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Directed Isoperimetry and Monotonicity Testing: A Dynamical Approach

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Abstract

This paper explores the connection between classical isoperimetric inequalities, their directed analogues, and monotonicity testing. We study the setting of real-valued functions $f : [0, 1]^d \to \mathbb{R}$ on the solid unit cube, where the goal is to test with respect to the L^p distance. Our goals are twofold: to further understand the relationship between classical and directed isoperimetry, and to give a monotonicity tester with sublinear query complexity in this setting.

Our main results are 1) an L^2 monotonicity tester for M-Lipschitz functions with query complexity $\widetilde{O}(\sqrt{d}M^2/\epsilon^2)$ and, behind this result, 2) the directed Poincaré inequality $\mathsf{dist}_2^{\mathsf{mono}}(f)^2 \leq C\,\mathbb{E}\left[|\nabla^- f|^2\right]$, where the "directed gradient" operator ∇^- measures the local violations of monotonicity of f.

To prove the second result, we introduce a partial differential equation (PDE), the *directed heat equation*, which takes a one-dimensional function f into a monotone function f^* over time and enjoys many desirable analytic properties. We obtain the directed Poincaré inequality by combining convergence aspects of this PDE with the theory of optimal transport. Crucially for our conceptual motivation, this proof is in complete analogy with the mathematical physics perspective on the classical Poincaré inequality, namely as characterizing the convergence of the standard heat equation toward equilibrium.

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1 Introduction

One of the central problems in the field of property testing is *monotonicity testing*: given a function f defined over some partially ordered domain, decide whether f is *monotone*, i.e. $f(x) \le f(y)$ whenever $x \preceq y$, or ϵ-*far* from any monotone function under a given distance metric. Since the introduction of this problem [GGLRS00] and especially over the last decade, a series of works has revealed striking connections between monotonicity testing and directed analogues of well-studied and ubiquitous isoperimetric inequalities such as Poincaré, Margulis, and Talagrand inequalities.

Let $p, q \ge 1$. Following the notation of [Fer23], we say a (classical) (L^p, ℓ^q)-*Poincaré inequality* is an inequality of the form

$$\mathsf{dist}_p^{\mathsf{const}}(f)^p \le C\,\mathbb{E}\left[\|\nabla f\|_q^p\right]$$

for, say, all functions $f : \{0,1\}^d \to \mathbb{R}$ on the Boolean cube, or all functions $f : [0,1]^d \to \mathbb{R}$ on the unit cube, or perhaps only the Boolean-valued functions on these domains. Here $\mathsf{dist}_p^{\mathsf{const}}(f)$ denotes the L^p distance of f to the closest constant function when the domain is given uniform probability measure. For example,

1. for functions $f : \{0,1\}^d \to \mathbb{R}$, the classical Poincaré inequality $\mathrm{Var}[f] \le C\,\mathrm{Inf}[f]$, where $\mathrm{Inf}[f]$ denotes the total influence of f, is an (L^2, ℓ^2)-Poincaré inequality (see e.g. [O'D14]);

2. in the same setting, the Talagrand inequality [Tal93] is an (L^1, ℓ^2) inequality;

3. for smooth functions $f : [0,1]^d \to \mathbb{R}$, the (L^2, ℓ^2) inequality is often called *the* Poincaré inequality, especially in mathematical analysis (see e.g. [BGL14]); and

4. in the same setting, the (L^1, ℓ^2) inequality was proved by Bobkov & Houdré [BH97].

A series of works on monotonicity testing has shown that many of these inequalities enjoy natural "directed analogues", as identified by [CS16]. Let $\mathsf{dist}_p^{\mathsf{mono}}(f)$ denote the L^p distance of f to the closest monotone function. Then a *directed* (L^p, ℓ^q)-*Poincaré inequality* is an inequality of the form

$$\mathsf{dist}_p^{\mathsf{mono}}(f)^p \le C\,\mathbb{E}\left[\|\nabla^- f\|_q^p\right] ,$$

where $\nabla^- f := \min\{0, \nabla f\}$, the *directed gradient* of f, captures the local violations of monotonicity[1]. For example (listing by type of inequality rather than in historical order),

1. an (L^1, ℓ^1) inequality was proved by [GGLRS00] for functions[2] $f : \{0,1\}^d \to \{0,1\}$;

2. [Fer23] gave the same inequality for Lipschitz functions $f : [0,1]^d \to \mathbb{R}$;

3. [KMS18] proved the stronger (L^1, ℓ^2) inequality for functions $f : \{0,1\}^d \to \{0,1\}$, in analogy with Talagrand's result in the classical setting; and

4. related directed isoperimetric statements, not in Poincaré form, include a directed analogue of the Margulis inequality by [CS16], and isoperimetric inequalities for real-valued functions on the Boolean cube [BKR24] and for Boolean functions on the hypergrid [BCS23; BKKM23].

[1] Note that both the classical and directed inequalities relate a local property (violations of the "constant" or "monotone" property, captured by the gradient) to a global one (distance to a constant or monotone function).

[2] For the purposes of exposition, we limit the present discussion to the Boolean cube and unit cube domains; in particular, we do not extend the notation and discussion to hypergrid domains.

What makes the story above especially compelling is that each new directed isoperimetric inequality also enabled an *algorithmic result*, namely a monotonicity tester with improved query complexity in the same setting. While we refrain from a full review and refer to [Bla23] instead, one central example is that, for functions $f : \{0,1\}^d \to \{0,1\}$, the celebrated work of [KMS18] essentially resolved the question for nonadaptive testers by giving a $\widetilde{O}(\sqrt{d}/\epsilon^2)$ query tester.

In this paper, we seek to further understand the connection between classical and directed isoperimetry, and the role of the latter in monotonicity testing. Building upon [Fer23], we pursue this goal by studying the fully continuous setting, namely functions $f : [0,1]^d \to \mathbb{R}$. Let us offer two reasons for this choice, which we view as connections along two conceptual "axes":

> **Classical versus directed.** The continuous setting is central to the study of isoperimetric phenomena. Ever since the original "isoperimetric problem" about shapes in Euclidean space, isoperimetric inequalities have enjoyed a rich and fruitful history with connections to mathematical physics, geometry, probability theory, diffusion processes, optimal transport, and so on; and closest to our subject, the Poincaré inequality itself first appeared in the study of partial differential equations arising in mathematical physics [Poi90]. Thus, if there are unifying principles underlying both classical and directed isoperimetric phenomena, it seems reasonable to expect such a principle to manifest itself in the continuous setting.

> **Discrete versus continuous.** Phenomena involving functions on Boolean and continuous domains are often intimately related[3], which we may interpret as a form of "robustness" of the phenomena. Thus, it is natural to ask about the full scope of this connection in the case of directed isoperimetry and monotonicity testing. While [Fer23] started to answer this question by giving an L^1 monotonicity tester with $O(d)$ query complexity via a directed (L^1, ℓ^1)-Poincaré inequality, many questions remain. For example, they left open the possibility of a tester with $O(\sqrt{d})$ query complexity, which would bring the continuous landscape closer to the discrete one.

Our main result in this paper is a directed (L^2, ℓ^2)-Poincaré inequality for sufficiently regular functions $f : [0,1]^d \to \mathbb{R}$ (in particular, Lipschitz continuity suffices; see Section 5.1):

Theorem 1.1 (Directed Poincaré inequality)**.** *There exists a universal constant $C > 0$ such that, for all $f \in H^1((0,1)^d)$,*

$$\mathsf{dist}_2^{\mathsf{mono}}(f)^2 \le C\,\mathbb{E}\left[\|\nabla^- f\|_2^2\right] . \tag{1}$$

We highlight two related aspects of this result. First, it takes the same form as the most classical form of the Poincaré inequality, namely the (L^2, ℓ^2) form. In contrast, [Fer23] gave an (L^1, ℓ^1) inequality, which does not have a natural classical counterpart[4].

Second, the main theme of our proof of this result is the study of the *convergence of a partial differential equation* (PDE), which is the original motivating problem for the Poincaré inequality. The idea is to take one of the central properties of the Poincaré inequality—that it characterizes the convergence of the *heat equation* to equilibrium—and modify that PDE in a natural way so that it converges to a monotone (rather than constant) equilibrium, then derive from its (exponential) convergence the directed inequality. In fact, plugging in the unmodified heat equation into our proof recovers a long but unsurprising proof of the classical Poincaré inequality. We contend that, in the continuous setting, exponential convergence of a PDE to its constant/monotone equilibrium is the unifying principle behind the classical and directed inequalities.

[3] See e.g. [O'D14, Chapter 11] and recent works such as [KOW16; CHHL19; DNS21; AHLVXY23; EMR23].

[4] In the classical settings, as noted above, we have the stronger (L^1, ℓ^2) inequality instead. Note that the ℓ^2 (Euclidean) norm enjoys many nice properties that the ℓ^1 norm does not (e.g. rotation invariance, being self-dual).

As an application, we obtain a monotonicity tester for Lipschitz functions $f : [0,1]^d \to \mathbb{R}$, with respect to the L^2 distance, using roughly $\sqrt{d}$ queries:

Theorem 1.2. *There exists a nonadaptive, directional derivative L^2 monotonicity tester for M-Lipschitz functions $f : [0,1]^d \to \mathbb{R}$ with query complexity $\widetilde{O}(\sqrt{d}M^2/\epsilon^2)$ and one-sided error.*

This result answers affirmatively (up to a logarithmic factor) the question asked by [Fer23]. We remark that the algorithm above is also an L^p tester for any $p \in [1,2]$, that the directional derivative queries may be replaced by value queries as long as f is sufficiently smooth, and that the dependence on d is optimal among a natural generalization of pair testers to the continuous setting; see the next section for definitions and details.

The rest of the introduction proceeds as follows. In Section 1.1, we define our testing model and describe our monotonicity tester; in Section 1.2 we give an overview of our proof of the directed Poincaré inequality; and in Section 1.3, we discuss our results and some open questions.

1.1 Monotonicity testing of Lipschitz functions

In this section, let $\Omega := [0,1]^d$ for simplicity of notation. For a function $f \in L^p(\Omega)$ and $p \geq 1$, we define the L^p distance to monotonicity of f as

$$\mathsf{dist}_p^{\mathsf{mono}}(f) := \inf \left\{ \|f - g\|_{L^p(\Omega)} : g \in L^p(\Omega) \text{ monotone} \right\} ,$$

where $\|f - g\|_{L^p(\Omega)} = \left(\int_\Omega (f-g)^p \, \mathrm{d}x \right)^{1/p}$ by definition.

As announced, our main algorithmic result is a monotonicity tester with respect to the L^2 distance, so we adopt the L^p testing model of [BRY14]. The specific problem we consider—L^p testing monotonicity of Lipschitz functions—is the same as [Fer23]. Let us formally define the model. We say that function f is M-Lipschitz if $|f(x) - f(y)| \leq M|x - y|$ for all pairs of points x, y. Then, the testing model is as follows:

Definition 1.3 (Monotonicity tester)**.** Let $p \geq 1$. Given parameters $d \in \mathbb{N}$ and $M, \epsilon > 0$, we say a randomized algorithm A is an L^p *monotonicity tester* for M-Lipschitz functions with query complexity $m(d, M, \epsilon)$ if, for every M-Lipschitz input function $f : [0,1]^d \to \mathbb{R}$, algorithm A makes $m(d, M, \epsilon)$ oracle queries to f and 1) accepts with probability at least $2/3$ if f is monotone; 2) rejects with probability at least $2/3$ if $\mathsf{dist}_p^{\mathsf{mono}}(f) \geq \epsilon$.

As usual, an algorithm is *nonadaptive* if it decides all its queries in advance before seeing any output from the oracle, and it has *one-sided error* if it accepts monotone functions with probability 1. As in [Fer23], we allow two types of oracles queries:

Value query: Given point $x \in \Omega$, the oracle outputs the value $f(x)$.

Directional derivative query: Given point $x \in \Omega$ and direction $v \in \mathbb{R}^d$, the oracle outputs the directional derivative $\nabla_v f(x) = v \cdot \nabla f(x)$, or the symbol $\perp$ if f is not differentiable at x.

We remark that Lipschitz functions on Ω are differentiable *almost everywhere* (i.e. outside a set of measure zero) by Rademacher's theorem, so non-differentiability is not a concern, and they are bounded and hence in $L^p(\Omega)$ for any $p \geq 1$, so $\mathsf{dist}_p^{\mathsf{mono}}(f)$ is always well-defined.

The algorithm. Our tester may be seen as the natural continuous analogue of the path tester of [KMS18] for the Boolean cube, which samples points $x \preceq y \in \{0,1\}^d$ connected by a path of length 2^k, for k sampled uniformly from $[\log d]$, and rejects if $f(x) > f(y)$. In our case, a path is replaced by a *direction* $v \in \{0,1\}^d$, and the condition $f(x) > f(y)$ is replaced by the condition $v \cdot \nabla f(x) < 0$, for a uniformly random point $x \in [0,1]^d$, via a directional derivative query. Such a tester accepts any monotone function, so the challenge is to ensure that it detects the case when $\mathsf{dist}_2^{\mathsf{mono}}(f) \geq \epsilon$.

In this case, hiding constant factors, Theorem 1.1 gives that $\mathbb{E}\left[\|\nabla^- f\|_2^2\right] \gtrsim \epsilon^2$. We would like the distribution of v to be such that, for any x, $\mathbb{P}_v\left[v \cdot \nabla^- f(x) < 0\right] \gtrsim \frac{\|\nabla^- f(x)\|_2^2}{\sqrt{d}\|\nabla f(x)\|_2^2}$. Note that $\|\nabla f(x)\|_2^2 \leq M^2$ because f is M-Lipschitz, so if we had such a distribution for v, the tester would reject with probability at least $\mathbb{E}_x\left[\frac{\|\nabla^- f(x)\|_2^2}{\sqrt{d}M^2}\right] \gtrsim \frac{\epsilon^2}{\sqrt{d}M^2}$, implying that $O(\sqrt{d}M^2/\epsilon^2)$ queries suffice.

A natural choice would be to sample $v \in \{0,1\}^d$ so that $v_i = 1$ corresponds to taking edge i in the path tester. We make this concrete as follows: sample p uniformly from $\left\{1, \frac{1}{2}, \dots, \frac{1}{2^{\lceil \log_2(4d) \rceil}}\right\}$, and sample $v \in \{0,1\}^d$ by letting each v_i be 1 independently with probability p. At a high level, Lemma 3.4 shows that this distribution satisfies the condition above (up to a logarithmic factor) by considering, for each *threshold* τ, the smallest value of p such that, informally, after issuing all its queries the algorithm should expect to sample at least one vector v whose support intersects with at least one entry i from $\nabla f(x)$ satisfying $(\nabla f(x))_i < -\tau$ (this idea allows us to "forget" about all the other negative entries of $\nabla f(x)$, and focus on a simpler "good event"). By ranging over all possible τ, we show that there exists τ and corresponding p such that, with good probability, the contribution of the positive entries of $\nabla f(x)$ to $v \cdot \nabla f(x)$ is smaller than τ, so that $v \cdot \nabla f(x) < 0$.

Remarks on the tester. As observed in [BRY14], for $1 \leq p \leq q$ and any (say) Lipschitz f and g, Jensen's inequality gives $\|f - g\|_{L^p(\Omega)} \leq \|f - g\|_{L^q(\Omega)}$. Thus any L^q monotonicity tester is also an L^p tester, so that the $\widetilde{O}(\sqrt{d}M^2/\epsilon^2)$ upper bound also holds for L^p testing with $p \in [1,2]$.

A second remark is that the directional derivative queries, while convenient for the analysis and in our opinion conceptually clean, can be replaced with value queries under reasonable assumptions. First, in Remark 3.3 we note that the rejection condition $v \cdot \nabla f(x) < 0$ may in fact be replaced with the more robust condition $v \cdot \nabla f(x) < -\Theta(\epsilon/d)$. Now, suppose the input function f is promised to be twice-differentiable with second derivatives bounded by any constant β. We may then replace any directional derivative query $v \cdot \nabla f(x)$ with the value queries $f(x)$ and $f(y)$, where $y = x + \alpha v$ for sufficiently small $\alpha > 0$ as a function of β, ϵ and d, and reject if $f(y) < f(x)$. This is possible because, if $v \cdot \nabla f(x) < -\Theta(\epsilon/d)$ and for such small α, the directional derivative $v \cdot \nabla f(z)$ remains negative on all points z in the line segment between x and y, and hence $f(y) < f(x)$. Conceptually, we view this as confirmation that directional derivative queries are not unreasonably powerful.

Finally, the $\sqrt{d}$ dependence in the query complexity is optimal for *derivative-pair testers*, which are randomized nonadaptive testers that, at each step, either sample points $x \preceq y$ and use value queries to reject if $f(x) > f(y)$, or sample point x and direction $v \succeq 0$ and use a directional derivative query to reject if $v \cdot \nabla f(x) < 0$. The proof is not difficult, and is given in Section 4.

1.2 Proof overview

Our guiding ideal is to prove Theorem 1.1 by identifying a robust approach to the classical Poincaré inequality such that, by "toggling" a single aspect of the approach, we can transform a proof of the classical statement into a proof of its directed counterpart.

1.2.1 Starting point: the heat equation

The first step is to identify the right classical starting point. For example, the Poincaré inequality can be proved using Fourier analysis, but since the directed problem is highly nonlinear (due to the ∇^- operator), this approach does not seem suitable. Instead, we take a physically-motivated approach. Let $u = u(t, x)$ where we think of the first variable as time, with each $f = u(t) = u(t, \cdot)$ a function over space. A remarkable property (see e.g. [BGL14, Chapter 4]) of the Poincaré inequality

$$\mathrm{Var}\left[f\right] \le C\,\mathbb{E}\left[\|\nabla f\|_2^2\right]$$

(where we have written $\mathrm{Var}\left[f\right]$ in the place of $\mathsf{dist}_2^{\mathsf{const}}(f)^2$, which is the same thing) is that it is equivalent to exponential decay of variance in the heat equation

$$\partial_t u = \Delta u\,, \tag{2}$$

where Δ denotes the Laplacian operator, given appropriate "no-flux" boundary conditions which we do not expand on at the moment. To see this equivalence, assume for simplicity the mean zero condition $\int u(0)\,\mathrm{d}x = \int u(0, x)\,\mathrm{d}x = 0$ (which is then preserved over time), compute $\partial_t \mathrm{Var}\left[u(t)\right]$, differentiate under the integral, apply (2) and integrate by parts to obtain

$$\partial_t \mathrm{Var}\left[u(t)\right] = \partial_t \int u(t)^2\,\mathrm{d}x = 2\int u(t)\partial_t u(t)\,\mathrm{d}x = 2\int u(t)\Delta u(t)\,\mathrm{d}x = -2\int \nabla u(t)\cdot\nabla u(t)\,\mathrm{d}x\,.$$

This means that $\partial_t \mathrm{Var}\left[u(t)\right] \le -C\,\mathrm{Var}\left[u(t)\right]$ if and only if $\mathrm{Var}\left[u(t)\right] \le \frac{2}{C}\int \|\nabla u(t)\|_2^2\,\mathrm{d}x$, i.e. exponential decay of the variance is equivalent to the Poincaré inequality.

We observe that solutions to the heat equation converge to a constant equilibrium (taking the average value of u), and the associated Poincaré inequality is a bound on the distance to a constant function. Accordingly, it seems intuitive that if we replace (2) with a PDE that converges to a *monotone* function, we might learn something about the distance to a monotone function instead.

1.2.2 The directed heat equation

One challenge we face is that analyzing directed analogues of (2) in the multidimensional setting proves challenging due to the interaction between nonlinearities in multiple dimensions. Hence, let us focus on the one-dimensional case for now. In one dimension, the heat equation is

$$\partial_t u = \partial_x \partial_x u\,.$$

The surface reading of this PDE is that, if we focus on the value of u at a single point x, the PDE tells us how this value changes over time. Another useful perspective is to think of $u(t, x)$ as the density of "particles" at each point, and ask about how these particles are "moving" over time; this idea can be made formal via the so-called *continuity equation*. Under this view, it turns out that the inner expression $\partial_x u$ represents (up to a sign change) the *flux* (or momentum) field, i.e. the total rate of particle movement at each point, and by taking the derivative of this field, i.e. $\partial_x \partial_x u$, we determine whether, on the balance, there are more "incoming" or "outgoing" particles—quantitatively, the rate of change $\partial_t u$.

With this perspective, a natural candidate PDE—which we call the *directed heat equation*—is

$$\partial_t u = \partial_x \partial_x^- u\,, \tag{3}$$

where $\partial_x^- u := \min\{0, \partial_x u\}$. The idea is that the new flux $\partial_x^- u$ is always nonpositive, which, up to a required sign change, means that particles are only allowed to "move to the right". Observe also that the RHS of (3) is zero for any monotone function, so monotone functions are stationary solutions, while for decreasing functions, this PDE behaves exactly as the heat equation. Thus, intuitively, this PDE seems to take a function $u(0)$ and move it toward a monotone limit over time.

We explained above that the Poincaré inequality is intimately connected to convergence of the heat equation, and we would like to show a similar property in the directed case. One option would be to study the rate of decay of the distance to monotonicity, $\mathsf{dist}_2^{\mathsf{mono}}$. Although this is possible, it does not seem to lead to a proof strategy; also note that, while in the case of the variance we also have tools such as Fourier analysis to reason about the rate of decay directly, here we only have the PDE to work with. Fortunately, there is another relevant quantity which decays exponentially under the heat equation, namely the *Dirichlet energy*

$$\mathcal{E}(f) = \frac{1}{2} \int (\partial_x f)^2 \, \mathrm{d}x \,.$$

Indeed, another formal computation shows that by differentiating under the integral, switching the partial derivatives, applying (2) and integrating by parts, we get

$$\begin{aligned}\partial_t \mathcal{E}(u(t)) &= \frac{1}{2}\partial_t \int (\partial_x u(t))^2 \, \mathrm{d}x = \int (\partial_x u(t))(\partial_t \partial_x u(t)) \, \mathrm{d}x = \int (\partial_x u(t))(\partial_x \partial_t u(t)) \, \mathrm{d}x \\ &= \int (\partial_x u(t))(\partial_x \partial_x \partial_x u(t)) \, \mathrm{d}x = -\int (\partial_x \partial_x u(t))^2 \, \mathrm{d}x \le -\frac{1}{C} \int (\partial_x u(t))^2 \, \mathrm{d}x = -\frac{2}{C} \mathcal{E}^-(u(t)) \,,\end{aligned}$$

where the inequality is an application of a version of the Poincaré inequality for functions that are zero on the boundary, which will be the case for $\partial_x u(t)$ by the no-flux boundary conditions.

Now, it seems reasonable to propose the following directed analogue of the Dirichlet energy:

$$\mathcal{E}^-(f) = \frac{1}{2} \int (\partial_x^- f)^2 \, \mathrm{d}x \,.$$

If $\mathcal{E}$ measures the "locally non-constant" activity of a function, $\mathcal{E}^-$ measures the "locally non-monotone" activity. Using $\mathcal{E}^-$ to recast the directed heat equation in the language of gradient flows and maximal monotone operators [Bre73], we can show that

1. this PDE has a solution (Proposition 5.17 and Corollary 5.20);
2. the directed Dirichlet energy decays exponentially in time (Proposition 5.43);
3. the solution converges to a monotone equilibrium as $t \to \infty$ (Proposition 5.65); and
4. the solution, including up to the limit above, satisfies several other essential analytic properties, such as nonexpansiveness (Proposition 5.69) and order preservation (Corollary 5.68).

1.2.3 Transport-energy inequality in one dimension

It is tempting, now, to try and use these results to conclude the one-dimensional directed Poincaré inequality. The problem is that this seems to lead to a dead end, because we do not know how to *tensorize* the one-dimensional inequality into a multidimensional one[5].

Instead, we keep leaning on what the PDEs naturally tell us. It turns out that there is a different notion of distance, the *Wasserstein* distance, which is much more closely connected to

[5] In the classical case, the *law of total variance* does allow such tensorization [BGL14, Chapter 4.3], but this aspect of the problem does not seem to be robust to passing to the directed setting. Also note that, if we only desired a one-dimensional directed Poincaré inequality, then [Fer23] offers a much shorter proof.

the theory of evolution equations and better suited to the dynamical approach we are undertaking. Informally, the squared Wasserstein distance $W_2^2(\varrho_0, \varrho_1)$ between probability measures ϱ_0 and ϱ_1 is the minimum total cost of a "transport plan" (coupling) moving particles from ϱ_0 to ϱ_1, where the cost of moving a particle from point x to point y is $|x-y|^2$. The connection to PDEs is via the *Benamou-Brenier formula*, which informally says that

$$W_2^2(\varrho_0, \varrho_1) = \min \left\{ \int_0^1 \|v_t\|^2_{L^2(\varrho_t)} \,\mathrm{d}t : v_t \text{ is a velocity field taking } \varrho_0 \text{ to } \varrho_1 \text{ from time 0 to 1} \right\} .$$

Without going into details, it follows that if we can upper bound the directed Dirichlet energy $\mathcal{E}^-(u(t))$, then we can upper bound the total (weighted) magnitude of the velocity field, i.e. $\|v_t\|^2_{L^2(\varrho_t)}$ (recall that the momentum of particles in the directed heat equation is essentially $\partial_x^- u$, so this connection is not arbitrary), and therefore upper bound the Wasserstein distance between the initial state ϱ_0—which informally corresponds to the function $f = u(0)$—and the final state ϱ_1—which informally corresponds to the monotone equilibrium $f^* = \lim_{t\to\infty} u(t)$.

This strategy yields, at least for bounded functions, the following result. Let $\mathcal{U}$ be a class of "reasonable" initial states which we do not define for now, and let the operator P_∞ map each initial state to its monotone equilibrium according to the directed heat equation. Then

Theorem 6.15 (Transport-energy inequality in one dimension)**.** *There exists a constant $C > 0$ such that the following holds. Let $u \in \mathcal{U}$ be positive, bounded away from zero, and satisfy $\int_{(0,1)} u \,\mathrm{d}x = 1$. Define the measures $\mathrm{d}\mu := u\,\mathrm{d}x$ and $\mathrm{d}\mu_\infty := (P_\infty u)\,\mathrm{d}x$. Then*

$$W_2^2(\mu, \mu_\infty) \le \frac{C}{\inf u} \mathcal{E}^-(u) .$$

1.2.4 Tensorizing the transport-energy inequality

The advantage of making this detour through the Wasserstein distance is that, modulo the required technical work, tensorization becomes relatively straightforward. Indeed, it is well-known in the theory of optimal transport (c.f. [Vil09, Remark 6.6]) that the W_2 distance scales well with the dimension and reflects useful geometric content. Here the main idea is that, since the cost function for the W_2 distance is the squared Euclidean distance $|x-y|^2$ between points $x, y \in [0,1]^d$, one way to extend the one-dimensional result above to, say, the unit square $[0,1]^2$ is to

1. apply the one-dimensional result to each row, making the row restrictions monotone while paying cost $W_2^2(\varrho_0, \varrho_1) = a^2$; and then

2. apply the one-dimensional result to each column, making the column restrictions monotone while paying cost $W_2^2(\varrho_1, \varrho_2) = b^2$.

Then, by combining the transport of particles along rows and columns into a single transport plan, we obtain via the Pythagorean theorem a plan with cost $a^2 + b^2$, so this quantity upper bounds the overall squared distance $W_2^2(\varrho_0, \varrho_2)$. The same idea extends to higher dimensions, and we prove:

Lemma 6.34 (Pythagorean composition of transport plans)**.** *Let $I, J \subseteq [d]$ be nonempty disjoint sets. Let $\mu, \varrho, \nu \in P(\mathbb{R}^d)$ be supported in bounded sets, and let $\gamma^+ \in \Pi_I(\mu \to \varrho)$ and $\gamma^- \in \Pi_J(\varrho \to \nu)$. Then there exists $\gamma \in \Pi_{I\cup J}(\mu \to \nu)$ satisfying $C_2(\gamma)^2 = C_2(\gamma^+)^2 + C_2(\gamma^-)^2$.*

In this statement, $C_2(\gamma)^2$ is the cost of transport plan γ, and $\Pi_I(\mu \to \nu)$ denotes the set of transport plans between probability measures μ to ν, under two additional restrictions: 1) particles can only move along "I-aligned" lines, e.g. along rows in the example; and 2) particles can only move "up" in the partial order on $\mathbb{R}^d$, i.e. a particle can move from x to y only if $x \preceq y$—we call such a transport plan *directed*. The second restriction is important because, when we finally recover a Poincaré inequality from a transport-energy inequality, the information that particles only moved up in the partial order will be reflected in the appearance of the directed gradient ∇^-.

Now, informally, Theorem 6.15 tells us that each step of the above strategy (e.g. moving particles along rows) incurs cost bounded by the directed Dirichlet energy, i.e. the integral $\int (\partial_i^- f)^2 \, \mathrm{d}x$ of the partial derivatives along the (say) rows. Repeating for every $i \in [d]$ via Lemma 6.34, we obtain the directed, multidimensional transport-energy inequality:

Theorem 6.51 (Transport-energy inequality)**.** *There exists a universal constant $C > 0$ such that the following holds. Let $a \in (0,1)$, and let $f \in \mathsf{Lip}$ satisfy $1-a \le f \le 1+a$ and $\int_{[0,1]^d} f \, \mathrm{d}x = 1$. Define the probability measures $\mathrm{d}\mu := f \, \mathrm{d}x$ and $\mathrm{d}\mu^* := f^* \, \mathrm{d}x$ on $[0,1]^d$. Then*

$$W_2^2(\mu \to \mu^*) \le \frac{C(1+a)^2}{(1-a)^3} \int_{[0,1]^d} \left|\nabla^- f\right|^2 \mathrm{d}x \, .$$

Moreover, by the order preservation property of our solution to our PDE, each operation along direction i preserves the monotonicity along the directions $j < i$, so the final function f^* in the statement above, which is the result after all d steps, is indeed monotone. The notation $W(\mu \to \mu^*)$ indicates that this bound holds even for directed transport plans, as explained above.

1.2.5 Recovering a Poincaré inequality via optimal transport duality

The final step is to recover, from the multidimensional transport-energy inequality, our desired directed Poincaré inequality. Fortunately, there is an intimate connection between transport and Poincaré inequalities (as well as Sobolev, logarithmic Sobolev, and other isoperimetric and concentration inequalities), and this theory is well established in the classical case (c.f. [Vil09, Chapters 21 and 22]). Closest to the present approach, [Liu20] proved, in the classical setting, the *equivalence* between transport-energy, Poincaré, and other related inequalities. Our task, therefore, is to obtain at least an implication in the directed setting.

The main ingredient toward this goal is the notion of *Kantorovich duality* for the Wasserstein distance. In the classical setting, the weak Kantorovich duality says that, given two probability measures μ and ν, if we can find "certificate functions" ϕ and ψ satisfying

$$\phi(y) - \psi(x) \le |x-y|^2 \qquad \text{for all points } x \text{ and } y, \tag{4}$$

then

$$W_2^2(\mu, \nu) \ge \int \phi(y) \, \mathrm{d}\nu(y) - \int \psi(x) \, \mathrm{d}\mu(x) \, . \tag{5}$$

Villani recounts the following analogy by Caffarelli: suppose you operate a consortium of bakeries and cafes in a city, and that $\mu(\mathrm{d}x)$ is the supply of bread by a bakery at point x while $\nu(\mathrm{d}y)$ is the demand by a cafe at point y. Then if transporting a unit of bread from x to y costs $|x-y|^2$, the minimum cost for transporting all the bread is by definition $W_2^2(\mu, \nu)$. Suppose a transportation company offers to take over the transportation job by buying bread from the bakeries at price $\psi(x)$ and selling it to cafes at price $\phi(y)$. If (4) holds then, for each x and y, hiring the transportation company is no more expensive than handling the transportation yourself. Therefore the cost $W_2^2(\mu, \nu)$ is at least as large as how much the bakery-cafe consortium pays the company to fulfill the supply and demand, which is the RHS of (5).

In the directed setting, where we can only transport mass from x to y if $x \preceq y$, we expect the transport to be in general more expensive, so "more" certificates should be valid. Indeed, it is not difficult to obtain the following natural directed version of weak duality (Lemma 7.1): if

$$\phi(y) - \psi(x) \le |x-y|^2 \qquad \text{for all points } x \preceq y,$$

then

$$W_2^2(\mu \to \nu) \ge \int \phi(y)\,\mathrm{d}\nu(y) - \int \psi(x)\,\mathrm{d}\mu(x)\,.$$

The final main ingredient is an operator which, given a candidate function ϕ, produces the best possible ψ for duality. This is accomplished by a directed analogue of the so-called *Hamilton-Jacobi operator*, defined as follows: for each function h on the unit cube,

$$(\vec{H}_t h)(x) := \begin{cases} h(x) & \text{if } t = 0 \\ \sup_{y \succeq x} \left\{ h(y) - \frac{1}{2t}|x-y|^2 \right\} & \text{otherwise.} \end{cases}$$

Then, for each h and setting $t = 1$, this operator yields the following (Proposition 7.4):

$$\tfrac{1}{2} W_2^2(\mu \to \nu) \ge \int h\,\mathrm{d}\nu - \int (\vec{H}_1 h)\,\mathrm{d}\mu\,. \tag{6}$$

Note that, as $t \to 0^+$, $(\vec{H}_t h)(x)$ intuitively seeks the direction $y - x \succeq 0$ of steepest ascent of h. This suggests a connection to the directed gradient, and indeed in Proposition 7.6 we show that

$$\limsup_{t \to 0^+} \frac{\vec{H}_t h(x) - h(x)}{t} \le \frac{|\nabla^+ h(x)|^2}{2}\,, \tag{7}$$

where $\nabla^+ h := \max\{0, \nabla h\}$.

At this point, instead of trying to reproduce the calculations from Section 7.2, let us give some intuition for how the pieces above recover a Poincaré inequality. By homogeneity, it suffices to consider mean-zero functions h. The key idea is to fix a small $t > 0$ and let h play two roles at the same time: 1) as the building block for measures $\mathrm{d}\mu(x) = (1 + th)\,\mathrm{d}x$ and $\mathrm{d}\mu^*(x) = (1 + th^*)\,\mathrm{d}x$, where h^* is the monotone function obtained via the coordinate-wise application of the directed heat equation as in the previous section; and 2) in the test function $-th$ for duality via (6).

From the fact that h plays both of these roles[6], (6) ends up producing the "interaction term" $\int (h^2 - hh^*)\,\mathrm{d}x$, which can be appropriately bounded by $\int (h - h^*)^2\,\mathrm{d}x$; and since h^* is monotone, the inequality $\int (h - h^*)\,\mathrm{d}x \ge \mathsf{dist}_2^{\mathsf{mono}}(h)^2$ explains why we should expect the distance to monotonicity to appear. Moreover, after the appropriate calculations, the term involving $\vec{H}$ in (6) gives rise to the expression in (7), so letting $t \to 0^+$ makes the directed gradient of h appear. Then $W_2^2(\mu, \mu^*)$ can be lower bounded by an expression involving $\mathsf{dist}_2^{\mathsf{mono}}(h)^2$ and $\int |\nabla^- h|^2\,\mathrm{d}x$. On the other hand, Theorem 6.51 gives that $W_2^2(\mu, \mu^*)$ is upper bounded by an expression involving $\int |\nabla^- h|^2\,\mathrm{d}x$. Therefore, chaining the inequalities, we have precisely the terms required to put (1) together.

1.3 Discussion and open questions

Conceptual and technical aspects. We consider the dynamical approach to Theorem 1.1—which establishes that the convergence properties of a PDE underlies both directed and classical isoperimetric statements—to be the main conceptual contribution of this work. The role played by

[6] We may think of test function $-th$ as the transportation company trying to profit from non-monotonicity of h.

optimal transport speaks to the intimate relation between optimal transport and such dynamical processes, and the fact that some of the optimal transport theory seems to find natural directed counterparts is also intriguing and could be of independent interest.

Much of our technical effort is in dealing with the nonlinear nature of the ∇^- operator. For example, in principle this rules out well-know Fourier analytic arguments. In fact, even many of the tools from nonlinear PDEs fail to apply at first, because they require some sort of coercivity that is not satisfied in our setting—at a very informal level, because $\partial_x^- u$ may remain at zero even as $\partial_x u$ grows arbitrarily large, which "opens the door" for pathological objects to obstruct the theory. We deal with this difficulty via a canonical decomposition $u = u\uparrow + u\downarrow$ of u into a nondecreasing $u\uparrow$ and a nonincreasing $u\downarrow$, so that we can isolate the phenomena we can control in $u\downarrow$ (and in fact recover a bit of linearity), and deal with the less well-behaved $u\uparrow$ only when necessary.

Comparison with [Fer23]. In prior work, [Fer23] gave an L^1 tester for functions f satisfying $\mathsf{Lip}_1(f) \le L$ with query complexity $O(dL/\epsilon)$, where $\mathsf{Lip}_1(f)$ is the Lipschitz constant of f with respect to the ℓ^1 metric. In contrast, in this paper we parameterize the problem by the more natural ℓ^2 (Euclidean) metric[7]. Our tester takes functions satisfying $\mathsf{Lip}_2(f) \le M$ and has query complexity $\widetilde{O}(\sqrt{d}M^2/\epsilon^2)$, and it is also a fortiori an L^1 tester as already remarked.

By monotonicity of ℓ^q norms and the Cauchy-Schwarz inequality, in general we have $\mathsf{Lip}_1(f) \le \mathsf{Lip}_2(f) \le \sqrt{d}\,\mathsf{Lip}_1(f)$, which allows rough comparisons between the two results. Since the Euclidean geometry is more natural than the ℓ^1 geometry, we find the first inequality more informative: it implies that the tester of [Fer23] is also an L^1 tester for functions satisfying $\mathsf{Lip}_2(f) \le M$ with query complexity $O(dM/\epsilon)$. Thus, for L^1 testing M-Lipschitz functions and ignoring logarithmic factors, Theorem 1.2 is better than the tester of [Fer23] when $\frac{\epsilon}{M} > \frac{1}{\sqrt{d}}$ and vice-versa. Curiously, something analogous is true of the testers of [KMS18] and [GGLRS00] for the Boolean setting.

[Fer23] asked about lower bounds (for general testers) in the present setting, and this question remains open. We also do not resolve Conjecture 1.8 of [Fer23], which asks for a directed (L^1, ℓ^2)-Poincaré inequality for Lipschitz $f : [0,1]^d \to \mathbb{R}$, i.e. an analogue to the inequalities of Talagrand and Bobkov & Houdré, but rather obtain our $\widetilde{O}(\sqrt{d}M^2/\epsilon^2)$ tester via the (L^2, ℓ^2) inequality.

Comparison with the path tester. As explained in Section 1.1, our directional derivative tester is essentially the natural continuous analogue of the path tester of [KMS18], where taking edge i in the path tester roughly corresponds to letting $v_i = 1$ in the directional derivative query $v \cdot \nabla f(x)$. This conceptual connection is intriguing, and we do not know whether there is a formal connection between the continuous and discrete problems that could explain it.

Further applications? Finally, we ask whether other problems in property testing—particularly in the case of continuous domain—can benefit from the techniques and ideas in the proof of Theorem 1.1. In property testing, the idea of comparing the input object f to some "ideal" object f^* that satisfies the property is very natural, and in this paper we offer techniques from partial differential equations and optimal transport as useful tools for reasoning about this comparison smoothly in time when the problem is continuous in nature. Therefore, it is plausible that these tools may have something to say about other property testing problems of continuous nature.

Organization. The rest of the paper is organized as follows. In Section 2, we introduce definitions and conventions used throughout the paper. In Section 3, we give our monotonicity tester and prove Theorem 1.2. In Section 4, we prove the query complexity lower bound for derivative-pair testers. Finally, Sections 5 to 7 establish Theorem 1.1, following the outline given in the proof overview.

[7] In fact, that a $\sqrt{d}$ tester would be parameterized in the ℓ^2 metric if it exists was already suggested in [Fer23].

2 Preliminaries

In this paper, $\mathbb{N}$ denotes the set of strictly positive integers $\{1, 2, \dots\}$. Throughout the paper, $d \in \mathbb{N}$ is an arbitrary natural number indicating the dimension of the ambient space $\mathbb{R}^d$ unless otherwise specified. For $m \in \mathbb{N}$, we write $[m]$ to denote the set $\{i \in \mathbb{N} : i \leq m\}$. For any $x \in \mathbb{R}$, we write x^+ for $\max\{0, x\}$ and x^- for $\max\{0, -x\}$, and we extend this notation to vectors $u \in \mathbb{R}^d$ in the natural way: $u^+, u^- \in \mathbb{R}^d$ are given by $u_i^+ := (u_i)^+$ and $u_i^- := (u_i)^-$. For real numbers a and b, we use the notation $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$.

For a vector $u \in \mathbb{R}^d$, we let $\mathrm{supp}(u) \subseteq [d]$ denote the set of indices where u is nonzero, and write $\|u\|_0 := |\mathrm{supp}(u)|$. For two points $x, y \in \mathbb{R}^d$, we write $x \preceq y$ if $x_i \leq y_i$ for every $i \in [d]$, and $y \succeq x$ if $x \preceq y$.

We denote the closure of a set $D \subset \mathbb{R}^d$ by $\overline{D}$. For a measure space (Ω, Σ, μ) and measurable function $f : \Omega \to \mathbb{R}$, we write $\int_\Omega f \, \mathrm{d}\mu$ for the Lebesgue integral of f over this space when it exists. Then for $1 \leq p < +\infty$, the space $L^p(\Omega)$ is the set of measurable functions f such that $|f|^p$ is Lebesgue integrable, i.e. $\int_\Omega |f|^p \, \mathrm{d}\mu < +\infty$, and we write the L^p norm of such functions as $\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p \, \mathrm{d}\mu\right)^{1/p}$. We usually write $\mathrm{d}x$ for the Lebesgue measure on $\mathbb{R}^d$ in the context of integration, i.e. for Lebesgue measurable set $\Omega \subset \mathbb{R}^d$, we write $\int_\Omega f \, \mathrm{d}x$ for the Lebesgue integral of integrable f. When we need to refer to the Lebesgue measure of a set Ω explicitly, we write $\mathcal{L}(\Omega)$.

Throughout the paper, all measures on $\mathbb{R}^d$ are Borel measures. We say that measure μ over $\Omega \subset \mathbb{R}^d$ is *absolutely continuous* if it is absolutely continuous with respect to the Lebesgue measure, i.e. if there exists a measurable function $f : \Omega \to \mathbb{R}_{\geq 0}$ satisfying $\mathrm{d}\mu = f \, \mathrm{d}x$. In this case, f is called the Radon-Nikodym derivative, or *density* of μ.

Given a set $\Omega \subset \mathbb{R}^d$ and $M > 0$, we say $f : \Omega \to \mathbb{R}$ is *M-Lipschitz* if $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \Omega$. The *Lipschitz constant* of f is the smallest M for which f is M-Lipschitz. We say that f is Lipschitz if it is M-Lipschitz for any $M > 0$.

We use the notation $a \overset{?}{\leq} b$, $a \overset{?}{=} b$, etc. within a proof to denote (in)equalities that have not yet been established.

Notation for directed partial derivatives and gradients. Let $\Omega \subset \mathbb{R}^d$ be an open set, and let $f : \Omega \to \mathbb{R}$ be Lipschitz. Then by Rademacher's theorem f is differentiable almost everywhere in Ω. For each $x \in \Omega$ where f is differentiable, let $\nabla f(x) = (\partial_1 f(x), \dots, \partial_d f(x))$ denote its gradient, where $\partial_i f(x)$ is the partial derivative of f along the i-th coordinate at x. Then, let $\partial_i^- := 0 \wedge \partial_i$, i.e. for every x where f is differentiable we have $\partial_i^- f(x) = -\left(\partial_i f(x)\right)^-$. We call ∂_i^- the *directed partial derivative* operator in direction i. Then we define the *directed gradient* operator by $\nabla^- := (\partial_1^-, \dots, \partial_d^-)$, again defined on every x where f is differentiable. Note that $\nabla^- f \preceq 0$. We also similarly define $\partial_i^+ := 0 \vee \partial_i$ and $\nabla_i^+ := (\partial_1^+, \dots, \partial_d^+)$.

3 Algorithm and upper bound

Definition 3.1 (Distribution of direction vector)**.** For each $p \in [0, 1]$, we define distribution $\mathcal{D}_p$ over $\{0, 1\}^d$ as follows. To sample $\boldsymbol{v} \sim \mathcal{D}_p$,

1. Sample $\boldsymbol{x}_i \sim \mathsf{Ber}(p)$ independently for each $i \in [d]$, where $\mathsf{Ber}(p)$ is the Bernoulli distribution;
2. Produce $\boldsymbol{v} = \sum_{i=1}^d \boldsymbol{x}_i e_i$, where e_i denotes the i-th standard basis vector.

Then, we define the distribution $\mathcal{D}$ over $\{0, 1\}^d$ as follows. Let $P := \left\{1, \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^{\lceil \log_2(4d) \rceil}}\right\}$. To sample $\boldsymbol{v} \sim \mathcal{D}$, we first sample $\boldsymbol{p}$ uniformly at random from P, and then sample $\boldsymbol{v}$ from $\mathcal{D}_{\boldsymbol{p}}$.

Algorithm 1 L^2 monotonicity tester for Lipschitz functions using directional derivative queries

Input: Directional derivative oracle access to M-Lipschitz function $f : [0,1]^d \to \mathbb{R}$.
Output: Accept if f is monotone, reject if $\mathsf{dist}_2^{\mathsf{mono}}(f) \ge \epsilon$.

procedure DirectionalDerivativeTester(f, d, M, ϵ)
 repeat $\Theta\left(\frac{\sqrt{d}M^2}{\epsilon^2}\log d\right)$ **times**
  Sample $x \in [0,1]^d$ uniformly at random.
  Sample $v \in \{0,1\}^d$ from distribution $\mathcal{D}$ given by Definition 3.1.
  Reject if $\nabla f(x) \cdot v < 0$.
 end repeat
 Accept.

Theorem 3.2 (Refinement of Theorem 1.2). *Algorithm 1 is a nonadaptive, directional derivative L^2 monotonicity tester for M-Lipschitz functions $f : [0,1]^d \to \mathbb{R}$ with query complexity $O\left(\frac{\sqrt{d}M^2}{\epsilon^2}\log d\right)$ and one-sided error.*

Proof. First note that, since f is Lipschitz and hence differentiable almost everywhere in $(0,1)^d$ by Rademacher's theorem, with probability 1 the algorithm only samples points x at which f is differentiable. Moreover, we have $\|\nabla f\|_2 \le M$ almost everywhere since f is M-Lipschitz.

The algorithm clearly accepts any monotone function; indeed if f is monotone, then $\nabla f \succeq 0$ and, since $\mathcal{D}$ is supported on $\{0,1\}^d$, we have $\nabla f \cdot \boldsymbol{v} \ge 0$ with probability 1. Now, suppose $\mathsf{dist}_2^{\mathsf{mono}}(f) \ge \epsilon$. Combining Lemma 3.4 and Theorem 1.1, we obtain that the probability that any single iteration of the tester rejects is

$$\begin{aligned}
\mathbb{P}\left[\text{Iteration rejects}\right] &= \underset{\substack{\boldsymbol{x}\in[0,1]^d \\ \boldsymbol{v}\sim\mathcal{D}}}{\mathbb{P}}\left[\nabla f(\boldsymbol{x}) \cdot \boldsymbol{v} < 0\right] \\
&\ge \underset{\substack{\boldsymbol{x}\in[0,1]^d \\ \boldsymbol{v}\sim\mathcal{D}}}{\mathbb{P}}\left[\nabla f(\boldsymbol{x}) \cdot \boldsymbol{v} < -\frac{\delta}{d}\|\nabla^- f(\boldsymbol{x})\|_2\right] \\
&= \int_{[0,1]^d} \underset{\boldsymbol{v}\sim\mathcal{D}}{\mathbb{P}}\left[\nabla f(x) \cdot \boldsymbol{v} < -\frac{\delta}{d}\|\nabla^- f(x)\|_2\right] \mathrm{d}x \\
&\ge \int_{[0,1]^d} c \cdot \frac{\|\nabla^- f(x)\|_2^2}{\sqrt{d}\log(d) M^2}\,\mathrm{d}x && (\text{Lemma 3.4}, \|\nabla f\|_2 \le M) \\
&= \frac{c}{M^2\sqrt{d}\log d}\int_{[0,1]^d} \|\nabla^- f(x)\|_2^2\,\mathrm{d}x \\
&\ge \frac{c}{M^2\sqrt{d}\log d} \cdot \frac{1}{C}\mathsf{dist}_2^{\mathsf{mono}}(f)^2 && (\text{Theorem 1.1}) \\
&\ge \frac{c}{C} \cdot \frac{\epsilon^2}{M^2\sqrt{d}\log d},
\end{aligned}$$

where δ, c are the constants from Lemma 3.4 and C is the constant from Theorem 1.1. Thus $\Theta\left(\frac{\sqrt{d}M^2}{\epsilon^2}\log d\right)$ iterations suffice to reject with probability at least $2/3$. □

Remark 3.3. A slight modification of the proof of Theorem 3.2 also shows that, if we replace the condition $\nabla f(x) \cdot v < 0$ in Algorithm 1 with the more demanding condition $\nabla f(x) \cdot v < -\frac{K\epsilon}{d}$ for some universal constant $K > 0$, then we still obtain a tester with the same guarantees; in particular, Algorithm 1 does not rely on arbitrary precision. Indeed, Theorem 1.1 gives that $\int \|\nabla^- f(x)\|_2^2 \, \mathrm{d}x \geq \frac{1}{C}\epsilon^2$ when f is ϵ-far from monotone, but the points x for which $\|\nabla^- f(x)\|_2^2 \leq \frac{\epsilon^2}{2C}$ can only contribute at most $\frac{\epsilon^2}{2C}$ to the integral. Therefore the points satisfying $\|\nabla^- f(x)\|_2^2 > \frac{\epsilon^2}{2C}$ must contribute at least $\frac{\epsilon^2}{2C}$ and, at each such point, Lemma 3.4 guarantees that $\nabla^- f(x) \cdot \boldsymbol{v} < -\frac{\delta}{d}\|\nabla^- f(x)\|_2 < -\frac{\delta}{d} \cdot \frac{\epsilon}{\sqrt{2C}}$ with at least the probability given in the lemma. Therefore a similar calculation to that of Theorem 3.2 shows that this modified tester also rejects with sufficient probability.

Lemma 3.4 (Detecting negative entries with dot products)**.** *There exist universal constants $c, \delta > 0$ such that the distribution $\mathcal{D}$ from Definition 3.1 has the following property: for any nonzero $u \in \mathbb{R}^d$, we have*

$$\mathop{\mathbb{P}}_{\boldsymbol{v} \sim \mathcal{D}}\left[u \cdot \boldsymbol{v} < -\frac{\delta}{d}\|u^-\|_2\right] \geq c \cdot \frac{\|u^-\|_2^2}{\sqrt{d}\log(d) \cdot \|u\|_2^2}\,.$$

Proof. We may assume that u contains at least one strictly negative entry, since otherwise $\|u^-\|_2^2 = 0$ and the claim is trivial. Let $\delta := 1/100$, and define $\delta_d := \delta/d$ for convenience.

Recall the distributions $\mathcal{D}$ and $\mathcal{D}_p$, as well as the set P, from Definition 3.1. Let $t := \frac{C\|u^-\|_2^2}{\sqrt{d} \cdot \|u\|_2^2}$, where we let $C := 1/40$. Note that $0 < t < 1$. Letting $c_1 := 1/10$, we will be done if we can show that there exists $p \in P$ such that

$$\mathop{\mathbb{P}}_{\boldsymbol{v} \sim \mathcal{D}_p}\left[u \cdot \boldsymbol{v} < -\delta_d\|u^-\|_2\right] \overset{?}{\geq} c_1 t\,,$$

Suppose for a contradiction that this is not the case, i.e. that for every $p \in P$,

$$\mathop{\mathbb{P}}_{\boldsymbol{v} \sim \mathcal{D}_p}\left[u \cdot \boldsymbol{v} < -\delta_d\|u^-\|_2\right] < c_1 t\,.$$

For convenience of notation, let $a := u^-$ and $b := u^+$. Then for every $p \in P$, letting $\boldsymbol{z}, \boldsymbol{w} \sim \mathcal{D}_p$ independently, we conclude that $u \cdot \boldsymbol{v}$ is distributed identically to $b \cdot \boldsymbol{w} - a \cdot \boldsymbol{z}$, and hence

$$\mathop{\mathbb{P}}_{\boldsymbol{z}, \boldsymbol{w} \sim \mathcal{D}_p}\left[a \cdot \boldsymbol{z} > b \cdot \boldsymbol{w} + \delta_d\|a\|_2\right] < c_1 t\,.$$

Fix any $\tau \in (2\delta_d\|a\|_2, \|a\|_\infty)$, where the interval is nonempty since $\|a\|_2 \leq \|a\|_1 \leq d\|a\|_\infty$, so that $2\delta_d\|a\|_2 = \frac{2\|a\|_2}{100d} < \|a\|_\infty$. Define $a_\tau \in \mathbb{R}^d$ as the vector obtained from a by only preserving entries that are strictly larger than τ, and zeroing out entries that are at most τ. Since the dot product $a \cdot \boldsymbol{z}$ only gets smaller if we omit some of its summands, we conclude that for every $p \in P$,

$$\mathop{\mathbb{P}}_{\boldsymbol{z}, \boldsymbol{w} \sim \mathcal{D}_p}\left[a_\tau \cdot \boldsymbol{z} > b \cdot \boldsymbol{w} + \delta_d\|a\|_2\right] < c_1 t\,.$$

Let E denote the event that $\mathrm{supp}(\boldsymbol{z}) \cap \mathrm{supp}(a_\tau) \neq \emptyset$, so that $a_\tau \cdot \boldsymbol{z} > \tau$ when E occurs and $a_\tau \cdot \boldsymbol{z} = 0$ otherwise. Fix the smallest $p = p(\tau) \in P$ satisfying $p \geq \frac{t}{\|a_\tau\|_0}$, which must exist because $\|a_\tau\|_0 \geq 1$ by the choice of range for τ and $t \leq 1$ as observed above, and therefore $\frac{t}{\|a_\tau\|_0} \leq 1 \in P$. Then

$$\mathop{\mathbb{P}}_{\boldsymbol{z} \sim D_{p(\tau)}}[E] = 1 - (1 - p(\tau))^{\|a_\tau\|_0} \geq 1 - e^{-p(\tau)\|a_\tau\|_0} \geq 1 - e^{-t} \geq \frac{t}{2}\,,$$

the last inequality since $e^{-x} \le 1 - x/2$ for (say) $0 \le x \le 1$. Therefore

$$\begin{aligned} c_1 t &> \mathop{\mathbb{P}}_{\boldsymbol{z},\boldsymbol{w}\sim\mathcal{D}_{p(\tau)}} [a_\tau \cdot \boldsymbol{z} > b \cdot \boldsymbol{w} + \delta_d \|a\|_2] \\ &= \mathop{\mathbb{P}}_{\boldsymbol{z}\sim D_{p(\tau)}} [E] \mathop{\mathbb{P}}_{\boldsymbol{z},\boldsymbol{w}\sim\mathcal{D}_{p(\tau)}} [a_\tau \cdot \boldsymbol{z} > b \cdot \boldsymbol{w} + \delta_d \|a\|_2 \quad | \quad E] \\ &\qquad + \mathop{\mathbb{P}}_{\boldsymbol{z}\sim D_{p(\tau)}} [\neg E] \mathop{\mathbb{P}}_{\boldsymbol{z},\boldsymbol{w}\sim\mathcal{D}_{p(\tau)}} [a_\tau \cdot \boldsymbol{z} > b \cdot \boldsymbol{w} + \delta_d \|a\|_2 \quad | \quad \neg E] \\ &\ge \frac{t}{2} \mathop{\mathbb{P}}_{\boldsymbol{w}\sim\mathcal{D}_{p(\tau)}} [\tau > b \cdot \boldsymbol{w} + \delta_d \|a\|_2] \,. \end{aligned}$$

We conclude that

$$\mathop{\mathbb{P}}_{\boldsymbol{w}\sim\mathcal{D}_{p(\tau)}} [b \cdot \boldsymbol{w} < \tau - \delta_d \|a\|_2] < 2c_1 \,.$$

We now claim that $p(\tau) \le \frac{2t}{\|a_\tau\|_0}$. Indeed if this were not the case, then the choice of $p(\tau)$ would imply that $p_{\min} := \min P$ satisfies $p_{\min} > \frac{2t}{\|a_\tau\|_0}$ with $p(\tau) = p_{\min} \le \frac{1}{4d}$ and hence, since $\tau > 2\delta_d \|a\|_2$,

$$\mathop{\mathbb{P}}_{\boldsymbol{w}\sim\mathcal{D}_{p(\tau)}} [b \cdot \boldsymbol{w} < \tau - \delta_d \|a\|_2] \ge \mathop{\mathbb{P}}_{\boldsymbol{w}\sim\mathcal{D}_{p(\tau)}} [b \cdot \boldsymbol{w} = 0] \ge \mathop{\mathbb{P}}_{\boldsymbol{w}\sim\mathcal{D}_{p(\tau)}} [\boldsymbol{w} = 0] = (1 - p_{\min})^d \ge 1 - d \cdot p_{\min} \ge \frac{3}{4} \,,$$

a contradiction. Thus $p(\tau) \le \frac{2t}{\|a_\tau\|_0}$. Now, the definition of $\mathcal{D}_{p(\tau)}$, linearity of expectation and inequality between ℓ^p-norms yield

$$\mathop{\mathbb{E}}_{\boldsymbol{w}\sim\mathcal{D}_{p(\tau)}} [b \cdot \boldsymbol{w}] = \sum_{i=1}^{d} \mathbb{P}[\boldsymbol{w}_i = 1]\, b_i = p(\tau) \cdot \|b\|_1 \le p(\tau)\sqrt{d}\|b\|_2 \,,$$

so by Markov's inequality,

$$\mathop{\mathbb{P}}_{\boldsymbol{w}\sim\mathcal{D}_{p(\tau)}} \left[b \cdot \boldsymbol{w} \ge 10 p(\tau)\sqrt{d}\|b\|_2 \right] \le \frac{1}{10} \,.$$

Now, the union bound implies that

$$\mathop{\mathbb{P}}_{\boldsymbol{w}\sim\mathcal{D}_{p(\tau)}} \left[\tau - \delta_d \|a\|_2 \le b \cdot \boldsymbol{w} < 10 p(\tau)\sqrt{d}\|b\|_2 \right] \ge 1 - 2c_1 - \frac{1}{10} > 0 \,,$$

so there exists $\boldsymbol{w}$ satisfying $\tau - \delta_d \|a\|_2 \le b \cdot \boldsymbol{w} < 10p(\tau)\sqrt{d}\|b\|_2$, and hence $\tau - \delta_d\|a\|_2 < 10p(\tau)\sqrt{d}\|b\|_2$. Now, using the fact that $p(\tau) \le \frac{2t}{\|a_\tau\|_0}$, recalling that $t = \frac{C\|u^-\|_2^2}{\sqrt{d}\cdot\|u\|_2^2}$, observing that $\|u\|_2^2 = \|u^-\|_2^2 + \|u^+\|_2^2$ while $b = u^+$ by definition, and using the inequality $2xy \le x^2 + y^2$, we obtain

$$\tau - \delta_d \|a\|_2 < \frac{20\sqrt{d}\|b\|_2 t}{\|a_\tau\|_0} = \frac{20C\sqrt{d}\|u^+\|_2 \|u^-\|_2^2}{\|a_\tau\|_0 \cdot \sqrt{d} \cdot \|u\|_2^2} \le \frac{10C \cdot \|u^-\|_2}{\|a_\tau\|_0} \,.$$

In summary, recalling that $a = u^-$ by definition, for all $\tau \in (2\delta_d\|a\|_2, \|a\|_\infty)$ we have

$$\frac{\tau}{2}\|a_\tau\|_0 \le (\tau - \delta_d\|a\|_2)\|a_\tau\|_0 < 10C\|a\|_2 \,,$$

so $\tau\|a_\tau\|_0 < 20C\|a\|_2$ for all $\tau \in (2\delta_d\|a\|_2, \|a\|_\infty)$. On the other hand, if $\tau \in (0, 2\delta_d\|a\|_2]$ then

$$\tau\|a_\tau\|_0 \le 2\delta_d\|a\|_2 \cdot d = 2\delta\|a\|_2 = \frac{1}{50}\|a\|_2 < \frac{1}{2}\|a\|_2 = 20C\|a\|_2 \,,$$

so in fact $\tau\|a_\tau\|_0 < 20C\|a\|_2$ for all $\tau \in (0, \|a\|_\infty)$. Integrating over τ and using Tonelli's theorem, we conclude that

$$\|a\|_\infty \cdot 20C\|a\|_2 = \int_{(0,\|a\|_\infty)} 20C\|a\|_2 \,\mathrm{d}\tau > \int_{(0,\|a\|_\infty)} \tau\|a_\tau\|_0 \,\mathrm{d}\tau = \int_{(0,\|a\|_\infty)} \sum_{i=1}^{d} (\tau \cdot \mathbb{1}[a_i > \tau]) \,\mathrm{d}\tau$$
$$= \sum_{i=1}^{d} \int_{(0,\|a\|_\infty)} (\tau \cdot \mathbb{1}[a_i > \tau]) \,\mathrm{d}\tau = \sum_{i=1}^{d} \int_{(0,a_i)} \tau \,\mathrm{d}\tau = \sum_{i=1}^{d} \frac{a_i^2}{2} = \frac{\|a\|_2^2}{2} .$$

Hence we obtain that $\|a\|_2 < 40C \cdot \|a\|_\infty = \|a\|_\infty$, a contradiction as desired. ☐

4 Lower bound

Definition 4.1. A *derivative-pair tester* for monotonicity of Lipschitz functions $f : [0,1]^d \to \mathbb{R}$ is described by a distribution $\mathcal{D}$ over *pair tests* and *directional derivative tests*, where

1. a pair test is an operation that performs a pair of value queries $f(x), f(y)$ for $x \preceq y$, and rejects if and only if $f(x) > f(y)$; and
2. a directional derivative test is an operation that performs a directional derivative query at point x and direction $v \succeq 0$, and rejects if and only if $\nabla f \cdot v < 0$.

The tester independently samples m tests from $\mathcal{D}$, rejects if any of these tests rejects, and accepts otherwise. By definition, every derivative-pair tester is nonadaptive and has one-sided error.

Theorem 4.2. *Every L^1 derivative-pair tester for monotonicity of M-Lipschitz functions f : $[0,1]^d \to \mathbb{R}$ must have query complexity $\Omega\left(\frac{\sqrt{d}M}{\epsilon}\right)$.*

Proof. It suffices to give a distribution over $O(M)$-Lipschitz functions $\boldsymbol{f}$ that are $\Omega(\epsilon)$-far from monotone in L^1 distance and such that any fixed pair test or directional derivative test has only an $O\left(\frac{\epsilon}{\sqrt{d}M}\right)$ probability of rejecting a random $\boldsymbol{f}$. Moreover, to obtain the asymptotic lower bound we may as well assume that (say) $\epsilon \le 1 \le M$.

For each $i \in [d]$, define the function $f_i : [0,1]^d \to \mathbb{R}$ by

$$f_i(x) := -\epsilon x_i + \sum_{j \in [d]\setminus\{i\}} \frac{M}{\sqrt{d}} x_j .$$

Then, define $\boldsymbol{f} := f_{\boldsymbol{i}}$ where $\boldsymbol{i}$ is sampled uniformly at random from $[d]$. Note that each f_i is linear and hence Lipschitz with Lipschitz constant

$$|\nabla f_i(x)| = \sqrt{\epsilon^2 + (d-1)\left(\frac{M}{\sqrt{d}}\right)^2} \le \sqrt{2}M = O(M) .$$

We claim that $\mathsf{dist}_1^{\mathsf{mono}}(f_i) = \Omega(\epsilon)$ for each $i \in [d]$; by symmetry it suffices to consider the case $i = d$. Any restriction $f_d(x_{-d}, \cdot)$ of f_d to an axis-aligned line in direction d (where $x_{-d} := (x_1, \dots, x_{d-1})$) is a linear function with slope $-\epsilon$, so it is $\Omega(\epsilon)$-far from monotone in L^1 distance (in fact, $\epsilon/4$-far, and the constant average-valued function achieves this). But by Tonelli's theorem, the distance from f_d to any monotone function g is the average of the distance over each such axis-aligned line:

$$\int_{[0,1]^d} |f_d - g| \,\mathrm{d}x = \int_{[0,1]^{d-1}} \mathrm{d}x_{-d} \int_{[0,1]} |f_d(x_{-d}, x_d) - g(x_{-d}, x_d)| \,\mathrm{d}x_d \ge \int_{[0,1]^{d-1}} \Omega(\epsilon) \,\mathrm{d}x_{-d} = \Omega(\epsilon) .$$

We now show that any pair test or directional derivative test rejects a random $\boldsymbol{f}$ only with probability $O\left(\frac{\epsilon}{\sqrt{d}M}\right)$. Note that, since each f_i is a linear function, any pair test on points $x \preceq y$ may be simulated by a directional derivative test on point x and direction $v = y - x \succeq 0$. Therefore it suffices to consider directional derivative tests. Moreover, since the gradient of each f_i is constant over the points x, only the direction v is relevant. For simplicity, we will say that direction v *rejects* function f_i if a directional derivative test with direction v rejects f_i.

Fix any direction $v \succeq 0$. Suppose there exists $j \in [d]$ such that

$$\epsilon v_j \le \frac{M}{\sqrt{d}} \sum_{k \in [d] \setminus \{j\}} v_k \,.$$

We then claim that the vector v' given by $v'_j := 0$ and $v'_k := v_k$ for $k \in [d] \setminus \{j\}$ rejects every f_i that v rejects. Indeed, suppose v rejects f_i. There are two cases. If $i = j$, then

$$0 > \nabla f_i(x) \cdot v = -\epsilon v_j + \sum_{k \in [d] \setminus \{j\}} \frac{M}{\sqrt{d}} v_k \,,$$

a contradiction, so this case cannot happen. On the other hand, if $i \neq j$, then

$$\nabla f(x) \cdot v' = -\epsilon v'_i + \sum_{k \in [d] \setminus \{i\}} \frac{M}{\sqrt{d}} v'_k \le -\epsilon v_i + \sum_{k \in [d] \setminus \{i\}} \frac{M}{\sqrt{d}} v_k = \nabla f(x) \cdot v < 0 \,,$$

the inequality since $v'_i = v_i$ and $v' \preceq v$. Hence v' also rejects f_i, as claimed. Thus v' rejects a random $\boldsymbol{f}$ with no less probability than v. Therefore we may assume without loss of generality that, for every $j \in [d]$, either $v_j = 0$ or

$$\epsilon v_j > \frac{M}{\sqrt{d}} \sum_{k \in [d] \setminus \{j\}} v_k \,.$$

Let $j^* \in [d]$ be such that v_{j^*} is a minimum nonzero entry of v, and let $n := \|v\|_0$ be the number of nonzero entries of v. Applying the inequality above to $j = j^*$ yields

$$\epsilon v_{j^*} > \frac{M}{\sqrt{d}} \sum_{k \in [d] \setminus \{j^*\}} v_k \ge \frac{M}{\sqrt{d}} \sum_{k \in [d] \setminus \{j^*\}} \mathbb{1}\left[v_k > 0\right] v_{j^*} = \frac{M}{\sqrt{d}} \cdot n v_{j^*} \,,$$

and hence $n < \frac{\sqrt{d}\epsilon}{M}$. Finally, note that v does not reject f_i if $v_i = 0$, so we conclude that

$$\mathop{\mathbb{P}}_{\boldsymbol{f}}\left[v \text{ rejects } \boldsymbol{f}\right] = \mathop{\mathbb{P}}_{\boldsymbol{i}}\left[v \text{ rejects } f_{\boldsymbol{i}}\right] \le \mathop{\mathbb{P}}_{\boldsymbol{i}}\left[v_{\boldsymbol{i}} > 0\right] = \frac{n}{d} \le \frac{\sqrt{d}\epsilon/M}{d} = \frac{\epsilon}{\sqrt{d}M} \,,$$

as desired. This concludes the proof. ☐

Remark 4.3. By the remark in Section 1.1, the lower bound above holds for all L^p testers, $p \ge 1$.

5 Directed heat semigroup

5.1 Preliminaries for PDE

We briefly outline some of the main concepts required to study our PDE, and refer the reader to e.g. [Eva10; Bre11] for detailed expositions.

Let $J = (a, b)$ be an open interval. The absolutely continuous (AC) functions $f : \overline{J} \to \mathbb{R}$ are precisely those for which there exists a function $g \in L^1(J)$ such that, for all $x \in \overline{J}$, $f(x) = f(a) + \int_{(a,x)} g \,\mathrm{d}x$. We call g a *weak derivative* of f, and write $\partial_x f$ for any weak derivative of f. The weak derivative is almost everywhere (a.e.) uniquely determined, and moreover, f is classically differentiable a.e. and its classical derivative agrees with $\partial_x f$ a.e. .

Let $\Omega := J^d$. Let $k \in \mathbb{Z}_{\geq 0}$ and $p \in [1, +\infty]$. The Sobolev space $W^{k,p}(\Omega)$ is the space of functions $f : \Omega \to \mathbb{R}$ which have weak derivatives up to order k in $L^p(\Omega)$. The definition of weak derivative in the multidimensional case is more involved than in the one-dimensional case, but the details are not relevant here. In one dimension, we may recursively define $W^{0,p}(J) := L^p(J)$ and, for $k \geq 1$, $W^{k,p}(J)$ as the set of functions $f \in W^{k-1,p}(J)$ such that $\partial_x f \in W^{k-1,p}(J)$.

We identify a.e. equal functions into equivalence classes in $W^{k,p}(\Omega)$, and we write "$f = g$ a.e." and "$f = g$ in $W^{k,p}(\Omega)$" interchangeably. Often, we are interested in a continuous representative of f, which is unique when it exists, and by abuse of notation write f for its continuous representative as well. With this interpretation, a standard fact is that $W^{1,\infty}(\Omega)$ is precisely the class of Lipschitz functions on Ω. If $1 \leq p \leq q \leq +\infty$, then $W^{k,q}(\Omega) \subset W^{k,p}(\Omega)$.

The case $p = 2$ is special because then $H^k(\Omega) := W^{k,2}(\Omega)$ is a Hilbert space. In one dimension, the inner product in $H^k(J)$ is

$$\langle f, g \rangle_{H^k(J)} := \sum_{i=0}^{k} \left\langle D^i f, D^i g \right\rangle_{L^2(J)} ,$$

where D^i above denotes the i-th weak derivative. When the space of integration is clear, we drop the subscript from the inner product notation. The inner product above also induces the norm

$$\|f\|_{H^k(J)} = \sqrt{\sum_{i=0}^{k} \|D^i f\|_{L^2(J)}^2} .$$

For a sequence $(f_n)_{n \in \mathbb{N}} \subset H^k(\Omega)$ and $f \in H^k(\Omega)$, the notation "$f \to g$ in $H^k(\Omega)$" means convergence in norm, i.e. $\|f_n - f\|_{H^k(\Omega)} \to 0$ as $n \to \infty$. We write "$f \rightharpoonup g$ weakly in $H^k(\Omega)$" to denote weak convergence in this space, which means that $\varphi(f_n) \to \varphi(f)$ as $n \to \infty$ for every φ in the dual space of $H^k(\Omega)$. The dual space of $L^2(\Omega)$ is $L^2(\Omega)$ itself with the $L^2(\Omega)$ inner product action $\varphi(f) = \langle \varphi, f \rangle$. The dual space of $H^1(\Omega)$ is larger, but contains $L^2(\Omega)$ with the same action.

It is often useful to approximate a function by a sequence of smooth functions, and for that we will use *mollification*. The following comes from [Eva10, Appendix C]. The *standard mollifier* $\eta \in C^\infty(\mathbb{R}^d)$ is given by

$$\eta(x) := \begin{cases} C \exp\left(\frac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \,, \end{cases}$$

for constant $C > 0$ chosen so that $\int_{\mathbb{R}^d} \eta \,\mathrm{d}x = 1$. Then for each $\epsilon > 0$, we let

$$\eta_\epsilon(x) := \frac{1}{\epsilon^d} \eta(x/\epsilon) \,,$$

which is a $C^\infty(\mathbb{R}^d)$ function satisfying $\int_{\mathbb{R}^d} \eta_\epsilon \,\mathrm{d}x = 1$ and $\eta(x) = 0$ for $x \notin B(0, \epsilon)$, where $B(0, \epsilon)$ is the open ball of radius ϵ centered at 0. We abuse language and also call η_ϵ a standard mollifier.

Let $U \subset \mathbb{R}^d$ be an open set and let $U_\epsilon := \{x \in U : \mathrm{dist}(x, \partial U) > \epsilon\}$, where ∂U is the boundary of U. For locally integrable $f : U \to \mathbb{R}$ (meaning that f is integrable on every compact $K \subset U$), we define the *mollification* $f^\epsilon : U_\epsilon \to \mathbb{R}$ of f by

$$f^\epsilon(x) := (\eta_\epsilon * f)(x) = \int_U \eta_\epsilon(x-y) f(y)\,\mathrm{d}y = \int_{B(0,\epsilon)} \eta(y) f(x-y)\,\mathrm{d}y\,.$$

We then have the following properties (see Theorem 7 of [Eva10, Appendix C]):

1. $f^\epsilon \in C^\infty(U_\epsilon)$.
2. $f^\epsilon \to f$ a.e. as $\epsilon \to 0$.
3. If $f \in C(U)$, then $f^\epsilon \to f$ uniformly on compact subsets of U.
4. If $1 \le p < \infty$ and $f \in L^p_{\mathrm{loc}}(U)$, then $f^\epsilon \to f$ in $L^p_{\mathrm{loc}}(U)$,

where $L^p_{\mathrm{loc}}(U)$ is the space of measurable functions whose every restriction to compact $K \subset U$ is in $L^p(K)$, and convergence in $L^p_{\mathrm{loc}}(U)$ means convergence in $L^p(K)$ for every compact $K \subset U$.

We also use the following abstract spaces. Let X be a real Banach space (for us, typically $L^2(J)$), let $1 \le p \le +\infty$ and let $T \in (0, +\infty]$. Then the *Bochner space* $L^p(0,T;X)$ is the space of Bochner measurable functions (whose precise definition is not important here) $\boldsymbol{u} : [0,T] \to X$ whose norm $\|\boldsymbol{u}\|_{L^p(0,T;X)}$ is finite, where

$$\|\boldsymbol{u}\|_{L^p(0,T;X)} := \begin{cases} \left(\int_{(0,T)} \|\boldsymbol{u}(t)\|_X^p \,\mathrm{d}t\right)^{1/p} & 1 \le p < +\infty \\ \operatorname{ess\,sup}_{t\in(0,T)} \|\boldsymbol{u}(t)\|_X & p = +\infty\,. \end{cases}$$

As usual, $C([0,T];X)$ denotes the set of continuous functions $\boldsymbol{u} : [0,T] \to X$. When X possesses the so-called Radon-Nikodym property, which in particular is the case for $X = L^2(J)$, and for $T < +\infty$, the characterization of absolutely continuous functions as those possessing an integrable weak derivative introduced earlier extends to functions $\boldsymbol{u} : [0,T] \to X$ (see [DU77, pp. 217–219]).

Above and hereafter, we use boldface symbols, e.g. $\boldsymbol{u}$, to denote Banach-valued functions whose domain we think of as the time in an evolution equation.

Hereafter, we write I for the unit interval $(0,1)$ unless otherwise specified, which will always be clearly indicated.

5.2 Neumann and gradient flow problems

Definition 5.1 (Down-H^1 functions)**.** We define the set $\mathcal{U}$ of *down-H^1 functions* as

$$\mathcal{U} := \{u \in L^2(I) \ :\ \mathcal{R}(u) \neq \emptyset\}\,,$$

where $\mathcal{R}(u)$ is the set of admissible *representations* for u as follows:

$$\mathcal{R}(u) := \Big\{ (u^{(1)}, u^{(2)}) \in L^2(I) \times H^1(I) \ : \\ u^{(1)} \text{ is nondecreasing, } u^{(2)} \text{ is nonincreasing, } \int_I u^{(2)}\,\mathrm{d}x = 0, \text{ and } u = u^{(1)} + u^{(2)} \text{ a.e. } \Big\}.$$

Remark 5.2. The properties “nondecreasing” and “nonincreasing” in the definition above technically do not apply directly to members of $L^2(I)$ and $H^1(I)$, which are equivalence classes of a.e. equal functions. Rather, we mean is that $u^{(1)}$ is a member of $L^2(I)$ which has a representative function $I \to \mathbb{R}$ that is nondecreasing, and likewise for $u^{(2)}$. More generally, when f is an object of any space that identifies a.e. equal functions, we abuse language and write “f is nonincreasing” (resp. nondecreasing) to mean that f has a nonincreasing (resp. nondecreasing) representative.

Remark 5.3. Is is easy to check that $H^1(I) \subset \mathcal{U}$, since for any $u \in H^1(I)$ we may separate the positive and negative parts of $\partial_x u$ into $u^{(1)}$ and $u^{(2)}$, respectively, and shift $u^{(1)}$ and $u^{(2)}$ by appropriate constants so that they satisfy the conditions of Definition 5.1. Then, since $H^1(I)$ is dense in $L^2(I)$, so is $\mathcal{U}$.

Definition 5.4 (Directed Dirichlet energy)**.** For each $u \in \mathcal{U}$ and $(u^{(1)}, u^{(2)}) \in \mathcal{R}(u)$, define

$$\mathcal{D}^-(u^{(1)}, u^{(2)}) := \frac{1}{2} \int_I \left(\partial_x u^{(2)}\right)^2 \mathrm{d}x \,.$$

Definition 5.5 (Optimal representations)**.** For each $u \in \mathcal{U}$, define $\mathcal{R}^*(u) \subseteq \mathcal{R}(u)$ as the set of representations minimizing $\mathcal{D}^-$, i.e.

$$\mathcal{R}^*(u) := \left\{ (u^{(1)}, u^{(2)}) \in \mathcal{R}(u) \ : \ \mathcal{D}^-(u^{(1)}, u^{(2)}) = \inf_{\mathcal{R}(u)} \mathcal{D}^- \right\} .$$

Proposition 5.6 (Existence and uniqueness of optimal representation)**.** *For all $u \in \mathcal{U}$, $\mathcal{R}^*(u)$ contains exactly one element.*

Proof of existence. Let $u \in \mathcal{U}$. Let $E := L^2(I) \oplus H^1(I)$, where $\oplus$ denotes the direct sum of Hilbert spaces, so that E is also a Hilbert space (and hence a reflexive Banach space). Note that $\mathcal{R}(u) \subseteq E$ by definition. Our goal is to show that $C = \mathcal{R}(u)$ and $f = \mathcal{D}^-$ satisfy the conditions of Lemma 5.7.

It is clear that $\mathcal{R}(u)$ is nonempty, because $u \in \mathcal{U}$. It is immediate to verify that $\mathcal{R}(u)$ is convex as well. We now check that $\mathcal{R}(u)$ is closed in E. Let $\big((u_n^{(1)}, u_n^{(2)})\big)_{n\in\mathbb{N}}$ be any sequence in $\mathcal{R}(u)$ that converges in E to some $(u^{(1)}, u^{(2)}) \in E$; we would like to show that $(u^{(1)}, u^{(2)}) \in \mathcal{R}(u)$. First, since $(u_n^{(1)}, u_n^{(2)}) \to (u^{(1)}, u^{(2)})$ in E, we have $u_n^{(1)} \to u^{(1)}$ in $L^2(I)$ and $u_n^{(2)} \to u^{(2)}$ in $H^1(I)$ and hence in $L^2(I)$, so that

$$\begin{aligned}
&\|u - (u^{(1)} + u^{(2)})\|_{L^2(I)} \\
&\quad = \lim_{n\to\infty} \left\| (u - (u_n^{(1)} + u_n^{(2)})) + (u_n^{(1)} - u^{(1)}) + (u_n^{(2)} - u^{(2)}) \right\|_{L^2(I)} \\
&\quad \le \lim_{n\to\infty} \left\| u - (u_n^{(1)} + u_n^{(2)}) \right\|_{L^2(I)} + \lim_{n\to\infty} \left\| u_n^{(1)} - u^{(1)} \right\|_{L^2(I)} + \lim_{n\to\infty} \left\| u_n^{(2)} - u^{(2)} \right\|_{L^2(I)} \\
&\quad = 0 \,,
\end{aligned}$$

where the last equality used the fact that $u = u_n^{(1)} + u_n^{(2)}$ a.e. for every n (since $(u_n^{(1)}, u_n^{(2)}) \in \mathcal{R}(u)$) and the two convergence observations above. Thus $u = u^{(1)} + u^{(2)}$ in $L^2(I)$ and hence almost everywhere. Moreover, by Lemma A.1, $u^{(1)}$ is nondecreasing and $u^{(2)}$ is nonincreasing. Finally, since $\int_I u_n^{(2)} \,\mathrm{d}x = 0$ for each n and $u_n^{(2)} \to u^{(2)}$ in $L^2(I)$, it follows that $\int_I u^{(2)} \,\mathrm{d}x = 0$ as well. We conclude that $(u^{(1)}, u^{(2)}) \in \mathcal{R}(u)$, and thus $\mathcal{R}(u)$ is closed.

Now, clearly $\mathcal{D}^-$ is proper since it is finitely valued by definition. It is also continuous (and thus lower semicontinuous) because for any $(u_n^{(1)}, u_n^{(2)}) \to (u^{(1)}, u^{(2)})$ in E, we have $u_n^{(2)} \to u^{(2)}$ in $H^1(I)$ and hence $\int_I \left(\partial_x u_n^{(2)}\right)^2 \mathrm{d}x \to \int_I \left(\partial_x u^{(2)}\right)^2 \mathrm{d}x$, so $\mathcal{D}^-(u_n^{(1)}, u_n^{(2)}) \to \mathcal{D}^-(u^{(1)}, u^{(2)})$. It is also immediate to check that that $\mathcal{D}^-$ is convex.

Finally, we need to show that $\mathcal{D}^-$ is coercive in the sense of Lemma 5.7. Let $\big((u_n^{(1)} u_n^{(2)})\big)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{R}(u)$ such that $\left\|(u_n^{(1)}, u_n^{(2)})\right\|_E^2 = \|u_n^{(1)}\|_{L^2(I)}^2 + \|u_n^{(2)}\|_{H^1(I)}^2 \to \infty$. We claim that $\|\partial_x u_n^{(2)}\|_{L^2(I)} \to \infty$. Suppose for a contradiction that this false. There are two cases. First, suppose

$\|u_n^{(2)}\|_{H^1(I)} \to \infty$. Then necessarily $\|u_n^{(2)}\|_{L^2(I)} \to \infty$, but since each $u_n^{(2)}$ has mean zero, the Poincaré-Wirtinger inequality gives that

$$\|\partial_x u_n^{(2)}\|_{L^2(I)} \geq \frac{1}{C}\|u_n^{(2)}\|_{L^2(I)}$$

for some constant $C > 0$. Thus $\|\partial_x u_n^{(2)}\|_{L^2(I)} \to \infty$, a contradiction. In the second case, $\|u_n^{(2)}\|_{H^1(I)}$ remains bounded, so we must have $\|u_n^{(1)}\|_{L^2(I)} \to \infty$. But since $u = u_n^{(1)} + u_n^{(2)}$ in $L^2(I)$, the reverse triangle inequality implies that $\|u_n^{(2)}\|_{L^2(I)} \geq \|u_n^{(1)}\|_{L^2(I)} - \|u\|_{L^2(I)} \to \infty$, so $\|u_n^{(2)}\|_{L^2(I)} \to \infty$. Then again Poincaré-Wirtinger implies that $\|\partial_x u_n^{(2)}\|_{L^2(I)} \to \infty$, so the claim holds. But since $\mathcal{D}^-\left(u_n^{(1)}, u_n^{(2)}\right) = \frac{1}{2}\|\partial_x u_n^{(2)}\|^2_{L^2(I)}$, we have $\mathcal{D}^-\left(u_n^{(1)}, u_n^{(2)}\right) \to +\infty$, so $\mathcal{D}^-$ is coercive. Thus $\mathcal{D}^-$ achieves its minimum on $\mathcal{R}(u)$ by Lemma 5.7, so $\mathcal{R}^*(u)$ is nonempty. □

Proof of uniqueness. We show uniqueness using strict convexity. Suppose $(u^{(1)}, u^{(2)}), (v^{(1)}, v^{(2)}) \in \mathcal{R}^*(u)$, and suppose for a contradiction that $(u^{(1)}, u^{(2)}) \neq (v^{(1)}, v^{(2)})$ in E (in the notation from the previous part of the proof; note that equality in E is equivalent to equality as tuples in $L^2(I) \times H^1(I)$).

We first claim that $\partial_x u^{(2)} \neq \partial_x v^{(2)}$ in $L^2(I)$. Indeed, suppose $\partial_x u^{(2)} = \partial_x v^{(2)}$ in $L^2(I)$. Note that $u^{(2)} \neq v^{(2)}$ in $L^2(I)$, because otherwise we would have $u^{(2)} = v^{(2)}$ in $H^1(I)$ and $u^{(1)} = u - u^{(2)} = u - v^{(2)} = v^{(1)}$ in $L^2(I)$, and thus $(u^{(1)}, u^{(2)}) = (v^{(1)}, v^{(2)})$ in E, a contradiction. Now, since $\partial_x u^{(2)} = \partial_x v^{(2)}$ and $u^{(2)} \neq v^{(2)}$ in $L^2(I)$, we conclude that $u^{(2)} = v^{(2)} + C$ in $L^2(I)$ for some constant $C \neq 0$. But this contradicts the fact that $\int_I u^{(2)}\,\mathrm{d}x = \int_I v^{(2)}\,\mathrm{d}x = 0$, which holds by the definition of $\mathcal{R}(u)$. Thus $\partial_x u^{(2)} \neq \partial_x v^{(2)}$ in $L^2(I)$.

Define the function $f : L^2(I) \to \mathbb{R}_{\geq 0}$ by $f(w) := \frac{1}{2}\|w\|^2_{L^2(I)}$, which is strictly convex, and note that $\mathcal{D}^-(r^{(1)}, r^{(2)}) = f(\partial_x r^{(2)})$ for all $r \in \mathcal{R}(u)$. Now, the element $(z^{(1)}, z^{(2)}) := \frac{1}{2}(u^{(1)}, u^{(2)}) + \frac{1}{2}(v^{(1)}, v^{(2)})$ is in $\mathcal{R}(u)$ by convexity of that set, and satisfies $\mathcal{D}^-(z^{(1)}, z^{(2)}) = f\left(\frac{1}{2}\partial_x u^{(2)} + \frac{1}{2}\partial_x v^{(2)}\right) < \frac{1}{2}f(\partial_x u^{(2)}) + \frac{1}{2}f(\partial_x v^{(2)}) = \frac{1}{2}\mathcal{D}^-(u^{(1)}, u^{(2)}) + \frac{1}{2}\mathcal{D}^-(v^{(1)}, v^{(2)})$, contradicting the fact that $(u^{(1)}, u^{(2)}), (v^{(1)}, v^{(2)}) \in \mathcal{R}^*(u)$. Thus $(u^{(1)}, u^{(2)}) = (v^{(1)}, v^{(2)})$ in E, as needed. □

Lemma 5.7 (See e.g. [Bre11, Corollary 3.23])**.** *Let E be a reflexive Banach space and let $C \subseteq E$ be nonempty, closed and convex. Suppose $f : C \to (-\infty, +\infty]$ is convex, proper, lower semicontinuous, and coercive in the sense that for every sequence $(x_n)_{n\in\mathbb{N}}$ in C with $\|x_n\|_E \to \infty$, we have $f(x_n) \to +\infty$. Then there exists $x^* \in C$ satisfying $f(x^*) = \inf_{x\in C} f(x)$.*

Owing to Proposition 5.6, we may define for each $u \in \mathcal{U}$ a canonical representation and directed Dirichlet energy (by a slight abuse of notation):

Definition 5.8 (Canonical representation)**.** For each $u \in \mathcal{U}$, the *canonical representation* of u is the unique element of $\mathcal{R}^*(u)$, denoted $(u\!\uparrow, u\!\downarrow)$. We define the *directed Dirichlet energy* of u by $\mathcal{D}^-(u) := \mathcal{D}^-(u\!\uparrow, u\!\downarrow)$.

The following definition, which extends the directed Dirichlet energy functional to all of $L^2(I)$ by assigning $+\infty$ to functions outside of $\mathcal{U}$, will enable us to find and study a solution to our PDE using tools from the theory of maximal monotone operators and gradient flows. Our main reference for this theory is [Bre73].

Definition 5.9 (Energy functional)**.** Define the functional $\mathcal{E}^- : L^2(I) \to [0, +\infty]$ by

$$\mathcal{E}^-(u) := \begin{cases} \mathcal{D}^-(u) & \text{if } u \in \mathcal{U} \\ +\infty & \text{otherwise.} \end{cases}$$

Proposition 5.10. *The functional $\mathcal{E}^- : L^2(I) \to [0, +\infty]$ is convex, proper and lower semicontinuous.*

Proof. Properness is trivial, since e.g. $0 \in \mathcal{U}$ and $\mathcal{E}^-(0) = 0$. Convexity also follows easily from the convexity of $\mathcal{U}$ and $\mathcal{D}^-$.

It remains to show lower semicontinuity, and since $L^2(I)$ is a metric space, it suffices to show sequential lower semicontinuity. Let $u \in L^2(I)$, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(I)$ converging to u in $L^2(I)$. We must show that

$$\mathcal{E}^-(u) \overset{?}{\leq} \liminf_{n\to\infty} \mathcal{E}^-(u_n)\,. \tag{8}$$

The only relevant case is when the RHS above is finite, so suppose there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \mathcal{E}^-(u_{n_k}) = A < +\infty$. By extracting a subsequence if necessary, we may assume that $\mathcal{E}^-(u_{n_k}) < +\infty$, and thus $u_{n_k} \in \mathcal{U}$, for every k.

We claim that $\big((u_{n_k}\!\uparrow, u_{n_k}\!\downarrow)\big)_k$ is bounded as a sequence in $L^2(I) \oplus H^1(I)$. First, since $\lim_{k\to\infty} \mathcal{E}^-(u_{n_k}) = A$, we have that $\big(\int_I (\partial_x u_{n_k}\!\downarrow)^2 \,\mathrm{d}x\big)_k$ is bounded. Then, since every $u_{n_k} \in \mathcal{U}$ and hence $\int_I u_{n_k}\!\downarrow \mathrm{d}x = 0$, the Poincaré-Wirtinger inequality implies that

$$\int_I (u_{n_k}\!\downarrow)^2 \,\mathrm{d}x \leq C \int_I (\partial_x u_{n_k}\!\downarrow)^2 \,\mathrm{d}x$$

for some $C > 0$, and hence $(u_{n_k}\!\downarrow)_k$ is bounded in $H^1(I)$. Now, since $u_{n_k} \to u$ in $L^2(I)$, we have that $\big(\|u_{n_k}\|_{L^2(I)}\big)_k$ is bounded, and since $\big(\|u_{n_k}\!\downarrow\|_{L^2(I)}\big)_k$ is bounded as a consequence of boundedness in $H^1(I)$, it follows that $\big(\|u_{n_k}\!\uparrow\|_{L^2(I)}\big)_k = \big(\|u_{n_k} - u_{n_k}\!\downarrow\|_{L^2(I)}\big)_k$ is also bounded, that is, $(u_{n_k}\!\uparrow)_k$ is bounded in $L^2(I)$. Hence $\big((u_{n_k}\!\uparrow, u_{n_k}\!\downarrow)\big)_k$ is bounded in $L^2(I) \oplus H^1(I)$ as claimed.

Since $L^2(I) \oplus H^1(I)$ is a Hilbert space, we conclude that $\big((u_{n_k}\!\uparrow, u_{n_k}\!\downarrow)\big)_k$ has a weakly convergent subsequence, which we denote by $\big((u_{n_{k_m}}\!\uparrow, u_{n_{k_m}}\!\downarrow)\big)_m$. Then there exist $v^{(1)} \in L^2(I)$ and $v^{(2)} \in H^1(I)$ such that $u_{n_{k_m}}\!\uparrow \rightharpoonup v^{(1)}$ weakly in $L^2(I)$ and $u_{n_{k_m}}\!\downarrow \rightharpoonup v^{(2)}$ weakly in $H^1(I)$. So letting $v := v^{(1)} + v^{(2)}$, we conclude that $u_{n_{k_m}} \rightharpoonup v$ weakly in $L^2(I)$; indeed, for every $f \in L^2(I)$, we have

$$\langle u_{n_{k_m}}, f\rangle = \langle u_{n_{k_m}}\!\uparrow + u_{n_{k_m}}\!\downarrow, f\rangle = \langle u_{n_{k_m}}\!\uparrow, f\rangle + \langle u_{n_{k_m}}\!\downarrow, f\rangle \to \left\langle v^{(1)}, f\right\rangle + \left\langle v^{(2)}, f\right\rangle = \langle v, f\rangle\ .$$

We now claim that $v \in \mathcal{U}$. Indeed, by Lemma A.1, $v^{(1)}$ is nondecreasing and $v^{(2)}$ is nonincreasing, and moreover $\int_I v^{(2)} \,\mathrm{d}x = 0$ by weak convergence; hence $(v^{(1)}, v^{(2)}) \in \mathcal{R}(v)$. Then, since $u_{n_{k_m}} \rightharpoonup v$ weakly in $L^2(I)$ and $u_{n_{k_m}} \to u$ in $L^2(I)$, we conclude that $u = v$, so $u \in \mathcal{U}$ with $(v^{(1)}, v^{(2)}) \in \mathcal{R}(u)$. In particular, this means that $\mathcal{E}^-(u) \leq \mathcal{D}^-(v^{(1)}, v^{(2)}) = \frac{1}{2}\int_I (\partial_x v^{(2)})^2 \,\mathrm{d}x$.

Now, the fact that $u_{n_{k_m}}\!\downarrow \rightharpoonup v^{(2)}$ weakly in $H^1(I)$ implies that $\|v^{(2)}\|_{H^1(I)} \leq \liminf_{m\to\infty} \|u_{n_{k_m}}\!\downarrow\|_{H^1(I)}$. Moreover, since $H^1(I)$ embeds compactly into $L^2(I)$ by the Rellich-Kondrachov theorem (see e.g. [Bre11, Theorem 9.16]), we have that $u_{n_{k_m}}\!\downarrow \to v^{(2)}$ in $L^2(I)$, so

$\|u_{n_{k_m}}\!\downarrow\|^2_{L^2(I)} \to \|v^{(2)}\|^2_{L^2(I)}$. Therefore we obtain

$$\begin{aligned}\liminf_{m\to\infty} \mathcal{E}^-(u_{n_{k_m}}) &= \frac{1}{2}\liminf_{m\to\infty}\int_I (\partial_x u_{n_{k_m}}\!\downarrow)^2 \,\mathrm{d}x = \frac{1}{2}\liminf_{m\to\infty}\left(\|u_{n_{k_m}}\!\downarrow\|^2_{H^1(I)} - \|u_{n_{k_m}}\!\downarrow\|^2_{L^2(I)}\right) \\ &= \left(\frac{1}{2}\liminf_{m\to\infty}\|u_{n_{k_m}}\!\downarrow\|^2_{H^1(I)}\right) - \frac{1}{2}\|v^{(2)}\|^2_{L^2(I)} \\ &\geq \frac{1}{2}\left(\|v^{(2)}\|^2_{H^1(I)} - \|v^{(2)}\|^2_{L^2(I)}\right) = \frac{1}{2}\int_I (\partial_x v^{(2)})^2 \,\mathrm{d}x \\ &\geq \mathcal{E}^-(u)\,,\end{aligned}$$

and thus (8) holds. □

Definition 5.11 (Static Neumann problem)**.** Let $u \in \mathcal{U}$ and $z \in L^2(I)$. We say that u, z form a *weak solution* to the *static Neumann problem*

$$\begin{cases} z = \partial_x\partial_x u\!\downarrow & \text{in } I \\ \partial_x u\!\downarrow = 0 & \text{on } \{0,1\} \end{cases} \tag{9}$$

if for all $\phi \in H^1(I)$ we have

$$\int_I z\phi \,\mathrm{d}x = -\int_I (\partial_x u\!\downarrow)(\partial_x \phi)\,\mathrm{d}x\,. \tag{10}$$

Remark 5.12. The equation $z = \partial_x\partial_x u\!\downarrow$ in (9) is also called the Poisson equation, while $\partial_x u\!\downarrow = 0$ on $\{0,1\}$ is called the (homogeneous) Neumann boundary condition.

Definition 5.13 (Nice evolution function)**.** Let $\boldsymbol{u} \in C([0,+\infty); L^2(I))$ and $\boldsymbol{u'} : (0,+\infty) \to L^2(I)$. We say that $\boldsymbol{u}$ is a *nice evolution function* (with weak derivative $\boldsymbol{u'}$) if 1) $\boldsymbol{u}(t) = \boldsymbol{u}(0) + \int_{(0,t)} \boldsymbol{u'}(s)\,\mathrm{d}s$ for all $t > 0$; 2) $\boldsymbol{u'}_{(\delta,+\infty)} \in L^\infty(\delta,+\infty; L^2(I))$ for all $\delta > 0$, where $\boldsymbol{u'}_J$ denotes the restriction of $\boldsymbol{u'}$ to domain $J \subseteq (0,+\infty)$; and 3) $\boldsymbol{u'}_{(0,\delta)} \in L^2(0,\delta; L^2(I))$ for all $\delta > 0$.

Remark 5.14. Definition 5.13 captures functions that are Lipschitz away from zero, and absolutely continuous with square-integrable weak derivative near zero.

Definition 5.15 (Neumann evolution problem)**.** Let $u_0 \in \mathcal{U}$ and let $\boldsymbol{u} \in C([0,+\infty); L^2(I))$ be a nice evolution function. We say $\boldsymbol{u}$ is a *weak solution* to the *Neumann evolution problem* with initial state (or initial data) u_0 if 1) $\boldsymbol{u}(0) = u_0$; 2) $\boldsymbol{u}(t) \in \mathcal{U}$ for all $t > 0$; and 3) $\boldsymbol{u}(t), \boldsymbol{u'}(t)$ form a weak solution to the static Neumann problem for a.e. $t > 0$.

Before introducing the gradient flow problem, we need some notation. Let $\varphi : L^2(I) \to [0,+\infty]$ be a convex, proper, lower semicontinuous function. We write $D(\varphi) := \{u \in L^2(I) : \varphi(u) < +\infty\}$ for the *domain* of φ. For any $u \in L^2(I)$, we write define the *subdifferential* of φ at u as

$$\partial\varphi(u) := \{z \in L^2(I) : \forall v \in L^2(I)\,.\,\varphi(v) \geq \varphi(u) + \langle z, v-u\rangle\}\,,$$

and we write $D(\partial\varphi) := \{u \in L^2(I) : \partial\varphi(u) \neq \emptyset\}$ for the *domain* of $\partial\varphi$. Using the fact that φ is proper, it is easy to check that $D(\partial\varphi) \subseteq D(\varphi)$. It is standard that $\partial\varphi$ is a *maximal monotone operator*; we will not use (the definition of) this property explicitly, but rather rely on the theory of such operators as presented in [Bre73].

We can now define the gradient flow problem:

Definition 5.16 (Gradient flow problem)**.** Let $u_0 \in \mathcal{U}$ and let $\boldsymbol{u}$ be a nice evolution function. We say $\boldsymbol{u}$ is a *solution* to the *gradient flow problem* with initial state (or initial data) u_0 if 1) $\boldsymbol{u}(0) = u_0$; 2) $\boldsymbol{u}(t) \in D(\partial\mathcal{E}^-)$ for all $t > 0$; and 3) $\boldsymbol{u}'(t) \in -\partial\mathcal{E}^-(\boldsymbol{u}(t))$ for a.e. $t > 0$.

Note that in both Definitions 5.15 and 5.16, the pointwise condition at $t = 0$ makes sense by the requirement that $\boldsymbol{u}$ be continuous.

The theory of maximal monotone operators and gradient flows immediately yields that the gradient flow problem has a unique solution, as follows:

Proposition 5.17. *Let $u_0 \in \mathcal{U}$. Then there exists a unique solution $\boldsymbol{u} \in C([0,+\infty); L^2(I))$ to the gradient flow problem. Moreover, for all $t > 0$ we have*

$$\mathcal{E}^-(u_0) - \mathcal{E}^-(\boldsymbol{u}(t)) = \int_{(0,t)} \|\boldsymbol{u}'(s)\|^2_{L^2(I)} \,\mathrm{d}s\,.$$

For $u_0, v_0 \in \mathcal{U}$, the corresponding solutions $\boldsymbol{u}, \boldsymbol{v}$ have $\|\boldsymbol{u}(t) - \boldsymbol{v}(t)\|_{L^2(I)} \leq \|u_0 - v_0\|_{L^2(I)}$ for all $t > 0$.

Proof. This is a direct application of [Bre73, Theorems 3.1 and 3.2 and Proposition 3.1]. □

Directed heat semigroup. As described in [Bre73], the nonexpansive property of solutions $\boldsymbol{u}$ allows us to define an operator $P_t : L^2(I) \to L^2(I)$ such that $P_t u = \boldsymbol{u}(t)$ for each initial state $u \in \mathcal{U}$ with corresponding solution $\boldsymbol{u}$, and with $P_t u$ defined by continuous extension when $u \in L^2(I) \setminus \mathcal{U}$ (recall that $\mathcal{U}$ is dense in $L^2(I)$). Then for every $u \in L^2(I)$ and $t > 0$ it holds that $P_t u \in D(\partial\mathcal{E}^-)$, and $(P_t)_{t \geq 0}$ forms a nonexpansive semigroup in $L^2(I)$, i.e.

1. $P_{t+s} = P_t P_s$ for all $t, s \in \mathbb{R}_{\geq 0}$, with P_0 the identity;
2. $P_t u \to u$ in $L^2(I)$ as $t \downarrow 0$, for all $u \in L^2(I)$; and
3. $\|P_t u - P_t v\|_{L^2(I)} \leq \|u - v\|_{L^2(I)}$ for all $u, v \in L^2(I)$ and $t \in \mathbb{R}_{\geq 0}$.

The last property also implies that, for each $t \geq 0$, $P_t : L^2(I) \to L^2(I)$ is continuous. We say that P_t is the semigroup *generated* by $-\partial\mathcal{E}^-$.

It turns out that the solution to the gradient flow problem is also a weak solution to the Neumann evolution problem, as the following results show.

Proposition 5.18. *Let $u \in D(\partial\mathcal{E}^-)$ and $z \in -\partial\mathcal{E}^-(u)$. Then u, z form a weak solution to the static Neumann problem.*

Proof. Note that we have $u \in D(\partial\mathcal{E}^-) \subseteq D(\mathcal{E}^-) = \mathcal{U}$. Let $\phi \in H^1(I)$; we will prove that (10) holds. Let $\alpha \neq 0$, let $\psi_\alpha := \alpha\phi$ and let $v_\alpha := u + \psi_\alpha$. Since $-z \in \partial\mathcal{E}^-(u)$, we have

$$\mathcal{E}^-(v_\alpha) \geq \mathcal{E}^-(u) + \langle -z, v_\alpha - u\rangle = \mathcal{E}^-(u) + \langle -z, \psi_\alpha\rangle\ . \tag{11}$$

Now, the fact that $u \in \mathcal{U}$ implies that $\mathcal{E}^-(u) = \mathcal{D}^-(u)$. We now claim that $v_\alpha \in \mathcal{U}$ as well, which will imply that $\mathcal{E}^-(v_\alpha) = \mathcal{D}^-(v_\alpha)$. In fact, we establish the following:

Claim 5.19. *We have $v_\alpha \in \mathcal{U}$. Moreover, $\mathcal{D}^-(v_\alpha) \leq \frac{1}{2}\int_I (\partial_x u\!\downarrow + \partial_x \psi_\alpha)^2 \,\mathrm{d}x$.*

Proof. We define a nondecreasing function $v^{(1)} \in L^2(I)$ and a nonincreasing function $v^{(2)} \in H^1(I)$ as follows: for each $x \in I$,

$$v^{(1)}(x) := u\!\uparrow\!(x) + \psi_\alpha(0) + \int_{(0,x)} \big(\partial_x \psi_\alpha(y) - |\partial_x u\!\downarrow\!(y)|\big)^+ \,\mathrm{d}y \qquad \text{and}$$

$$v^{(2)}(x) := u\!\downarrow\!(x) + \int_{(0,x)} \big(\partial_x \psi_\alpha(y) \wedge |\partial_x u\!\downarrow\!(y)|\big) \,\mathrm{d}y\,.$$

It is clear that $v^{(1)}, v^{(2)} \in L^2(I)$ with $v^{(1)}$ nondecreasing. Since $u\!\downarrow \in H^1(I)$ by definition of $\mathcal{R}(u)$, while $\psi_\alpha \in H^1(I)$ because $\phi \in H^1(I)$, we also obtain that $v^{(2)} \in H^1(I)$. To see that $v^{(2)}$ is nonincreasing, note that for a.e. $x \in I$ we have

$$\partial_x v^{(2)}(x) = \partial_x u\!\downarrow\!(x) + \big(\partial_x \psi_\alpha(x) \wedge |\partial_x u\!\downarrow\!(x)|\big) \le 0\,.$$

Observe also that $v^{(1)} + v^{(2)} = u + \psi_\alpha = v_\alpha$. Finally, by translating $v^{(1)}$ and $v^{(2)}$ by a constant if necessary, we can ensure that $\int_I v^{(2)} \,\mathrm{d}x = 0$ without invalidating the other properties. Thus $v_\alpha \in \mathcal{U}$.

To show the second part of the claim, we note that for a.e. $x \in I$,

$$\Big|\underbrace{\partial_x v^{(2)}(x)}_{\le 0}\Big| = \Big|\underbrace{\partial_x u\!\downarrow\!(x)}_{\le 0} + \underbrace{\big(\partial_x \psi_\alpha(x) \wedge |\partial_x u\!\downarrow\!(x)|\big)}_{\le |\partial_x u\downarrow(x)|}\Big| \le \Big|\partial_x u\!\downarrow\!(x) + \partial_x \psi_\alpha(x)\Big|\,,$$

where the inequality follows from inspecting the cases $\partial_x \psi_\alpha(x) \le |\partial_x u\!\downarrow\!(x)|$ and $\partial_x \psi_\alpha(x) > |\partial_x u\!\downarrow\!(x)|$. We conclude that

$$\mathcal{D}^-(v_\alpha) \le \mathcal{D}^-(v^{(1)}, v^{(2)}) = \frac{1}{2}\int_I (\partial_x v^{(2)})^2 \,\mathrm{d}x \le \frac{1}{2}\int_I (\partial_x u\!\downarrow\!(x) + \partial_x \psi_\alpha(x))^2 \,\mathrm{d}x\,. \qquad \square$$

Now, putting together (11) and the claim, we get

$$\frac{1}{2}\int_I (\partial_x u\!\downarrow)^2 \,\mathrm{d}x - \int_I z\psi_\alpha \,\mathrm{d}x \le \mathcal{D}^-(v_\alpha) \le \frac{1}{2}\int_I (\partial_x u\!\downarrow + \partial_x \psi_\alpha)^2 \,\mathrm{d}x\,.$$

Simplifying and substituting $\psi_\alpha = \alpha\phi$, we obtain

$$\alpha \int_I z\phi \,\mathrm{d}x \ge -\frac{1}{2}\alpha^2 \int_I (\partial_x \phi)^2 \,\mathrm{d}x - \alpha \int_I (\partial_x u\!\downarrow)(\partial_x \phi) \,\mathrm{d}x\,.$$

Taking the limits $\alpha \to 0^+$ and $\alpha \to 0^-$, we conclude that

$$\int_I z\phi \,\mathrm{d}x = -\int_I (\partial_x u\!\downarrow)(\partial_x \phi) \,\mathrm{d}x\,,$$

which is (10) as needed. $\square$

Corollary 5.20. *Let $u_0 \in \mathcal{U}$ and suppose $\boldsymbol{u}$ is a solution to the gradient flow problem. Then $\boldsymbol{u}$ is also a weak solution to the Neumann evolution problem.*

Proof. The condition $\boldsymbol{u}(0) = u_0$ holds by definition of solution to the gradient flow problem, and for all $t > 0$ we have $\boldsymbol{u}(t) \in D(\partial\mathcal{E}^-) \subseteq D(\mathcal{E}^-) = \mathcal{U}$. Finally, for a.e. $t > 0$ we have $\boldsymbol{u}'(t) \in -\partial\mathcal{E}^-(\boldsymbol{u}(t))$, which by Proposition 5.18 implies that $\boldsymbol{u}(t), \boldsymbol{u}'(t)$ form a solution to the static Neumann problem. Hence $\boldsymbol{u}$ is a weak solution to the Neumann evolution problem. $\square$

5.3 Auxiliary results for studying regularity of solutions

The following elliptic regularity result is standard, and essentially shows that weak solutions to the static Neumann problem are in fact strong solutions whose regularity is two degrees higher than that of z; in particular, even if we only have $z \in L^2(I)$, we gain one degree of regularity by obtaining $u\!\downarrow \in H^2(I)$ when we only assumed $u\!\downarrow \in H^1(I)$. Recall that $H^0(I)$ is the same as $L^2(I)$.

Lemma 5.21 (Elliptic regularity; see [Mik78, Chapter IV Section 2, Theorems 3 and 4])**.** *Let $k \geq 0$ be an integer and let $u \in \mathcal{U}, z \in H^k(I)$ form a weak solution to the static Neumann problem*

$$\begin{cases} z = \partial_x \partial_x u\!\downarrow & \text{in } I \\ \partial_x u\!\downarrow = 0 & \text{on } \{0,1\} \,. \end{cases}$$

Then $u\!\downarrow \in H^{k+2}(I)$, and moreover $\partial_x u\!\downarrow = 0$ on $\{0,1\}$ and $z = \partial_x \partial_x u\!\downarrow$ a.e. in I.

Lemma 5.22 (Functions in $D(\partial\mathcal{E}^-)$ are well-behaved)**.** *Suppose $u \in D(\partial\mathcal{E}^-)$. Let $(a,b) \in I$ be a nonempty interval and suppose that $\partial_x u\!\downarrow(x) < 0$ for all $x \in (a,b)$. Then $u\!\uparrow$ is constant in (a,b).*

Proof. Let $z \in L^2(I)$ be such that $-z \in \partial\mathcal{E}^-(u)$, which is nonempty by hypothesis. By Proposition 5.18 and Lemma 5.21, we conclude that $u\!\downarrow \in H^2(I)$ (which in particular justifies writing the condition that $\partial_x u\!\downarrow < 0$ on (a,b), which we take to mean in terms of the continuous representative of $\partial_x u\!\downarrow \in H^1(I)$).

By the continuity of $\partial_x u\!\downarrow$ and the boundary conditions $\partial_x u\!\downarrow(0) = \partial_x u\!\downarrow(1) = 0$ (which follows from Lemma 5.21), we may assume without loss of generality (by extending the interval (a,b) if necessary) that $\partial_x u\!\downarrow(a) = \partial_x u\!\downarrow(b) = 0$.

Let

$$Z := \begin{cases} \inf_{[a,b)} u\!\uparrow & \text{if } a > 0 \\ \frac{1}{b-a} \int_{(a,b)} u\!\uparrow \, \mathrm{d}x & \text{if } a = 0 \,, \end{cases}$$

which is finite in the first case because the infimum must be no smaller than (say) $u\!\uparrow(\frac{a}{2})$, and in the second case because $u\!\uparrow \in L^2(I)$. Next, define $v \in L^2(I)$ by

$$v(x) := \begin{cases} u(x) & \text{if } x \in (0,a) \cup [b,1) \\ u\!\downarrow(x) + Z & \text{if } x \in [a,b) \,. \end{cases}$$

We first observe that $v \in \mathcal{U}$, since $v = v^{(1)} + v^{(2)}$ with $v^{(2)} := u\!\downarrow$ and $v^{(1)} \in L^2(I)$ given by

$$v^{(1)}(x) := \begin{cases} u\!\uparrow(x) & \text{if } x \in (0,a) \cup [b,1) \\ Z & \text{if } x \in [a,b) \,, \end{cases}$$

which is nondecreasing because $u\!\uparrow$ is and by the definition of Z. Therefore $v \in D(\mathcal{E}^-)$ and

$$\mathcal{E}^-(v) \leq \mathcal{D}^-(v^{(1)}, v^{(2)}) = \frac{1}{2}\int_I (\partial_x v^{(2)})^2 \, \mathrm{d}x = \frac{1}{2}\int_i (\partial_x u\!\downarrow)^2 \, \mathrm{d}x = \mathcal{D}^-(u\!\uparrow, u\!\downarrow) = \mathcal{E}^-(u) \,.$$

On the other hand, we claim that $\mathcal{E}^-(v) \geq \mathcal{E}^-(u)$ as well. Indeed, let $(w^{(1)}, w^{(2)}) \in \mathcal{R}(v)$ be arbitrary. Construct $u^{(1)} \in L^2(I)$ and $u^{(2)} \in H^1(I)$ by $u^{(2)} := w^{(2)}$ and

$$u^{(1)}(x) := \begin{cases} w^{(1)}(x) & \text{if } x \in (0,a) \cup [b,1) \\ u(x) - u^{(2)}(x) & \text{if } x \in [a,b) \,, \end{cases}$$

That $u = u^{(1)} + u^{(2)}$ is clear by construction. To show that $(u^{(1)}, u^{(2)}) \in \mathcal{R}(u)$, it remains to show that $u^{(1)}$ is nondecreasing. Certainly it is nondecreasing on $(0,a)\cup[b,1)$ since $w^{(1)}$ is nondecreasing. Note that for $x \in [a,b)$, we have

$$u^{(1)}(x) = u(x) - w^{(2)}(x) - w^{(1)}(x) + w^{(1)}(x) = u(x) - v(x) + w^{(1)}(x) = u{\uparrow}(x) + w^{(1)}(x) - Z\,.$$

Thus $u^{(1)}$ is nondecreasing on $[a,b)$, since $u{\uparrow}$ and $w^{(1)}$ are. For $x \in (0,a)$ and $y \in [a,b)$, which only applies when $a > 0$, the inequality $u^{(1)}(x) \le u^{(1)}(y)$ follows from the nondecreasing monotonicity of $w^{(1)}$ and the inequality $u{\uparrow}(y) \ge Z$, which holds by the (first case of the) definition of Z.

Now, we must show that $u^{(1)}(x) \le u^{(1)}(y)$ if $x \in [a,b)$ and $y \in [b,1)$. In fact, it suffices to consider $y \in (b,1)$, because if the inequality holds in all such cases, then $u^{(1)}$ can be made monotone on all of I by possibly changing its value at b, which does not affect the a.e. equality $u = u^{(1)} + u^{(2)}$. Therefore let $x \in [a,b)$ and $y \in (b,1)$, which in particular only applies when $b < 1$. Recalling that every monotone function has limits from the left and from the right at every point, the key observation is that

$$\begin{aligned}
\lim_{s\to b^+} w^{(1)}(s) - \lim_{s\to b^-} w^{(1)}(s) &= \left[\lim_{s\to b^+} v(s) - w^{(2)}(s)\right] - \left[\lim_{s\to b^-} v(s) - w^{(2)}(s)\right] \\
&= \left[\lim_{s\to b^+} u{\uparrow}(s) + u{\downarrow}(s)\right] - w^{(2)}(b) - \left[\lim_{s\to b^-} u{\downarrow}(s) + Z\right] + w^{(2)}(b) \\
&= \lim_{s\to b^+} u{\uparrow}(s) - Z\,,
\end{aligned}$$

where we used the fact that $(w^{(1)}, w^{(2)}) \in \mathcal{R}(v)$ in the first equality, the definition of v and continuity of $w^{(2)}$ in the second equality, and the continuity of $u{\downarrow}$ in the third equality. Now, we have

$$\begin{aligned}
u^{(1)}(x) \le u^{(1)}(y) &\iff u{\uparrow}(x) + u{\downarrow}(x) - w^{(2)}(x) + Z - Z \le w^{(1)}(y) \\
&\iff u{\uparrow}(x) + w^{(1)}(x) - Z \le w^{(1)}(y) \\
&\Longleftarrow u{\uparrow}(x) + w^{(1)}(x) - Z \le \lim_{s\to b^+} w^{(1)}(s) \\
&\iff u{\uparrow}(x) + w^{(1)}(x) - Z \le \lim_{s\to b^-} w^{(1)}(s) + \lim_{s\to b^+} u{\uparrow}(s) - Z \\
&\Longleftarrow u{\uparrow}(x) \le \lim_{s\to b^+} u{\uparrow}(s) \quad \text{and} \quad w^{(1)}(x) \le \lim_{s\to b^-} w^{(1)}(s)\,,
\end{aligned}$$

which is true because $u{\uparrow}$ and $w^{(1)}$ are nondecreasing and $x < b$. This establishes that $u^{(1)}$ is nondecreasing and thus $(u^{(1)}, u^{(2)}) \in \mathcal{R}(u)$. It follows that

$$\mathcal{E}^-(u) \le \mathcal{D}^-(u^{(1)}, u^{(2)}) = \frac{1}{2}\int_I (\partial_x u^{(2)})^2 \,\mathrm{d}x = \frac{1}{2}\int_I (\partial_x w^{(2)})^2 \,\mathrm{d}x = \mathcal{D}^-(w^{(1)}, w^{(2)})\,,$$

and thus $\mathcal{E}^-(u) \le \mathcal{E}^-(v)$ as claimed. Thus $\mathcal{E}^-(u) = \mathcal{E}^-(v)$, which in particular implies that $\mathcal{E}^-(v) = \mathcal{D}^-(v^{(1)}, v^{(2)})$ and hence $(v{\uparrow}, v{\downarrow}) = (v^{(1)}, v^{(2)})$.

By definition of subdifferential, the fact that $-z \in \partial\mathcal{E}^-(u)$ implies that

$$\frac{1}{2}\int_I (\partial_x v{\downarrow})^2 \,\mathrm{d}x = \mathcal{E}^-(v) \ge \mathcal{E}^-(u) + \langle -z, v - u\rangle = \frac{1}{2}\int_I (\partial_x u{\downarrow})^2 \,\mathrm{d}x - \langle z, v - u\rangle\;.$$

Since $v = u$ on $I \setminus [a,b]$ and $v{\downarrow} = u{\downarrow}$, we conclude that

$$0 \le \langle z, v - u\rangle = \langle z, v - u\rangle_{L^2(a,b)} = \langle z, Z - u{\uparrow}\rangle_{L^2(a,b)}\;.$$

Now, since $z = \partial_x\partial_x u{\downarrow}$ a.e. and $\partial_x u{\downarrow}(a) = \partial_x u{\downarrow}(b) = 0$, we have

$$\langle z, Z\rangle_{L^2(a,b)} = Z\,\langle \partial_x\partial_x u{\downarrow}, 1\rangle_{L^2(a,b)} = Z(\partial_x u{\downarrow}(b) - \partial_x u{\downarrow}(a)) = 0\,,$$

and hence

$$\langle z, u{\uparrow}\rangle_{L^2(a,b)} \le 0\,.$$

Now, suppose for a contradiction that $u{\uparrow}$ is not constant in (a,b). Then since $u{\uparrow}$ is nondecreasing, there must exist a', b' with $a < a' < b' < b$ and $u{\uparrow}(a') < u{\uparrow}(b')$, i.e. $\delta := u{\uparrow}(b') - u{\uparrow}(a') > 0$. Let $\alpha := -\sup_{(a',b')}\partial_x u{\downarrow}$, and note that $\alpha > 0$ by the extreme value theorem together with the continuity of $\partial_x u{\downarrow}$ and the fact that $\partial_x u{\downarrow} < 0$ in $[a', b']$. Then Lemma 5.24 applied to $f = u{\uparrow}$, $g = \partial_x u{\downarrow}$ and $a', b' \in J = (a,b)$ implies that $\langle z, u{\uparrow}\rangle_{L^2(a,b)} = \langle \partial_x\partial_x u{\downarrow}, u{\uparrow}\rangle_{L^2(a,b)} \ge \alpha\delta > 0$, which is the desired contradiction. $\square$

Lemma 5.23. *Let $J \subset \mathbb{R}$ be a finite, nonempty open interval. Let $f \in L^\infty(J)$ be nondecreasing and let $g \in H^1(J)$ be such that $g < 0$ in J and $g = 0$ on ∂J, i.e. the endpoints of J. Let $[a,b] \subset J$, let $\delta := f(b) - f(a)$ and let $\alpha := -\sup_{(a,b)} g$. Then $\langle f, \partial_x g\rangle_{L^2(J)} \ge \alpha\delta$.*

Proof. Without loss of generality, we may assume that $J = I$. Now, for each sufficiently small $\epsilon > 0$, recall that $I_\epsilon = (\epsilon, 1-\epsilon)$ and let $f_\epsilon \in C^\infty(I_\epsilon)$ be the mollification of f. Let $f^*_\epsilon : I \to \mathbb{R}$ be given by

$$f^*_\epsilon(x) := \begin{cases} f_{\epsilon/2}(\epsilon) & \text{if } x \in (0,\epsilon) \\ f_{\epsilon/2}(x) & \text{if } x \in [\epsilon, 1-\epsilon] \\ f_{\epsilon/2}(1-\epsilon) & \text{if } x \in (1-\epsilon, 1)\,. \end{cases}$$

Note that each $f^*_\epsilon \in H^1(I)$, in particular because the piecewise definition is continuous and $f_{\epsilon/2}$ is smooth on $[\epsilon, 1-\epsilon]$. Also, each $f_{\epsilon/2}$ is nondecreasing and hence so is each f^*_ϵ. Moreover, $f^*_\epsilon \to f$ a.e. as $\epsilon \to 0$ since this is true of $(f_\epsilon)_{\epsilon>0}$. Finally, we have $f^*_\epsilon \to f$ in $L^2_{\mathrm{loc}}(I)$ since $f_\epsilon \to f$ in $L^2_{\mathrm{loc}}(\Omega)$, and since $\|f^*_\epsilon\|_{L^\infty(I)} \le \|f_{\epsilon/2}\|_{L^\infty(I)} \le \|f\|_{L^\infty(I)}$, we have that $(f^*_\epsilon)_{\epsilon>0}$ is bounded in $L^2(I)$ and Lemma A.2 implies that $f^*_\epsilon \rightharpoonup f$ weakly in $L^2(I)$ as $\epsilon \to 0$.

Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ be two sequences such that $a_n \uparrow a$, $b_n \downarrow b$, and moreover, for every $n \in \mathbb{N}$, $f^*_\epsilon(a_n) \to f(a_n)$ and $f^*_\epsilon(b_n) \to f(b_n)$ as $\epsilon \to 0$; the existence of such sequences is guaranteed by the fact that $f^*_\epsilon \to f$ almost everywhere. For each $n \in \mathbb{N}$, let $\delta_n := f(b_n) - f(a_n)$ and $\alpha_n := -\sup_{(a_n,b_n)} g$. Note that $\delta_n \ge \delta$ because f is nondecreasing while $a_n \le a$ and $b \le b_n$, and that $\alpha_n \to \alpha$ as $n \to \infty$ by the continuity of g.

For each $\epsilon > 0$, integration by parts gives

$$\langle f^*_\epsilon, \partial_x g\rangle = \int_I f^*_\epsilon(\partial_x g)\,\mathrm{d}x = f^*_\epsilon g|_0^1 - \int_I (\partial_x f^*_\epsilon) g\,\mathrm{d}x\,.$$

Recall that $g = 0$ on $\{0,1\}$. Moreover, since f^*_ϵ is nondecreasing while $g < 0$ in I, the integrand in the RHS above is nonpositive. Hence we can only make the RHS smaller by restricting the range of integration. Thus, fixing any $n \in \mathbb{N}$ and using the definition of α_n,

$$\langle f^*_\epsilon, \partial_x g\rangle \ge -\int_{(a_n,b_n)} (\partial_x f^*_\epsilon) g\,\mathrm{d}x \ge -\int_{(a_n,b_n)} (\partial_x f^*_\epsilon)(-\alpha_n)\,\mathrm{d}x = \alpha_n\, f^*_\epsilon|_{a_n}^{b_n}\,.$$

Since $f^*_\epsilon \rightharpoonup f$ weakly in $L^2(I)$, we obtain

$$\langle f, \partial_x g\rangle = \lim_{\epsilon\to 0}\langle f^*_\epsilon, \partial_x g\rangle \ge \lim_{\epsilon\to 0}\alpha_n\, f^*_\epsilon|_{a_n}^{b_n} = \alpha_n\delta_n \ge \alpha_n\delta\,,$$

the second equality by the choice of sequences $(a_n)_n, (b_n)_n$ and definition of δ_n. We conclude that

$$\langle f, \partial_x g\rangle \ge \lim_{n\to\infty}\alpha_n\delta = \alpha\delta\,. \qquad\square$$

Lemma 5.24. *The statement of Lemma 5.23 still holds if we replace the condition $f \in L^\infty(J)$ with $f \in L^2(J)$.*

Proof. Again let $J = I$ without loss of generality. We proceed by an approximation argument. Let $(a_n)_{n\in\mathbb{N}}$ be a strictly decreasing sequence satisfying $a_1 = a$ and $a_n \to 0$. Similarly, let $(b_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence satisfying $b_1 = b$ and $b_n \to 1$. For each $n \in \mathbb{N}$, define $f_n \in L^\infty(I)$ by

$$f_n(x) := \begin{cases} \inf_{[a_n,b_n]} f & \text{if } x \in (0, a_n) \\ f(x) & \text{if } x \in [a_n, b_n] \\ \sup_{[a_n,b_n]} f & \text{if } x \in (b_n, 1)\,. \end{cases}$$

Note that the infimum and supremum above are finite by virtue of the monotonicity of f and the observation that each $[a_n, b_n] \subset I$; thus we indeed have $f_n \in L^\infty(I)$. Moreover, each f_n is nondecreasing and, since $a_n \le a < b \le b_n$, we have $f_n(a) = f(a)$ and $f_n(b) = f(b)$. Lemma 5.23 implies that $\langle f_n, \partial_x g\rangle \ge \alpha\delta$ for α as in that statement and $\delta = f_n(b) - f_n(a) = f(b) - f(a)$. Finally, we have $f_n \to f$ in $L^2(I)$; indeed, letting $c := \frac{a+b}{2}$ for convenience and using the monotonicity of f,

$$\begin{aligned} \|f - f_n\|^2_{L^2(I)} &= \int_{(0,a_n)} \left(f(x) - \inf_{[a_n,b_n]} f\right)^2 \mathrm{d}x + \int_{(b_n,1)} \left(f(x) - \sup_{[a_n,b_n]} f\right)^2 \mathrm{d}x \\ &\le \int_{(0,a_n)} (f(x) - f(c))^2 \,\mathrm{d}x + \int_{(b_n,1)} (f(x) - f(c))^2 \,\mathrm{d}x \\ &\le 2\left[\|f\|^2_{L^2(0,a_n)} + \|f\|^2_{L^2(b_n,1)} + (a_n + 1 - b_n) f(c)^2\right] \to 0\,, \end{aligned}$$

the last step by the continuity of the functions $x \to \|f\|^2_{L^2(0,x)}$ and $x \to \|f\|^2_{L^2(x,1)}$ and the fact that $a_n + (1 - b_n) \to 0$ as $n \to \infty$. Hence $\langle f_n, \partial_x g\rangle \to \langle f, \partial_x g\rangle$, and the conclusion follows. $\square$

Proposition 5.25. *Let $1 \le p \le \infty$, and let $u \in \mathcal{U} \cap W^{1,p}(I)$. Then $u{\uparrow}, u{\downarrow} \in W^{1,p}(I)$ and, moreover, we have $\partial_x u{\uparrow} = \partial_x^+ u$ and $\partial_x u{\downarrow} = \partial_x^- u$ a.e. in I.*

Proof. Let $u^{(1)}, u^{(2)} : I \to \mathbb{R}$ be given by

$$u^{(1)}(y) := u(0) + C + \int_{(0,y)} \partial_x^+ u(x)\,\mathrm{d}x \qquad \text{and} \qquad u^{(2)}(y) := -C + \int_{(0,y)} \partial_x^- u(x)\,\mathrm{d}x$$

for each $y \in I$, where $C \in \mathbb{R}$ is implicitly defined so as to satisfy

$$\int_I u^{(2)}\,\mathrm{d}x = 0\,.$$

Then $u^{(1)}$ is nondecreasing, $u^{(2)}$ is nonincreasing, both are absolutely continuous with $\partial_x u^{(1)} = \partial_x^+ u$ and $\partial_x u^{(2)} = \partial_x^- u$ a.e. in I, and

$$(u^{(1)} + u^{(2)})(y) = u(0) + \int_{(0,y)} \left(\partial_x^+ u(x) + \partial_x^- u(x)\right) \mathrm{d}x = u(x)$$

for all $y \in I$ by the absolute continuity of u. It is also clear that $u^{(1)}, u^{(2)} \in L^\infty(I) \subset L^p(I)$, since they are pointwise upper bounded in magnitude by $|u(0)| + |C| + \|\partial_x u\|_{L^1(I)} < +\infty$. Moreover, since $u \in W^{1,p}(I)$, we have $\partial_x u \in L^p(I)$ and hence $\partial_x^+ u, \partial_x^- u \in L^p(I)$, yielding that $u^{(1)}, u^{(2)} \in W^{1,p}(I)$.

The proof will be concluded if we show that $(u{\uparrow}, u{\downarrow}) = (u^{(1)}, u^{(2)})$. However, at this point we cannot even state that $(u^{(1)}, u^{(2)}) \in \mathcal{R}(u)$ because we have not established that $u^{(2)} \in H^1(I) = W^{1,2}(I)$ (unless $p \geq 2$, of course). However, if we can show that any $(v^{(1)}, v^{(2)}) \in \mathcal{R}(u)$ satisfies

$$\int_I (\partial_x v^{(2)})^2 \,\mathrm{d}x \overset{?}{\geq} \int_I (\partial_x u^{(2)})^2 \,\mathrm{d}x \,, \tag{12}$$

then using the assumption that $u \in \mathcal{U}$ and the definition of $u{\uparrow}, u{\downarrow}$, we will conclude that indeed $(u{\uparrow}, u{\downarrow}) = (u^{(1)}, u^{(2)})$, as needed.

Let $(v^{(1)}, v^{(2)}) \in \mathcal{R}(u)$. We claim that $\left|\partial_x v^{(2)}(x)\right| \geq \left|\partial_x u^{(2)}(x)\right|$ for a.e. $x \in I$, which will imply (12). Recall that $v^{(2)} \in H^1(I) \subset W^{1,1}(I)$. Since $u = v^{(1)} + v^{(2)}$ a.e. and $u \in W^{1,p}(I) \subset W^{1,1}(I)$, we conclude that $v^{(1)} \in W^{1,1}(I)$. Hence $u, u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}$ are all absolutely continuous. The fundamental theorem of calculus for the Lebesgue integral implies that, almost everywhere in I, these functions are all differentiable, and their classical and weak derivatives agree with $\partial_x u = \partial_x u^{(1)} + \partial_x u^{(2)} = \partial_x v^{(1)} + \partial_x v^{(2)}$. On any such point x, the monotonicity of $u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}$ implies that $\partial_x u^{(1)}(x), \partial_x v^{(1)}(x) \geq 0$ and $\partial_x u^{(2)}(x), \partial_x v^{(2)}(x) \leq 0$. Therefore

$$\begin{aligned}\left|\partial_x v^{(2)}(x)\right| &= -\partial_x v^{(2)}(x) = -\partial_x u(x) + \partial_x v^{(1)}(x) = -\partial_x u^{(2)}(x) - \partial_x u^{(1)}(x) + \partial_x v^{(1)}(x) \\ &= \left|\partial_x u^{(2)}(x)\right| - \partial_x u^{(1)}(x) + \partial_x v^{(1)}(x) \,.\end{aligned}$$

Now, if $\partial_x u(x) \geq 0$, then $\partial_x u^{(2)}(x) = 0$ by definition, so $\left|\partial_x v^{(2)}(x)\right| \geq \left|\partial_x u^{(2)}(x)\right|$ holds trivially. Otherwise, we conversely have $\partial_x u^{(1)}(x) = 0$ while $\partial_x v^{(1)} \geq 0$, and hence, by the above,

$$\left|\partial_x v^{(2)}(x)\right| \geq \left|\partial_x u^{(2)}(x)\right| ,$$

which concludes the proof. □

We will also need the following standard facts.

Fact 5.26 (See e.g. [EG15, Theorem 4.4])**.** *Let $1 \leq p < \infty$ and let $f \in W^{1,p}(I)$. Then $\partial_x f = 0$ a.e. on $\{f = 0\}$.*

The following fact is an immediate application of the Sobolev embedding theorem:

Fact 5.27. *Let $f \in H^2(I)$. Then $f \in C^{1,1/2}(\overline{I})$. In particular, f is continuously differentiable.*

Fact 5.28 (See e.g. [Bar76, Chapter II, Corollary 2.1])**.** *The set $D(\partial\mathcal{E}^-)$ is a dense subset of $D(\mathcal{E}^-) = \mathcal{U}$.*

Observation 5.29. *Since $\mathcal{U}$ contains $H^1(I)$, which is dense in $L^2(I)$, Fact 5.28 implies that*

$$\overline{D(\partial\mathcal{E}^-)} = \overline{D(\mathcal{E}^-)} = \overline{\mathcal{U}} = L^2(I) \,.$$

5.4 Preservation of H^1 regularity

We wish to show that if the initial state u is in $H^1(I)$, then $P_t u$ remains in $H^1(I)$ for all times $t > 0$. To that end, define $\varphi : L^2(I) \to [0, +\infty]$ by

$$\varphi(u) := \begin{cases} \|\partial_x u\|^2_{L^2(I)} & \text{if } u \in H^1(I) \\ +\infty & \text{otherwise.} \end{cases}$$

The theory of maximal monotone operators gives us a recipe to establish that $t \mapsto \varphi(P_t u)$ is nonincreasing. The key ingredients are Claim 5.30 and Lemma 5.31.

Claim 5.30. *The functional φ is convex, proper and lower semicontinuous.*

Proof. Convexity and properness are straightforward; it remains to verify lower semicontinuity. Since $L^2(I)$ is a metric space, it suffices to check sequential lower semicontinuity. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(I)$ such that $u_n \to u$ in $L^2(I)$. We need to show that

$$\varphi(u) \overset{?}{\leq} \liminf_{n\to\infty} \varphi(u_n)\,.$$

The only relevant case is when the RHS above is finite, so suppose there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \varphi(u_{n_k}) = A < +\infty$. By extracting a subsequence if necessary, we may assume that $\varphi(u_{n_k}) < +\infty$, and thus $u_{n_k} \in H^1(I)$, for every k.

We claim that $(u_{n_k})_k$ is bounded in $H^1(I)$. Indeed suppose this is not the case. Then since $\|u_{n_k}\|^2_{H^1(I)} = \|u_{n_k}\|^2_{L^2(I)} + \|\partial_x u_{n_k}\|^2_{L^2(I)}$ and $\|\partial_x u_{n_k}\|^2_{L^2(I)}$ remains bounded due to the fact that $\varphi(u_{n_k}) \to A$, we conclude that $\|u_{n_k}\|_{L^2(I)}$ gets arbitrarily large as $k \to \infty$. But this contradicts the fact that $u_{n_k} \to u$ in $L^2(I)$, so the claim holds.

It follows that we may extract a weakly convergent subsequence $(u_{n_{k_\ell}})_{\ell\in\mathbb{N}}$, and by uniqueness of weak limits, we obtain that $u \in H^1(I)$ and $u_{n_{k_\ell}} \rightharpoonup u$ weakly in $H^1(I)$. By weak lower semicontinuity the norm in $H^1(I)$,

$$\|u\|^2_{L^2(I)} + \|\partial_x u\|^2_{L^2(I)} = \|u\|^2_{H^1(I)} \leq \liminf_{\ell\to\infty} \|u_{n_{k_\ell}}\|^2_{H^1(I)} = \liminf_{\ell\to\infty} \|u_{n_{k_\ell}}\|^2_{L^2(I)} + \|\partial_x u_{n_{k_\ell}}\|^2_{L^2(I)}\,.$$

Since $u_{n_{k_\ell}} \to u$ in $L^2(I)$, we have $\|u_{n_{k_\ell}}\|^2_{L^2(I)} \to \|u\|^2_{L^2(I)}$ and hence

$$\varphi(u) = \|\partial_x u\|^2_{L^2(I)} \leq -\|u\|^2_{L^2(I)} + \liminf_{\ell\to\infty} \|u_{n_{k_\ell}}\|^2_{L^2(I)} + \|\partial_x u_{n_{k_\ell}}\|^2_{L^2(I)} = \liminf_{\ell\to\infty} \varphi(u_{n_{k_\ell}})\,. \qquad \square$$

Lemma 5.31. *Let $u \in D(\partial\mathcal{E}^-)$ and let $z \in -\partial\mathcal{E}^-(u)$. Then for all $\lambda > 0$, $\varphi(u - \lambda z) \geq \varphi(u)$.*

Proof. By Proposition 5.18, z is a weak solution to the static Neumann problem, and by Lemma 5.21 we have $u{\downarrow} \in H^2(I)$ with $\partial_x u{\downarrow} = 0$ on $\{0,1\}$ and $z = \partial_x\partial_x u{\downarrow}$ in $L^2(I)$.

Note that the result holds trivially if $u - \lambda z \notin H^1(I)$, in which case $\varphi(u - \lambda z) = +\infty$. Therefore assume that $u - \lambda z \in H^1(I)$. We consider four cases: $u \in H^1(I)$; $u \in W^{1,1}(I) \setminus H^1(I)$, i.e. u is AC but not in $H^1(I)$; u is continuous but not AC; and u is not continuous.[8]

Case 1. Suppose $u \in H^1(I)$. Note that in this case the assumption that $u - \lambda z \in H^1(I)$ implies that $z \in H^1(I)$ as well, so in particular $\partial_x u, \partial_x z \in L^2(I)$. Additionally, Lemma 5.21 also yields $u{\downarrow} \in H^3(I)$. We have

$$\varphi(u - \lambda z) = \int_I (\partial_x(u - \lambda z))^2 \,\mathrm{d}x = \varphi(u) + \lambda^2 \|\partial_x z\|^2_{L^2(I)} - 2\lambda \int_I (\partial_x u)(\partial_x z)\,\mathrm{d}x\,.$$

Therefore it suffices to show that

$$\int_I (\partial_x u)(\partial_x z)\,\mathrm{d}x \overset{?}{\leq} 0\,. \tag{13}$$

[8] As usual, phrases such as “u is continuous” should be understood as “the object $u \in L^2(I)$ has a continuous representative”, and in particular the condition “u is continuous but not AC” makes sense because the continuous representative, if it exists, is unique.

Recalling that $u = u\uparrow + u\downarrow$ with $u\uparrow, u\downarrow \in H^1(I) = W^{1,2}(I)$ by Proposition 5.25, we have

$$\int_I (\partial_x u)(\partial_x z)\,\mathrm{d}x = \int_I (\partial_x u\uparrow)(\partial_x z)\,\mathrm{d}x + \int_I (\partial_x u\downarrow)(\partial_x z)\,\mathrm{d}x\,,$$

with the second term in the RHS satisfying

$$\int_I (\partial_x u\downarrow)(\partial_x z)\,\mathrm{d}x = (\partial_x u\downarrow)z|_0^1 - \int_I (\partial_x\partial_x u\downarrow)z\,\mathrm{d}x = -\|\partial_x\partial_x u\downarrow\|^2_{L^2(I)} \le 0\,.$$

By Proposition 5.25, $\partial_x u\uparrow = \partial_x^+ u$ and $\partial_x u\downarrow = \partial_x^- u$ in $L^2(I)$. Since $z = \partial_x\partial_x u\downarrow$ in $L^2(I)$, we get

$$\int_I (\partial_x u\uparrow)(\partial_x z)\,\mathrm{d}x = \int_I (\partial_x^+ u)(\partial_x\partial_x\partial_x u\downarrow)\,\mathrm{d}x\,.$$

We claim that the quantity above is zero. Indeed, fixing any representative of $\partial_x u \in L^2(I)$, let $S := \{x \in I : \partial_x u > 0\}$. First, we have $(\partial_x^+ u)(\partial_x\partial_x\partial_x^- u) = \partial_x^+ u = 0$ on $I \setminus S$. Second, we have $\partial_x u\downarrow = \partial_x^- u = 0$ on S, so applying Fact 5.26 twice (recall that $u\downarrow \in H^3(I)$) gives that $\partial_x\partial_x\partial_x u\downarrow = 0$ a.e. in S, thus establishing the claim. Thus (13) indeed holds, which concludes the proof in Case 1.

Case 2. Suppose $u \in W^{1,1}(I) \setminus H^1(I)$. We will derive a contradiction, showing that this case cannot happen. Note that, since $u - \lambda z \in H^1(I)$ by assumption, we conclude that $z \in W^{1,1}(I) \setminus H^1(I)$, and in particular u, z are AC with $\partial_x u, \partial_x z \in L^1(I)$. As in the previous case, we have

$$\varphi(u - \lambda z) = \int_I \left(\partial_x(u - \lambda z)\right)^2 \mathrm{d}x \tag{14}$$

$$= \int_I \left(\partial_x u - \lambda\partial_x z\right)^2 \mathrm{d}x \tag{15}$$

$$= \int_I \left((\partial_x u)^2 - 2\lambda(\partial_x u)(\partial_x z) + \lambda^2(\partial_x z)^2\right) \mathrm{d}x\,. \tag{16}$$

We claim that the function $(\partial_x u)(\partial_x z) \in L^1(I)$. First, by Proposition 5.25 we have $u\uparrow \in W^{1,1}(I)$ (while $u\downarrow \in H^1(I)$ since $u\downarrow \in H^2(I)$), as well as $\partial_x u\uparrow = \partial_x^+ u$ and $\partial_x u\downarrow = \partial_x^- u$ a.e. in I. Hence

$$\int_I |(\partial_x u)(\partial_x z)|\,\mathrm{d}x = \int_I |(\partial_x u\uparrow)(\partial_x z) + (\partial_x u\downarrow)(\partial_x z)|\,\mathrm{d}x \le \int_I \Big[|(\partial_x u\uparrow)(\partial_x z)| + |(\partial_x u\downarrow)(\partial_x z)|\Big]\,\mathrm{d}x\,.$$

We claim that $(\partial_x u\uparrow)(\partial_x z), (\partial_x u\downarrow)(\partial_x z) \in L^1(I)$. First, we again have that $(\partial_x u\uparrow)(\partial_x z) = (\partial_x u\uparrow)(\partial_x\partial_x\partial_x u\downarrow) = 0$ a.e. as in the previous case, where in particular we are allowed to apply Fact 5.26 twice to $\partial_x u\downarrow$ because $\partial_x u\downarrow \in W^{2,1}(I)$ by virtue of the fact that $\partial_x\partial_x u\downarrow = z \in W^{1,1}(I)$ in the current case. Hence $(\partial_x u\uparrow)(\partial_x z) \in L^1(I)$. Second, note that $\partial_x u\downarrow$ is AC and hence bounded, while $\partial_x z \in L^1(I)$ since z is AC. Therefore

$$\int_I |(\partial_x u\downarrow)(\partial_x z)|\,\mathrm{d}x \le \|\partial_x u\downarrow\|_{L^\infty(I)}\|\partial_x z\|_{L^1(I)} < +\infty\,,$$

and hence $(\partial_x u\downarrow)(\partial_x z) \in L^1(I)$. Hence $|(\partial_x u\uparrow)(\partial_x z)| + |(\partial_x u\downarrow)(\partial_x z)| \in L^1(I)$ and

$$\int_I |(\partial_x u)(\partial_x z)|\,\mathrm{d}x \le \int_I |(\partial_x u\uparrow)(\partial_x z)|\,\mathrm{d}x + \int_I |(\partial_x u\downarrow)(\partial_x z)|\,\mathrm{d}x < +\infty\,,$$

so $(\partial_x u)(\partial_x z) \in L^1(I)$ as claimed. Now, since $\varphi(u - \lambda z) < +\infty$ by assumption, (16) shows that $(\partial_x u)^2 - 2\lambda(\partial_x u)(\partial_x z) + \lambda^2(\partial_x z)^2 \in L^1(I)$, while we have just established that $2\lambda(\partial_x u)(\partial_x z) \in L^1(I)$. We conclude that $(\partial_x u)^2 + \lambda^2(\partial_x z)^2 \in L^1(I)$, i.e.

$$\int_I \left((\partial_x u)^2 + \lambda^2(\partial_x z)^2\right) \mathrm{d}x < +\infty\,.$$

On the other hand, the fact that $u, z \in W^{1,1}(I) \setminus H^1(I)$ implies that

$$\int_I (\partial_x u)^2 \,\mathrm{d}x = +\infty \qquad \text{and} \qquad \int_I (\partial_x z)^2 \,\mathrm{d}x = +\infty\,,$$

which is the desired contradiction. This concludes the proof in Case 2.

Case 3. Suppose u is continuous but not AC (and hence the same is true of $u{\uparrow}$). Then by the definition of absolute continuity, there exists $\epsilon > 0$ such that, for all $\delta > 0$, there exists a set of pairwise disjoint intervals $((a_i, b_i))_{i\in[k]}$ in I such that $\sum_{i=1}^k (b_i - a_i) < \delta$ and $\sum_{i=1}^k |u(a_i) - u(b_i)| > \epsilon$.

We claim that, moreover, the sequences $((a_i, b_i))_{i\in[k]}$ above can always be taken to satisfy $u(a_i) < u(b_i)$ for every $i \in [k]$. Indeed, let $\epsilon > 0$ be as in the paragraph above, let $\delta > 0$, and let $((a_i, b_i))_{i\in[k]}$ be the corresponding sequence. Let $S := \{i \in [k] : u(a_i) \geq u(b_i)\}$. Then for each $i \in S$, we have

$$|u(a_i) - u(b_i)| = u(a_i) - u(b_i) = \underbrace{\Big[u{\downarrow}(a_i) - u{\downarrow}(b_i)\Big]}_{\geq 0} + \underbrace{\Big[u{\uparrow}(a_i) - u{\uparrow}(b_i)\Big]}_{\leq 0} \leq |u{\downarrow}(a_i) - u{\downarrow}(b_i)|\,.$$

Now, since $u{\downarrow}$ is AC, let $\delta > 0$ be small enough so that

$$\sum_{i\in S} |u{\downarrow}(a_i) - u{\downarrow}(b_i)| \leq \frac{\epsilon}{2}\,.$$

It follows that

$$\epsilon < \sum_{i=1}^k |u(a_i) - u(b_i)| \leq \sum_{i\in S} |u{\downarrow}(a_i) - u{\downarrow}(b_i)| + \sum_{i\in[k]\setminus S} |u(a_i) - u(b_i)| \leq \frac{\epsilon}{2} + \sum_{i\in[k]\setminus S} |u(a_i) - u(b_i)|\,,$$

and hence

$$\sum_{i\in[k]\setminus S} |u(a_i) - u(b_i)| > \frac{\epsilon}{2}\,,$$

and of course $u(a_i) < u(b_i)$ for each $i \in [k] \setminus S$ and $\sum_{i\in[k]\setminus S} (b_i - a_i) < \delta$. This establishes the claim.

Let $v := u - \lambda z$ for convenience, and fix any sequence $((a_i, b_i))_{i\in[k]}$ of pairwise disjoint intervals satisfying $u(a_i) < u(b_i)$ for each $i \in [k]$. Using Lemma 5.32, we map each interval (a_i, b_i) into an interval $(a_i', b_i') \subseteq (a_i, b_i)$ such that

$$\left|v(a_i') - v(b_i')\right| \geq \frac{|u(a_i) - u(b_i)|}{2}\,.$$

This implies that v is not AC and hence $\varphi(v) = +\infty$, thus concluding the proof in this case.

Case 4. Suppose u is not continuous. By Lemma 5.33, $u - \lambda z$ is not continuous, which implies that $u - \lambda z \notin H^1(I)$ and $\varphi(u - \lambda z) = +\infty$, concluding the proof. $\square$

Lemma 5.32. *Let $u \in D(\partial\mathcal{E}^-)$, suppose u is continuous, and let $z \in -\partial\mathcal{E}^-(u)$. Let $\lambda > 0$ and let $v := u - \lambda z$. Let $0 < a < b < 1$ and suppose $u(a) < u(b)$. Then for all $\epsilon > 0$, there exist a', b' with $a \le a' < b' \le b$ such that*

$$v(b') - v(a') \ge |u(a) - u(b)| - \epsilon\,.$$

Proof. Recall that, by Proposition 5.18, u, z form a weak solution to the static Neumann problem, and by Lemma 5.21 we have $u{\downarrow} \in H^2(I)$ with $\partial_x u{\downarrow} = 0$ on $\{0, 1\}$ and $z = \partial_x\partial_x u{\downarrow}$ in $L^2(I)$. In particular, $u{\downarrow}$ is continuously differentiable by Fact 5.27.

We first observe that it cannot be the case that $\partial_x u{\downarrow}(x) < 0$ for all $x \in (a, b)$, since otherwise Lemma 5.22 would imply that $u{\uparrow}$ is constant in (a, b), which by continuity would imply that $u{\uparrow}(a) = u{\uparrow}(b)$ and hence $u(a) \ge u(b)$, a contradiction.

We first construct $a' \in (a, b)$. In particular, we wish a' to satisfy

$$u(a') \overset{?}{\le} u(a) + \frac{\epsilon}{2} \qquad \text{and} \qquad z(a') \overset{?}{\ge} 0\,. \tag{17}$$

Let $x^* \in [a, b)$ be given by

$$x^* := \inf\{x \in (a, b) : \partial_x u{\downarrow}(x) = 0\}\,,$$

which is well-defined by the observation above and the fact that $\partial_x u{\downarrow} \le 0$ in all of I (since $u{\downarrow}$ is nonincreasing and continuously differentiable). By the continuity of $\partial_x u{\downarrow}$, we have that $\partial_x u{\downarrow}(x^*) = 0$. Since $\partial_x u{\downarrow}(a) \le 0$, it must be the case that either $a = x^*$ or

$$\mathcal{L}\,\{x \in (a, x^*) : \partial_x\partial_x u{\downarrow}(x) \ge 0\} > 0\,. \tag{18}$$

We consider each case separately. First, suppose $a = x^*$. We claim that for all $\delta > 0$, there exists $x \in (a, a + \delta)$ such that $z(x) \ge 0$. Suppose for a contradiction that this is not the case, and fix $\delta > 0$ such that for all $x \in (a, a + \delta)$, $z(x) < 0$. Then since $z = \partial_x\partial_x u{\downarrow}$ almost everywhere, we conclude that $\partial_x\partial_x u{\downarrow} < 0$ a.e. in $(a, a + \delta)$. Hence $\partial_x u{\downarrow}(x) < 0$ for all $x \in (a, a + \delta)$, contradicting the assumption that $a = x^*$ given the definition of x^*. Thus the claim holds. Now, using the continuity of u, choose $\delta > 0$ small enough and choose $a' \in (a, a + \delta)$ so that $z(a') \ge 0$ and moreover $u(a') \le u(a) + \epsilon/2$. This choice of a' satisfies (17).

Second, suppose $a < x^*$ and (18) holds. Since $z = \partial_x\partial_x u{\downarrow}$ almost everywhere, choose $a' \in (a, x^*)$ such that $z(a') \ge 0$, which is possible by (18). Now, since $\partial_x u{\downarrow} < 0$ for all $x \in (a, a') \subset (a, x^*)$ by the choice of x^*, Lemma 5.22 implies that $u{\uparrow}$ is constant in (a, a'). By the continuity of $u{\uparrow}$, we conclude that $u{\uparrow}(a) = u{\uparrow}(a')$ and hence, since $u{\downarrow}$ is nonincreasing, we have

$$u(a') = u{\uparrow}(a') + u{\downarrow}(a') \le u{\uparrow}(a) + u{\downarrow}(a) = u(a)\,,$$

and again (17) is satisfied. This concludes the choice of a'.

Now, we may assume without loss of generality that $\epsilon < |u(a) - u(b)|/2$. Therefore our choice of a' yields an interval (a', b) which, using (17) and recalling that $u(a) < u(b)$, satisfies $u(a') < u(b)$. Therefore, repeating a symmetric version of the argument above yields a choice of $b' \in (a', b)$ satisfying

$$u(b') \ge u(b) - \frac{\epsilon}{2} \qquad \text{and} \qquad z(b') \le 0\,. \tag{19}$$

Combining (17) and (19), we conclude that

$$v(b') - v(a') = \Big[u(b') - u(a')\Big] - \lambda \underbrace{\Big[z(b') - z(a')\Big]}_{\le 0} \ge u(b) - u(a) - \epsilon = |u(a) - u(b)| - \epsilon\,. \qquad \square$$

Lemma 5.33. *Let $u \in D(\partial\mathcal{E}^-)$, suppose u is not continuous, and let $z \in -\partial\mathcal{E}^-(u)$. Let $\lambda > 0$. Then $u - \lambda z$ is not continuous.*

Proof. Recall that, by Proposition 5.18, u, z form a weak solution to the static Neumann problem, and by Lemma 5.21 we have $u\!\downarrow \in H^2(I)$ with $\partial_x u\!\downarrow = 0$ on $\{0,1\}$ and $z = \partial_x\partial_x u\!\downarrow$ in $L^2(I)$. In particular, $u\!\downarrow$ is continuously differentiable by Fact 5.27.

Since $u\!\downarrow \in H^2(I)$ is continuous while u is not continuous, we conclude that $u\!\uparrow$ is not continuous and, since it is monotone, it contains only jump discontinuities. Let $x_0 \in I$ be a point of jump discontinuity of $u\!\uparrow$, i.e. a point such that $L_- < L_+$ where

$$L_- := \lim_{x\to x_0^-} u\!\uparrow(x) \qquad \text{and} \qquad L_+ := \lim_{x\to x_0^+} u\!\uparrow(x)\,.$$

We first claim that $\partial_x u\!\downarrow(x_0) = 0$. Indeed, it is clear that $\partial_x u\!\downarrow \le 0$ since $u\!\downarrow$ is nonincreasing and continuously differentiable. If we had $\partial_x u\!\downarrow(x_0) < 0$, then by continuity $\partial_x u\!\downarrow$ would be strictly negative in a neighbourhood of x_0, in which case Lemma 5.22 would imply that $u\!\uparrow$ is constant in a neighbourhood of x_0, contradicting the fact that $L_- < L_+$. Hence the claim holds.

We now claim that, for all $\delta > 0$, there exists $x \in (x_0, x_0 + \delta)$ such that $z(x) \le 0$. Suppose for a contradiction that there exists $\delta > 0$ such that for all $x \in (x_0, x_0 + \delta)$, $z(x) > 0$. Then, since $z = \partial_x\partial_x u\!\downarrow$ a.e. and $\partial_x u\!\downarrow(x_0) = 0$, we conclude that $\partial_x u\!\downarrow > 0$ in $(x_0, x_0 + \delta)$, contradicting the fact that $\partial_x u\!\downarrow \le 0$. Hence the claim holds. By the same reasoning, we conclude that for all $\delta > 0$ there exists $x \in (x_0 - \delta, x_0)$ such that $z(x) \ge 0$.

Thus, for all $\delta > 0$ there exist points $x_- \in (x_0-\delta, x_0)$, $x_+ \in (x_0, x_0+\delta)$ such that $z(x_+) - z(x_-) \le 0$ and hence

$$\begin{aligned}(u-\lambda z)(x_+) - (u-\lambda z)(x_-) &= \Big[u\!\uparrow(x_+) - u\!\uparrow(x_-)\Big] + \Big[u\!\downarrow(x_+) - u\!\downarrow(x_-)\Big] - \lambda\Big[z(x_+) - z(x_-)\Big] \\ &\ge \Big[u\!\uparrow(x_+) - u\!\uparrow(x_-)\Big] + \Big[u\!\downarrow(x_+) - u\!\downarrow(x_-)\Big]\,.\end{aligned}$$

Since $u\!\uparrow \to L_+$ as $x \downarrow x_0$ and $u\!\uparrow \to L_-$ as $x \uparrow x_0$ and $u\!\uparrow$ is nondecreasing, we have $u\!\uparrow(x_+) - u\!\uparrow(x_-) \ge L_+ - L_-$. By the continuity of $u\!\downarrow$, we can let $\delta > 0$ be small enough so that $u\!\downarrow(x_+) - u\!\downarrow(x_-) \ge -\frac{L_+ - L_-}{2}$. We conclude that, for all sufficiently small $\delta > 0$, there exist points $x_- \in (x_0-\delta, x_0)$, $x_+ \in (x_0, x_0 + \delta)$ such that

$$(u-\lambda z)(x_+) - (u-\lambda z)(x_-) \ge \frac{L_+ - L_-}{2}\,,$$

and thus $u - \lambda z$ is not continuous. ☐

Lemma 5.34 (Specialization of [Bre73, Theorem 4.4])**.** *Let H be a Hilbert space and let $\tau : H \to [0, +\infty]$ be a convex, proper, and lower semicontinuous functional such that $\tau(\mathrm{Proj}_{\overline{D(A)}} x) \le \tau(x)$ for all $x \in H$. Let $A : H \to 2^H$ be a maximal monotone operator and let S_t be the semigroup generated by $-A$. Then the following are equivalent:*

1. *$\tau((I + \lambda A)^{-1}x) \le \tau(x)$ for all $x \in H$ and $\lambda > 0$; and*
2. *$\tau(S_t x) \le \tau(x)$ for all $x \in \overline{D(A)}$ and $t \ge 0$.*

In the statement above, $(I + \lambda A)^{-1} : H \to D(A)$ is the *resolvent* of A (also denoted by J_λ, see e.g. [Eva10, p. 563]).

Proposition 5.35 (φ-monotonicity of solutions)**.** *Let $u_0 \in \mathcal{U}$ and let $\boldsymbol{u} \in C([0, +\infty); L^2(I))$ be the solution to the gradient flow problem with initial data u_0. Then for all $0 \le t_1 \le t_2 < +\infty$, we have $\varphi(\boldsymbol{u}(t_1)) \ge \varphi(\boldsymbol{u}(t_2))$.*

Proof. This is a direct consequence of Lemma 5.34. Indeed, letting $H := L^2(I)$ and $A := \partial\mathcal{E}^-$ (which is maximal monotone as remarked earlier), we first observe that indeed $\varphi(\mathrm{Proj}_{\overline{D(A)}} f) \leq \varphi(f)$ for all $f \in L^2(I)$, since by Observation 5.29 we have $\mathrm{Proj}_{\overline{D(A)}} f = f$. Thus it suffices to show that for all $f \in L^2(I)$ and all $\lambda > 0$,

$$\varphi((I + \lambda A)^{-1} f) \stackrel{?}{\leq} \varphi(f)\,.$$

But this is equivalent to showing that, for all $u \in D(A)$, $-z \in A(u)$ and $\lambda > 0$,

$$\varphi(u) \stackrel{?}{\leq} \varphi(u - \lambda z)\,,$$

which is precisely Lemma 5.31. ☐

Corollary 5.36 (Preservation of H^1 regularity)**.** *Suppose $u_0 \in H^1(I)$, and let $\boldsymbol{u} \in C([0, +\infty); L^2(I))$ be the solution to the gradient flow problem with initial data u_0. Then $\boldsymbol{u}(t) \in H^1(I)$ for all $t > 0$.*

Proof. This is an immediate consequence of Proposition 5.35 and the definition of φ. ☐

5.5 Preservation of Lipschitz regularity

It will also be useful to control the Lipschitz regularity of solutions. At a high level, we follow a similar strategy as in Section 5.4. Recall that $W^{1,\infty}(I)$ is equivalent to the space of Lipschitz real-valued functions on I, up to identification of almost everywhere equal functions. Define $\psi : L^2(I) \to [0, +\infty]$ by

$$\psi(u) := \begin{cases} \|\partial_x u\|_{L^\infty(I)} & \text{if } u \in W^{1,\infty}(I) \\ +\infty & \text{otherwise.} \end{cases}$$

Fact 5.37. *Let $u \in L^2(I)$ and let $M \in \mathbb{R}_{\geq 0}$. Then $\psi(u) \leq M$ if and only if $u = f$ a.e. for some M-Lipschitz function $f : I \to \mathbb{R}$. Moreover, in this case f is the (unique) continuous representative of u and $\psi(u)$ is the Lipschitz constant of f.*

Claim 5.38. *The functional ψ is convex, proper and lower semicontinuous.*

Proof. Convexity and properness are straightforward; it remains to verify lower semicontinuity. Since $L^2(I)$ is a metric space, it suffices to check sequential lower semicontinuity. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(I)$ such that $u_n \to u$ in $L^2(I)$. We need to show that

$$\psi(u) \stackrel{?}{\leq} \liminf_{n\to\infty} \psi(u_n)\,.$$

The only relevant case is when the RHS above is finite, so suppose there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \psi(u_{n_k}) = M < +\infty$. By extracting a subsequence if necessary, we may assume that $\psi(u_{n_k}) < +\infty$, and thus $u_{n_k} \in W^{1,\infty}(I)$, for every k. For simplicity and using Fact 5.37, fix for each n_k the continuous representative of u_{n_k}, which we denote by the same name. Then each u_{n_k} is M_k-Lipschitz with $M_k \to M$.

Since $u_{n_k} \to u$ in $L^2(I)$, it is standard that we may extract a subsequence that converges to u pointwise almost everywhere. Denote this further subsequence again by $(u_{n_k})_k$, and let $N \subset I$ be a measure zero set such that $u_{n_k} \to u$ pointwise in $I \setminus N$. Then for each $x \neq y$ in $I \setminus N$, we have

$$|u(x) - u(y)| = \lim_{k\to\infty} |u_{n_k}(x) - u_{n_k}(y)| \leq \lim_{k\to\infty} M_k|x - y| = M|x - y|\,.$$

By Lemma A.3, u is a.e. equal to an M-Lipschitz function, so $\psi(u) \leq M$ by Fact 5.37, as needed. ☐

Definition 5.39. For any $w : I \to \mathbb{R}$ and distinct $x, y \in I$, let

$$\mathsf{slope}_w(x,y) := \frac{w(y) - w(x)}{y - x}\,.$$

Lemma 5.40. *Let $u \in D(\partial\mathcal{E}^-)$ and let $z \in -\partial\mathcal{E}^-(u)$. Then for all $\lambda > 0$, $\psi(u - \lambda z) \geq \psi(u)$.*

Proof. We follow the proof outline from Lemma 5.31, but some of the technical details are different. As in that proof, we have $u{\downarrow} \in H^2(I)$ with $\partial_x u{\downarrow} = 0$ on $\{0, 1\}$ and $z = \partial_x \partial_x u{\downarrow}$ in $L^2(I)$.

We may assume that $u - \lambda z \in W^{1,\infty}(I)$, since otherwise $\psi(u - \lambda z) = +\infty$ and there is nothing to prove. In particular, this establishes that $u - \lambda z$ is continuous.

When u is not continuous, Lemma 5.33 yields that $u - \lambda z$ is not continuous, a contradiction. Therefore we may assume that u is continuous. Since both u and $u - \lambda z$ are continuous, we conclude that z is also continuous. Let $v := u - \lambda z$. By Fact 5.37, it suffices to show that for every $(a, b) \subset I$ and all $\epsilon > 0$, there exists $(a', b') \subset I$ such that

$$\left|\mathsf{slope}_v(a', b')\right| \overset{?}{\geq} |\mathsf{slope}_u(a, b)| - \epsilon\,. \tag{20}$$

If $u(a) < u(b)$, then the interval $(a', b') \subseteq (a, b)$ given by Lemma 5.32 with parameter $\epsilon(b - a)$ satisfies (20); and if $u(a) = u(b)$, then any interval will do. Therefore suppose $u(a) > u(b)$, so in particular $\mathsf{slope}_u(a, b) < 0$.

Note that a sufficient condition for (20) is that $|\mathsf{slope}_u(a', b')| \geq |\mathsf{slope}_u(a, b)| - \epsilon$ with $u(a') > u(b')$, $z(a') \leq 0$ and $z(b') \geq 0$, since in that case,

$$\begin{aligned}\left|\mathsf{slope}_v(a', b')\right| &= \frac{|(u - \lambda z)(b') - (u - \lambda z)(a')|}{b' - a'} = \Big|\underbrace{u(b') - u(a')}_{<0} - \lambda\underbrace{(z(b') - z(a'))}_{\geq 0}\Big| \left(\frac{1}{b' - a'}\right) \\ &\geq \frac{|u(b') - u(a')|}{b' - a'} = \left|\mathsf{slope}_u(a', b')\right| \geq |\mathsf{slope}_u(a, b)| - \epsilon\,,\end{aligned}$$

which is (20). We now find a' and b' satisfying the aforementioned conditions.

For each point $x \in I$, say x is *left-favourable* if every neighbourhood $(x - \delta, x + \delta)$ contains a point x' such that $z(x') \leq 0$. Similarly, say x is *right-favourable* if every neighbourhood $(x - \delta, x + \delta)$ contains a point x' such that $z(x') \geq 0$.

We claim that there exist points $a^* < b^*$ in I such that a^* is left-favourable, b^* is right-favourable, and $|\mathsf{slope}_u(a^*, b^*)| \geq |\mathsf{slope}_u(a, b)|$ with $u(a^*) > u(b^*)$. Let us first show that this claim yields the desired points a' and b', and then proceed to prove the claim. Suppose we have a^* and b^* as claimed. Then by the continuity of u, we may fix sufficiently small $\delta > 0$ and find points $a' \in (a^* - \delta, a^* + \delta)$ and $b' \in (b^* - \delta, b^* + \delta)$ such that $a' < b'$, $z(a') \leq 0$, $z(b') \geq 0$, $u(a') > u(b')$, and, for $\alpha := \frac{\epsilon}{2} \cdot \frac{b^* - a^*}{u(a^*) - u(b^*)}$ and $\beta := \frac{\epsilon(b^* - a^*)}{2}$,

$$\begin{aligned}\left|\mathsf{slope}_u(a', b')\right| &= \frac{u(a') - u(b')}{b' - a'} \geq \frac{u(a^*) - u(b^*) - \beta}{(b^* - a^*)(1 + \alpha)} \geq \frac{u(a^*) - u(b^*)}{b^* - a^*}(1 - \alpha) - \frac{\epsilon/2}{1 + \alpha} \\ &\geq \frac{u(a^*) - u(b^*)}{b^* - a^*} - \epsilon = |\mathsf{slope}_u(a^*, b^*)| - \epsilon \geq |\mathsf{slope}_u(a, b)| - \epsilon\,,\end{aligned}$$

as desired, where we used the inequality $\frac{1}{1+\alpha} \geq 1 - \alpha$. We now establish the existence of a^* and b^*.

We first find a^*. If a is left-favourable, then choose $a^* = a$; note that, of course, we have $u(a^*) > u(b)$ and $|\mathsf{slope}_u(a^*, b)| \geq |\mathsf{slope}_u(a, b)|$. Otherwise, a is not left-favourable, which gives some $\delta > 0$ such that $z(x) > 0$ for every $x \in (a - \delta, a + \delta)$, and hence $\partial_x \partial_x u{\downarrow} > 0$ a.e. in $(a - \delta, a + \delta)$. We now consider two cases. Recall that $\mathsf{slope}_u(a, b) < 0$.

First, suppose $\partial_x u{\downarrow}(a) \geq \mathsf{slope}_u(a, b)$. Note that it cannot be the case that $\partial_x u{\downarrow}(x) > \mathsf{slope}_u(a, b)$ for all $x \in (a, b)$, since otherwise we would have

$$u(b) = u{\uparrow}(b) + u{\downarrow}(b) \geq u{\uparrow}(a) + u{\downarrow}(a) + \int_{(a,b)} \partial_x u{\downarrow} \, \mathrm{d}x > u(a) + (b - a) \, \mathsf{slope}_u(a, b) = u(b) \,,$$

a contradiction. Therefore we may let

$$a^* := \inf \left\{ x \in (a, b) : \partial_x u{\downarrow}(x) \leq \mathsf{slope}_u(a, b) \right\} \,.$$

By the continuity of $\partial_x u{\downarrow}$ (recall that $u{\downarrow} \in H^2(I)$ is continuously differentiable), we conclude that $\partial_x u{\downarrow}(a^*) = \mathsf{slope}_u(a, b)$. We also observe that we must have $a^* > a$ since, as noted above, we have $\partial_x \partial_x u{\downarrow}(x) > 0$ a.e. in some neighbourhood $(a - \delta, a + \delta)$, which implies that $\partial_x u{\downarrow}(x) > \partial_x u{\downarrow}(a) \geq \mathsf{slope}_u(a, b)$ for all $x \in (a, a + \delta)$. We claim that a^* is left-favourable. Indeed, otherwise there would be some $\delta > 0$ such that $z(x) > 0$ for all $x \in (a^* - \delta, a^* + \delta)$, and since $z = \partial_x \partial_x u{\downarrow}$ a.e. we would conclude that $\partial_x u{\downarrow}(x) < \partial_x u{\downarrow}(a^*) = \mathsf{slope}_u(a, b)$ for $x \in (a^* - \delta, a^*)$, contradicting the choice of a^*. Hence we have found a left-favourable a^* in this case; we claim that a^* also satisfies $u(a^*) > u(b)$ and $|\mathsf{slope}_u(a^*, b)| \geq |\mathsf{slope}_u(a, b)|$. The first inequality holds since

$$\begin{aligned}
u(a^*) - u(b) &= u{\downarrow}(a^*) + u{\uparrow}(a^*) - \big[u(a) + (b - a) \, \mathsf{slope}_u(a, b)\big] \\
&\geq u{\downarrow}(a) + \int_{(a,a^*)} \partial_x u{\downarrow} \, \mathrm{d}x + u{\uparrow}(a) - u{\uparrow}(a) - u{\downarrow}(a) - (b - a) \, \mathsf{slope}_u(a, b) \\
&= \int_{(a,a^*)} \left(\underbrace{\partial_x u{\downarrow}(x)}_{> \mathsf{slope}_u(a,b)} - \mathsf{slope}_u(a, b) \right) \mathrm{d}x - (b - a^*) \underbrace{\mathsf{slope}_u(a, b)}_{<0} \\
&> 0 \,,
\end{aligned}$$

and the second inequality holds since

$$\begin{aligned}
&(b - a^*)(b - a)\Big(|\mathsf{slope}_u(a^*, b)| - |\mathsf{slope}_u(a, b)|\Big) \\
&\quad = (b - a^*)(b - a) \left(\frac{u(a^*) - u(b)}{b - a^*} - \frac{u(a) - u(b)}{b - a} \right) \\
&\quad = \Big(u(a^*) - u(b)\Big)(b - a) - \Big(u(a) - u(b)\Big)(b - a^*) \\
&\quad = \left(u{\uparrow}(a^*) + u{\downarrow}(a) + \int_{(a,a^*)} \underbrace{\partial_x u{\downarrow}(x)}_{> \mathsf{slope}_u(a,b)} \, \mathrm{d}x - u(b) \right)(b - a) - \Big(u(a) - u(b)\Big)(b - a^*) \\
&\quad > \Big(u{\uparrow}(a) + u{\downarrow}(a) + (a^* - a) \, \mathsf{slope}_u(a, b) - u(b)\Big)(b - a) - \Big(u(a) - u(b)\Big)(b - a^*) \\
&\quad = \Big(u(a) + (a^* - a) \frac{u(b) - u(a)}{b - a} - u(b)\Big)(b - a) - \Big(u(a) - u(b)\Big)(b - a^*) \\
&\quad = \Big(u(a) - u(b)\Big)\Big((b - a) - (b - a^*) - (a^* - a)\Big) \\
&\quad = 0 \,.
\end{aligned}$$

Therefore in the first case we have found a left-favourable $a^* \in (a, b)$ such that $u(a^*) > u(b)$ and $|\mathsf{slope}_u(a^*, b)| \geq |\mathsf{slope}_u(a, b)|$.

Second, suppose $\partial_x u{\downarrow}(a) < \mathsf{slope}_u(a,b)$. We proceed similarly, but with points to the left of a instead. Namely, it cannot be the case that $\partial_x u{\downarrow}(x) < \mathsf{slope}_u(a,b)$ for all $x \in (0,a)$, since by continuity this would imply that $\partial_x u{\downarrow}(0) \le \mathsf{slope}_u(a,b) < 0$, a contradiction. Hence we may define

$$a^* := \sup\{x \in (0,a) : \partial_x u{\downarrow}(x) \ge \mathsf{slope}_u(a,b)\}\ .$$

By continuity, we conclude that $\partial_x u{\downarrow}(a^*) = \mathsf{slope}_u(a,b)$, which also implies that $a^* < a$. We claim that a^* is left-favourable. Indeed, if it was not, then for some $\delta > 0$ we would have $z(x) > 0$ for all $x \in (a^* - \delta, a^* + \delta)$, and since $z = \partial_x \partial_x u{\downarrow}$ a.e. we would conclude that $\partial_x u{\downarrow}(x) > \partial_x u{\downarrow}(a^*) = \mathsf{slope}_u(a,b)$ for all $x \in (a^*, a^* + \delta)$, contradicting the choice of a^*. We now claim that a^* also satisfies $u(a^*) > u(b)$ and $|\mathsf{slope}_u(a^*,b)| \ge |\mathsf{slope}_u(a,b)|$.

To prove the first inequality, we first observe that $\partial_x u{\downarrow}(x) < \mathsf{slope}_u(a,b) < 0$ for all $x \in (a^*, a)$, which by Lemma 5.22 implies that $u{\uparrow}$ is constant in (a^*, a). Since $u{\uparrow}$ is continuous (because u and $u{\downarrow}$ are), we conclude that $u{\uparrow}(a^*) = u{\uparrow}(a)$. Therefore we have

$$u(a^*) - u(a) = u{\downarrow}(a^*) - u{\downarrow}(a) \ge 0$$

since $u{\downarrow}$ is nonincreasing, and hence $u(a^*) \ge u(a) > u(b)$. As for the second inequality, we have

$$\begin{aligned}
&(b-a^*)(b-a)\Big(|\mathsf{slope}_u(a^*,b)| - |\mathsf{slope}_u(a,b)|\Big)\\
&\quad = (b-a^*)(b-a)\left(\frac{u(a^*)-u(b)}{b-a^*} - \frac{u(a)-u(b)}{b-a}\right)\\
&\quad = \Big(u(a^*)-u(b)\Big)(b-a) - \Big(u(a)-u(b)\Big)(b-a^*)\\
&\quad = \left(u{\uparrow}(a) + u{\downarrow}(a) - \int_{(a^*,a)} \underbrace{\partial_x u{\downarrow}(x)}_{<\mathsf{slope}_u(a,b)}\,\mathrm{d}x - u(b)\right)(b-a) - \Big(u(a)-u(b)\Big)(b-a^*)\\
&\quad > \left(u(a) - (a-a^*)\frac{u(b)-u(a)}{b-a} - u(b)\right)(b-a) - \Big(u(a)-u(b)\Big)(b-a^*)\\
&\quad = \Big(u(a)-u(b)\Big)\Big((b-a) - (b-a^*) + (a-a^*)\Big)\\
&\quad = 0\,.
\end{aligned}$$

Therefore in any case we have found a left-favourable $a^* \in (0,b)$ such that $u(a^*) > u(b)$ and $|\mathsf{slope}_u(a^*,b)| \ge |\mathsf{slope}_u(a,b)|$.

Repeating an analogous argument for the right endpoint of the interval (a^*,b), we find a right-favourable $b^* \in (a^*,1)$ such that $u(a^*) > u(b^*)$ and $|\mathsf{slope}_u(a^*,b^*)| \ge |\mathsf{slope}_u(a^*,b)|$, which concludes the proof as explained above. ☐

Proposition 5.41 (ψ-monotonicity of solutions)**.** *Let $u_0 \in \mathcal{U}$ and let $\boldsymbol{u} \in C([0,+\infty); L^2(I))$ be the solution to the gradient flow problem with initial data u_0. Then for all $0 \le t_1 \le t_2 < +\infty$, we have $\psi(\boldsymbol{u}(t_1)) \ge \psi(\boldsymbol{u}(t_2))$.*

Proof. As in the proof of Proposition 5.35, this follows from Lemmas 5.34 and 5.40. ☐

Corollary 5.42 (Preservation of Lipschitz regularity)**.** *Suppose $u_0 \in W^{1,\infty}(I)$, and let $\boldsymbol{u} \in C([0,+\infty); L^2(I))$ be the solution to the gradient flow problem with initial data u_0. Then $\boldsymbol{u}(t) \in W^{1,\infty}(I)$ for all $t > 0$.*

Proof. This is an immediate consequence of Proposition 5.41 and the definition of ψ. ☐

5.6 Exponential decay of directed Dirichlet energy

Proposition 5.43. *There exists a constant $K > 0$ such that the following holds. Let $u \in \mathcal{U}$. Then for all $t > 0$,*

$$\mathcal{E}^-(P_t u) \le e^{-Kt}\mathcal{E}^-(u)\,.$$

Proof. Let $u \in \mathcal{U}$ and let $\boldsymbol{u}(t) = P_t u$ be the corresponding solution to the gradient flow problem. Recall that Proposition 5.17 gives, for all $t > 0$,

$$\mathcal{E}^-(\boldsymbol{u}(t)) = \mathcal{E}^-(u) + \int_{(0,t)} -\|\boldsymbol{u}'(s)\|^2_{L^2(I)}\,\mathrm{d}s\,.$$

It follows that $t \mapsto \mathcal{E}^-(\boldsymbol{u}(t))$ is absolutely continuous on every interval $[0,T]$ with weak derivative $\partial_t \mathcal{E}^-(\boldsymbol{u}(t)) = -\|\boldsymbol{u}'(t)\|^2_{L^2(I)}$. Moreover for a.e. $t > 0$ we have that $\boldsymbol{u}(t), \boldsymbol{u}'(t)$ form a weak solution to the static Neumann problem, and thus $\boldsymbol{u}'(t) = \partial_x\partial_x\boldsymbol{u}(t)\!\downarrow$ a.e. in I and $\partial_x\boldsymbol{u}(t)\!\downarrow = 0$ on $\{0,1\}$ by Lemma 5.21. Therefore the Poincaré inequality (for zero-on-the-boundary functions) yields

$$\mathcal{E}^-(\boldsymbol{u}(t)) = \frac{1}{2}\|\partial_x\boldsymbol{u}(t)\!\downarrow\|^2_{L^2(I)} \le \frac{1}{2}C\|\partial_x\partial_x\boldsymbol{u}(t)\!\downarrow\|^2_{L^2(I)} = \frac{C}{2}\|\boldsymbol{u}'(t)\|^2_{L^2(I)}$$

for some constant $C > 0$. It follows that for a.e. $t > 0$, we have

$$\partial_t\mathcal{E}^-(\boldsymbol{u}(t)) = -\|\boldsymbol{u}'(t)\|^2_{L^2(I)} \le -\frac{2}{C}\mathcal{E}^-(\boldsymbol{u}(t))\,,$$

which implies that for all $t > 0$,

$$\mathcal{E}^-(\boldsymbol{u}(t)) \le e^{-2t/C}\mathcal{E}^-(u)\,.$$

□

The exponential decay of $t \mapsto \mathcal{E}^-(P_t u)$ allows us to find a Cauchy sequence in $(P_t u)_{t\ge 0}$, and thus establish its strong convergence to some limit in $L^2(I)$. Later on, we will say more about this limit by reasoning about the weak convergence of $P_t u$ (to the same limit).

Lemma 5.44 (Cauchy sequence)**.** *Let $u \in \mathcal{U}$, and let $(u_k)_{k\in\mathbb{N}}$ be given by $u_k := P_k u$. Then $(u_k)_k$ is Cauchy as a sequence in $L^2(I)$. As a consequence, $u_k \to u^*$ in $L^2(I)$ for some $u^* \in L^2(I)$.*

Proof. The existence of a strong limit from the Cauchy property follows from the fact that $L^2(I)$ is a complete normed space. Let us now establish the Cauchy property. Fix any $n > m$ in $\mathbb{N}$. Then

$$\begin{aligned}
\|u_n - u_m\|_{L^2(I)} &= \left\|\int_{(m,n)} \boldsymbol{u}'(t)\,\mathrm{d}t\right\|_{L^2(I)} && \text{(Absolute continuity)}\\
&\le \sum_{j=m}^{n-1}\int_{(j,j+1)} \|\boldsymbol{u}'(t)\|_{L^2(I)}\,\mathrm{d}t && \text{(Triangle inequality)}\\
&\le \sum_{j=m}^{n-1}\left(\int_{(j,j+1)} \|\boldsymbol{u}'(t)\|^2_{L^2(I)}\,\mathrm{d}t\right)^{1/2} && \text{(Jensen's inequality)}\\
&= \sum_{j=m}^{n-1}\left(\mathcal{E}^-(u_j) - \mathcal{E}^-(u_{j+1})\right)^{1/2} && \text{(Proposition 5.17)}\\
&\le \sum_{j=m}^{\infty}\mathcal{E}^-(u_j)^{1/2} && (\mathcal{E}^- \text{ is nonnegative})
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{j=m}^{\infty} \left(e^{-Kj}\mathcal{E}^-(u)\right)^{1/2} && \text{(Proposition 5.43)} \\
&= \mathcal{E}^-(u)^{1/2}\frac{e^{-Km/2}}{1-e^{-K/2}} && \text{(Geometric series)} \\
&= A\mathcal{E}^-(u)^{1/2}e^{-Bm}
\end{aligned}
$$

for constants $A, B > 0$ that only depend on the constant K from Proposition 5.43. It follows that, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for all $n > m > N$,

$$
\|u_n - u_m\|_{L^2(I)} \leq A\mathcal{E}^-(u)^{1/2}e^{-Bm} \leq A\mathcal{E}^-(u)^{1/2}e^{-BN} < \epsilon\,,
$$

so $(u_k)_k$ is Cauchy. $\square$

Lemma 5.45 (Strong convergence)**.** *Let $u \in \mathcal{U}$. Then $P_t u$ converges in $L^2(I)$ to some $u^* \in L^2(I)$ as $t \to \infty$.*

Proof. Let u^* be such that $(P_k u)_{k\in\mathbb{N}}$ converges to u^* in $L^2(I)$ as $k \to \infty$, as given by Lemma 5.44. We claim that $P_t u \to u^*$. Let $\epsilon > 0$, and let $N \in \mathbb{N}$ be such that $\|P_k u - u^*\|_{L^2(I)} < \epsilon$ for all $k \geq N$, as given by the convergence of the sequence $(P_k u)_k$. Then for any $t \geq N$, letting $j := \lfloor t \rfloor$, we have

$$
\|P_t u - u^*\|_{L^2(I)} \leq \|P_j u - u^*\|_{L^2(I)} + \|P_t u - P_j u\|_{L^2(I)}
$$

by the triangle inequality. We have $\|P_j u - u^*\|_{L^2(I)} < \epsilon$ by the choice of N, and on the other hand,

$$
\begin{aligned}
\|P_t u - P_j u\|_{L^2(I)} &= \left\| \int_{(j,t)} \boldsymbol{u}'(s)\,\mathrm{d}s \right\|_{L^2(I)} && \text{(Absolute continuity)} \\
&\leq \int_{(j,t)} \|\boldsymbol{u}'(s)\|_{L^2(I)}\,\mathrm{d}s && \text{(Triangle inequality)} \\
&\leq (t-j)^{1/2}\left(\int_{(j,t)} \|\boldsymbol{u}'(s)\|_{L^2(I)}^2\,\mathrm{d}s\right)^{1/2} && \text{(Jensen's inequality)} \\
&\leq \left(\mathcal{E}^-(P_j u) - \mathcal{E}^-(P_t u)\right)^{1/2} && (t-j \in [0,1),\ \text{Proposition 5.17}) \\
&\leq e^{-Kj/2}\mathcal{E}^-(u)^{1/2} && (\mathcal{E}^- \geq 0,\ \text{Proposition 5.43})\,.
\end{aligned}
$$

Thus by letting N be large enough, we can ensure that

$$
\|P_t u - P_j u\|_{L^2(I)} \leq e^{-Kj/2}\mathcal{E}^-(u)^{1/2} \leq e^{-KN/2}\mathcal{E}^-(u)^{1/2} < \epsilon\,,
$$

and hence $\|P_t u - u^*\|_{L^2(I)} < 2\epsilon$. Thus $P_t u \to u^*$ in $L^2(I)$. $\square$

5.7 Nonexpansiveness and order preservation

Our goal in this section is to establish that the semigroup P_t is nonexpansive and order preserving. These properties will help us show that applying P_t to line restrictions in the multidimensional setting behaves as expected by "making progress" toward monotonicity with each application.

In the context of PDEs, one desirable way to show that a property is preserved through time is to differentiate in time, and then pass the derivative inside the integral to exploit the definition of the PDE. However, we need to establish some technical results before we can justify such calculations.

We start with the following lemma, slightly adapted from [CT80], which reveals a close connection between order preservation and nonexpansiveness (here, in the supremum norm) of operators. Hence our strategy will be to establish order preservation, and conclude nonexpansiveness.

Lemma 5.46. *Let (Ω, Σ, μ) be a finite measure space, and let $1 \leq p \leq +\infty$. Let $L^\infty(\Omega) \subset C \subset L^p(\Omega)$ have the property that for all $f \in C$ and $r \in \mathbb{R}$, $f + r \in C$. Let $T : C \to L^p(\Omega)$ be continuous as a map from $L^p(\Omega)$ to $L^p(\Omega)$ (the continuity requirement may be dropped if $p = \infty$) satisfying*

$$T(f + r) = T(f) + r \text{ a.e.} \tag{21}$$

for every $f \in C$ and $r \in \mathbb{R}$. Then the following are equivalent:

(a) For all $f, g \in C$, if $f \leq g$ a.e. then $Tf \leq Tg$ a.e. .

(b) For all $f, g \in C$, $(Tf - Tg)^+ \leq \operatorname{ess\,sup}(f - g)^+$ a.e. .

(c) For all $f, g \in C$, $|Tf - Tg| \leq \operatorname{ess\,sup}|f - g|$ a.e. .

Proof. We show (a) $\implies$ (b) $\implies$ (c) $\implies$ (a). First, assume (a) holds, and let $f, g \in C$. Suppose $r := \operatorname{ess\,sup}(f - g)^+ < +\infty$, since otherwise there is nothing to prove. Note that $g + r \geq f$ and $g + r \geq g$ both a.e. . Thus by (a) and (21), we have

$$Tg + (Tf - Tg)^+ = Tf \vee Tg \leq T(g + r) = Tg + r \text{ a.e. },$$

so $(Tf - Tg)^+ \leq r$ a.e. as needed. Next, suppose (b) holds, and let $f, g \in C$. Let $r_1 := \operatorname{ess\,sup}(f - g)^+$ and $r_2 := \operatorname{ess\,sup}(g - f)^+$; note that $\operatorname{ess\,sup}|f - g| = \max\{r_1, r_2\}$, so we may assume that $r_1, r_2 < +\infty$, since otherwise there is nothing to show. Then indeed, using (b),

$$|Tf - Tg| = \max\left\{(Tf - Tg)^+, (Tg - Tf)^+\right\} \leq \max\{r_1, r_2\} = \operatorname{ess\,sup}|f - g| \text{ a.e. },$$

as needed. Finally, suppose (c) holds, and let $f, g \in C$ satisfy $f \leq g$ a.e. . First, suppose $f, g \in L^\infty(\Omega)$. Let $r := \operatorname{ess\,sup}(g - f)$, so that $0 \leq r < +\infty$ by assumption. Then, by (c) and (21),

$$\begin{aligned}\operatorname{ess\,sup}\{Tf - Tg + r\} &= \operatorname{ess\,sup}\{T(f + r) - Tg\} \leq \operatorname{ess\,sup}|T(f + r) - Tg| \\ &\leq \operatorname{ess\,sup}|f - g + r| \leq r,\end{aligned}$$

so $Tf \leq Tg$ a.e. as needed.

Now, for general $f, g \in C$ with $f \leq g$ a.e. , we may approximate them by sequences $(f_n)_n, (g_n)_n$ in $L^\infty(\Omega) \subset C$ such that $f_n \to f$ and $g_n \to g$ in $L^p(\Omega)$ and moreover $f_n \leq g_n$ a.e. for each n; indeed, letting $f_n := \max\{-n, \min\{n, f\}\}$ and likewise for g_n, it is immediate that $f_n, g_n \in L^\infty(\Omega)$ with $f_n \leq g_n$ a.e. , and the convergences $f_n \to f$ and $g_n \to g$ in $L^p(\Omega)$ follow from the dominated convergence theorem: since $|f - f_n|^p \to 0$ pointwise and $|f - f_n| \leq |f|$ pointwise for each n, we have $\lim_{n\to\infty} \int_\Omega |f_n - f|^p \,\mathrm{d}\mu = 0$.

The previous case yields $Tf_n \leq Tg_n$ a.e. for each n, and we claim that the continuity of T implies that $Tf \leq Tg$ a.e. as well. Indeed, we have $Tf_n \to Tf$ an $Tg_n \to Tg$ in $L^p(\Omega)$ and thus

$$\begin{aligned}\left\|(Tf - Tg)^+ - (Tf_n - Tg_n)^+\right\|_{L^p(\Omega)} &\leq \|(Tf - Tg) - (Tf_n - Tg_n)\|_{L^p(\Omega)} \\ &\leq \|Tf - Tf_n\|_{L^p(\Omega)} + \|Tg - Tg_n\|_{L^p(\Omega)} \to 0,\end{aligned}$$

which means that $(Tf_n - Tg_n)^+ \to (Tf - Tg)^+$ in $L^p(\Omega)$. But since $(Tf_n - Tg_n)^+ = 0$ a.e. and hence in $L^p(\Omega)$ for each n, we conclude that $(Tf - Tg)^+ = 0$ a.e. , i.e. $Tf \leq Tg$ a.e. as needed. $\square$

The following lemma is a specific formulation of the well-known Leibniz rule for differentiating under the integral sign. It is a specialization of the version stated in [Che], and can be proved by a standard argument using the Fubini-Tonelli theorem.

Lemma 5.47 (Differentiating under the integral sign)**.** *Let (Ω, Σ, μ) be a measure space and let $[a, b] \subset \mathbb{R}$. Let $f : [a, b] \times \Omega \to \mathbb{R}$ be a jointly measurable function satisfying the following:*

1. *$f(t, \cdot) \in L^1(\Omega)$ for a. e. $t \in (a, b)$.*
2. *$f(\cdot, x)$ is AC for a. e. $x \in \Omega$. (Its a. e. defined partial derivative is denoted by $\partial_t f$ as usual.)*
3. *It holds that*
$$\int_{(a,b)} \int_\Omega |\partial_t f(t, x)| \,\mathrm{d}\mu \,\mathrm{d}t < +\infty \,.$$

Then the function $t \mapsto \int_\Omega f(t, x) \,\mathrm{d}\mu$ is AC and

$$\partial_t \int_\Omega f(t, x) \,\mathrm{d}\mu = \int_\Omega \partial_t f(t, x) \,\mathrm{d}\mu \qquad \text{for a. e. } t \in (a, b).$$

The following fact is a standard consequence of the definition of absolute continuity:

Fact 5.48. *Let $J \subset \mathbb{R}$ be a compact interval. If $f, g : J \to \mathbb{R}$ are AC, then $f \vee g$ is AC.*

Fact 5.49 (See e.g. [EG15, Theorem 4.4])**.** *Let $J \subset \mathbb{R}$ be a compact interval. Let $1 \leq p < \infty$ and let $f \in W^{1,p}(J)$. Then $f^+ \in W^{1,p}(J)$ and $\partial_x(f^+) = \chi_{\{f>0\}}\partial_x f$ a. e. in J.*

We also use the following standard formulation of a chain rule for Sobolev functions. This version follows e.g. from [EG15, Theorem 4.4], which is stated for globally Lipschitz functions F, by using the fact that the image of f on J is bounded, so that the local Lipschitz condition suffices (e.g. extend F linearly outside the image of f to obtain a $C^1(\mathbb{R})$, globally Lipschitz function $\tilde{F}$).

Fact 5.50. *Let $J \subset \mathbb{R}$ be a compact interval. Let $1 \leq p < \infty$, let $f \in W^{1,p}(J)$, and let $F \in C^1(\mathbb{R})$ be locally Lipschitz. Then $F \circ f \in W^{1,p}(J)$ and $\partial_x(F \circ f)(x) = F'(f(x))\partial_x f(x)$ for a. e. $x \in J$.*

The following lemma essentially says that, given an element of a Bochner space, we can get a handle on a concrete jointly measurable function "representing" that element in a precise sense. This makes concrete the intuitive expectation that a solution $\boldsymbol{u}$ to our PDEs, which maps each point in time to an element of $L^2(I)$, should also give us a specific value at each point in time and space "$\boldsymbol{u}(t, x)$".

Lemma 5.51. *[DS58, Theorem 17, p. 198] Let (S, Σ_S, μ) and (T, Σ_T, λ) be measure spaces which are either both finite or both positive and σ-finite, and let (R, Σ_R, ρ) be their product. Let $1 \leq p \leq \infty$ and let F be a μ-integrable function on S to $L^p(T, \Sigma_T, \lambda, \mathcal{X})$ where $\mathcal{X}$ is a real or complex Banach space. Then there is a ρ-measurable function f on R to $\mathcal{X}$, which is uniquely determined except for a set of ρ-measure zero, such that $f(s, \cdot) = F(s)$ for μ-almost all s in S. Moreover $f(\cdot, t)$ is μ-integrable on S for λ-almost all t and the integral $\int_S f(s, t)\mu(\mathrm{d}s)$, as a function of t is equal to the element $\int_S F(s)\mu(\mathrm{d}s)$ of $L^p(T, \Sigma_T, \lambda, \mathcal{X})$.*

Corollary 5.52. *Let $[a, b] \subset \mathbb{R}$ be a compact interval endowed with the Lebesgue measure, and let $F \in L^1(a, b; L^2(I))$. Then there exists a jointly measurable function $f^* : [a, b] \times I \to \mathbb{R}$ satisfying*

1. $f^*(s,\cdot) = F(s)$ *in* $L^2(I)$ *for a. e.* $s \in (a,b)$.
2. $f^*(\cdot,x) \in L^1(a,b)$ *for all* $x \in I$.
3. *For all* $s \in [a,b]$, *the functions* $x \mapsto \int_{(a,s)} f^*(r,x)\,\mathrm{d}r$ *and* $\int_{(a,s)} F(r)\,\mathrm{d}r$ *are equal in* $L^2(I)$.

Proof. Let condition $\tilde{2}$ denote condition 2 with "all $x \in I$" replaced by "a.e. $x \in I$". We first apply Lemma 5.51 with $S = [a,b]$ and $T = I$, both endowed with the Lebesgue measure, to obtain a function f satisfying conditions 1 and $\tilde{2}$, as well condition 3 for $s = b$.

We now show that condition 3 also holds for other values of $s \in [a,b)$ using the a.e. uniqueness given by Lemma 5.51. For any $s \in [a,b)$, we may apply that lemma with $\tilde{S} = [a,s]$ instead to obtain a function $\tilde{f}$ satisfying $x \mapsto \int_{(a,s)} \tilde{f}(r,x)\,\mathrm{d}r = \int_{(a,s)} F(r)\,\mathrm{d}r$ in $L^2(I)$. But since both $g = \tilde{f}$ and $g = f$ satisfy that $g(r,\cdot) = F(r)$ in $L^2(I)$ for a.e. $r \in (a,s)$, Lemma 5.51 implies that $\tilde{f} = f$ except for a subset of $[a,s] \times I$ of joint measure zero. We conclude that, for a.e. $x \in I$, we have $\tilde{f}(\cdot,x) = f(\cdot,x)$ a.e. in (a,s) and hence $\int_{(a,s)} f(r,x)\,\mathrm{d}r = \int_{(a,s)} \tilde{f}(r,x)\,\mathrm{d}r$. It follows that $x \mapsto \int_{(a,s)} f(r,x)\,\mathrm{d}r = x \mapsto \int_{(a,s)} \tilde{f}(r,x)\,\mathrm{d}r = \int_{(a,s)} F(r)$ in $L^2(I)$, as claimed.

The final step is to construct a jointly measurable function $f^* : [a,b] \times I \to \mathbb{R}$ satisfying condition 2 rather than just $\tilde{2}$, while preserving conditions 1 and 3. Let $N \subset I$ be a measure zero set such that $f(\cdot,x) \in L^1(a,b)$ for all $x \in I \setminus N$. Define f^* by

$$f^*(s,x) := \begin{cases} f(s,x) & \text{if } x \in I \setminus N \\ 0 & \text{otherwise.} \end{cases}$$

We note that f^* is jointly measurable; indeed, this follows from the facts that $f = f^*$ on $[a,b] \times (I \setminus N)$ and that the set $[a,b] \times (I \setminus N)$ is jointly measurable. By construction, f^* satisfies condition 2, and it is clear that f^* also satisfies conditions 1 and 3 since f does and N is a null set. $\square$

Lemma 5.53. *Let* $0 \le a < b < +\infty$, *and let* $\boldsymbol{u} \in C([a,b]; L^2(I))$ *be the restriction to domain* $[a,b]$ *of any solution to the gradient flow problem, with* $\boldsymbol{u'} : [a,b] \to L^2(I)$ *its weak derivative restricted in the same way. Then there exists a jointly measurable function* $\tilde{\boldsymbol{u}}' : [a,b] \times I \to \mathbb{R}$ *satisfying*

1. $\tilde{\boldsymbol{u}}'(t,\cdot) = \boldsymbol{u'}(t)$ *in* $L^2(I)$ *for a. e.* $t \in (a,b)$.
2. $\tilde{\boldsymbol{u}}'(\cdot,x) \in L^1(a,b)$ *for all* $x \in I$.
3. *For all* $t \in [a,b]$, *the functions* $x \mapsto \int_{(a,t)} \tilde{\boldsymbol{u}}'(s,x)\,\mathrm{d}s$ *and* $\int_{(a,t)} \boldsymbol{u'}(s)\,\mathrm{d}s$ *are equal in* $L^2(I)$.

Moreover, fixing any representative of $\boldsymbol{u}(a) \in L^2(I)$, *the function* $\tilde{\boldsymbol{u}} : [a,b] \times I \to \mathbb{R}$ *given by*

$$\tilde{\boldsymbol{u}}(t,x) := \boldsymbol{u}(a)(x) + \int_{(a,t)} \tilde{\boldsymbol{u}}'(s,x)\,\mathrm{d}s$$

is jointly measurable and satisfies

4. *For each* $t \in [a,b]$, $\tilde{\boldsymbol{u}}(t,\cdot) = \boldsymbol{u}(t)$ *in* $L^2(I)$.
5. *For each* $x \in I$, $\tilde{\boldsymbol{u}}(\cdot,x)$ *is absolutely continuous with weak derivative* $\partial_t \tilde{\boldsymbol{u}}(t,x) = \tilde{\boldsymbol{u}}'(t,x)$.

Proof. Apply Corollary 5.52 to $\boldsymbol{u'}$, which is integrable since it is the weak derivative of $\boldsymbol{u}$, to obtain jointly measurable $\tilde{\boldsymbol{u}}' : [a,b] \times I \to \mathbb{R}$ satisfying properties 1–3. We now verify that $\tilde{\boldsymbol{u}}$ satisfies properties 4 and 5. Note that these two properties then imply that $\tilde{\boldsymbol{u}}$ is a Carathéodory function and hence jointly measurable (see e.g. [AB06, Lemma 4.51]).

For each $t \in [a,b]$, the definition of $\boldsymbol{u}$ (in particular its absolute continuity) implies that

$$\boldsymbol{u}(t) - \boldsymbol{u}(a) = \left[\boldsymbol{u}(0) + \int_{(0,t)} \boldsymbol{u}'(s)\,\mathrm{d}s\right] - \left[\boldsymbol{u}(0) + \int_{(0,a)} \boldsymbol{u}'(s)\,\mathrm{d}s\right] = \int_{(a,t)} \boldsymbol{u}'(s)\,\mathrm{d}s$$

in $L^2(I)$. Property 3 implies that, for a.e. $x \in I$,

$$\boldsymbol{u}(t)(x) = \boldsymbol{u}(a)(x) + \int_{(a,t)} \tilde{\boldsymbol{u}}'(s,x)\,\mathrm{d}s = \tilde{\boldsymbol{u}}(t,x)\,,$$

which is property 4. Finally, property 5 is an immediate consequence of the definition of $\tilde{\boldsymbol{u}}$. □

We are now prepared to differentiate in time in order to establish that P_t is order preserving.

Proposition 5.54. *Let $u_0, v_0 \in H^1(I)$, and let $\boldsymbol{u}, \boldsymbol{v}$ be the solutions to the gradient flow problem with initial data u_0, v_0 respectively. Then the function $\Delta : [0,+\infty) \to [0,+\infty)$ given by*

$$\Delta(t) := \frac{1}{2}\int_I \left[(\boldsymbol{u}(t) - \boldsymbol{v}(t))^+\right]^2 \mathrm{d}x$$

is nonincreasing.

Proof. Let $0 \le a < b < +\infty$, so that it suffices to show that $\Delta(a) \ge \Delta(b)$. Apply Lemma 5.53 to $\boldsymbol{u}$ and $\boldsymbol{v}$ to obtain functions $\tilde{\boldsymbol{u}}'$, $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{v}}'$, and $\tilde{\boldsymbol{v}}$ with the properties stated in that lemma. Define the function $f : [a,b] \times I \to \mathbb{R}$ by

$$f(t,x) := \frac{1}{2}\left[(\tilde{\boldsymbol{u}}(t,x) - \tilde{\boldsymbol{v}}(t,x))^+\right]^2 .$$

Note that f is jointly measurable, since both $\tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{v}}$ are. We also obtain that, for each $t \in [a,b]$,

$$\Delta(t) = \int_I f(t,x)\,\mathrm{d}x\,.$$

We now verify that f satisfies the conditions of Lemma 5.47. Let us verify the first condition. Since $\tilde{\boldsymbol{u}}(t,\cdot), \tilde{\boldsymbol{v}}(t,\cdot) \in L^2(I)$, it follows that $f(t,\cdot) \in L^1(I)$, as desired.

We now verify the second condition. We already have that, for all $x \in I$, $\tilde{\boldsymbol{u}}(\cdot,x)$ and $\tilde{\boldsymbol{v}}(\cdot,x)$ are AC. Fact 5.49 implies that, for all $x \in I$, the function $t \mapsto (\tilde{\boldsymbol{u}}(\cdot,x) - \tilde{\boldsymbol{v}}(\cdot,x))^+$ is AC. But the function $y \mapsto y^2$ is locally Lipschitz, so Fact 5.50 implies that $f(\cdot,x)$ is AC for each x, so the second condition is satisfied. Moreover, using Facts 5.49 and 5.50, its weak derivative is

$$\begin{aligned}
\partial_t f(t,x) &= \left[(\tilde{\boldsymbol{u}}(t,x) - \tilde{\boldsymbol{v}}(t,x))^+\right] \partial_t \left[(\tilde{\boldsymbol{u}}(t,x) - \tilde{\boldsymbol{v}}(t,x))^+\right] \\
&= \left[(\tilde{\boldsymbol{u}}(t,x) - \tilde{\boldsymbol{v}}(t,x))^+\right] \chi_{\{\tilde{\boldsymbol{u}}(t,x) > \tilde{\boldsymbol{v}}(t,x)\}} \left[\partial_t \tilde{\boldsymbol{u}}(t,x) - \partial_t \tilde{\boldsymbol{v}}(t,x)\right] \\
&= \chi_{\{\tilde{\boldsymbol{u}}(t,x) > \tilde{\boldsymbol{v}}(t,x)\}} \left[\tilde{\boldsymbol{u}}(t,x) - \tilde{\boldsymbol{v}}(t,x)\right] \left[\tilde{\boldsymbol{u}}'(t,x) - \tilde{\boldsymbol{v}}'(t,x)\right] .
\end{aligned}$$

We claim that $\int_{(a,b)}\int_I |\partial_t f(t,x)|\,\mathrm{d}x\,\mathrm{d}t < +\infty$. Indeed, we have

$$\begin{aligned}
&\int_{(a,b)}\int_I |\partial_t f(t,x)|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_{(a,b)}\int_I \Big|\chi_{\{\tilde{\boldsymbol{u}}(t,x)>\tilde{\boldsymbol{v}}(t,x)\}}\left[\tilde{\boldsymbol{u}}(t,x)-\tilde{\boldsymbol{v}}(t,x)\right]\left[\tilde{\boldsymbol{u}}'(t,x)-\tilde{\boldsymbol{v}}'(t,x)\right]\Big|\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \frac{1}{2}\int_{(a,b)}\int_I (\tilde{\boldsymbol{u}}(t,x)-\tilde{\boldsymbol{v}}(t,x))^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_{(a,b)}\int_I (\tilde{\boldsymbol{u}}'(t,x)-\tilde{\boldsymbol{v}}'(t,x))^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \int_{(a,b)}\int_I \tilde{\boldsymbol{u}}(t,x)^2\,\mathrm{d}x\,\mathrm{d}t + \int_{(a,b)}\int_I \tilde{\boldsymbol{v}}(t,x)^2\,\mathrm{d}x\,\mathrm{d}t + \int_{(a,b)}\int_I \tilde{\boldsymbol{u}}'(t,x)^2\,\mathrm{d}x\,\mathrm{d}t + \int_{(a,b)}\int_I \tilde{\boldsymbol{v}}'(t,x)^2\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}$$

We claim that each of the four terms above is finite. First, using Tonelli's theorem, the definition of $\tilde{\boldsymbol{u}}$ and Jensen's inequality, and letting $L := b-a$, we have

$$\begin{aligned}
\int_{(a,b)}\int_I \tilde{\boldsymbol{u}}(t,x)^2\,\mathrm{d}x\,\mathrm{d}t &= \int_I\int_{(a,b)} \left(\boldsymbol{u}(a)(x) + \int_{(a,t)} \tilde{\boldsymbol{u}}'(s,x)\,\mathrm{d}s\right)^2 \mathrm{d}t\,\mathrm{d}x \\
&\le 2\int_I\int_{(a,b)} \boldsymbol{u}(a)(x)^2\,\mathrm{d}t\,\mathrm{d}x + 2L\int_I\int_{(a,b)}\int_{(a,t)} \tilde{\boldsymbol{u}}'(s,x)^2\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x \\
&\le 2\int_{(a,b)}\int_I \boldsymbol{u}(a)(x)^2\,\mathrm{d}x\,\mathrm{d}t + 2L\int_{(a,b)}\int_{(a,b)}\int_I \tilde{\boldsymbol{u}}'(s,x)^2\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}t \\
&= 2L\|\boldsymbol{u}(a)\|^2_{L^2(I)} + 2L^2\int_{(a,b)}\int_I \tilde{\boldsymbol{u}}'(s,x)^2\,\mathrm{d}x\,\mathrm{d}s\,,
\end{aligned}$$

where in the last line we write $\|\boldsymbol{u}(a)\|^2_{L^2(I)}$, which is finite, since $\boldsymbol{u}(a) \in L^2(I)$. Hence, since the argument for the $\boldsymbol{v}$ terms proceeds identically, all we need to show is that

$$\int_{(a,b)}\int_I \tilde{\boldsymbol{u}}'(t,x)^2\,\mathrm{d}x\,\mathrm{d}t \overset{?}{<} +\infty\,.$$

Since $\tilde{\boldsymbol{u}}'(t,\cdot) = \boldsymbol{u}'(t)$ in $L^2(I)$ for a.e. $t \in (a,b)$, this is equivalent to showing that

$$\int_{(a,b)} \|\boldsymbol{u}'(t)\|^2_{L^2(I)}\,\mathrm{d}t \overset{?}{<} +\infty\,. \tag{22}$$

Recall that, by Definition 5.13, $\boldsymbol{u}' \in L^2(0,b;L^2(I))$, so in particular $\boldsymbol{u}' \in L^2(a,b;L^2(I))$, which by definition implies (22). Hence the claim holds and the third condition of Lemma 5.47 is satisfied.

Therefore, Lemma 5.47 implies that $\Delta(\cdot) = \int_I f(\cdot,x)\,\mathrm{d}x$ is AC and, for a.e. $t \in (a,b)$,

$$\begin{aligned}
\partial_t\Delta(t) &= \partial_t\int_I f(t,x)\,\mathrm{d}x = \int_I \partial_t f(t,x)\,\mathrm{d}x \\
&= \int_I \chi_{\{\tilde{\boldsymbol{u}}(t,x)>\tilde{\boldsymbol{v}}(t,x)\}}\left[\tilde{\boldsymbol{u}}(t,x)-\tilde{\boldsymbol{v}}(t,x)\right]\left[\tilde{\boldsymbol{u}}'(t,x)-\tilde{\boldsymbol{v}}'(t,x)\right]\mathrm{d}x && \text{(Shown above)} \\
&= \int_I \chi_{\{\boldsymbol{u}(t)>\boldsymbol{v}(t)\}}\left[\boldsymbol{u}(t)-\boldsymbol{v}(t)\right]\left[\boldsymbol{u}'(t)-\boldsymbol{v}'(t)\right]\mathrm{d}x && \text{(By Lemma 5.53)} \\
&= \int_I \left[\boldsymbol{u}(t)-\boldsymbol{v}(t)\right]^+ \partial_x\partial_x\left[\boldsymbol{u}(t)\!\downarrow - \boldsymbol{v}(t)\!\downarrow\right]\mathrm{d}x && \text{(By Lemma 5.21).}
\end{aligned}$$

By Corollary 5.36, $\boldsymbol{u}(t), \boldsymbol{v}(t) \in H^1(I)$ and hence are AC, and by Fact 5.49, $[\boldsymbol{u}(t) - \boldsymbol{v}(t)]^+$ is AC. Therefore we can integrate by parts and, using Fact 5.49 and the boundary condition $\partial_x \boldsymbol{u}(t){\downarrow} = \partial_x \boldsymbol{v}(t){\downarrow} = 0$ on $\{0, 1\}$ from Lemma 5.21, we obtain

$$\begin{aligned}\partial_t \Delta(t) &= -\int_I \chi_{\{\boldsymbol{u}(t) > \boldsymbol{v}(t)\}} \left[\partial_x(\boldsymbol{u}(t) - \boldsymbol{v}(t))\right] \left[\partial_x(\boldsymbol{u}(t){\downarrow} - \boldsymbol{u}(t){\downarrow})\right] \\ &= -\int_I \chi_{\{\boldsymbol{u}(t) > \boldsymbol{v}(t)\}} \left[\partial_x \boldsymbol{u}(t) - \partial_x \boldsymbol{v}(t)\right] \left[\partial_x^- \boldsymbol{u}(t) - \partial_x^- \boldsymbol{u}(t)\right] \qquad \text{(By Proposition 5.25).}\end{aligned}$$

But for any numbers $\alpha, \beta \in \mathbb{R}$, it is the case that $\alpha \geq \beta \implies (\alpha \wedge 0) \geq (\beta \wedge 0)$ and $\alpha \leq \beta \implies (\alpha \wedge 0) \leq (\beta \wedge 0)$. Hence the integrand above is pointwise nonnegative, and we conclude that $\partial_t \Delta(t) \leq 0$ for a.e. $t \in (a, b)$. Hence $\Delta(a) \geq \Delta(b)$, as needed. □

Corollary 5.55 ("Directed nonexpansiveness" of P_t)**.** *Let $u, v \in L^2(I)$. Then for all $t > 0$,*

$$\int_I \left[(P_t u - P_t v)^+\right]^2 \mathrm{d}x \leq \int_I \left[(u - v)^+\right]^2 \mathrm{d}x\,.$$

Proof. This is an immediate consequence of Proposition 5.54 when $u, v \in H^1(I)$, and the general case follows by approximating u and v by $H^1(I)$ functions and using the continuity of the map $w \mapsto P_t w$ from $L^2(I)$ to $L^2(I)$. □

Corollary 5.56 (P_t is order preserving)**.** *Let $u, v \in L^2(I)$, and suppose $u \leq v$ a.e. . Then $P_t u \leq P_t v$ a.e. for all $t > 0$.*

Proof. This is an immediate consequence of Corollary 5.55. □

Lemma 5.57. *Every nondecreasing $u \in L^2(I)$ is a stationary point of P_t, i.e. $P_t u = u$ for all t.*

Proof. By definition, $u = u{\uparrow} + u{\downarrow}$ with $u{\uparrow} = u$ and $u{\downarrow} = 0$, so $\mathcal{E}^-(u) = 0$. Thus for every $v \in L^2(I)$,

$$\mathcal{E}^-(v) \geq 0 = \mathcal{E}^-(u) + \langle 0, v - u \rangle\ ,$$

so $0 \in \partial\mathcal{E}^-(u)$. Thus $\boldsymbol{u}(t) = u$ is the solution to the gradient flow problem with initial state u. □

Observation 5.58. *Corollary 5.56 in particular implies that if $u \geq a$ a.e. for some $a \in \mathbb{R}$, then $P_t u \geq P_t a = a$ a.e. for all $t > 0$, the equality by Lemma 5.57.*

We also observe below that P_t is degree one positively homogeneous and additive when one argument is a constant function, as the following results show.

Lemma 5.59. *Let $u \in \mathcal{U}$, $\beta \in \mathbb{R}$, and $v := u + \beta$. Then $v \in \mathcal{U}$ with $v{\uparrow} = u{\uparrow} + \beta$ and $v{\downarrow} = u{\downarrow}$, and thus $\mathcal{E}^-(v) = \mathcal{E}^-(u)$.*

Proof. This is a straightforward consequence of the definition of $v{\uparrow}, v{\downarrow}$. □

Lemma 5.60. *Let $u \in \mathcal{U}$, $\alpha > 0$, and $v := \alpha u$. Then $v \in \mathcal{U}$ with $v{\uparrow} = \alpha u{\uparrow}$ and $v{\downarrow} = \alpha u{\downarrow}$, and thus $\mathcal{E}^-(v) = \alpha^2 \mathcal{E}^-(u)$.*

Proof. This is a straightforward consequence of the definitions of $v{\uparrow}, v{\downarrow}$ and $\mathcal{E}^-$. □

Lemma 5.61. *Let $u \in D(\partial\mathcal{E}^-)$, $\beta \in \mathbb{R}$, and $v := u + \beta$. Then $\partial\mathcal{E}^-(u) = \partial\mathcal{E}^-(v)$.*

Proof. By symmetry, it suffices to prove that $\partial\mathcal{E}^-(u) \subseteq \partial\mathcal{E}^-(v)$. Let $z \in \partial\mathcal{E}^-(u)$. Then for all $w \in \mathcal{U}$, using Lemma 5.59 twice and the definition of subdifferential,

$$\mathcal{E}^-(w) = \mathcal{E}^-(w-\beta) \geq \mathcal{E}^-(u) + \langle z, w-\beta-u\rangle = \mathcal{E}^-(v) + \langle z, w-v\rangle \ ,$$

and hence $z \in \partial\mathcal{E}^-(v)$ as desired. ☐

Lemma 5.62. *Let $u \in D(\partial\mathcal{E}^-)$, $\alpha > 0$, and $v := \alpha u$. Then $\partial\mathcal{E}^-(v) = \alpha\partial\mathcal{E}^-(u)$.*

Proof. By symmetry, it suffices to prove that $\alpha\partial\mathcal{E}^-(u) \subseteq \partial\mathcal{E}^-(v)$. Let $z \in \partial\mathcal{E}^-(u)$; we claim that $\alpha z \in \partial\mathcal{E}^-(v)$. Indeed for all $w \in \mathcal{U}$, using Lemma 5.60 twice and the definition of subdifferential,

$$\mathcal{E}^-(w) = \alpha^2\mathcal{E}^-\left(\tfrac{1}{\alpha}w\right) \geq \alpha^2\left(\mathcal{E}^-(u) + \langle z, \tfrac{1}{\alpha}w - u\rangle\right) = \mathcal{E}^-(v) + \langle \alpha z, w-v\rangle \ ,$$

and hence $\alpha z \in \partial\mathcal{E}^-(v)$ as desired. ☐

Proposition 5.63 (Effect of certain affine transformations on P_t)**.** *Let $u \in L^2(I)$, $\alpha > 0$, and $\beta \in \mathbb{R}$. Then $P_t(\alpha u + \beta) = \alpha P_t u + \beta$.*

Proof. We suffices to prove the statement for $u \in \mathcal{U}$; the general case then follows by approximating $\mathcal{U}$ by $H^1(I)$ functions and the continuity of P_t from $L^2(I)$ to $L^2(I)$. For $v := u+\beta$, we observe that $\boldsymbol{v}(t) := \boldsymbol{u}(t) + \beta$ with $\boldsymbol{v}'(t) := \boldsymbol{u}'(t)$ is a solution to the gradient flow problem with initial state v, since for each $t > 0$ for which $-\boldsymbol{u}'(t) \in \partial\mathcal{E}^-(\boldsymbol{u}(t))$, Lemma 5.61 implies that $-\boldsymbol{v}'(t) \in \partial\mathcal{E}^-(\boldsymbol{v}(t))$.

Similarly, let $w := \alpha u$. Then $\boldsymbol{w}(t) := \alpha\boldsymbol{u}(t)$ with $\boldsymbol{w}'(t) := \alpha\boldsymbol{u}'(t)$ is the solution to the gradient flow problem with initial state w, since for each $t > 0$ for which $-\boldsymbol{u}'(t) \in \partial\mathcal{E}^-(\boldsymbol{u}(t))$, Lemma 5.62 implies that $-\boldsymbol{w}'(t) \in \partial\mathcal{E}^-(w)$. ☐

5.8 Convergence to monotone equilibrium

Since the directed Dirichlet energy $\mathcal{E}^-(P_t u)$ decays over time and $P_t u$ converges to some limit as $t \to \infty$ (by Lemma 5.45), we expect this limit to be a monotone function. Let us establish this fact and other properties of that limit.

Lemma 5.64. *Let $u \in \mathcal{U}$ satisfy $\mathcal{E}^-(u) = 0$. Then u is nondecreasing.*

Proof. Since $\mathcal{E}^-(u) = 0$, we have $\partial_x u{\downarrow} = 0$ a.e. in I. Hence $u{\downarrow}$ is a constant function, while $u{\uparrow}$ is nondecreasing by definition. ☐

Proposition 5.65. *Let $u \in L^2(I)$. Then there exists a nondecreasing $u^* \in \mathcal{U}$, unique as an element of $L^2(I)$, such that $P_t u \to u^*$ in $L^2(I)$ as $t \to \infty$.*

Proof. Since $\mathcal{E}^-$ achieves its minimum (namely 0, on e.g. constant functions) and $P_t u \in D(\partial\mathcal{E}^-)$ for all $t > 0$, [AC84, Theorem 2, p. 160] implies that there exists a minimizer u^* of $\mathcal{E}^-$ such that $P_t u \rightharpoonup u^*$ weakly in $L^2(I)$. This means that $u^* \in \mathcal{U}$ with $\mathcal{E}^-(u^*) = 0$, so by Lemma 5.64 u^* is nondecreasing. By Lemma 5.45 (which we may apply because, fixing any $t_0 > 0$, we have $P_{t_0} u \in D(\partial\mathcal{E}^-) \subseteq D(\mathcal{E}^-) = \mathcal{U}$), $P_t u$ also converges strongly in $L^2(I)$, and it is standard that the weak and strong limits agree and that this limit is unique. ☐

Therefore the following definition is justified:

Definition 5.66 (Monotone equilibrium)**.** Let $P_\infty : L^2(I) \to \mathcal{U}$ be the operator mapping each $u \in L^2(I)$ to the unique (as an element of $L^2(I)$) nondecreasing $u^* \in \mathcal{U}$ such that $P_t u \to u^*$ in $L^2(I)$ as $t \to \infty$. We call $P_\infty u$ the *monotone equilibrium* of u.

We now pass to the limit P_∞ some useful properties of P_t.

Proposition 5.67 ("Directed nonexpansiveness" of P_∞)**.** *Let $u, v \in L^2(I)$. Then*

$$\int_I \left[(P_\infty u - P_\infty v)^+\right]^2 \mathrm{d}x \le \int_I \left[(u - v)^+\right]^2 \mathrm{d}x \,.$$

Proof. This follows by a standard limit argument as follows. Let $u, v \in L^2(I)$. By Corollary 5.55, $\|(P_t u - P_t v)^+\|_{L^2(I)} \le \|(u-v)^+\|_{L^2(I)}$ for all $t > 0$. By definition of P_∞, we have $P_t u \to P_\infty u$ and $P_t v \to P_\infty v$ in $L^2(I)$ as $t \to \infty$. Using the triangle inequality, we obtain

$$\begin{aligned}\|(P_t u - P_t v)^+ - (P_\infty u - P_\infty v)^+\|_{L^2(I)} &\le \|(P_t u - P_t v) - (P_\infty u - P_\infty v)\|_{L^2(I)} \\ &\le \|P_t u - P_\infty u\|_{L(I)} + \|P_t v - P_\infty v\|_{L^2(I)} \to 0\,,\end{aligned}$$

so $(P_t u - P_t v)^+ \to (P_\infty u - P_\infty v)^+$ in $L^2(I)$ as $t \to \infty$. Since $\|(P_t u - P_t v)^+\|_{L^2(I)} \le \|(u-v)^+\|_{L^2(I)}$ for all $t > 0$, we conclude that $\|(P_\infty u - P_\infty v)^+\|_{L^2(I)} \le \|(u-v)^+\|_{L^2(I)}$, which gives the conclusion. ☐

Corollary 5.68 (P_∞ is order preserving)**.** *Let $u, v \in L^2(I)$, and suppose $u \le v$ a.e. . Then $P_\infty u \le P_\infty v$ a.e. .*

Proof. This is a direct consequence of Proposition 5.67. ☐

Proposition 5.69. *P_∞ is nonexpansive, and therefore continuous, as an $L^2(I) \to L^2(I)$ map.*

Proof. Let $u, v \in L^2(I)$. In the proof of Proposition 5.67, we showed that $\|(P_t u - P_t v) - (P_\infty u - P_\infty v)\|_{L^2(I)} \to 0$, which implies that $\|P_t u - P_t v\|_{L^2(I)} \to \|P_\infty u - P_\infty v\|_{L^2(I)}$. But $\|P_t u - P_t v\|_{L^2(I)} \le \|u - v\|_{L^2(I)}$ for all $t > 0$ since $(P_t)_{t \ge 0}$ is a nonexpansive semigroup, so the conclusion follows. ☐

Lemma 5.70. *For every nondecreasing $u \in L^2(I)$, we have $P_\infty u = u$.*

Proof. This follows from Lemma 5.57 along with the convergence $P_t u \to P_\infty u$ in $L^2(I)$. ☐

Strong convergence to the monotone equilibrium, together with the preservation of regularity results from Section 5.4, allows us to obtain regularity of the monotone equilibrium as well:

Proposition 5.71 (H^1 regularity of the monotone equilibrium)**.** *Let $u \in H^1(I)$. Then $P_\infty u \in H^1(I)$ with $\varphi(P_\infty u) \le \varphi(u)$.*

Proof. Recall the functional φ from Section 5.4. Since $u \in H^1(I)$, $\varphi(u) < +\infty$. By Proposition 5.35, we conclude that $\varphi(P_t u) \le \varphi(u) < +\infty$ for all $t > 0$. Since φ is lower semicontinuous by Claim 5.30 and $P_t u \to P_\infty u$ in $L^2(I)$ by Lemma 5.45, we conclude that $\varphi(P_\infty u) \le \varphi(u) < +\infty$. Hence $P_\infty u \in H^1(I)$. ☐

Proposition 5.72 (Lipschitz regularity of the monotone equilibrium)**.** *Let $u \in W^{1,\infty}(I)$. Then $P_\infty u \in W^{1,\infty}(I)$ with $\psi(P_\infty u) \le \psi(u)$.*

Proof. Recall the functional ψ from Section 5.5. Since $u \in W^{1,\infty}(I)$, $\psi(u) < +\infty$. By Proposition 5.41, we conclude that $\psi(P_t u) \le \psi(u) < +\infty$ for all $t > 0$. Since ψ is lower semicontinuous by Claim 5.38

and $P_t u \to P_\infty u$ in $L^2(I)$ by Lemma 5.45, we conclude that $\psi(P_\infty u) \le \psi(u) < +\infty$. Hence $P_\infty u \in W^{1,\infty}(I)$. $\square$

We observe that P_∞ also behaves nicely under the appropriate class of affine transformations.

Proposition 5.73 (Effect of certain affine transformations on P_∞)**.** *Let $u \in L^2(I)$, $\alpha > 0$, and $\beta \in \mathbb{R}$. Then $P_\infty(\alpha u + \beta) = \alpha P_\infty u + \beta$.*

Proof. Since $P_t v \to P_\infty v$ as $t \to \infty$ for each $v \in L^2(I)$, applying Proposition 5.63 yields

$$P_\infty(\alpha u + \beta) = \lim_{t\to\infty} P_t(\alpha u + \beta) = \lim_{t\to\infty} [\alpha P_t u + \beta] = \alpha \left[\lim_{t\to\infty} P_t u\right] + \beta = \alpha P_\infty u + \beta\,. \qquad \square$$

We can finally conclude, via Lemma 5.46, that P_∞ is nonexpansive in the L^∞ norm.

Proposition 5.74 (P_∞ is nonexpansive in L^∞ norm)**.** *Let $u, v \in L^2(I)$. Then $|P_\infty u - P_\infty v| \le \operatorname{ess\,sup}|u - v|$ a.e. in I.*

Proof. We verify the conditions of Lemma 5.46 with (Ω, Σ, μ) the set I endowed with the Lebesgue measure, $p = 2$, $C = L^2(I)$, and $T = P_\infty$. It is clear that for all $f \in L^2(I)$ and $r \in \mathbb{R}$, $f + r \in L^2(I)$. By Proposition 5.69, P_∞ is a continuous $L^2(I) \to L^2(I)$ map. Moreover, for all $f \in L^2(I)$ and $r \in \mathbb{R}$, the condition $P_\infty(f + r) = P_\infty f + r$ holds by Proposition 5.73. Finally, for all $f, g \in L^2(I)$ with $f \le g$ a.e. , we have that $P_\infty f \le P_\infty g$ a.e. by Corollary 5.68. The conclusion follows. $\square$

6 Directed transport-energy inequality

In this section, we establish a connection between the PDE studied above and the Wasserstein distance between the initial state u and its monotone equilibrium $P_\infty u$, via the dynamical approach embodied by the *Benamou-Brenier formula*. We follow the presentation and formalism of [San15], and also refer to [AGS05; Vil09].

6.1 Preliminaries for optimal transport

We start by introducing notation and definitions relevant to the theory of optimal transport.

Projections. Say $X \times Y$ is a product space and $z = (x, y)$ is an element in this space. Then the projection operator onto the first coordinate, denoted interchangeably by π_1 or π_x depending on the context, is given by $\pi_1(z) = \pi_x(z) = x$. Similarly, $\pi_2(z) = \pi_y(z) = y$. We extend this definition in the natural way projections from larger product spaces onto smaller product spaces. For example, given the product space $X_1 \times \cdots \times X_d$ and index set $I \subset [d]$, $I = \{i_1, \dots, i_n\}$, the operator π_I projects any element in this space down to an element of $X_{i_1} \times \cdots \times X_{i_n}$. We also use the shorthand $\pi_{-I} := \pi_{[d]\setminus I}$.

Now, let $\Omega \subset \mathbb{R}^d$ be a Borel set and let $I \subseteq [d]$. For a point $x \in \Omega$, we write the projections $x_I := \pi_I(x)$ and $x_{-I} := \pi_{-I}(x)$, and we also write $x = (x_I, x_{-I})$. Then, for the set Ω, we write the projections $\Omega_I := \{x_I : x \in \Omega\}$ and $\Omega_{-I} := \{x_{-I} : x \in \Omega\}$. For small sets I or $[d] \setminus I$, we also use shorthand notation such as $x_i := x_{\{i\}}$ and $\Omega_{-i-j} := \Omega_{-\{i,j\}}$.

Pushforward measure. Given measurable spaces (X, Σ_X) and (Y, Σ_Y), measure μ on X, and measurable map $T : X \to Y$, the *pushforward measure* $T_\# \mu$ on Y is the measure satisfying

$$(T_\#\mu)(B) = \mu(T^{-1}(B)) \qquad \text{for all } B \in \Sigma_Y, \text{ or equivalently,}$$
$$\int_Y \phi(y)\,\mathrm{d}(T_\#\mu)(y) = \int_X \phi(T(x))\,\mathrm{d}\mu(x) \qquad \text{for all measurable } \phi : Y \to \mathbb{R}.$$

For a measure γ on product space $X \times Y$, we say that $(\pi_1)_\#\gamma$ and $(\pi_2)_\#\gamma$, which are measures on X and Y respectively, are the first and second *marginals* of γ, respectively.

If (Ω_3, Σ_3) is another measurable space and $S : Y \to Z$ a measurable map, then it holds that $(S \circ T)_\#\mu = S_\#(T_\#\mu)$.

Transport plans. For two probability spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$, we write $\Pi(\mu_1, \mu_2)$ for the set of *couplings*, or *transport plans*, between μ_1 and μ_2, namely probability measures γ on the product space $\Omega_1 \times \Omega_2$ whose first and second marginals are μ_1 and μ_2, respectively.

Space of probability measures. For any Borel set $\Omega \subset \mathbb{R}^d$, let $P(\Omega)$ denote the space of all (Borel) probability measures on Ω. We endow $P(\Omega)$ with the weak topology, which is the topology of weak convergence with respect to bounded continuous functionals. Namely, we say μ_n converges weakly to μ in $P(\Omega)$, and write $\mu_n \rightharpoonup \mu$, if $\int_\Omega \phi\,\mathrm{d}\mu_n \to \int_\Omega \phi\,\mathrm{d}\mu$ for all bounded continuous $\phi : \Omega \to \mathbb{R}$.

Wasserstein distances. Let $\Omega \subset \mathbb{R}^d$ be a *bounded* Borel set and let $p \in [1, +\infty)$. Let $\mu, \nu \in P(\Omega)$. Given transport plan $\gamma \in \Pi(\mu, \nu)$, we define the *cost* $C_p(\gamma)$ by

$$C_p(\gamma) := \left(\int_{\Omega\times\Omega} |x-y|^p\,\mathrm{d}\gamma(x,y)\right)^{1/p}.$$

We then define the *p-Wasserstein* distance between μ and ν by

$$W_p(\mu,\nu) := \inf_{\gamma\in\Pi(\mu,\nu)} C_p(\gamma)\,,$$

and we also often refer to the quantity $W_p^p(\mu, \nu) := W_p(\mu, \nu)^p$. It is standard that $W_p(\cdot, \cdot)$ is a distance metric on $P(\Omega)$.

Fact 6.1 (Wasserstein distance metrizes weak convergence; see e.g. [San15, Theorems 5.10 and 5.11])**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Borel set, and let $p \in [1, \infty)$. Then for a sequence $(\mu_n)_n$ in $P(\Omega)$ and $\mu \in P(\Omega)$, we have $\mu_n \rightharpoonup \mu$ if and only if $W_p(\mu_n, \mu) \to 0$.*

Corollary 6.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Borel set, and let $p \in [1, \infty)$. Then for sequences $(\mu_n)_n$ and $(\nu_n)_n$ in $P(\Omega)$ with $\mu_n \rightharpoonup \mu$ and $\nu_n \rightharpoonup \nu$ for $\mu, \nu \in P(\Omega)$, we have $W_p(\mu_n, \nu_n) \to W_p(\mu, \nu)$. In other words, the p-Wasserstein distance is continuous in (the weak topology on) $P(\Omega)$.*

Proof. By the triangle inequality, we have

$$W_p(\mu,\nu) - W_p(\mu,\mu_n) - W_p(\nu_n,\nu) \le W_p(\mu_n,\nu_n) \le W_p(\mu_n,\mu) + W_p(\mu,\nu) + W_p(\nu,\nu_n)\,.$$

By Fact 6.1, the LHS and RHS converge to $W_p(\mu, \nu)$ as $n \to \infty$, so the conclusion follows. ☐

6.2 Optimal transport via Benamou-Brenier

Let $\{\varrho_t : t \in [0,T]\}$ be a family of measures on $\overline{I}$ and $\{v_t : t \in [0,T]\}$ be a family of *velocity fields* such that $v_t \in L^1(\varrho_t)$ for each t. We start by defining what it means for the family (ϱ_t, v_t) to solve the *continuity equation*

$$\partial_t \varrho_t + \partial_x(\varrho_t v_t) = 0\,.$$

Definition 6.3 (Weak solution; see [San15, Section 4.1.2])**.** Let (ϱ_t, v_t) be a family of measure/velocity field pairs indexed by $t \in [0,T]$ such that $v_t \in L^1(\varrho)$ for each t. We say that (ϱ_t, v_t) is a *weak solution to the continuity equation* if, for every test function $\psi \in C^1(\overline{I})$, the function $t \mapsto \int_{\overline{I}} \psi \,\mathrm{d}\varrho_t$ is absolutely continuous in t and, for a.e. t, we have

$$\partial_t \int_{\overline{I}} \psi \,\mathrm{d}\varrho_t = \int_{\overline{I}} (\partial_x \psi) v_t \,\mathrm{d}\varrho_t\,. \tag{23}$$

In this case, we call ϱ_0 and ϱ_T the *initial* and *final states* of the solution, respectively (this makes sense because the above implies that $t \mapsto \varrho_t$ is continuous for the weak convergence of measures).

Remark 6.4. We will only work with absolutely continuous measures $\mathrm{d}\varrho_t = u\,\mathrm{d}x$, $u \in L^2(I)$.

Remark 6.5. Applying (23) with a constant test function ψ shows that every solution to the continuity equation is *mass conserving*. In particular, Proposition 6.8 below implies that this is true of the semigroup P_t; see Corollary 6.9.

Proposition 6.6 (Benamou-Brenier formula; see e.g. [San15, Theorem 5.28])**.** *Let μ, ν be probability measures on $\overline{I}$. Then*

$$W_2^2(\mu,\nu) = \min\left\{ \int_0^1 \|v_t\|^2_{L^2(\varrho_t)} \,\mathrm{d}t : \partial_t \varrho_t + \partial_x(\varrho_t v_t) = 0, \varrho_0 = \mu, \varrho_1 = \nu \right\},$$

where the constraint above means that $(\varrho_t, v_t)_{t\in[0,1]}$ is a weak solution to the continuity equation with initial state μ and final state ν.

Condition 6.7. $u \in \mathcal{U}$ is a.e. positive and bounded away from zero, and satisfies $\int_I u\,\mathrm{d}x = 1$.

Proposition 6.8. *Let $u \in \mathcal{U}$ satisfy Condition 6.7. Let $\boldsymbol{u}(t)$ be the solution to the gradient flow problem with initial state u, and $\boldsymbol{u}'(t)$ its weak derivative for each $t \geq 0$. Let $T > 0$ and define the measure/velocity field family $(\varrho_t, v_t)_{t\in[0,T]}$ by*

$$\mathrm{d}\varrho_t := \boldsymbol{u}(t)\,\mathrm{d}x \quad \textit{and} \quad v_t := -\frac{\partial_x \boldsymbol{u}(t)\!\downarrow}{\boldsymbol{u}(t)}$$

for each $t \in [0,T]$. Then (ϱ_t, v_t) is a weak solution to the continuity equation with initial state $\boldsymbol{u}(0)\,\mathrm{d}x$ and final state $\boldsymbol{u}(T)\,\mathrm{d}x$.

Proof. We first note that, by Observation 5.58, each $\boldsymbol{u}(t)$ is a.e. positive and bounded away from zero, uniformly in t. In particular, this justifies the denominator in the definition of v_t. Furthermore, recall that $\boldsymbol{u}(t) \in \mathcal{U}$ for all $t \geq 0$ by Corollary 5.20 and Definition 5.15, so the numerator in the definition of v_t is defined up to sets of measure zero for each t. (In fact, by Definition 5.16, Proposition 5.18, and Lemma 5.21 we have $\boldsymbol{u}(t)\!\downarrow \in H^2(I)$ for a.e. $t > 0$, in which case the numerator is even pointwise well-defined.)

We now verify that (ϱ_t, v_t) satisfies Definition 6.3. The initial and final states are as claimed by construction. Let $\psi \in C^1(\overline{I})$. Let $\tilde{\boldsymbol{u}}', \tilde{\boldsymbol{u}} : [0,T] \times I \to \mathbb{R}$ be obtained by applying Lemma 5.53 to $\boldsymbol{u}$ with time domain $[0,T]$. Define the (jointly measurable) function $f : [0,T] \times I \to \mathbb{R}$ by

$$f(t,x) := \psi(x)\tilde{\boldsymbol{u}}(t,x)\,.$$

We claim that f satisfies the properties of Lemma 5.47. It is clear that $f(t,\cdot) \in L^2(I) \subset L^1(I)$ for each $t \in [0,T]$ since $\boldsymbol{u}(t,\cdot) \in L^2(I)$ while ψ is bounded, so the first property is satisfied. Also, for each $x \in I$ we have that $\tilde{\boldsymbol{u}}(\cdot, x)$ is AC and hence so is $f(\cdot, x)$, so the second property is satisfied. Finally, by properties 1 and 5 of Lemma 5.53 and the Cauchy-Schwarz inequality,

$$\int_{(0,T)} \int_I |\partial_t f(t,x)| \,\mathrm{d}x \,\mathrm{d}t = \int_{(0,T)} \int_I \big|\psi(x)\tilde{\boldsymbol{u}}'(t,x)\big| \,\mathrm{d}x \,\mathrm{d}t \le \|\psi\|_{L^2(I)} \int_{(0,T)} \|\boldsymbol{u}'(t)\|_{L^2(I)} \,\mathrm{d}t < +\infty\,,$$

the last inequality since $\psi \in C^1(\overline{I})$ while $\boldsymbol{u}' \in L^2(0,T;L^2(I))$ by Definition 5.13. Hence the third property is satisfied and Lemma 5.47 applies. Thus the function mapping each $t \in [0,T]$ to

$$\int_{\overline{I}} \psi \,\mathrm{d}\varrho_t = \int_I \psi \boldsymbol{u}(t) \,\mathrm{d}x = \int_I \psi(x)\tilde{\boldsymbol{u}}(t,x) \,\mathrm{d}x = \int_I f(t,x) \,\mathrm{d}x$$

is absolutely continuous and, for a.e. $t \in (0,T)$,

$$\begin{aligned}
\partial_t \int_{\overline{I}} \psi \,\mathrm{d}\varrho_t &= \int_I \partial_t f(t,x) \,\mathrm{d}x = \int_I \psi(x)\tilde{\boldsymbol{u}}'(t,x) \,\mathrm{d}x && \text{(Property 5 of Lemma 5.53)}\\
&= \int_I \psi \boldsymbol{u}'(t) \,\mathrm{d}x && \text{(Property 1 of Lemma 5.53)}\\
&= \int_I \psi \partial_x \partial_x \boldsymbol{u}(t)\!\downarrow \mathrm{d}x && \text{(Proposition 5.18 and Lemma 5.21)}\\
&= -\int_I (\partial_x \psi)(\partial_x \boldsymbol{u}(t)\!\downarrow) \,\mathrm{d}x && \text{(Integration by parts, Lemma 5.21)}\\
&= \int_I (\partial_x \psi) \left(-\frac{\partial_x \boldsymbol{u}(t)\!\downarrow}{\boldsymbol{u}(t)} \right) \boldsymbol{u}(t) \,\mathrm{d}x = \int_{\overline{I}} (\partial_x \psi) v_t \,\mathrm{d}\varrho_t\,. && \square
\end{aligned}$$

Corollary 6.9 (P_t is mass conserving)**.** *Let $u \in L^2(I)$. Then for all $t > 0$, $\int_I P_t u \,\mathrm{d}x = \int_I u \,\mathrm{d}x$.*

Proof. If u satisfies Condition 6.7, then this follows from Proposition 6.8 by taking any constant test function ψ in Definition 6.3.

If $u \in H^1(I)$, then it is bounded, so let $\alpha > 0$, $\beta \in \mathbb{R}$ be such that $v := \alpha u + \beta \in H^1(I)$ satisfies Condition 6.7. Then $u = \frac{1}{\alpha} v - \frac{\beta}{\alpha}$ and, by Proposition 5.63 and the above,

$$\int_I P_t u \,\mathrm{d}x = \int_I \left(\frac{1}{\alpha} P_t v - \frac{\beta}{\alpha} \right) \mathrm{d}x = -\frac{\beta}{\alpha} + \frac{1}{\alpha} \int_I P_t v \,\mathrm{d}x = -\frac{\beta}{\alpha} + \frac{1}{\alpha} \int_I v \,\mathrm{d}x = \int_I \left(\frac{1}{\alpha} v - \frac{\beta}{\alpha} \right) \mathrm{d}x = \int_I u \,\mathrm{d}x\,.$$

Finally, let $u \in \mathcal{U}$ be arbitrary. Since $H^1(I)$ is dense in $L^2(I)$, let $(u_n)_n$ be a sequence in $H^1(I)$ such that $u_n \to u$ in $L^2(I)$. By the continuity of P_t from $L^2(I)$ to $L^2(I)$, and using the above,

$$\int_I P_t u \,\mathrm{d}x = \lim_{n\to\infty} \int_I P_t u_n \,\mathrm{d}x = \lim_{n\to\infty} \int_I u_n \,\mathrm{d}x = \int_I u \,\mathrm{d}x\,. \qquad \square$$

By passing to the limit $P_t u \to P_\infty u$ in $L^2(I)$, we also conclude

Corollary 6.10 (P_∞ is mass conserving)**.** *Let $u \in L^2(I)$. Then $\int_I P_\infty u \, \mathrm{d}x = \int_I u \, \mathrm{d}x$.*

The definition of the velocity field v_t in Proposition 6.8, along with the fact that $\partial_x \boldsymbol{u}(t)\!\downarrow \le 0$, gives $v_t \ge 0$, which intuitively says that "particles only move to the right". This suggests that the solution $\boldsymbol{u}(t)$ can only lose mass in any prefix of the interval $(0,1)$ over time. The following items make this observation rigorous.

Lemma 6.11. *Let $\delta \in I$ and let $J := (0, \delta)$. Let $u \in L^2(I)$. Then for all $t > 0$,*

$$\int_J (P_t u) \, \mathrm{d}x \le \int_J u \, \mathrm{d}x \, .$$

Proof. Suppose u satisfies Condition 6.7; the general case will then follow by the same arguments as in the proof of Corollary 6.9. Let $J' := (-\infty, \delta)$, let $\chi_{J'} : \mathbb{R} \to [0,1]$ be the characteristic function for the set J', and for each $\epsilon > 0$, let $\chi_{J'}^\epsilon$ be its mollification by the standard mollifier. Note that each $\chi_{J'}^\epsilon$ is nonincreasing. Also note that $\chi_{J'} = \chi_J$ pointwise in I. Now, fix $T > 0$. Since the family $(\varrho_t, v_t)_{t \in [0,T]}$ given by Proposition 6.8 solves the continuity equation, applying (23) with test function $\chi_{J'}^\epsilon \in C^\infty(\overline{I})$ gives that $t \mapsto \int_I \chi_{J'}^\epsilon \boldsymbol{u}(t) \, \mathrm{d}x$ is AC and, for a.e. $t \in [0,T]$,

$$\partial_t \int_I \chi_{J'}^\epsilon \boldsymbol{u}(t) \, \mathrm{d}x = \int_I \underbrace{(\partial_x \chi_{J'}^\epsilon)}_{\le 0} \underbrace{v_t}_{\ge 0} \underbrace{\boldsymbol{u}(t)}_{\ge 0} \, \mathrm{d}x \le 0 \, .$$

Hence $t \mapsto \int_I \chi_{J'}^\epsilon \boldsymbol{u}(t) \, \mathrm{d}x$ is nonincreasing, and obtain

$$\langle \chi_{J'}^\epsilon, u \rangle_{L^2(I)} = \int_I \chi_{J'}^\epsilon \boldsymbol{u}(0) \, \mathrm{d}x \ge \int_I \chi_{J'}^\epsilon \boldsymbol{u}(T) \, \mathrm{d}x = \langle \chi_{J'}^\epsilon, P_T u \rangle_{L^2(I)} \, .$$

Now, since $\chi_{J'} \in L^2_{\mathrm{loc}}(\mathbb{R})$, we have that $\chi_{J'}^\epsilon \to \chi_{J'}$ in $L^2_{\mathrm{loc}}(\mathbb{R})$, so in particular $\chi_{J'}^\epsilon \to \chi_{J'} = \chi_J$ in $L^2(I)$. Thus we have

$$\int_J (P_T u) \, \mathrm{d}x = \langle \chi_J, P_T u \rangle_{L^2(I)} = \lim_{\epsilon \to 0} \langle \chi_{J'}^\epsilon, P_T u \rangle_{L^2(I)} \le \lim_{\epsilon \to 0} \langle \chi_{J'}^\epsilon, u \rangle_{L^2(I)} = \langle \chi_J, u \rangle_{L^2(I)} = \int_J u \, \mathrm{d}x \, .$$

□

Corollary 6.12. *Let $\delta \in I$ and let $J := (0, \delta)$. Let $u \in L^2(I)$. Then*

$$\int_J (P_\infty u) \, \mathrm{d}x \le \int_J u \, \mathrm{d}x \, .$$

Proof. This follows from Lemma 6.11 along with the convergence $P_t u \to P_\infty u$ in $L^2(I)$. □

Definition 6.13. Let μ and ν be probability measures on $\mathbb{R}$. We say μ *dominates* ν, and write $\mu \succeq \nu$, if for every $x \in \mathbb{R}$ we have $\mu(-\infty, x) \ge \nu(-\infty, x)$.

Corollary 6.14. *Let $u \in \mathcal{U}$ satisfy Condition 6.7. Define the measures $\mathrm{d}\mu := u \, \mathrm{d}x$ and $\mathrm{d}\mu_\infty := (P_\infty u) \, \mathrm{d}x$. Then $\mu \succeq \mu_\infty$.*

Proof. We may view μ and μ_∞ as absolutely continuous measures on all of $\mathbb{R}$, taking zero outside $[0,1]$. Now, for each $x \in [0,1]$, Corollary 6.12 gives

$$\mu(-\infty, x) = \int_{(0,x)} u \, \mathrm{d}y \ge \int_{(0,x)} (P_\infty u) \, \mathrm{d}y = \mu_\infty(-\infty, x) \, .$$

On the other hand, for $x < 0$ both sides are zero, and for $x > 1$, both sides are 1 by Corollary 6.10. □

We can now upper bound the Wasserstein distance between $u \in \mathcal{U}$ and its monotone equilibrium by combining the Benamou-Brenier formula with the exponential decay of the directed Dirichlet energy, at least as long as u is positive and bounded away from zero.

Theorem 6.15 (Transport-energy inequality in one dimension)**.** *There exists a constant $C > 0$ such that the following holds. Let $u \in \mathcal{U}$ be positive, bounded away from zero, and satisfy $\int_{(0,1)} u \,\mathrm{d}x = 1$. Define the measures $\mathrm{d}\mu := u \,\mathrm{d}x$ and $\mathrm{d}\mu_\infty := (P_\infty u)\,\mathrm{d}x$. Then*

$$W_2^2(\mu, \mu_\infty) \leq \frac{C}{\inf u} \mathcal{E}^-(u)\,.$$

Proof. Let $a := \inf u$. By Corollaries 6.9 and 5.56, $P_t u$ satisfies Condition 6.7 with $P_t u \geq a$ a.e. for all $t \geq 0$. For each $t \geq 0$, define the measure $\mathrm{d}\varrho_t := (P_t u)\,\mathrm{d}x$. Let $\ell \in \mathbb{Z}_{\geq 0}$. Then Proposition 6.8 applies to time interval $[\ell, \ell+1]$ (we refrain from introducing excessive notation for such relabeling of intervals) with initial/final states ϱ_ℓ and $\varrho_{\ell+1}$, respectively, so that Proposition 6.6 gives

$$\begin{aligned}
W_2^2(\varrho_\ell, \varrho_{\ell+1}) &\leq \int_\ell^{\ell+1} \int_I \left(-\frac{\partial_x \boldsymbol{u}(t){\downarrow}}{\boldsymbol{u}(t)} \right)^2 \boldsymbol{u}(t)\,\mathrm{d}x\,\mathrm{d}t = \int_\ell^{\ell+1} \int_I \frac{(\partial_x \boldsymbol{u}(t){\downarrow})^2}{\boldsymbol{u}(t)}\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \frac{1}{a} \int_\ell^{\ell+1} \int_I (\partial_x \boldsymbol{u}(t){\downarrow})^2\,\mathrm{d}x\,\mathrm{d}t = \frac{2}{a} \int_\ell^{\ell+1} \mathcal{E}^-(P_t u)\,\mathrm{d}t \\
&\leq \frac{2}{a}\mathcal{E}^-(u) \int_\ell^{\ell+1} e^{-Kt}\,\mathrm{d}t < \frac{2}{a}\mathcal{E}^-(u) \int_\ell^{\infty} e^{-Kt}\,\mathrm{d}t = \frac{2e^{-K\ell}}{aK}\mathcal{E}^-(u)\,,
\end{aligned}$$

where the penultimate inequality and the constant K come from Proposition 5.43. Now, by the triangle inequality (on the distance W_2, not the squared distance W_2^2), for every $\ell \in \mathbb{Z}_{\geq 1}$ we have

$$W_2(\varrho_0, \varrho_\ell) \leq \sum_{j=0}^{\ell-1} W_2(\varrho_j, \varrho_{j+1}) \leq \left(\frac{2}{aK}\mathcal{E}^-(u) \right)^{1/2} \sum_{j=0}^{\infty} e^{-Kj/2} = \left(\frac{2}{aK}\mathcal{E}^-(u) \right)^{1/2} \frac{1}{1 - e^{-K/2}}\,.$$

Therefore, extracting the appropriate constant C from the terms involving K, we obtain

$$W_2(\varrho_0, \varrho_\ell) \leq \frac{C}{a^{1/2}} (\mathcal{E}^-(u))^{1/2}\,.$$

Recall that $P_t u \rightharpoonup P_\infty u$ weakly in $L^2(I)$ by Proposition 5.65. Hence, defining the absolutely continuous measure $\mathrm{d}\varrho_\infty := (P_\infty u)\,\mathrm{d}x$, we conclude that $\varrho_t \rightharpoonup \varrho_\infty$ as $t \to \infty$. Then by Corollary 6.2,

$$W_2(\varrho_0, \varrho_\infty) = \lim_{\ell \to \infty} W_2(\varrho_0, \varrho_\ell) \leq \frac{C}{a^{1/2}} (\mathcal{E}^-(u))^{1/2}\,. \qquad \square$$

6.3 Directed optimal transport

We now introduce directed versions of basic optimal transport concepts and theory, with the goal of combining a directed version of weak Kantorovich duality with a multidimensional directed version of the transport-energy inequality from Theorem 6.15 to obtain our desired directed Poincaré inequality via a perturbation argument. In the interest of space, we refrain starting a systematic study of directed optimal transport, but rather limit ourselves to the results we require.

Definition 6.16 (Quasimetric)**.** Let X be a set. An extended real-valued function $d : X \times X \to [0, +\infty]$, not necessarily symmetric, is called a *quasimetric* on X if it satisfies the following:

1. For all $x, y \in X$, $d(x,y) \geq 0$ with $d(x,y) = 0$ if and only if $x = y$.
2. For all $x, y, z \in X$, $d(x,z) \leq d(x,y) + d(y,z)$.

Definition 6.17. Let $\Omega \subset \mathbb{R}^d$ be a Borel set, and let μ and ν be probability measures on Ω. We define the set $\Pi(\mu \to \nu)$ of *directed couplings*, or *directed transport plans*, from μ to ν as

$$\Pi(\mu \to \nu) := \left\{ \gamma \in \Pi(\mu, \nu) : \int_{\Omega \times \Omega} \chi_{\{x \not\preceq y\}} \, \mathrm{d}\gamma(x,y) = 0 \right\} ,$$

where the integral is well-defined because the set $\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x \not\preceq y\}$ is open and hence Borel measurable. Note that the condition could also be written as $\gamma(\{x \not\preceq y\}) = 0$, or $\gamma(\{x \preceq y\}) = 1$.

Definition 6.18 (Directed Wasserstein distance)**.** Let $\Omega \subset \mathbb{R}^d$ be a bounded Borel set and let $p \in [1, \infty)$. Given two probability distributions μ, ν over Ω, we define the *directed p-Wasserstein distance from μ to ν* by

$$W_p(\mu \to \nu) := \inf_{\gamma \in \Pi(\mu \to \nu)} C_p(\gamma) ,$$

and we write $W_p^p(\mu \to \nu) := W_p(\mu \to \nu)^p$. Note that we may have $W_p(\mu \to \nu) = +\infty$, and that $W_p(\cdot \to \cdot)$ is not symmetric in general.

Observation 6.19. *For all measures μ and ν, we have $\Pi(\mu \to \nu) \subset \Pi(\mu, \nu)$ and hence $W_p(\mu, \nu) \leq W_p(\mu \to \nu)$.*

Lemma 6.20 (Gluing lemma; see e.g. [San15, Lemma 5.5])**.** *Let $\Omega \subset \mathbb{R}^d$ be a Borel set. Let μ, ϱ, ν be probability measures on Ω, and let $\gamma^+ \in \Pi(\mu, \varrho)$ and $\gamma^- \in \Pi(\varrho, \nu)$. Then there exists a probability measure σ on $\Omega \times \Omega \times \Omega$ such that $(\pi_{x,y})_{\#}\sigma = \gamma^+$ and $(\pi_{y,z})_{\#}\sigma = \gamma^-$.*

The proof of that $W_p(\cdot \to \cdot)$ is a quasimetric follows the presentation of [San15, Lemma 5.4], with a simple additional argument to handle *directed* couplings.

Lemma 6.21 (Composition of directed transport plans)**.** *In Lemma 6.20, if $\gamma^+ \in \Pi(\mu \to \varrho)$ and $\gamma^- \in \Pi(\varrho \to \nu)$, then $(\pi_{x,z})_{\#}\sigma \in \Pi(\mu \to \nu)$.*

Proof. Let $\gamma := (\pi_{x,z})_{\#}\sigma$. First, since $(\pi_x)_{\#}\gamma = (\pi_x \circ \pi_{x,z})_{\#}\sigma = (\pi_x \circ \pi_{x,y})_{\#}\sigma = (\pi_x)_{\#}\gamma^+ = \mu$, and similarly $(\pi_z)_{\#}\gamma = \nu$, we have $\gamma \in \Pi(\mu, \nu)$. Moreover, by definition of pushforward measure we have

$$\begin{aligned}
\int_{\Omega \times \Omega} \chi_{\{x \not\preceq z\}} \, \mathrm{d}\gamma(x,z) &= \int_{\Omega \times \Omega \times \Omega} \chi_{\{x \not\preceq z\}} \, \mathrm{d}\sigma(x,y,z) \leq \int_{\Omega \times \Omega \times \Omega} \left(\chi_{\{x \not\preceq y\}} + \chi_{\{y \not\preceq z\}} \right) \mathrm{d}\sigma(x,y,z) \\
&= \int_{\Omega \times \Omega \times \Omega} \chi_{\{x \not\preceq y\}} \, \mathrm{d}\sigma(x,y,z) + \int_{\Omega \times \Omega \times \Omega} \chi_{\{y \not\preceq z\}} \, \mathrm{d}\sigma(x,y,z) \\
&= \int_{\Omega \times \Omega} \chi_{\{x \not\preceq y\}} \, \mathrm{d}\gamma^+(x,y) + \int_{\Omega \times \Omega} \chi_{\{y \not\preceq z\}} \, \mathrm{d}\gamma^-(y,z) = 0 ,
\end{aligned}$$

the last equality since γ^+ and γ^- are directed couplings. Hence $\gamma \in \Pi(\mu \to \nu)$ as claimed. $\square$

Proposition 6.22. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Borel set and $p \in [1, \infty)$. Then $W_p(\cdot \to \cdot)$ is a quasimetric on $P(\Omega)$.*

Proof. It is clear that $W_p(\mu \to \nu) \geq 0$ always. If $\mu = \nu$ then the identity coupling shows that $W_p(\mu \to \nu) = 0$, and conversely if $W_p(\mu \to \nu) = 0$ then, by Observation 6.19, $W_p(\mu, \nu) = 0$ and hence $\mu = \nu$. It remains to show that $W_p(\cdot \to \cdot)$ satisfies the triangle inequality.

Let μ, ϱ, ν be probability measures on Ω. Let $\gamma^+ \in \Pi(\mu \to \varrho)$ and $\gamma^- \in \Pi(\varrho \to \nu)$. Then since we also have $\gamma^+ \in \Pi(\mu, \varrho)$ and $\gamma^- \in \Pi(\varrho, \nu)$, apply Lemma 6.20 to obtain a probability measure σ on $\Omega \times \Omega \times \Omega$ such that $(\pi_{x,y})_{\#}\sigma = \gamma^+$ and $(\pi_{y,z})_{\#}\sigma = \gamma^-$. Let $\gamma := (\pi_{x,z})_{\#}\sigma$, so that $\gamma \in \Pi(\mu \to \nu)$ by Lemma 6.21. Then

$$
\begin{aligned}
W_p(\mu \to \nu) &\leq \left(\int_{\Omega \times \Omega} |x - z|^p \, \mathrm{d}\gamma(x, z) \right)^{1/p} = \left(\int_{\Omega \times \Omega \times \Omega} |x - z|^p \, \mathrm{d}\sigma(x, y, z) \right)^{1/p} \\
&= \| |x - z| \|_{L^p(\sigma)} \leq \| |x - y| + |y - z| \|_{L^p(\sigma)} \leq \| |x - y| \|_{L^p(\sigma)} + \| |y - z| \|_{L^p(\sigma)} \\
&= \left(\int_{\Omega \times \Omega \times \Omega} |x - y|^p \, \mathrm{d}\sigma(x, y, z) \right)^{1/p} + \left(\int_{\Omega \times \Omega \times \Omega} |y - z|^p \, \mathrm{d}\sigma(x, y, z) \right)^{1/p} \\
&= \left(\int_{\Omega \times \Omega} |x - y|^p \, \mathrm{d}\gamma^+(x, y) \right)^{1/p} + \left(\int_{\Omega \times \Omega} |y - z|^p \, \mathrm{d}\gamma^-(y, z) \right)^{1/p} .
\end{aligned}
$$

Since $\gamma^+ \in \Pi(\mu \to \varrho), \gamma^- \in \Pi(\varrho \to \nu)$ were arbitrary, $W_p(\mu \to \nu) \leq W_p(\mu \to \varrho) + W_p(\varrho \to \nu)$. □

The directed Wasserstein distance arises naturally in the one-dimensional case when one probability measure dominates the other in the sense of Definition 6.13, as we now show.

Proposition 6.23 (Specialization of [San15, Theorem 2.9])**.** *Let $p \in (1, +\infty)$. Let μ and ν be two absolutely continuous probability measures on $\overline{I}$ with strictly positive densities. Then there exists a unique $\gamma \in \Pi(\mu, \nu)$ attaining $C_p(\gamma) = W_p(\mu, \nu)$, and in fact $\gamma = (\mathsf{id}, T_{\mathrm{mon}})_{\#}\mu$ where $T_{\mathrm{mon}} : \overline{I} \to \overline{I}$ is a nondecreasing function that is a. e. uniquely determined, and id denotes the identity map.*

Corollary 6.24. *If in Proposition 6.23 we have $\mu \succeq \nu$, then in fact $\gamma \in \Pi(\mu \to \nu)$. As a consequence, $W_p(\mu \to \nu) = W_p(\mu, \nu)$.*

Proof. We claim that $T_{\mathrm{mon}}(x) \geq x$ for every $x \in I$. Indeed, suppose $T_{\mathrm{mon}}(x) < x$ for some $x \in I$. Then

$$
\nu(0, T_{\mathrm{mon}}(x)) = (T_{\mathrm{mon}\#}\mu)(0, T_{\mathrm{mon}}(x)) = \mu(T_{\mathrm{mon}}^{-1}(0, T_{\mathrm{mon}}(x))) \geq \mu(0, x) ,
$$

the inequality because certainly $(0, x) \subseteq T_{\mathrm{mon}}^{-1}(0, T_{\mathrm{mon}}(x))$, but we do not rule out at this point that T remains constant for a while after x. But since $T_{\mathrm{mon}}(x) < x$ and ν has strictly positive density by assumption, we conclude that

$$
\nu(0, x) > \nu(0, T_{\mathrm{mon}}(x)) \geq \mu(0, x) ,
$$

contradicting the assumption that $\mu \succeq \nu$. Hence $T_{\mathrm{mon}}(x) \geq x$ for every $x \in I$ as claimed. We now show that $\gamma \in \Pi(\mu \to \nu)$. Let $S := \{(x, y) \in \overline{I} \times \overline{I} : x \leq y\}$. Since $\gamma = (\mathsf{id}, T_{\mathrm{mon}})_{\#}\mu$, we have

$$
\gamma(S) = \mu((\mathsf{id}, T_{\mathrm{mon}})^{-1}(S)) = \mu(\{x \in \overline{I} : x \leq T_{\mathrm{mon}}(x)\}) = \mu(\overline{I}) = 1 ,
$$

so $\gamma \in \Pi(\mu \to \nu)$ as claimed. □

We now introduce the formal language for two related operations: constructing a probability measure by its marginal and conditional distributions, and the inverse process of extracting marginal and conditionals from a probability measure, which is also called *disintegration*. We refer the reader to [AGS05, Section 5.3] for an overview of these ideas from a measure-theoretic perspective.

We use the following definition from [AGS05]. For X and Y separable metric spaces and $x \in X \mapsto \mu_x \in P(Y)$ a measure-valued map, we say μ_x is a Borel map if $x \mapsto \mu_x(B)$ is a Borel map for any Borel set $B \subset Y$, or equivalently if this holds for any open set $A \subset Y$. In this case we also have that

$$x \mapsto \int_Y f(x,y)\,\mathrm{d}\mu_x(y)$$

is Borel for any bounded (or nonnegative) Borel function $f : X \times Y \to \mathbb{R}$.

Definition 6.25 (Construction and disintegration)**.** Let S be a finite set and let $I \subset S$. Let $x_{S\setminus I} \in \mathbb{R}^{S\setminus I} \mapsto \mu_{|x_{S\setminus I}} \in P(\mathbb{R}^I)$ be a Borel map (the *conditionals*). Then for any bounded (or nonnegative) Borel function $f : \mathbb{R}^S \to \mathbb{R}$, the $\mathbb{R}^{S\setminus I} \to \mathbb{R}$ map

$$x_{S\setminus I} \mapsto \int_{\mathbb{R}^I} f(x_{S\setminus I}, x_I)\,\mathrm{d}\mu_{|x_{S\setminus I}}(x_I) \tag{24}$$

is Borel. Therefore, for any $\mu_{S\setminus I} \in P(\mathbb{R}^{S\setminus I})$ (the *marginal*), we define $\mu \in P(\mathbb{R}^S)$ implicitly by

$$\int_{\mathbb{R}^S} f\,\mathrm{d}\mu = \int_{\mathbb{R}^{S\setminus I}} \mathrm{d}\mu_{S\setminus I}(x_{S\setminus I}) \int_{\mathbb{R}^I} f(x_{S\setminus I}, x_I)\,\mathrm{d}\mu_{|x_{S\setminus I}}(x_I)\,, \tag{25}$$

and we formally write $\mu = \int_{\mathbb{R}^{S\setminus I}} \mu_{|x_{S\setminus I}}\,\mathrm{d}\mu_{S\setminus I}$.

Conversely, given $\mu \in P(\mathbb{R}^S)$, we let $\mu_{S\setminus I} := (\pi_{S\setminus I})_{\#}\mu \in P(\mathbb{R}^{S\setminus I})$ and write $(\mu_{|x_{S\setminus I}})_{x_{S\setminus I} \in \mathbb{R}^{S\setminus I}} \subset P(\mathbb{R}^I)$ for the $\mu_{S\setminus I}$-a.e. uniquely determined Borel family of probability measures such that $\mu = \int_{\mathbb{R}^{S\setminus I}} \mu_{|x_{S\setminus I}}\,\mathrm{d}\mu_{S\setminus I}$, and we call this decomposition a *disintegration* of μ.

Examples and simplified notational conventions. Since the index notation in the definition above can be somewhat laborious to parse, let us briefly give two concrete settings we will use below, and introduce simplified notational conventions for those settings. The simplifications are supposed to be mnemonic for what we have already defined for projections.

1. For a probability measure $\mu \in P(\mathbb{R}^d)$ and index $i \in [d]$, we may disintegrate μ into a marginal on index set $[d] \setminus \{i\}$ and conditionals supported along i-th coordinate, and write $\mu = \int_{\mathbb{R}^{[d]\setminus\{i\}}} \mu_{|x_{[d]\setminus\{i\}}}\,\mathrm{d}\mu_{[d]\setminus\{i\}}$. We simplify this notation by writing $\mu = \int_{\mathbb{R}^d_{-i}} \mu_{|x_{-i}}\,\mathrm{d}\mu_{-i}$.

2. For a transport plan $\gamma \in P(\mathbb{R}^d \times \mathbb{R}^d)$, we write the index set as $[2d] = S = S_x \cup S_y$ for $S_x := [d]$ and $S_y := [2d] \setminus [d]$, so we write each element in the support of γ as a tuple (x, y) with $x \in \mathbb{R}^{S_x}$ and $y \in \mathbb{R}^{S_y}$. Then, given index set $I \subset [d]$, we may disintegrate γ into a marginal on index set $S_M := (S_x \setminus I_x) \cup (S_y \setminus I_y)$ for $I_x := I$ and $I_y := \{d + i : i \in I\}$, so that we think of the marginal as determining all but the I-indexed coordinates of the points x and y; and we view the conditionals as supported along the $I_x \cup I_y$ directions. This disintegration is $\gamma = \int_{\mathbb{R}^{S_M}} \gamma_{|x_{S_M}}\,\mathrm{d}\gamma_{S_M}$. To be clear, here we have $\gamma_{S_M} \in P(\mathbb{R}^{S_M})$ and $\gamma_{|x_{S_M}} \in P(\mathbb{R}^{I_x \cup I_y})$ for each $x_{S_M} \in \mathbb{R}^{S_M}$. We make the notation somewhat more mnemonic by writing $\gamma = \int_{\mathbb{R}^d_{-I} \times \mathbb{R}^d_{-I}} \gamma_{|x_{-I}, y_{-I}}\,\mathrm{d}\gamma_{-I_x - I_y}$. When we are considering a singleton $I = \{i\}$, we further simplify the notation by directly writing i in the place of I in this formula.

For absolutely continuous probability measures on $[0,1]^d$, the following more familiar characterization of disintegrations will be useful:

Proposition 6.26. *Let μ be an absolutely continuous probability measure whose density is Borel measurable and supported in $[0,1]^d$. Namely, write $\mathrm{d}\mu = u\,\mathrm{d}x$ where $u : [0,1]^d \to [0,+\infty)$ is a Borel map. Let $i \in [d]$. Then the disintegration $\mu = \int_{\mathbb{R}^d_{-i}} \mu_{|x_{-i}} \mathrm{d}\mu_{-i}$ is as follows: μ_{-i} and each $\mu_{|x_{-i}}$ are absolutely continuous probability measures with Borel densities supported in $[0,1]^d_{-i}$ and $[0,1]$ respectively; and writing $\mathrm{d}\mu_{-i} = u_{-i}\,\mathrm{d}x_{-i}$, and $\mathrm{d}\mu_{|x_{-i}} = u_{|x_{-i}}\,\mathrm{d}x_i$ for each x_{-i}, where the (Borel) density functions are $u_{-i} : [0,1]^d_{-i} \to [0,+\infty)$ and $u_{|x_{-i}} : [0,1] \to [0,+\infty)$ for each $x_{-i} \in [0,1]^d_{-i}$, we have (marginal density)*

$$u_{-i}(x_{-i}) = \int_I u(x_{-i}, x_i)\,\mathrm{d}x_i$$

for each $x_{-i} \in [0,1]^d_{-i}$, and (conditional density)

$$u_{|x_{-i}}(x_i) = \frac{u(x)}{u_{-i}(x_{-i})}$$

for each $x \in [0,1]^d$ where the denominator is nonzero, or 1 by convention otherwise.

Proof. First, note that it is standard that u_{-i} is a Borel map since it is the integral of the section $u(x_{-i}, \cdot)$ at each point x_{-i}. It is hence immediate that each $u_{|x_{-i}}$ is also Borel. Thus μ_{-i} and all $\mu_{|x_{-i}}$ are indeed absolutely continuous probability measures.

We now check the conditions of Definition 6.25. We claim that the map $x_{-i} \in \mathbb{R}^d_{-i} \mapsto \mu_{|x_{-i}} \in P(\mathbb{R})$ is Borel. Let $B \subset \mathbb{R}$ be a Borel set; we may assume that $B \subset [0,1]$ without loss of generality, since all measures assign zero outside this interval. Then the map $x_{-i} \in \mathbb{R}^d_{-i} \mapsto \mu_{|x_{-i}}(B)$ takes value zero outside $[0,1]^d_{-i}$, and for $x_{-i} \in [0,1]^d_{-i}$, it is

$$x_{-i} \mapsto \frac{1}{u_{-i}(x_{-i})} \int_B u(x_{-i}, x_i)\,\mathrm{d}x_i\,.$$

This is again, at each point, the integral of a section over the subspace B, so it is standard that this map is Borel. Hence the claim holds.

It remains to verify (25). Let $f : \mathbb{R}^d \to [0,+\infty)$ be a nonnegative Borel function (and the result for bounded f will also follow). By the definition of the measures and Tonelli's theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^{-i}} \mathrm{d}\mu_{-i}(x_{-i}) \int_{\mathbb{R}} f(x_{-i}, x_i)\,\mathrm{d}u_{|x_{-i}}(x_i) &= \int_{[0,1]^d_{-i}} u_{-i}(x_{-i})\,\mathrm{d}x_{-i} \int_I f(x_{-i}, x_i) u_{|x_{-i}}(x_i)\,\mathrm{d}x_i \\ &= \int_{[0,1]^d} f(x) u_{-i}(x_{-i}) u_{|x_{-i}}(x_i)\,\mathrm{d}x = \int_{[0,1]^d} f(x) u(x)\,\mathrm{d}x\,,\end{aligned}$$

where the last equality holds because $u(x) = u_{-i}(x_{-i}) u_{|x_{-i}}(x_i)$ whenever $u_{-i}(x_{-i}) > 0$, and if $u_{-i}(x_{-i}) = 0$ then $u(x_{-i}, \cdot) = 0$ almost everywhere, so $\left\{x \in [0,1]^d : u(x) > 0 \text{ and } u_{-i}(x_{-i}) = 0\right\}$ is Borel and has measure zero (by another application of Tonelli's theorem). $\square$

Definition 6.27 (Aligned transport plans)**.** Let $I \subsetneq [d]$ be nonempty and let $\mu, \nu \in P(\mathbb{R}^d)$. Let $\gamma \in \Pi(\mu, \nu)$ and write its disintegration $\gamma = \int_{\mathbb{R}^d_{-I} \times \mathbb{R}^d_{-I}} \gamma_{|x_{-I}, y_{-I}}\,\mathrm{d}\gamma_{-I_x - I_y}$. We say γ is *I-aligned* if its marginal $\gamma_{-I_x - I_y} \in P(\mathbb{R}^d_{-I} \times \mathbb{R}^d_{-I})$ satisfies the following:

$$\gamma_{-I_x - I_y}\left(\left\{(x_{-I}, y_{-I}) \in \mathbb{R}^d_{-I} \times \mathbb{R}^d_{-I} : x_{-I} \neq y_{-I}\right\}\right) = 0\,.$$

Note that this condition is well-defined because the set being measured is open and hence Borel. By convention, we say that every $\gamma \in \Pi(\mu, \nu)$ is $[d]$-aligned. For any nonempty $I \subseteq [d]$, we denote the set of I-aligned transport plans by $\Pi_I(\mu, \nu)$. When we have a singleton $I = \{i\}$, we write i directly in the place of I in this definition.

Definition 6.28. For nonempty $I \subseteq [d]$ and $\mu, \nu \in P(\mathbb{R}^d)$, write $\Pi_I(\mu \to \nu) := \Pi(\mu \to \nu) \cap \Pi_I(\mu, \nu)$.

Lemma 6.29 (Alternative characterization of I-aligned plans)**.** *Let $I \subsetneq [d]$ be nonempty and let $\mu, \nu \in P(\mathbb{R}^d)$. Then $\gamma \in \Pi(\mu, \nu)$ is I-aligned if and only if*

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \chi_{\{x_{-I} \neq y_{-I}\}} \, \mathrm{d}\gamma(x, y) = 0 \,.$$

Proof. Writing the disintegration $\gamma = \int_{\mathbb{R}^d_{-I} \times \mathbb{R}^d_{-I}} \gamma_{|x_{-I}, y_{-I}} \, \mathrm{d}\gamma_{-I_x - I_y}$, we have

$$\begin{aligned}
\gamma_{-I_x - I_y}(\{x_{-I} \neq y_{-I}\}) &= \int_{\mathbb{R}^d_{-I} \times \mathbb{R}^d_{-I}} \chi_{\{x_{-I} \neq y_{-I}\}} \, \mathrm{d}\gamma_{-I_x - I_y} \underbrace{\int_{\mathbb{R}^I \times \mathbb{R}^I} \mathrm{d}\gamma_{|x_{-I}, y_{-I}}(x_I, y_I)}_{=1} \\
&= \int_{\mathbb{R}^d_{-I} \times \mathbb{R}^d_{-I}} \mathrm{d}\gamma_{-I_x - I_y} \int_{\mathbb{R}^I \times \mathbb{R}^I} \chi_{\{x_{-I} \neq y_{-I}\}} \, \mathrm{d}\gamma_{|x_{-I}, y_{-I}}(x_I, y_I) \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi_{\{x_{-I} \neq y_{-I}\}} \, \mathrm{d}\gamma \,. \qquad \square
\end{aligned}$$

While we defined I-aligned transport plans for probability measures on all of $\mathbb{R}^d$, we extend the definition to measures on subsets $\Omega \subset \mathbb{R}^d$ (e.g. on the cube) by canonically extending any such measure to all of $\mathbb{R}^d$ by assigning zero outside of Ω.

Lemma 6.30 (Composition of aligned transport plans)**.** *Let $I, J \subseteq [d]$ be nonempty. Then in Lemma 6.20, if $\gamma^+ \in \Pi_I(\mu, \varrho)$ and $\gamma^- \in \Pi_J(\varrho, \nu)$, then $(\pi_{x,z})_\# \sigma \in \Pi_{I \cup J}(\mu, \nu)$.*

Proof. Let $\gamma := (\pi_{x,z})_\# \sigma$ and $K := I \cup J$. As in the proof of Lemma 6.21, we do have $\gamma \in \Pi(\mu, \nu)$, so it remains to show that γ is K-aligned (and we may assume that $K \subsetneq [n]$, otherwise there is nothing to prove). We have

$$\begin{aligned}
&\int_{\mathbb{R}^d \times \mathbb{R}^d} \chi_{\{x_{-K} \neq z_{-K}\}} \, \mathrm{d}\gamma \\
&\quad = \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \chi_{\{x_{-K} \neq z_{-K}\}}(x, z) \, \mathrm{d}\sigma(x, y, z) && \text{(Pushforward)} \\
&\quad \leq \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \big(\chi_{\{x_{-K} \neq y_{-K}\}}(x, y) + \chi_{\{y_{-K} \neq z_{-K}\}}(y, z)\big) \, \mathrm{d}\sigma(x, y, z) \\
&\quad = \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \chi_{\{x_{-K} \neq y_{-K}\}} \, \mathrm{d}\sigma + \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \chi_{\{y_{-K} \neq z_{-K}\}} \, \mathrm{d}\sigma \\
&\quad = \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi_{\{x_{-K} \neq y_{-K}\}} \, \mathrm{d}\gamma^+ + \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi_{\{y_{-K} \neq z_{-K}\}} \, \mathrm{d}\gamma^- && \text{(Pushforward)} \\
&\quad \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi_{\{x_{-I} \neq y_{-I}\}} \, \mathrm{d}\gamma^+ + \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi_{\{y_{-J} \neq z_{-J}\}} \, \mathrm{d}\gamma^- && (I, J \subseteq K) \\
&\quad = 0 \,,
\end{aligned}$$

the last step since γ^+ is I-aligned and γ^- is J-aligned and by Lemma 6.29; and again by the latter, γ is K-aligned. $\square$

The following results let us find aligned transport plans by transporting mass only within i-aligned lines.

Lemma 6.31 (Measurable selection; specialization of [Vil09, Corollary 5.22])**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded set. Let $p \in [1,\infty)$. Let A be a measurable space and let $a \mapsto (\mu_a, \nu_a)$ be a measurable function $A \to P(\mathbb{R}) \times P(\mathbb{R})$ with each μ_a and ν_a having bounded support. Then there is a measurable choice $a \mapsto \gamma_a$ such that, for each $a \in A$, $\gamma_a \in \Pi(\mu_a, \nu_a)$ and $C_p(\gamma_a) = W_p(\mu_a, \nu_a)$.*

Lemma 6.32. *Let $p \in [1,\infty)$, $i \in [d]$, and let $\mu, \nu \in P(\mathbb{R}^d)$ be supported inside some bounded set. Suppose $\mu_{-i} = \nu_{-i}$. Then there exists $\gamma \in \Pi_i(\mu,\nu)$ satisfying*

$$C_p(\gamma)^p = \int_{\mathbb{R}^d_{-i}} W_p^p(\mu_{|x_{-i}}, \nu_{|x_{-i}}) \,\mathrm{d}\mu_{-i}(x_{-i}) \,.$$

Proof. Let $A := \mathbb{R}^d_{-i}$ be equipped with the Borel σ-algebra on $\mathbb{R}^{[d]\setminus\{i\}}$. Then the $A \to P(\mathbb{R}) \times P(\mathbb{R})$ function $x_{-i} \mapsto (\mu_{|x_{-i}}, \nu_{|x_{-i}})$ is Borel measurable because, by disintegration, both $x_{-i} \mapsto \mu_{|x_{-i}}$ and $x_{-i} \mapsto \nu_{|x_{-i}}$ are Borel measurable. By Lemma 6.31, we obtain a Borel measurable map $x_{-i} \mapsto \gamma_{x_{-i}}$ such that, for each $x_{-i} \in \mathbb{R}^d_{-i}$, $\gamma_{x_{-i}} \in \Pi(\mu_{|x_{-i}}, \nu_{|x_{-i}})$ and $C_p(\gamma_{x_{-i}})^p = W_p^p(\mu_{|x_{-i}}, \nu_{|x_{-i}})$.

Define $\gamma \in P(\mathbb{R}^d \times \mathbb{R}^d)$ by its disintegration $\gamma = \int_{\mathbb{R}^d_{-i} \times \mathbb{R}^d_{-i}} \gamma_{|x_{-i}, y_{-i}} \,\mathrm{d}\gamma_{-i_x - i_y}$ as follows. First, for the marginal $\gamma_{-i_x - i_y} \in P(\mathbb{R}^d_{-i} \times \mathbb{R}^d_{-i})$, for each Borel $Z \subset \mathbb{R}^d_{-i} \times \mathbb{R}^d_{-i}$ we set

$$\gamma_{-i_x - i_y}(Z) := \int_{\mathbb{R}^d_{-i}} \chi_Z(x_{-i}, x_{-i}) \,\mathrm{d}\mu_{-i}(x_{-i}) \,,$$

where the integral is well-defined because the set $\{x_{-i} \in \mathbb{R}^d_{-i} : (x_{-i}, x_{-i}) \in Z\}$ is Borel by standard arguments. We remark that $\gamma_{-i_x - i_y}$ is indeed a probability measure because it is nonnegative, it assigns zero to the empty set, it is additive over countable disjoint unions, and

$$\gamma_{-i_x - i_y}(\mathbb{R}^d_{-i} \times \mathbb{R}^d_{-i}) = \int_{\mathbb{R}^d_{-i}} \mathrm{d}\mu_{-i} = 1 \,.$$

It follows that for every bounded (or nonnegative) Borel function $f : \mathbb{R}^d_{-i} \times \mathbb{R}^d_{-i} \to \mathbb{R}$, we have

$$\int_{\mathbb{R}^d_{-i} \times \mathbb{R}^d_{-i}} f(x_{-i}, y_{-i}) \,\mathrm{d}\gamma_{-i_x - i_y}(x_{-i}, y_{-i}) = \int_{\mathbb{R}^d_{-i}} f(x_{-i}, x_{-i}) \,\mathrm{d}\mu_{-i}(x_{-i}) \,. \tag{26}$$

Second, for the conditionals, we simply set $\gamma_{|x_{-i}, y_{-i}} := \gamma_{x_{-i}}$ for each $(x_{-i}, y_{-i}) \in \mathbb{R}^d_{-i} \times \mathbb{R}^d_{-i}$.

We claim that $\gamma \in \Pi_i(\mu, \nu)$. The i-aligned condition holds by construction, so it remains to show that $\gamma \in \Pi(\mu, \nu)$. We first verify that $(\pi_1)_\# \gamma = \mu$. It suffices to verify that these measures agree on each rectangle $X := X_{-i} \times X_i$ of Borel sets $X_{-i} \subset \mathbb{R}^d_{-i}$, $X_i \subset \mathbb{R}$. We have

$$\begin{aligned}
&((\pi_1)_\# \gamma)(X) \\
&\quad = \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi_X(x) \,\mathrm{d}\gamma(x, y) && \text{(Pushforward)} \\
&\quad = \int_{\mathbb{R}^d_{-i} \times \mathbb{R}^d_{-i}} \mathrm{d}\gamma_{-i_x - i_y}(x_{-i}, y_{-i}) \int_{\mathbb{R} \times \mathbb{R}} \chi_X(x_{-i}, x_i) \,\mathrm{d}\gamma_{|x_{-i}, y_{-i}}(x_i, y_i) && \text{(Disintegration)} \\
&\quad = \int_{\mathbb{R}^d_{-i} \times \mathbb{R}^d_{-i}} \chi_{X_{-i}}(x_{-i}) \,\mathrm{d}\gamma_{-i_x - i_y}(x_{-i}, y_{-i}) \int_{\mathbb{R} \times \mathbb{R}} \chi_{X_i}(x_i) \,\mathrm{d}\gamma_{x_{-i}}(x_i, y_i) && \text{(Definition of } X, \gamma_{|x_{-i}, y_{-i}}) \\
&\quad = \int_{\mathbb{R}^d_{-i} \times \mathbb{R}^d_{-i}} \chi_{X_{-i}}(x_{-i}) ((\pi_1)_\# \gamma_{x_{-i}})(X_i) \,\mathrm{d}\gamma_{-i_x - i_y}(x_{-i}, y_{-i}) && \text{(Pushforward)}
\end{aligned}$$

$$= \int_{\mathbb{R}^d_{-i}\times\mathbb{R}^d_{-i}} \chi_{X_{-i}}(x_{-i})\mu_{|x_{-i}}(X_i)\,\mathrm{d}\gamma_{-i_x-i_y}(x_{-i},y_{-i}) \qquad (\gamma_{x_{-i}}\in\Pi(\mu_{|x_{-i}},\nu_{|x_{-i}}))$$
$$= \int_{\mathbb{R}^d_{-i}} \chi_{X_{-i}}(x_{-i})\mu_{|x_{-i}}(X_i)\,\mathrm{d}\mu_{-i}(x_{-i}) \qquad \text{(Application of (26))}$$
$$= \int_{\mathbb{R}^d_{-i}} \chi_{X_{-i}}(x_{-i})\,\mathrm{d}\mu_{-i}(x_{-i})\int_{\mathbb{R}} \chi_{X_i}(x_i)\,\mathrm{d}\mu_{|x_{-i}}(x_i)$$
$$= \int_{\mathbb{R}^d_{-i}} \mathrm{d}\mu_{-i}(x_{-i})\int_{\mathbb{R}} \chi_X(x_{-i},x_i)\,\mathrm{d}\mu_{|x_{-i}}(x_i) \qquad \text{(Definition of } X\text{)}$$
$$= \int_{\mathbb{R}^d} \chi_X\,\mathrm{d}\mu = \mu(X) \qquad \text{(Disintegration)}\,,$$

as desired. Thanks to the hypothesis that $\mu_{-i} = \nu_{-i}$, an analogous calculation on the second variable yields that $(\pi_2)_{\#}\gamma = \nu$ and hence $\gamma \in \Pi(\mu,\nu)$. Thus $\gamma \in \Pi_i(\mu,\nu)$ as claimed.

Finally, we compute the cost of the plan γ. Again using its disintegration and (26), we have

$$\begin{aligned}
C_p(\gamma)^p &= \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p\,\mathrm{d}\gamma(x,y)\\
&= \int_{\mathbb{R}^d_{-i}\times\mathbb{R}^d_{-i}} \mathrm{d}\gamma_{-i_x-i_y}(x_{-i},y_{-i})\int_{\mathbb{R}\times\mathbb{R}} |(x_{-i},x_i)-(y_{-i},y_i)|^p\,\mathrm{d}\gamma_{|x_{-i},y_{-i}}(x_i,y_i)\\
&= \int_{\mathbb{R}^d_{-i}} \mathrm{d}\mu_{-i}(x_{-i})\int_{\mathbb{R}\times\mathbb{R}} |(x_{-i},x_i)-(x_{-i},y_i)|^p\,\mathrm{d}\gamma_{|x_{-i},x_{-i}}(x_i,y_i)\\
&= \int_{\mathbb{R}^d_{-i}} \mathrm{d}\mu_{-i}(x_{-i})\int_{\mathbb{R}\times\mathbb{R}} |x_i-y_i|^p\,\mathrm{d}\gamma_{x_{-i}}(x_i,y_i)\\
&= \int_{\mathbb{R}^d_{-i}} \mathrm{d}\mu_{-i}(x_{-i})C_p(\gamma_{x_{-i}})^p = \int_{\mathbb{R}^d_{-i}} W_p^p(\mu_{|x_{-i}},\nu_{|x_{-i}})\,\mathrm{d}\mu_{-i}(x_{-i})\,.
\end{aligned}$$

□

We may combine the above to find *directed* transport plans by transporting mass within axis-aligned lines, as long as the first probability distribution dominates the second in each such line.

Lemma 6.33. *Let $p \in (1,+\infty)$, and let $\mu,\nu \in P([0,1]^d)$ be absolutely continuous probability measures with strictly positive densities. Let $i \in [d]$ and suppose that 1) $\mu_{-i} = \nu_{-i}$ and 2) $\mu_{|x_{-i}} \succeq \nu_{|x_{-i}}$ for each $x_{-i} \in [0,1]^d_{-i}$. Then there exists $\gamma \in \Pi_i(\mu \to \nu)$ satisfying*

$$C_p(\gamma)^p = \int_{[0,1]^d_{-i}} W_p^p(\mu_{|x_{-i}},\nu_{|x_{-i}})\,\mathrm{d}\mu_{-i}(x_{-i})\,. \tag{27}$$

Proof. Let $\gamma \in \Pi_i(\mu,\nu)$ be the plan obtained from Lemma 6.32, which satisfies (27). After recalling the definition of γ in Lemma 6.32, Corollary 6.24 implies that $\gamma_{|x_{-i},y_{-i}} \in \Pi(\mu_{|x_{-i}} \to \nu_{|x_{-i}})$ for each

$(x_{-i}, y_{-i}) \in [0,1]^d_{-i} \times [0,1]^d_{-i}$. It remains to confirm that $\gamma \in \Pi(\mu \to \nu)$. Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}^d \times \mathbb{R}^d} \chi_{\{x \not\leq y\}} \,\mathrm{d}\gamma(x,y) \\
&\quad = \int_{\mathbb{R}^d_{-i} \times \mathbb{R}^d_{-i}} \mathrm{d}\gamma_{-i_x - i_y}(x_{-i}, y_{-i}) \int_{\mathbb{R}\times\mathbb{R}} \chi_{\{x \not\leq y\}}((x_{-i}, x_i), (y_{-i}, y_i)) \,\mathrm{d}\gamma_{|x_{-i}, y_{-i}}(x_i, y_i) \quad \text{(Disintegration)} \\
&\quad \leq \int_{\mathbb{R}^d_{-i} \times \mathbb{R}^d_{-i}} \mathrm{d}\gamma_{-i_x - i_y}(x_{-i}, y_{-i}) \int_{\mathbb{R}\times\mathbb{R}} \chi_{\{x_{-i} \not\leq y_{-i}\}}(x_{-i}, y_{-i}) \,\mathrm{d}\gamma_{|x_{-i}, y_{-i}}(x_i, y_i) \\
&\qquad\quad + \int_{\mathbb{R}^d_{-i} \times \mathbb{R}^d_{-i}} \mathrm{d}\gamma_{-i_x - i_y}(x_{-i}, y_{-i}) \underbrace{\int_{\mathbb{R}\times\mathbb{R}} \chi_{\{x_i > y_i\}}(x_i, y_i) \,\mathrm{d}\gamma_{|x_{-i}, y_{-i}}(x_i, y_i)}_{=0 \text{ since } \gamma_{|x_{-i}, y_{-i}} \text{ is a directed plan}} \\
&\quad = \int_{\mathbb{R}^d_{-i} \times \mathbb{R}^d_{-i}} \chi_{\{x_{-i} \not\leq y_{-i}\}} \,\mathrm{d}\gamma_{-i_x - i_y} \underbrace{\int_{\mathbb{R}\times\mathbb{R}} \mathrm{d}\gamma_{|x_{-i}, y_{-i}}}_{=1} = \int_{\mathbb{R}^d_{-i}} \underbrace{\chi_{\{x_{-i} \not\leq x_{-i}\}}}_{=0} \,\mathrm{d}\mu_{-i}(x_{-i}) \qquad \text{(By (26))} \\
&\quad = 0\,.
\end{aligned}$$

□

Now, we show that the directed, aligned 2-Wasserstein distance enjoys a nice "Pythagorean" composition property: composing I-aligned and J-aligned directed transport plans, for I and J disjoint, yields an $I \cup J$-aligned directed transport plan whose cost is obtained from the costs of the two other plans via the Pythagorean theorem. Arguments of this nature are well-known in the undirected case, e.g. a similar statement appears in [Vil09, p. 572].

Lemma 6.34 (Pythagorean composition of transport plans)**.** *Let $I, J \subseteq [d]$ be nonempty disjoint sets. Let $\mu, \varrho, \nu \in P(\mathbb{R}^d)$ be supported in bounded sets, and let $\gamma^+ \in \Pi_I(\mu \to \varrho)$ and $\gamma^- \in \Pi_J(\varrho \to \nu)$. Then there exists $\gamma \in \Pi_{I \cup J}(\mu \to \nu)$ satisfying $C_2(\gamma)^2 = C_2(\gamma^+)^2 + C_2(\gamma^-)^2$.*

Proof. Using Lemma 6.20, we obtain $\sigma \in P(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ such that $(\pi_{1,2})_\# \sigma = \gamma^+$ and $(\pi_{2,3})_\# \sigma = \gamma^-$, and by Lemma 6.21 and Lemma 6.30, we have $\gamma := (\pi_{1,3})_\# \sigma \in \Pi_{I \cup J}(\mu \to \nu)$. To compute the cost of γ, we first claim that σ assigns zero measure to points x, y, z such that $x_{-I} \neq y_{-I}$ or $y_{-J} \neq z_{-J}$. Indeed,

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} (1 - \chi_{\{x_{-I} = y_{-I} \text{ and } y_{-J} = z_{-J}\}}) \,\mathrm{d}\sigma(x,y,z) \\
&\leq \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \chi_{\{x_{-I} \neq y_{-I}\}} \,\mathrm{d}\sigma(x,y,z) + \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \chi_{\{y_{-J} \neq z_{-J}\}} \,\mathrm{d}\sigma(x,y,z) \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi_{\{x_{-I} \neq y_{-I}\}} \,\mathrm{d}\gamma^+(x,y) + \int_{\mathbb{R}^d \times \mathbb{R}^d} \chi_{\{y_{-J} \neq z_{-J}\}} \,\mathrm{d}\gamma^-(y,z) = 0\,,
\end{aligned}$$

the last step by Lemma 6.29 since γ^+ and γ^- are I- and J-aligned, respectively. Hence $\int f \,\mathrm{d}\sigma = \int f \chi_{\{x_{-I} = y_{-I} \text{ and } y_{-J} = z_{-J}\}} \,\mathrm{d}\sigma$ for any bounded or nonnegative Borel function f. Therefore, using the assumption that I and J are disjoint to apply the Pythagorean theorem, we obtain

$$\begin{aligned}
C_2(\gamma)^2 &= \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - z|^2 \,\mathrm{d}\gamma(x,z) = \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} |x - z|^2 \,\mathrm{d}\sigma(x,y,z) \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \chi_{\{x_{-I} = y_{-I} \text{ and } y_{-J} = z_{-J}\}} |x - z|^2 \,\mathrm{d}\sigma(x,y,z) \\
&= \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} \chi_{\{x_{-I} = y_{-I} \text{ and } y_{-J} = z_{-J}\}} \left(|x - y|^2 + |y - z|^2\right) \mathrm{d}\sigma(x,y,z)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} \left(|x-y|^2 + |y-z|^2\right) \mathrm{d}\sigma(x,y,z) \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \,\mathrm{d}\sigma(x,y,z) + \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d} |y-z|^2 \,\mathrm{d}\sigma(x,y,z) \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 \,\mathrm{d}\gamma^+(x,y) + \int_{\mathbb{R}^d\times\mathbb{R}^d} |y-z|^2 \,\mathrm{d}\gamma^-(y,z) \\
&= C_2(\gamma^+)^2 + C_2(\gamma^-)^2 \,.
\end{aligned}
$$ □

Lemma 6.35 (Induction over coordinates)**.** *Let $(\mu^{(i)})_{i\in[d+1]}$ be a family of absolutely continuous probability measures on $[0,1]^d$ with strictly positive densities. Suppose that, for each $i \in [d]$, we have 1) ${\mu^{(i)}}_{-i} = {\mu^{(i+1)}}_{-i}$ and 2) ${\mu^{(i)}}_{|x_{-i}} \succeq {\mu^{(i+1)}}_{|x_{-i}}$ for each $x_{-i} \in [0,1]^d_{-i}$. Then*

$$
W_2^2(\mu^{(1)} \to \mu^{(d+1)}) \le \sum_{i=1}^{d} \int_{[0,1]^d_{-i}} W_2^2({\mu^{(i)}}_{|x_{-i}}, {\mu^{(i+1)}}_{|x_{-i}}) \,\mathrm{d}{\mu^{(i)}}_{-i}(x_{-i}) \,.
$$

Proof. For each $k \in [d+1]$, let $A(k)$ be the following proposition: there exists $\gamma \in \Pi_{[k]}(\mu^{(1)} \to \mu^{(k+1)})$ satisfying

$$
C_2(\gamma)^2 = \sum_{i=1}^{k} \int_{[0,1]^d_{-i}} W_2^2({\mu^{(i)}}_{|x_{-i}}, {\mu^{(i+1)}}_{|x_{-i}}) \,\mathrm{d}{\mu^{(i)}}_{-i}(x_{-i}) \,.
$$

Note that $A(d)$ implies the result we want to prove, while $A(1)$ holds by Lemma 6.33. Now let $2 \le k \le d$. Suppose $A(k-1)$ holds, and thus let $\gamma^+ \in \Pi_{[k-1]}(\mu^{(1)} \to \mu^{(k)})$ satisfy

$$
C_2(\gamma^+)^2 = \sum_{i=1}^{k-1} \int_{[0,1]^d_{-i}} W_2^2({\mu^{(i)}}_{|x_{-i}}, {\mu^{(i+1)}}_{|x_{-i}}) \,\mathrm{d}{\mu^{(i)}}_{-i}(x_{-i}) \,.
$$

Apply Lemma 6.33 with $i = k$ to obtain a plan $\gamma^- \in \Pi_k(\mu^{(k)} \to \mu^{(k+1)})$ satisfying

$$
C_2(\gamma^-)^2 = \int_{[0,1]^d_{-k}} W_2^2({\mu^{(k)}}_{|x_{-k}}, {\mu^{(k+1)}}_{|x_{-k}}) \,\mathrm{d}{\mu^{(k)}}_{-k}(x_{-k}) \,.
$$

Then, use Lemma 6.34 with $I = [k-1]$ and $J = \{k\}$ to obtain $\gamma \in \Pi_{[k]}(\mu^{(1)} \to \mu^{(k+1)})$ satisfying

$$
C_2(\gamma)^2 = C_2(\gamma^+)^2 + C_2(\gamma^-)^2 = \sum_{i=1}^{k} \int_{[0,1]^d_{-i}} W_2^2({\mu^{(i)}}_{|x_{-i}}, {\mu^{(i+1)}}_{|x_{-i}}) \,\mathrm{d}{\mu^{(i)}}_{-i}(x_{-i}) \,,
$$

which implies that $A(k)$ holds. Thus $A(d)$ follows by induction. □

6.4 From one-dimensional PDE to optimal transport in the cube

We now tie most of the foregoing theory together by showing how to inductively transform a function on the unit cube into a monotone function, one coordinate at a time via the directed heat semigroup machinery, keeping track of the cost using the directed optimal transport machinery.

For this part of the proof, it is convenient to restrict our attention to Lipschitz functions. As we will see, the Lipschitz property of $f : [0,1]^d \to \mathbb{R}$ is preserved under taking the monotone equilibrium along each axis-aligned direction, which enables an inductive argument over the coordinates.

Denote the set of Lipschitz functions $f : [0,1]^d \to \mathbb{R}$ by Lip (the dimension d will always be clear from context).

Observation 6.36. *A function $f : [0,1]^d \to \mathbb{R}$ is Lipschitz if and only if its restriction to every axis-aligned line is uniformly Lipschitz, i.e. there exists some $M > 0$ such that, for every $i \in [d]$ and $x_{-i} \in [0,1]^d_{-i}$, $f(x_{-i}, \cdot) : [0,1] \to \mathbb{R}$ is M-Lipschitz.*

Definition 6.37 (i- and I-monotonicity)**.** Let $i \in [d]$. We say $f : [0,1]^d \to \mathbb{R}$ is *i-monotone* if its every restriction along direction i is nondecreasing, i.e. if for every $x_{-i} \in [0,1]^d_{-i}$, $f(x_{-i}, \cdot)$ is nondecreasing. For $I \subseteq [d]$, we say f is *I-monotone* if it is i-monotone for every $i \in I$. We write Mon_I for the set of I-monotone functions.

Observation 6.38. $\mathsf{Mon}_{[d]}$ *is simply the set of monotone functions on $[0,1]^d$.*

The following useful characterization of i-monotone functions in terms of k-aligned lines follows by definition:

Observation 6.39. *Let $i, k \in [d]$ be distinct. Then for all $f : [0,1]^d \to \mathbb{R}$, f is i-monotone if and only if the following holds: for every $z_{-i-k} \in [0,1]^d_{-i-k}$, and for all $x_i \le y_i$ in $[0,1]$, $f(z_{-i-k}, x_i, \cdot) \le f(z_{-i-k}, y_i, \cdot)$.*

Definition 6.40 (Monotone equilibrium operator)**.** Let $k \in [d]$. We define the operator $\mathcal{M}_k : \mathsf{Lip} \to ([0,1]^d \to \mathbb{R})$ as follows: for each function $f \in \mathsf{Lip}$, $x_{-k} \in [0,1]^d_{-k}$ and $x_k \in [0,1]$, the function $\mathcal{M}_k f : [0,1]^d \to \mathbb{R}$ satisfies

$$(\mathcal{M}_k f)(x_{-k}, x_k) := (P_\infty f(x_{-k}, \cdot))(x_k)\,,$$

where the application of P_∞ is well-defined because each $f(x_{-k}, \cdot)$ is Lipschitz and hence in $L^2(I)$.

The following lemma shows that $\mathcal{M}_k$ preserves Lipschitzness, confirming that $\mathcal{M}_k f$ is well-defined pointwise as a real-valued function, as opposed to only defined up to sets of measure zero.

Lemma 6.41. *Let $k \in [d]$. Let $f \in \mathsf{Lip}$. Then $\mathcal{M}_k f \in \mathsf{Lip}$.*

Proof. Let $M > 0$ be such that every axis-aligned line restriction of f is M-Lipschitz. We claim that every axis-aligned line restriction of $\mathcal{M}_k f$ is M-Lipschitz. We first check the k-aligned lines. For any $x_{-k} \in [0,1]^d_{-k}$ Proposition 5.72 implies that

$$\psi((\mathcal{M}_k f)(x_{-k}, \cdot)) = \psi(P_\infty f(x_{-k}, \cdot)) \le \psi(f(x_{-k}, \cdot)) \le M\,,$$

the last inequality and the conclusion that $(\mathcal{M}_k f)(x_{-k}, \cdot)$ is M-Lipschitz by Fact 5.37.

It remains to check the other line restrictions. Let $i \in [d] \setminus \{k\}$ and let $z_{-i-k} \in [0,1]^d_{-i-k}$. Let $x_i \ne y_i \in [0,1]$. Since the axis-aligned line restrictions of f (in particular, in direction i) are M-Lipschitz, we have

$$\sup_{w_k \in [0,1]} |f(z_{-i-k}, x_i, w_k) - f(z_{-i-k}, y_i, w_k)| \le M|x_i - y_i|\,.$$

Proposition 5.74 implies that

$$|P_\infty f(z_{-i-k}, x_i, \cdot) - P_\infty f(z_{-i-k}, y_i, \cdot)| \le \operatorname*{ess\,sup}_{w_k \in [0,1]} |f(z_{-i-k}, x_i, w_k) - f(z_{-i-k}, y_i, w_k)| \le M|x_i - y_i| \tag{28}$$

a.e. in I, and since $P_\infty f(z_{-i-k}, x_i, \cdot)$ and $P_\infty f(z_{-i-k}, y_i, \cdot)$ are (Lipschitz) continuous as observed above, we conclude that (28) holds pointwise. Since (28) holds for every $i \in [d] \setminus \{k\}$ and $z_{-i-k} \in [0,1]^d_{-i-k}$, and since

$$|(\mathcal{M}_k f)(z_{-i-k}, x_i, \cdot) - (\mathcal{M}_k f)(z_{-i-k}, y_i, \cdot)| = |P_\infty f(z_{-i-k}, x_i, \cdot) - P_\infty f(z_{-i-k}, y_i, \cdot)|$$

pointwise, we get that every axis-aligned line restriction of $\mathcal{M}_k f$ is M-Lipschitz, and $\mathcal{M}_k f \in \mathsf{Lip}$. $\square$

Lemma 6.42 ($\mathcal{M}_k$ application makes progress)**.** *Let $k \in [d]$, and let $I \subseteq [d] \setminus \{k\}$. Let $f \in \mathsf{Lip} \cap \mathsf{Mon}_I$. Then $\mathcal{M}_k f \in \mathsf{Lip} \cap \mathsf{Mon}_{I \cup \{k\}}$.*

Proof. The fact that $\mathcal{M}_k f \in \mathsf{Lip}$ is given by Lemma 6.41, and the fact that $\mathcal{M}_k f$ is k-monotone follows from the definition of $\mathcal{M}_k$ and the fact that the monotone equilibrium $P_\infty f(x_{-k}, \cdot)$ is nondecreasing. Now, let $i \in I$, so that f is i-monotone. By Observation 6.39, we have that for every $z_{-i-k} \in [0,1]^d_{-i-k}$, and for all $x_i \le y_i$ in $[0,1]$, $f(z_{-i-k}, x_i, \cdot) \le f(z_{-i-k}, y_i, \cdot)$. By Corollary 5.68, we obtain that $P_\infty f(z_{-i-k}, x_i, \cdot) \le P_\infty f(z_{-i-k}, y_i, \cdot)$ a.e. in I, and since these two functions are (Lipschitz) continuous, this inequality holds pointwise. Thus $(\mathcal{M}_k f)(z_{-i-k}, x_i, \cdot) \le (\mathcal{M}_k f)(z_{-i-k}, y_i, \cdot)$, and again by Observation 6.39, $\mathcal{M}_k f$ is i-monotone. This concludes the proof. $\square$

Lemma 6.43 ($\mathcal{M}_k$ preserves bounds)**.** *Let $f \in \mathsf{Lip}$. Let $a \le b$ be real numbers and suppose $a \le f \le b$. Then $a \le \mathcal{M}_k f \le b$.*

Proof. This follows by the definition of $\mathcal{M}_k$ together with Corollary 5.68 and Lemma 5.70. $\square$

Lemma 6.44 ($\mathcal{M}_k$ is nonexpansive in L^2)**.** *Let $f, g \in \mathsf{Lip}$. Then*

$$\int_{[0,1]^d} (\mathcal{M}_k f - \mathcal{M}_k g)^2 \, \mathrm{d}x \le \int_{[0,1]^d} (f-g)^2 \, \mathrm{d}x \,.$$

Proof. This follows from Proposition 5.69 and Tonelli's theorem, as follows:

$$\begin{aligned}
\int_{[0,1]^d} (\mathcal{M}_k f - \mathcal{M}_k g)^2 \, \mathrm{d}x &= \int_{[0,1]^d_{-k}} \mathrm{d}x_{-k} \int_I \left[(\mathcal{M}_k f)(x_{-k}, x_k) - (\mathcal{M}_k g)(x_{-k}, x_k)\right]^2 \mathrm{d}x_k \\
&= \int_{[0,1]^d_{-k}} \mathrm{d}x_{-k} \int_I \left[(P_\infty f(x_{-k}, \cdot))(x_k) - (P_\infty g(x_{-k}, \cdot))(x_k)\right]^2 \mathrm{d}x_k \\
&\le \int_{[0,1]^d_{-k}} \mathrm{d}x_{-k} \int_I \left[f(x_{-k}, x_k) - g(x_{-k}, x_k)\right]^2 \mathrm{d}x_k = \int_{[0,1]^d} (f-g)^2 \, \mathrm{d}x \,.
\end{aligned}$$
$\square$

Similarly, combining Corollary 6.10 with Fubini's theorem yields

Lemma 6.45. *Let $f \in \mathsf{Lip}$. Then*

$$\int_{[0,1]^d} (\mathcal{M}_k f) \, \mathrm{d}x = \int_{[0,1]^d} f \, \mathrm{d}x \,.$$

The following result is essentially an L^2 version of [Fer23, Proposition 3.14].

Lemma 6.46 (Effect of $\mathcal{M}_k$ on the directed Dirichlet energy)**.** *Let $i, k \in [d]$ be distinct and let $f \in \mathsf{Lip}$. Then*

$$\int_{[0,1]^d_{-i}} \mathcal{E}^-((\mathcal{M}_k f)(x_{-i}, \cdot)) \, \mathrm{d}x_{-i} \le \int_{[0,1]^d_{-i}} \mathcal{E}^-(f(x_{-i}, \cdot)) \, \mathrm{d}x_{-i} \,.$$

Proof. For any $g \in \mathsf{Lip}$, $i \in [d]$ and $h \in \mathbb{R} \setminus \{0\}$, define the $Q_{g,i,h} : [0,1]^d \to \mathbb{R}$ by

$$Q_{g,i,h}(x) := \begin{cases} \frac{g(x_{-i}, x_i+h) - g(x_{-i}, x_i)}{h} & \text{if } x_i + h \in (0,1) \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

By Rademacher's theorem on the open domain $(0,1)^d$, g is differentiable almost everywhere. In particular, the function $D_{g,i} : [0,1]^d \to \mathbb{R}$ given by

$$D_{g,i}(x) := \begin{cases} \lim_{h \to 0} Q_{g,i,h}(x) & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases}$$

gives the partial derivative of g in direction i for a.e. $x \in [0,1]^d$, and is measurable as the limit of the measurable functions $Q_{g,i,h}$. Moreover, letting $M > 0$ be such that the axis-aligned line restrictions of g are M-Lipschitz, we have $|Q_{g,i,h}| \le M$ pointwise. Finally, fixing any $x_{-i} \in [0,1]^d_{-i}$, $g(x_{-i}, \cdot)$ is (Lipschitz and hence) in $H^1(I)$, and the partial derivative $D_{g,i}(x_{-i}, \cdot)$ agrees with any weak derivative $\partial_{x_i} g(x_{-i}, \cdot)$ a.e. in $[0,1]$. In particular, all of these considerations apply to the functions f and $\mathcal{M}_k f$ by Lemma 6.41. We conclude that, for $g = f$ or $g = \mathcal{M}_k f$ and each $x_{-i} \in [0,1]^d_{-i}$,

$$\begin{aligned} \mathcal{E}^-(g(x_{-i}, \cdot)) &= \frac{1}{2} \int_I \left[\partial^-_{x_i} g(x_{-i}, x_i)\right]^2 \mathrm{d}x_i = \frac{1}{2} \int_I \left[D_{g,i}(x_{-i}, x_i)^-\right]^2 \mathrm{d}x_i \\ &= \frac{1}{2} \int_I \lim_{h \to 0} \left[Q_{g,i,h}(x_{-i}, x_i)^-\right]^2 \mathrm{d}x_i \,, \end{aligned}$$

where for simplicity we write the limit in the last expression with the understanding that it is only defined almost everywhere in $[0,1]$. Note that the measurability of the last integrand over $[0,1]^d$ justifies writing the integrals in the statement of the lemma via Tonelli's theorem (as the computation below shows). Define for each $h \in \mathbb{R}$ the set

$$I_h := \begin{cases} (0, 1-h) & \text{if } h \ge 0 \\ (-h, 1) & \text{otherwise,} \end{cases}$$

so that $Q_{g,i,h}(x)$ is defined by the first case of (29) if and only if $x_i \in I_h$. Then, by repeated applications of Tonelli's theorem and the dominated convergence theorem, we obtain

$$\begin{aligned} &\int_{[0,1]^d_{-i}} \mathcal{E}^-((\mathcal{M}_k f)(x_{-i}, \cdot)) \, \mathrm{d}x_{-i} = \frac{1}{2} \int_{[0,1]^d_{-i}} \mathrm{d}x_{-i} \int_I \lim_{h \to 0} \left[Q_{\mathcal{M}_k f,i,h}(x_{-i}, x_i)^-\right]^2 \mathrm{d}x_i \\ &= \frac{1}{2} \int_{[0,1]^d} \lim_{h \to 0} \underbrace{\left[Q_{\mathcal{M}_k f,i,h}(x)^-\right]^2}_{\le M^2} \mathrm{d}x = \frac{1}{2} \lim_{h \to 0} \int_{[0,1]^d} \left[Q_{\mathcal{M}_k f,i,h}(x)^-\right]^2 \mathrm{d}x \\ &= \frac{1}{2} \lim_{h \to 0} \int_{[0,1]^d_{-i-k}} \mathrm{d}x_{-i-k} \int_I \mathrm{d}x_i \int_I \left[Q_{\mathcal{M}_k f,i,h}(x_{-i-k}, x_i, x_k)^-\right]^2 \mathrm{d}x_k \\ &= \frac{1}{2} \lim_{h \to 0} \int_{[0,1]^d_{-i-k}} \mathrm{d}x_{-i-k} \int_{I_h} \mathrm{d}x_i \int_I \left[\left(\frac{(\mathcal{M}_k f)(x_{-i-k}, x_i+h, x_k) - (\mathcal{M}_k f)(x_{-i-k}, x_i, x_k)}{h}\right)^-\right]^2 \mathrm{d}x_k \\ &= \frac{1}{2} \lim_{h \to 0} \int_{[0,1]^d_{-i-k}} \mathrm{d}x_{-i-k} \int_{I_h} \mathrm{d}x_i \int_I \left[\left(\frac{(P_\infty f(x_{-i-k}, x_i+h, \cdot))(x_k) - (P_\infty f(x_{-i-k}, x_i, \cdot))(x_k)}{h}\right)^-\right]^2 \mathrm{d}x_k \\ &\le \frac{1}{2} \lim_{h \to 0} \int_{[0,1]^d_{-i-k}} \mathrm{d}x_{-i-k} \int_{I_h} \mathrm{d}x_i \int_I \left[\left(\frac{f(x_{-i-k}, x_i+h, x_k) - f(x_{-i-k}, x_i, x_k)}{h}\right)^-\right]^2 \mathrm{d}x_k \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\lim_{h\to 0}\int_{[0,1]^d_{-i-k}} \mathrm{d}x_{-i-k}\int_I \mathrm{d}x_i \int_I \left[Q_{f,i,h}(x_{-i-k},x_i,x_k)^-\right]^2 \mathrm{d}x_k \\
&= \frac{1}{2}\lim_{h\to 0}\int_{[0,1]^d} \underbrace{\left[Q_{f,i,h}(x)^-\right]^2}_{\le M^2} \mathrm{d}x \;=\; \frac{1}{2}\int_{[0,1]^d}\lim_{h\to 0}\left[Q_{f,i,h}(x)^-\right]^2 \mathrm{d}x \\
&= \frac{1}{2}\int_{[0,1]^d_{-i}} \mathrm{d}x_{-i}\int_I \lim_{h\to 0}\left[Q_{f,i,h}(x_{-i},x_i)^-\right]^2 \mathrm{d}x_i = \int_{[0,1]^d_{-i}} \mathcal{E}^-(f(x_{-i},\cdot))\,\mathrm{d}x_{-i}\,,
\end{aligned}
$$

the inequality by Proposition 5.67 via the identity $\left[(a-b)^+\right]^2 = \left[(b-a)^-\right]^2$. □

Definition 6.47 (Coordinate-wise monotone equilibrium)**.** For each $f \in \mathsf{Lip}$, define $f^* \in \mathsf{Lip}$, the *coordinate-wise monotone equilibrium* of f, by

$$
f^* := \mathcal{M}_d\mathcal{M}_{d-1}\cdots\mathcal{M}_1 f\,.
$$

Note that indeed f^* is indeed Lipschitz by Lemma 6.41, and it is monotone:

Proposition 6.48. *Let $f \in \mathsf{Lip}$. Then f^* is monotone.*

Proof. This follows by repeatedly applying Lemma 6.42, and recalling that $\mathsf{Mon}_{[d]}$ is the set of monotone functions on $[0,1]^d$. □

By Proposition 5.73 and the definition of $\mathcal{M}_k$, we also observe that the coordinate-wise monotone equilibrium behaves nicely with respect to certain affine transformations:

Observation 6.49. *Let $f \in \mathsf{Lip}$. Let $\alpha > 0$ and $\beta \in \mathbb{R}$. Then $(\alpha f + \beta)^* = \alpha f^* + \beta$.*

Similarly, Lemma 6.44 implies that taking the coordinate-wise monotone equilibrium is a non-expansive operation:

Observation 6.50. *Let $f, g \in \mathsf{Lip}$. Then $\|f^* - g^*\|_{L^2((0,1)^d)} \le \|f - g\|_{L^2((0,1)^d)}$. Since $0^* = 0$ by Lemma 5.70 (where we write 0 for the constant zero function), we conclude in particular that $\|f^*\|_{L^2((0,1)^d)} \le \|f\|_{L^2((0,1)^d)}$.*

We are now ready to combine the ingredients from the previous sections and establish the transport-energy inequality. Note that this inequality is most effective when the function f is pointwise bounded close to 1.

Theorem 6.51 (Transport-energy inequality)**.** *There exists a universal constant $C > 0$ such that the following holds. Let $a \in (0,1)$, and let $f \in \mathsf{Lip}$ satisfy $1 - a \le f \le 1 + a$ and $\int_{[0,1]^d} f\,\mathrm{d}x = 1$. Define the probability measures $\mathrm{d}\mu := f\,\mathrm{d}x$ and $\mathrm{d}\mu^* := f^*\,\mathrm{d}x$ on $[0,1]^d$. Then*

$$
W_2^2(\mu \to \mu^*) \le \frac{C(1+a)^2}{(1-a)^3}\int_{[0,1]^d}\left|\nabla^- f\right|^2 \mathrm{d}x\,.
$$

Proof. Define the family $(f^{(i)})_{i\in[d+1]} \subset \mathsf{Lip}$ by $f^{(1)} := f$ and $f^{(i+1)} := \mathcal{M}_i f^{(i)}$ for each $i \in [d]$, so that $f^{(d+1)} = f^*$. Note that we have $1 - a \le f^{(i)} \le 1 + a$ and $\int_{[0,1]^d} f^{(i)}\,\mathrm{d}x = 1$ for each $i \in [d+1]$ by Lemmas 6.43 and 6.45. Then, define the family $(\mu^{(i)})_{i\in[d+1]}$ of probability measures on $[0,1]^d$ by $\mathrm{d}\mu^{(i)} := f^{(i)}\,\mathrm{d}x$, so that $\mu^{(1)} = \mu$ and $\mu^{(d+1)} = \mu^*$.

We claim $(\mu^{(i)})_{i\in[d+1]}$ satisfies the conditions of Lemma 6.35. First, they have strictly positive densities because $1-a \le f^{(i)} \le 1+a$ for each $i \in [d+1]$ as observed above. They are also absolutely continuous with Borel measurable density because each $f^{(i)}$ is continuous and hence Borel.

Recall that, by Proposition 6.26, for each $i, j \in [d+1]$ we may characterize the disintegration $\mu^{(j)} = \int_{\mathbb{R}^d_{-i}} \mu^{(j)}{}_{|x_{-i}} \, \mathrm{d}\mu^{(j)}{}_{-i}$ as follows: the marginal and all conditional measures are both absolutely continuous supported in $[0,1]^d_{-i}$ and $[0,1]$ respectively, and we have $\mathrm{d}\mu^{(j)}{}_{-i} = f^{(j)}{}_{-i} \, \mathrm{d}x_{-i}$ where

$$f^{(j)}{}_{-i}(x_{-i}) = \int_I f^{(j)}(x_{-i}, x_i) \, \mathrm{d}x_i \tag{30}$$

for each $x_{-i} \in [0,1]^d_{-i}$, and $\mathrm{d}\mu^{(j)}{}_{|x_{-i}} = f^{(j)}{}_{|x_{-i}} \, \mathrm{d}x_i$ where

$$f^{(j)}{}_{|x_{-i}}(x_i) = \frac{f^{(j)}(x)}{f^{(j)}{}_{-i}(x_{-i})} \tag{31}$$

for each $x \in [0,1]^d$ (note that the denominator is nonzero since $f^{(j)} \ge 1-a$).

Now, fix any $i \in [d]$. Then the condition $\mu^{(i)}{}_{-i} = \mu^{(i+1)}{}_{-i}$ is implied by the pointwise condition

$$\int_I f^{(i)}(x_{-i}, x_i) \, \mathrm{d}x_i = \int_I f^{(i+1)}(x_{-i}, x_i) \, \mathrm{d}x_i \qquad \forall x_{-i} \in [0,1]^d_{-i}\,, \tag{32}$$

which in turn holds by the definitions of $f^{(i+1)}$ and $\mathcal{M}_i$, and Corollary 6.10. Finally, fixing $x_{-i} \in [0,1]^d_{-i}$, we verify the condition $\mu^{(i)}{}_{|x_{-i}} \succeq \mu^{(i+1)}{}_{|x_{-i}}$. By Corollary 6.14, it suffices to show that

$$f^{(i+1)}{}_{|x_{-i}} \stackrel{?}{=} P_\infty f^{(i)}{}_{|x_{-i}}\,. \tag{33}$$

By (30) and (31), this is equivalent to

$$\frac{1}{\int_I f^{(i+1)}(x_{-i}, x_i) \, \mathrm{d}x_i} f^{(i+1)}(x_{-i}, \cdot) \stackrel{?}{=} P_\infty \left(\frac{1}{\int_I f^{(i)}(x_{-i}, x_i) \, \mathrm{d}x_i} f^{(i)}(x_{-i}, \cdot) \right).$$

Since $f^{(i+1)}(x_{-i}, \cdot) = P_\infty f^{(i)}(x_{-i}, \cdot)$, and applying Proposition 5.73, the above is equivalent to

$$\frac{1}{\int_I f^{(i+1)}(x_{-i}, x_i) \, \mathrm{d}x_i} P_\infty f^{(i)}(x_{-i}, \cdot) \stackrel{?}{=} \frac{1}{\int_I f^{(i)}(x_{-i}, x_i) \, \mathrm{d}x_i} P_\infty f^{(i)}(x_{-i}, \cdot)\,,$$

which is true again by (32), so indeed $\mu^{(i)}{}_{|x_{-i}} \succeq \mu^{(i+1)}{}_{|x_{-i}}$. Thus, by Lemma 6.35, we have

$$\begin{aligned} W_2^2(\mu \to \mu^*) &\le \sum_{i=1}^d \int_{[0,1]^d_{-i}} W_2^2(\mu^{(i)}{}_{|x_{-i}}, \mu^{(i+1)}{}_{|x_{-i}}) \, \mathrm{d}\mu^{(i)}{}_{-i}(x_{-i}) \\ &= \sum_{i=1}^d \int_{[0,1]^d_{-i}} W_2^2(\mu^{(i)}{}_{|x_{-i}}, \mu^{(i+1)}{}_{|x_{-i}}) f^{(i)}{}_{-i}(x_{-i}) \, \mathrm{d}x_{-i} \\ &\le (1+a) \sum_{i=1}^d \int_{[0,1]^d_{-i}} W_2^2(\mu^{(i)}{}_{|x_{-i}}, \mu^{(i+1)}{}_{|x_{-i}}) \, \mathrm{d}x_{-i}\,, \end{aligned}$$

the last inequality because $1-a \le f^{(i)} \le 1+a$ also implies that $1-a \le f^{(i)}{}_{-i} \le 1+a$. Now, applying Theorem 6.15 via (33), we have

$$W_2^2(\mu^{(i)}{}_{|x_{-i}}, \mu^{(i+1)}{}_{|x_{-i}}) \le \frac{C}{\inf f^{(i)}{}_{|x_{-i}}} \mathcal{E}^-(f^{(i)}{}_{|x_{-i}})$$

for each $i \in [d]$ and $x_{-i} \in [0,1]^d_{-i}$. Note that $f^{(i)}{}_{|x_{-i}} \geq \frac{1-a}{1+a}$ by the pointwise bounds on $f^{(i)}$ and $f^{(i)}{}_{-i}$, and that $\mathcal{E}^-(\alpha g) = \alpha^2 \mathcal{E}^-(g)$ for any $\alpha \geq 0$ and (say) Lipschitz g by the definition of $\mathcal{E}^-$. Therefore, applying (31), we obtain

$$W_2^2(\mu^{(i)}{}_{|x_{-i}}, \mu^{(i+1)}{}_{|x_{-i}}) \leq \frac{C(1+a)}{1-a} \left(\frac{1}{f^{(i)}{}_{-i}(x_{-i})} \right)^2 \mathcal{E}^-(f^{(i)}(x_{-i}, \cdot)) \leq \frac{C(1+a)}{(1-a)^3} \mathcal{E}^-(f^{(i)}(x_{-i}, \cdot)) \,.$$

Hence we have

$$W_2^2(\mu \to \mu^*) \leq \frac{C(1+a)^2}{(1-a)^3} \sum_{i=1}^{d} \int_{[0,1]^d_{-i}} \mathcal{E}^-(f^{(i)}(x_{-i}, \cdot)) \, \mathrm{d}x_{-i} \,.$$

Recall that $f^{(1)} = f$ while $f^{(i)} = \mathcal{M}_{i-1} \cdots \mathcal{M}_1 f$ for each $i = 2, \dots, d+1$. Hence, by inductively applying Lemma 6.46 for each $i \geq 2$, and by Tonelli's theorem and the definition of $\mathcal{E}^-$, we obtain

$$\begin{aligned}
W_2^2(\mu \to \mu^*) &\leq \frac{C(1+a)^2}{(1-a)^3} \sum_{i=1}^{d} \int_{[0,1]^d_{-i}} \mathcal{E}^-(f(x_{-i}, \cdot)) \, \mathrm{d}x_{-i} \\
&= \frac{C(1+a)^2}{2(1-a)^3} \sum_{i=1}^{d} \int_{[0,1]^d_{-i}} \mathrm{d}x_{-i} \int_I \left(\partial^-_{x_i} f(x_{-i}, x_i) \right)^2 \mathrm{d}x_i \\
&= \frac{C(1+a)^2}{2(1-a)^3} \int_{[0,1]^d} \sum_{i=1}^{d} \left(\partial^-_i f \right)^2 \mathrm{d}x \\
&= \frac{C(1+a)^2}{2(1-a)^3} \int_{[0,1]^d} \left| \nabla^- f \right|^2 \mathrm{d}x \,.
\end{aligned}$$

□

7 Directed Poincaré from Wasserstein and Kantorovich

In this section, we establish a directed Poincaré inequality (Theorem 1.1) by combining the transport-energy inequality from Theorem 6.51 with a directed version of (weak) Kantorovich duality via a perturbation argument. The general theme of going between transport inequalities and Poincaré inequalities, as well as the spirit of the technical arguments presented here, are well-known in the undirected case; we refer the reader to [Vil09, Chapters 7 and 22] for a comprehensive presentation and primary references, and to [Liu20] for an undirected version of the precise reduction we instantiate here (that is, from a transport-energy inequality to a Poincaré inequality).

For simplicity, we do not attempt to initiate a systematic study or to state results in the broadest possible generality, but rather limit ourselves to what is required for the aforementioned goal.

7.1 Directed weak Kantorovich duality and Hamilton-Jacobi operator

Kantorovich duality plays a fundamental role in the study of Wasserstein distance. The weak part of this duality allows one to lower bound the Wasserstein distance between two distributions by means of "test functions" satisfying a certain constraint. Here, we translate a small part of this rich theory to the directed setting.

Lemma 7.1 (Weak duality). *Let $\Omega \subset \mathbb{R}^d$ be a bounded Borel set and let $\mu, \nu \in P(\Omega)$. Suppose $\phi \in L^1(\nu)$ and $\psi \in L^1(\mu)$ satisfy*

$$\phi(y) - \psi(x) \leq |x-y|^2 \qquad \forall x \preceq y \text{ in } \Omega \,.$$

Then

$$W_2^2(\mu \to \nu) \geq \int_\Omega \phi \, \mathrm{d}\nu - \int_\Omega \psi \, \mathrm{d}\mu \,.$$

Proof. Suppose $\gamma \in \Pi(\mu \to \nu)$. Then

$$\begin{aligned}
C_2(\gamma)^2 &= \int_{\Omega\times\Omega} |x-y|^2 \, \mathrm{d}\gamma(x,y) \\
&= \int_{\Omega\times\Omega} \chi_{\{x \preceq y\}} |x-y|^2 \, \mathrm{d}\gamma(x,y) && \text{(Since } \gamma \in \Pi(\mu \to \nu)) \\
&\geq \int_{\Omega\times\Omega} \chi_{\{x \preceq y\}} (\phi(y) - \psi(x)) \, \mathrm{d}\gamma(x,y) && \text{(Hypothesis)} \\
&= \int_{\Omega\times\Omega} (\phi(y) - \psi(x)) \, \mathrm{d}\gamma(x,y) && \text{(Since } \gamma \in \Pi(\mu \to \nu)) \\
&= \int_\Omega \phi \, \mathrm{d}\nu - \int_\Omega \psi \, \mathrm{d}\mu && \text{(Since } (\pi_1)_\# \gamma = \mu, (\pi_2)_\# \gamma = \nu) \,.
\end{aligned}$$

Since this holds for every $\gamma \in \Pi(\mu \to \nu)$, the result follows. □

This duality results motivates the following definition. Let $C_b(\Omega)$ denote the set of bounded continuous functions $\Omega \to \mathbb{R}$.

Definition 7.2 (Directed Hamilton-Jacobi operator)**.** Let $\Omega \subset \mathbb{R}^d$. For each $t \geq 0$, define the *directed Hamilton-Jacobi operator* $\vec{H}_t : C_b(\Omega) \to (\Omega \to \mathbb{R})$ as follows: for each $h \in C_b(\Omega)$ and $x \in \Omega$, we set

$$(\vec{H}_t h)(x) := \begin{cases} h(x) & \text{if } t = 0 \\ \sup_{y \succeq x} \left\{ h(y) - \frac{1}{2t}|x-y|^2 \right\} & \text{otherwise.} \end{cases} \tag{34}$$

Remark 7.3. The definition above is a directed analogue of the so-called (backward) *Hamilton-Jacobi-Hopf-Lax-Oleinik semigroup.* We do not claim that $\vec{H}_t$ forms a semigroup.

Proposition 7.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Borel set, and let $\mu, \nu \in P(\Omega)$. Let $h \in C_b(\Omega)$. Then*

$$\tfrac{1}{2} W_2^2(\mu \to \nu) \geq \int_\Omega h \, \mathrm{d}\nu - \int_\Omega (\vec{H}_1 h) \, \mathrm{d}\mu \,.$$

Proof. Let $\phi := 2h$ and $\psi := 2\vec{H}_1 h$. Note that $\phi \in L^1(\nu)$ since h is bounded. The fact that h is bounded also implies that $\vec{H}_1 h$ is bounded, as can be verified from its definition, and hence $\psi \in L^1(\mu)$. Finally, for each $x \preceq y$ in Ω we have

$$\begin{aligned}
\phi(y) - \psi(x) &= 2h(y) - 2 \sup_{y' \succeq x} \left\{ h(y') - \frac{1}{2}|x-y'|^2 \right\} \\
&= |x-y|^2 + \underbrace{2 \left[\left(h(y) - \frac{1}{2}|x-y|^2 \right) - \sup_{y' \succeq x} \left\{ h(y') - \frac{1}{2}|x-y'|^2 \right\} \right]}_{\leq 0} \leq |x-y|^2 \,.
\end{aligned}$$

Thus Lemma 7.1 gives the conclusion. □

The following simple lemma plays a key role in the perturbation argument.

Lemma 7.5. *Let $\Omega \subset \mathbb{R}^d$. Let $t > 0$ and let $h \in C_b(\Omega)$. Then $\vec{H}_1(th) = t\vec{H}_t h$.*

Proof. Indeed, for each $x \in \Omega$ we have

$$(\vec{H}_1(th))(x) = \sup_{y \succeq x} \left\{ th(y) - \frac{1}{2}|x-y|^2 \right\} = t \sup_{y \succeq x} \left\{ h(y) - \frac{1}{2t}|x-y|^2 \right\} = t((\vec{H}_t h)(x)) . \qquad \square$$

Notation. For $\Omega \subset \mathbb{R}^d$ an open set and $k \in \mathbb{N} \cup \{+\infty\}$, write $C^k(\overline{\Omega})$ for the set of restrictions to $\overline{\Omega}$ of functions in $C^k(\mathbb{R}^d)$.

For $\Omega \subset \mathbb{R}^d$, point $x \in \Omega$, and $r > 0$, let $B^\circ_\Omega(x,r) := B^\circ(x,r) \cap \Omega$, where $B^\circ(x,r) := \{y \in \mathbb{R}^d : 0 < |x-y| < r\}$ is the open ball of radius r centered at x with x itself excluded, and let $B^+_\Omega(x,r) := B^\circ_\Omega(x,r) \cap \{y \in \mathbb{R}^d : y \succeq x\}$.

The following lemma gives a directed analogue to some of the properties of the Hamilton-Jacobi semigroup, as presented in [Vil09, Theorem 22.16].[9]

Proposition 7.6 (Some properties of the directed Hamilton-Jacobi operator)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set. Let $h \in C^1(\overline{\Omega})$.*

(a) Let $C := \sup h - \inf h$, which is well-defined because h is continuous over the compact set $\overline{\Omega}$ and thus bounded. Then for each $t > 0$ and $x \in \Omega$, the supremum in (34) may be taken over the set $B^+_\Omega(x, \sqrt{2Ct})$.

(b) For each $x \in \Omega$,

$$\limsup_{t \to 0^+} \frac{\vec{H}_t h(x) - h(x)}{t} \le \frac{|\nabla^+ h(x)|^2}{2} .$$

(c) The quotient

$$\frac{\vec{H}_t h(x) - h(x)}{t}$$

is nonnegative and bounded, uniformly in $t > 0$ and $x \in \Omega$.

Proof. Let us first show (a). For any $x \preceq y$ in Ω and $t > 0$, if $|x-y| \ge \sqrt{2Ct}$ then

$$h(y) - \frac{1}{2t}|x-y|^2 \le h(x) + C - \frac{1}{2t}2Ct = h(x) ,$$

so y is irrelevant to the supremum in (34). Moreover, by the continuity of h, we may drop x itself from the supremum. Thus y is irrelevant to the supremum whenever $y \notin B^+_\Omega(x, \sqrt{2Ct})$, as claimed.

We now show (b) using (a). Let $x \in \Omega$ and $t > 0$. We have

$$\begin{aligned}
\frac{\vec{H}_t h(x) - h(x)}{t} &= \frac{\sup_{y \in B^+_\Omega(x,\sqrt{2Ct})} \left\{ h(y) - \frac{1}{2t}|x-y|^2 \right\} - h(x)}{t} \\
&= \sup_{y \in B^+_\Omega(x,\sqrt{2Ct})} \left\{ \frac{h(y) - h(x)}{t} - \frac{1}{2t^2}|x-y|^2 \right\} \\
&\le \sup_{y \in B^+_\Omega(x,\sqrt{2Ct})} \left\{ \frac{(h(y) - h(x))^+}{t} - \frac{1}{2t^2}|x-y|^2 \right\}
\end{aligned}$$

[9] Note that our use of the notation $\nabla^+ h = 0 \vee \nabla h$, $\nabla^- h = 0 \wedge \nabla h$ is unrelated to the use of similar notation in [Vil09], where it denotes a notion of norm of the gradient for functions in more general spaces where the usual derivative may not be defined. In particular, their definition agrees with the norm of the gradient for differentiable functions, while the point is the our definition (the directed gradient) does not.

$$= \sup_{y \in B_\Omega^+(x,\sqrt{2Ct})} \left\{ \frac{(h(y)-h(x))^+}{|x-y|} \frac{|x-y|}{t} - \frac{1}{2}\left(\frac{|x-y|}{t}\right)^2 \right\}.$$

Using the inequality $\alpha\beta - \frac{1}{2}\beta^2 \leq \frac{1}{2}\alpha^2$, we obtain

$$\frac{\vec{H}_t h(x) - h(x)}{t} \leq \frac{1}{2} \sup_{y \in B_\Omega^+(x,\sqrt{2Ct})} \left(\frac{(h(y)-h(x))^+}{|x-y|}\right)^2. \tag{35}$$

Using Lemma 7.7 and the fact that the term inside the supremum is nonnegative, we conclude that

$$\limsup_{t\to 0^+} \frac{\vec{H}_t h(x) - h(x)}{t} \leq \frac{1}{2}\left(\lim_{t\to 0^+} \sup_{y \in B_\Omega^+(x,\sqrt{2Ct})} \frac{(h(y)-h(x))^+}{|x-y|}\right)^2 = \frac{|\nabla^+ h(x)|^2}{2},$$

as desired. We may also obtain (c) from (35) as follows. First note that $\vec{H}_t h(x) \geq h(x)$ for all $t \geq 0$ and $x \in \Omega$. It is standard that C^1 functions are Lipschitz on compact sets, so h is Lipschitz on $\overline{\Omega}$. Let $M > 0$ be its Lipschitz constant. Then from (35) we conclude that, for each $x \in \Omega$ and $t > 0$,

$$0 \leq \frac{\vec{H}_t h(x) - h(x)}{t} \leq \frac{1}{2} \sup_{y \in B_\Omega^+(x,\sqrt{2Ct})} \left(\frac{M|x-y|}{|x-y|}\right)^2 = \frac{M^2}{2},$$

which establishes (c). $\square$

Lemma 7.7. *Let $\Omega \subset \mathbb{R}^d$ be an open set. Let $h \in C^1(\overline{\Omega})$. Then for each $x \in \Omega$,*

$$|\nabla h(x)| = \lim_{r\to 0^+} \sup_{y \in B_\Omega^\circ(x,r)} \frac{(h(y)-h(x))^+}{|x-y|}$$

and

$$|\nabla^+ h(x)| = \lim_{r\to 0^+} \sup_{y \in B_\Omega^+(x,r)} \frac{(h(y)-h(x))^+}{|x-y|}.$$

Proof. The first part of the statement is known, but let us give a proof for completeness. Let $x \in \Omega$. We have

$$\lim_{r\to 0^+} \sup_{y \in B_\Omega^\circ(x,r)} \frac{h(y)-h(x)}{|x-y|} = \lim_{r\to 0^+} \sup_{y \in B_\Omega^\circ(x,r)} \frac{h(y)-h(x)-\nabla h(x)^\top (y-x) + \nabla h(x)^\top (y-x)}{|x-y|}.$$

By definition of derivative, we have

$$\lim_{y\to x} \frac{\left|h(y)-h(x)-\nabla h(x)^\top (y-x)\right|}{|x-y|} = 0,$$

and hence, by ℓ^2 norm duality,

$$\lim_{r\to 0^+} \sup_{y \in B_\Omega^\circ(x,r)} \frac{h(y)-h(x)}{|x-y|} = \lim_{r\to 0^+} \sup_{y \in B_\Omega^\circ(x,r)} \frac{\nabla h(x)^\top (y-x)}{|x-y|} = \sup_{v\in\mathbb{R}^d : |v|=1} \nabla h(x)^\top v = |\nabla h(x)|.$$

Similarly, we also have

$$\begin{aligned}\lim_{r\to 0^+}\sup_{y\in B^\circ_\Omega(x,r)}\frac{|h(y)-h(x)|}{|x-y|} &= \lim_{r\to 0^+}\sup_{y\in B^\circ_\Omega(x,r)}\frac{\left|h(y)-h(x)-\nabla h(x)^\top(y-x)+\nabla h(x)^\top(y-x)\right|}{|x-y|}\\ &\le \lim_{r\to 0^+}\sup_{y\in B^\circ_\Omega(x,r)}\frac{\left|h(y)-h(x)-\nabla h(x)^\top(y-x)\right|+\left|\nabla h(x)^\top(y-x)\right|}{|x-y|}\\ &= \lim_{r\to 0^+}\sup_{y\in B^\circ_\Omega(x,r)}\frac{\left|\nabla h(x)^\top(y-x)\right|}{|x-y|} = \sup_{v\in\mathbb{R}^d:|v|=1}\left|\nabla h(x)^\top v\right| = |\nabla h(x)|\,.\end{aligned}$$

Since $h(y)-h(x)\le (h(y)-h(x))^+\le |h(y)-h(x)|$, the first part of the statement follows.

We now establish the second part. Write the partition $[d]=P\cup N$ for

$$\begin{aligned}P &:= \{i\in[d]:\partial_i h(x)>0\}\,,\\ N &:= \{i\in[d]:\partial_i h(x)\le 0\}\,.\end{aligned}$$

Assume for a moment that $P\neq\emptyset$[10]. Define $\Omega^P\subset\mathbb{R}^P$ by

$$\Omega^P := \{y_P : y\in\Omega \text{ satisfies } \operatorname{supp}(y-x)\subseteq P\}\,,$$

and let $h^P\in C^1(\overline{\Omega^P})$ be given by $h^P(y_P):=h(y_P,x_{-P})$ for each $y_P\in\Omega_P$, i.e. h^P is the restriction of h to the space obtained by fixing the N-coordinates of inputs to those of x. Note that $\partial_i h(x)=\partial_i h^P(x_P)$ for each $i\in P$, and hence $|\nabla^+ h(x)|=|\nabla h^P(x_P)|$. Applying the first part of the statement to Ω^P, h^P and x_P gives

$$|\nabla^+ h(x)| = |\nabla h^P(x_P)| = \lim_{r\to 0^+}\sup_{y_P\in B^\circ_{\Omega^P}(x_P,r)}\frac{(h^P(y_P)-h^P(x_P))^+}{|x_P-y_P|} \tag{36}$$

We claim that

$$\lim_{r\to 0^+}\sup_{y_P\in B^\circ_{\Omega^P}(x_P,r)}\frac{(h^P(y_P)-h^P(x_P))^+}{|x_P-y_P|} \stackrel{?}{=} \lim_{r\to 0^+}\sup_{y\in B^+_\Omega(x,r)}\frac{(h(y)-h(x))^+}{|x-y|}\,, \tag{37}$$

with the LHS replaced by 0 if $P=\emptyset$, which will conclude the proof.

For each set $A\subseteq[d]$, let $X_A:=\{y\in\Omega: y_A\succeq x_A\}$ and $Y_A:=\{y\in\Omega: y_A\preceq x_A\}$. Note that for all $r>0$, $B^+_\Omega(x,r)=B^\circ_\Omega(x,r)\cap X_P\cap X_N$ and moreover, defining $B'_\Omega(x,r):=B^\circ_\Omega(x,r)\cap X_N\cap Y_N$, when $P\neq\emptyset$ we have

$$\lim_{r\to 0^+}\sup_{y_P\in B^\circ_{\Omega^P}(x_P,r)}\frac{(h^P(y_P)-h^P(x_P))^+}{|x_P-y_P|} = \lim_{r\to 0^+}\sup_{y\in B'_\Omega(x,r)}\frac{(h(y)-h(x))^+}{|x-y|} \tag{38}$$

by definition of the objects involved.

Note that, when $P\neq\emptyset$, $\min_{i\in P}\partial_i h(x)>0$ by the definition of P. Since h is continuously differentiable, it follows that there exists $r_0>0$ such that, for all $y\in B^\circ_\Omega(x,r_0)$ and every $i\in P$, we have $\partial_i h(y)>0$. Since Ω is open, we can also let r_0 be small enough so that $B(x,r_0)\subset\Omega$.

Now, suppose $y\in B^\circ_\Omega(x,r_0)$ and $i\in P$ are such that $y_i<x_i$. Then define $z\in\mathbb{R}^d$ by $z_i:=y_i+2(x_i-y_i)>x_i$, and $z_j:=y_j$ for $j\in[d]\setminus\{i\}$. Then $|x-z|=|x-y|$, and in particular $z\in B^\circ(x,r_0)\subset\Omega$. Also, since $i\in P$ and z agrees with y on all $j\neq i$, we have that $y\in X_N$ (resp.

[10] We track the edge case $P=\emptyset$ as we go along the proof; the main idea is the same.

Y_N) if and only if $z \in X_N$ (resp. Y_N). Moreover, since $\partial_i h(w) > 0$ for all w in the line segment connecting y and z (which is contained in $B^\circ_\Omega(x, r_0) \cup \{x\}$), the fundamental theorem of calculus implies that $h(z) > h(y)$.

Repeating this argument inductively for each index $i \in P$ for which $y_i < x_i$, we conclude that for all $r \in (0, r_0)$ and $y \in B'_\Omega(x, r)$, there exists $z \in B'_\Omega(x, r) \cap X_P$ such that $|x - z| = |x - y|$ and $h(z) > h(y)$. It follows that, when $P \neq \emptyset$,

$$\lim_{r \to 0^+} \sup_{y \in B'_\Omega(x,r)} \frac{(h(y) - h(x))^+}{|x - y|} = \lim_{r \to 0^+} \sup_{y \in B'_\Omega(x,r) \cap X_P} \frac{(h(y) - h(x))^+}{|x - y|} . \tag{39}$$

Combining (37), (38) and (39), we see that it remains to show that

$$\lim_{r \to 0^+} \sup_{y \in B'_\Omega(x,r) \cap X_P} \frac{(h(y) - h(x))^+}{|x - y|} \stackrel{?}{=} \lim_{r \to 0^+} \sup_{y \in B^+_\Omega(x,r)} \frac{(h(y) - h(x))^+}{|x - y|} , \tag{40}$$

again with the LHS replaced by 0 if $P = \emptyset$. Note that for each $r > 0$ it holds by definition that $B'_\Omega(x, r) \cap X_P \subseteq B^+_\Omega(x, r)$, namely $B'_\Omega(x, r) \cap X_P = B^+_\Omega(x, r) \cap Y_N$. Thus we already have

$$\lim_{r \to 0^+} \sup_{y \in B'_\Omega(x,r) \cap X_P} \frac{(h(y) - h(x))^+}{|x - y|} \leq \lim_{r \to 0^+} \sup_{y \in B^+_\Omega(x,r)} \frac{(h(y) - h(x))^+}{|x - y|} ,$$

which also holds when the LHS is replaced by 0, and it remains to prove the reverse inequality. Let $\epsilon > 0$. As before, recalling that $\partial_i h(x) \leq 0$ for every $i \in N$ and using the fact that h is continuously differentiable, we can let $r_0 > 0$ be small enough so that for all $y \in B^\circ(x, r_0)$ and every $i \in N$, we have $\partial_i h(y) \leq \epsilon$. Again since Ω is open, we may also let r_0 be small enough so that $B(x, r_0) \subset \Omega$.

Now, let $r \in (0, r_0)$ and $y \in B^+_\Omega(x, r)$. Define $y^P \in \mathbb{R}^d$ as follows: $y^P_i := y_i$ for each $i \in P$, and $y^P_j := x_j$ for each $j \in N$. Then $\mathrm{supp}(y^P - x) \subseteq P$ by construction, so $y^P \in X_N \cap Y_N$. We also have $y^P \succeq x$ and $|x - y^P| \leq |x - y|$, and hence $y^P \in (B^+(x, r) \cap Y_N) \cup \{x\} = (B'_\Omega(x, r) \cap X_P) \cup \{x\}$.

By a standard multivariate version of the mean value theorem, there exists a point z in the line segment connecting y^P and y such that

$$h(y) - h(y^P) = \nabla h(z)^\top (y - y^P) .$$

By construction, we have $\mathrm{supp}(y - y^P) \subseteq N$. Moreover, since $y \in B^+_\Omega(x, r)$ and hence $y \succeq x$, while $y^P_j = x_j$ for each $j \in N$, we have that $y - y^P \succeq \vec{0}$. Finally, it is clear from the construction of y^P that $|y - y^P| \leq |x - y|$. We conclude that

$$\begin{aligned}
h(y) - h(y^P) &= \nabla h(z)^\top (y - y^P) \\
&\leq \nabla^+ h(z)^\top (y - y^P) && \text{(Since } y - y^P \succeq \vec{0}\text{)} \\
&= \sum_{i \in N} (\partial^+_i h(z))(y_i - y^P_i) && \text{(Since } \mathrm{supp}(y - y^P) \subseteq N\text{)} \\
&\leq \left(\sum_{i \in N} (\partial^+_i h(z))^2 \right)^{1/2} |y - y^P| && \text{(Cauchy-Schwarz inequality)} \\
&\leq \left(\sum_{i \in N} \epsilon^2 \right)^{1/2} |x - y| && \text{(Choice of } r_0 \text{ and last observation)} \\
&\leq \epsilon \sqrt{d} |x - y| .
\end{aligned}$$

Therefore, when $P \neq \emptyset$ we have

$$\begin{aligned}\lim_{r\to 0^+} \sup_{y\in B^+_\Omega(x,r)} \frac{(h(y)-h(x))^+}{|x-y|} &\le \lim_{r\to 0^+} \sup_{y\in B^+_\Omega(x,r)} \frac{\left(h(y^P)-h(x)+\epsilon\sqrt{d}|x-y|\right)^+}{|x-y|} \\ &\le \epsilon\sqrt{d} + \lim_{r\to 0^+} \sup_{y\in B'_\Omega(x,r)\cap X_P} \frac{(h(y)-h(x))^+}{|x-y|}\,,\end{aligned}$$

the second inequality since $|x-y^P| \le |x-y|$ and since we showed that, for all $r < r_0$ and $y \in B^+_\Omega(x,r)$, we have $y^P \in (B'_\Omega(x,r) \cap X_P) \cup \{x\}$, and moreover $(h(y^P)-h(x))^+ = 0$ when $y^P = x$. Similarly, when $P = \emptyset$ we have $y^P = x$ always, so in this case we obtain the upper bound $\epsilon\sqrt{d}$, i.e. the last limit superior may indeed be replaced by 0. Since this inequality holds for every $\epsilon > 0$, (40) follows, and this concludes the proof. □

7.2 Perturbation argument

We can now obtain the main result, first for $C^1([0,1]^d)$ functions and then for all of $H^1((0,1)^d)$ by an approximation argument. The key idea of the perturbation argument is to use the same (mean zero) function h in two different roles: 1) to construct absolutely continuous probability measures from h and its coordinate-wise monotone equilibrium, namely $\mathrm{d}\mu = (1+th)\,\mathrm{d}x$ and $\mathrm{d}\mu^* = (1+th^*)\,\mathrm{d}x$, for small $t > 0$; and 2) in the "test function" $-th$ for weak duality on $W_2^2(\mu \to \mu^*)$ via the directed Hamilton-Jacobi operator.

Theorem 7.8. *There exists a universal constant $C > 0$ such that the following holds. Define $\Omega := (0,1)^d$, and let $h \in C^1(\overline{\Omega})$. Then*

$$\mathsf{dist}_2^{\mathsf{mono}}(h)^2 \le \int_\Omega (h-h^*)^2\,\mathrm{d}x \le C\int_\Omega |\nabla^- h|^2\,\mathrm{d}x\,,$$

where h^* is the coordinate-wise monotone equilibrium of h.

Proof. Note that h^* is well-defined because C^1 functions are Lipschitz on compact sets. We first show that we may assume without loss of generality that h is bounded and has mean zero. For any $\alpha > 0$ and $\beta \in \mathbb{R}$, we have that $\alpha g + \beta$ is monotone if and only if g is monotone, so it follows that

$$\mathsf{dist}_2^{\mathsf{mono}}(\alpha h+\beta)^2 = \alpha^2 \mathsf{dist}_2^{\mathsf{mono}}(h)^2\,.$$

Moreover, Observation 6.49 gives that $(\alpha h+\beta)^* = \alpha h^* + \beta$, and clearly $|\nabla^-(\alpha h+\beta)|^2 = \alpha^2|\nabla^- h|^2$ pointwise. Thus as long as we show the result for h satisfying $\int_\Omega h\,\mathrm{d}x = 0$ and (say) $-0.1 \le h \le 0.1$, then we may write any other function in $C^1(\overline{\Omega})$ as $\alpha h + \beta$ for some h satisfying these conditions, and conclude the result. Hence assume that $\int_\Omega h\,\mathrm{d}x = 0$ and that $-0.1 \le h \le 0.1$ pointwise.

Let $t \in (0,1)$, and define $f \in C^1(\overline{\Omega})$ by $f := 1 + th$. Note that f is Lipschitz and bounded between 0.9 and 1.1, and it satisfies $\int_\Omega f\,\mathrm{d}x = 1$. Define the absolutely continuous probability measures $\mathrm{d}\mu := f\,\mathrm{d}x$ and $\mathrm{d}\mu^* := f^*\,\mathrm{d}x$. Theorem 6.51 then implies that, for some universal constant $C > 0$,

$$W_2^2(\mu \to \mu^*) \le C\int_\Omega \left|\nabla^- f\right|^2 \mathrm{d}x = Ct^2 \int_\Omega \left|\nabla^- h\right|^2 \mathrm{d}x = Ct^2\int_\Omega \left|\nabla^+(-h)\right|^2 \mathrm{d}x\,,$$

the first equality by the definition of f and the fact that $t>0$. On the other hand, noting that $-th \in C^1(\overline{\Omega})$ and hence $-th \in C_b(\Omega)$ as well, we have

$$\begin{aligned}
\tfrac{1}{2}W_2^2(\mu \to \mu^*) &\geq \int_\Omega (-th)\,\mathrm{d}\mu^* - \int_\Omega (\vec{H}_1(-th))\,\mathrm{d}\mu && \text{(Proposition 7.4)}\\
&= -t\int_\Omega h(1+th)^*\,\mathrm{d}x - \int_\Omega (\vec{H}_1(-th))(1+th)\,\mathrm{d}x && \text{(Definition of } \mu,\mu^*).
\end{aligned}$$

Recalling that $\int_\Omega h\,\mathrm{d}x = 0$ and that $(1+th)^* = 1+th^*$ by Observation 6.49, the first term in the last line above is

$$-t\int_\Omega h(1+th)^*\,\mathrm{d}x = -t\int_\Omega h(1+th^*)\,\mathrm{d}x = -t^2\int_\Omega hh^*\,\mathrm{d}x\,.$$

Recalling that $1+th$ is bounded between 0.9 and 1.1, the second term is

$$\begin{aligned}
&-\int_\Omega (\vec{H}_1(-th))(1+th)\,\mathrm{d}x\\
&\quad= -\int_\Omega (t\vec{H}_t(-h))(1+th)\,\mathrm{d}x && \text{(Lemma 7.5)}\\
&\quad= -t^2\int_\Omega \left(\frac{\vec{H}_t(-h)-(-h)-h}{t}\right)(1+th)\,\mathrm{d}x\\
&\quad= -t^2\int_\Omega \underbrace{\left(\frac{\vec{H}_t(-h)-(-h)}{t}\right)}_{\geq 0 \text{ by Proposition 7.6(c)}}\underbrace{(1+th)}_{\leq 2}\,\mathrm{d}x + t\int_\Omega h(1+th)\,\mathrm{d}x\\
&\quad\geq -2t^2\int_\Omega \frac{\vec{H}_t(-h)-(-h)}{t}\,\mathrm{d}x + t\underbrace{\int_\Omega h\,\mathrm{d}x}_{=0} + t^2\int_\Omega h^2\,\mathrm{d}x\\
&\quad= -2t^2\int_\Omega \frac{\vec{H}_t(-h)-(-h)}{t}\,\mathrm{d}x + t^2\int_\Omega h^2\,\mathrm{d}x\,.
\end{aligned}$$

Putting all of the above together, we conclude that

$$\tfrac{C}{2}t^2\int_\Omega \left|\nabla^+(-h)\right|^2\mathrm{d}x \geq \tfrac{1}{2}W_2^2(\mu\to\mu^*) \geq -t^2\int_\Omega hh^*\,\mathrm{d}x - 2t^2\int_\Omega \frac{\vec{H}_t(-h)-(-h)}{t}\,\mathrm{d}x + t^2\int_\Omega h^2\,\mathrm{d}x$$

and hence, since $t>0$,

$$\int_\Omega h^2\,\mathrm{d}x - \int_\Omega hh^*\,\mathrm{d}x \leq \tfrac{C}{2}\int_\Omega \left|\nabla^+(-h)\right|^2\mathrm{d}x + 2\int_\Omega \frac{\vec{H}_t(-h)-(-h)}{t}\,\mathrm{d}x\,.$$

Since this holds for all sufficiently small $t>0$, we may pass the inequality to the limit superior as $t\to 0^+$. Since $\frac{\vec{H}_t(-h)-(-h)}{t}$ is uniformly bounded by Proposition 7.6(c), we may apply the reverse Fatou lemma and Proposition 7.6(b) to obtain

$$\begin{aligned}
\int_\Omega h^2\,\mathrm{d}x - \int_\Omega hh^*\,\mathrm{d}x &\leq \tfrac{C}{2}\int_\Omega \left|\nabla^+(-h)\right|^2\mathrm{d}x + 2\limsup_{t\to 0^+}\int_\Omega \frac{\vec{H}_t(-h)-(-h)}{t}\,\mathrm{d}x\\
&\leq \tfrac{C}{2}\int_\Omega \left|\nabla^+(-h)\right|^2\mathrm{d}x + 2\int_\Omega \limsup_{t\to 0^+}\frac{(\vec{H}_t(-h))(x)-(-h)(x)}{t}\,\mathrm{d}x\\
&\leq \tfrac{C}{2}\int_\Omega \left|\nabla^+(-h)\right|^2\mathrm{d}x + \int_\Omega |\nabla^+(-h)(x)|^2\,\mathrm{d}x = \left(1+\tfrac{C}{2}\right)\int_\Omega \left|\nabla^- h\right|^2\mathrm{d}x\,.
\end{aligned}$$

By Observation 6.50, we have that $\int_\Omega (h^*)^2\,\mathrm{d}x \le \int_\Omega h^2\,\mathrm{d}x$ and hence, since h^* is monotone,

$$\begin{aligned}\tfrac{1}{2}\mathsf{dist}_2^{\mathsf{mono}}(h)^2 &\le \tfrac{1}{2}\int_\Omega (h-h^*)^2\,\mathrm{d}x = \tfrac{1}{2}\int_\Omega h^2\,\mathrm{d}x + \tfrac{1}{2}\int_\Omega (h^*)^2\,\mathrm{d}x - \int_\Omega hh^*\,\mathrm{d}x \\ &\le \int_\Omega h^2\,\mathrm{d}x - \int_\Omega hh^*\,\mathrm{d}x \le \left(1+\tfrac{C}{2}\right)\int_\Omega \left|\nabla^- h\right|^2\mathrm{d}x\,. \end{aligned}$$ □

The following denseness result is an immediate application of [Maz11, Theorem 1, p. 10].

Fact 7.9. *Define $\Omega := (0,1)^d$. The space $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$.*

Thus we may promote the main result to all of $H^1(\Omega)$ by an approximation argument:

Theorem 1.1 (Directed Poincaré inequality)**.** *There exists a universal constant $C > 0$ such that, for all $f \in H^1((0,1)^d)$,*

$$\mathsf{dist}_2^{\mathsf{mono}}(f)^2 \le C\,\mathbb{E}\left[\|\nabla^- f\|_2^2\right]\,. \tag{1}$$

Proof. By Fact 7.9, we may find a sequence $(f_n)_{n\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ such that $f_n \to f$ in $H^1(\Omega)$. Since each f_n belongs in particular to $C^1(\overline{\Omega})$, Theorem 7.8 gives that, for each $n \in \mathbb{N}$,

$$\|f_n - f_n^*\|_{L^2(\Omega)}^2 \le C\int_\Omega |\nabla^- f_n|^2\,\mathrm{d}x\,.$$

Let $\epsilon > 0$. Since $f_n \to f$ in $H^1(\Omega)$ implies in particular that $f_n \to f$ in $L^2(\Omega)$, we have that, for all sufficiently large n, $\|f - f_n\|_{L^2(\Omega)} \le \epsilon$ and hence, by the triangle inequality,

$$\mathsf{dist}_2^{\mathsf{mono}}(f) \le \|f - f_n^*\|_{L^2(\Omega)} \le \|f - f_n\|_{L^2(\Omega)} + \|f_n - f_n^*\|_{L^2(\Omega)} \le \epsilon + \sqrt{C\int_\Omega |\nabla^- f_n|^2\,\mathrm{d}x}\,. \tag{41}$$

Now, since the norm in $H^1(\Omega)$ is given by

$$\|u\|_{H^1(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \sum_{i=1}^d \int_\Omega |\partial_i u|^2\,\mathrm{d}x\,,$$

the fact that $f_n \to f$ in $H^1(\Omega)$ implies that $\|f - f_n\|_{H^1(\Omega)} \to 0$ and hence

$$0 = \lim_{n\to\infty}\sum_{i=1}^d \int_\Omega |\partial_i(f - f_n)|^2\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^d \int_\Omega |(\partial_i f) - (\partial_i f_n)|^2\,\mathrm{d}x\,.$$

Since $0 \le |(a\wedge 0) - (b \wedge 0)| \le |a - b|$ for any $a, b \in \mathbb{R}$, we conclude that

$$\lim_{n\to\infty}\sum_{i=1}^d \int_\Omega |(\partial_i^- f) - (\partial_i^- f_n)|^2\,\mathrm{d}x = 0\,. \tag{42}$$

Observing that $\partial_i^- f, \partial_i^- f_n \in L^2(\Omega)$ for each $i \in [d]$ and $n \in \mathbb{N}$ (because $\partial_i f, \partial_i f_n \in L^2(\Omega)$ by the definition of $H^1(\Omega)$, and if $u \in L^2(\Omega)$ then $u \wedge 0 \in L^2(\Omega)$), (42) implies that $\partial_i^- f_n \to \partial_i^- f$ in $L^2(\Omega)$ for each $i \in [d]$, and hence

$$\int_\Omega |\nabla^- f|^2\,\mathrm{d}x = \sum_{i=1}^d \int_\Omega |\partial_i^- f|^2\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^d \int_\Omega |\partial_i^- f_n|^2\,\mathrm{d}x = \lim_{n\to\infty}\int_\Omega |\nabla^- f_n|^2\,\mathrm{d}x\,.$$

Thus, for all sufficiently large n we have

$$\int_\Omega |\nabla^- f_n|^2 \, \mathrm{d}x \le \epsilon + \int_\Omega |\nabla^- f|^2 \, \mathrm{d}x \,.$$

Combining with (41) we obtain, for all sufficiently large n,

$$\begin{aligned}\mathsf{dist}_2^{\mathsf{mono}}(f)^2 &\le \left(\epsilon + \sqrt{C \int_\Omega |\nabla^- f_n|^2 \, \mathrm{d}x}\right)^2 \le 2\epsilon^2 + 2C \int_\Omega |\nabla^- f_n|^2 \, \mathrm{d}x \\ &\le 2\epsilon^2 + 2C\left(\epsilon + \int_\Omega |\nabla^- f|^2 \, \mathrm{d}x\right) = 2\epsilon^2 + 2C\epsilon + 2C \int_\Omega |\nabla^- f|^2 \, \mathrm{d}x \,.\end{aligned}$$

Since this holds for all $\epsilon > 0$, we conclude that

$$\mathsf{dist}_2^{\mathsf{mono}}(f)^2 \le 2C \int_\Omega |\nabla^- f|^2 \, \mathrm{d}x \,. \qquad \square$$

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References

[AB06] Charalambos D Aliprantis and Kim C Border. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer Berlin, Heidelberg, 2006.

[AC84] Jean-Pierre Aubin and Arrigo Cellina. *Differential Inclusions: Set-Valued Maps and Viability Theory*. Springer Berlin, Heidelberg, 1984.

[AGS05] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient Flows in Metric Spaces and in the Space of Probability Measures*. Birkhäuser Basel, 2005.

[AHLVXY23] Nima Anari, Yizhi Huang, Tianyu Liu, Thuy-Duong Vuong, Brian Xu, and Katherine Yu. "Parallel discrete sampling via continuous walks". In: *Proceedings of the ACM SIGACT Symposium on Theory of Computing (STOC)*. 2023, pp. 103–116.

[Bar76] Viorel Barbu. *Nonlinear semigroups and differential equations in Banach spaces*. Springer Dordrecht, 1976.

[BCS23] Hadley Black, Deeparnab Chakrabarty, and C Seshadhri. "Directed Isoperimetric Theorems for Boolean Functions on the Hypergrid and an $\widetilde{O}(n\sqrt{d})$ Monotonicity Tester". In: *Proceedings of the ACM SIGACT Symposium on Theory of Computing (STOC)*. 2023, pp. 233–241.

[BGL14] Dominique Bakry, Ivan Gentil, and Michel Ledoux. *Analysis and Geometry of Markov Diffusion Operators*. Springer Cham, 2014.

[BH97] S G Bobkov and C Houdré. "Isoperimetric constants for product probability measures". In: *The Annals of Probability* 25.1 (1997), pp. 184–205.

[BKKM23] Mark Braverman, Subhash Khot, Guy Kindler, and Dor Minzer. "Improved Monotonicity Testers via Hypercube Embeddings". In: *Proceedings of the Innovations in Theoretical Computer Science Conference (ITCS)*. 2023.

[BKR24] Hadley Black, Iden Kalemaj, and Sofya Raskhodnikova. "Isoperimetric inequalities for real-valued functions with applications to monotonicity testing". In: *Random Structures & Algorithms* (2024).

[Bla23] Hadley Black. "Testing and Learning in High-Dimensions: Monotonicity Testing, Directed Isoperimetry, and Convex Sets". PhD thesis. University of California, Los Angeles, 2023.

[Bre11] Haïm Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer New York, 2011.

[Bre73] Haïm Brezis. *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*. Elsevier, 1973.

[BRY14] Piotr Berman, Sofya Raskhodnikova, and Grigory Yaroslavtsev. "L_p-Testing". In: *Proceedings of the ACM SIGACT Symposium on Theory of Computing (STOC)*. 2014, pp. 164–173.

[Che] Steve Cheng. *Differentiation under the Integral Sign*. PlanetMath. https://planetmath.org/diff (version: 2024-03-19).

[CHHL19] Eshan Chattopadhyay, Pooya Hatami, Kaave Hosseini, and Shachar Lovett. "Pseudorandom generators from polarizing random walks". In: *Theory of Computing* 15.1 (2019), pp. 1–26.

[CS16] D Chakrabarty and C Seshadhri. "An $o(n)$ Monotonicity Tester for Boolean Functions over the Hypercube". In: *SIAM Journal on Computing* 45.2 (2016), pp. 461–472.

[CT80] Michael G Crandall and Luc Tartar. "Some relations between nonexpansive and order preserving mappings". In: *Proceedings of the American Mathematical Society* 78.3 (1980), pp. 385–390.

[DNS21] Anindya De, Shivam Nadimpalli, and Rocco A Servedio. "Quantitative Correlation Inequalities via Semigroup Interpolation". In: *Proceedings of the Innovations in Theoretical Computer Science Conference (ITCS)*. 2021.

[DS58] Nelson Dunford and Jacob T Schwartz. *Linear Operators, Part 1: General Theory*. Interscience Publishers, 1958.

[DU77] J Diestel and J J Uhl Jr. *Vector Measures*. American Mathematical Society, 1977.

[EG15] Lawrence C Evans and Ronald F Gariepy. *Measure Theory and Fine Properties of Functions, Revised Edition*. Chapman and Hall/CRC, 2015.

[EMR23] Ronen Eldan, Dan Mikulincer, and Prasad Raghavendra. "Noise stability on the Boolean hypercube via a renormalized Brownian motion". In: *Proceedings of the ACM SIGACT Symposium on Theory of Computing (STOC)*. 2023, pp. 661–671.

[Eva10] Lawrence C Evans. *Partial Differential Equations*. Second Edition. American Mathematical Society, 2010.

[Fen] Feng. *Boundedness + convergence in L^2_{loc} implies weak convergence?* Mathematics Stack Exchange. https://math.stackexchange.com/q/4495940 (version: 2022-07-19).

[Fer23] Renato Ferreira Pinto Jr. "Directed Poincaré Inequalities and L^1 Monotonicity Testing of Lipschitz Functions". In: *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM)*. 2023.

[GGLRS00] Oded Goldreich, Shafi Goldwasser, Eric Lehman, Dana Ron, and Alex Samorodnitsky. “Testing Monotonicity”. In: *Combinatorica* 3.20 (2000), pp. 301–337.

[KMS18] Subhash Khot, Dor Minzer, and Muli Safra. “On monotonicity testing and boolean isoperimetric-type theorems”. In: *SIAM Journal on Computing* 47.6 (2018), pp. 2238–2276.

[KOW16] Guy Kindler, Ryan O’Donnell, and David Witmer. *Remarks on the Most Informative Function Conjecture at fixed mean.* 2016. arXiv: 1506.03167 [cs.IT].

[Liu20] Yuan Liu. “The Poincaré inequality and quadratic transportation-variance inequalities”. In: *Electron. J. Probab.* 25.1 (2020), pp. 1–16.

[Maz11] Vladimir Maz’ya. *Sobolev Spaces with Applications to Elliptic Partial Differential Equations.* 2nd ed. Springer Berlin, Heidelberg, 2011.

[Mik78] V P Mikhailov. *Partial Differential Equations.* Mir Publishers, Moscow, 1978.

[O’D14] Ryan O’Donnell. *Analysis of Boolean Functions.* Cambridge University Press, 2014.

[Poi90] Henri Poincaré. “Sur les équations aux dérivées partielles de la physique mathématique”. In: *American Journal of Mathematics* (1890), pp. 211–294.

[San15] Filippo Santambrogio. *Optimal Transport for Applied Mathematicians: Calculus of Variations, PDEs, and Modeling.* Birkhäuser Cham, 2015.

[Tal93] Michel Talagrand. “Isoperimetry, logarithmic Sobolev inequalities on the discrete cube, and Margulis’ graph connectivity theorem”. In: *Geometric & Functional Analysis* 3.3 (1993), pp. 295–314.

[Vil09] Cédric Villani. *Optimal Transport: Old and New.* Springer Berlin, Heidelberg, 2009.

A Technical lemmas

Lemma A.1. *Define $\Omega := (0,1)^d$. Suppose $f_n \rightharpoonup f$ weakly in $L^2(\Omega)$, and each f_n is monotone nondecreasing (resp. monotone nonincreasing). Then f is monotone nondecreasing (resp. monotone nonincreasing).*

Proof. Recall that for each $\epsilon > 0$ and $g \in L^2(\Omega)$, $g^\epsilon = \eta_\epsilon * g$ is the mollification of g, defined on $\Omega_\epsilon = (\epsilon, 1-\epsilon)^d$.

Let $\epsilon \in (0, 1/3)$. We claim that $f_n^\epsilon \to f^\epsilon$ pointwise in Ω_ϵ. Indeed, fix any $x \in \Omega_\epsilon$ and define $\eta_{\epsilon,x} \in C^\infty(\mathbb{R}^d)$ by $\eta_{\epsilon,x}(y) := \eta_\epsilon(x-y)$. Since $f_n \rightharpoonup f$ weakly in $L^2(\Omega)$, we obtain

$$f^\epsilon(x) = \int_\Omega \eta_\epsilon(x-y) f(y)\,\mathrm{d}y = \langle \eta_{\epsilon,x}, f\rangle = \lim_{n\to\infty} \langle \eta_{\epsilon,x}, f_n\rangle = \lim_{n\to\infty} \int_\Omega \eta_\epsilon(x-y) f_n(y)\,\mathrm{d}y = \lim_{n\to\infty} f_n^\epsilon(x)\,.$$

Now, suppose without loss of generality that each f_n is monotone nondecreasing. Let $\epsilon \in (0, 1/3)$. It is immediate that each f_n^ϵ is also monotone nondecreasing (in Ω_ϵ). Since $f_n^\epsilon \to f^\epsilon$ pointwise in Ω_ϵ, we conclude that f^ϵ is monotone nondecreasing as well.

The conclusion follows the fact that $f^\epsilon \to f$ almost everywhere as $\epsilon \to 0$. □

The following lemma is essentially standard, and below we present the proof sketched in [Fen].

Lemma A.2. *Let $\Omega \subset \mathbb{R}^d$ be an open set, let $u \in L^2(\Omega)$ and let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^2(\Omega)$ such that $u_n \to u$ in $L^2_{\mathrm{loc}}(\Omega)$. Then $u_n \rightharpoonup u$ weakly in $L^2(\Omega)$.*

Proof. It suffices to show that every subsequence $(u_{n_k})_{k\in\mathbb{N}}$ has a subsequence that weakly converges to u. Fix any subsequence $(u_{n_k})_{k\in\mathbb{N}}$. First, since $u_{n_k} \to u$ in $L^2_{\mathrm{loc}}(\Omega)$, we have that $\langle u_{n_k}, \phi\rangle \to \langle u, \phi\rangle$ for all $\phi \in C_c^\infty(\Omega)$. Moreover, since $L^2(\Omega)$ is a Hilbert space and $(u_{n_k})_{k\in\mathbb{N}}$ is bounded, let $(u_{n_{k_\ell}})_{\ell\in\mathbb{N}}$ be a subsequence such that $u_{n_{k_\ell}} \rightharpoonup w$ weakly in $L^2(\Omega)$ for some $w \in L^2(\Omega)$. Now, let $v \in L^2(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in $L^2(\Omega)$, let $(\phi_m)_{m\in\mathbb{N}}$ be a sequence in $C_c^\infty(\Omega)$ with $\phi_m \to v$ in $L^2(\Omega)$. Then

$$\begin{aligned}
\langle u, v\rangle &= \lim_{m\to\infty} \langle u, \phi_m\rangle && \text{(Since } \phi_m \to v \text{ in } L^2(\Omega))\\
&= \lim_{m\to\infty}\lim_{k\to\infty} \langle u_{n_k}, \phi_m\rangle && \text{(As observed above)}\\
&= \lim_{m\to\infty}\lim_{\ell\to\infty} \left\langle u_{n_{k_\ell}}, \phi_m\right\rangle && \text{(Taking subsequence preserves the limit)}\\
&= \lim_{m\to\infty} \langle w, \phi_m\rangle && \text{(Since } u_{n_{k_\ell}} \rightharpoonup w \text{ weakly in } L^2(\Omega))\\
&= \langle w, v\rangle && \text{(Since } \phi_m \to v \text{ in } L^2(\Omega))\\
&= \lim_{\ell\to\infty} \left\langle u_{n_{k_\ell}}, v\right\rangle && \text{(Since } u_{n_{k_\ell}} \rightharpoonup w \text{ weakly in } L^2(\Omega)),
\end{aligned}$$

so $u_{n_{k_\ell}} \rightharpoonup u$ weakly in $L^2(\Omega)$ as needed. □

Lemma A.3 (From "almost Lipschitz" to Lipschitz)**.** *Let $M \in \mathbb{R}_{\geq 0}$ and let $N \subset I$ be a measure zero set. Suppose $f : I \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in I \setminus N$. Then there exists a M-Lipschitz function $g : I \to \mathbb{R}$ such that $f = g$ in $I \setminus N$.*

Proof. Let $\overline{f} : I \to (-\infty, +\infty]$ be given by $\overline{f} := f$ in $I \setminus N$ and $\overline{f} := +\infty$ in N. Define $g : I \to \mathbb{R}$ by

$$g(x) := \begin{cases} f(x) & \text{if } x \in I \setminus N\\ \liminf_{z\to x} \overline{f}(z) & \text{otherwise.}\end{cases}$$

Note that g is real-valued because, when $x \in N$, we may find a sequence $(z_n)_{n\in\mathbb{N}} \subset I \setminus N$ such that $z_n \to x$ and then, by the definition of lim inf by subsequential limits and the assumption on f,

$$g(x) = \liminf_{z\to x} \overline{f}(z) \leq \lim_{n\to\infty} \overline{f}(z_n) = \lim_{n\to\infty} f(z_n) \leq \lim_{n\to\infty} \left[f(z_1)+M|z_n-z_1|\right] = f(z_1)+M|x-z_1| < +\infty\,.$$

Clearly $g = f$ in $I \setminus N$. We claim that g is M-Lipschitz. By the assumption on f, we do have $|g(x) - g(y)| \leq M|x - y|$ for all $x, y \in I \setminus N$. Now, let $x \in N$ and $y \in I \setminus N$. By the definition of g, fix a sequence $(z_n)_{n\in\mathbb{N}} \subset I \setminus N$ such that $z_n \to x$ and moreover $\lim_{n\to\infty} f(z_n) = g(x)$. Then

$$|g(x) - g(y)| = \left|\lim_{n\to\infty} f(z_n) - f(y)\right| = \lim_{n\to\infty} |f(z_n) - f(y)| \leq \lim_{n\to\infty} M|z_n - y| = M|x - y|\,,$$

as desired. The case when $x, y \in N$ follows by the triangle inequality: suppose $x < y$ without loss of generality, choose any $z \in (x, y) \setminus N$, which is possible since N has measure zero, and use the above to conclude that

$$|g(x) - g(y)| \leq |g(x) - g(z)| + |g(z) - g(y)| \leq M|x - z| + M|z - y| = M|x - y|\,. \qquad \square$$

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