j_2} in degree at most $2d$, which completes the proof. To justify this assertion, let $t_i \in P_{j_1}$ without loss of generality (every term in P_i appears in either P_{j_1} or P_{j_2}). Then degree of $t_i t_{j_1}$ is bounded by the Quadratic degree of P_{j_1} and hence by d . Additionally, if $t_{j_2} \in P_i$, then the degree of $t_i t_{j_2}$ is bounded by the quadratic degree of P_i and is also bounded by d . In the case where $t_{j_2} \notin P_i$, it means that it was cancelled in the sum and therefore $t_{j_2} \in P_{j_1}$ and so degree of $t_i t_{j_2}$ is bounded by the Quadratic degree of P_{j_1} and is again bounded by d .

Thus all lines in Π' can be derived from previous lines and axioms of F in degree at most $2 \max(d, d_0)$. Since the last line of Π' is 1, we get that Π' can be successfully transformed into a valid proof of F of degree $2 \max(d, d_0)$. □

References

- [AR01] Michael Alekhovich and Alexander A. Razborov. Lower bounds for Polynomial Calculus: Non-binomial case. In *42nd Annual Symposium on Foundations of Computer Science, FOCS 2001, 14-17 October 2001, Las Vegas, Nevada, USA*, pages 190–199. IEEE Computer Society, 2001.
- [BG01] Maria Luisa Bonet and Nicola Galesi. Optimality of size-width tradeoffs for resolution. *computational complexity*, 10:261–276, 2001.
- [BGIP01] Sam Buss, Dima Grigoriev, Russell Impagliazzo, and Toniann Pitassi. Linear gaps between degrees for the Polynomial Calculus modulo distinct primes. *Journal of Computer and System Sciences*, 62(2):267–289, 2001.
- [CEI96] Matthew Clegg, Jeffery Edmonds, and Russell Impagliazzo. Using the Groebner basis algorithm to find proofs of unsatisfiability. In *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing*, pages 174–183, 1996.
- [DMM23] Yogesh Dahiya, Meena Mahajan, and Sasank Mouli. New lower bounds for polynomial calculus over non-boolean bases. *Electron. Colloquium Comput. Complex.*, TR23-132, 2023.
- [dRLNS21] Susanna F de Rezende, Massimo Lauria, Jakob Nordström, and Dmitry Sokolov. The power of negative reasoning. In *36th Computational Complexity Conference (CCC 2021)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2021.
- [GL10] Nicola Galesi and Massimo Lauria. Optimality of size-degree tradeoffs for Polynomial Calculus. *ACM Transactions on Computational Logic (TOCL)*, 12(1):1–22, 2010.
- [IMP20] Russell Impagliazzo, Sasank Mouli, and Toniann Pitassi. The surprising power of constant depth algebraic proofs. In *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 591–603, 2020.
- [IMP23] Russell Impagliazzo, Sasank Mouli, and Toniann Pitassi. Lower bounds for Polynomial Calculus with extension variables over finite fields. In *38th Computational Complexity Conference (CCC 2023)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2023.
- [IPS99] Russell Impagliazzo, Pavel Pudlák, and Jiri Sgall. Lower bounds for the Polynomial Calculus and the Gröbner basis algorithm. *Computational Complexity*, 8:127–144, 1999.
- [Kri85] Balakrishnan Krishnamurthy. Short proofs for tricky formulas. *Acta informatica*, 22:253–275, 1985.
- [MN15] Mladen Miksa and Jakob Nordström. A generalized method for proving Polynomial Calculus degree lower bounds. In *30th Conference on Computational Complexity, CCC 2015*, volume 33 of *LIPICs*, pages 467–487, 2015.

- [Raz] Alexander Razborov. Alexander razborov: Pedagogical. <http://people.cs.uchicago.edu/~razborov/teaching/>.
- [Raz98] Alexander A Razborov. Lower bounds for the Polynomial Calculus. *Computational Complexity*, 7:291–324, 1998.
- [Sok20] Dmitry Sokolov. (Semi)Algebraic proofs over $\{\pm 1\}$ variables. In *Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing*, pages 78–90, 2020.
- [Stå96] Gunnar Stålmårck. Short resolution proofs for a sequence of tricky formulas. *Acta Informatica*, 33:277–280, 1996.