



A Note on Zero-Knowledge for \mathbf{NP} and One-Way Functions

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September 6, 2024

Abstract

We present a simple alternative exposition of the recent result of Hirahara and Nanashima (STOC'24) showing that one-way functions exist if (1) every language in \mathbf{NP} has a zero-knowledge proof/argument (i.e., $\mathbf{NP} \subseteq \mathbf{ZKA}$) and (2) \mathbf{ZKA} contains non-trivial languages (i.e., $\mathbf{ZKA} \not\subseteq \mathbf{iP/poly}$). Our presentation does not rely on meta-complexity and we hope it may be useful for didactic purposes. We also remark that the same result hold for (imperfect) \mathbf{iO} for 3CNF, or Witness Encryption for \mathbf{NP} .

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1 Introduction

Zero-knowledge (ZK) proofs, introduced by Goldwasser, Micali and Rackoff [GMR89] are paradoxical constructs that enables a Prover to convince a Verifier that some instance x belongs to a language L without revealing any additional information. In the case where the soundness condition only holds against polynomial-size attacker, the proof systems is instead referred to as a *zero-knowledge argument* [BCC88]; let **ZKA** denote the class of languages having ZK arguments. While classic results in the early 1990’s established that, assuming the existence of *one-way functions (OWFs)*, every language in **NP** has a ZK proof [GMW91; HILL99; Nao91], it remains an open problem whether the existence of *non-trivial ZK proofs/arguments* also imply the existence of OWFs.

Seminal results by Ostrovsky [Ost91], and Ostrovsky and Wigderson (OW) [OW93] from the 1990’s, however, show that non-trivial ZK “almost” implies OWFs. In particular, they show that if **ZKA** $\not\subseteq$ **ioP/poly**, then a relaxation of OWFs, referred to as a *auxiliary-input OWF (ai-OWF)* exist.¹ Additionally OW shows that if **ZKA** contains a language that is *average-case hard*, then (standard) OWFs exist.²

Thus, the only “gap” is between ai-OWFs and (standard) OWFs, or between worst-case and average-case hardness for a language in **ZKA**. A recent elegant paper by Hirahara and Nanashima (HN) [HN24] closes this gap under the assumption that all of **NP** (or even just a specific meta-complexity language) has zero-knowledge arguments (i.e., **NP** \subseteq **ZKA**). In other words, they show:

Theorem 1.1 (Main). *Assume that $\mathbf{NP} \subseteq \mathbf{ZKA}$, and that $\mathbf{ZKA} \not\subseteq \mathbf{ioP/poly}$. Then one-way functions exist.*

In this note, we provide a somewhat alternative presentation of the proof of the HN result. The proof elements are very similar, but our exposition is more direct and we dispense of the use of meta-complexity. As such, we hope that their results becomes easier to appreciate (without any background on meta-complexity).

Proof Overview The proof proceeds in two steps:

1. **Errorless average-case hardness of ZKA implies OWFs.** As mentioned, OW already showed that average-case hardness of **ZKA** implies OWFs. We observe (following the approach in [HN24]) that essentially a direct combination of a characterization of Vadhan [Vad06] together with the results of Ostrovsky [Ost91] yields the stronger statement that *errorless*³ average-case hardness suffices (i.e., that **ZKA** $\not\subseteq$ **ioAvgBPP/poly**).⁴ In essence, the reason why errorless average-case hardness suffices is that given a statement x , the reduction provided in these earlier works either correctly decides x , or fails to invert some ai-OWF candidate f_x ; but since the latter event is checkable, the reduction can easily be made errorless.

¹Roughly speaking, an ai-OWF is family of functions f_i such that no polynomial-size attacker A can invert *every* function in the family. That is, there exists some i on which A fails to invert f_i .

²In essence, when just assuming worst-case hardness of some language in **ZKA**, the index i is selected as an instance in the language on which the attacker will fail to decide the language, whereas in the case of average-case hardness, the index can be efficiently sampled so we get a standard OWF.

³That is, we consider hardness against algorithms that either give the right answer or \perp , and that only output \perp with small probability.

⁴Coincidentally, OV actually stated their result assuming that **ZKA** $\not\subseteq$ **ioAvgBPP/poly**, but while the notation *Avg* typically denotes errorless average-case hardness in the literature, they defined it as two-sided error average-case hardness, so this is a strict strengthening of their result.

2. **Auxiliary-input OWFs imply average-case hardness of NP.** We observe that the PRG construction of HILL [HILL99] yields an average-case hardness language in **NP** even when instantiated with an ai-OWF: The language consists of all strings in the range of the PRG, and average-case hardness holds over the uniform distribution. The argument (which can be traced back to [Hir18], and made explicit for general PRGs in [LP21b]), is simple: an errorless algorithm for the uniform distribution needs to output 1 with high probability when given a uniform sample (since with high probability those strings are not in the range of the PRG), and output either 0 or \perp when given a sample in the range of the PRG, so simply interpret \perp as a 0 and we have a PRG distinguisher that works with high probability, which using the HILL reduction can be turned into an inverter for the (ai)-OWFs.⁵

So, by OW, non-trivial \mathcal{ZK} implies ai-OWF, which by step (2), and the assumption that $\mathbf{NP} \subseteq \mathbf{ZKA}$ implies that $\mathbf{ZKA} \not\subseteq \mathbf{ioAvgBPP/poly}$ which by step (1) implies OWFs. (Note that this results also hold just for honest-verifier zero-knowledge; in any case, by [OV07], a language has zero-knowledge argument if and only if it has honest-verifier zero-knowledge argument.)

1.1 Corollaries to iO and WE

We note an interesting new corollary of Theorem 1.1 (i.e., the main result of [HN24]) with respect to *indistinguishability obfuscation (iO)* [Bar+12] and *witness encryption for NP (WE)* [GGSW13]. [Kom+14] showed that (imperfect) iO for polynomial-size circuits and the assumption that **NP** is worst-case hard yield the existence of one-way functions. They further showed that iO for just the class of 3CNF formulas, or even WE for **NP** (which is implied by iO for 3CNFs) together with *average-case hardness* of **NP** yield one-way functions, but left open the problem of whether worst-case hardness of **NP** suffices. We observe that Theorem 1.1 can be used to solve this (in the non-uniform setting).

In particular, [Kom+14] show that these primitives can be used to construct statistical zero-knowledge arguments for **NP** (see Theorem 5.2 and the following remark in [Kom+14]); next, by relying on Theorem 1.1, we can conclude the existence of one-way functions assuming just (non-uniform) worst-case hardness of **NP**.⁶

Corollary 1.2. *Assume that $\mathbf{NP} \not\subseteq \mathbf{ioP/poly}$. If there exists an (imperfect) iO for 3CNF formulas, or WE for **NP**, then one-way functions exist.*

In fact, it suffices to assume WE for MCSP (using [HN24]) or $\mathbf{MK}^t\mathbf{P}$ (by Corollary A.3.)

2 Preliminaries and Definitions

Let **ZKA** denote set of languages having zero-knowledge arguments [GMR89; BCC88], and let **SZKP** denote the set of languages having statistical zero-knowledge proofs (see e.g., [SV97]). We proceed to defining the notion of errorless average-case hardness.

⁵In fact, essentially the same argument shows errorless average-case hardness of the *Minimum time-bounded Kolmogorov complexity problem* [Ko86], see Appendix A. We note that HN used a similar but more complicated argument to show average-case hardness of the Minimum Circuit Size problem [KC00] relying on the construction of a PRF [GGM84] from OWFs.

⁶In contrast, whereas [Kom+14] required average-case hardness, their result also applied in the uniform setting.

Definition 2.1 (**ioAvgBPP/poly**). A pair $(\mathcal{L}, \mathcal{D})$ of a language \mathcal{L} and a samplable distribution $\mathcal{D} = \{\mathcal{D}_n\}_{n \in \mathbb{N}}$ is in **ioAvgBPP/poly** if there exists a non-uniform PPT A such that the following holds for infinitely many $n \in \mathbb{N}$:

- For every $x \in \text{Supp}(\mathcal{D}_n)$, $\Pr[A(x) \in \{\perp, \mathcal{L}(x)\}] \geq 0.9$
- $\Pr_{x \leftarrow \mathcal{D}_n}[A(x) = \perp] \leq 1/4$.

We recall that notion of an ai-OWF:

Definition 2.2 (ai-OWF). A function family $\{f_a: \{0, 1\}^{m(n)} \rightarrow \{0, 1\}^{m(n)}\}_{a \in \{0, 1\}^n}$ is an auxiliary-input one-way function (ai-OWF) if for every non-uniform PPT \mathcal{A} there exists a negligible function μ such that for every $n \in \mathbb{N}$, there exists some $a \in \{0, 1\}^n$ such that

$$\Pr_{x \leftarrow \{0, 1\}^{m(n)}}[\mathcal{A}(a, f_a(x)) \in f_a^{-1}(f_a(x))] \leq \mu(n) \quad (1)$$

If a is simply 0^n , we refer to the family as simply a one-way function (OWF). We say that $\{f_a\}$ is almost-everywhere hard on a set $\mathcal{I} \subset \{0, 1\}^*$ if Equation (1) holds for every $a \in \mathcal{I} \cap \{0, 1\}^n$.

Finally, let us recall the classic result by Ostrovsky and Wigderson [OW93]:

Theorem 2.3 (\mathcal{ZK} to ai-OWF, [OW93]). Assume that **ZKA** $\not\subseteq$ **ioP/poly**. Then ai-OWF exists.

3 Proof of Main Theorem

3.1 Step 1: **ZKA** $\not\subseteq$ **ioAvgBPP/poly** \Rightarrow OWFs

Recall that OV showed that *two-sided* error average-case hardness of **ZKA** implies OWF; we here show the same by starting with just *errorless* average-case hardness of **ZKA**.⁷

Lemma 3.1 (Errorless Avg-Hardness of \mathcal{ZK} to OWFs). Assume that there exists a language $\mathcal{L} \in \mathbf{ZKA}$ and samplable distribution \mathcal{D} such that $(\mathcal{L}, \mathcal{D}) \notin \mathbf{ioAvgBPP/poly}$. Then OWFs exist.

Towards proving this, we will rely on the following characterization of **ZKA** of Vadhan [Vad06].

Definition 3.2 (SZK/OWF [Vad06]). A promise problem $\Pi = (\Pi_{\mathcal{Y}}, \Pi_{\mathcal{N}})$ satisfies the SZKP/OWF Condition if there is $\mathcal{I} \subseteq \Pi_{\mathcal{Y}} \cup \Pi_{\mathcal{N}}$ such that:

- The promise problem $(\Pi_{\mathcal{Y}} \setminus \mathcal{I}, \Pi_{\mathcal{N}} \setminus \mathcal{I})$ is in **SZKP**.
- There exists an auxiliary-input one-way function which is almost everywhere hard on \mathcal{I} .

Theorem 3.3 ([OV07]). If $(\mathcal{Y}, \mathcal{N}) \in \mathbf{ZKA}$ then $(\mathcal{Y}, \mathcal{N})$ satisfies the SZKP/OWF Condition.

We will also rely on the following version of the results by Ostrovsky (as explicitly stated in [Vad06].)

Theorem 3.4 ([Ost91] (c.f. [Vad06, Theorem 7.5])). Let $\Pi = (\Pi_{\mathcal{Y}}, \Pi_{\mathcal{N}}) \in \mathbf{SZKP}$. Then there exists a function family $\{f_x\}_{x \in \{0, 1\}^*}$ and an oracle-aided PPT R , such that for every $x \in \Pi_{\mathcal{Y}} \cup \Pi_{\mathcal{N}}$ and any algorithm A that inverts f_x with probability at least 0.01, $\Pr[R^A(x) = \Pi(x)] \geq 0.99$.

⁷This result also easily follows from two theorem statements in [HN24], but was not explicit stated as far as we can tell.

We now turn to the proof of Lemma 3.1.

Proof of Lemma 3.1. Let $(\Pi = (\mathcal{L}, \bar{\mathcal{L}}), \mathcal{D}) \in (\mathbf{ZKA}) \setminus (\mathbf{ioAvgBPP/poly})$. Let \mathcal{I} and $\{h_x\}_{x \in \{0,1\}^*}$ be the set and the auxiliary-input one-way function promised by Theorem 3.3 and the SZK/OWF condition for Π . Let $\{g_x\}_{x \in \{0,1\}^*}$ and R be the auxiliary-input one-way function and reduction promised by Theorem 3.4 for $\Pi' = (\Pi_{\mathcal{Y}} \setminus \mathcal{I}, \Pi_{\mathcal{N}} \setminus \mathcal{I})$.

Let $f(r, y_1, y_2) = \mathcal{D}(r) \| h_{\mathcal{D}(r)}(y_1) \| g_{\mathcal{D}(r)}(y_2)$. We claim that f is a weak one-way function. Indeed, assume towards a contradiction that an efficient algorithm A inverts f with probability 0.99 for infinitely many input lengths n , and fix such large enough n . Let A' be the algorithm that uses A to invert g_x : given an input $x, g_x(y)$, A' samples y_1 and executes $A(x \| h_x(y_1) \| g_x(y))$ to get a pre-image of $g_x(y)$. Let B be the algorithm that given input x , samples random y_1, y_2 , and runs A on $z = x \| h_x(y_1) \| g_x(y_2)$. If A failed in inverting f on z , B outputs \perp . Otherwise, B outputs $R^{A'}(x)$. We next show that B contradicts the assumption that $\Pi \notin \mathbf{ioAvgBPP/poly}$.

Observe that for every r such that $\mathcal{D}(r) \in \mathcal{I}$, A' can only invert $g_{\mathcal{D}(r)}$ with negligible probability and consequently, A inverts $f(r, \cdot)$ only with negligible probability. Thus B outputs \perp on every (sufficiently large) such input with probability 0.99. On the other hand, for every r such that $\mathcal{D}(r) \notin \mathcal{I}$, B outputs either \perp or the right answer with probability at least 0.99. Thus B outputs the wrong answer with probability at most 0.01 for any x . This implies the first item in Definition 2.1.

Next, to see that the second item holds, observe that for any r such that

$$\Pr_y[A(f(r, y) \in f^{-1}(f(r, y)))] \geq 0.9,$$

B outputs a (non- \perp) right answer with probability at least 0.9 (probability of running B) - 0.01 (probability that B outputs an incorrect answer) = 0.89. Moreover, by an averaging argument,

$$\Pr_r[\Pr_y[A(f(r, y) \in f^{-1}(f(r, y)))] \geq 0.9] \geq 0.9.$$

(since otherwise, it cannot be that A inverts f with probability 0.99). Thus, B outputs the right answer with probability at least $0.89 \cdot 0.9 \geq 3/4$. \square

3.2 Step 2: io-OWF \Rightarrow NP $\not\subseteq$ ioAvgBPP/poly

We turn to observing that io-OWFs imply (errorless) average-case hardness of **NP**; this observation may be folklore in the community but, as far as we can tell, was first explicitly stated in [HN24] using a somewhat more complicated proof for a stronger statement (in particular, they proved not only that **NP** is average-case hard but also that the particular MCSP problem [KC00] is so).⁸

Lemma 3.5 (ai-OWF to AvgBPP hardness). *Assume that ai-OWF exists. Then there exists a language $\mathcal{L} \in \mathbf{NP}$ such that $(\mathcal{L}, \{U_{m(n)}\}_{n \in \mathbb{N}}) \notin \mathbf{ioAvgBPP/poly}$ for some $m \in \text{poly}$.*

Proof. Let $\{f_x\}_{x \in \{0,1\}^*}$ be an ai-OWF. For every $x \in \{0,1\}^*$, let $G_x: \{0,1\}^{m(|x|)} \rightarrow \{0,1\}^{2m(|x|)}$ be the PRG construction of HILL from f_x with $m \in \text{poly}$ such that $m(|x|) > 2|x|$. [HILL99]. By [HILL99] it holds that (1) $G_x(s)$ can be efficiently computed given x, s , and (2) there is a reduction from distinguishing the output of G_x from uniform and inverting f_x . Let

$$\mathcal{L}_{\text{HILL}} = \left\{ y: \exists x \in \{0,1\}^*, s \in \{0,1\}^{m(|x|)} \text{ s.t. } G_x(s) = y \right\},$$

⁸As mentioned, our proof directly extends also to showing that the MK^tP problem [Ko86] also is average-case hard—see Appendix A for more details—but this is not of relevance for proving the main result.

and let $\mathcal{D} = \{\mathcal{D}_n = U_{2m(n)}\}_{n \in \mathbb{N}}$. We claim that $(\mathcal{L}_{\text{HILL}}, \mathcal{D}) \in \mathbf{ioAvgBPP/poly}$. To see this, assume toward contradiction that $(\mathcal{L}_{\text{HILL}}, \mathcal{D}) \notin \mathbf{ioAvgBPP/poly}$, and let A be the algorithm that decides $\mathcal{L}_{\text{HILL}}$ with good probability over the \mathcal{D}_n for infinite many n 's. We show that A can be used to invert $\{f_x\}$ for every $x \in \{0, 1\}^n$ and for infinitely many n 's.

Indeed, fix n such that A succeed on \mathcal{D}_n , and fix $x \in \{0, 1\}^n$. Observe that $G_x(s) \in \mathcal{L}_{\text{HILL}}$ for any $s \in \{0, 1\}^{m(n)}$. Thus, A outputs 1 or \perp on x with probability at least 0.9 on every output of G_x . On the other hand,

$$|\mathcal{L}_{\text{HILL}} \cap \{0, 1\}^{2m(n)}| \leq 2^n \cdot 2^{m(n)} \leq 2^{2m(n)-n}.$$

Therefore, it holds that $\Pr_{y \leftarrow U_{2m(n)}}[y \in \mathcal{L}_{\text{HILL}}] \leq 2^{-n}$, and thus

$$\Pr_{y \leftarrow U_{2m(n)}}[A(y) \neq 0] \leq 2^{-n} + 0.1 + 1/4 \leq 1/2.$$

We get that A distinguishes the output of G_x from random with advantage at least $0.9 - 0.5 = 0.4$. \square

3.3 Concluding the Proof of Theorem 1.1

Proof of Theorem 1.1. Assume that $\mathbf{ZKA} \not\subseteq \mathbf{ioP/poly}$. By Theorem 2.3 ai-OWF exist. By Lemma 3.5, there exists $\mathcal{L} \in \mathbf{NP}$ and samplable distribution \mathcal{D} such that $(\mathcal{L}, \mathcal{D}) \notin \mathbf{ioAvgBPP/poly}$. Finally, since by assumption $\mathbf{NP} \subseteq \mathbf{ZKA}$, we get by Lemma 3.1 that OWFs exist. \square

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A On Average-case Hardness of MK^tP

We here briefly observe that ai-OWF imply average-case hardness of the Minimum Kolmogorov complexity problem, MK^tP [Kol68; Ko86]; this result may be folklore in the community but as far as we know has not been explicitly stated (although a close variant of it can be found in [LP21a], following the approach in [Hir18]).

Recall that for a threshold $s = s(n)$, $\text{MK}^t\text{P}[s]$ is the language of all strings $x \in \{0,1\}^*$ with $K^t(x) \leq s(|x|)$. Similarly, $\text{MK}^t\text{P}[s_0, s_1]$ is the promise problem in which the Yes instances are the strings with $K^t(x) \leq s_0(|x|)$, and the No instances are the strings with $K^t(x) > s_1(|x|)$. We prove the following theorem.

Theorem A.1. *Let $\epsilon > 0$ be a constant. Assuming that ai-OWF exist, $(\text{MK}^t\text{P}[\epsilon n, (1-\epsilon)n], U_{m(n)}) \notin \mathbf{ioAvgBPP}/\mathbf{poly}$ for any $t(n) \geq n^{1+\epsilon}$ and some $m \in \mathbf{poly}$.*

Proof. Assume toward contradiction that $(\text{MK}^t\text{P}[\epsilon n, (1-\epsilon)n], U_{m(n)}) \in \mathbf{ioAvgBPP}/\mathbf{poly}$ for any $m \in \mathbf{poly}$. We claim that there is no ai-OWF. Indeed, we can use the MK^tP solver to invert any function family $\{f_x\}_{x \in \{0,1\}^*}$ on every x of length n , for infinite many n 's.

To see that, we use each function f_x to construct a PRG $G_x: \{0,1\}^{\epsilon \cdot m(n)/2} \rightarrow \{0,1\}^{m(n)}$ for some polynomial $m(n)$, such that (1) $\epsilon \cdot m(n)/2 \geq 2n$, (2) $G_x(z)$ can be computed in time at most $(m(n))^{1+\epsilon}$ given x and an input z , and (3) there is a reduction from distinguishing the output of G_x from uniform and inverting f_x .

Let A be the zero-error algorithm that for $(\text{MK}^t\text{P}[\epsilon n, (1-\epsilon)n], U_{m(n)})$. Using the family $\{G_x\}$ and the algorithm A we can invert $\{f_x\}$ almost everywhere by the observation that A distinguish between the output of G_x and uniform $m(n)$ bit string for every choice of x of length n . Indeed, by the correctness of A , A must output No with probability at least $3/4 - \text{neg}(n)$ over the uniform distribution. On the other hand, $K^t(G_x(z)) \leq |z| + |x| + O(\log n) \leq \epsilon \cdot m(n)/2 + n + (\log n) \leq \epsilon \cdot m(n)$ for any z, x . Thus, A must output \perp or Yes on any output of G_x .

Finally, to construct G_x we use the PRG construction of HILL from the one-way function f_x to get a function $G': \{0,1\}^{\epsilon \cdot m'(n)/2} \rightarrow \{0,1\}^{m'(n)}$. Let p' be a polynomial that bound the running time of G' , and let $G_x(z_1, \dots, z_{q(n)}) = G'_x(z_1) \parallel \dots \parallel G'_x(z_{q(n)})$. Then G' can be computed in time roughly $q(n) \cdot p'(n)$, and the output length of G' is $m(n) := q(n) \cdot m'(n)$. By taking $q(n) \geq \max\{(p'(n))^{1/\epsilon}, 2n/\epsilon\}$ we get that G_x can be computed in time at most $(m(n))^{1+\epsilon}$. \square

We directly get the following corollaries; the second one using the proof of Theorem 1.1.

Corollary A.2. *Let $\epsilon > 0$ be a constant. Assuming that ai-OWF exist, $(\text{MK}^t\text{P}[(1-\epsilon)n], U_{m(n)}) \notin \mathbf{ioAvgBPP}/\mathbf{poly}$ for any $t(n) \geq n^{1+\epsilon}$ and some $m \in \mathbf{poly}$.*

Corollary A.3. *Assume that $\text{MK}^t\text{P}[\epsilon n, (1-\epsilon)n] \in \mathbf{ZKA}$ for some constant ϵ , and that $\mathbf{ZKA} \notin \mathbf{ioP}/\mathbf{poly}$. Then one-way functions exist.*