Electronic Colloquium on Computational Complexity, Revision 2 of Report No. 53 (2024)

Gap MCSP is not (Levin) **NP**-complete in Obfustopia

Noam Mazor * Rafael Pass †

May 3, 2026

Abstract

We demonstrate that under believable cryptographic hardness assumptions, Gap versions of standard meta-complexity problems, such as the Minimum Circuit Size Problem (MCSP) and the Minimum Time-Bounded Kolmogorov Complexity problem ($\mathrm{MK^tP}$) are not **NP**-hard w.r.t. Levin (i.e., witness-preserving many-to-one) reductions.
In more detail:

- Assuming the existence of indistinguishability obfuscation, and subexponentially-secure one-way functions, an appropriate Gap version of MCSP is not **NP**-hard under randomized Levin-reductions.
- Assuming the existence of subexponentially-secure indistinguishability obfuscation, subexponentially-secure one-way functions and injective PRGs, an appropriate Gap version of $\mathrm{MK^tP}$ is not **NP**-hard under randomized Levin-reductions.

*Tel Aviv University. E-mail: noammaz@gmail.com. Research partly supported by NSF CNS-2149305 and DARPA under Agreement No. HR00110C0086.

†Tel-Aviv University and Cornell Tech. E-mail: rafaelp@tau.ac.il. Supported in part by AFOSR Award FA9550-23-1-0387, AFOSR Award FA9550-23-1-0312, an Algorand Foundation grant, and ERC Advanced Grant KolmoCrypt - 101142322.. This material is based upon work supported by DARPA under Agreement No. HR00110C0086. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the United States Government, DARPA, AFOSR or the Algorand Foundation.

ISSN 1433-8092

1 Introduction

As described by Trakhtenbrot [Tra84], starting in the 1960s, there has been an on-going effort studying the computational complexity of so-called "meta-complexity" problems; notably (a) the *Minimum Circuit Size problem* (MCSP) [KC00; Tra84]—determining the size of the smallest Boolean circuit that computes the function corresponding to a given truth table x, and (b) the *Time-Bounded Kolmogorov Complexity Problem (*$\mathrm{MK^tP}$*)* [Kol68; Sol64; Cha69; Ko86; Har83; Sip83]—determining the the length, denoted $\mathrm{K}^t(x)$ of the shortest program (evaluated on some particular Universal Turing machine U) that generates a given string x, within time t, where $t = poly(|x|)$ is a polynomial. In particular, a major problem since the 1960s is whether these problems, or the Gap versions of them (where the goal is to determine whether the size is above a threshold s_2 or below a threshold s_1) are **NP**-hard. Indeed, as recounted by [AKRR11; Ila20; Ila23], Levin is said to have delayed the publication of his theory of **NP**-completeness [Lev73] in order to show **NP**-completeness of MCSP.

In the following decades, there has been a lot of amazing progress—providing evidence pointing towards *both* a positive and a negative answer:

Towards NP-hardness: While it is still unknown whether the original problems are **NP**-hard, several generalizations of them have been proven to be **NP**-hard. Most notably, Ilango first demonstrated this for an oracle version of MCSP [Ila20]; this was subsequently extended to a multi-bit version of MCSP referred to as Multi-MCSP [ILCO20], to a conditional version of the $\mathrm{MK^tP}$ problem, $\mathrm{McK^tP}$ [LP22], and to other variants [Hir22a]. [HIR23] recently improved the parameters of the reduction to $\mathrm{McK^tP}$ [LP22], assuming that witness encryption scheme exists. Additionally, Ilango [Ila23] very recently demonstrates the **NP**-hardness of a variant of MCSP and $\mathrm{MK^tP}$ where the programs are allowed to access a random oracle, yielding a *heuristic* **NP**-hardness result for these problems under Karp (i.e., many-one) reductions (if instantiating the random oracle with a concrete hash function). Finally, a recent work by Impagliazzo, Kabanets, and Volkovich [IKV23b] provides various different results that can be interpreted as giving evidence that MCSP is **NP**-hard with respect to randomized reductions.

Towards Non NP-hardness: There is also evidence pointing towards non **NP** hardness: Allender and Hirahara [AH19] showed that assuming one-way functions, the gap version of MCSP is not **NP** hard for super-polynomial gap. Ko [Ko91] showed that a version of $\mathrm{MK^tP}$ is not **NP** hard with respect to an oracle, and Ren and Santhanam [RS22] gave an oracle with respect to which MCSP is not **NP** hard. Other works prove limitations on the structure of reduction to meta-complexity problems. Murray and Williams [MW17] prove that MCSP is not **NP** hard under so-called *local reductions*. Kabanets and Cai [KC00] and Saks and Santhanam [SS20] show that the **NP**-hardness of MCSP under Turing reductions with certain properties implies circuit lower bounds. For example if MCSP is **NP**-hard under so-called *parametric honest* Turing reductions, then $\mathbf{E} \not\subseteq \mathbf{SIZE}(\mathrm{poly})$. More recently, Saks and Santhanam [SS22] gave evidence that the running time of any randomized non-adaptive reduction from SAT to K^t approximation must grow with the time parameter t. These results, however, only rule out quite limited types of reductions.

Despite this progress, the original question, however, remains wide open.

1.1 Our Results

The current paper provides strong evidence that the Gap versions of MCSP and $\mathrm{MK^tP}$ are not **NP**-hard w.r.t. *Levin reductions*—that is *witness-preserving* many-to-one reductions. In particular, we demonstrate that under somewhat strong, but generally believed, cryptographic hardness assumptions, the Gap version of MCSP is not **NP**-hard w.r.t. Levin reductions.

Levin Reductions: The three original ways [Coo71; Kar72; Lev73] of defining **NP** completeness differ in how reductions from a language L to a language L' are defined (see e.g., [Gol08] for a discussion). Cook [Coo71] considers the most permissive notion: a Turing machine deciding L having oracle access to a decider for L'. Karp's notion—called a *Karp reduction* (or *many-one reduction*) is more restrictive: it requires efficiently mapping an instance x into an instance x' such that $x \in L$ iff $x' \in L'$. Levin's notion, called a *Levin reduction* (or a *witness preserving many-one reduction*) is the most restrictive: it additionally requires *efficiently* mapping any witness w for x into a witness for x', and furthermore any witness w' for x' into a witness w for x. While Karp reductions are most commonly used, as far as we are aware, most natural **NP**-completeness reductions are actually of the Levin type as well. Furthermore, for *constructive applications* of **NP**-completeness, **NP**-hardness demonstrated using a Levin reduction is typically what is needed: In particular, for cryptographic application to interactive proofs (e.g., demonstrating that every language in **NP** has a zero-knowledge proof of knowledge [FFS87], or that every language in **NP** has a succinct argument [BG09], the notion of a Levin reduction is crucial (see e.g., [BG09] that in particular notes that even the most sophisticated **NP** hardness reductions, as those provided by the PCP theorem [ALMSS98; AS98], are Levin reductions). Our focus here is on such Levin reductions; in particular, we will present the (conditional) impossibility of Levin reductions for demonstrating **NP**-hardness; in fact, our impossibility will apply not only to deterministic but also *randomized* Levin reductions (where the reduction is allowed to fail with some small constant probability).

We mention that e.g,. the **NP**-hardness results of [Ila23] and [LP22] rely on the **NP**-hardness of approximation for the *Set-Cover* problem [DGKR03; Tre01]. In both works, the reductions from Set-Cover to the GapMCSP and $\mathrm{Gap_pMK^tP}$ (or the conditional version in the case of [LP22]) are (randomized) Levin reductions (see Appendix A for a discussion of the result of [Ila23]). The Set-cover **NP**-hardness itself relies on a long sequence of the reductions that we have not been able to verify whether they are all Levin (although, as mentioned above, the main technical core, the PCP theorem, is).

Our Cryptographic Hardness Assumptions: Indistinguishability Obfuscation: We will rely on the existence of *indistinguishability obfuscation (iO)* for circuits [Bar+01]. Roughly speaking, an indistinguishability obfuscator is an efficient algorithm iO that given a circuit C outputs an "obfuscated" version of C having the property that obfuscations of any two functionally equivalent circuits are indistinguishable. Following the ground-breaking work of [Gar+16], several heuristic candidates were proposed, as well as provably secure constructions based on various assumptions [PST14; GLSW15; Lin16; WW21; LT17; LV16; Lin17; AJS18; JLMS19; JLMS19; AJLMS19; GJLS21; APM20; Agr19]. Most notable, the recent breakthrough result presents a construction based on several well-founded (and generally believed) hardness assumption [JLS21]. (Constructions based on less standard, but seemingly quantum-safe, "circular-security" assumptions also appear in [BDGM23; GP21; BDGM20]).

For our main results on MCSP, we will simply rely on indistinguishability obfuscation and subexponentially-secure one-way function. For our results on $\mathrm{MK^tP}$, we will rely on *iO* with subexponential security as well as other standard cryptographic assumptions such as injective pseudorandom generators (PRGs), that e.g., are implied by the existence of one-way permutations.

Main Theorem We present the following main result:

- Assuming the existence of indistinguishability obfuscation and subexponentially-secure one-way function, an appropriate Gap version of MCSP is not **NP**-hard under randomized Levin-reductions.
- Assuming the existence of subexponentially-secure indistinguishability obfuscation, subexponentially-secure one-way function and injective PRGs, an appropriate Gap version of $\mathrm{MK^tP}$ is not **NP**-hard under randomized Levin-reductions.

In more detail, let $\mathrm{GapMCSP}[s_0, s_1]$ be the promise problem in which given a truth table x we need to distinguish between the following two cases:[1]

- **Yes instances**: There exists a circuit C of size at most $s_0(|x|)$ that computes x.
- **No Instances**: There is no circuit of size $s_1(|x|)$ that computes x.

Our first theorem states that when the gap between s_0 and s_1 is large enough, and under cryptographic assumptions, $\mathrm{GapMCSP}[s_0, s_1]$ is not **NP**-hard with respect to Levin reductions.

Theorem 1.1. *Assume that iO and subexponentially-secure one-way functions exist. Then there exists a polynomial p, such that for any pair of efficiently computable functions $s_0, s_1 \colon \mathbb{N} \to \mathbb{N}$ for which $s_1(n) > p(s_0(n))$, it holds that* $\mathrm{GapMCSP}[s_0(n), s_1(n)]$ *is not* **NP** *hard with respect to Levin reductions.*

We remark that if all of the assumed cryptographic primitives are secure against *sub-exponential adversaries* (in contrast to just polynomial adversaries), then our results rule out also randomized Levin reductions that run in sub-exponential time.

Additionally, the assumption of subexponentially-secure one-way functions in Theorem 1.1 is only to handle so-called non *honest* reductions: A Karp reduction f is to be *honest* if for every $x \in \{0,1\}^*$, $|f(x)| \geq |x|^\delta$ for some constant $\delta > 0$ (i.e., the mapping from statements x to x' is polynomially preserving).

If we only want to exclude honest reductions, it is enough to assume one-way function with polynomial security. Such one-way functions are known to exist assuming iO and the minimal assumption that $\mathbf{NP} \not\subseteq \mathbf{ioBPP}$ [Kom+14]. We get the following theorem.

Theorem 1.2. *Assume that iO exists, and that* $\mathbf{NP} \not\subseteq \mathbf{ioBPP}$*. Then there exists a polynomial p, such that for every $\epsilon > 0$, for any pair of efficiently computable functions $s_0, s_1 \colon \mathbb{N} \to \mathbb{N}$ for which $s_1(n) > p(s_0(n))$ and $s_0(n) > n^\epsilon$, it holds that* $\mathrm{GapMCSP}[s_0(n), s_1(n)]$ *is not* **NP** *hard with respect to honest Levin reductions.*

[1] Nnote Added:Note that the threshold parameters s_0, s_1 in GapMCSP are sometimes defined to be functions of $\log|x|$ (the number of input bits of the circuit) instead of functions of $|x|$. We choose to define it as functions of $|x|$ for ease of notation, but our results hold for the mentioned, equivalent definition.

Our second result is a similar result for the $\mathrm{Gap_pMK^tP}$ problem. Recall that $\mathrm{K}^t(x)$ is the minimal length of a program that outputs x within $t(|x|)$ steps. For polynomials t and p, let $\mathrm{Gap_pMK^tP}[s_0, s_1]$ be the promise problem in which given a string x we need to distinguish between the following two cases:

- **Yes instances**: $\mathrm{K}^t(x) \leq s_0(|x|)$
- **No Instances**: $\mathrm{K}^{p(t)}(x) > s_1(|x|)$.

We prove the following theorem.

Theorem 1.3. *Assume that subexponentially-secure iO, subexponentially-secure one-way functions and injective PRGs exist. Then there exist a polynomial q such that for any $t \in$ poly and any efficiently computable functions $s_0, s_1 \colon \mathbb{N} \to \mathbb{N}$ for which $s_1(n) > q(\log t(n), s_0(n))$, and for every large enough polynomial p, it holds that $\mathrm{Gap_pMK^tP}[s_0, s_1]$ is not* **NP** *hard with respect to Levin reductions.*

Achieving a smaller gap under stronger assumptions As discussed above, several generalizations of the GapMCSP and $\mathrm{Gap_pMK^tP}$ problem have been proven **NP** hard. The work of [Ila23] showed that the same problems we consider here are **NP** hard relative to a random oracle. There, the gap between the Yes and No instances is a multiplicative $(1+\epsilon)$ gap, for a small constant $\epsilon > 0$ while in the theorems above we need the gap to be larger. Similarly, [LP22] showed that deciding a *conditional* version of $\mathrm{MK^tP}$ is **NP**-hard, and their result can be generalized to a gap problem with a larger constant multiplicative factor. Hirahara [Hir22b] used a reduction from the Minimum Monotone Satisfying Assignment problem to $\mathrm{McK^tP}$, resulting with a **NP**-hardness of the $\mathrm{Gap\,McK^tP}$ with a larger multiplicative gap, but still sub polynomial in the input length $(n^{o(1)})$.

The polynomial p in Theorems 1.1 and 1.2 is the *overhead* of the iO algorithm we use. By assuming a stronger assumption—that iO with a small overhead exists—we can improve the gap. For example, we say that iO has additive overhead if on input C and security parameter λ, the size of the obfuscated circuit is $|C| + \mathrm{poly}(\lambda)$. If we assume iO with additive overhead, we would get the hardness of GapMCSP also for the additive gap case. Unfortunately, such iO constructions are currently not known (but as far as we know, there are also no results indicating that this should be impossible). However, if we consider slightly stronger assumptions, we can get iO for TM with a factor $2+\epsilon$ overhead (for any constant $\epsilon > 0$) [AJS17], yielding the following theorem.[2]

Theorem 1.4. *Assume subexponential-secure iO, and subexponentially-secure one-way function exist and assume subexponential DDH or LWE. Then for every constant $\epsilon > 0$, for every large enough polynomial p, and for every efficiently computable function s_0 it holds that $\mathrm{Gap_pMK^tP}[s_0, (2+\epsilon)s_0(n)]$ is not* **NP** *hard with respect to Levin reductions.*

Proof Overview In this proof outline, we will for simplicity focus on ruling out deterministic Levin reductions for GapMCSP. Additionally, on top of the existence of iO, we will here assume the

[2] In a previous version of this paper, we claimed a similar result for GapMCSP using iO for circuits with a factor $2+\epsilon$ overhead. iO with such small overhead w.r.t. circuits does not appear to be known; while [AJS17] claim an iO where the size of an obfuscation of a circuit C is of length $2|C| + \mathrm{poly}(\lambda)$, it appears that this "program" may need to be further interpreted, which may result in larger circuit size.

existence of a collision-resistant hash function; that is, the existence of a family $\mathcal{H}$ of compressing functions such that for a randomly sampled $h \leftarrow \mathcal{H}$, it is infeasible to find two inputs x_1, x_2 that "collide" (i.e., $h(x_1) = h(x_2)$) although such collision exists. (In our actual proof, we instead rely on the weaker primitive of a target collision-resistant hash function (TCR; also known as, universal one-way hash function [NY89]) which can be constructed from one-way functions [Rom90]). Finally, let us start by assuming that the reduction is "honest" (i.e., mapping instances x to instances x' of polynomially-related length).

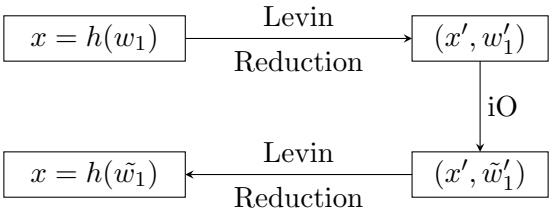

Figure 1: The proof overview. Given a witness w_1 such that $h(w_1) = x$, we use the Levin reduction to get a witness for MCSP. Then we use the iO to get a new witness for MCSP, and use the Levin reduction again to get back a witness $\tilde{w}_1$ such that $h(\tilde{w}_1) = x$.

The key idea will be to use the Levin reduction and the iO in order to find a collision for h. Roughly speaking, we start by sampling some w_1 and compute $x = h(w_1)$; we think of x as a instance for the language of images of h, and of w_1 as the witness for x. We next use the Levin reduction to get an MCSP instance x' and its corresponding witness w_1'. Note that the witness w_1' is a circuit computing x'. We then *obfuscate* w_1' using the iO to get a *new* witness $\tilde{w}_1'$ for x'. Using the Levin reduction, we can finally turn $\tilde{w}_1'$ into a (hopefully new) witness $\tilde{w}_1$ for x, which is a collision for h.

We want to show that $\tilde{w}_1 \neq w_1$ with a good probability. Indeed, the key point is that if we had started with a different preimage $w_2 \neq w_1$ for $x = h(w_1)$ and done the same process, then w_2' would become a functionally equivalent circuit to w_1' (as x' would be the same in both case, and both w_1', w_2' are circuits computing the function described by x). Thus, by the security of the iO, the distributions of $\tilde{w}_2'$ and $\tilde{w}_1'$ are computationally indistinguishable, and by data-processing, also the distributions of $\tilde{w}_2$ and $\tilde{w}_1$. In particular, it must be the case that either $w_1 \neq \tilde{w}_1$ or $w_2 \neq \tilde{w}_2$, as otherwise we can easily distinguish between $\tilde{w}_1$ and $\tilde{w}_2$. We conclude that for a random w_1, it follows that $\tilde{w}_1 \neq w_1$ with probability at least $1/2$, and we thus find a collision with the same probability.

Note that we here rely on the **NP**-hardness of the Gap version of the MCSP problem since when applying the iO we get a new witness for x' but this witness (i.e, the circuit) is *bigger* than the original one. In particular, the overhead of the iO translates into the gap of the problem—for instance, if the overhead of the iO is only linear, we can handle a linear gap, and if it has polynomial overhead then we can only rule out reductions for the polynomial gap version of the problem.

Dealing with Non-honest Reductions If the reduction is not honest, the instance x' could be a lot shorter than x; the problem then becomes if we run the iO on a security parameter that is polynomially related to $|x'|$ (which we require to ensure that we stay within the promise), we may no longer have security with respect to an attacker who runs in time polynomial in $|x| = n$ (which is required to ensure that we find a collision). However, if we start off with a collision-resistant hash

function with sub-exponential security (i.e., 2^{n^ϵ} security), we can resolve this problem using a case-analysis. If $|x'| \leq n^\epsilon$, then we simply find a new witness $\tilde{w}'$ using *brute-force search*, and otherwise use the iO. This ensures that we only run the iO in case the reduction behaves "honestly"; on the other hand, when the reduction chooses a short x', we still contradict the subexponential security of the collision-resistant hash function.

Extensions for $\mathrm{Gap_pMK^tP}$. We next generalize the above proof for the $\mathrm{Gap_pMK^tP}$ problem. To be able to do so, we need a way to move from one $\mathrm{Gap_pMK^tP}$ witness to another, when a $\mathrm{Gap_pMK^tP}$ witness is a t-time TM P of size at most $s_0(|x|)$ that outputs x. A naive approach is to first convert the TM P into a circuit, then apply the iO for circuits, and lastly, convert the circuit back to a TM. The problem in this approach is that since the program P outputs x, the time bound t must be at least $|x|$. This means that the circuit we construct from P will have a trivial size, and we will not be able to get back a non-trivial program that outputs x.

Luckily, we can use iO for TMs directly on P, or even it suffices to rely on a weaker primitive of a *randomized encoding*. Randomized encoding for TMs is known to exist assuming subexponential-secure iO for circuits and injective PRGs [KLW15; LPST15].

1.2 Discussion.

The results presented give evidence that GapMCSP and $\mathrm{Gap_pMK^tP}$ are not **NP**-hard w.r.t. Levin reductions even when the gap is small. These results thus provide (in our eyes) convincing cryptographic evidence that the original task set out by Levin is impossible (since he indeed defined **NP**-hardness through the notion of what today is referred to as a Levin reduction.)

Of course, it could still be that a weaker notion of a reduction (e.g., a Karp) reduction can be used to prove **NP**-hardness of these problems. In particular, consider the results of [Ila23], which shows **NP**-hardness of GapMCSP in the random oracle model. While, as discussed, his reduction from (approximate) Set-Cover to GapMCSP is a Levin reductions (see Appendix A), the witness preserving part of the reduction relies on the random oracle—in particular, the witness reconstruction step relies on observing the queries to the random oracle performed by the circuit $\tilde{w}'$ (i.e., the witness for the transformed instance x').[3] If instantiating the random oracle with a concrete hash function h, it is no longer clear how to perform this task—in particular if the circuit has been obfuscated so that it (intuitively) becomes hard to find the code of h in the description of the circuit. As such, when instantiating the random oracle with a hash function, the reduction most likely is no longer a Levin reduction, but conceivably it could still be a Karp reduction.

Obfuscation and NP-hardness of MCSP. In this work we show that assuming iO, GapMCSP is not **NP**-hard under Levin reductions. In contrast, as was shown in [IKV23a], if iO exists and MCSP $\in$ **BPP** (and using similar ideas, even if GapMCSP or $\mathrm{Gap\,MK^tP}$ with polynomial gap are in **BPP**), then **NP** $\subseteq$ **BPP**. Indeed, if GapMCSP$[n^\epsilon, n^{1-\epsilon}] \in$ **BPP** then (infinitely-often) one-way functions do not exist, and thus by the result of [Kom+14], **NP** $\subseteq$ **BPP**. This result gives, assuming obfuscation, a randomized reduction from **NP** to GapMCSP. This reduction however is not a Karp (or Levin) reduction. As mentioned above, [HIR23] showed that, under the related

[3] Interestingly, a similar method of observing the queries to the random oracle was used by [GR14] to show that there is no obfuscation for circuits with oracle access to a random oracle.

assumption of (subexponentially-secure) witness encryption, the conditional version of GapMK$^{\mathrm{t}}$P, Gap McK$^{\mathrm{t}}$P, is **NP**-hard under Karp reductions.

Comparison with [AH19]. [AH19] showed, assuming that MCSP $\notin$ **BPP** (which is implied for example by the existence of one-way functions), GapMCSP with super-polynomial gap is not **NP**-hard.[4] Their result rule out **NP**-hardness under much larger class of reductions, that is, non-uniform Cook reductions.[5] On the other hand, assuming stronger cryptographic assumptions, in this work we rule out the **NP**-hardness of GapMCSP with polynomial gap, but under a weaker class of reductions.

Understanding whether MK$^{\mathrm{t}}$P and MCSP are **NP**-hard with no gap or with a small gap is motivated for example from its connection to cryptography and complexity. [LP20] showed that the average-case hardness of MK$^{\mathrm{t}}$P$[n-1]$ is equivalent to the existence of one-way functions, and showing the **NP**-completeness of this problem is a step towards basing the existence of one-way functions on the assumption that **P** $\neq$ **NP**.

2 Preliminaries

2.1 Notations

All logarithms are taken in base 2. We use calligraphic letters to denote sets and distributions, uppercase for random variables, and lowercase for values and functions. Given a set $\mathcal{S} \subseteq \{0,1\}^*$, we let $\overline{\mathcal{S}} = \{0,1\}^* \setminus \mathcal{S}$. Let poly stand for the set of all polynomials. Let PPT stand for probabilistic poly-time, and n.u.-poly-time stand for non-uniform poly-time. An n.u.-poly-time algorithm A is equipped with a (fixed) poly-size advice string set $\{z_n\}_{n\in\mathbb{N}}$. A function $\mu\colon \mathbb{N} \to [0,1]$ is noticeable if $\mu(n) \geq p(n)$ for some polynomial p and for every large enough n. Let neg stand for a negligible function. For a SAT formula ϕ over n variables and an assignment $v \in \{0,1\}^n$, we use $\phi[v] \in \{0,1\}$ to denote the truth value of the evaluation of ϕ on v.

2.2 Distributions and Random Variables

When unambiguous, we will naturally view a random variable as its marginal distribution. The support of a finite distribution $\mathcal{P}$ is defined by $\mathrm{Supp}(\mathcal{P}) := \{x\colon \Pr_{\mathcal{P}}[x] > 0\}$. For a (discrete) distribution $\mathcal{P}$, let $x \leftarrow \mathcal{P}$ denote that x was sampled according to $\mathcal{P}$. Similarly, for a set $\mathcal{S}$, let $x \leftarrow \mathcal{S}$ denote that x is drawn uniformly from $\mathcal{S}$.

2.3 Kolmogorov Complexity

Roughly speaking, the *t-time-bounded Kolmogorov complexity*, $\mathrm{K}^t(x)$, of a string $x \in \{0,1\}^*$ is the length of the shortest program $\Pi = (M, y)$ such that, when simulated by a universal Turing machine, Π outputs x in $t(|x|)$ steps. Here, a program Π is simply an encoding of a pair of a Turing Machine M and an input y, where the output of Π is defined as the output of $M(y)$, and we use

[4] We note that in the definition of GapMCSP in [AH19], the threshold s is part of the input and not a parameter of the problem. Our result also holds with respect to this definition.

[5] Note that GapMCSP$[s_0, s_1]$ can be solved by brute-force in time $2^{s_0(n)}\mathrm{poly}(n)$. Thus, under SETH, GapMCSP with sup-polynomial s_0 cannot be **NP**-complete. The main challenge in [AH19] is to weakening the hardness assumption.

$|\Pi|$ to denote the length of the encoding. When there is no running time bound (i.e., the program can run in an arbitrary number of steps), we obtain the notion of Kolmogorov complexity.

In the following, let $\mathsf{U}(\Pi, 1^t)$ denote the output of Π when emulated on U for t steps. We now define the notion of Kolmogorov complexity with respect to the universal TM U.

Definition 2.1. *Let $t \in \mathbb{N}$ be a number. For all $x \in \{0,1\}^*$, define*

$$\mathrm{K}^t_{\mathsf{U}}(x) = \min_{\Pi \in \{0,1\}^*} \{|\Pi| : \mathsf{U}(\Pi, 1^t) = x\}$$

where $|\Pi|$ is referred to as the description length *of Π.*

It is well known that for every x, $K^t(x) \leq |x| + c$, for some constant c depending only on the choice of the universal TM U.

Fact 2.2. *For every universal TM U, there exists a constant c such that for every $x \in \{0,1\}^*$, and for every t such that $t(n) > 0$, $K^t_{\mathsf{U}}(x) \leq |x| + c$.*

In the following we fix some universal TM U and omit it from the notation, as our results hold for any universal TM U.

2.4 Levin Reductions

For a relation $\mathcal{R} \subseteq \{0,1\}^* \times \{0,1\}^*$, let $\mathcal{L}(\mathcal{R}) = \{x \in \{0,1\}^* \colon \exists w \in \{0,1\}^* \text{ s.t. } (x,w) \in \mathcal{R}\}$. We say that a relation $\mathcal{R}$ is the witness relation of a language $\mathcal{L} \subseteq \{0,1\}^*$ if $\mathcal{L}(\mathcal{R}) = \mathcal{L}$.

Definition 2.3 (Levin reduction)**.** *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be relations. A triplet of efficiently computable functions (f, g, h) is a* Levin reduction *from $\mathcal{R}_1$ to $\mathcal{R}_2$ if*

- *For every $(x, w) \in \mathcal{R}_1$, $(f(x), g(x, w)) \in \mathcal{R}_2$.*
- *If $(f(x), w) \in \mathcal{R}_2$ then $(x, h(x, w)) \in \mathcal{R}_1$.*

Remark 2.4. *Notice that if (f, g, h) a Levin reduction from $\mathcal{R}_1$ to $\mathcal{R}_2$, then f is a Karp reduction from $\mathcal{L}(\mathcal{R}_1)$ to $\mathcal{L}(\mathcal{R}_2)$. Indeed, the first item above implies that if $x \in \mathcal{L}(\mathcal{R}_1)$ then $f(x) \in \mathcal{L}(\mathcal{R}_2)$, and the second item implies the other direction.*

A Levin reduction (f, g, h) is *honest* if there exists a constant $\delta > 0$ such that for every large enough $n \in \mathbb{N}$ and every $x \in \{0,1\}^n$, $f(x) \geq n^\delta$.

When for two languages $\mathcal{L}_1$ and $\mathcal{L}_2$ we fix canonical relations $\mathcal{R}_1$ and $\mathcal{R}_2$, we say that there is a Levin reduction from $\mathcal{L}_1$ to $\mathcal{L}_2$ if there is a Levin reduction from $\mathcal{R}_1$ to $\mathcal{R}_2$. We say that $\mathcal{L} \in \mathbf{NP}$ is **NP**-hard under Levin reductions if there exists a Levin reduction from SAT to $\mathcal{L}$, where the canonical relation for SAT is

$$\mathcal{R}_{\mathrm{SAT}} = \{(\phi, v) \colon \phi \text{ is a SAT formula and } \phi[v] = 1\}.$$

We also define Levin reductions for promise problems. In the following, we consider a promise problem $(\mathcal{Y}, \mathcal{N})$ that is associated with two relations $(\mathcal{R}_{\mathcal{Y}}, \mathcal{R}_{\overline{\mathcal{N}}})$ such that $\mathcal{R}_{\mathcal{Y}} \subseteq \mathcal{R}_{\overline{\mathcal{N}}}$, where $\mathcal{R}_{\mathcal{Y}}$ is the witness relation for $\mathcal{Y}$, and $\mathcal{R}_{\overline{\mathcal{N}}}$ is the witness relation for $\overline{\mathcal{N}}$. That is, $(\mathcal{Y}, \mathcal{N}) = (\mathcal{L}(\mathcal{R}_{\mathcal{Y}}), \overline{\mathcal{L}(\mathcal{R}_{\overline{\mathcal{N}}})})$.

Definition 2.5 (Levin reduction, promise problems)**.** *Let* $(\mathcal{R}^1_{\mathcal{Y}}, \mathcal{R}^1_{\overline{\mathcal{N}}})$ *and* $(\mathcal{R}^2_{\mathcal{Y}}, \mathcal{R}^2_{\overline{\mathcal{N}}})$ *be pairs of relations such that* $\mathcal{R}^1_{\mathcal{Y}} \subseteq \mathcal{R}^1_{\overline{\mathcal{N}}}$ *and* $\mathcal{R}^2_{\mathcal{Y}} \subseteq \mathcal{R}^2_{\overline{\mathcal{N}}}$*. A triplet of efficiently computable functions* (f, g, h) *is a* Levin reduction *from* $(\mathcal{R}^1_{\mathcal{Y}}, \mathcal{R}^1_{\overline{\mathcal{N}}})$ *to* $(\mathcal{R}^2_{\mathcal{Y}}, \mathcal{R}^2_{\overline{\mathcal{N}}})$ *if*

- *For every* $(x, w) \in \mathcal{R}^1_{\mathcal{Y}}$*,* $(f(x), g(x, w)) \in \mathcal{R}^2_{\mathcal{Y}}$*.*
- *If* $(f(x), w) \in \mathcal{R}^2_{\overline{\mathcal{N}}}$ *then* $(x, h(x, w)) \in \mathcal{R}^1_{\overline{\mathcal{N}}}$*.*

Note that we can define reductions from language to promise problem by taking $\mathcal{R}_{\mathcal{Y}} = \mathcal{R}_{\overline{\mathcal{N}}}$. Lastly, our results hold even when the reductions are allowed to be randomized. In this case, $f(x; r)$ can be a randomized function (that uses randomness r), and both g, h get access to r (and possibly use more randomness). We then only require that the above requirements hold with high probability over r.

Definition 2.6 (Randomized Levin reduction, promise problems)**.** *Let* $(\mathcal{R}^1_{\mathcal{Y}}, \mathcal{R}^1_{\overline{\mathcal{N}}})$ *and* $(\mathcal{R}^2_{\mathcal{Y}}, \mathcal{R}^2_{\overline{\mathcal{N}}})$ *be pairs of relations such that* $\mathcal{R}^1_{\mathcal{Y}} \subseteq \mathcal{R}^1_{\overline{\mathcal{N}}}$ *and* $\mathcal{R}^2_{\mathcal{Y}} \subseteq \mathcal{R}^2_{\overline{\mathcal{N}}}$*. A triplet of efficiently computable functions* (f, g, h) *is a* randomized Levin reduction with ϵ-error *from* $(\mathcal{R}^1_{\mathcal{Y}}, \mathcal{R}^1_{\overline{\mathcal{N}}})$ *to* $(\mathcal{R}^2_{\mathcal{Y}}, \mathcal{R}^2_{\overline{\mathcal{N}}})$ *if for some* $\ell \in$ poly

- *For every* $x \in \mathcal{L}(\mathcal{R}^1_{\mathcal{Y}})$*, with probability at least* $1 - \epsilon$ *over the choice of* $r_1 \leftarrow \{0,1\}^{\ell(|x|)}$ *the following holds:*
  1. $(f(x; r_1), g(x, w; r_1)) \in \mathcal{R}^2_{\mathcal{Y}}$*, and,*
  2. *for every* w' *such that* $(f(x; r_1), w') \in \mathcal{R}^2_{\overline{\mathcal{N}}}$ *it holds that*

$$\Pr_{r_2 \leftarrow \{0,1\}^{\ell(|x|)}}\left[(x, h(x, w'; r_1, r_2)) \in \mathcal{R}^1_{\overline{\mathcal{N}}}\right] \geq 1 - \epsilon.$$

- *For every* $x \notin \mathcal{L}(\mathcal{R}^1_{\overline{\mathcal{N}}})$ *it holds that* $\Pr_{r_1 \leftarrow \{0,1\}^{\ell(|x|)}}\left[f(x; r_1) \in \mathcal{L}(\mathcal{R}^2_{\overline{\mathcal{N}}})\right] \leq \epsilon$*.*

Note that in the above definition h receive additional randomness r_2. One can consider a definition in which f, g and h gets the same randomness r_1, but the above definition is stronger, as the set of "good" randomness on which h successfully output a witness for x can be depend on the specific witness w'.

2.5 Cryptographic Primitives

In this part we define the cryptographic tools we will use. We start with the definition of one-way functions.

Definition 2.7 (One-way function)**.** *A polynomial-time computable function* $f : \{0,1\}^* \to \{0,1\}^*$ *is called a* one-way function *if for every* PPT *algorithm* A*, there is a negligible function* $\mu : \mathbb{N} \to [0,1]$ *such that for every* $n \in \mathbb{N}$

$$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{A}(f(x)) \in f^{-1}(f(x))\right] \leq \mu(n).$$

A one-way function is subexponentially-secure *if there exists a constant* $\delta > 0$ *such that for every* 2^{n^δ} *time algorithm* A*, and for every large enough* $n \in \mathbb{N}$

$$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{A}(f(x)) \in f^{-1}(f(x))\right] \leq 2^{-n^\delta}.$$

Next, we define iO.

Definition 2.8 (indistinguishability obfuscation)**.** *An efficient randomized algorithm iO is an* indistinguishability obfuscator *if for every $\lambda, n \in \mathbb{N}$ and any circuit $C\colon \{0,1\}^n \to \{0,1\}$,*

$$\Pr_{\widehat{C} \leftarrow iO(1^\lambda, C), x \leftarrow \{0,1\}^n}\left[C(x) = \widehat{C}(x)\right] = 1,$$

and for every $s \in \text{poly}$ and every n.u.-poly-time algorithm $\mathcal{A}$, there exists a negligible function μ, such that for every $\lambda \in \mathbb{N}$ and every two circuits $C, C'\colon \{0,1\}^n \to \{0,1\}$ with $|C| = |C'| \leq s(\lambda)$ and $n \leq \lambda$,

$$\left|\Pr\left[\mathcal{A}(1^\lambda, iO(1^\lambda, C)) = 1\right] - \Pr\left[\mathcal{A}(1^\lambda, iO(1^\lambda, C')) = 1\right]\right| \leq \mu(\lambda).$$

We say that iO has overhead *p if for every C and λ, $\left|iO(1^\lambda, C)\right| \leq p(|C|, \lambda)$ with probability 1.*

Next we define Target collision-resistant hash functions, also known as universal one-way hash functions.

Definition 2.9 (Target collision resistant hash)**.** *An efficiently computable function*

$$T\colon \{0,1\}^n \to \{0,1\}^{n-s(n)}$$

is a Target collision resistant hash function (TCR) *if $s(n) \geq 1$ and for every PPT algorithm $\mathcal{A}$,*

$$\Pr_{x \leftarrow \{0,1\}^n}\left[x' \leftarrow \mathcal{A}(x); T(x) = T(x') \text{ and } x \neq x'\right] \leq \text{neg}(n).$$

We say that a TCR is secure against subexponential adversaries *if there exists a constant $\delta > 0$ such that for every 2^{n^δ}-time algorithm* A*,*

$$\Pr_{x \leftarrow \{0,1\}^n}\left[x' \leftarrow \mathcal{A}(x); T(x) = T(x') \text{ and } x \neq x'\right] \leq \text{neg}(n).$$

Rompel [Rom90] showed that TCR can be constructed from one-way functions.

Theorem 2.10 ([Rom90])**.** *Assume that one-way functions exist. Then TCR $T\colon \{0,1\}^n \to \{0,1\}^{n-s(n)}$ with $s(n) \in \omega(\log n)$ exists.*

Since the proof of the theorem above is black-box, the same holds for subexponential adversaries.

Theorem 2.11. *Assume that subexponentially-secure one-way functions exist. Then there exists a TCR $T\colon \{0,1\}^n \to \{0,1\}^{n-s(n)}$ secure against subexponential adversaries, with $s(n) \in \omega(\log n)$.*

We will also use the following theorem, by [Kom+14].

Theorem 2.12 ([Kom+14])**.** *Assume that iO exists and* $\mathbf{NP} \not\subseteq \mathbf{ioBPP}$*. Then one-way functions exist.*

Lastly, we will also use the fact that a TCR is a one-way function.

Claim 2.13. *Let $T\colon \{0,1\}^n \to \{0,1\}^{n-s(n)}$ be a TCR with $s(n) \in \omega(\log n)$. Then T is a one-way function. That is, for every PPT algorithm* A*,*

$$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathsf{A}(f(x)) \in T^{-1}(T(x))\right] \leq \text{neg}(n).$$

Moreover, if secure against subexponential adversaries, the above holds for any algorithm $\mathcal{A}$ with running time at most 2^{n^δ}, for some constant δ.

We sketch the proof here.

Proof. Assume that algorithm $\mathcal{A}$ can invert T with non-negligible probability. We claim that $\mathcal{A}$ can be used to find a collision with non-negligible probability. Indeed, let $X \leftarrow \{0,1\}^n$ be a uniformly distributed random variable. Let $\mathcal{A}'$ be the algorithm that given random input X, executes $\mathcal{A}(T(X))$ and outputs its output.

Conditioned on that $\mathcal{A}(T(X))$ found a pre-image x' of $T(X)$, it holds that the input of $\mathcal{A}'$, X, is uniformly distributed over the set $T^{-1}(T(x'))$. Since the size of $T^{-1}(T(x'))$ is large with high probability (the probability that $\left|T^{-1}(T(x'))\right| \leq k$ is at most $k \cdot 2^{-s(n)}$), with high probability it holds that $x \neq X$, and thus $\mathcal{A}'$ found a collision. □

3 GapMCSP is not NP-hard under Levin Reductions

In this section we prove our main result for GapMCSP. We first define GapMCSP$[s_0, s_1]$. In the following, a circuit C computes a string x if the truth table of C is x.

Definition 3.1. *For two functions $s_0, s_1 \colon \mathbb{N} \to \mathbb{N}$, let* GapMCSP$[s_0, s_1]$ *denote the following promise problem.*

- $\mathcal{Y} = \left\{x \in \{0,1\}^n \colon n = 2^k \text{ for } k \in \mathbb{N} \text{ and there exists a circuit } C \text{ of size at most } s_0(n) \text{ that computes } x\right\}$
- $\mathcal{N} = \left\{x \in \{0,1\}^n \colon n = 2^k \text{ for } k \in \mathbb{N} \text{ and there is no circuit of size } s_1(n) \text{ that computes } x\right\}$

We define the relations $\mathcal{R}_{\mathcal{Y}}$ and $\mathcal{R}_{\mathcal{N}}$ for GapMCSP$[s_0, s_1]$ in the natural way:

$$\mathcal{R}_{\mathcal{Y}} = \{(x, C) \colon C \text{ is a circuit of size at most } s_0(n) \text{ that computes } x\},$$

and,

$$\mathcal{R}_{\overline{\mathcal{N}}} = \{(x, C) \colon C \text{ is a circuit of size at most } s_1(n) \text{ that computes } x\}.$$

We start with the following theorem for deterministic reductions. In Section 3.2 we prove a similar theorem for randomized Levin reductions.

Theorem 3.2. *Let $p \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function. Assume that there exists iO with overhead p, and subexponentially-secure one-way functions. Then for any constant $\alpha > 0$ and for any pair of efficiently computable functions $s_0, s_1 \colon \mathbb{N} \to \mathbb{N}$ for which $s_1(n) > p(s_0(n), (s_0(n))^\alpha)$, it holds that* GapMCSP$[s_0(n), s_1(n)]$ *is not* **NP** *hard with respect to Levin reductions.*

Since iO is an efficient algorithm, the overhead of any iO is polynomial. Combining this observation with Theorem 3.2 yields Theorem 1.1.

3.1 Proving Theorem 3.2

To prove Theorem 3.2, let iO be an indistinguishability obfuscator, and let $p \in \text{poly}$ be the overhead of iO. Let $T \colon \{0,1\}^n \to \{0,1\}^{n - \omega(\log n)}$ be a TCR with security against subexponential algorithms.

Consider the following distribution ensemble $\mathcal{D} = \{\mathcal{D}_n\}_{n \in \mathbb{N}}$ over SAT formulas and assignments (ϕ, v). For every $n \in \mathbb{N}$, to sample from $\mathcal{D}_n$: sample a random $x \in \{0,1\}^n$. Let $\phi_{T(x)}$ be a formula such that $\phi_{T(x)}[x'] = 1$ if and only if $T(x') = T(x)$. Output $(\phi_{T(x)}, x)$. We remark that $\phi_{T(x)}$ only depends on the value of $T(x)$ and not on x itself.

We start with the following claim.

Claim 3.3. *The following hold for every $n \in \mathbb{N}$:*

- $\Pr_{(\phi,v)\leftarrow\mathcal{D}_n}[\phi[v]=1]=1$;
- $\Pr_{(\phi,v)\leftarrow\mathcal{D}_n}[\exists v' \text{ s.t. } v\neq v' \text{ and } \phi[v']=1]\geq 1-\mathrm{neg}(n)$, *and,*
- *for every* PPT *algorithm* $\mathcal{A}$

$$\Pr_{(\phi,v)\leftarrow\mathcal{D}_n}\left[\mathcal{A}(\phi,v)=v'; v\neq v' \text{ and } \phi[v']=1\right]\leq \mathrm{neg}(n).$$

Proof. The first and last items follow directly from the definition of the distribution $\mathcal{D}$ and the definition of TCR. The second item holds since T is shrinking by $\omega(\log n)$ bits. □

We also prove the following claim, which states that for any reduction f from SAT to GapMCSP, the output of f on inputs samples from $\mathcal{D}_n$ must have length polynomial in n. Here we need the subexponential security of T.

Claim 3.4. *Let (f,g,h) be a Levin reduction from SAT to* GapMCSP$[s_0,s_1]$. *Then there exists a constant $\delta>0$ such that*

$$\Pr_{(\phi,v)\leftarrow\mathcal{D}_n}\left[s_0(|f(\phi)|)\geq n^{\delta}\right]\geq 1-\mathrm{neg}(n)$$

Remark 3.5. *Claim 3.4 is the only place in which we use the subexponential security assumption. We need it to make sure that (with high probability over $\mathcal{D}$) $s_0(|f(\phi)|)$ is not too small. While we can require that $s_0(n)\geq n^{\epsilon}$ for some $\epsilon>0$, the reduction f itself can return short outputs.*

When the reduction f is honest (that is, $|f(x)|\geq|x|^{\alpha}$ for all inputs x and for some $\alpha>0$), we can replace the assumption on exponentially-secure one-way function with the above requirement that $s_0(n)\geq n^{\epsilon}$, and minimal assumption that **NP** $\not\subseteq$ **ioBPP**. *The latter assumption is known to imply (together with iO) one-way function (see Theorem 2.12). Using the same proof as follows we get Theorem 1.2.*

Proof. Assume toward a contradiction that this is not the case for all constants $\delta>0$. We will show how to invert T. That is, we will show an algorithm $\mathcal{A}$ that runs in time $2^{n^{c\cdot\delta}}$ for some constant c such that

$$\Pr_{x\leftarrow\{0,1\}^n}\left[\mathcal{A}(T(x))\in T^{-1}(T(x))\right]\geq\Pr_{(\phi,v)\leftarrow\mathcal{D}_n}\left[s_0(|f(\phi)|)<n^{\delta}\right].$$

The claim will then follow by Claim 2.13, as by assumption $\Pr_{(\phi,v)\leftarrow\mathcal{D}_n}\left[s_0(|f(\phi)|)<n^{\delta}\right]$ is non-negligible for all choices of $\delta>0$ (and for infinitely many n's).

Let $\mathcal{A}$ be the algorithm that given $y=T(x)$, constructs the formula ϕ_y, and then uses brute force to find a minimal circuit C of size at most n^{δ} that computes $f(\phi_y)$. Lastly, if such C exists, $\mathcal{A}$ outputs $h(\phi_y,C)$.

It is not hard to see that $\mathcal{A}$ runs in time $2^{\mathrm{poly}(n^{\delta})}$. By the definition of Levin reductions, when $s_0(\left|f(\phi_{T(x)})\right|)<n^{\delta}$, $\mathcal{A}$ always outputs x' such that $T(x')=T(x)$. Lastly, observe that the distribution of ϕ_y for $y=T(x)$ when $x\leftarrow\{0,1\}^n$, is exactly the distribution of ϕ when $(\phi,v)\leftarrow\mathcal{D}_n$. □

The next lemma shows it is possible to use iO to find collisions in the TCR.

Lemma 3.6. *Let iO be an indistinguishability obfuscator with overhead p, and let s_0 and s_1 as in Theorem 3.2. Assume that there exists a Levin reduction from SAT to* $\mathrm{GapMCSP}[s_0, s_1]$. *Then there exists an efficient algorithm $\mathcal{A}$ such that for every large enough $n \in \mathbb{N}$*

$$\Pr_{(\phi,v)\leftarrow\mathcal{D}_n}\left[\mathcal{A}(\phi,v) = v'; v \neq v' \text{ and } \phi[v'] = 1\right] > 1/4.$$

Proof. We start with the definition of $\mathcal{A}$. Let f, g, h be the Levin reduction between SAT to $\mathrm{GapMCSP}[s_0, s_1]$. Define $\mathcal{A}(\phi, v) = h(\phi, iO(1^{|g(\phi,v)|^\alpha}, g(\phi, v)))$. In the following we omit the security parameter $1^{|g(\phi,v)|^\alpha}$ from the notation.

Next, we show that $\mathcal{A}(\phi, v)$ returns $v' \neq v$ that satisfies ϕ with probability at least $1/4$. By Claim 3.3, such v' exists with all but negligible probability over a random sample $(\phi, v) \leftarrow \mathcal{D}_n$. For the constant $\delta > 0$ from Claim 3.4 let $\mathcal{G}$ be the set of all (ϕ, v) such that $s_0(|f(\phi)|) \geq n^\delta$ and that there exists $v' \neq v$ with $\phi[v'] = 1$. By Claim 3.4, $\Pr_{(\phi,v)\leftarrow\mathcal{D}_n}[(\phi, v) \in \mathcal{G}] \geq 1 - \mathrm{neg}(n)$. In the following, fix $n \in \mathbb{N}$, and fix $(\phi, v) \in \mathcal{G}$, and $v' \neq v$ with $\phi[v'] = 1$.

By the correctness of f and g, $g(\phi, v)$ and $g(\phi, v')$ are two circuits with size at most $s_0(|f(\phi)|)$ with the same truth table $f(\phi)$. We assume without loss of generality that $|g(\phi, v)| = |g(\phi, v')| = s_0(|f(\phi)|)$. By the assumption on the overhead time of the obfuscator iO, we get that the size of the output of $iO(g(\phi, v))$ and $iO(g(\phi, v'))$ are at most

$$p(|g(\phi, v)|, |g(\phi, v)|^\alpha) = p(s_0(|f(\phi)|), (s_0(|f(\phi)|))^\alpha) < s_1(|f(\phi)|).$$

Thus, the output $iO(g(\phi, v))$ is a witness that $f(\phi)$ is not a No instance of $\mathrm{GapMCSP}[s_0, s_1]$, and by the definition of h, $h(\phi, iO(g(\phi, v)))$ returns a witness that $\phi \in \mathrm{SAT}$. Similarly, the same holds for v': $h(\phi, iO(g(\phi, v')))$ returns a witness that $\phi \in \mathrm{SAT}$.

Lastly, we use the security of iO to claim that $h(\phi, iO(g(\phi, v))) \neq v$ with a good probability (over the randomness of iO). By the security of the obfuscator, and since $g(\phi, v)$ and $g(\phi, v')$ are circuits computing the same function $f(\phi)$ the output distributions of $iO(g(\phi, v))$ and $iO(g(\phi, v'))$ are indistinguishable. Moreover, since the iO is secure against non-uniform algorithms, the above distributions are indistinguishable also given (ϕ, v, v') (importantly, the size of (ϕ, v, v') is polynomial in the security parameter and in the size of the circuit $g(\phi, v)$ when $s_0(|f(x)|) \geq n^\delta$). In particular, by data processing, the distributions $h(\phi, iO(g(\phi, v)))$ and $h(\phi, iO(g(\phi, v')))$ must be indistinguishable.

By the definition of $\mathcal{A}$, we get that

$$\Pr[\mathcal{A}(\phi, v) = v] \leq \Pr\left[\mathcal{A}(\phi, v') = v\right] + \mu(s_0(|f(\phi)|))$$

for some negligible function μ, where the probability is over the randomness of $\mathcal{A}$ (that is, the randomness of iO). Since $(\phi, v) \in \mathcal{G}$, for every large enough n we get that

$$\Pr[\mathcal{A}(\phi, v) = v] \leq \Pr\left[\mathcal{A}(\phi, v') = v\right] + \mu(s_0(|f(\phi)|)) \leq \Pr\left[\mathcal{A}(\phi, v') \neq v'\right] + 1/3,$$

where the last inequality holds since $v \neq v'$. This implies that

$$1 - \Pr[\mathcal{A}(\phi, v) \neq v] \leq \Pr\left[\mathcal{A}(\phi, v') \neq v'\right] + 1/3,$$

or that

$$1/2 \cdot (\Pr[\mathcal{A}(\phi, v) \neq v] + \Pr\left[\mathcal{A}(\phi, v') \neq v'\right]) \geq 1/3. \tag{1}$$

To finish the proof, consider the distribution $\mathcal{D}'_n$, in which we sample $(\phi, v) \leftarrow \mathcal{D}_n$, and then if $(\phi, v) \in \mathcal{G}$, we sample a random $v' \neq v$ such that $\phi[v'] = 1$ (or let $v' = v$ if $(\phi, v) \notin \mathcal{G}$). We then output (ϕ, v, v').

We get that

$$\begin{aligned}
&\Pr_{(\phi,v)\leftarrow\mathcal{D}_n}[\mathcal{A}(\phi, v) \neq v] \\
&\geq \Pr_{(\phi,v)\leftarrow\mathcal{D}_n}[\mathcal{A}(\phi, v) \neq v \mid (\phi, v) \in \mathcal{G}] \cdot \Pr_{(\phi,v)\leftarrow\mathcal{D}_n}[(\phi, v) \in \mathcal{G}] \\
&\geq \Pr_{(\phi,v)\leftarrow\mathcal{D}_n}[\mathcal{A}(\phi, v) \neq v \mid (\phi, v) \in \mathcal{G}] \cdot (1 - \mathrm{neg}(n)) \\
&= \Pr_{(\phi,v_0,v_1)\leftarrow\mathcal{D}'_n}[\mathcal{A}(\phi, v_0) \neq v_0 \mid (\phi, v_0) \in \mathcal{G}] \cdot (1 - \mathrm{neg}(n)) \\
&= \Pr_{(\phi,v_0,v_1)\leftarrow\mathcal{D}'_n, b\leftarrow\{0,1\}}[\mathcal{A}(\phi, v_b) \neq v_b \mid (\phi, v_b) \in \mathcal{G}] \cdot (1 - \mathrm{neg}(n)) \\
&= 1/2 \cdot \sum_{b\in\{0,1\}} \Pr_{(\phi,v_0,v_1)\leftarrow\mathcal{D}'_n}[\mathcal{A}(\phi, v_b) \neq v_b \mid (\phi, v_b) \in \mathcal{G}] \cdot (1 - \mathrm{neg}(n)) \\
&\geq 1/3 - \mathrm{neg}(n).
\end{aligned}$$

where the third equality holds since the distribution of (ϕ, v_0) and (ϕ, v_1) are identical for $(\phi, v_0, v_1) \leftarrow \mathcal{D}'_n$, and the last inequality by Equation (1). □

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. Assume that iO and subexponential one-way functions exist. By Theorem 2.11, there exists a TCR with security against subexponential adversaries.

Assume there exists Levin reduction from SAT to GapMCSP$[s_0, s_1]$, and let $\mathcal{D}$ be the distribution defined above. By Claim 3.3, there is no efficient algorithm that given a random sample (ϕ, v) from $\mathcal{D}_n$ finds $v' \neq v$ such that $\phi[v'] = 1$ with non-negligible probability. But by Lemma 3.6, there exists such an algorithm that succeeds with probability $1/4$, which is a contradiction. □

3.2 Randomized Levin Reductions

In this part we generalize Theorem 3.2 to hold with respect to randomized reductions. We prove the following theorem.

Theorem 3.7. *Let $0 \leq \epsilon \leq 1/30$ be a constant, and let $p\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function. Assume that there exist iO with overhead p and subexponentially-secure one-way function. Then for any constant $\alpha > 0$ and for any pair of efficiently computable functions $s_0, s_1\colon \mathbb{N} \to \mathbb{N}$ for which $s_1(n) > p(s_0(n), (s_0(n))^\alpha)$, it holds that* GapMCSP$[s_0(n), s_1(n)]$ *is not* **NP***-hard with respect to randomized Levin reductions with ϵ-error.*

Theorem 1.1 (for randomized reductions) directly follows by Theorem 3.7 and the observation that the overhead p is always bounded by polynomial. The proof of Theorem 3.7 is similar to the proof of Theorem 3.2. Let *iO* be an indistinguishability obfuscator with overhead p, and $T\colon \{0,1\}^n \to \{0,1\}^{n-\omega(\log n)}$ be a TCR secure against subexponential adversaries. Let $\mathcal{D} = \{\mathcal{D}_n\}_{n\in\mathbb{N}}$ be the same distribution as defined in the proof of Theorem 3.2.

The following claim is the analog of Claim 3.4 for randomized reductions.

Claim 3.8. *Let (f, g, h) be a randomized Levin reduction with ϵ-error from SAT to* GapMCSP$[s_0, s_1]$. *Then there exists a constant $\delta > 0$ such that*

$$\Pr_{(\phi,v)\leftarrow\mathcal{D}_n, r_1\leftarrow\{0,1\}^{\ell(|\phi|)}}\left[s_0(|f(\phi; r_1)|) \geq n^\delta\right] \geq 1 - 2\epsilon - \mathrm{neg}(n)$$

Proof. The proof follows the same lines as the proof of Claim 3.4. Specifically, let $\delta > 0$, $\mathcal{A}$ be the algorithm described in the proof of Claim 3.4. We will show that

$$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathcal{A}(T(x)) \in T^{-1}(T(x))\right] \geq \Pr_{(\phi,v)\leftarrow\mathcal{D}_n, r_1 \leftarrow \{0,1\}^{\ell(|\phi|)}}\left[s_0(|f(\phi)|) < n^{\delta}\right] - 2\epsilon.$$

The claim will then follow by Claim 2.13.

By the definition of randomized Levin reductions, with probability at least $1-\epsilon$ over the choice of r_1, it holds that h succeeds to convert a witness for $f(\phi; r_1)$ to a witness for ϕ with probability at least $1-\epsilon$. By the union bound, with probability at least

$$\Pr_{(\phi,v)\leftarrow\mathcal{D}_n, r_1 \leftarrow \{0,1\}^{\ell(|\phi|)}}\left[s_0(|f(\phi; r_1)|) < n^{\delta}\right] - \epsilon$$

over the choice of $(\phi, v) \leftarrow \mathcal{D}_n$ and r_1, it holds that both $s_0(|f(\phi; r_1)|) < n^{\delta}$, and h converts witnesses for $f(\phi; r_1)$ to witnesses for ϕ with probability at least $1-\epsilon$. In this case, $\mathcal{A}$ finds a witness for $f(\phi; r_1)$ and outputs a pre-image of T with probability $1-\epsilon$.

Using the union bound again, we get that $\mathcal{A}$ finds such a pre-image with probability at least

$$\Pr_{(\phi,v)\leftarrow\mathcal{D}_n, r_1 \leftarrow \{0,1\}^{\ell(|\phi|)}}\left[s_0(|f(\phi; r_1)|) < n^{\delta}\right] - 2\epsilon$$

as claimed. □

The next lemma generalizes Lemma 3.6, showing that it is possible to use iO and randomized Levin reduction to find collisions in the TCR.

Lemma 3.9. *Let iO be indistinguishability obfuscator with overhead p, and let ϵ, s_0 and s_1 as in Theorem 3.7. Assume that there exists a randomized Levin reduction with ϵ-error from SAT to* GapMCSP$[s_0, s_1]$. *Then there exists an efficient algorithm $\mathcal{A}$ such that for every large enough $n \in \mathbb{N}$*

$$\Pr_{(\phi,v)\leftarrow\mathcal{D}_n}\left[\mathcal{A}(\phi, v) = v'; v \neq v' \text{ and } \phi[v'] = 1\right] > 1/4 - 7\epsilon.$$

Proof. We start with the definition of $\mathcal{A}$. Let f, g, h be the Levin reduction from SAT to GapMCSP$[s_0, s_1]$, and define $\mathcal{A}$ to be the algorithm that on input ϕ, v, outputs

$$h(\phi, iO(1^{|g(\phi,v;r_1)|^{\alpha}}, g(\phi, v; r_1)); r_1, r_2),$$

for a random choice of randomness r_1, r_2 for g, h (additionally to the randomness of iO). In the following we omit the security parameter $1^{|g(\phi,v;r_1)|^{\alpha}}$ from the notation.

Next, we show that $\mathcal{A}(\phi, v)$ returns $v' \neq v$ that satisfies ϕ with probability at least $1/4 - 7\epsilon$. Let $\mathcal{G}$ be the set of all SAT formulas ϕ such that there are $v \neq v'$ such that $\phi[v] = \phi[v'] = 1$.

Let $\delta > 0$ be the constant from Claim 3.8. In the following, we say that a randomness r_1 is *good* for a formula ϕ and a satisfying assignment v, if it holds that (1) $s_0(|f(\phi; r_1)|) \geq n^{\delta}$, (2) $g(\phi, v; r_1)$ is a circuit of size at most $s_0(|f(\phi; r_1)|)$ that computes $f(\phi; r_1)$, and (3) for any circuit C of size less than $s_1(|f(\phi; r_1)|)$ which computes $f(\phi; r_1)$, it holds that $h(\phi, C; r_1, r_2)$ is a satisfying assignment for ϕ with probability at least $1-\epsilon$ over the choice of r_2. That is, r_1 is good if the output of $f(\phi; r_1)$ is not too short, and if the reduction succeed in converting witnesses from SAT to GapMCSP using the randomness r_1.

By the definition of Levin reductions with ϵ-error a random r_1 fulfils the last two requirements with probability at least $1-\epsilon$. Using Claim 3.8 and the union bound, we get that a random r_1 is good for a random pair $(\phi, v) \leftarrow \mathcal{D}_n$ with probability at least $1 - 3\epsilon - \mathrm{neg}(n)$.

For $\phi \in \mathcal{G}$, and two satisfying assignments $v \neq v'$, let $\mathcal{R}_{\phi,v,v'}$ be the set of all random strings r_1 such that r_1 is good for both (ϕ, v) and for (ϕ, v'). Let $\mathcal{D}'_n$ be the distribution in which we sample $(\phi, v) \leftarrow \mathcal{D}_n$, and then if $\phi \in \mathcal{G}$, we sample a random $v' \neq v$ such that $\phi[v'] = 1$ (otherwise we let $v' = v$). We then output (ϕ, v, v'). Observe that the marginal distribution of both (ϕ, v) and (ϕ, v') is identical to $\mathcal{D}_n$. Using the union bound again, together with Claim 3.3, we get that

$$\Pr_{(\phi,v,v') \leftarrow \mathcal{D}'_n, r_1 \leftarrow \{0,1\}^{\ell(|\phi|)}} \left[r_1 \in \mathcal{R}_{\phi,v,v'} \wedge \phi \in \mathcal{G} \right] \geq 1 - 6\epsilon - \mathrm{neg}(n). \tag{2}$$

In the following, fix $\phi \in \mathcal{G}$ two satisfying assignments $v \neq v'$, and randomness $r_1 \in \mathcal{R}_{\phi,v,v'}$. We now continue as in the proof of Lemma 3.6, where towards the end of the proof we will take into account the probability over ϕ, v and r_1. First, we show that $\mathcal{A}$ outputs a witness for ϕ with high probability.

By the definition of $\mathcal{R}_{\phi,v,v'}$, $g(\phi, v; r_1)$ and $g(\phi, v'; r_1)$ are two circuits with size at most $s_0(f(\phi))$ with the same truth table $f(\phi; r_1)$. We assume without loss of generality that $|g(\phi, v)| = |g(\phi, v')| = s_0(|f(\phi)|)$. As in the proof of Lemma 3.6, by the assumption on the overhead of the obfuscator iO, we get that the size of the output of $iO(g(\phi, v; r_1))$ and $iO(g(\phi, v'; r_1))$ is less than $s_1(|f(\phi; r_1)|)$. Thus, the output $iO(g(\phi, v; r_1))$ is a witness that $f(\phi; r_1)$ is not a No instance of GapMCSP$[s_0, s_1]$, and by the definition of h and $\mathcal{R}_{\phi,v,v'}$, $h(\phi, iO(g(\phi, v; r_1, r_2)))$ returns a witness that $\phi \in \mathrm{SAT}$ with probability at least $1 - \epsilon$ over the choice of r_2. Similarly, the same holds for v': $h(\phi, iO(g(\phi, v')))$ returns a witness that $\phi \in \mathrm{SAT}$ with the same probability.

Next, we use the security of iO to claim that $h(\phi, iO(g(\phi, v; r_1); r_1, r_2)$ outputs a satisfying assignment to ϕ which is not equal to v with a good probability. By the security of the obfuscator, and since $g(\phi, v; r_1)$ and $g(\phi, v'; r_1)$ computes the same function $f(\phi; r_1)$ the output distributions of $iO(g(\phi, v; r_1))$ and $iO(g(\phi, v'; r_1))$ are indistinguishable. Moreover, by the non-uniform security, the above distributions are indistinguishable also given (x, v, v', r_1). In particular, by data processing, the distributions $h(\phi, iO(g(\phi, v; r_1)); r_1, r_2)$ and $h(\phi, iO(g(\phi, v'; r_1)); r_1, r_2)$ must be indistinguishable. Let $\mathcal{A}(\phi, v; r_1)$ be the output of $\mathcal{A}(\phi, v)$ when we fix the randomness $\mathcal{A}$ uses for g to be r_1. In the following we assume without loss of generality that whenever $\mathcal{A}$ does not output a satisfying assignment for ϕ, it outputs $\perp$. By the definition of $\mathcal{A}$, when $r_1 \in \mathcal{R}_{\phi,v,v'}$ we get that

$$\Pr[\mathcal{A}(\phi, v; r_1) = v] \leq \Pr\left[\mathcal{A}(\phi, v'; r_1) = v\right] + \mu(s_0(|f(\phi)|))$$

for some negligible function μ. As in the proof of Lemma 3.6, this implies that

$$1/2 \cdot (\Pr[\mathcal{A}(\phi, v; r_1) \neq v] + \Pr\left[\mathcal{A}(\phi, v'; r_1) \neq v'\right]) \geq 1/3. \tag{3}$$

Since h fails with probability at most ϵ, we get that

$$1/2 \cdot (\Pr[\mathcal{A}(\phi, v; r_1) \notin \{v, \perp\}] + \Pr\left[\mathcal{A}(\phi, v'; r_1) \notin \left\{v', \perp\right\}\right]) \geq 1/3 - \epsilon. \tag{4}$$

We get that

$$\begin{aligned}
&\Pr_{(\phi,v)\leftarrow\mathcal{D}_n, r_1\leftarrow\{0,1\}^{\ell(|\phi|)}}[\mathcal{A}(\phi,v;r_1)\notin\{v,\perp\}]\\
&=\Pr_{(\phi,v_0,v_1)\leftarrow\mathcal{D}'_n, r_1\leftarrow\{0,1\}^{\ell(|\phi|)}}[\mathcal{A}(\phi,v_0;r_1)\notin\{v_0,\perp\}]\\
&\geq \Pr_{\substack{(\phi,v_0,v_1)\leftarrow\mathcal{D}'_n,\\ r_1\leftarrow\{0,1\}^{\ell(|\phi|)}}}[\mathcal{A}(\phi,v_0;r_1)\notin\{v_0,\perp\}\mid \phi\in\mathcal{G}, r_1\in\mathcal{R}_{\phi,v_0,v_1}]\\
&\qquad\cdot\Pr[r_1\in\mathcal{R}_{\phi,v_0,v_1}\wedge\phi\in\mathcal{G}]\\
&\geq \Pr_{\substack{(\phi,v_0,v_1)\leftarrow\mathcal{D}'_n,\\ r_1\leftarrow\{0,1\}^{\ell(|\phi|)}}}[\mathcal{A}(\phi,v_0;r_1)\notin\{v_0,\perp\}\mid \phi\in\mathcal{G}, r_1\in\mathcal{R}_{\phi,v_0,v_1}]\\
&\qquad\cdot(1-6\epsilon-\mathrm{neg}(n))\\
&\geq \Pr_{\substack{(\phi,v_0,v_1)\leftarrow\mathcal{D}'_n,\\ r_1\leftarrow\{0,1\}^{\ell(|\phi|)}\\ b\leftarrow\{0,1\}}}[\mathcal{A}(\phi,v_b;r_1)\notin\{v_b,\perp\}\mid \phi\in\mathcal{G}, r_1\in\mathcal{R}_{\phi,v_0,v_1}]\\
&\qquad\cdot(1-6\epsilon-\mathrm{neg}(n))\\
&\geq (1/3-\epsilon)\cdot(1-6\epsilon-\mathrm{neg}(n))\\
&\geq 1/4-7\epsilon.
\end{aligned}$$

where the second inequality holds by Equation (2), the third equality holds since the distribution of (ϕ, v_0) and (ϕ, v_1) are identical for $(\phi, v_0, v_1) \leftarrow \mathcal{D}'_n$, by a similar argument as in the proof of Lemma 3.6, and the last inequality holds for large enough n and for a small enough constant ϵ. □

We are now ready to prove Theorem 3.7.

Proof of Theorem 3.7. Assume that iO and subexponentially-secure one-way function exist. By Theorem 2.11, there exists a TCR with security against subexponential adversaries.

Assume there exists a Levin reduction from SAT to GapMCSP$[s_0, s_1]$, and let $\mathcal{D}$ be the distribution defined above. By Claim 3.3, there is no efficient algorithm that given a random sample (ϕ, v) from $\mathcal{D}_n$ finds $v' \neq v$ such that $\phi[v'] = 1$ with non-negligible probability. But by Lemma 3.9, there exists such an algorithm that succeeds with probability $1/4 - 7\epsilon$, which is a contradiction when $\epsilon < 1/28$. □

4 $\mathrm{Gap_pMK^tP}$ is not NP-hard under Levin Reductions

In this section we prove our result for $\mathrm{MK^tP}$. That is, we prove that (under cryptographic assumptions) there is no Levin reduction from SAT to the following promise problem. For $p, t \in \mathrm{poly}$, let $\mathrm{Gap_pMK^tP}[s_0, s_1]$ be the following promise problem:

- $\mathcal{Y} = \left\{x \in \{0,1\}^n \colon \mathrm{K}^{t(n)}(x) \leq s_0(n)\right\}$
- $\mathcal{N} = \left\{x \in \{0,1\}^n \colon \mathrm{K}^{p(t(n))}(x) > s_1(n)\right\}$

We define the relations $\mathcal{R}_\mathcal{Y}$ and $\mathcal{R}_\mathcal{N}$ for $\mathrm{Gap_pMK^tP}[s_0, s_1]$ in the natural way:

$$\mathcal{R}_\mathcal{Y} = \left\{(x, P) \colon P \text{ is a program of length at most } s_0(n) \text{ such that } \mathsf{U}(P, 1^{t(|x|)}) = x\right\},$$

and,

$$\mathcal{R}_{\overline{\mathcal{N}}} = \Big\{(x,P)\colon P \text{ is a program of length at most } s_1(n) \text{ such that } \mathsf{U}(P, 1^{p(t(|x|))}) = x\Big\}.$$

The proof follows the same line as the proof of Theorem 3.2, where we replace the *iO* with randomized encoding for Turing machines with indistinguishability-based security [AJ15].

Definition 4.1 (Randomized encoding for TM)**.** *A pair of efficient randomized algorithms* (Enc, Dec) *is* randomized encoding *for TMs if the following holds: Let* M *be a TM and* $x \in \{0,1\}^*$ *be an input,* $\lambda \in \mathbb{N}$ *be a security parameter and let* $T \in \mathbb{N}$ *be a bound on the running time of* $M(x)$*. Then*

1. *(Correctness:)* $\Pr\big[Dec(Enc(1^\lambda, M, x, T)) = M(x)\big] = 1$
2. *(Efficiency:)* $Enc(1^\lambda, M, x, T)$ *runs in time* $\mathrm{poly}(\lambda, |M|, |x|, \log T)$ *and* $Dec(\widehat{M(x)})$ *runs in time* $\mathrm{poly}(\lambda, |M|, |x|, t)$ *for* $\widehat{M(x)} \leftarrow Enc(1^\lambda, M, x, T)$ *and where* $t \le T$ *is the running time of* $M(x)$*, and,*
3. *(Security:) For every* $n.u. - poly - time$ *algorithm* $\mathcal{A}$ *and every* $s \in \mathrm{poly}$ *there exists a negligible function* μ*, such that for every TM* M *and two inputs* x_0, x_1 *such that* $M(x_0) = M(x_1)$*,* $|M| \le s(\lambda), |x_0| \le s(\lambda), |x_1| \le s(\lambda)$ *and the running time of* M *on* x_0 *at most* $s(\lambda)$ *and is the same as the running time of* M *on* x_1*, the following holds:*

$$\Big|\Pr\Big[\mathcal{A}(Enc(1^\lambda, M, x_0, T)) = 1\Big] - \Pr\Big[\mathcal{A}(Enc(1^\lambda, M, x_1, T)) = 1\Big]\Big| = \mu(\lambda).$$

We say that (Enc, Dec) *has overhead* p *if* $\big|\mathsf{Enc}(1^\lambda, M, x, T)\big| \le p(|M|, |x|, T, \lambda)$ *with probability* 1*.*

Using randomized encoding, we get the following theorem.

Theorem 4.2. *Let* $0 \le \epsilon \le 1/30$ *be a constant. Assume that randomized encoding for TMs with overhead* q*, and subexponentially-secure one-way function exists. Then there exists a constant* $c \in \mathbb{N}$ *such that for every constant* $\alpha > 0$*, for any* $t \in \mathrm{poly}$ *and any efficiently computable functions* $s_0, s_1 \colon \mathbb{N} \to \mathbb{N}$ *for which*

$$s_1(n) > q(c, s_0(n) + c\log(t(n)) + c\log(s_0(n)), \log t(n), (s_0(n))^\alpha),$$

and for every large enough polynomial p*, it holds that* $\mathrm{Gap_pMK^tP}[s_0, s_1]$ *is not* **NP** *hard with respect to randomized Levin reductions with* ϵ*-error.*

By the results of [LPST15; KLW15] such randomized encoding with polynomial overhead q for poly-time TMs can be constructed assuming one-way functions, subexponentially-secure iO for circuits and injective PRG (that can be constructed from one-way permutation). Together with Theorem 4.2 we get Theorem 1.3. As in Theorem 3.2, we can relax the requirement for subexponentially-secure one-way function if we only want to exclude honest reductions.

[AJS17] constructed *iO* for TM with multiplicative overhead. By combining the construction of randomized encoding for TMs of [LPST15] with the *iO* of [AJS17], we get randomized encoding with multiplicative overhead.

Theorem 4.3. *Assuming subexponentially-secure iO and subexponentially secure rerandomizable encryption schemes, there exists a randomized encoding for TMs scheme with overhead* $q(|M|, |x|, T, \lambda) = 2(|M| + |x|) + \mathrm{poly}(\lambda, \log T)$*.*

We get the following corollary.

Corollary 4.4. *Let $0 \leq \epsilon \leq 1/30$ be a constant. Assume subexponential-secure iO, and subexponentially-secure one-way function exist and assume subexponential DDH or LWE. Then for every constant $\alpha > 0$, and for any efficiently computable function s_0, it holds that $\mathrm{Gap_pMK^tP}[s_0(n), (2+\alpha)s_0(n)]$ is not* **NP** *hard with respect to randomized Levin reductions with ϵ-error.*

Proof of Theorem 4.2. For ease of notation, we explain how to modify the proof of Theorem 3.2 to get the proof of Theorem 4.2 for deterministic reductions. Similar changes to the proof of Theorem 3.7 yield the result for randomized reductions.

We only need to change the proof of Lemma 3.6. Let (f, g, h) be the Levin reduction from SAT to $\mathrm{Gap_pMK^tP}[s_0, s_1]$, and assume that for every (ϕ, v) in the support of $\mathcal{D}$, $g(\phi, v)$ output a program of length exactly $s_0(|f(\phi)|)$ that runs in time exactly $t(|f(\phi)|)$ (this can be assume by adding $O(\log t(n) + \log s_0(n))$ bits to the description of $g(\phi, v)$). Let U be a universal TM and (Enc, Dec) be randomized encoding for TMs. Consider the algorithm

$$\mathcal{A}(\phi, v) = h(\phi, \widehat{g(\phi, v)})),$$

where $\widehat{g(\phi, v)}$ is a program that runs Dec on $\widehat{P}$ for $\widehat{P} \leftarrow Enc(1^{|g(\phi,v)|^\alpha}, \mathsf{U}, g(\phi, v), t(|f(\phi)|))$. That is, we replace the iO in the construction of $\mathcal{A}$ from the proof of Lemma 3.6, with a randomized encoding of $\mathsf{U}(g(\phi, v))$. Since for every two witnesses v, v' of ϕ it holds that $\mathsf{U}(g(\phi, v)) = \mathsf{U}(g(\phi, v')) = f(\phi)$, we get that $\widehat{g(\phi, v)}$ and $\widehat{g(\phi, v')}$ are indistinguishable.

By the overhead of the randomized encoding scheme,

$$\left|\widehat{g(\phi, v')}\right| \leq q(|\mathsf{U}|, s_0(n) + O(\log(t(n)) + \log(s_0(n)), \log t(n), |g(\phi, v)|^\alpha).$$

By the efficiency of Dec, the running time of $\widehat{g(\phi, v')}$ is at most $\mathrm{poly}(s_0(|f(\phi)|), t(|f(\phi)|)) = \mathrm{poly}(t(|f(\phi)|))$, where the equality holds since $s_0(|f(\phi)|) \leq |f(\phi)| + O(1)$ or the $\mathrm{Gap_pMK^tP}[s_0, s_1]$ problem is trivial. Thus, by taking p be a polynomial that bound the running time of $\widehat{g(\phi, v')}$, we get that $\widehat{g(\phi, v')}$ is a witness that $f(\phi)$ is not a No instance. The proof continues along the same lines as the proof of Lemma 3.6. □

Acknowledgment

We thank the reviewers for their very valuable comments.

References

[Agr19] Shweta Agrawal. "Indistinguishability obfuscation without multilinear maps: new methods for bootstrapping and instantiation". In: *Advances in Cryptology–EUROCRYPT 2019: 38th Annual International Conference on the Theory and Applications of Cryptographic Techniques, Darmstadt, Germany, May 19–23, 2019, Proceedings, Part I 38*. Springer. 2019, pp. 191–225 (cit. on p. 3).

[AH19] Eric Allender and Shuichi Hirahara. "New insights on the (non-) hardness of circuit minimization and related problems". In: *ACM Transactions on Computation Theory (ToCT)* 11.4 (2019), pp. 1–27 (cit. on pp. 2, 8).

[AJ15] Prabhanjan Ananth and Abhishek Jain. "Indistinguishability obfuscation from compact functional encryption". In: *Annual Cryptology Conference*. Springer. 2015, pp. 308–326 (cit. on p. 19).

[AJLMS19] Prabhanjan Ananth, Aayush Jain, Huijia Lin, Christian Matt, and Amit Sahai. "Indistinguishability obfuscation without multilinear maps: new paradigms via low degree weak pseudorandomness and security amplification". In: *Annual International Cryptology Conference*. Springer. 2019, pp. 284–332 (cit. on p. 3).

[AJS17] Prabhanjan Ananth, Abhishek Jain, and Amit Sahai. "Indistinguishability obfuscation for Turing machines: constant overhead and amortization". In: *Annual International Cryptology Conference*. Springer. 2017, pp. 252–279 (cit. on pp. 5, 19).

[AJS18] Prabhanjan Ananth, Aayush Jain, and Amit Sahai. "Indistinguishability obfuscation without multilinear maps: iO from LWE, bilinear maps, and weak pseudorandomness". In: *Cryptology ePrint Archive* (2018) (cit. on p. 3).

[AKRR11] Eric Allender, Michal Kouckỳ, Detlef Ronneburger, and Sambuddha Roy. "The pervasive reach of resource-bounded Kolmogorov complexity in computational complexity theory". In: *Journal of Computer and System Sciences* 77.1 (2011), pp. 14–40 (cit. on p. 2).

[ALMSS98] Sanjeev Arora, Carsten Lund, Rajeev Motwani, Madhu Sudan, and Mario Szegedy. "Proof verification and the hardness of approximation problems". In: *Journal of the ACM (JACM)* 45.3 (1998), pp. 501–555 (cit. on p. 3).

[APM20] Shweta Agrawal and Alice Pellet-Mary. "Indistinguishability obfuscation without maps: Attacks and fixes for noisy linear FE". In: *Annual International Conference on the Theory and Applications of Cryptographic Techniques*. Springer. 2020, pp. 110–140 (cit. on p. 3).

[AS98] Sanjeev Arora and Shmuel Safra. "Probabilistic checking of proofs: A new characterization of NP". In: *Journal of the ACM (JACM)* 45.1 (1998), pp. 70–122 (cit. on p. 3).

[Bar+01] Boaz Barak, Oded Goldreich, Rusell Impagliazzo, Steven Rudich, Amit Sahai, Salil Vadhan, and Ke Yang. "On the (im) possibility of obfuscating programs". In: *Annual international cryptology conference*. Springer. 2001, pp. 1–18 (cit. on p. 3).

[BDGM20] Zvika Brakerski, Nico Döttling, Sanjam Garg, and Giulio Malavolta. "Factoring and pairings are not necessary for iO: Circular-secure LWE suffices". In: *Cryptology ePrint Archive* (2020) (cit. on p. 3).

[BDGM23] Zvika Brakerski, Nico Döttling, Sanjam Garg, and Giulio Malavolta. "Candidate iO from homomorphic encryption schemes". In: *Journal of Cryptology* 36.3 (2023), p. 27 (cit. on p. 3).

[BG09] Boaz Barak and Oded Goldreich. "Universal arguments and their applications". In: *SIAM Journal on Computing* 38.5 (2009), pp. 1661–1694 (cit. on p. 3).

[Cha69] Gregory J. Chaitin. "On the Simplicity and Speed of Programs for Computing Infinite Sets of Natural Numbers". In: *J. ACM* 16.3 (1969), pp. 407–422 (cit. on p. 2).

[Coo71] Stephen A. Cook. "The Complexity of Theorem-Proving Procedures". In: *Annual ACM Symposium on Theory of Computing (STOC)*. 1971, pp. 151–158 (cit. on p. 3).

[DGKR03] Irit Dinur, Venkatesan Guruswami, Subhash Khot, and Oded Regev. "A new multilayered PCP and the hardness of hypergraph vertex cover". In: *Proceedings of the thirty-fifth annual ACM symposium on Theory of computing*. 2003, pp. 595–601 (cit. on p. 3).

[FFS87] Uriel Fiege, Amos Fiat, and Adi Shamir. "Zero knowledge proofs of identity". In: *Proceedings of the nineteenth annual ACM symposium on Theory of computing*. 1987, pp. 210–217 (cit. on p. 3).

[Gar+16] Sanjam Garg, Craig Gentry, Shai Halevi, Mariana Raykova, Amit Sahai, and Brent Waters. "Candidate indistinguishability obfuscation and functional encryption for all circuits". In: *SIAM Journal on Computing* 45.3 (2016), pp. 882–929 (cit. on p. 3).

[GJLS21] Romain Gay, Aayush Jain, Huijia Lin, and Amit Sahai. "Indistinguishability obfuscation from simple-to-state hard problems: New assumptions, new techniques, and simplification". In: *Annual International Conference on the Theory and Applications of Cryptographic Techniques*. Springer. 2021, pp. 97–126 (cit. on p. 3).

[GLSW15] Craig Gentry, Allison Bishop Lewko, Amit Sahai, and Brent Waters. "Indistinguishability obfuscation from the multilinear subgroup elimination assumption". In: *2015 IEEE 56th Annual Symposium on Foundations of Computer Science*. IEEE. 2015, pp. 151–170 (cit. on p. 3).

[Gol08] Oded Goldreich. *Computational Complexity: A Conceptual Perspective*. 2008 (cit. on p. 3).

[GP21] Romain Gay and Rafael Pass. "Indistinguishability obfuscation from circular security". In: *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*. 2021, pp. 736–749 (cit. on p. 3).

[GR14] Shafi Goldwasser and Guy N Rothblum. "On best-possible obfuscation". In: *Journal of Cryptology* 27.3 (2014), pp. 480–505 (cit. on p. 7).

[Har83] J. Hartmanis. "Generalized Kolmogorov complexity and the structure of feasible computations". In: *24th Annual Symposium on Foundations of Computer Science (sfcs 1983)*. 1983, pp. 439–445. DOI: 10.1109/SFCS.1983.21 (cit. on p. 2).

[Hir22a] Shuichi Hirahara. "NP-hardness of learning programs and partial MCSP". In: *2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE. 2022, pp. 968–979 (cit. on p. 2).

[Hir22b] Shuichi Hirahara. "Symmetry of information from meta-complexity". In: *37th Computational Complexity Conference (CCC 2022)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik. 2022 (cit. on p. 5).

[HIR23] Yizhi Huang, Rahul Ilango, and Hanlin Ren. "NP-Hardness of Approximating Meta-Complexity: A Cryptographic Approach". In: *Cryptology ePrint Archive* (2023) (cit. on pp. 2, 7).

[IKV23a] Russell Impagliazzo, Valentine Kabanets, and Ilya Volkovich. "Synergy Between Circuit Obfuscation and Circuit Minimization". In: *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM 2023)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik. 2023 (cit. on p. 7).

[IKV23b] Russell Impagliazzo, Valentine Kabanets, and Ilya Volkovich. "The power of natural properties as oracles". In: *computational complexity* 32.2 (2023), p. 6 (cit. on p. 2).

[Ila20] Rahul Ilango. "Approaching MCSP from Above and Below: Hardness for a Conditional Variant and $AC^0[p]$". In: *11th Innovations in Theoretical Computer Science Conference (ITCS 2020)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik. 2020 (cit. on p. 2).

[Ila23] Rahul Ilango. "SAT Reduces to the Minimum Circuit Size Problem with a Random Oracle". In: *2023 IEEE 64th Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE. 2023, pp. 733–742 (cit. on pp. 2, 3, 5, 7, 25, 26).

[ILCO20] Rahul Ilango, Bruno Loff, and Igor Carboni Oliveira. "NP-hardness of circuit minimization for multi-output functions". In: *CCC'20: Proceedings of the 35th Computational Complexity Conference*. 2020, pp. 1–36 (cit. on p. 2).

[JLMS19] Aayush Jain, Huijia Lin, Christian Matt, and Amit Sahai. "How to Leverage Hardness of Constant-Degree Expanding Polynomials over R R to build i O i O". In: *Advances in Cryptology–EUROCRYPT 2019: 38th Annual International Conference on the Theory and Applications of Cryptographic Techniques, Darmstadt, Germany, May 19–23, 2019, Proceedings, Part I 38*. Springer. 2019, pp. 251–281 (cit. on p. 3).

[JLS21] Aayush Jain, Huijia Lin, and Amit Sahai. "Indistinguishability obfuscation from well-founded assumptions". In: *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*. 2021, pp. 60–73 (cit. on p. 3).

[Kar72] Richard M. Karp. "Reducibility among Combinatorial Problems". In: *Complexity of Computer Computations*. Ed. by J. W. Thatcher and R. E. Miller. Plenum Press, Inc., 1972, pp. 85–103 (cit. on p. 3).

[KC00] Valentine Kabanets and Jin-yi Cai. "Circuit minimization problem". In: *Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing, May 21-23, 2000, Portland, OR, USA*. 2000, pp. 73–79 (cit. on p. 2).

[KLW15] Venkata Koppula, Allison Bishop Lewko, and Brent Waters. "Indistinguishability obfuscation for turing machines with unbounded memory". In: *Proceedings of the forty-seventh annual ACM symposium on Theory of Computing*. 2015, pp. 419–428 (cit. on pp. 7, 19).

[Ko86] Ker-I Ko. "On the Notion of Infinite Pseudorandom Sequences". In: *Theor. Comput. Sci.* 48.3 (1986), pp. 9–33. DOI: 10.1016/0304-3975(86)90081-2. URL: https://doi.org/10.1016/0304-3975(86)90081-2 (cit. on p. 2).

[Ko91] Ker-I Ko. "On the complexity of learning minimum time-bounded Turing machines". In: *SIAM Journal on Computing* 20.5 (1991), pp. 962–986 (cit. on p. 2).

[Kol68] A. N. Kolmogorov. "Three approaches to the quantitative definition of information". In: *International Journal of Computer Mathematics* 2.1-4 (1968), pp. 157–168 (cit. on p. 2).

[Kom+14] Ilan Komargodski, Tal Moran, Moni Naor, Rafael Pass, Alon Rosen, and Eylon Yogev. "One-way functions and (im) perfect obfuscation". In: *2014 IEEE 55th Annual Symposium on Foundations of Computer Science*. IEEE. 2014, pp. 374–383 (cit. on pp. 4, 7, 11).

[Lev73] Leonid A. Levin. "Universal'nyĭe perebornyĭezadachi (Universal search problems : in Russian)". In: *Problemy Peredachi Informatsii* (1973), pp. 265–266 (cit. on pp. 2, 3).

[Lin16] Huijia Lin. "Indistinguishability obfuscation from constant-degree graded encoding schemes". In: *Advances in Cryptology–EUROCRYPT 2016: 35th Annual International Conference on the Theory and Applications of Cryptographic Techniques, Vienna, Austria, May 8-12, 2016, Proceedings, Part I 35*. Springer. 2016, pp. 28–57 (cit. on p. 3).

[Lin17] Huijia Lin. "Indistinguishability obfuscation from SXDH on 5-linear maps and locality-5 PRGs". In: *Annual International Cryptology Conference*. Springer. 2017, pp. 599–629 (cit. on p. 3).

[LP20] Yanyi Liu and Rafael Pass. "On one-way functions and Kolmogorov complexity". In: *2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE. 2020, pp. 1243–1254 (cit. on p. 8).

[LP22] Yanyi Liu and Rafael Pass. "On One-Way Functions from NP-Complete Problems". In: *37th Computational Complexity Conference*. 2022 (cit. on pp. 2, 3, 5).

[LPST15] Huijia Lin, Rafael Pass, Karn Seth, and Sidharth Telang. "Output-compressing randomized encodings and applications". In: *Theory of Cryptography Conference*. Springer. 2015, pp. 96–124 (cit. on pp. 7, 19).

[LT17] Huijia Lin and Stefano Tessaro. "Indistinguishability obfuscation from trilinear maps and block-wise local PRGs". In: *Annual International Cryptology Conference*. Springer. 2017, pp. 630–660 (cit. on p. 3).

[Lup58] Lupanov. *On a method of circuit synthesis.* Izvestia VUZ Radiofizika, 1(1):120– 140. 1958 (cit. on p. 25).

[LV16] Huijia Lin and Vinod Vaikuntanathan. "Indistinguishability obfuscation from DDH-like assumptions on constant-degree graded encodings". In: *2016 IEEE 57th Annual Symposium on Foundations of Computer Science (FOCS)*. IEEE. 2016, pp. 11–20 (cit. on p. 3).

[MW17] Cody D Murray and R Ryan Williams. "On the (non) NP-hardness of computing circuit complexity". In: *Theory of Computing* 13.1 (2017), pp. 1–22 (cit. on p. 2).

[NY89] Moni Naor and Moti Yung. "Universal One-Way Hash Functions and their Cryptographic Applications". In: *Proceedings of the 21st Annual ACM Symposium on Theory of Computing (STOC)*. 1989, pp. 33–43 (cit. on p. 6).

[PST14] Rafael Pass, Karn Seth, and Sidharth Telang. "Indistinguishability obfuscation from semantically-secure multilinear encodings". In: *Advances in Cryptology–CRYPTO 2014: 34th Annual Cryptology Conference, Santa Barbara, CA, USA, August 17-21, 2014, Proceedings, Part I 34*. Springer. 2014, pp. 500–517 (cit. on p. 3).

[Rom90] John Rompel. "One-Way Functions are Necessary and Sufficient for Secure Signatures". In: *Annual ACM Symposium on Theory of Computing (STOC)*. 1990, pp. 387–394 (cit. on pp. 6, 11).

[RS22] Hanlin Ren and Rahul Santhanam. "A relativization perspective on meta-complexity". In: *39th International Symposium on Theoretical Aspects of Computer Science (STACS 2022)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik. 2022 (cit. on p. 2).

[Sip83] Michael Sipser. "A Complexity Theoretic Approach to Randomness". In: *Proceedings of the 15th Annual ACM Symposium on Theory of Computing (STOC)*. 1983, pp. 330–335 (cit. on p. 2).

[Sol64] R.J. Solomonoff. "A formal theory of inductive inference. Part I". In: *Information and Control* 7.1 (1964), pp. 1 –22. ISSN: 0019-9958. DOI: https://doi.org/10.1016/S0019-9958(64)90223-2 (cit. on p. 2).

[SS20] Michael Saks and Rahul Santhanam. "Circuit lower bounds from NP-hardness of MCSP under Turing reductions". In: *LIPIcs* 169 (2020) (cit. on p. 2).

[SS22] Michael Saks and Rahul Santhanam. "On randomized reductions to the random strings". In: *37th Computational Complexity Conference (CCC 2022)*. Schloss Dagstuhl-Leibniz-Zentrum für Informatik. 2022 (cit. on p. 2).

[Tra84] Boris A Trakhtenbrot. "A survey of Russian approaches to perebor (brute-force searches) algorithms". In: *Annals of the History of Computing* 6.4 (1984), pp. 384–400 (cit. on p. 2).

[Tre01] Luca Trevisan. "Non-approximability results for optimization problems on bounded degree instances". In: *Proceedings of the thirty-third annual ACM symposium on Theory of computing*. 2001, pp. 453–461 (cit. on p. 3).

[WW21] Hoeteck Wee and Daniel Wichs. "Candidate obfuscation via oblivious LWE sampling". In: *Annual International Conference on the Theory and Applications of Cryptographic Techniques*. Springer. 2021, pp. 127–156 (cit. on p. 3).

A [Ila23]'s Reduction is a Levin Reduction (in the ROM)

Ilango [Ila23] show a reduction from τ-Frequency Set Cover to both GapMCSP and $\mathrm{Gap_pMK^tP}$ with a random oracle $\mathcal{O}$. Here we explain why this reduction is a Levin reduction.

Given a witness to the τ-Frequency Set Cover, the construction of a witness for the output of the reduction form $\mathrm{Gap_pMK^tP}$ is straightforward, and the construction of the witness for GapMCSP uses the construction of [Lup58] that can be made efficient (recall that the running time can be polynomial in the truth-table of the circuit).

We briefly explain how the proof in [Ila23] implies that given a witness for the GapMCSP or $\mathrm{Gap_pMK^tP}$ instances that we get from the reduction, we can reconstruct a witness for the τ-Frequency Set Cover instance. Specifically, this can be done by considering the set of queries made by the GapMCSP or $\mathrm{Gap_pMK^tP}$ witnesses to the random oracle $\mathcal{O}$. For concreteness, we focus on the reduction for GapMCSP (the proof for $\mathrm{Gap_pMK^tP}$ is of the same lines).

We start with a short description of the reduction. Given a instance $\phi = (\mathcal{S}_1, \ldots, \mathcal{S}_m \subseteq [n])$, recall that we want to find a small subset of $\mathcal{J} \subseteq [m]$ such that $\cup_{j \in \mathcal{J}} \mathcal{S}_j = [n]$. The reduction samples for each such set $\mathcal{S}_j$ a secret key sk_j, and for every element in $i \in [n]$ a random value v_i. It then finds for each i and for each j such that $i \in \mathcal{S}_j$, a value $c_{i,j}$ such that $\mathcal{O}(i, sk_j, c_{i,j}) = v_i$. Then the truth table that the reduction outputs is the concatenation of $c_{i,j}$ and v_i for all $i \in [n], j \in [m]$.

The hope is that any circuit that computes this truth table will have the values of sk_j hardcoded to it for every j in the minimal cover $\mathcal{J}$. While this is not true, we explain below that (with high probability over the oracle $\mathcal{O}$) we can extract an approximation of $\mathcal{J}$ using the oracle calls the

circuit makes to $\mathcal{O}$. Specifically, for the gap problem used in [Ila23], we need to find a set cover of size smaller than $n/3$.

Let ϕ be a τ-Frequency Set Cover instance, and let $x = f(\phi)$ be the output of the reduction. [Ila23] shows that when ϕ is a No instance, the probability over the choice of $\mathcal{O}$ that any fixed algorithm that makes bounded number of queries to $\mathcal{O}$ can output x is exponentially small in the length of x. Then, by the union bound over all possible small circuits (or program), [Ila23] shows that no such circuit that outputs x exists (with high probability over $\mathcal{O}$). We observe that with the same exponentially small probability, if an algorithm can output x, then we can extract from it a set cover of size smaller than $n/3$. By the same union bound over all circuits, we get that we can extract such a solution from all of the small circuits that output x.

The way the probability of a algorithm A to output x is bounded in [Ila23] by considering the set $skHit$ of all the indexes $j \in [m]$ such that A queried $\mathcal{O}$ on (i, sk_j, c) for some i and c. Then, let $Missed = [n] \setminus \cup_{j \in skHit} \mathcal{S}_j$. Now, if the total size of $skHit$ and $Missed$ is less than $n/3$, we can take $skHit$ together with some trivial cover of $Missed$ as our set cover, and we are done (impotently, $skHit$ and $Missed$ can be computed from the algorithm). We thus left to show that for any algorithm A, the probability that A outputs x and $|skHit|+|Missed| \geq n/3$ is exponentially small.

This follows by the proof in [Ila23]: In the proof of Proposition 37, we can just remove from the first sum terms with $|skHit| + |Missed| < n/3$. Note that by the the information revealed by the third step in the proof, we can compute the sets $skHit$ and $Missed$, and thus we can check if $|skHit| + |Missed| < n/3$ without revealing any new information.

ECCC ISSN 1433-8092

https://eccc.weizmann.ac.il