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Relations between monotone complexity measures based on decision tree complexity

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Abstract. In a recent result, Knop, Lovett, McGuire and Yuan (STOC 2021) proved the log-rank conjecture for communication complexity, up to $\log n$ factor, for any Boolean function composed with AND function as the inner gadget. One of the main tools in this result was the relationship between monotone analogues of well-studied Boolean complexity measures like block sensitivity and certificate complexity. The relationship between the standard measures has been a long line of research, with a landmark result by Huang (Annals of Mathematics 2019), finally showing that sensitivity is polynomially related to all other standard measures.
In this article, we study the monotone analogues of standard measures like block sensitivity ($\mathsf{mbs}(f)$), certificate complexity ($\mathsf{MCC}(f)$) and fractional block sensitivity ($\mathsf{fmbs}(f)$); and study the relationship between these measures given their connection with AND-decision tree and sparsity of a Boolean function. We show the following results:

- Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$, the ratio $\frac{\mathsf{fmbs}(f^l)}{\mathsf{mbs}(f^l)}$ is bounded by a function of n (and not l). A similar result was known for the corresponding standard measures (Tal, ITCS 2013). This result allows us to extend any upper bound by a *well behaved* measure on monotone block sensitivity to monotone fractional block sensitivity.
- The question of the best possible upper bound on monotone block sensitivity by the logarithm of sparsity is equivalent to the natural question of best upper bound by degree on sensitivity. One side of this relationship was used in the proof by Knop, Lovett, McGuire and Yuan (STOC 2021).
- For two natural classes of functions, symmetric and monotone, hitting set complexity (MCC) is equal to monotone sensitivity.

1 Introduction

Decision tree complexity is one of the simplest complexity measure for a Boolean function where the complexity of an algorithm only takes into account the

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number of queries to the input. Various complexity measures based on decision tree complexity (like quantum query complexity, randomized query complexity, certificate complexity, sensitivity, block sensitivity and many more) have been introduced to study Boolean functions (functions from a subset of $\{0,1\}^n$ to $\{0,1\}$) [6,2,16]. Understanding the relations between these complexity measures of Boolean function has been a central area of research in computational complexity theory for at least 30 years. Refer to [6] for an introduction to this area.

Two such complexity measures M_1 and M_2 are said to be polynomially related if there exists constants c_1 and c_2 such that $M_1 = O(M_2^{c_1})$ and $M_2 = O(M_1^{c_2})$. Recently, Huang [9] resolved a major open problem in this area known as the "sensitivity conjecture", showing the polynomial relationship between the two complexity measures sensitivity ($\mathsf{s}(f)$) and block sensitivity ($\mathsf{bs}(f)$) for a Boolean function f (implying sensitivity is polynomially related to almost all other complexity measures too).

Once two complexity measures have been shown to be polynomially related, it is natural to ask if the relationships are tight or not. This means, if we can show $\forall\ f,\ M_1(f) = O(M_2(f))^\alpha$, then is there an example that witnesses the same gap? In other words, does there exists a function f for which $M_1(f) = \Omega(M_2(f))^\alpha$? Figuring out tight relations between complexity measures based on decision trees has become the central goal of this research area. ([3] compiled an excellent table with the best-known relationships between these different measures.)

Additionally, many new related complexity measures have been introduced in diverse areas, sometimes to understand these relations better [10,4,7]. Recently, monotone analogues of such combinatorial measures have been explored in [11] for studying the celebrated *log-rank conjecture* in communication complexity (for definitions of these monotone measures, see section 2).

In particular, Knop et. al. [11] resolved the log rank conjecture (up to $\log(n)$ factor) for any Boolean function f composed with AND function as the inner gadget. For such functions, the rank is equal to the sparsity of the function (denoted by $\mathsf{spar}(f)$) in its polynomial representation (with range $\{0,1\}$). So the log-rank conjecture amounts to proving a polynomial upper bound of $\log(\mathsf{spar}(f))$ on the deterministic communication complexity of $(f \circ \wedge_2)$ i.e. $D^{cc}(f \circ \wedge_2)$. As mentioned before, the proof provided in [11] utilized monotone analogues of the standard combinatorial measures (block sensitivity, fractional block sensitivity etc.). The reason for considering monotone measures was due to the observation that the deterministic communication complexity of such functions is related to their fractional monotone block sensitivity (fmbs)

$$D^{cc}(f \circ \wedge_2) \leq \mathsf{fmbs}(f) \log(\mathsf{spar}(f)) \log(n).$$

Fractional block sensitivity (and its relation with block sensitivity) has been studied before [1,12,19,8]. It seems natural to look at the monotone analogues of fractional block sensitivity (fmbs) and block sensitivity (mbs), and see if they can used to upper bound fmbs with logarithm of sparsity. They precisely do this, and show that for every Boolean function $f : \{0,1\}^n \to \{0,1\}$:

- $\mathsf{mbs}(f) = O(\log^2(\mathsf{spar}(f)))$,

– $\mathsf{fmbs}(f) = O(\mathsf{mbs}^2(f))$.

This implies that a sparse Boolean function has small fractional monotone block sensitivity and in turn small deterministic communication complexity,

$$\mathsf{fmbs}(f) = O(\log^4(\mathsf{spar}(f))) \Rightarrow D^{cc}(f \circ \wedge_2) = O(\log^5(\mathsf{spar}(f)) \log n).$$

Here $\mathsf{spar}(f)$ denotes the sparsity of f as a polynomial with range $\{0, 1\}$.

The above proof technique gives rise to a natural question, can these relationships between monotone and related measures be improved? The main objective of this article is to explore this question. Specifically, we look at the following variants.

– Is it possible to improve the exponent in the relationship $\mathsf{mbs}(f) = O(\log^2(\mathsf{spar}(f)))$?
– Can we translate the bound on $\mathsf{fmbs}(f)$ by *well behaved quantities* using their bound on $\mathsf{mbs}(f)$. (Similar to the result of Tal [19] which allows us to lift upper bounds on block sensitivity to fractional block sensitivity for many well behaved measures.)
– Are there specific class of functions for which monotone analogues have a better dependence on sparsity?

Ideally, we would like to compile a table similar to [3] for monotone measures. We start by giving a preliminary table in Appendix E.

1.1 Our Results

We study the monotone analogues of standard complexity measures like block sensitivity, certificate complexity and their relations with standard complexity measures.

It is natural to ask if it is possible to improve upper bounds on these monotone measures? One very interesting approach for improving bounds on $\mathsf{fbs}(f)$ (standard measure) is by Tal [19]. He showed that the ratio of $\mathsf{bs}(f^l)$ and $\mathsf{fbs}(f^l)$ is bounded by a quantity independent of l; this allowed him to lift *any* upper bound on $\mathsf{bs}(f)$ by a measure which is *well behaved with respect to composition* to $\mathsf{fbs}(f)$. We prove a similar result for $\mathsf{mbs}(f)$ and $\mathsf{fmbs}(f)$.

Theorem 1. *Consider a Boolean function $f : \{0, 1\}^n \to \{0, 1\}$. For a sufficiently large n, the ratio*

$$\frac{\mathsf{fmbs}(f^l)}{\mathsf{mbs}(f^l)} \leq p(n)$$

for all $l \geq 1$, where $p(n)$ is a function in n independent of l.

As mentioned earlier, there is a nice implication of this behaviour (as shown in [13] for standard measures): given a measure M which *behaves well under composition* and an upper bound on monotone block sensitivity in terms of measure M, we can lift the same upper bound to fractional monotone block sensitivity.

Corollary 1. *Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function. Let $M(.)$ be a complexity measure such that for all $l \geq 2$, $M(f^l) \leq M(f)M(f^{l-1})$. If $\mathsf{mbs}(f) \leq M(f)^\alpha$ then $\mathsf{fmbs}(f) = O(M(f)^{2\alpha})$. Furthermore, if $M(1-f) = O(M(f))$ then $\mathsf{fmbs}(f) = O(M(f)^\alpha)$.*

We need another extra condition on M, $M(1-f) = O(M(f))$, as compared to Kulkarni and Tal [13]. However, most of the complexity measures should satisfy this condition trivially.

It is tempting to apply this corollary on $\log(\mathsf{spar}(f))$ and try to improve the upper bound on $\mathsf{fmbs}(f)$ in terms of $\log(\mathsf{spar}(f))$. We show a negative result here: $\log(\mathsf{spar}(f))$ does not behave well under composition, indeed the sparsity of a composed function $f \circ g$ can depend on the degree of f which can be much larger than the logarithm of the sparsity. Hence, Corollary 1 can not be used to improve the upper bound on $\mathsf{fmbs}(f)$. For the counterexample, please see Section 3.1.

Although our attempt to improve the bound on $\mathsf{fmbs}(f)$ did not bear success; we asked, is it possible to improve the log-sparsity upper bound on $\mathsf{mbs}(f)$? For the question of improving the relation $\mathsf{mbs}(f) = O(\log(\mathsf{spar}(f))^2)$, we show that it will improve the upper bound on sensitivity in terms of degree (a central question in this field).

Theorem 2. *If there exists an α s.t. for every Boolean function $f : \{0,1\}^n \to \{0,1\}$, $\mathsf{mbs}(f) = O(\log^\alpha \mathsf{spar}(f))$, then for every Boolean function $f : \{0,1\}^n \to \{0,1\}$ $\mathsf{s}(f) = O(\deg^\alpha(f))$.*

The converse of this result follows from the proof of $\mathsf{mbs}(f) = O(\log(\mathsf{spar}(f))^2)$ in [11]. Nisan and Szegedy [16] showed that $\mathsf{s}(f) = O(\deg(f)^2)$. However, the best possible separation known is due to Kushilevitz (described in [17]) giving a function f such that $\mathsf{s}(f) = \Omega(\deg(f)^{1.63})$. So, our result implies that the best possible bound on monotone block sensitivity in terms of logarithm of sparsity cannot be better than $\mathsf{mbs}(f) = O(\log(\mathsf{spar}(f))^{1.63})$.

Going further, we ask if these bounds can be improved for a class of functions instead of a generic Boolean function? Buhrman and de Wolf [5] proved that the log-rank conjecture holds when the outer function is monotone or symmetric. It turns out that all these monotone measures are equal for these classes of functions.

Theorem 3. *If $f : \{0,1\}^n \to \{0,1\}$ is either symmetric or monotone, then*

$$\mathsf{ms}(f) = \mathsf{mbs}(f) = \mathsf{fmbs}(f) = \mathsf{MCC}(f).$$

This implies an upper bound of $O(\log^2(\mathsf{spar}(f)))$ on MCC for these functions. For symmetric functions, this bound can be improved to $O(\log \mathsf{spar}(f))$ by combining the above relation with the upper bound on communication complexity for the corresponding AND-functions [5]. Moreover, Buhrman and de Wolf [5] showed that $\mathsf{mbs}(f) = \Omega(\log(\mathsf{spar}(f))/\log n)$, which implies that the upper bound is essentially tight.

Organization: In Section 2 we recall the definitions of standard Boolean complexity measures as well as state their monotone analogues. In Section 3 we will give the proof ideas of our results. This section also contains the counterexample which shows that the relationship between $\mathsf{fmbs}f$ and $\log(\mathsf{spar}(f))$ can't be improved using this method (Section 3.1). Section 4 contains the conclusion and some related open problems to pursue.

Appendix A and Appendix B contain the complete proof of Theorem 1. Appendix C contains the equivalence between the problem of upper bounding $\mathsf{mbs}(f)$ in terms of $\log(\mathsf{spar}(f))$ and the well-studied problem of upper bounding $\mathsf{s}(f)$ in terms of $\deg(f)$ (Theorem 2). Finally, Appendix D shows that for the common classes of symmetric and monotone Boolean functions MCC and mbs are the same (Theorem 3). In Appendix E, we give an overview of the present scenario of the relationships between monotone measures.

2 Preliminaries

For the rest of the paper, f denotes a Boolean function $f : \{0,1\}^n \to \{0,1\}$ if not stated otherwise. We start by introducing the following notations that will be used in the paper:

- $[n]$ denotes the $\{1, 2, ..., n\}$. For a set $C \subseteq [n]$, $|C|$ denotes its cardinality.
- For a string $x \in \{0,1\}^n$, its support is defined as $\text{supp}(x) := \{i : x_i = 1\}$ and $|x| := |\text{supp}(x)|$ denotes its Hamming weight .
- For a string $x \in \{0,1\}^n$, $x^{\oplus i}$ denotes the string obtained by flipping the i^{th} bit of the string x.
- For a string $x \in \{0,1\}^n$ and a $B \subseteq [n]$, x^B denotes the string obtained by flipping the input bits of x that correspond to B.
- Every Boolean function $f : \{0,1\}^n \to \{0,1\}$ can be expressed as a polynomial over $\mathbb{R}$, $f(x) = \sum_{S \subseteq [n]} \alpha_S \prod_{i \in S} x_i$. The sparsity of f is defined as $\mathsf{spar}(f) := |\{S \neq \emptyset : \alpha_S \neq 0\}|$ and the degree of f is defined as $\deg(f) := \max\limits_{S \subseteq [n] : \alpha_S \neq 0} |S|$.

Having introduced the notations, we now recall the definitions of standard Boolean complexity measures.

Definition 1 (Sensitivity). *For an input $x \in \{0,1\}^n$ the i^{th} bit is said to be sensitive for x if $f(x^{\oplus i}) \neq f(x)$. The sensitivity of x w.r.t f is defined as*

$$\mathsf{s}(f, x) := |\{i \in [n] : f(x^{\oplus i}) \neq f(x)\}|,$$

while the sensitivity of f is defined as

$$\mathsf{s}(f) := \max_{x \in \{0,1\}^n} \mathsf{s}(f, x).$$

Definition 2 (Block Sensitivity). *For an input $x \in \{0,1\}^n$, a subset $B \subseteq [n]$ is said to be a sensitive block for x w.r.t f if $f(x^B) \neq f(x)$. The block sensitivity of f at x, denoted by $\mathsf{bs}(f, x)$, is defined as*

$$\mathsf{bs}(f, x) = \max\{k | \exists B_1, \ldots, B_k \textit{ with } B_i \cap B_j = \emptyset \textit{ for } i \neq j \textit{ and } f(x^{B_i}) \neq f(x)\}.$$

Block sensitivity of f is defined as:

$$\mathsf{bs}(f) := \max_{x \in \{0,1\}^n} \mathsf{bs}(f,x).$$

Fractional block sensitivity (fbs) is obtained by allowing fractional weights on sensitive blocks.

Definition 3 (Fractional Block Sensitivity). *Let $W(f,x) := \{B \subseteq [n] : f(x^B) \neq f(x)\}$ denote the set of all sensitive blocks for the input $x \in \{0,1\}^n$. The fractional block sensitivity of f at x, denoted by $\mathsf{fbs}(f,x)$ is the value of the linear program:*

$$\mathsf{fbs}(f,x) := \max \sum_{w \in W(f,x)} b_w$$

s.t.

$$\forall i \in [n], \sum_{w \in W(f,x): i \in w} b_w \leq 1$$

and

$$\forall w \in W(f,x),\ b_w \in [0,1].$$

The fractional block sensitivity of f is defined as:

$$\mathsf{fbs}(f) := \max_{x \in \{0,1\}^n} \mathsf{fbs}(f,x).$$

Note that restricting the linear program for $\mathsf{fbs}(f,x)$ to only integral values gives $\mathsf{bs}(f,x)$.

Definition 4 (Certificate Complexity). *For a function f and an input $x \in \{0,1\}^n$, a subset $C \subset [n]$ is said to be a certificate for x if for all $y \in \{y \in \{0,1\}^n : \forall i \in C,\ x_i = y_i\}$ we have $f(x) = f(y)$. For a function f and an input $x \in \{0,1\}^n$ the certificate complexity of f at x, denoted by $\mathsf{C}(f,x)$, is defined as:*

$$\mathsf{C}(f,x) := \min_{C: C \text{ is a certificate for } x} |C|.$$

The certificate complexity of f is defined as:

$$\mathsf{C}(f) := \max_{x \in \{0,1\}^n} \mathsf{C}(f,x).$$

The fractional measures fbs and FC were introduced in [19]. There it was observed that for all $x \in \{0,1\}^n$ we have : $\mathsf{fbs}(f,x) = \mathsf{FC}(f,x)$ since the linear program for Fractional Certificate Complexity and Fractional Block Sensitivity are the primal-dual of each other and are also feasible.

For each of these standard measures, the analogous monotone versions can be defined by restricting functions f to the positions in the support of a given input $x \in \{0,1\}^n$. Formally, for a function $f : \{0,1\}^n \to \{0,1\}$ and an input $x \in \{0,1\}^n$ let f_x denote the function f obtained by restricting f to the set $\{y \in \{0,1\}^n : \forall i \in \mathrm{supp}(x),\ y_i = 1\}$.

Definition 5 (Monotone Sensitivity). *The monotone sensitivity for x is defined as* $\mathsf{ms}(f,x) := \mathsf{s}(f_x, 0^{n-|x|})$ *while the monotone sensitivity for f is defined as*

$$\mathsf{ms}(f) := \max_{x\in\{0,1\}^n} \mathsf{ms}(f,x).$$

Definition 6 (Monotone Block Sensitivity). *The monotone block sensitivity of a function f at an input $x \in \{0,1\}^n$ is defined as* $\mathsf{mbs}(f,x) := \mathsf{bs}(f_x, 0^{n-|x|})$ *while the monotone block sensitivity of f is defined as:*

$$\mathsf{mbs}(f) = \max_{x\in\{0,1\}^n} \mathsf{mbs}(f,x).$$

Similar to block sensitivity, fractional block sensitivity of f can be extended to the monotone setting by defining the linear program over the sensitive monotone blocks i.e. sensitive blocks containing only 0's.

Definition 7 (Fractional Monotone Block Sensitivity). *For a function f the fractional monotone block sensitivity at an input $x \in \{0,1,\}^n$ is defined as:* $\mathsf{fmbs}(f,x) := \mathsf{fbs}(f_x, 0^{n-|x|})$ *and the fractional monotone block sensitivity of f is defined as:*

$$\mathsf{fmbs}(f) := \max_{x\in\{0,1\}^n} \mathsf{fmbs}(f,x).$$

Certificate complexity can also be extended to the monotone setting by counting only the zero entries in the certificate. The monotone analogue of certificate complexity was introduced in [11] as hitting set complexity (it can be viewed as a hitting set for system of monomials). Formally,

Definition 8 (Monotone Certificate Complexity/Hitting Set Complexity). *For a function f and an input $x \in \{0,1\}^n$ the hitting set complexity for x is defined as:*

$$\mathsf{MCC}(f,x) := \mathsf{C}(f_x, 0^{n-|x|}),$$

while the hitting set complexity of the function f is defined as:

$$\mathsf{MCC}(f) := \max_{x\in\{0,1\}^n} \mathsf{MCC}(f,x).$$

Since bs allows only integer solutions to fbs linear program, and C only allows integer solutions to the dual linear program [19],

$$\mathsf{bs}(f_x, 0^{n-|x|}) \le \mathsf{fbs}(f_x, 0^{n-|x|}) \le \mathsf{C}(f_x, 0^{n-|x|}),$$

By similar arguments,

$$\mathsf{mbs}(f,x) \le \mathsf{fmbs}(f,x) \le \mathsf{MCC}(f,x).$$

Instead of taking maximum over all inputs, these measures can be defined for a certain output too. In other words, for a complexity measure $M \in \{\mathsf{s}, \mathsf{bs}, \mathsf{fbs}, \mathsf{C}, \mathsf{ms}, \mathsf{mbs}, \mathsf{fmbs}, \mathsf{MCC}\}$ and $b \in \{0,1\}$,

$$M^b(f) := \max_{x\in f^{-1}(b)} M(f,x).$$

3 Proof Outline

First, we outline the ideas for the proofs of Theorem 2 and Theorem 3. Subsequently, we will give proof outline for our main result, Theorem 1.

Proof idea of Theorem 2 We would like to prove that $\mathsf{s}(f) = O(\deg^{\alpha}(f))$ for any Boolean function f (given that $\mathsf{mbs}(g) = O(\log^{\alpha} \mathsf{spar}(g))$ for all Boolean functions g). The idea is to convert f into $\widetilde{f}$ by shifting the point with maximum sensitivity to 0^n; this transformation can only decrease the degree and $\mathsf{mbs}(f)$ is higher than $\mathsf{s}(\widetilde{f})$.

The rest is accomplished by using the fact that sparsity is at most exponential in degree. This is shown for Boolean functions with $\{-1, 1\}$ domain first using Parseval's identity, and then it can be translated for Boolean functions with $\{0, 1\}$ domain.

For the interest of space we will present the proof of Theorem 2 in Appendix C.

Proof idea of Theorem 3 We deal with the cases of monotone and symmetric boolean functions separately.

For monotone boolean functions, the idea for showing equality between the monotone versions of the standard boolean complexity measures is similar to the approach used for the standard complexity measures i.e. we consider a string x which achieves the hitting set complexity $\mathsf{MCC}(f) = \mathsf{MCC}(f, x)$ with C as one of its witness. Now, using x and C we construct another input $x^{'}$ with $\text{supp}(x) \subseteq \text{supp}(x^{'})$ and $\text{supp}(x^{'}) \cap C = \phi$ s.t. every bit $i \in C$ is sensitive for f_x at $x^{'}$. Hence leading to $\mathsf{MCC}(f, x) \leq \mathsf{ms}(f, x^{'}) \leq \mathsf{ms}(f)$.

Now for the case of symmetric boolean functions, we show that there exists an input $z \in \{0, 1\}^n$ s.t. $\mathsf{MCC}(f) = \mathsf{MCC}(f, z)$ and $\mathsf{MCC}(f, z) = n - |z|$, where $|z|$ is the Hamming weight of z i.e. $|z| = \text{supp}(f)$. But this implies $\mathsf{ms}(f, z) = \mathsf{s}(f_z, 0^{n-|z|}) = n - |z| = \mathsf{MCC}(f, z) = \mathsf{MCC}(f)$.

We will give a complete proof in Appendix D.

3.1 fmbs versus mbs

Let us move to the proof idea of Theorem 1, we essentially follow the same proof outline as [19]. Theorem 1 proves that for any $f : \{0, 1\}^n \to \{0, 1\}$, the ratio $\frac{\mathsf{fmbs}(f^l)}{\mathsf{mbs}(f^l)}$ is bounded above by a function of just n (and independent of l). In other words, composition make fmbs and mbs equal in the asymptotic sense.

We will be considering the case of monotone functions and non-monotone functions separately. While the case for monotone functions is handled easily due to Theorem 3 (we have a stronger relation $\frac{\mathsf{fmbs}(f^l)}{\mathsf{mbs}(f^l)} = 1$), most of the work is done for the case when f is non-monotone.

Proof outline of Theorem 1

From the discussion above, assume that $f : \{0,1\}^n \to \{0,1\}$ is a non-monotone Boolean function. We want to show that

$$\mathsf{mbs}(f^l) \geq p(n)\ \mathsf{fmbs}(f^l),$$

for some function $p(n)$ and big enough l.

Similar to $\mathsf{fbs}(f)$, $\mathsf{fmbs}(f)$ can also be written as a fractional relaxation (linear program) of an integer program for $\mathsf{mbs}(f)$. The proof converts a feasible solution of the linear program for $\mathsf{fmbs}(f^{l+1})$ into a feasible solution of $\mathsf{mbs}(f^{l+1})$ without much loss in the objective value, bounding $\frac{\mathsf{mbs}(f^{l+1})}{\mathsf{fmbs}(f^{l+1})}$ in terms of $\frac{\mathsf{mbs}(f^{l})}{\mathsf{fmbs}(f^{l})}$:

$$\mathsf{mbs}(f^{l+1}) \geq \mathsf{fmbs}(f^{l+1}) \frac{\mathsf{mbs}(f^l)}{\mathsf{fmbs}(f^l)} \alpha_l, \tag{1}$$

where α_l s.t. $\prod_{l=1}^{\infty} \alpha_l = \Omega(1)$. This finishes the proof by taking large enough l. We are left with proving Equation 1 for some α_l's.

Remember, the idea is to convert a solution of $\mathsf{fmbs}(f^{l+1})$ into a solution of $\mathsf{mbs}(f^{l+1})$. Let $x := (x^1, x^2, ..., x^n) \in \{0,1\}^{n^{l+1}}$ be the input s.t. $\mathsf{fmbs}(f^{l+1}) = \mathsf{fmbs}(f^{l+1}, x)$ where $x^1, x^2, ..., x^n \in \{0,1\}^{n^l}$. The input $y \in \{0,1\}^n$ be the n-bit string corresponding to x i.e. $\forall\ i \in [n],\ y_i := f^l(x^i)$. We know that $f^{l+1}(x) = f(y)$.

Let $\{B_1, ..., B_k\}$ be the set of all minimal monotone blocks for f at y. A minimal monotone block, say $B = \{i_1, i_2, \cdots, i_k\}$, of y gives minimal monotone blocks for f^l at inputs $x^{i_1}, x^{i_2}, \cdots, x^{i_k}$. Observe that the total weight contributed by any block B_i in the linear program for f^{l+1} will become feasible for the following linear program:

$$\max \sum_{i=1}^{k} w_i,$$

s.t.

$$\sum_{j : i \in B_j} w_j \leq \mathsf{fmbs}(f^l, x^i),\ \forall\ i \in [n],$$

$$w_j \geq 0, \forall j \in [k].$$

A small modification to these weights (multiplying by a quantity closely related to $\frac{\mathsf{mbs}(f^l)}{\mathsf{fmbs}(f^l)}$ and taking their integer part) gives the solution of the following integer program (notice that $\mathsf{mbs}(f^l)$ is taken over another suitable input $\widehat{x}$):

$$\max \sum_{i=1}^{k} w_i,$$

s.t.

$$\sum_{j : i \in B_j} w_j \leq \mathsf{mbs}(f^l, \widehat{x}^i),\ \forall\ i \in [n],$$

$$w_j \in \{0, 1, 2, \ldots, \mathsf{mbs}(f^l)\}, \forall j \in [k].$$

Let $\{w_i^{'}\}$ be the solution of the program above. Using this assignment $w_i^{'}$ we can construct $\sum_{i=1}^{k} w_i^{'}$ many disjoint monotone sensitive blocks of f^{l+1} (see Appendix A for this construction).

It can be shown that the objective value of the obtained solution satisfies,

$$\mathsf{mbs}(f^{l+1}) \geq \mathsf{fmbs}(f^{l+1}) \frac{\mathsf{mbs}(f^l)}{\mathsf{fmbs}(f^l)} - 2^n.$$

Here, the term 2^n appears because we take the integer part of a fractional solution to construct $\{w_i^{'}\}$. This inequality can be converted into Equation 1 by using properties of composition of fractional monotone block sensitivity and some minor assumptions on $\mathsf{mbs}(f)$. We present the complete proof of Theorem 1 in Appendix A.

Implications of Theorem 1

One of the reason Theorem 1 is interesting because it provides a way of lifting upper bounds on $\mathsf{mbs}(f)$ to upper bounds on $\mathsf{fmbs}(f)$. This was observed by [13] for the standard setting (bs and fbs), using which they showed the quadratic relation between $\mathsf{fbs}(f)$ and $\deg(f)$ i.e. $\mathsf{fbs}(f) = O(\deg^2(f))$. This was an improvement over $\mathsf{fbs}(f) \leq \mathsf{C}(f) = O(\deg^3(f))$ [15].

Similarly, we can do the lifting for $\mathsf{mbs}(f)$ and $\mathsf{fmbs}(f)$ which we have stated in Corollary 1. We present the proof of Corollary 1 in section A.

We now give an example showing that $\log \mathsf{spar}(f^2)$ may be exponentially larger than $(\log \mathsf{spar}(f))^2$ and so Corollary 1 cannot be applied to $\log \mathsf{spar}$. For any Boolean functions f and g, $\mathsf{spar}(f \circ g) \geq (\mathsf{spar}(g) - 1)^{\deg(f)}$ (see, for instance, [14][5]). In particular, when $\mathsf{spar}(g) \geq 3$, $\log \mathsf{spar}(f \circ g) \geq \deg(f)$. So any function f satisfying $\mathsf{spar}(f) \geq 3$ and $\deg(f) = 2^{\Omega(\log \mathsf{spar}(f))}$ gives us the desired separation. For instance, we may take,

$$f(x_1, x_2, \ldots, x_n) = \mathsf{OR}(\mathsf{AND}(x_1, x_2, \ldots, x_{n/2}), \mathsf{AND}(x_{n/2+1}, x_{n/2+2}, \ldots, x_n))$$

which has degree n and sparsity only 3.

4 Conclusion

In the present work we studied the behaviour of different monotone complexity measures and their relation with one another. The relations between these measures are natural questions by themselves; on top of that, they can potentially be used to improve the upper bound on deterministic communication complexity in terms of logarithm of sparsity.

[5] Their proof is stated for sparsity in the Fourier representation, but is readily seen to work for block composition of arbitrary multilinear polynomials.

To summarize our results, we were able to show a better upper bound on $\mathsf{MCC}(f)$ in terms of $\log(\mathsf{spar}(f))$ for monotone and symmetric Boolean functions. It will be interesting to find other class of functions for which the upper bound can be improved. Our result that the mbs vs. $\log(\mathsf{spar})$ question is equivalent to the s vs. deg question, might give another direction to attack this old open question.

This work also showed that the ratio $\frac{\mathsf{fmbs}(f^l)}{\mathsf{mbs}(f^l)}$ is independent of the iteration number l. Even though we were not able to use it to show $\mathsf{fmbs} = O(\log(\mathsf{spar}(f))^2)$, this results seems to be of independent interest in terms of behavior of these monotone measures.

Some of the other open questions from this work are listed below.

Open question 4 *Can we prove that for any Boolean function* $f : \{0,1\}^n \to \{0,1\}$*,* $\mathsf{fmbs}(f) = O(\log(\mathsf{spar}(f))^2)$*?*

Another possible open question in this direction is asking about best possible separation between fmbs and $\log(\mathsf{spar})$. Right now it is known that for all Boolean function $\mathsf{fmbs}(f) = O(\log(\mathsf{spar}(f))^4)$ and the best known separation is due to Kushilevitz (described in [17]), giving a function f such that $\mathsf{s}(f) = \Omega(\deg(f)^{1.63})$. Can we give a better separation for monotone measures?

Open question 5 *Does there exist a function* f *for which,* $\mathsf{fmbs}(f) = \Omega(\log(\mathsf{spar}(f))^{\alpha})$ *for some* $\alpha > 1.63$*?*

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A Proof of Theorem 1 and Corollary 1

Proof (Proof of Theorem 1).
If $f : \{0,1\}^n \to \{0,1\}$ is monotone then f^l is also monotone. Hence from Theorem 3 it follows that $\mathsf{mbs}(f^l) = \mathsf{MCC}(f^l)$ which gives $\mathsf{mbs}(f^l) = \mathsf{fmbs}(f^l)$.

Now we consider f to be a non-monotone Boolean function. We consider two sub cases $\mathsf{mbs}(f) = 1$ and $\mathsf{mbs}(f) \geq 2$ separately. The sub case of f being non-monotone with $\mathsf{mbs}(f) = 1$ does not arise in the proof of fbs and bs ratio. This is because $\mathsf{bs}(f) \geq 2$ for every non-monotone function f.

If $\mathsf{mbs}(f) = 1$ and if $\mathsf{mbs}(f^l) = 1$ for all $l \geq 1$ then using the fact that $\mathsf{fmbs}(f) = O(\mathsf{mbs}^2(f))$ we get $\mathsf{fmbs}(f^l)/\mathsf{mbs}(f^l) = O(1)$. If the aforementioned condition does not hold i.e. there exists a $k \in \mathbb{N}$ s.t. $\mathsf{mbs}(f^k) \geq 2$ then what remains to show is that $\mathsf{fmbs}(f^l)/\mathsf{mbs}(f^l) \leq p(n)$ for all $l \geq k$. It follows that the argument for this part is similar to the case when f is non monotone and $\mathsf{mbs}(f) \geq 2$.

To prove the theorem for non-monotone functions and $\mathsf{mbs}(f) \geq 2$, we will need several lemmas about the behaviour of these monotone complexity measures under composition.

Lemma 1 ([19]). *For Boolean functions $f : \{0,1\}^n \to \{0,1\}$ and $g : \{0,1\}^m \to \{0,1\}$, if $f(z^n) = g(z^m) = z$ for $z \in \{0,1\}$ then:*

$$\mathsf{fbs}(f \circ g, z^{nm}) \geq \mathsf{fbs}(f, z^n)\mathsf{fbs}(g, z^m).$$

The above observation can be adapted to fmbs^0.

Lemma 2. *If $f : \{0,1\}^n \to \{0,1\}$ and $g : \{0,1\}^m \to \{0,1\}$ are Boolean functions with $\mathsf{fmbs}^0(f) = \mathsf{fmbs}(f,x)$ and $\mathsf{fmbs}^0(g) = \mathsf{fmbs}(g,y)$ then:*

$$\mathsf{fmbs}^0(f \circ g) \geq \mathsf{fmbs}^0(f)\mathsf{fmbs}^0(g).$$

Proof. Consider the inputs $x \in \{0,1\}^n$ and $y \in \{0,1\}^m$ s.t. $\mathsf{fmbs}^0(f) = \mathsf{fmbs}(f,x)$ and $\mathsf{fmbs}^0(g) = \mathsf{fmbs}(g,y)$. As $f_x(0^{n-|x|}) = g_y(0^{n-|y|}) = 0$ hence by Lemma 1 it follows that:

$$\mathsf{fmbs}(f_x \circ g_y, \mathbf{0}) \geq \mathsf{fmbs}^0(f)\mathsf{fmbs}^0(g),$$

where $\mathbf{0}$ is the all zero string in $\{0,1\}^{(n-|x|)(n-|y|)}$.

Fix any $z \in g^{-1}(1)$. (If g is the constant 0 function, then the lemma holds since $\mathsf{fmbs}^0(g) = 0$.) Now, consider the input $\gamma := (\gamma_1, \gamma_2, ..., \gamma_m)$ with $\gamma_1, ..., \gamma_n \in \{0,1\}^m$ defined as:

$$\gamma_i := \begin{cases} z, \ if \ x_i = 1 \\ y, \ otherwise \end{cases}$$

Observe that $\mathsf{fmbs}(f \circ g, \gamma) \geq \mathsf{fmbs}(f_x \circ g_y, \mathbf{0})$, hence giving us the result:

$$\mathsf{fmbs}^0(f \circ g) \geq \mathsf{fmbs}^0(f)\mathsf{fmbs}^0(g).$$

The remaining lemmas given below are proved in the Appendix B.

Lemma 3. *Let f, g be two Boolean function where f is non-monotone and $z \in \{0,1\}$ then,*
1. $\mathsf{mbs}^z(f \circ g) \geq \mathsf{mbs}(g)$,
2. $\mathsf{fmbs}^z(f \circ g) \geq \mathsf{fmbs}(g)$.

Lemma 4. *For Boolean functions $f : \{0,1\}^n \to \{0,1\}$ and $g : \{0,1\}^m \to \{0,1\}$ we have:*

$$\mathsf{mbs}^z(f \circ g) \geq \max\{\mathsf{mbs}^z(f)\mathsf{mbs}^0(g), \mathsf{bs}^z(f)\min\{\mathsf{mbs}^0(g), \mathsf{mbs}^1(g)\}\}.$$

Corollary 2. *Let f be a non-monotone Boolean function with $z \in \{0,1\}$ then, the sequence $\{\mathsf{mbs}^z(f^l)\}_{l\in\mathbb{N}}$ is monotone increasing and if $\mathsf{mbs}(f) \geq 2$ then for every $z \in \{0,1\}$ the sequence $\{\mathsf{mbs}^z(f^l)\}_{l\in\mathbb{N}}$ tends to infinity.*

We are now in a position to prove Theorem 1. To recall, Theorem 1 states that for any function $f : \{0,1\}^n \to \{0,1\}$ the ratio $\frac{\mathsf{fmbs}(f^l)}{\mathsf{mbs}(f^l)}$ is independent of l.

Remember that we are left with the case when f is not monotone and $\mathsf{mbs}(f) \geq 2$. If we show that there exists a sequence $\{r_l\}_{l\geq 1}$ s.t. for all $l \geq 1$ we have:

$$\frac{\mathsf{mbs}(f^l)}{\mathsf{fmbs}(f^l)} \geq r_l \geq 1/p(n),$$

then we will be done.

Now consider the sequence:

$$r_l := \min\{r_l^0,\ r_l^1\},$$

where $r_l^z := \frac{\mathsf{mbs}^z(f^l)}{\mathsf{fmbs}^z(f^l)}$ for $z \in \{0,1\}$. Taking $z' \in \{0,1\}$ as $\mathsf{fmbs}(f^l) = \mathsf{fmbs}^{z'}(f^l)$, we get:

$$\frac{\mathsf{mbs}(f^l)}{\mathsf{fmbs}(f^l)} = \frac{\mathsf{mbs}(f^l)}{\mathsf{fmbs}^{z'}(f^l)} \geq \frac{\mathsf{mbs}^{z'}(f^l)}{\mathsf{fmbs}^{z'}(f^l)} = r_l^{z'} \geq r_l,$$

i.e. $\frac{\mathsf{mbs}(f^l)}{\mathsf{fmbs}(f^l)}$ has r_l as its lower bound. What remains to show is that for all $l \geq 1$:

$$r_l \geq 1/p(n).$$

Now, notice it is sufficient to show that for $l \geq l_0$, $r_l \geq 1/p(n)$, where l_0 is a parameter we fix later.

To this effect, we show that for $l \geq l_0$:

$$r_{l+1} \geq r_l(1 - 2^{-1-\lfloor\frac{l-(l_0+1)}{2}\rfloor}). \tag{2}$$

Equation 2 will complete the proof because it implies that for all $s \geq l_0$:

$$r_s \geq r_{l_0} \cdot \prod_{i=1}^{\infty}(1-2^{-i})^2 \underset{\textit{Proposition 1}}{\geq} r_{l_0} \cdot 1/e^4 \geq 1/e^4 \cdot q(n),$$

where $q(n)$ is any function of n s.t. $r_{l_0} \geq q(n)$.

For the rest of the proof, our aim will be to show Equation 2. We do this by rounding a solution of the linear program for $\mathsf{fmbs}(f^{l+1})$ to a feasible solution for the integer program corresponding to $\mathsf{mbs}(f^{l+1})$. To accomplish this goal, we look at the composed function f^{l+1} as $f \circ f^l$, which seems natural given the fact that we have a better understanding of the function f.

Let $x := (x^1, x^2, ..., x^n) \in \{0,1\}^{n^{l+1}}$ be an input for which $\mathsf{fmbs}^z(f^{l+1}) = \mathsf{fmbs}(f^{l+1}, x)$ where $x^1, x^2, ..., x^n \in \{0,1\}^{n^l}$ and let $y \in \{0,1\}^n$ be the n-bit string corresponding to x i.e. $\forall\ i \in [n],\ y_i := f^l(x^i)$. As already mentioned, we will convert the optimal solution x for $\mathsf{fmbs}(f^{l+1})$ to a feasible solution $\widehat{x}$ (mentioned later) for $\mathsf{mbs}(f^{l+1})$.

Let $\{B_1, ..., B_k\}$ be the set of all minimal sensitive blocks for y. What now needs to be observed is that any minimal monotone sensitive block for f^{l+1} corresponds to a minimal sensitive block B_i for f. The consequence of this observation is that every feasible solution of $\mathsf{fmbs}(f^{l+1})$ is a feasible solution of the following linear program:

$$\max \sum_{i=1}^{k} w_i,$$

s.t.

$$\sum_{j:i \in B_j} w_j \le \mathsf{fmbs}(f^l, x^i), \ \forall\, i \in [n],$$

$$w_j \ge 0, \forall j \in [k].$$

We now convert an optimal solution of $\mathsf{fmbs}(f^{l+1})$ to a feasible solution of $\mathsf{mbs}(f^{l+1})$. Let $\{w_j^*\}_{j \in [k]}$ be an optimal assignment of weights for the above linear program and let $w_j^{'} := \lfloor w_j^* \cdot r_l \rfloor$. Define $\widehat{x} := (\widehat{x}^1, \widehat{x}^2, ..., \widehat{x}^n)$, where $\mathsf{mbs}(f^l, \widehat{x}^i) := \mathsf{mbs}^{y_i}(f^l)$. It can be observed that for all $j \in [k]$ we have:

$$\sum_{j:i \in B_j} w_j^{'} \le \sum_{j:i \in B_j} w_j^* \cdot r_l \le \mathsf{fmbs}(f^l, x^i) r_l \le \mathsf{fmbs}(f^l, x^i) \cdot \frac{\mathsf{mbs}^{y_i}(f^l)}{\mathsf{fmbs}^{y_i}(f^l)} \le \mathsf{mbs}(f^l, \widehat{x}^i),$$

where the last inequality follows from the fact that $f^l(x^i) = f^l(\widehat{x}^i) = y_i$.

Now consider the following integer program:

$$\max \sum_{i=1}^{k} w_i,$$

s.t.

$$\sum_{j:i \in B_j} w_j \le \mathsf{mbs}(f^l, \widehat{x}^i), \ \forall\, i \in [n],$$

$$w_j \in \{0, 1, 2, \dots, \mathsf{mbs}(f^l)\}, \forall j \in [k].$$

Clearly, $w^{'}$ forms a feasible solution for the above mentioned integer linear program.

We claim that using the assignment $w_i^{'}$ defined above we can construct $\sum_{i=1}^{k} w_i^{'}$ many disjoint monotone sensitive blocks for $\widehat{x}$, which would imply $\sum_{i=1}^{k} w_i^{'} \le \mathsf{mbs}(f^{l+1}, \widehat{x})$.

We argue this as follows, consider the minimal monotone sensitive block B_1 for y and to simplify the discussion assume that $B_1 := \{i_1,\ i_2\}$. Now pick the i_1^{th} copy of f^l. Consider $w_1^{'}$ many disjoint monotone blocks for x^{i_1} and denote them by $B_{i_1,1}^1,\ B_{i_1,2}^1, \cdots, B_{i_1,w_1^{'}}^1$. Similarly consider $w_1^{'}$ many disjoint monotone sensitive blocks for x^{i_2}. Observe that each of the monotone blocks

$B^1_{i_1,1} \cup B^1_{i_2,1},\ B^1_{i_1,2} \cup B^1_{i_2,2}, ...,\ B^1_{i_1,w_1'} \cup B^1_{i_2,w_1'}$ are sensitive for the input $\widehat{x}$ and are pairwise disjoint.

This implies:

$$\mathsf{fmbs}^z(f^{l+1})r_l \leq \sum_{i=1}^{k} w_i^* r_l \leq \sum_{i=1}^{k}(w_i^{'}+1) \leq \mathsf{mbs}(f^{l+1}, \widehat{x}) + 2^n \leq \mathsf{mbs}^z(f^{l+1}) + 2^n,$$

where the last inequality follows from the fact that $f^{l+1}(\widehat{x}) = z$ and by the fact that monotone blocks for y are subsets of $[n]$.

Using the aforementioned inequality, we get:

$$\begin{aligned} r^z_{l+1} &= \frac{\mathsf{mbs}^z(f^{l+1})}{\mathsf{fmbs}^z(f^{l+1})} \geq r_l - \frac{2^n}{\mathsf{fmbs}^z(f^{l+1})} \\ &= r_l\left(1 - \frac{2^n}{\mathsf{fmbs}^z(f^{l+1})} \cdot \frac{\mathsf{fmbs}^{z^{'}}(f^l)}{\mathsf{mbs}^{z^{'}}(f^l)}\right) \underset{Lemma\ 3}{\geq} r_l\left(1 - \frac{2^n}{\mathsf{mbs}^{z^{'}}(f^l)}\right) \end{aligned} \quad (3)$$

where $z^{'} = \arg\min_{z\in\{0,1\}} r^z_l$.

We fix l_0 to be the minimum integer s.t. $\mathsf{mbs}(f^{l_0}) \geq 2.2^n$. This gives us:

$$\begin{aligned} \mathsf{mbs}^{z^{'}}(f^l) &\underset{Lemma\ 4}{\geq} \mathsf{mbs}^{z^{'}}(f^{l-(l_0+1)}) \cdot \min_{b\in\{0,1\}} \mathsf{mbs}^b(f^{l_0+1}) \\ &\underset{Corollary\ 2}{\geq} 2^{\lfloor \frac{l-(l_0+1)}{2} \rfloor} \cdot (2 \cdot 2^n). \end{aligned}$$

Putting the value of $\mathsf{mbs}^{z^{'}}(f^l)$ in Equation 3 gives us Equation 2, completing the proof.

Proof (Proof of Corollary 1).

We will derive the lifting for fmbs by using a lifting for fmbs^0. Formally, if $\mathsf{mbs}(f) = O(M(f)^\alpha)$, where the complexity measure $M(.)$ composes, then $\mathsf{fmbs}^0(f) = M(f)^\alpha$.

Let $\mathsf{fmbs}^0(f) > M(f)^\alpha$ i.e. $\mathsf{fmbs}^0(f) = M(f)^\alpha + \epsilon$ for some $\epsilon > 0$. This implies,

$$\mathsf{fmbs}^0(f)^l = (M(f)^\alpha)^l(1+\epsilon^{'})^l,$$

where $\epsilon^{'} := \epsilon/M(f)^\alpha$,

$$\mathsf{fmbs}^0(f^l) \geq \mathsf{fmbs}^0(f)^l \geq (M(f)^\alpha)^l(1+\epsilon^{'})^l.$$

Using Lemma 2 and Theorem 1, we get:

$$p(n)M(f^l)^\alpha \geq p(n)\mathsf{mbs}(f^l) \geq \mathsf{fmbs}^0(f^l) \geq M(f)^{l\alpha}(1+\epsilon^{'})^l.$$

This implies,

$$p(n) \geq (1+\epsilon^{'})^l$$

Which is a contradiction for a fixed n and a sufficiently large l.

Now using the above lifting for $\mathsf{fmbs}^0(f)$ we derive the lifting for $\mathsf{fmbs}(f)$.

Let $\mathsf{fmbs}(f) > M(f)^{2\alpha}$ i.e. $\mathsf{fmbs}(f) = M(f)^{2\alpha} + \epsilon$ for some $\epsilon > 0$. This implies,

$$\mathsf{fmbs}(f)^l = (M(f)^{2\alpha})^l (1+\epsilon^{'})^l,$$

where $\epsilon^{'} := \epsilon / M(f)^{2\alpha}$. Now by Lemma 3 it follows that:

$$\mathsf{fmbs}^0(f^2)^l \geq \mathsf{fmbs}(f)^l \geq (M(f)^{2\alpha})^l (1+\epsilon^{'})^l.$$

Using Lemma 2 and Theorem 1, we get:

$$p(n)M(f)^{2l\alpha} \geq p(n)M(f^{2l})^{\alpha} \geq p(n)\mathsf{mbs}(f^{2l}) \geq \mathsf{fmbs}^0(f^{2l}) \geq \mathsf{fmbs}(f)^l \geq M(f)^{2l\alpha}(1+\epsilon^{'})^l.$$

This implies,

$$p(n) \geq (1+\epsilon^{'})^l$$

Which is a contradiction for a fixed n and a sufficiently large l.

Now, if the complexity measure M satisfies the condition $M(1-f) = O(M(f))$ then using the fact that $\mathsf{fmbs}^1(f) = \mathsf{fmbs}^0(1-f)$, we have:

$$\mathsf{fmbs}(f) = \max\{\mathsf{fmbs}^0(f),\ \mathsf{fmbs}^1(f)\} = O(M(f)^{\alpha}).$$

B Results needed for the proof of Theorem 1

We basically give an analog of the identities that hold for standard complexity measures for monotone complexity measures presented in [19]. We start by showing how $\mathsf{MCC}(f)$ and $\mathsf{mbs}(f)$ are related.

Theorem 6. *For a Boolean function $f : \{0,1\}^n \to \{0,1\}$ we have:*

$$\mathsf{MCC}^z(f) \leq \mathsf{mbs}^z(f)\mathsf{ms}^{1-z}(\widetilde{f}),$$

where $\widetilde{f}(x) := f(1-x_1, 1-x_2, ..., 1-x_n)$.

Proof. Let $\mathsf{MCC}(f) = \mathsf{MCC}(f,x)$. We now consider the function f_x at the input $0^{n-|x|}$ along with a set of disjoint minimal sensitive blocks $\{B_1, B_2, ..., B_k\}$ with $k = \mathsf{mbs}(f,x)$. We claim that for every $i \in [k]$, f_x is sensitive on $0^{\oplus B_i}$ at each index $j \in B_i$. If this was not the case then there would exist $B \subsetneq B_i$ s.t. B is a sensitive block for f_x at $0^{n-|x|}$, contradicting the claim that B_i is minimal.

Now, we claim that the set $\cup_{i\in[k]}B_i$ is a certificate for f_x at $0^{n-|x|}$. If this was not the case then we would have obtained a sensitive block B at $0^{n-|x|}$ s.t. $B \cap B_i = \emptyset$ for all $i \in [k]$. This would have contradicted the assumption that $\{B_i : i \in [k]\}$ is a witness for $\mathsf{mbs}(f,x)$. Hence from the above discussion we obtain:

$$\mathsf{MCC}(f) = \mathsf{MCC}(f,x) \leq \sum_{i\in[k]} |B_i| \leq \mathsf{mbs}(f,x)\mathsf{ms}(\widetilde{f}) \leq \mathsf{mbs}(f)\mathsf{ms}(\widetilde{f}).$$

In the following theorem we show that the composition result for Boolean functions f, g (see [19]), can also be extended to mbs.

Theorem 7. *Let $f : \{0,1\}^n \to \{0,1\}$ and $g : \{0,1\}^m \to \{0,1\}$ be Boolean functions then $\mathsf{mbs}(f \circ g) \leq \mathsf{fbs}(f).\mathsf{mbs}(g)$.*

Proof. Let $f : \{0,1\}^n \to \{0,1\}$ and $g : \{0,1\}^m \to \{0,1\}$ be two Boolean functions. Now consider an input $(x^1, x^2, ..., x^n) \in (\{0,1\}^m)^n$ where $x^i := (x^i_1, ..., x^i_m)$ for $i \in [n]$.

Now we wish to calculate $\mathsf{mbs}(f \circ g, x)$. Any minimal monotone block for $f \circ g$ at x is a union of minimal monotone blocks for different g^i where $i \in \mathcal{I} \subseteq [n]$ where $\mathcal{I}$ is a minimal sensitive block for f at $y := (g(x^1), .., g(x^n))$.

If we assume that $\mathcal{B}_1, .., \mathcal{B}_k$ is the set of all minimal sensitive blocks for f at y and by m_j let us denote the number of minimal monotone sensitive blocks for $f \circ g$ that intersects with $\mathcal{B}_j$. As we want to obtain a collection of disjoint monotone sensitive blocks for $f \circ g$ at x hence we have that $\forall i \in [n]$, $\sum_{j:i\in\mathcal{B}_j} m_j \leq \mathsf{mbs}(g, x^i)$. Hence $\mathsf{mbs}(f \circ g, x)$ is equal to the optimal value of the following integer program:

$$\max \sum_{i=1}^{k} m_i,$$

s.t.

$$\sum_{j:i\in\mathcal{B}_j} m_j \leq \mathsf{mbs}(g, x^i),\ \forall i \in [n],$$

$$m_i \in \{0, 1, .., \mathsf{mbs}(g)\}$$

Now, relaxing the above integer program to linear program we get a feasible solution for the linear program corresponding to $\mathsf{fbs}(f, y)$ by simply dividing it by $\mathsf{mbs}(g)$.: This implies,

$$\mathsf{fbs}(f,y) \geq \frac{OPT(LP(\mathsf{mbs}(f \circ g, x)))}{\mathsf{mbs}(g)} \geq \frac{OPT(IP(\mathsf{mbs}(f \circ g, x)))}{\mathsf{mbs}(g)} = \frac{\mathsf{mbs}(f \circ g, x)}{\mathsf{mbs}(g)}$$

Taking x to be the input s.t. $\mathsf{mbs}(f \circ g) = \mathsf{mbs}(f \circ g, x)$ then we get our desired result.

The next lemma shows that how

Proof (Proof of Lemma 3). 1. Let $\mathsf{mbs}(g) = \mathsf{mbs}(g, x)$ and assume that $g(x) = 0$. Now as we know that f is non-monotone hence there exists inputs x^1, x^2, x^3, x^4 s.t. $x^1 < x^2$ and $x^3 < x^4$ and $f(x^1) \neq f(x^2)$ and $f(x^3) \neq f(x^4)$. In fact we can consider the stronger assumption $|x^1 - x^2| = |x^3 - x^4| = 1$ i.e. have hamming distance 1.

For ease of discussion let us assume $f(x^1) = f(x^4) = 0$ and $f(x^2) = f(x^3) = 1$.

Now consider inputs $y^i \equiv (y^i_1, ..., y^i_n)$, $i \in \{1, 2, 3, 4\}$, for $f \circ g$ which are defined as follows:

$$y^i_j := \begin{cases} x & \text{, if } x^i_j = 0 \\ \alpha & \text{, o.w.} \end{cases},$$

where α is any string in $g^{-1}(1)$.

What we claim is that $\mathsf{mbs}(f \circ g, y^1) \geq \mathsf{mbs}(g)$. This is because we can convert the string x^1 to x^2 by flipping the corresponding bits in y^1. As $f \circ g(y^1) = f(x^1) = 0$ hence we have $\mathsf{mbs}^0(f \circ g) \geq \mathsf{mbs}(f \circ g, y^1) \geq \mathsf{mbs}(g)$. Similarly, we can convert the string x^3 to x^4 by flipping the corresponding bits in y^3 to obtain $\mathsf{mbs}^1(f \circ g) \geq \mathsf{mbs}(f \circ g, y^3) \geq \mathsf{mbs}(g)$.

If it was the case that $g(x) = 1$ then the definition of y^i for $i \in \{1,\ 2,\ 3,\ 4\}$ would have been as follows:

$$y^i_j := \begin{cases} x & \text{, if } x^i_j = 1 \\ \alpha & \text{, o.w.} \end{cases},$$

where α is any string in $g^{-1}(0)$. Using a similar argument as done for the case when $g(x) = 0$ we would have obtained the same inequality.

2. Let $\mathsf{fmbs}(f) = \mathsf{fmbs}(f, x)$ and let $g(x) = 0$. Let $x^1, x^2, x^3, x^4 \in \{0, 1\}^n$ and $y^1, y^2, y^3, y^4 \in (\{0, 1\}^n)^m$ be the strings defined in part 1.

Now consider the linear program for $\mathsf{fmbs}(f \circ g, y^1)$ i.e.:

$$\mathsf{fmbs}(f \circ g, y^1) := \sum_{w \in \mathcal{W}(f \circ g)} b_w,$$

s.t.

$$\forall (i, j) \in [n] \times [m], \sum_{w \in \mathcal{W}(f \circ g, y^1) : (i,j) \in w} b_w$$

and,

$$\forall w \in \mathcal{W}(f \circ g, y^1),\ b_w \in [0, 1].$$

Now, as $|x^1 - x^2| = 1$ hence let us consider the copy of g in $f \circ g$, call it $j \in [n]$, which corresponds to the bit where x^1 and x^2 differ. As $f(x^1) \neq f(x^2)$ and $|x^2 - x^1| = 1$ hence all the monotone blocks for the $j - th$ copy of g are monotone blocks for $f \circ g$. Now, let $\{\widehat{b}_{w'} : w' \in \mathcal{W}(g, x)\}$ be a feasible solution corresponding to the linear program for $\mathsf{fmbs}(g, x)$.

Consider the following assignment to weights b_w:

$$b_w := \begin{cases} \widehat{b}_w, & \text{if w} \in \mathcal{W}(g, x) \\ 0, & \text{otherwise} \end{cases}$$

It is easy to verify that that the above assignment for b_w forms a feasible solution for the LP corresponding to $\mathsf{fmbs}(f \circ g, y^1)$. Hence

$$\mathsf{fmbs}^0(f \circ g) \geq \mathsf{fmbs}(f \circ g, y^1) \geq \mathsf{fmbs}(g).$$

Similarly, we can say that:

$$\mathsf{fmbs}^1(f \circ g) \geq \mathsf{fmbs}(f \circ g, y^3) \geq \mathsf{fmbs}(g).$$

Proof (Proof of Lemma 4). Let p^0, p^1 be the inputs for which $\mathsf{mbs}^z(g) = \mathsf{mbs}(g, p^z)$, for $z \in \{0,1\}$. Let $\mathsf{mbs}^z(f) = \mathsf{mbs}(f, y)$. Now consider the input $x \equiv (x^1, x^2, ..., x^n)$ defined as:

$$x^i := p^{y_i}.$$

The $\mathsf{mbs}^z(f \circ g, x)$ is clearly $\geq \mathsf{mbs}^z(f)\mathsf{mbs}^0(g)$. This is because for every disjoint monotone sensitive block corresponding to $\mathsf{mbs}^z(f)$ we have $\mathsf{mbs}^0(g)$ many disjoint sensitive blocks.

Similarly, for every disjoint sensitive block corresponding to $\mathsf{bs}^z(f)$, we have $\min\{\mathsf{mbs}^0(g), \mathsf{mbs}^1(g)\}$ many monotone sensitive blocks. This gives $\mathsf{mbs}^z(f \circ g) \geq \mathsf{bs}^z(f) \min\{\mathsf{mbs}^0(g), \mathsf{mbs}^1(g)\}$.

Proof (Proof of Corollary 2). The monotone increasing part of the lemma is obtained using part 1 of Lemma 3 as follows:

$$\mathsf{mbs}^z(f^l) \geq \mathsf{mbs}(f^{l-1}) \geq \mathsf{mbs}^z(f^{l-1}).$$

Now we show that the sequence diverges if $\mathsf{mbs}(f) \geq 2$. In particular, we show that for all $l \geq 2$ and for all $z \in \{0,1\}$, we have:

$$\mathsf{mbs}^z(f^l) \geq 2^{\lfloor l/2 \rfloor}.$$

We prove it via induction on l. The base case is for $l = 2$. Using Lemma 4 and the assumption that $\mathsf{mbs}(f) \geq 2$ we obtain:

$$\mathsf{mbs}^z(f^2) \geq \mathsf{mbs}(f) \geq 2$$

Now for the inductive step consider $l = k$. Using part 1 of Lemma 3 and we get,

$$\mathsf{mbs}^z(f^k) \geq \mathsf{mbs}^z((f^2)^{\lfloor k/2 \rfloor}) \geq \mathsf{mbs}^z(f^2)\mathsf{mbs}^0(f^{2\lfloor k/2 \rfloor - 2}).$$

Hence by using the induction hypothesis we get that:

$$\mathsf{mbs}^z(f^k) \geq \mathsf{mbs}^z(f^2)\mathsf{mbs}^0(f^{2\lfloor k/2 \rfloor - 2}) \geq 2.2^{\lfloor k/2 \rfloor - 1} = 2^{\lfloor k/2 \rfloor}$$

A natural question to ask at this point is do we have a relation for the ratio between $\mathsf{MCC}(f^l)$ and $\mathsf{mbs}(f^l)$ similar to $\mathsf{fmbs}(f^l)$ and $\mathsf{mbs}(f^l)$? That might be an interesting problem to look at but what we do have is the following simple corollary which follows from the fact that $\mathsf{MCC}(f) = O(\mathsf{fmbs}(f) \log(\mathsf{spar}(f)))$.

Corollary 3. *For a Boolean function $f : \{0,1\}^n \to \{0,1\}$ we have that for all $l \geq 1$:*

$$\frac{\mathsf{MCC}(f^l)}{\mathsf{mbs}(f^l)\log(\mathsf{spar}(f^l))} \leq C(n),$$

where $C(n)$ is a function independent of l.

Proof. Using the fact that $\mathsf{MCC}(f) = O(\mathsf{fmbs}(f)\log(\mathsf{spar}(f)))$ and by Theorem 1 we get:

$$\frac{\mathsf{MCC}(f^l)}{\mathsf{fmbs}(f^l)\log(\mathsf{spar}(f^l))} \cdot \frac{\mathsf{fmbs}(f^l)}{\mathsf{mbs}(f^l)} \leq \frac{\mathsf{MCC}(f^l)}{\mathsf{mbs}(f^l)\log(\mathsf{spar}(f^l))} = O(p(n)).$$

Another interesting observation from Theorem 1 is the following corollary.

Corollary 4. *For any Boolean function f and constant $c > 0$, there exists a $l_0 \in \mathbb{N}$ such that for all $l \geq l_0$,*

$$\mathsf{fmbs}(f^l) \leq \mathsf{mbs}^{1+c}(f^l).$$

Proof. Let us assume that no such l_0 exists i.e. there exists an infinite sequence of integers, say $\{m_i\}_{i\geq 1}$ s.t.:

$$\mathsf{fmbs}(f^{m_i}) > \mathsf{mbs}(f^{m_i})\mathsf{mbs}^c(f^{m_i}),$$

for all $i \geq 1$.

In other words, this implies that:

$$p(n) \geq \frac{\mathsf{fmbs}(f^{m_i})}{\mathsf{mbs}(f^{m_i})} > \mathsf{mbs}^c(f^{m_i}),$$

for all $i \geq 1$.

This is a contradiction to the fact that the sequence $\{\mathsf{mbs}(f^l)\}$ diverges.

Finally, we also mention the following inequality which has been used in the proof of Theorem 1:

Proposition 1. $\prod_{i=1}^{\infty}(1-2^{-i}) \geq 1/e^2$

Proof. We prove the inequality by applying A.M-G.M. inequality on positive real nos. $\{a_1, ..., a_N\}$ where $a_i := \frac{1}{1-2^{-i}}$ followed by taking the limit $N \to \infty$.

Applying A.M.-G.M. inequality on $\{a_1, ..., a_N\}$, we get:

$$(a_1 a_2 ... a_N)^{1/N} \leq \frac{\sum_{i=1}^{N} a_i}{N}$$

This implies,

$$\left(\prod_{i=1}^{N}(1-2^{-i})^{-1}\right)^{1/N} \leq \frac{\sum_{i=1}^{N}\frac{2^i}{2^i-1}}{N}.$$

Now simplifying the above inequality we obtain the following set of inequalities:

$$\left(\prod_{i=1}^{N}(1-2^{-i})^{-1}\right)^{1/N} \leq 1+\frac{\sum_{i=1}^{N}2^{-(i-1)}}{N}$$
$$= 1+\frac{2}{N}(1-2^{-N}) \leq 1+2/N.$$

This implies,

$$\prod_{i=1}^{N}(1-2^{-i})^{-1} \leq (1+2/N)^{N}.$$

Taking $N \to \infty$ we get the desired inequality.

B.1 Characterization of Boolean functions with $\mathsf{mbs}(f) = 1$

In this section, we provide another noticeable difference in the behaviour of bs and mbs. We already know from [19] that for all non-monotone functions $\mathsf{bs}(f) \geq 2$. Interestingly, the same is not true for mbs i.e. there are non-monotone Boolean functions for which $\mathsf{mbs}(f) = 1$. For example, the function:

$$ODD{-}MAX{-}BIT(X_{S_1}, X_{S_2}, ..., X_{S_k}) := X_{S_1} - X_{S_1}X_{S_2} + X_{S_1}X_{S_2}X_{S_3} - ... + (-1)^k \prod_{i=1}^{k} X_{S_i},$$

where the product $X_S X_T := X_{S\cap T} X_{S\setminus T \cup T\setminus S}$, has $\mathsf{mbs}(f) = 1$.

What we now show is that the $ODD-MAX-BIT$ is in fact the "only" function with $\mathsf{mbs}(f) = 1$. To prove the aforementioned result we need the following claim about the structure of Boolean functions with $\mathsf{mbs}(f) \leq 1$.

Claim. If $f : \{0,1\}^n \to \{0,1\}$ is a Boolean function with $\mathsf{mbs}(f) = 1$ and $f(0^n) = 0$ then there exists an input $x \in \{0,1\}^n$ s.t. for all $y \in \{0,1\}^n$, if $y \not\geq x$ then $f(y) = 0$.

Proof. Consider a Boolean function $f : \{0,1\}^n \to \{0,1\}$ with $\mathsf{mbs}(f) = 1$ and $f(0^n) = 0$ and let k be the smallest integer s.t. $f(x) = 1$ and $|x| = k$. Now if we assume that there exists a $y \not\geq x$ with $f(y) = 1$ then we have $\mathsf{mbs}(f, x \wedge y) \geq 2$ which contradicts the assumption of $\mathsf{mbs}(f) = 1$.

Lemma 5. *If $f : \{0,1\}^n \to \{0,1\}$ is a Boolean function with $\mathsf{mbs}(f) \leq 1$ then f can be expressed as $ODD-MAX-BIT(X_{S_1}, X_{S_2}, ..., X_{S_k})$ or 1-($ODD-MAX-BIT(X_{S_1}, X_{S_2}, ..., X_{S_k})$) where X_{S_i} are monomials corresponding to set $S_i \subseteq [n]$.*

Proof. We prove the lemma by applying induction on the arity of the Boolean function f i.e. n. For the base case let $n = 1$.

Now if f is a constant function then $f(x) = X_\phi$ or $1 - X_\phi$ where $X_\phi = 1$. If it is not then $f(x) = x_1$ or $1 - x_1$. We see that the condition is satisfied for the base case of $n = 1$.

Now for the inductive step assume that $f(0^n) = 0$ with $n = k$ and $\mathsf{mbs}(f) \leq 1$. Using subsection B.1 we have that there exists a $x \in \{0,1\}^n$ s.t. $f(x) = 1$ and for all $y \ngeq x$ we have $f(y) = 0$. This implies,

$$f(x) = X_S g(x),$$

where $S := supp(x)$ and $g : \{0,1\}^{n-|x|} \to \{0,1\}$ is the restriction of f on the support of x i.e. $g := f_x$. As g is the restriction of f on x hence $\mathsf{mbs}(g) \leq 1$. For the case when $\mathsf{mbs}(g) = 0$ i.e. g is constant, we have $f(x) = 0 = 1 - X_\phi$ or $f(x) = X_S{=}OMB(X_S)$ for the case when $g(x) = 0$ or 1 respectively. This implies we can assume g to be a non-constant function i.e. $\mathsf{mbs}(g) = 1$. Now using the induction hypothesis we have:

$$g(x) = X_1 - X_{S_1}X_{S_2} + ... + (-1)^k X_{S_1}X_{S_2}...X_{S_k} = OMB(X_{S_1}, X_{S_2}, ..., X_{S_k}),$$

for some $S_1, ..., S_k \subseteq [n]$. Note that over here we could have also assumed that $g(x) = 1 - OMB(X_{S_1}, ..., X_{S_k})$

Hence $f(x) = X_{S\cup S_1} - X_{S\cup S_1}X_{S_2} + ... + (-1)^k X_{S\cup S_1}X_{S_2}...X_{S_k} = OMB(X_{S\cup S_1}, X_{S_2}, ..., X_{S_k})$.

C Relation between $\mathsf{mbs}(f)$ and $\log(\mathsf{spar}(f))$

Sensitivity $\mathsf{s}(f)$ and Fourier degree $\deg(f)$ are two very well studied complexity measures on Boolean functions. Huang, in his landmark result [9], explicitly proved $\deg(f) \leq s(f)^2$ to show sensitivity is polynomially related to other complexity measures. In the other direction, Nisan and Szegedy [16] showed that $\mathsf{s}(f) \leq \deg(f)^2$ around thirty years ago (we still don't know if this relation is tight). The article [11] used this relation to show $\mathsf{mbs}(f) = O(\log^2(\mathsf{spar}(f)))$. We show that improving upper bound $\mathsf{mbs}(f) = O(\log^2(\mathsf{spar}(f)))$ is indeed equivalent to improving the upper bound on degree in terms of sensitivity (a long standing open question).

Recall that Theorem 2 states: suppose, there exists a constant α such that for every Boolean function $f : \{0,1\}^n \to \{0,1\}$, $\mathsf{mbs}(f) = O(\log^\alpha \mathsf{spar}(f))$. Then for every Boolean function $f : \{0,1\}^n \to \{0,1\}$, $\mathsf{s}(f) = O(\deg^\alpha(f))$.

To prove Theorem 2, we will need the following known relation between Fourier degree and Fourier sparsity (see e.g. the proof of Fact 5.1 in [18]). A proof is included for completeness.

Claim. For $f : \{-1,1\}^n \to \{-1,1\}$, $\mathsf{spar}(f) \leq 4^{\deg(f)}$.

Proof. Let $f : \{-1,1\}^n \to \{-1,1\}$. Define $g : \{0,1\}^n \to \{-1,1\}$ by $g(x_1, x_2, \ldots, x_n) = f(1-2x_1, 1-2x_2, \ldots, 1-2x_n)$ (notice that $\deg(g) = \deg(f)$).

Let $g(x) = \sum_{S\subseteq[n]} \alpha_S \prod_{i\in S} x_i$ be its polynomial representation. Since g is integer-valued, all α_S's are integers [6].

[6] This can be seen by induction, using the fact that α_S is an integer linear combination of $f(S)$ (where we interpret S as its indicator vector) and α_T for $T \subsetneq S$.

Using the polynomial representation of g,

$$f(y) = g(\frac{1-y_1}{2}, ..., \frac{1-y_n}{2}) = \sum_{S \subseteq [n]} \alpha_S \prod_{i \in S} \left(\frac{1-y_i}{2}\right).$$

From this representation of f, every Fourier coefficient of f is an integer multiple of $1/2^{\deg(f)}$. Say $\widehat{f}(S) = \beta_S/2^{\deg(f)}$ for some $\beta_S \in \mathbb{Z}$. Using Parseval, $\sum_{S \subseteq [n]} \beta_S^2 = 4^{\deg(f)}$. Since β_S's are integers, this implies that sparsity is at most $4^{\deg(f)}$.

Corollary 5. *For $g : \{0,1\}^n \to \{0,1\}$, $\mathsf{spar}(g) \leq 8^{\deg(g)}$.*

Proof. Let $f : \{-1,1\}^n \to \{-1,1\}$ be defined by $f(x) = 1 - 2g(\frac{1-x_1}{2}, ..., \frac{1-x_n}{2})$ similar to what was done in the previous claim. This is equivalent to $g(x_1, ..., x_n) = (1 - f(1-2x_1, ..., 1-2x_n))/2$. By the above claim,

$$\mathsf{spar}(g) \leq 2^{\deg(f)}\mathsf{spar}(f) \leq 8^{\deg(g)}.$$

$\square$

Now we can prove Theorem 2.

Proof (Proof of Theorem 2). Let $w \in \{0,1\}^n$ be such that $\mathsf{s}(f) = \mathsf{s}(f,w)$. Consider the function $\widetilde{f}$ defined by

$$\widetilde{f}(x) := f(x \oplus w),$$

where $x \oplus w$ denotes the bitwise XOR of x and w.

Observe that $\mathsf{s}(\widetilde{f}, 0^n) = \mathsf{s}(f,w)$. Also, $\deg(\widetilde{f}) \leq \deg(f)$ since performing an affine substitution cannot increase the degree.

Using the given condition, $\mathsf{mbs}(\widetilde{f}) = O(\log^\alpha \mathsf{spar}(\widetilde{f}))$ and the fact $\mathsf{mbs}(\widetilde{f}, 0) = \mathsf{bs}(\widetilde{f}, 0) \geq \mathsf{s}(\widetilde{f}, 0) = \mathsf{s}(f,w)$ it follows that $\mathsf{s}(f) = O(\log^\alpha \mathsf{spar}(\widetilde{f}))$. Finally, by Corollary 5 and $\deg(\widetilde{f}) \leq \deg(f)$, we get $\mathsf{s}(f) = O(\deg^\alpha(\widetilde{f})) = O(\deg^\alpha(f))$ as desired.

Hence improving the bound on $\mathsf{mbs}(f)$ in terms of $\log(\mathsf{spar}(f))$ is equivalent to improving the upper bound on $\mathsf{s}(f)$ in terms of $\deg(f)$. The other possible approach of improving the bound on $D^{0-dt}(f)$ is to improve the upper bound on $\mathsf{fmbs}(f)$ in terms of $\log(\mathsf{spar}(f))$.

It turns out that for the class of symmetric and monotone Boolean functions $\mathsf{mbs}(f) = \mathsf{fmbs}(f) = \mathsf{MCC}(f)$, giving a much better upper bound on $\mathsf{fmbs}(f)$ in terms of sparsity of f.

D Boolean functions with $\mathsf{ms}(f) = \mathsf{MCC}(f)$

In this section, we look at classes of Boolean functions for which $\mathsf{mbs}(f) = \mathsf{MCC}(f)$. We get that for such class of functions $\mathsf{MCC}(f) = O(\log^2(\mathsf{spar}(f)))$ (an improvement over the relationship $\mathsf{MCC}(f) = O(\log^5 \mathsf{spar}(f))$ proved in [11]).

Theorem 3 states that if $f : \{0,1\}^n \to \{0,1\}$ is monotone or symmetric Boolean function, then

$$\mathsf{ms}(f) = \mathsf{mbs}(f) = \mathsf{fmbs}(f) = \mathsf{MCC}(f).$$

Proof (Proof of Theorem 3).

We prove the two cases separately.

Case 1 (f is monotone): it suffices to show that for monotone Boolean functions $\mathsf{MCC}(f) \le \mathsf{ms}(f)$. Let x be an input for which $\mathsf{MCC}(f,x) = \mathsf{MCC}(f)$.

Without loss of generality assume that f is monotonically increasing and $f(x) = 0$. Let C be a minimal certificate for f_x at the corresponding all $0^{n-|x|}$ string s.t. $|C| = \mathsf{MCC}(f,x)$. Now consider the input y defined as follows:

$$y_i := \begin{cases} 1 & i \notin \text{C} \\ 0 & i \in \text{C} \end{cases}$$

As y agrees with the all zero string $0^{n-|x|}$ at the indices in C we have that $f_x(0) = f_x(y)$. Now we claim that each of the indices in C is sensitive for f_x at y. If it wasn't the case then there exists an $i \in C$ s.t. $f_x(y^{\oplus i}) = f_x(y)$. As f_x is also monotone hence for all $z \in \{0,1\}^{n-|x|}$ for which $y^{\oplus i} \ge z$ we have $f_x(y^{\oplus i}) \ge f_x(z)$.

For any string $z^{'} \in \{0,1\}^{n-|x|}$ that agrees with $0^{n-|x|}$ at the bits in $C \setminus i$ notice that $z^{'} \le y^{\oplus i}$. Hence $0 = f_x(y^{\oplus i}) \ge f_x(z^{'}) = 0$. But this implies that $C \setminus \{i\}$ is a certificate for f_x at $0^{n-|x|}$ which is a contradiction.

Hence,

$$\mathsf{MCC}(f) = \mathsf{MCC}(f,x) = \mathsf{ms}(f_x, y) \le \mathsf{ms}(f_x) \le \mathsf{ms}(f).$$

Case 2 (f is symmetric): again, it suffices to show that if $f : \{0,1\}^n \to \{0,1\}$ is symmetric then

$$\mathsf{MCC}(f) \le \mathsf{ms}(f).$$

Let $x \in \{0,1\}^n$ be one of the inputs for which $\mathsf{MCC}(f) = \mathsf{MCC}(f,x) = C(f_x, 0^{n-|x|})$. Now, let C be the witness for $\mathsf{MCC}(f,x)$. If $|C| = n - |x|$ then this would imply that all the bits of $0^{n-|x|}$ are sensitive for f_x. For the other case, i.e. $|C| < n - |x|$,what we claim is that $\mathsf{MCC}(f,x) = \mathsf{MCC}(f,z)$ where z is the input s.t.supp$(z) = \text{supp}(x) \cup \{i : i \in [n] \setminus (\text{supp}(x) \cup C)\}$ i.e. z is set to 1 at the bits lying in supp(x) and all the bits not lying in C.

We claim that $\mathsf{MCC}(f,z) = C(f_z, 0^{n-|z|}) = |C| = n - |z|$ i.e. all the bits of 0^{n-z} are sensitive for f_z. If we assume this is not the case i.e. $C^{'} \subsetneq C$ is a certificate for $0^{n-|z|}$ with $|C^{'}| = |C| - 1$ then it would imply that $C^{'}$ is also a certificate for $0^{n-|x|}$.

To argue this, say $C^{'} = C \setminus i$. Now for any input $x < y < z$ that agrees with $0^{n-|x|}$ at the bits of $C^{'}$ we have that $f_x(0^{n-|x|}) = f_x(y)$. This is from using the fact that f_x is symmetric, hence we can always swap the $i - th$ bit of y with a zero bit in y not lying in C.

Hence by the above argument we would have that $C^{'}$ is a certificate for $0^{n-|x|}$ as well. But this contradicts the condition that $\mathsf{MCC}(f,x) = |C|$

Using Theorem 3, we get the following corollary.

Corollary 6. *Consider a Boolean function $f : \{0,1\}^n \to \{0,1\}$.*

1. *If f is monotone,* $\mathsf{MCC}(f) = \mathsf{fmbs}(f) = \mathsf{mbs}(f) = O(\log^2(\mathsf{spar}(f)))$.
2. *If f is symmetric,* $\mathsf{MCC}(f) = \mathsf{fmbs}(f) = \mathsf{mbs}(f) = (1 + o(1)) \log(\mathsf{spar}(f))$.

Proof. For the statement about monotone functions, we combine Theorem 3 with the relation $\mathsf{mbs}(f) = O(\log^2(\mathsf{spar}(f)))$ [11,5].

For symmetric functions, we use the relations $D^{cc}(f \circ \wedge_2) \leq (1+o(1)) \log(\mathsf{spar}(f))$ [5] and $\mathsf{mbs}(f) \leq D^{cc}(f \circ \wedge_2)$ [11].

Note that the bound above for symmetric functions is tight as can be seen by considering the OR function which has sparsity $2^n - 1$ and monotone sensitivity n.

E Examples of separations for monotone measures

	ms	mbs	fmbs	HSC	log(spar)
ms	■	1	1	1	2
mbs	?	■	1	1	2
fmbs	?	2	■	1	4
HSC	?	?	?	■	5

Table 1. Known relations for monotone measures: Entry b in row A and column B represents, for any function f, $A(f) = O(B(f))^{b+o(1)}$,

Here, we note some observations about monotone measures that follow from previous work. We have listed all the known relationships between complexity measures in Table 1. All non-trivial relationships follow from the results of Knop et. al. [11]. A similar table for standard measures was compiled in [3]. The relationships between standard measures do not seem to straightforwardly imply relationships between monotone combinatorial measures. On the bright side, almost all the existing separations between classical complexity measures can be lifted for a monotone analogue of complexity measures, essentially in the same way as the proof of Theorem 2 as we explain later. For the relation part, it is natural and interesting to ask if monotone measures are polynomially related at all. Note that a row with $\log(\mathsf{spar})$ doesn't make sense since it is not polynomially related to other monotone complexity measures.

Why we don't need a row for $\log(\mathsf{spar})$*?* The function $(\mathsf{AND}_n \circ \mathsf{OR}_2)$ (example 2.18 in Knop et al [11]) has sparsity exponential in n but constant MCC. So $log(\mathsf{spar})$ can not be bounded by any polynomial power of the monotone complexity measures.

Lifting separations between classical measures for monotone measures: Let $M_1, M_2 \in \{\mathsf{s}, \mathsf{bs}, \mathsf{fbs}, \mathsf{C}\}$ and let mM_1, mM_2 denote their respective monotone analogues. Suppose f achieves a separation $M_1(f) \geq \Omega(M_2(f)^c)$ and suppose y is the input where $M_1(f) = M_1(f, y)$. Consider the shifted function g which maps x to $f(xXORy)$. Then $mM_1(g, 0^n) = M_1(f, y) = M_1(f) \geq \Omega(M_2(f)^c) = \Omega(M_2(g)^c) \geq \Omega(mM_2(g)^c)$.

We will give a precise example of the fact that classical separations can be lifted easily for monotone measures.

For example in terms of monotone measures [11] proved that $\mathsf{fmbs}(f) = O(\mathsf{mbs}(f)^2)$ for all Boolean function f. It is natural to ask if the relation is tight or not. Consequently it comes to the best known separations between fbs and bs and to check if that example works for monotone measures as well. There exist classes of functions given by [8] that gives separations between fbs and bs. Let us denote the function introduced by [8] by GSS.

Theorem 8 ([8]). *There exists a family of Boolean functions* GSS *for which* $\mathsf{fbs}(\mathsf{GSS}) = \Omega(\mathsf{bs}(\mathsf{GSS})^{\frac{3}{2}})$.

Now GSS function is such that $\mathsf{fbs}(\mathsf{GSS}(0^n)) = \Omega(n^{\frac{3}{4}})$ and $\mathsf{bs}(\mathsf{GSS}) = O(n^{\frac{1}{2}})$. Now, from the definition of monotone measures, it follows that $\mathsf{fmbs}(\mathsf{GSS}(0^n)) = \Omega(n^{\frac{3}{4}})$ and $\mathsf{mbs}(\mathsf{GSS}) = O(\mathsf{bs}(\mathsf{GSS})) = O(n^{\frac{1}{2}})$. Consequently, we have the following lemma,

Lemma 6. *There exists Boolean function for which,* $\mathsf{fmbs}(\mathsf{GSS}) = \Omega(\mathsf{mbs}(\mathsf{GSS})^{\frac{3}{2}})$

Note that the above separation is not tight but it matches the best-known separations for standard measures fbs and bs.

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