

Formula Size-Depth Tradeoffs for Iterated Sub-Permutation Matrix Multiplication

Benjamin Rossman
Duke University

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Abstract

We study the formula complexity of *Iterated Sub-Permutation Matrix Multiplication*, the logspace-complete problem of computing the product of k n -by- n Boolean matrices with at most a single 1 in each row and column. For all $d \leq \log k$, this problem is solvable by $n^{O(dk^{1/d})}$ size monotone formulas of two distinct types: (unbounded fan-in) AC^0 formulas of depth $d+1$ and (semi-unbounded fan-in) SAC^0 formulas of \wedge -depth d and \wedge -fan-in $k^{1/d}$. The results of this paper give

- matching $n^{\Omega(dk^{1/d})}$ lower bounds for monotone AC^0 and SAC^0 formulas for all $k \leq \log \log n$, as well as
- slightly weaker $n^{\Omega(dk^{1/2^d})}$ lower bounds for non-monotone AC^0 and SAC^0 formulas.

These size-depth tradeoffs converge at $d = \log k$ to tight $n^{\Omega(\log k)}$ lower bounds for both unbounded-depth monotone formulas [21] and bounded-depth non-monotone formulas [23]. Our non-monotone lower bounds extend to the more restricted *Iterated Permutation Matrix Multiplication* problem, improving the previous $n^{k^{1/\exp(O(d))}}$ tradeoff for this problem [4].

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1 Introduction

Motivated by the question of NC^1 vs. L (polynomial-size formulas vs. branching programs), we study the formula complexity of *Iterated Sub-Permutation Matrix Multiplication* (SUB-PMM $_{n,k}$), the logspace-complete problem of computing the product of k n -by- n sub-permutation matrices. Our results give asymptotically tight size-depth tradeoffs for monotone (unbounded fan-in) AC^0 and (semi-unbounded fan-in) SAC^0 formulas solving this problem, as well as slightly weaker tradeoffs for non-monotone formulas.

Our lower bounds are based on a method of reducing AC^0 formulas for SUB-PMM $_{n,k}$ to simpler formulas called *join trees*, which construct the length- k path graph from its single-edge subgraphs via the union operation. A large part of this paper (§3–5) is devoted to the combinatorial theory of join trees. We establish tradeoffs between a few “size” and “depth” measures on join trees, which we are able to lift to size-depth tradeoffs for AC^0 and SAC^0 formulas by extending the Pathset Framework introduced in papers [21, 23].

The rest of this introduction is organized as follows. §1.1 introduces iterated matrix multiplication problems IMM $_{n,k}$, BMM $_{n,k}$, PMM $_{n,k}$ and SUB-PMM $_{n,k}$. §1.2 discusses the formula complexity of these problems in relation to the NC^1 vs. L question. §1.3 reviews some upper bounds on the AC^0 and SAC^0 formula size of SUB-PMM $_{n,k}$. §1.4 states our main results. §1.5 discusses prior related work. §1.6 briefly outlines the rest of the paper.

1.1 Iterated matrix multiplication problems

Iterated Matrix Multiplication (IMM $_{n,k}$) is the task, given $n \times n$ matrices $M^{(1)}, \dots, M^{(k)}$ (over some specified field), of computing the 1, 1-entry of the matrix product $M^{(1)} \cdots M^{(k)}$. In algebraic complexity theory, this problem is represented by a formal polynomial of kn^2 variables (the entries of matrices $M^{(i)}$):

$$\text{IMM}_{n,k}(\vec{M}) = \sum_{a_1, \dots, a_{k-1} \in [n]} M_{1,a_1}^{(1)} M_{a_1,a_2}^{(2)} \cdots M_{a_{k-1},1}^{(k)}.$$

Iterated Boolean Matrix Multiplication (BMM $_{n,k}$) is the corresponding problem for matrices with entries in the Boolean ring $\{0, 1\}$ with addition \vee and multiplication \wedge . This problem is represented by a monotone Boolean function $\{0, 1\}^{kn^2} \rightarrow \{0, 1\}$ described by the formula

$$\text{BMM}_{n,k}(\vec{M}) = \bigvee_{a_1, \dots, a_{k-1} \in [n]} M_{1,a_1}^{(1)} \wedge M_{a_1,a_2}^{(2)} \wedge \cdots \wedge M_{a_{k-1},1}^{(k)}.$$

A special case of this problem arises when the input $M^{(1)}, \dots, M^{(k)}$ are promised to be *permutation matrices*, or more generally *sub-permutation matrices* with at most a single 1 in each row and column. (For such matrices, BMM $_{n,k}$ coincides with the function computed by IMM $_{n,k}$ over any field.) These restricted promise problems are called *Iterated (Sub-)Permutation Matrix Multiplication* and denoted by PMM $_{n,k}$ and SUB-PMM $_{n,k}$. Note that there is a significant difference in computational complexity: whereas BMM $_{n,k}$ and IMM $_{n,k}$ are complete for nondeterministic logspace NL and its algebraic analogue VBP (polynomial-size algebraic branching programs), both PMM $_{n,k}$ and SUB-PMM $_{n,k}$ are complete for deterministic logspace L [8].

This paper focuses on the problem SUB-PMM $_{n,k}$ rather than PMM $_{n,k}$, since like BMM $_{n,k}$ it is naturally represented by a *monotone* Boolean function (whose domain is the downward-closed subset of $\{0, 1\}^{kn^2}$ representing k -tuples of sub-permutation matrices). It is natural to study the complexity of SUB-PMM $_{n,k}$ in both monotone and non-monotone models of computation. In contrast, PMM $_{n,k}$ is an inherently non-monotone “slice function” whose relevant inputs (k -tuples of permutation matrices) have equal Hamming weight kn ; for such functions, it is well-known that monotone and non-monotone formula complexity coincide up to a polynomial factor [6].

1.2 Formula complexity and the question of NC^1 vs. L

A *formula* is a rooted tree whose leaves (“inputs”) are labeled by constants (0 or 1) or literals (X_i or \bar{X}_i) and whose non-leaves (“gates”) are labeled by \wedge or \vee . Understanding the formula complexity of iterated matrix multiplication problems is a major challenge at the frontier of lower bounds in Circuit Complexity. Recall that

NC^1 is — by one of its equivalent definitions — the complexity class of languages that are decidable by a sequence of polynomial-size formulas. It is known that $AC^0 \subset AC^0[\oplus] \subset NC^1 \subseteq L \subseteq NL \subseteq P \subseteq NP$, but open whether NC^1 is a proper subclass of L , or even NP . At present, no lower bound better than $n^{3-o(1)}$ is known on the minimum formula size of any *explicit* sequence of Boolean functions (describing a language in P , or even NP) [27].

Historically, the first lower bounds against many weak circuit classes have targeted functions of barely higher complexity. Consider the classical results that $\text{PARITY} \in AC^0[\oplus] \setminus AC^0$ [1, 9] and $\text{MAJ} \in NC^1 \setminus AC^0[\oplus]$ [20, 26]. Even the best $n^{3-o(1)}$ formula lower bound [12, 27] targets a function (Andreev’s function) with $O(n^3)$ size formulas. From this perspective, *Iterated Sub-Permutation Matrix Multiplication* $\text{SUB-PMM}_{n,k}$ is the ideal target for a super-polynomial lower bound against NC^1 . The smallest known formulas for $\text{SUB-PMM}_{n,k}$ have size $n^{O(\log k)}$, which is widely believed to be optimal. A matching $n^{\Omega(\log k)}$ lower bound on the formula size of $\text{SUB-PMM}_{n,k}$ (respectively: $\text{BMM}_{n,k}$, $\text{IMM}_{n,k}$) for any jointly super-constant range of k and n would separate NC^1 from L (respectively: NC^1 from NL , VNC^1 from VBP).

Motivated by this goal, previous work of the author [21, 23] took aim at this lower bound challenge in two restricted classes of formulas:

- (+) monotone unbounded-depth formulas, and
- (−) non-monotone bounded-depth formulas.

Papers [21, 23] established tight $n^{\Omega(k)}$ lower bounds in settings (+) and (−) with respect to an average-case version of $\text{BMM}_{n,k}$ (on random matrices with i.i.d. Bernoulli($\frac{1}{n}$) entries). The following theorem restates these results as lower bounds for problems $\text{SUB-PMM}_{n,k}$ and $\text{PMM}_{n,k}$, which follow from randomized AC^0 reductions from average-case $\text{BMM}_{n,k}$.

Theorem 1.1 ([21, 23]). *For all $k \leq \log \log n$, size $n^{\Omega(\log k)}$ is required by both*

- (+) *monotone formulas solving $\text{SUB-PMM}_{n,k}$,*
- (−) *formulas of depth $\frac{\log n}{(\log \log n)^{O(1)}}$ solving $\text{PMM}_{n,k}$.*

Both of these lower bounds come “close” to showing $NC^1 \neq L$. To separate these complexity classes, it suffices to extend (+) from the monotone function $\text{SUB-PMM}_{n,k}$ to the slice function $\text{PMM}_{n,k}$. It also suffices to extend (−) from depth $\frac{\log n}{(\log \log n)^{O(1)}}$ to depth $\log n$.

1.3 Upper bounds

Lower bounds (+) and (−) of Theorem 1.1 are simultaneously shown to be tight by bounded-depth monotone formulas implementing the standard divide-and-conquer (a.k.a. recursive doubling) technique of Savitch [25].

Proposition 1.2. *$\text{SUB-PMM}_{n,k}$ is solvable by monotone formulas of size $n^{O(\log k)}$ and depth $O(\log k)$ for all n and k .*

This raises the interesting question: what about formulas of very small depths 2, 3, 4, . . . below $O(\log k)$? Here the best known upper bounds exhibit a typical size-depth tradeoff. In fact, there are two different tradeoffs given by distinct families of (unbounded fan-in) AC^0 formulas and (semi-unbounded fan-in) SAC^0 formulas.¹

Proposition 1.3. *When $k^{1/d}$ is an integer, $\text{SUB-PMM}_{n,k}$ is solvable by size $kn^{dk^{1/d}}$ monotone formulas of the following types:*

- (I) Σ_{2d} formulas with \wedge -fan-in $k^{1/d}$ (and \vee -fan-in $n^{k^{1/d}}$),
- (II) Σ_{d+1} as well as Π_{d+1} formulas (with fan-in $k^{1/d}n^{k^{1/d}}$).

¹See §2.2 for definitions of classes Σ_d and Π_d of depth- d AC^0 formulas. Consistently throughout this paper, “ SAC^0 ” refers to AC^0 formulas where \wedge -gates have fan-in $O(k^{1/d})$ in the context of *upper bounds* and $O(n^{1/k})$ in *lower bounds*. Note that $k^{1/d} \ll n^{1/k}$ for the range of parameters we consider.

To illustrate constructions (I) and (II), Examples 1.4 and 1.5 below concretely present formulas (I) $_{\Sigma_{2d}}$, (II) $_{\Sigma_{d+1}}$ and (II) $_{\Pi_{d+1}}$ in the cases $(d, k) = (1, 5)$ and $(d, k) = (2, 5^2)$.

Example 1.4 (The case $d = 1$ and $k = 5$ of Proposition 1.3). For sub-permutation matrices $M^{(1)}, \dots, M^{(5)} \in \{0, 1\}^{n \times n}$ and indices $a_0, a_5 \in [n]$, the a_0, a_5 -entry of the product $M^{(1)} \dots M^{(5)}$ is computed by the following monotone Σ_2 and Π_2 (a.k.a. DNF and CNF) formulas:

$$\begin{aligned} (\mathbf{D}_{a_0, a_5}) \quad & \bigvee_{a_1, a_2, a_3, a_4} \left(M_{a_0, a_1}^{(1)} \wedge M_{a_1, a_2}^{(2)} \wedge M_{a_2, a_3}^{(3)} \wedge M_{a_3, a_4}^{(4)} \wedge M_{a_4, a_5}^{(5)} \right), \\ (\mathbf{C}_{a_0, a_5}) \quad & \bigwedge_{a_1, a_2, a_3, a_4} \left(\bigvee_{b_1 \neq a_1} M_{a_0, b_1}^{(1)} \vee \bigvee_{b_2 \neq a_2} M_{a_1, b_2}^{(2)} \vee \bigvee_{b_3 \neq a_3} M_{a_2, b_3}^{(3)} \vee \bigvee_{b_4 \neq a_4} M_{a_3, b_4}^{(4)} \vee M_{a_4, a_5}^{(5)} \right). \end{aligned}$$

Note that \mathbf{D}_{a_0, a_5} computes $(M^{(1)} \dots M^{(5)})_{a_0, a_5}$ for arbitrary Boolean matrices $M^{(1)}, \dots, M^{(5)} \in \{0, 1\}^{n \times n}$, whereas \mathbf{C}_{a_0, a_5} relies on the assumption that $M^{(1)}, \dots, M^{(5)}$ that have at most a single 1 in each row. Setting $a_0 = a_5 = 1$, monotone Σ_2 and Π_2 formulas $\mathbf{D}_{1,1}$ and $\mathbf{C}_{1,1}$ compute the function $\text{SUB-PMM}_{n,5}$. This provides the case $d = 1$ and $k = 5$ of Proposition 1.3: namely, $\mathbf{D}_{1,1}$ furnishes upper bounds (I) $_{\Sigma_2}$ and (II) $_{\Sigma_2}$, while $\mathbf{C}_{1,1}$ furnishes (II) $_{\Pi_2}$.

Example 1.5 (The case $d = 2$ and $k = 25$ of Proposition 1.3). $\text{SUB-PMM}_{n,25}$ is computed by the following monotone Σ_4 , Σ_3 and Π_3 formulas, built from sub-formulas $\mathbf{D}_{a_5(i-1), a_5 i}$ and $\mathbf{C}_{a_5(i-1), a_5 i}$ over matrices $M^{(5i+1)}, \dots, M^{(5i)}$:

$$\begin{aligned} (\mathbf{I})_{\Sigma_4} \quad & \bigvee_{a_5, a_{10}, a_{15}, a_{20}} \left(\mathbf{D}_{1, a_5} \wedge \mathbf{D}_{a_5, a_{10}} \wedge \mathbf{D}_{a_{10}, a_{15}} \wedge \mathbf{D}_{a_{15}, a_{20}} \wedge \mathbf{D}_{a_{20}, 1} \right), \\ (\mathbf{II})_{\Sigma_3} \quad & \bigvee_{a_5, a_{10}, a_{15}, a_{20}} \left(\mathbf{C}_{1, a_5} \wedge \mathbf{C}_{a_5, a_{10}} \wedge \mathbf{C}_{a_{10}, a_{15}} \wedge \mathbf{C}_{a_{15}, a_{20}} \wedge \mathbf{C}_{a_{20}, 1} \right), \\ (\mathbf{II})_{\Pi_3} \quad & \bigwedge_{a_5, a_{10}, a_{15}, a_{20}} \left(\bigvee_{b_5 \neq a_5} \mathbf{D}_{1, b_5} \vee \bigvee_{b_{10} \neq a_{10}} \mathbf{D}_{a_5, b_{10}} \vee \bigvee_{b_{15} \neq a_{15}} \mathbf{D}_{a_{10}, b_{15}} \vee \bigvee_{b_{20} \neq a_{20}} \mathbf{D}_{a_{15}, b_{20}} \vee \mathbf{D}_{a_{20}, 1} \right). \end{aligned}$$

Note that (I) $_{\Sigma_4}$ has \wedge -fan-in 5 ($= k^{1/d}$) and all three formulas have size at most $25n^{10}$ ($= kn^{dk^{1/d}}$) as required. A similar recurrence produces formulas (I) $_{\Sigma_{2d}}$, (II) $_{\Sigma_{d+1}}$ and (II) $_{\Pi_{d+1}}$ for all $d \geq 3$.

Formulas (I) $_{\Sigma_{2d}}$ (and their algebraic counterparts with \sum, \prod replacing \vee, \wedge) implement a standard divide-and-conquer algorithm for problems $\text{BMM}_{n,k}$ and $\text{IMM}_{n,k}$. This algorithm recursively partitions the matrix product $M^{(1)} \dots M^{(k)}$ as $N^{(1)} \dots N^{(k^{(d-1)/d})}$ where $N^{(i)}$ is the sub-product $M^{((i-1)k^{1/d}+1)} \dots M^{(ik^{1/d})}$. Formulas (II) $_{\Sigma_{d+1}}$ and (II) $_{\Pi_{d+1}}$ for $\text{SUB-PMM}_{n,k}$ utilize a similar recurrence, which generalizes the well-known Σ_{d+1} and Π_{d+1} formulas for PARITY_k (which are essentially the $n = 2$ case of formulas (II)).

Notice that Proposition 1.3 implies the $n^{O(\log k)}$ upper bound of Proposition 1.2 when $d = \log k$. In contrast, we remark that the circuit versions of constructions (I) and (II) have size $O(kn^{k^{1/d}})$, which shrinks to merely polynomial $O(kn^2)$ at $d = \log k$.

1.4 Our tradeoffs for AC^0 and SAC^0 formulas

Lower bounds (+) and (-) of Theorem 1.1 were proved using a technique, known as the Pathset Framework, which reduces formulas computing $\text{SUB-PMM}_{n,k}$ to simpler combinatorial objects called *join trees* (binary trees of \cup -gates which construct the length- k path graph from its single-edge subgraphs). This reduction provides a means of lifting lower bounds for join trees to lower bounds for formulas computing $\text{SUB-PMM}_{n,k}$.

A curious feature of lower bound (-) for AC^0 formulas in [23] is that the reduction to join trees starts by converting AC^0 formulas an equivalent DeMorgan formulas with fan-in 2. The resulting $n^{\Omega(\log k)}$ lower bound applies to AC^0 formulas up to depth $\frac{\log n}{(\log \log n)^{O(1)}}$. Significantly, the depth d does not appear as a parameter in the lower bound, in contrast to typical AC^0 lower bounds that use switching lemmas or the polynomial

approximation method in bottom-up fashion. This is a powerful feature of the result, which is tight above depth $O(\log k)$. However, the lower bound technique does not (in any obvious way) yield lower bounds stronger than $n^{\Omega(\log k)}$ for formulas of depths $2, 3, 4, \dots$ below $O(\log k)$.

The results of this paper expand the Pathset Framework of [21, 23] in order to extend lower bounds (+) and (-) of Theorem 1.1 to size-depth tradeoffs for both AC^0 and SAC^0 formulas. We obtain tradeoffs for monotone formulas which match the upper bounds of Proposition 1.3. We are able to extend these tradeoffs to non-monotone formulas, at the expense of a slightly weaker lower bound and smaller range of k .

Theorem 1.6 (Size-depth tradeoffs for AC^0 and SAC^0 formulas computing $\text{SUB-PMM}_{n,k}$). *For all $k \leq \log \log n$ and $d \leq \log k$, size $n^{\Omega(dk^{1/d})}$ is required by both*

- (I)⁺ *monotone SAC^0 formulas of \wedge -depth d and \wedge -fan-in $n^{1/k}$, and*
- (II)⁺ *monotone AC^0 formulas of depth $d + 1$.*

In the non-monotone setting, for all $k \leq \log^ n$ and $d \leq \log k$, size $n^{\Omega(dk^{1/2d})}$ is required by both*

- (I)⁻ *SAC^0 formulas of \wedge -depth d and \wedge -fan-in $n^{1/k}$, and*
- (II)⁻ *AC^0 formulas of depth $d + 1$.*

Note that the maximum \wedge -fan-in $n^{1/k}$ in SAC^0 lower bounds (I)⁺ and (I)⁻ exceeds the \wedge -fan-in $k^{1/d}$ of formulas (I) _{Σ_{2d}} of Proposition 1.3. Monotone lower bounds (I)⁺ and (II)⁺ are thus tight up to the constant in the exponent of n , in light of the upper bounds given by Proposition 1.3. Also observe that AC^0 tradeoffs (II)⁺ and (II)⁻ converge at $d = \log k$ to the tight $n^{\Omega(\log k)}$ lower bounds (+) and (-) of Theorem 1.1.

Below we remark on a few corollaries and extensions of Theorem 1.6.

Tradeoffs for circuits. Formula lower bounds $n^{\Omega(dk^{1/d})}$ and $n^{\Omega(dk^{1/2d})}$ of Theorem 1.6 directly imply $n^{\Omega(k^{1/d})}$ and $n^{\Omega(k^{1/2d})}$ lower bounds for the corresponding $(S)AC^0$ circuits. This follows from the observation that every size s AC^0 circuit of depth $d + 1$, or SAC^0 circuit of \wedge -depth d and \wedge -fan-in $s^{O(1/d)}$, expands to an $(S)AC^0$ formula with size at most $s^{d+O(1)}$.

The range of k . Examination of the proof in §7 shows that range $k \leq \log^* n$ of non-monotone lower bounds (I)⁻ and (II)⁻ can be improved to at least $k \leq 0.99 \log \log \log \log n$ (see Remark 7.35). The statement of Theorem 1.6 confines the range to $k \leq \log^* n$ for simplicity sake and to emphasize that we are happy with any super-constant k . With some effort, we believe the range $k \leq \log \log n$ of monotone lower bounds (I)⁻ and (II)⁻ can also be improved somewhat, though probably not to $k = n^{\Omega(1)}$ using our methods.

Tradeoff for $\text{PMM}_{n,k}$. Lower bound (II)⁻ on the AC^0 formula size of $\text{SUB-PMM}_{n,k}$ implies a slightly weaker $n^{\Omega(dk^{1/(2d+O(1))})}$ lower bound on the AC^0 formula size of the more restricted *Iterated Permutation Matrix Multiplication* problem $\text{PMM}_{n,k}$, whose AC^0 complexity was previously studied in [2, 4, 5]. This extension of lower bound (II)⁻ follows from the existence of a randomized AC^0 reduction from $\text{SUB-PMM}_{n,k}$ to $\text{PMM}_{n,k}$, using the method of [18, 10] to generate uniform random permutations in AC^0 (see also [29]).

1.5 Related work

1.5.1 Previous lower bounds

Prior to the results of this paper, the strongest size-depth tradeoff for $\text{PMM}_{n,k}$ (or $\text{SUB-PMM}_{n,k}$) was $n^{k^{\exp(-O(d))}}$, proved by Beame, Impagliazzo and Pitassi [4] using a special purpose *switching lemma*. This tradeoff, which is non-trivial to depth $O(\log \log k)$, improved previous tradeoffs of [2, 5] which were only non-trivial to depth $O(\log^* k)$.

With respect to the significantly more general $\text{BMM}_{n,k}$ problem, Chen, Oliveira, Servedio and Tan [7] established a nearly tight tradeoff $n^{\Omega((1/d)k^{1/d})}$ lower bound for depth- d AC^0 circuits. This lower bound is non-trivial

to depth $O(\frac{\log k}{\log \log k})$, and for depths $d = o(\frac{\log k}{\log \log k})$ it is quantitatively stronger than the $n^{\Omega(k^{1/(2d-1)})}$ lower bound implied by Theorem 1.6 for depth- d AC^0 circuits. However, the result of [7] is fundamentally a size-depth trade-off for *Sipser functions* (read-once formulas with a carefully chosen fan-in sequence), combined with a reduction from the NC^1 -complete SIPSER problem to $BMM_{n,k}$. In contrast to [4] the present paper, this result says nothing about the complexity of $PMM_{n,k}$ or the questions of NC^1 vs. L or NC^1 vs. NL .

1.5.2 Different regimes of parameters n and k

Our focus in this paper is on the formula complexity of $SUB\text{-}PMM_{n,k}$ in the regime where k is an arbitrarily slow-growing but super-constant function of n . We have not attempted to optimize the ranges of $k \leq \log \log n$ and $k \leq \log^* n$ in Theorem 1.6. We remark that the circuit lower bound of Beame *et al* [4] and Chen *et al* [7] extend to larger ranges $k \leq \log n$ and $k \leq n^{1/5}$, respectively.

In the opposite parameter regime where $n = 2$ and k is unbounded, all three problems $BMM_{n,k}$, $SUB\text{-}PMM_{n,k}$ and $PMM_{n,k}$ are equivalent to $PARITY_k$ under AC^0 reductions. Here tight $2^{\Omega(k^{1/d})}$ and $2^{\Omega(dk^{1/d})}$ lower bounds for depth $d + 1$ AC^0 circuits and formulas computing $PARITY_k$ were shown by Håstad [11] and the author [22] respectively.

The complexity jumps at $n = 5$, where all three problems are complete for NC^1 by Barrington's Theorem [3]. The complexity of $BMM_{n,k}$ (resp. $SUB\text{-}PMM_{n,k}$ and $PMM_{n,k}$) then jumps again at $k = n$ to being complete for NL (resp. L). (See [19, 30] concerning TC^0 and ACC^0 circuits for $BMM_{n,k}$.)

1.5.3 Algebraic circuit classes

The Iterated Matrix Multiplication polynomial ($IMM_{n,k}$) is computable by set-multilinear formulas of product-depth d and size $n^{O(dk^{1/d})}$ by construction (I) of Proposition 1.3 with \sum, \prod replacing \wedge, \vee . A recent breakthrough of Limaye, Srinivasan and Tavenas [17] showed that arithmetic formulas of product-depth d computing $IMM_{n,k}$ require size $n^{k^{1/\exp(O(d))}}$. This tradeoff, quantitatively similar to [4], is the first super-polynomial lower bound for constant-depth arithmetic formulas. A related paper of Tavenas, Limaye, and Srinivasan [28] gives a different $(\log n)^{\Omega(dk^{1/d})}$ lower bound for set-multilinear formulas. In the further restricted class of set-multilinear formulas with the *few parse trees* property, Legarde, Limaye and Srinivasan [16] established a tight $n^{\Omega(dk^{1/d})}$ tradeoff.

He and Rossman [13] studied the complexity of $PMM_{n,k}$ in the hybrid Boolean-algebraic model of AC^0 formula that are invariant under a group action of $\text{Sym}(n)^{k-1}$. This is the class of AC^0 formulas on kn^2 variables (encoding permutation matrices $M^{(1)}, \dots, M^{(k)}$) which are syntactically invariant under a certain action of $\text{Sym}(n)^{k-1}$, where the i th symmetric group $\text{Sym}(n)$ acts by permuting both the columns of $M^{(i)}$ and rows of $M^{(i+1)}$. For instance, construction (II) of Proposition 1.3 produces $\text{Sym}(n)^{k-1}$ -invariant AC^0 formulas of depth $d + 1$ and size $n^{dk^{1/d} + O(1)}$ which solve $PMM_{k,n}$ for all $k \leq n$ and $d \leq \log k$. The paper [13] proves a sharply matching $n^{dk^{1/d} - O(1)}$ lower bound in the $\text{Sym}(n)^{k-1}$ -invariant setting. This lower bound even extends to $\text{Sym}(n)^{k-1}$ -invariant TC^0 formulas.

1.5.4 Smallest known formulas for $SUB\text{-}PMM_{n,k}$

Construction (II) of Proposition 1.3 in the limit $d = \log_2 k$ produces AC^0 formulas for $SUB\text{-}PMM_{n,k}$ of depth $O(\log k)$ and size $n^{\log_2 k + O(1)}$. These formulas are moreover $\text{Sym}(n)^{k-1}$ -invariant, monotone, and uniform. However, these formulas are not the smallest known. For all $k \leq n$, Rossman [23] showed that there exist (non- $\text{Sym}(n)^{k-1}$ -invariant, non-monotone, non-uniform) AC^0 formulas for $SUB\text{-}PMM_{n,k}$ of depth $O(\log k)$ and size $n^{\frac{1}{2} \log_2 k + O(1)}$. This was subsequently improved to $n^{\frac{1}{3} \log_\varphi k + O(1)}$ where $\varphi = \frac{1+\sqrt{5}}{2}$ by Kush and Rossman [15], who have conjectured that $\frac{1}{3} \log_\varphi k \approx 0.49 \log_2 k$ is optimal in the exponent of n .

1.6 Outline of the paper

The rest of this paper is organized as follows. §2 gives several definitions pertaining to AC^0 and SAC^0 formulas, subgraphs of Path_k , and join trees. §3 states our combinatorial main result (Theorem 3.6) consisting of two distinct tradeoffs (I) and (II) for join trees. Parts (I) and (II) of Theorem 3.6 are proved in the next two sections §4 and §5. §6 reviews the Pathset Framework (introduced in papers [21, 23]) and establishes new lower bounds for pathset complexity (Theorem 6.11), which follow from our tradeoffs for join trees. Finally, our size-depth tradeoffs for AC^0 and SAC^0 formulas (Theorem 1.6) are proved in §7 via reductions to our lower bounds for pathset complexity.

We advise the reader that §2–5 of this paper may be read as a self-contained work dealing exclusively with combinatorics of join trees for the graph Path_k . On the other hand, readers mainly interested in the application to Circuit Complexity (Theorem 1.6) might wish to skip §4–5 (the proof of Theorem 3.6) on an initial reading, in order to first understand our method of reducing $(S)AC^0$ formulas computing $\text{SUB-PMM}_{n,k}$ to pathset formulas computing dense relations $\mathcal{A} \subseteq [n]^k$ and finally to join trees computing the graph Path_k .

2 Preliminaries

\mathbb{Z} is the set of integers, and $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of nonnegative integers. Throughout this paper, d, k, m, n represent positive integers, and h, i, j, l, r, s, t represent (usually nonnegative) integers. $[i]$ stands for the set $\{1, \dots, i\}$. In particular, $[0]$ is the empty set \emptyset .

$\ln(\cdot)$ and $\log(\cdot)$ denote the base-e and base-2 logarithms. $\log^*(\cdot)$ is the number of times the $\log(\cdot)$ function must be iterated until the result is less than 1.

2.1 Graphs and trees

Graphs in this paper are simple graphs without isolated vertices. That is, a graph G consists of sets $V(G)$ and $E(G) \subseteq \binom{V(G)}{2}$ such that $V(G) = \bigcup_{e \in E(G)} e$. The empty graph is denoted by the empty set symbol \emptyset . With the exception of the infinite path graph $\text{Path}_{\mathbb{Z}}$, all other graphs considered in this paper are finite.

Trees in this paper are finite rooted trees. *Binary trees* are trees in which each non-leaf node has exactly two children, designed “left” and “right”. For binary trees T_1 and T_2 , we denote by $\langle T_1, T_2 \rangle$ the binary tree consisting of a root with T_1 and T_2 as its left and right subtrees.

2.2 AC^0 and SAC^0 formulas

An AC^0 formula on variables X_1, \dots, X_N is a rooted tree whose leaves (called “inputs”) are labeled by a constant 0 or 1 or literals X_i or \overline{X}_i , and whose non-leaves (called “gates”) are labeled by \wedge or \vee . An AC^0 formula is *monotone* if no input is labeled by a negative literal \overline{X}_i . Every [monotone] AC^0 formula computes a [monotone] Boolean function $\{0, 1\}^N \rightarrow \{0, 1\}$ in the usual way.

We measure the *size* of an AC^0 formula by the number of leaves that are labeled by literals. *Depth* (resp. \wedge -*depth*) is the maximum number of gates (resp. \wedge -gates) on a root-to-leaf-branch. *Fan-in* (resp. \wedge -*fan-in*) is the maximum number of children of any gate (resp. \wedge -gate).

Depth 0 AC^0 formulas (i.e., literals) are also known as Σ_0 formulas and Π_0 formulas. For $d \geq 1$, a Σ_d formula (resp. Π_d formula) is either a Π_{d-1} formula (resp. Σ_{d-1} formula) or a depth d AC^0 formula whose output gate is labeled \vee (resp. \wedge). Note that Σ_{2d+1} formulas are precisely the class of AC^0 formulas with \wedge -depth d .

SAC^0 formulas (where “S” stands for “semi-unbounded fan-in”) are AC^0 formulas with bounded \wedge -fan-in. This usually refers to \wedge -fan-in 2 (i.e., formulas with unbounded fan-in \vee gates and fan-in 2 \wedge gates). However, when speaking of problems $\text{BMM}_{n,k}$, $\text{SUB-PMM}_{n,k}$ and $\text{PMM}_{n,k}$, we allow \wedge -fan-in $O(k^{1/d})$ in the context of upper bounds, and at most $n^{1/k}$ in the context of lower bounds. Note that $k^{1/d} \ll n^{1/k}$ in the regime of parameters that we consider (i.e., $k \leq \log n$ and $d \leq \log k$).

2.3 Subgraphs of paths

The graphs that we will mainly consider are *disjoint unions of paths*. It is convenient to regard these as finite subgraphs of the infinite path graph $\text{Path}_{\mathbb{Z}}$ with vertex set \mathbb{Z} and edge set $\{\{i-1, i\} : i \in \mathbb{Z}\}$.

For integers $s < t$, let $\text{Path}_{s,t}$ denote subgraph of $\text{Path}_{\mathbb{Z}}$ with vertex set $\{i \in \mathbb{Z} : s \leq i \leq t\}$ and edge set $\{\{i-1, i\} \in \mathbb{Z} : s < i \leq t\}$. For positive integers k , we write Path_k for $\text{Path}_{0,k}$.

Note that every finite subgraph of $\text{Path}_{\mathbb{Z}}$ is a union of paths $\text{Path}_{s_1, t_1} \cup \dots \cup \text{Path}_{s_c, t_c}$ for unique integers $c \geq 0$ and $s_1 < t_1 < s_2 < \dots < t_{c-1} < s_c < t_c$.

The next three definitions introduce notation that will be heavily used throughout the paper.

Definition 2.1 (Parameters $\|G\|$, $\Delta(G)$, $\lambda(G)$). For a finite graph $G \subset \text{Path}_{\mathbb{Z}}$, let

$$\|G\| := \#\{\text{edges of } G\},$$

$$\Delta(G) := \#\{\text{connected components of } G\} = \#\{\text{vertices of } G\} - \#\{\text{edges of } G\},$$

$$\lambda(G) := \max_{\text{connected components } C \text{ of } G} \|C\| = \text{length of the longest path in } G.$$

Note that if $G = \text{Path}_{s_1, t_1} \cup \dots \cup \text{Path}_{s_c, t_c}$ where $s_1 < t_1 < s_2 < \dots < t_{c-1} < s_c < t_c$, then

$$\|G\| = \sum_{i=1}^c (t_i - s_i), \quad \Delta(G) = c, \quad \lambda(G) = \max_{i \in \{1, \dots, c\}} (t_i - s_i).$$

Definition 2.2 (The operation $G \ominus F$). For graphs $G, F \subset \text{Path}_{\mathbb{Z}}$, we denote by $G \ominus F$ the subgraph of G comprised of the connected components of G that are vertex-disjoint from F .

Definition 2.3. For a sequence of finite graphs $G_1, \dots, G_m \subset \text{Path}_{\mathbb{Z}}$, let

$$\vec{\Delta}(G_1, \dots, G_m) := \sum_{j \in [m]} \Delta(G_j \ominus (G_1 \cup \dots \cup G_{j-1})),$$

$$\vec{\lambda}(G_1, \dots, G_m) := \sum_{j \in [m]} \lambda(G_j \ominus (G_1 \cup \dots \cup G_{j-1})),$$

$$\vec{\lambda\Delta}(G_1, \dots, G_m) := \sum_{j \in [m]} \lambda(G_j \ominus (G_1 \cup \dots \cup G_{j-1})) \cdot \Delta(G_j \ominus (G_1 \cup \dots \cup G_{j-1})).$$

For $F \subset \text{Path}_{\mathbb{Z}}$, we also introduce a convenient “conditional $\vec{\Delta}$ ” notation:

$$\vec{\Delta}(G_1, \dots, G_m \mid F) := \sum_{j \in [m]} \Delta(G_j \ominus (F \cup G_1 \cup \dots \cup G_{j-1})).$$

The value of $\vec{\Delta}(G_1, \dots, G_m)$ is sensitive to the order of G_1, \dots, G_m , as the following example shows.

Example 2.4. Let $E_1, E_2, E_3, \dots, E_{25}$ be the single-edge subgraphs of Path_{25} in the standard order. We have

$$\vec{\Delta}(E_1, E_2, E_3, \dots, E_{25}) = 1,$$

since $\vec{\Delta}(E_1) = 1$ and $\vec{\Delta}(E_j \ominus (E_1 \cup \dots \cup E_{j-1})) = 0$ for $j = 2, \dots, 25$. However, we can increase the $\vec{\Delta}$ -value by permuting the sequence E_1, \dots, E_{25} as follows:

$$\vec{\Delta}(\underbrace{E_1, E_3, E_5, \dots, E_{25}}_{\text{odd indices}}, \underbrace{E_2, E_4, E_6, \dots, E_{24}}_{\text{even indices}}) = 13.$$

It is not hard to check that 13 is the maximum possible $\vec{\Delta}$ -value among all permutations of E_1, \dots, E_{25} . Generalizing this example from 25 to any k , we get a maximum $\vec{\Delta}$ -value of $\lceil k/2 \rceil$.

For later reference, the following lemma records some basic properties of $\vec{\Delta}(\cdot | \cdot)$, which follow directly from definitions.

Lemma 2.5. *Let $F, G_1, \dots, G_m, H_1, \dots, H_m$ be finite subgraphs of $\text{Path}_{\mathbb{Z}}$.*

(a) $\vec{\Delta}(G_1, \dots, G_m | F) \leq \vec{\Delta}(G_1, \dots, G_m | F_0)$ for all $F_0 \subseteq F$.

(b) $\vec{\Delta}(G_1, \dots, G_m | \emptyset) = \vec{\Delta}(G_1, \dots, G_m)$. *More generally,*

$$\vec{\Delta}(G_1, \dots, G_m | F) = \vec{\Delta}(F, G_1, \dots, G_m) - \Delta(F).$$

(c) $\Delta(G_1 \cup \dots \cup G_m) \leq \vec{\Delta}(G_1, \dots, G_m)$. *More generally,*

$$\Delta((G_1 \cup \dots \cup G_m) \ominus F) \leq \vec{\Delta}(G_1, \dots, G_m | F) \leq \vec{\Delta}(G_1 \ominus F, \dots, G_m \ominus F).$$

(d) $\vec{\Delta}(\cdot | \cdot)$ decomposes via the following “chain rule”:

$$\vec{\Delta}(G_1, \dots, G_m, H_1, \dots, H_n | F) = \vec{\Delta}(G_1, \dots, G_m | F) + \vec{\Delta}(H_1, \dots, H_n | F \cup G_1 \cup \dots \cup G_m).$$

(e) $\vec{\Delta}(G_1, G_2, G_3, \dots, G_m) = \vec{\Delta}(G_1, G_1 \cup G_2, G_1 \cup G_2 \cup G_3, \dots, G_1 \cup \dots \cup G_m)$. □

2.4 Join trees

We define the notion of *join trees* with respect to arbitrary finite graphs G . However, our focus will soon narrow to join trees for subgraphs of Path_k .

Definition 2.6 (Join tree). For any finite graph G , a G -*join tree* is a binary tree T together a labeling of nodes of T by subgraphs of G such that

- each leaf is labeled by a single-edge subgraph of G or the empty graph,
- each non-leaf is labeled by the union of the graphs labeling its children, and
- the root is labeled by G .

Note that the labeling function is induced by its value on leaves of T , given by a surjective partial function from $\{\text{leaves of } T\}$ to $E(G)$.

We consider non-standard notions of “depth” and “size” for join trees. Our combinatorial main theorem gives tradeoffs between these parameters, which we define next.

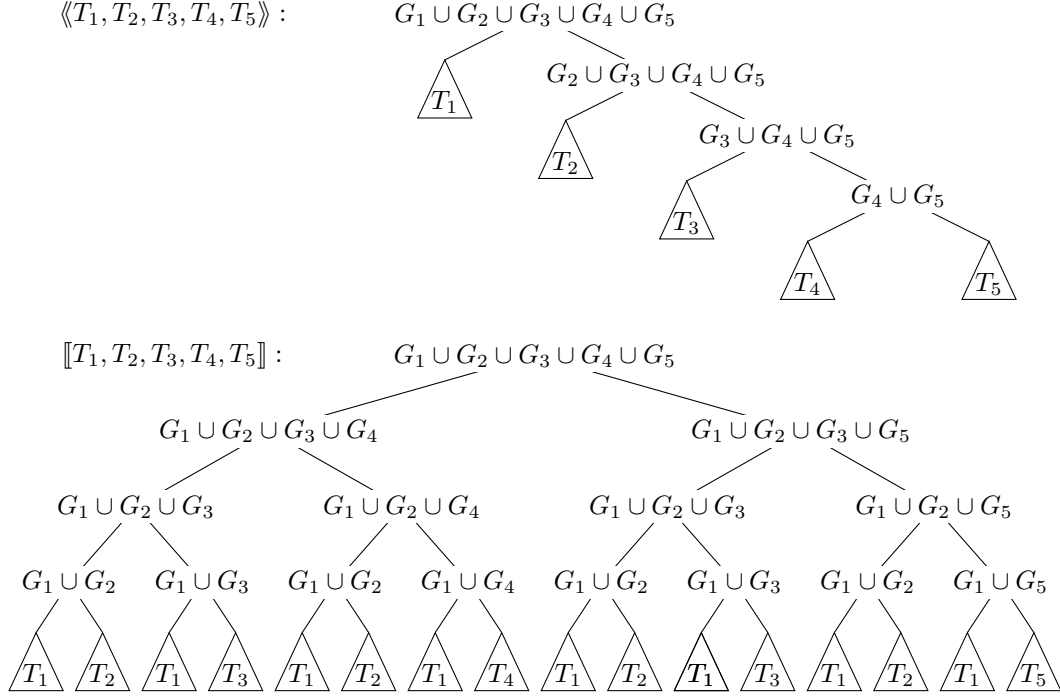
Definition 2.7 (Operations $\langle T_1, T_2 \rangle$, $\langle\langle T_1, \dots, T_m \rangle\rangle$ and $\llbracket T_1, \dots, T_m \rrbracket$). If T_1 is a G_1 -join tree and T_2 is a G_2 -join tree, then we denote by $\langle T_1, T_2 \rangle$ the $G_1 \cup G_2$ -join tree consisting of a root with left and right subtrees T_1 and T_2 with the naturally induced labeling of nodes.

Operations $\langle\langle \cdot \rangle\rangle$ and $\llbracket \cdot \rrbracket$ are defined inductively on finite sequences of join trees. In the base case $m = 1$, we have $\langle\langle T_1 \rangle\rangle = \llbracket T_1 \rrbracket := T_1$. For $m \geq 2$, we define

$$\langle\langle T_1, \dots, T_m \rangle\rangle := \langle T_1, \langle\langle T_2, \dots, T_m \rangle\rangle \rangle = \langle T_1, \langle T_2, \dots \langle T_{m-2}, \langle T_{m-1}, T_m \rangle \rangle \dots \rangle \rangle,$$

$$\llbracket T_1, \dots, T_m \rrbracket := \langle \llbracket T_1, \dots, T_{m-1} \rrbracket, \llbracket T_1, \dots, T_{m-2}, T_m \rrbracket \rangle = \llbracket \langle T_1, T_m \rangle, \langle T_2, T_m \rangle, \dots, \langle T_{m-1}, T_m \rangle \rrbracket.$$

Note that $\langle\langle T_1, T_2 \rangle\rangle = \llbracket T_1, T_2 \rrbracket = \langle T_1, T_2 \rangle$ in the case $m = 2$. However, note that operations $\langle\langle \cdot \rangle\rangle$ and $\llbracket \cdot \rrbracket$ are different for $m \geq 3$. For visual reference, the case $m = 5$ is picture below where T_1, \dots, T_5 are G_1, \dots, G_5 -join trees:



Definition 2.8 (Depth measures). We define $\langle \rangle$ -depth, $\langle \langle \rangle$ -depth and $\llbracket \rrbracket$ -depth as the pointwise minimum functions $\{\text{join trees}\} \rightarrow \{0, 1, 2, \dots\}$ satisfying

$$\begin{aligned} \text{depth}_{\langle \rangle}(\langle T_1, T_2 \rangle) &\leq 1 + \max\{\text{depth}_{\langle \rangle}(T_1), \text{depth}_{\langle \rangle}(T_2)\}, \\ \text{depth}_{\langle \langle \rangle}(\langle \langle T_1, \dots, T_m \rangle \rangle) &\leq 1 + \max\{\text{depth}_{\langle \langle \rangle}(T_1), \dots, \text{depth}_{\langle \langle \rangle}(T_m)\}, \\ \text{depth}_{\llbracket \rrbracket}(\llbracket T_1, \dots, T_m \rrbracket) &\leq 1 + \max\{\text{depth}_{\llbracket \rrbracket}(T_1), \dots, \text{depth}_{\llbracket \rrbracket}(T_m)\}. \end{aligned}$$

That is, the $\langle \rangle$ -depth (resp. $\langle \langle \rangle$, $\llbracket \rrbracket$ -depth) of T is the minimum nesting depth of binary \cup operations (resp. unbounded $\langle \langle \rangle$, $\llbracket \rrbracket$ operations) required to express T in terms of individual edges.

$\langle \rangle$ -depth is the standard notion of “depth” for binary trees, that is, the maximum length (= number of parent-to-child descents) of a root-to-leaf branch. $\langle \langle \rangle$ -depth is what might be called “left-depth”, that is, the maximum number of left descents on a root-to-leaf-branch. $\llbracket \rrbracket$ -depth is perhaps a new notion, which arises in connection to our tradeoffs for unbounded fan-in AC^0 formulas.

In order to define our “size” measure for join trees, we first introduce the notion of *branch coverings*.

Definition 2.9 (Branch coverings). Let T be a G -join tree. We associate each root-to-leaf branch (b_1, \dots, b_ℓ) in T (where b_1 is the root and b_ℓ is a leaf) with the sequence of graphs B_1, \dots, B_ℓ where

- B_j labels the sibling of b_{j+1} (i.e., the opposite-side child of b_j) for each $j \in \{1, \dots, \ell - 1\}$, and
- B_ℓ labels the leaf b_ℓ (note that $\|B_\ell\| \leq 1$).

Note that $B_1 \cup \dots \cup B_\ell = G$, that is, B_1, \dots, B_ℓ is a covering of G . We refer to sets $\{B_1, \dots, B_\ell\}$ arising from root-to-leaf branches as T -branch coverings of G .

One standard way of measuring the “size” of a binary tree is by the number of leaves (= number of root-to-leaf branches). For join trees T with graph $G \subset \text{Path}_k$, we consider a completely different “size” measure defined as the maximum of a certain complexity measure over all T -branch coverings $\{B_1, \dots, B_\ell\}$ of G . This complexity

measure is itself the maximum $\vec{\Delta}$ -value among all orderings $B_{\pi(1)}, \dots, B_{\pi(\ell)}$. This rather complicated “size” measure, denoted by Ψ , arises naturally in the context of *pathset complexity* (see §6 and in particular Corollary 6.7 and Lemma 6.10, which explain the roles of $\vec{\Delta}$ and Ψ).

Definition 2.10 (Ψ -size of join trees). Let T be a G -join tree where $G \subset \text{Path}_{\mathbb{Z}}$. We define the Ψ -size of T by

$$\Psi(T) := \max_{\substack{T\text{-branch coverings } \{B_1, \dots, B_\ell\} \text{ of } G \\ \text{and permutations } \pi : [\ell] \xrightarrow{\cong} [m]}} \vec{\Delta}(B_{\pi(1)}, \dots, B_{\pi(\ell)}).$$

Stated differently, $\Psi(T)$ is the maximum value of $\vec{\Delta}(C_1, \dots, C_m)$ over all T -branch covering \mathcal{C} of G and enumerations C_1, \dots, C_m of \mathcal{C} . This alternative description of $\Psi(T)$ emphasizes the fact that we treat T -branch coverings as (unordered) sets.

3 Tradeoffs for join trees

In this section we state the combinatorial main theorem of this paper, Theorem 3.6, which gives tight tradeoffs between the Ψ -size and the $\langle\langle\rangle\rangle$ - and $\llbracket\rrbracket$ -depth of Path_k -join trees. We also state two auxiliary results, Lemmas 3.4 and 3.5, which concern coverings of Path_k . Proofs of these results are postponed to §4 and §5 (the combinatorial core of the paper, which may be read independently of §6 and §7). We also state and prove a key numerical inequality, Lemma 3.9, which will be used later on in the inductive proofs of our tradeoffs for join trees.

3.1 Shift permutations

Given a sequence $G_1, \dots, G_m \subset \text{Path}_{\mathbb{Z}}$, we will be interested in the question of maximizing $\vec{\Delta}(G_{\pi(1)}, \dots, G_{\pi(m)})$ over permutations π . We will also be studying a variant of this question over a restricted class of permutations, which we call “shift permutations”. (The author is unaware of any standard terminology for this class of permutations.)

Definition 3.1. A *shift permutation* is a permutation $\sigma : [m] \xrightarrow{\cong} [m]$ satisfying $\sigma(j) \geq j - 1$ for all $j \in [m]$. (For clarity of notation, we will consistently write σ for shift permutations and π for general permutations.)

There are exactly 2^{m-1} shift permutations, which we will index via a bijection

$$I \mapsto \sigma_I : \{\text{subsets of } [m] \text{ containing } m\} \rightarrow \{\text{shift permutations}\}.$$

Suppose $I = \{i_1, \dots, i_p\}$ with $0 =: i_0 \leq i_1 < \dots < i_p = m$. Then we define $\sigma_I : [m] \xrightarrow{\cong} [m]$ by

$$\sigma_I : (1, \dots, m) \mapsto \underbrace{(i_1, 1, 2, \dots, i_1 - 1)}_{\text{first } i_1}, \underbrace{(i_2, i_1 + 1, i_2 + 2, \dots, i_2 - 1)}_{\text{next } i_2 - i_1}, \dots, \underbrace{(i_p, i_{p-1} + 1, i_{p-1} + 2, \dots, m - 1)}_{\text{last } m - i_{p-1}}.$$

That is,

$$\sigma_I(j) := \begin{cases} i_h & \text{if } j = i_{h-1} + 1 \text{ for some } h \in [p], \\ j - 1 & \text{otherwise.} \end{cases}$$

Note that $\sigma_{[m]}$ is the identity permutation and $\sigma_{\{m\}}$ is the m -cycle $(1, \dots, m) \mapsto (m, 1, \dots, m - 1)$. Also note that inverse of the bijection $I \mapsto \sigma_I$ is given by

$$\sigma \mapsto \left\{ j \in [m] : \{\sigma(1), \sigma(2), \dots, \sigma(j)\} = \{1, 2, \dots, j\} \right\} \quad \left(= \{\sigma(j) : j \in [m] \text{ such that } \sigma(j) \geq j\} \right).$$

Example 3.2. Again consider the sequence $E_1, E_2, E_3, \dots, E_{25}$ of the single-edge subgraphs of Path_{25} . Example 2.4 observes that $\vec{\Delta}(E_{\pi(1)}, \dots, E_{\pi(25)}) = 13$ where π is the permutation

$$\pi : (1, \dots, 25) \mapsto (\underbrace{1, 3, 5, \dots, 25}, \underbrace{2, 4, 6, \dots, 24}).$$

Note that π is not a shift permutation. However, we can achieve the same value $\vec{\Delta}(E_{\sigma(1)}, \dots, E_{\sigma(25)}) = 13$ via the shift permutation $\sigma = \sigma_{\{1,3,5,\dots,25\}}$. That is, we have

$$\vec{\Delta}(E_1, \underbrace{E_3, E_2}, \underbrace{E_5, E_4}, \underbrace{E_{25}, E_{24}}) = 13.$$

Generalizing from 25 to any odd k , the shift permutation $\sigma_{\{1,3,5,\dots,k\}}$ produces a $\vec{\Delta}$ -value of $\frac{k+1}{2}$.

Example 3.3. Suppose that instead of E_1, \dots, E_{25} in the usual order, we are given the sequence

$$\underbrace{E_1, E_6, E_{11}, E_{16}, E_{21}}_{\text{Group 1}}, \underbrace{E_2, E_7, E_{12}, E_{17}, E_{22}}_{\text{Group 2}}, \underbrace{E_3, E_8, E_{13}, E_{18}, E_{23}}_{\text{Group 3}}, \underbrace{E_4, E_9, E_{14}, E_{19}, E_{24}}_{\text{Group 4}}, \underbrace{E_5, E_{10}, E_{15}, E_{20}, E_{25}}_{\text{Group 5}}.$$

Note that the $\vec{\Delta}$ -value of this sequence equals 5.

What is the best $\vec{\Delta}$ -value we can obtain by applying a shift permutation to this sequence of graphs? It turns out that the answer is 7. Among many possibilities, this is achieved by the shift permutation $\sigma_{\{15,25\}}$, which produces the sequence

$$E_{23}, \underbrace{E_1, E_6, E_{11}, E_{16}, E_{21}}_{\text{Group 1}}, \underbrace{E_2, E_7, E_{12}, E_{17}, E_{22}}_{\text{Group 2}}, \underbrace{E_3, E_8, E_{13}, E_{18}, E_{25}}_{\text{Group 3}}, \underbrace{E_4, E_9, E_{14}, E_{19}, E_{24}}_{\text{Group 4}}, \underbrace{E_5, E_{10}, E_{15}, E_{20}}_{\text{Group 5}}.$$

Generalizing this example from 25 to any odd square k , we get an ordering $E_1, E_{\sqrt{k}+1}, E_{2\sqrt{k}+1}, \dots, E_k$ where applying the best shift permutation increases the $\vec{\Delta}$ -value from \sqrt{k} to $\sqrt{k} + \frac{\sqrt{k}-1}{2}$.

3.2 Combinatorial results of this paper

We now state the main combinatorial results of this paper, whose proofs are given in §4 and §5. Parts (I) and (II) in the lemmas and theorem below yield lower bounds which match upper bounds (I) and (II) of Proposition 1.3.

Lemma 3.4 (Pre-Main Lemma). *Suppose that $G_1 \cup \dots \cup G_m = \text{Path}_k$.*

- (I) *If $\max_{j \in [m]} \lambda(G_j) = 1$, then there exists a **permutation** π such that $\vec{\Delta}(G_{\pi(1)}, \dots, G_{\pi(m)}) \geq k/6$.*
- (II) *If $\vec{\Delta}(G_1, \dots, G_m) = 1$, then there exists a **shift permutation** σ s.t. $\vec{\lambda}(G_{\sigma(1)}, \dots, G_{\sigma(m)}) \geq k/4$.*

Lemma 3.5 (Main Lemma). *Suppose that $G_1 \cup \dots \cup G_m = \text{Path}_k$.*

- (I) *There exists a **permutation** π such that $\vec{\lambda}\vec{\Delta}(G_{\pi(1)}, \dots, G_{\pi(m)}) \geq k/30$.*
- (II) *There exists a **shift permutation** σ s.t. $\vec{\lambda}\vec{\Delta}(G_{\sigma(1)}, \dots, G_{\sigma(m)}) \geq \sqrt{k/8}$.*

Lemma 3.4 is used to prove Lemma 3.5, which is in turn used to prove the main combinatorial result of this paper:

Theorem 3.6. *Let T be any Path_k -join tree.*

- (I) $\Psi(T) \geq \frac{1}{30e} dk^{1/d} - d$ where d is the $\langle\langle\rangle\rangle$ -depth of T .
- (II) $\Psi(T) \geq \frac{1}{\sqrt{32e}} dk^{1/2d} - d$ where d is the $\llbracket\rrbracket$ -depth of T .

Note that inequalities (I) and (II) of Theorem 3.6 are respectively $\Omega(dk^{1/d})$ and $\Omega(dk^{1/2d})$ up to some $d = O(\log k)$, yet both inequalities become trivial for some slightly larger $d = \Omega(\log k)$. However, this doesn't bother us since a different bound $\Phi(T) \geq 0.35 \log k$ is known by a result of [15] for different parameter $\Phi(T)$ satisfying $\Phi(T) \geq \Psi(T)$. As we describe later in Corollary 6.12, we end up with tradeoffs $n^{\Omega(d(k^{1/d}-1))}$ and $n^{\Omega(d(k^{1/2d}-1))}$ for pathset complexity, which are tight for all d . In particular, these tradeoffs both converge to $n^{\Omega(\log k)}$ as $d \rightarrow \infty$.

3.3 Matching upper bounds

The following Examples 3.7 and 3.8 show that the lower bounds of Theorem 3.6 are tight up to a constant factor.

Example 3.7 (Tightness of Theorem 3.6(I)). Let P be a path of length k , where we assume $\ell := k^{1/d}$ is an integer. Consider the covering $P_1 \cup P_2 \cup \dots \cup P_\ell = P$ where P_1, P_2, \dots, P_ℓ are consecutive edge-disjoint paths of length k/ℓ ($= k^{(d-1)/d}$).

Let $T = \langle\langle T_1, T_2, \dots, T_k \rangle\rangle$ be the P -join tree of $\langle\langle \rangle\rangle$ -depth d where each T_i is the P_i -join tree of $\langle\langle \rangle\rangle$ -depth $d-1$ constructed recursively by the same process. Note that $\Psi(T) \leq \ell/2 + \max_i \Psi(T_i)$. It follows by induction that $\Psi(T) \leq d\ell/2 = dk^{1/d}/2$.

Example 3.8 (Tightness of Theorem 3.6(II)). Let P be a path of length k , where we assume $\ell := k^{1/2d}$ is an integer. Consider the covering $\bigcup_{(i,j) \in [\ell]^2} P_{i,j} = P$ where

$$\underbrace{P_{1,1}, P_{1,2}, \dots, P_{1,\ell}}_{\text{paths}}, \underbrace{P_{2,1}, P_{2,2}, \dots, P_{2,\ell}}_{\text{paths}}, \dots, \underbrace{P_{\ell,1}, P_{\ell,2}, \dots, P_{\ell,\ell}}_{\text{paths}}$$

are consecutive edge-disjoint paths of length k/ℓ^2 ($= k^{(d-1)/d}$). Note that $P_{i,1} \cup P_{i,2} \cup \dots \cup P_{i,\ell}$ is a path of length k/ℓ , while $P_{1,j} \cup P_{2,j} \cup \dots \cup P_{\ell,j}$ is a vertex-disjoint union of ℓ paths of length k/ℓ^2 .

Let

$$T = \langle\langle \underbrace{T_{1,1}, T_{2,1}, \dots, T_{\ell,1}}_{\text{trees}}, \underbrace{T_{1,2}, T_{2,2}, \dots, T_{\ell,2}}_{\text{trees}}, \dots, \underbrace{T_{1,\ell}, T_{2,\ell}, \dots, T_{\ell,\ell}}_{\text{trees}} \rangle\rangle$$

be the P -join tree of $\langle\langle \rangle\rangle$ -depth d where each $T_{i,j}$ is the $P_{i,j}$ -join tree of $\langle\langle \rangle\rangle$ -depth $d-1$ constructed recursively by the same process. Note that $\Psi(T) \leq 2\ell + \max_{i,j} \Psi(T_{i,j})$. It follows by induction that $\Psi(T) \leq 2d\ell = 2dk^{1/2d}$.

3.4 Numerical inequality used in the induction

Both inequalities of Theorem 3.6 are proved by induction on the parameter d . The following numerical inequality plays a key role in the induction step.

Lemma 3.9. For all real numbers $x_1, \dots, x_m, y_1, \dots, y_m \geq 0$ and $d > 1$,

$$\max_{j \in \{1, \dots, m\}} \left((d-1)x_j^{1/(d-1)} + \sum_{i=1}^j y_i \right) \geq d \left(\frac{1}{e} \sum_{j=1}^m x_j y_j \right)^{1/d}.$$

Proof. Let

$$Y_j := \sum_{i=1}^j y_i \quad \text{for } j \in \{0, \dots, m\},$$

$$Z := x_1 y_1 + \dots + x_m y_m.$$

Assume that $Z > 0$, since otherwise the inequality is trivially valid.

Toward a contradiction, assume that for all $j \in \{1, \dots, m\}$, we have

$$(d-1)x_j^{1/(d-1)} + Y_j < d \left(\frac{Z}{e} \right)^{1/d}.$$

It follows that

$$\begin{aligned} x_j y_j &\leq \frac{1}{(d-1)^{d-1}} \left(d \left(\frac{Z}{e} \right)^{1/d} - Y_j \right)^{d-1} y_j \\ &= \frac{1}{(d-1)^{d-1}} \int_{Y_{j-1}}^{Y_j} \left(d \left(\frac{Z}{e} \right)^{1/d} - Y_j \right)^{d-1} dw \\ &\leq \frac{1}{(d-1)^{d-1}} \int_{Y_{j-1}}^{Y_j} \left(d \left(\frac{Z}{e} \right)^{1/d} - w \right)^{d-1} dw. \end{aligned}$$

We now get a contradiction $Z < Z$ as follows:

$$\begin{aligned} Z = \sum_{j=1}^m x_j y_j &\leq \frac{1}{(d-1)^{d-1}} \int_0^{Y_m} \left(d \left(\frac{Z}{e} \right)^{1/d} - w \right)^{d-1} dw \\ &\leq \frac{1}{(d-1)^{d-1}} \int_0^{d(Z/e)^{1/d}} \left(d \left(\frac{Z}{e} \right)^{1/d} - w \right)^{d-1} dw && \text{(since } Y_m \leq d(Z/e)^{1/d} \text{ by} \\ &&& \text{case } j = m \text{ of our assumption)} \\ &= \frac{1}{(d-1)^{d-1}} \frac{1}{d} \left(d \left(\frac{Z}{e} \right)^{1/d} \right)^d && \text{(since } \int_0^C (C-w)^{d-1} dw = \frac{1}{d} C^{1/d} \\ &&& \text{for } C \geq 0) \\ &= \left(1 + \frac{1}{d-1} \right)^{d-1} \frac{Z}{e} \\ &< Z && \text{(since } 1 + C \leq \exp(C) \text{ for } C \geq 0). \quad \square \end{aligned}$$

Lemma 3.9 is invoked once and twice, respectively, in the induction step of lower bounds (I) and (II) of Theorem 3.6. This accounts for the different values $1/d$ vs. $1/2d$ in the exponent of k in these two bounds.

We remark that in the simplest case $m = 1$ of Lemma 3.9, one can show a stronger bound (without the constant $1/e$) by elementary calculus. We record this claim as a separate lemma, since we will use it later on (as one of the two applications of Lemma 3.9 in the proof of Theorem 3.6(II)).

Lemma 3.10 (The $m = 1$ case of Lemma 3.9). *For all real numbers $x, y \geq 0$ and $d > 1$, we have*

$$(d-1)x^{1/(d-1)} + y \geq d(xy)^{1/d}.$$

We remark that Lemma 3.10 is a common inequality that shows up in formula size-depth tradeoffs including [16, 22]. We suspect that the more general inequality given by Lemma 3.9 might find applications beyond the present paper to additional formula tradeoffs which involve the function $dk^{1/d}$.

4 Join tree tradeoff (I)

In this section we prove inequalities (I) of Lemmas 3.4–3.5 and Theorem 3.6, which we eventually use to prove tradeoffs (I)⁺ and (I)[−] of Theorem 1.6 for monotone and non-monotone SAC^0 circuits.

4.1 Proof of Pre-Main Lemma 3.4(I)

The following notion of $\vec{\Delta}$ -greedy sequences of subgraphs of $\text{Path}_{\mathbb{Z}}$ plays a key role in our proof of Lemma 3.4(I).

Definition 4.1 ($\vec{\Delta}$ -greedy sequence). For finite subgraphs F and G_1, \dots, G_m of $\text{Path}_{\mathbb{Z}}$, we say that the sequence (G_1, \dots, G_m) is $\vec{\Delta}$ -greedy over F if

$$\Delta(G_j \ominus (F \cup G_1 \cup \dots \cup G_{j-1})) \geq \Delta(G_l \ominus (F \cup G_1 \cup \dots \cup G_{j-1})) \text{ for all } 1 \leq j < l \leq m.$$

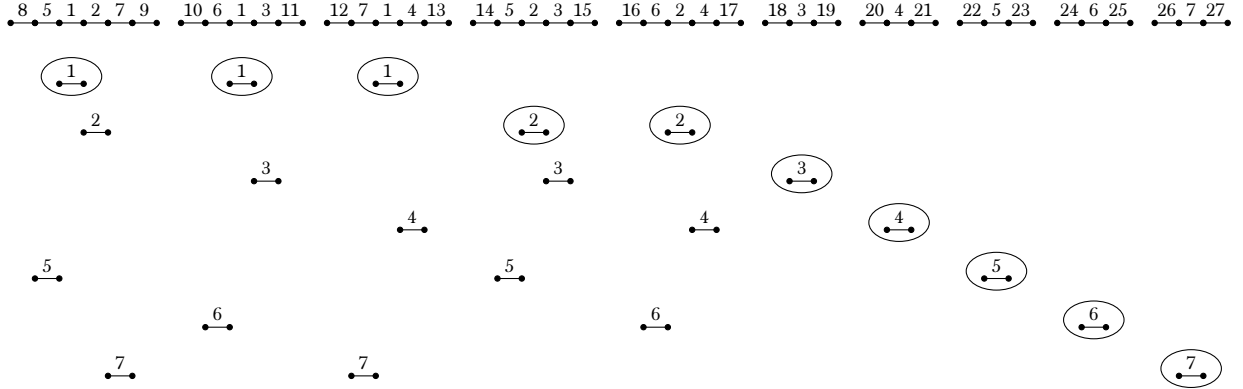
In the case $F = \emptyset$, we simply say that (G_1, \dots, G_m) is $\vec{\Delta}$ -greedy.

Note that for any F and finite family \mathcal{G} of finite subgraphs of $\text{Path}_{\mathbb{Z}}$, there exists an enumeration of (G_1, \dots, G_m) of \mathcal{G} which is $\vec{\Delta}$ -greedy with respect to F . We next point out that if (G_1, \dots, G_m) is $\vec{\Delta}$ -greedy over F , then

$$\Delta(G_1 \ominus F) \geq \Delta(G_2 \ominus (F \cup G_1)) \geq \Delta(G_3 \ominus (F \cup G_1 \cup G_2)) \geq \dots \geq \Delta(G_m \ominus (F \cup G_1 \cup \dots \cup G_{m-1})).$$

That is, $\vec{\Delta}$ -greediness implies that the sequence (t_1, \dots, t_m) is non-decreasing where $t_j = \Delta(G_j \ominus (F \cup G_1 \cup \dots \cup G_{j-1}))$. However, this is not sufficient, since Δ -greediness also requires that no transposition of two graphs in the sequence G_1, \dots, G_m increases the value of (t_1, \dots, t_m) under the lexicographic order.

Example 4.2. Illustrated below is a graph $P \subset \text{Path}_{\mathbb{Z}}$ (on top) with 41 edges, formed by a union of edge-disjoint subgraphs G_1, \dots, G_{27} . The sequence G_1, \dots, G_{27} is $\vec{\Delta}$ -greedy and has $\Delta(G_1, \dots, G_{27}) = 10$. Graphs G_1, \dots, G_7 (below P) each have three edges, while G_8, \dots, G_{27} each have a single edge. The components of $\Delta(G_j \ominus (G_1 \cup \dots \cup G_{j-1}))$, $j \in \{1, \dots, 7\}$, that contribute to $\Delta(G_1, \dots, G_{27})$ are circled.



To check that (G_1, \dots, G_{27}) is $\vec{\Delta}$ -greedy, it may be seen by inspection that

$$\begin{aligned} 3 = \Delta(G_1) &\geq \Delta(G_j) && \text{for } 2 \leq j \leq 27, \\ 2 = \Delta(G_2 \ominus G_1) &\geq \Delta(G_j \ominus G_1) && \text{for } 3 \leq j \leq 27, \\ 1 = \Delta(G_3 \ominus (G_1 \cup G_2)) &\geq \Delta(G_j \ominus (G_1 \cup G_2)) && \text{for } 4 \leq j \leq 27, \\ 1 = \Delta(G_4 \ominus (G_1 \cup G_2 \cup G_3)) &\geq \Delta(G_j \ominus (G_1 \cup G_2 \cup G_3)) && \text{for } 5 \leq j \leq 27, \end{aligned}$$

and so on.

This example is the $t = 3$ case of a more general construction. For every positive integer t , there is a $\vec{\Delta}$ -greedy sequence G_1, \dots, G_m with $\lambda(G_1) = \dots = \lambda(G_m) = 1$ and $\Delta(G_1) = t$ and

$$\begin{aligned} m &= t + (t+1) + (t+2)(t+1) && = t^2 + 5t + 3, \\ \|G_1 \cup \dots \cup G_m\| &= \sum_{j=1}^m \|G_j\| = \underbrace{t + \dots + t}_{t \text{ times}} + \underbrace{t + \dots + t}_{t+1 \text{ times}} + \underbrace{1 + \dots + 1}_{(t+2)(t+1) \text{ times}} && = 3t^2 + 4t + 2, \\ \Delta(G_1, \dots, G_m) &= t + (t-1) + (t-2) + \dots + 2 + 1 + \underbrace{1 + \dots + 1}_{t+1 \text{ times}} + \underbrace{0 + \dots + 0}_{(t+2)(t+1) \text{ times}} && = \frac{(t+2)(t+1)}{2}. \end{aligned}$$

As the following theorem shows, this construction achieves the maximum possible ratio of $\|G_1 \cup \dots \cup G_m\|$ to $\Delta(G_1, \dots, G_m)$ for any given t .

Theorem 4.3. *Suppose G_1, \dots, G_m are graphs such that $\lambda(G_1) = \dots = \lambda(G_m) = 1$ and (G_1, \dots, G_m) is $\vec{\Delta}$ -greedy over \emptyset . Let $t = \Delta(G_1)$. Then*

$$\|G_1 \cup \dots \cup G_m\| \leq \frac{2(3t^2 + 4t + 2)}{(t+2)(t+1)} \vec{\Delta}(G_1, \dots, G_m).$$

Moreover, this inequality is tight for all $t \geq 1$.

Note that Lemma 3.4(I) follows directly from this theorem, since $\frac{2(3t^2+4t+2)}{(t+2)(t+1)} > 6$ for all $t \geq 1$.

Proof. The theorem is proved by analyzing a linear program with $t+1$ constraints and $C_1 + \dots + C_{t+1} = \Theta(t^{-3/2}4^t)$ variables, where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n th Catalan number. This linear program turns out to have an integral solution that corresponds to the construction given in Example 4.2.

Without loss of generality, assume that for all $j \in [m]$, each edge of G_j has at most one endpoint in $V(G_1 \cup \dots \cup G_{j-1})$. In particular, G_1, \dots, G_m are edge-disjoint and no two connected components of $G_1 \cup \dots \cup G_{j-1}$ belong to the same connected component of $G_1 \cup \dots \cup G_j$.

For $j \in [m]$ and $s \in \{0, \dots, t\}$, we say that a graph G_j has *level s* if $\Delta(G_j \ominus (G_1 \cup \dots \cup G_{j-1})) = t - s$. In the sequence G_1, \dots, G_m , the level 0 graphs come first, followed by level 1, etc. For a graph G_j of level s , we define the *profile* of G_j as the sequence $(a_1, \dots, a_s) \in \mathbb{N}^s$ where a_r is the number of vertices that G_j shares with graphs of level $s - r$. For example, in Example 4.2 we have $t = 3$ and graphs G_1, G_2, G_3, G_4 have profiles $(), (1), (1, 1), (1, 1)$ respectively.

For $s \in \{0, \dots, t\}$ and $(a_1, \dots, a_s) \in \mathbb{N}^s$, let

$$w_{a_1, \dots, a_s} = \#\{\text{graphs of level } s \text{ with profile } (a_1, \dots, a_s)\}.$$

Note that

$$\begin{aligned} \vec{\Delta}(G_1, \dots, G_m) &= \sum_{s=0}^t (t-s) \cdot \#\{\text{graphs of level } s\} = \sum_{s=0}^t \sum_{a_1, \dots, a_s} (t-s) w_{a_1, \dots, a_s}, \\ \|G_1 \cup \dots \cup G_m\| &= \sum_{s=0}^t \sum_{a_1, \dots, a_s} (t-s + a_1 + \dots + a_s) w_{a_1, \dots, a_s}. \end{aligned}$$

s

Let $\gamma = \gamma(t)$ be the constant

$$\gamma := 5 + \frac{2}{t+1} - \frac{12}{t+2} \quad \left(= \frac{2(3t^2 + 4t + 2)}{(t+2)(t+1)} - 1 \right).$$

To prove the theorem (i.e., the inequality $\|G_1 \cup \dots \cup G_m\| \leq (\gamma + 1)\Delta(G_1, \dots, G_m)$), it suffices to show that

$$\sum_{s=0}^t \sum_{a_1, \dots, a_s} (t-s + a_1 + \dots + a_s) w_{a_1, \dots, a_s} \leq \sum_{s=0}^t \sum_{a_1, \dots, a_s} (\gamma + 1)(t-s) w_{a_1, \dots, a_s}.$$

The variables in our program will be indexed by Dyck sequences of length $\leq t$. We say that $(a_1, \dots, a_s) \in \mathbb{N}^s$ is a *Dyck sequence* if $a_1 + \dots + a_r \leq r$ for all $r \in [s]$. (Note: The number of Dyck sequences of length s is the Catalan number C_{s+1} .)

For all $s \in \{0, \dots, t\}$ and $(a_1, \dots, a_s) \in \mathbb{N}^s$, we claim that

$$(1) \quad w_{a_1, \dots, a_s} \neq 0 \implies (a_1, \dots, a_s) \text{ is a Dyck sequence.}$$

To see why, assume for contradiction that $w_{a_1, \dots, a_s} \neq 0$ and $a_1 + \dots + a_r \geq r + 1$ for some $r \in [s]$. Fix the minimum such r and note that $a_1 + \dots + a_{r-1} \leq r - 1$ and $a_r \geq 2$. Since $w_{a_1, \dots, a_s} \neq 0$, there exists $j \in [m]$ such that G_j has level s and profile (a_1, \dots, a_r) . Since a_r is positive, there exists at least $i \in [m]$ such that G_i has level $s - r$. Now fix the minimum such i and note that $i < j$. Observe that

$$\Delta(G_j \ominus (G_1 \cup \dots \cup G_{i-1})) = \Delta(G_j \ominus (G_1 \cup \dots \cup G_{j-1})) + a_1 + \dots + a_r,$$

since each edge of G_j that contributes to the sum $a_1 + \dots + a_r$ (i.e., shares an endpoint with a graph of level between $s - 1$ and $s - r$) contributes 0 to $\Delta(G_j \ominus (G_1 \cup \dots \cup G_{i-1}))$ and 1 to $\Delta(G_j \ominus (G_1 \cup \dots \cup G_{j-1}))$. Since $a_1 + \dots + a_r \geq r + 1$, it follows that

$$\begin{aligned} \Delta(G_j \ominus (G_1 \cup \dots \cup G_{i-1})) &\geq \Delta(G_j \ominus (G_1 \cup \dots \cup G_{j-1})) + r + 1 \\ &= t - s + r + 1 \\ &> t - s + r \\ &= \Delta(G_i \ominus (G_1 \cup \dots \cup G_{i-1})). \end{aligned}$$

However, this contradicts Δ -greediness of the sequence (G_1, \dots, G_m) , proving the implication (1).

We next observe $t + 1$ linear inequalities among the w -numbers:

$$(*_0) \quad w_{()} \geq 1,$$

$$(*_r) \quad \sum_{s=r}^t \sum_{a_1, \dots, a_s} a_{s-r} w_{a_1, \dots, a_s} \leq \sum_{a_1, \dots, a_{r-1}} \left(a_1 + \dots + a_{r-1} + 2(t - r + 1) \right) w_{a_1, \dots, a_{r-1}} \quad \text{for } r \in [t].$$

Here constraint $(*_0)$ expresses that “ $\Delta(G_1) \geq t$ ”. For $r \in [t]$, the constraint $(*_r)$ expresses “the number of endpoints of graphs of level $\geq r$ that match with endpoints of graph of level $r - 1$ is at most the number of *available* endpoints of level s ” (i.e., $2(t - r + 1)$ for each graph of level $r - 1$, plus 1 for each edge of level $r - 1$ that has an endpoint in level $\leq r - 2$).

Let us now forget the definition of numbers $w_{\vec{a}}$ in terms of graph G_1, \dots, G_m and instead view $w_{\vec{a}}$ as a family of real-valued variables indexed by Dyck sequences of length $\leq t$. Consider the linear program

$$(LP) \quad \max_{w_{\vec{a}} \in \mathbb{R}_{\geq 0} \text{ for each Dyck sequence } \vec{a} \text{ of length } \leq t: (*_0), \dots, (*_t)} \sum_{s=0}^t \sum_{a_1, \dots, a_s} \left(a_1 + \dots + a_s - (t - s)\gamma \right) w_{a_1, \dots, a_s}.$$

Our goal is to show that this (LP) has value 0.

In fact, an optimal solution is given by the following integer-valued w -vector. Let $\vec{\alpha}_0, \dots, \vec{\alpha}_t$ be the following Dyck sequences of length $0, \dots, t$ respectively:

$$\vec{\alpha}_0 = (), \quad \vec{\alpha}_1 = (1), \quad \vec{\alpha}_2 = (1, 1), \quad \dots, \quad \vec{\alpha}_{t-1} = \underbrace{(1, \dots, 1)}_{t-1}, \quad \vec{\alpha}_t = \underbrace{(1, 0, \dots, 0)}_{t-1}.$$

Then w is given by

$$\begin{aligned} w_{\vec{\alpha}_0} &= w_{\vec{\alpha}_1} = \dots = w_{\vec{\alpha}_{t-2}} = 1, \\ w_{\vec{\alpha}_{t-1}} &= t + 1, \\ w_{\vec{\alpha}_t} &= (t + 2)(t + 1), \\ w_{\vec{\beta}} &= 0 \quad \text{for all } \vec{\beta} \notin \{\vec{\alpha}_0, \dots, \vec{\alpha}_k\}. \end{aligned}$$

This vector corresponds to the greedy construction given in Example 4.2. A simple calculation shows that (LP) has value 0 under this w .

To prove optimality, we consider the dual linear program given by

$$(\text{dual LP}) \quad \min_{y_0, \dots, y_t \in \mathbb{R}_{\geq 0}: (*_{\vec{a}}) \text{ for each Dyck sequence } \vec{a} \text{ of length } \leq t} -y_0$$

where constraints $(\star_{\vec{a}})$ are as follows:

$$\begin{aligned}
(\star_{()}) \quad & \mathbf{y}_0 + 2t\mathbf{y}_1 \leq t\gamma, \\
(\star_{a_1, \dots, a_t}) \quad & \sum_{r=1}^t a_{t-r}\mathbf{y}_r \geq a_1 + \dots + a_t, \\
(\star_{a_1, \dots, a_s}) \quad & \sum_{r=1}^s a_{s-r}\mathbf{y}_r - \left(a_1 + \dots + a_s + 2(t-s) \right) \mathbf{y}_{s+1} \geq a_1 + \dots + a_s - (t-s)\gamma \quad \text{for } s \in [t-1].
\end{aligned}$$

We claim that the following vector $\mathbf{y} = (\mathbf{y}_0, \dots, \mathbf{y}_t) \in \mathbb{R}_{\geq 0}^{t+1}$ is an optimal solution to the dual LP:

$$\begin{aligned}
\mathbf{y}_0 &= 0, \\
\mathbf{y}_r &= \frac{\gamma}{2} - \frac{(r-1)(4t+2-r)}{2(2t+2-r)(2t+1-r)} \quad \text{for } r \in [t].
\end{aligned}$$

To see this, we first note that

$$\frac{5}{2} > \frac{\gamma}{2} = \mathbf{y}_1 > \mathbf{y}_2 > \dots > \mathbf{y}_t = 1.$$

Next, note that constraint $(\star_{()})$ clearly holds with equality (i.e., $\mathbf{y}_0 + 2t\mathbf{y}_1 = 0 + 2t\frac{\gamma}{2} = \gamma$). To check that \mathbf{y} satisfies $(\star_{a_1, \dots, a_t})$ for Dyck sequences of length t , we have

$$\begin{aligned}
\sum_{r=1}^t a_{t-r}\mathbf{y}_r - (a_1 + \dots + a_t) &= \sum_{s=1}^t (\mathbf{y}_{t-s+1} - 1)a_s \\
&= \sum_{s=1}^t \left(\frac{\gamma}{2} - \frac{(t-s)(3t+s+1)}{2(t+s+1)(t+s)} - 1 \right) a_s \\
&\geq \sum_{s=1}^t \left(\frac{\gamma}{2} - \frac{(t-s)(3t+s+1)}{2(t+s+1)(t+s)} - 1 \right) \\
&= \frac{t^2(t-1)}{(t+1)(t+2)} \\
&\geq 0.
\end{aligned}$$

The first inequality above follows from Karamata's inequality, using the fact that $(1, \dots, 1)$ majorizes (a_1, \dots, a_t) and the rational function

$$\rho(x) := \frac{(t-x)(3t+x+1)}{2(t+x+1)(t+x)}$$

is convex over $x \in [0, t]$. (Karamata's inequality implies that $\sum_{s=1}^t \rho(s)a_s \leq \sum_{s=1}^t \rho(s)$ for every convex function ρ .) This shows that \mathbf{y} satisfies $(\star_{a_1, \dots, a_t})$. The argument that \mathbf{y} satisfies $(\star_{a_1, \dots, a_s})$ for each Dyck sequence of length $s < t$ uses a similar application of Karamata's inequality.

Finally, note that the objective function $-\mathbf{y}_0$ of the dual LP at the feasible point \mathbf{y} equals 0. Therefore, both the primal and dual LPs have value 0, which completes the proof. \square

Remark 4.4. Optimality of \mathbf{w} and \mathbf{y} may be seen by considering the primal LP in standard matrix form with columns restricted to the support of \mathbf{w} (i.e., with columns $\vec{\alpha}_0, \dots, \vec{\alpha}_t$). This looks like

$$\max_{\mathbf{w}_{\vec{\alpha}_0}, \dots, \mathbf{w}_{\vec{\alpha}_t} \geq 0} f^\top \begin{pmatrix} \mathbf{w}_{\vec{\alpha}_0} \\ \mathbf{w}_{\vec{\alpha}_1} \\ \vdots \\ \mathbf{w}_{\vec{\alpha}_t} \end{pmatrix} \quad \text{subject to} \quad M \begin{pmatrix} \mathbf{w}_{\vec{\alpha}_0} \\ \mathbf{w}_{\vec{\alpha}_1} \\ \vdots \\ \mathbf{w}_{\vec{\alpha}_t} \end{pmatrix} \leq \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where

$$f = \begin{pmatrix} -t\gamma \\ 1 - (t-1)\gamma \\ 2 - (t-2)\gamma \\ \vdots \\ t-2-2\gamma \\ t-1-\gamma \\ 1 \end{pmatrix}, \quad M = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2t & 1 & 1 & \cdots & 1 & 1 & 0 \\ 0 & -2t+1 & 1 & \cdots & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -t-2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -t-1 & 1 \end{pmatrix}.$$

Optimality of w and y is then certified by checking that

$$M \begin{pmatrix} w_{\bar{\alpha}_0} \\ w_{\bar{\alpha}_1} \\ \vdots \\ w_{\bar{\alpha}_t} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad M^\top \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_t \end{pmatrix} = f, \quad f^\top \begin{pmatrix} w_{\bar{\alpha}_0} \\ w_{\bar{\alpha}_1} \\ \vdots \\ w_{\bar{\alpha}_t} \end{pmatrix} = -y_0 = 0.$$

Corollary 4.5. *If (G_1, \dots, G_m) is $\vec{\Delta}$ -greedy over F , then*

$$\frac{\|(G_1 \cup \dots \cup G_m) \ominus F\|}{\max_{j \in [m]} \lambda(G_j \ominus F)} \leq 6\vec{\Delta}(G_1, \dots, G_m \mid F).$$

Proof. Suppose (G_1, \dots, G_m) is $\vec{\Delta}$ -greedy over F . We can easily modify these graphs to obtain a sequence (G'_1, \dots, G'_m) which is $\vec{\Delta}$ -greedy over \emptyset and satisfies

$$\begin{aligned} \lambda(G'_1) &= \dots = \lambda(G'_m) = 1, \\ \vec{\Delta}(G'_1, \dots, G'_m) &= \Delta(G_1, \dots, G_m \mid F), \\ \|G_1 \cup \dots \cup G_m\| &\leq \|G'_1 \cup \dots \cup G'_m\| \cdot \max_{j \in [m]} \lambda(G_j \ominus F). \end{aligned}$$

The idea is simple: we form G'_j by contracting each component of $G_j \ominus (F \cup G_1 \cup \dots \cup G_{j-1})$ that is contained in $(G_1 \cup \dots \cup G_m) \ominus F$ to a single edge. The bound then follows from Theorem 4.3. \square

Corollary 4.5 is not exactly what we need for our proof of Lemma 3.5(I) in the next subsection. Instead of a bound on $\|(G_1 \cup \dots \cup G_m) \ominus F\|$, we require a bound on the potentially larger quantity $\|(G \ominus F) \cup \dots \cup (G_m \ominus F)\|$. Fortunately, LP duality again yields a lower bound.

Lemma 4.6 (Strengthening of Lemma 3.4(I)). *If (G_1, \dots, G_m) is $\vec{\Delta}$ -greedy over F , then*

$$\frac{\|(G_1 \ominus F) \cup \dots \cup (G_m \ominus F)\|}{\max_{j \in [m]} \lambda(G_j \ominus F)} \leq 5\Delta(F) + 6\vec{\Delta}(G_1, \dots, G_m \mid F).$$

Proof. We slightly modify the linear program considered in Theorem 4.3. For $t = \Delta(G_1 \ominus F)$, we now index variables by sequences $(a_1, \dots, a_s; b)$ where a_1, \dots, a_s is a Dyck sequence of length $\leq t$ and $b \in \{0, \dots, 2\Delta(F)\}$ (representing the number of endpoints of F that a given G_j covers). In addition to (appropriately modified) constraints $(*_0), \dots, (*_t)$, we introduce an extra constraint

$$(*_{t+1}) \quad \sum_{a_1, \dots, a_s} \sum_b w_{a_1, \dots, a_s; b} \leq 2\Delta(F).$$

We maximize essentially the same objective function as before:

$$\sum_{s=0}^t \sum_{a_1, \dots, a_s} \sum_b (a_1 + \dots + a_s - (t-s)\gamma) w_{a_1, \dots, a_s; b}$$

for the same constant $\gamma := 5 + \frac{2}{t+1} - \frac{12}{t+2} (< 5)$. The value of this LP gives an upper bound on

$$(\gamma + 1) \cdot \vec{\Delta}(G_1, \dots, G_m \mid F) - \frac{\|(G_1 \ominus F) \cup \dots \cup (G_m \ominus F)\|}{\max_{j \in [m]} \lambda(G_j \ominus F)}.$$

It turns out that this value is $\gamma \cdot \Delta(F)$. However, unlike the LP of Theorem 4.3 (or this modified version in the case $\Delta(F) = 0$), the primal optimal solution is no longer integral. Nevertheless, we can show that $\gamma \cdot \Delta(F)$ is an upper bound by considering the dual LP. The dual optimal solution $\mathbf{y} = (\mathbf{y}_0, \dots, \mathbf{y}_{t+1})$ happens to coincide on $\mathbf{y}_0, \dots, \mathbf{y}_t$ with the dual optimal solution of Theorem 4.3; the extra coordinate \mathbf{y}_{t+1} equals $\gamma/2$. We omit the calculations showing this, as the details are similar to Theorem 4.3. \square

4.2 Proof of Main Lemma 3.5(I)

We shall now prove:

Main Lemma 3.5(I) (rephrased). *For every covering \mathcal{G} of Path_k , there exists a sequence G_1, \dots, G_m of graphs in \mathcal{G} (w.l.o.g. an enumeration of \mathcal{G}) such that*

$$\vec{\lambda} \Delta(G_1, \dots, G_m) \geq \frac{k}{30}.$$

Proof. Let $r = \lceil \log(k+1) \rceil$. We greedily construct integers $m_1, \dots, m_r \geq 0$ and a sequence

$$G_{1,1}, \dots, G_{1,m_1}, G_{2,1}, \dots, G_{2,m_2}, \dots, G_{r,1}, \dots, G_{r,m_r} \in \mathcal{G}$$

such that, letting $F_0 = \emptyset$ and

$$\begin{aligned} F_i &:= F_{i-1} \cup G_{i,1} \cup \dots \cup G_{i,m_i}, \\ G^{i,j} &:= G_{i,j} \ominus (F_{i-1} \cup G_{i,1} \cup \dots \cup G_{i,j-1}), \end{aligned}$$

the following properties hold for all $i \in [r]$ and $j \in [m_i]$:

- (i) $\lambda(G \ominus F_i) < 2^{r-i}$ for all $G \in \mathcal{G}$,
- (ii) $\lambda(G^{i,j}) \geq 2^{r-i}$,
- (iii) $\Delta(G^{i,j}) = \max_{G \in \mathcal{G} : \lambda(G \ominus (F_{i-1} \cup G_{i,1} \cup \dots \cup G_{i,j-1})) \geq 2^{r-i}} \Delta(G \ominus (F_{i-1} \cup G_{i,1} \cup \dots \cup G_{i,j-1}))$.

To see that such a sequence exists, assume for $i \in [r]$ and $j \geq 1$ that we have constructed integers m_1, \dots, m_{i-1} (but not yet m_i) and graphs $G_{1,1}, \dots, G_{i,j-1}$ (but not $G_{i,j}$) which so far satisfy properties (i), (ii), (iii). There are now two possibilities for the next step in the construction:

- Suppose that there exists $G \in \mathcal{G}$ such that $\lambda(G \ominus (F_{i-1} \cup G_{i,1} \cup \dots \cup G_{i,j-1})) \geq 2^{r-i}$.
In this case, we pick $G_{i,j}$ to be any such G which maximizes $\Delta(G \ominus (F_{i-1} \cup G_{i,1} \cup \dots \cup G_{i,j-1}))$.
Note that properties (ii) and (iii) are satisfied with respect to i and j .
- Suppose that for all $G \in \mathcal{G}$, we have $\lambda(G \ominus (F_{i-1} \cup G_{i,1} \cup \dots \cup G_{i,j-1})) < 2^{r-i}$.
In this case, we set $m_i := j - 1$.
Note that property (i) is satisfied with respect to i .

For the base case $i = j = 1$ of this inductive procedure, note that $\lambda(G \ominus F_0) = \lambda(G) \leq k < 2^r$ for all $G \in \mathcal{G}$.

Having constructed the sequence $G_{1,1}, \dots, G_{r,m_r}$, we turn to showing that

$$\vec{\lambda\Delta}(G_{1,1}, \dots, G_{r,m_r}) \left(= \sum_{i \in [r]} \sum_{j \in [m_i]} \lambda(G^{i,j}) \Delta(G^{i,j}) \right) \geq \frac{k}{30}.$$

Note that by property (ii), for all $i \in [r]$,

$$(2) \quad 2^{r-i} \vec{\Delta}(G_{i,1}, \dots, G_{i,m_i} \mid F_i) = 2^{r-i} \sum_{j \in [m_i]} \Delta(G^{i,j}) \leq \sum_{j \in [m_i]} \lambda(G^{i,j}) \Delta(G^{i,j}).$$

We next prove two inequalities:

$$(3) \quad \sum_{i \in [r]} 2^{r-i} \Delta(F_i) \leq 2 \sum_{i \in [r]} \sum_{j \in [m_i]} \lambda(G^{i,j}) \Delta(G^{i,j}),$$

$$(4) \quad \sum_{i \in [r]} \left\| \bigcup_{j \in [m_i]} (G_{i,j} \ominus F_{i-1}) \right\| \leq 22 \sum_{i \in [r]} \sum_{j \in [m_i]} \lambda(G^{i,j}) \Delta(G^{i,j}).$$

For inequality (3), noting that $\emptyset = F_0 \subseteq F_1 \subseteq \dots \subseteq F_r$, we have

$$\begin{aligned} \sum_{i \in [r]} 2^{r-i} \Delta(F_i) &= \sum_{i \in [r]} 2^{r-i} \sum_{h \in [i]} \Delta(F_h \ominus F_{h-1}) \\ &\leq 2 \sum_{i \in [r]} 2^{r-i} \Delta(F_i \ominus F_{i-1}) \\ &= 2 \sum_{i \in [r]} 2^{r-i} \Delta((G_{i,1} \cup \dots \cup G_{i,m_i}) \ominus F_{i-1}) \\ &\leq 2 \sum_{i \in [r]} 2^{r-i} \Delta(G_{i,1}, \dots, G_{i,m_i} \mid F_{i-1}) && \text{(by Lemma 2.5 (a))} \\ &\leq 2 \sum_{i \in [r]} \sum_{j \in [m_i]} \lambda(G^{i,j}) \Delta(G^{i,j}) && \text{(by (2)).} \end{aligned}$$

For inequality (4), note that condition (iii) implies that $(G_{i,1}, \dots, G_{i,m_i})$ is $\vec{\Delta}$ -greedy over F_{i-1} for all $i \in [r]$. Therefore, by Lemma 4.6 and property (i),

$$\left\| \bigcup_{j \in [m_i]} (G_{i,j} \ominus F_{i-1}) \right\| \leq 2^{r-i+1} \left(5\Delta(F_{i-1}) + 6\Delta(G_{i,1}, \dots, G_{i,m_i} \mid F_{i-1}) \right).$$

Inequality (4) now follows from (2) and (3).

In the next part of the argument, we consider the 2^{r-i+1} -neighborhood of each graph F_i . To define this formally: for a finite graph $P \subset \text{Path}_{\mathbb{Z}}$, let us write $\text{Nbd}_1(P)$ for the graph consisting of all edges $\text{Path}_{\mathbb{Z}}$ incident to a vertex of P . For $t \in \mathbb{N}$, let

$$\text{Nbd}_t(P) := \underbrace{\text{Nbd}_1(\dots \text{Nbd}_1(P) \dots)}_{t \text{ times}}.$$

Note the inequalities $\Delta(\text{Nbd}_t(P)) \leq \Delta(P)$ and $\|\text{Nbd}_t(P)\| \leq 2t\Delta(P) + \|P\|$.

For finite graphs $P, Q \subset \text{Path}_{\mathbb{Z}}$, let $P \setminus Q$ denote the graph with edge set $E(P) \setminus E(Q)$ (and no isolated vertices). Here is another obvious inequality, numbered for later reference:

$$(5) \quad \|\text{Nbd}_t(P) \setminus Q\| \leq 2t\Delta(P) + \|P \setminus Q\|.$$

Note that $P \ominus Q \subseteq P \setminus Q$. In the other direction, note that $P \setminus Q_1 \subseteq P \ominus Q_2$ if and only if $(C \subseteq Q_1$ or $V(C) \cap V(Q_2) = \emptyset)$ for every connected component C of P .

Now comes a key claim: for all $i \in [r]$ and $G \in \mathcal{G}$, we have

$$(6) \quad G \setminus \bigcup_{h \in [i]} \text{Nbd}_{2^{r-h+1}}(F_h) \subseteq G \ominus F_i.$$

To see why, suppose C is a connected component of G such that $V(C) \cap V(F_i)$ is nonempty. We must show that $C \subseteq \bigcup_{h \in [i]} \text{Nbd}_{2^{r-h+1}}(F_h)$. There are two cases:

- If $\|C\| < 2^{r-i+1}$, then clearly $C \subseteq \text{Nbd}_{2^{r-i+1}}(F_i)$ since C is connected and shares a vertex with F_i .
- If $\|C\| \geq 2^{r-i+1}$, then we consider the unique $h \in [i-1]$ such that $2^{r-h} \leq \|C\| < 2^{r-h+1}$. (Note that h is well-defined since $\|C\| \leq k < 2^r$.)

By property (i), we have $\lambda(G \ominus F_h) < 2^{r-h}$. It follows that $V(C) \cap V(F_h)$ is nonempty, since otherwise $C \subseteq G \ominus F_h$ and we would have a contradictory inequality $2^{r-h} \leq \|C\| \leq \lambda(G \ominus F_h) < 2^{r-h}$. Since C is connected of length $< 2^{r-h+1}$ and shares a vertex with F_h , we conclude that $C \subseteq \text{Nbd}_{2^{r-h+1}}(F_h)$.

We next observe that for all $G \in \mathcal{G}$, case $i = r$ of property (i) implies that $\lambda(G \ominus F_r) < 1$ and hence $G \ominus F_r = \emptyset$. Case $i = r$ of the containment (6) now yields

$$G \setminus \bigcup_{i \in [r]} \text{Nbd}_{2^{r-i+1}}(F_i) \subseteq G \ominus F_r = \emptyset, \quad \text{that is,} \quad G \subseteq \bigcup_{i \in [r]} \text{Nbd}_{2^{r-i+1}}(F_i).$$

Since $\bigcup_{G \in \mathcal{G}} G = \text{Path}_k$, it follows that $\text{Path}_k \subseteq \bigcup_{i \in [r]} \text{Nbd}_{2^{r-i+1}}(F_i)$.

The proof now finishes as follows:

$$\begin{aligned} k &\leq \left\| \bigcup_{i \in [r]} \text{Nbd}_{2^{r-i+1}}(F_i) \right\| \\ &\leq \sum_{i \in [r]} \left\| \text{Nbd}_{2^{r-i+1}}(F_i) \setminus \bigcup_{h \in [i-1]} \text{Nbd}_{2^{r-h+1}}(F_h) \right\| \\ &\leq \sum_{i \in [r]} \left(2^{r-i+2} \Delta(F_i) + \left\| F_i \setminus \bigcup_{h \in [i-1]} \text{Nbd}_{2^{r-h+1}}(F_h) \right\| \right) && \text{(by (5))} \\ &= 4 \sum_{i \in [r]} 2^{r-i} \Delta(F_i) + \sum_{i \in [r]} \left\| \bigcup_{j \in [m_j]} (G_{i,j} \setminus \bigcup_{h \in [i-1]} \text{Nbd}_{2^{r-h+1}}(F_h)) \right\| \\ &\leq 4 \sum_{i \in [r]} 2^{r-i} \Delta(F_i) + \sum_{i \in [r]} \left\| \bigcup_{j \in [m_j]} (G_{i,j} \ominus F_i) \right\| && \text{(by (6))} \\ &\leq 30 \sum_{i \in [r]} \sum_{j \in [m_i]} \lambda(G^{i,j}) \Delta(G^{i,j}) && \text{(by (3) and (4))} \\ &= 30 \overline{\lambda \Delta}(G_{1,1}, \dots, G_{r,m_r}). \end{aligned} \quad \square$$

4.3 Proof of Theorem 3.6(I)

We will actually prove the following slightly more general version of Theorem 3.6(I), whose statement is better suited to induction on d .

Main Theorem 3.6(I) (slightly more general restatement). *Let P be a finite subgraph of $\text{Path}_{\mathbb{Z}}$ and let T be a P -join tree of $\langle\langle \rangle\rangle$ -depth d . Then*

$$\Psi(T) \geq \frac{1}{30e} d \lambda(P)^{1/d} + \Delta(P) - d.$$

Proof. We argue by induction on d . To clarify the exposition, we give headers to the various steps in the argument.

The base case $d = 1$

(We could actually treat $d = 0$ as the base case of the induction, but instead choose $d = 1$ since it will help to illustrate the reduction to the case $\Delta(P) = 1$ in the next step.)

For the base case $d = 1$, let \mathcal{G} be the rightmost branch covering of T . Note that \mathcal{G} is the set of single-edge subgraphs of P .

The graph P is a disjoint union of paths P_1, \dots, P_c where $c := \Delta(P)$ and $\lambda(P) = \|P_1\|$ (without loss of generality). Let S_1 be a set consisting of $\lceil \lambda(P)/2 \rceil$ vertex-disjoint edges in P_1 , and let S_2 be a set containing one edge from each of P_2, \dots, P_c . Let e_1, \dots, e_m be any enumeration of the edges of P with respect to which the edges in $S_1 \cup S_2$ come before edges outside this set, and let G_1, \dots, G_m be the corresponding enumeration of \mathcal{G} . We have

$$\vec{\Delta}(G_1, \dots, G_m) \geq |S_1| + |S_2| \geq \frac{1}{2}\lambda(P) + c = \frac{1}{2}d\lambda(P)^{1/d} + \Delta(P) - d,$$

which proves the base case since $1/2 > 1/30e$.

Induction step: reduction to the case $\Delta(P) = 1$

For the induction step, let $d \geq 2$ and assume that the theorem holds for $d - 1$. We argue that it suffices to prove the induction step in the case that $\Delta(P) = 1$. To see this, we again decompose P as a disjoint union of paths P_1, \dots, P_c where $c := \Delta(P)$ and $\lambda(P) = \|P_1\|$.

Let T_1 be the P_1 -join tree obtained from T by relabeling to \emptyset (the empty graph) any leaf originally labeled by an edge of $P_2 \cup \dots \cup P_c$. Assume that we are given a T_1 -branch covering \mathcal{G}_1 of P_1 and an enumeration H_1, \dots, H_m of \mathcal{G}_1 such that

$$\vec{\Delta}(H_1, \dots, H_m) \geq \frac{1}{30e}d\lambda(P_1)^{1/d} - d + 1.$$

Let G_1, \dots, G_m be the graphs corresponding to H_1, \dots, H_m in the original tree T , and let $\mathcal{G} = \{G_1, \dots, G_m\}$. Note that \mathcal{G} is a T -branch covering of P . Also note that $H_j = G_j \cap P_1$ for all $j \in [m]$.

Writing G^j for $G_j \ominus (G_1 \cup \dots \cup G_{j-1})$ and H^j for $H_j \ominus (G_1 \cup \dots \cup G_{j-1})$, we next observe that

$$\Delta(G^j) \geq \Delta(H^j) + \#\{i \in \{2, \dots, c\} : j \text{ is the first index such that } G^j \cap P_i \text{ is nonempty}\}.$$

It follows that

$$\Psi(T) \geq \vec{\Delta}(G_1, \dots, G_m) = \sum_{j \in [m]} \Delta(G^j) \geq \sum_{j \in [m]} \Delta(H^j) + c - 1 = \vec{\Delta}(H_1, \dots, H_m) + \Delta(P) - 1.$$

Therefore, we have $\vec{\Delta}(G_1, \dots, G_m) \geq \frac{1}{30e}d\lambda(P)^{1/d} + \Delta(P) - d$ as required.

Having established that it suffices to prove the induction step in the case that $\Delta(P) = 1$, let us proceed under the assumption P is connected. Without loss of generality, let $P = \text{Path}_k$ where $k = \lambda(P)$. Thus, we assume that T is a Path_k -join tree of $\langle\langle\rangle\rangle$ -depth d . Our goal is to find a T -branch covering \mathcal{G} of Path_k together with an enumeration G_1, \dots, G_m of \mathcal{G} satisfying

$$(7) \quad \vec{\Delta}(G_1, \dots, G_m) \geq \frac{1}{30e}dk^{1/d} - d + 1.$$

The right spine of T

As a first step, we consider the ‘‘right spine’’ $T = \langle\langle T_1, \dots, T_\ell \rangle\rangle$ where T_ℓ is a leaf and each T_j is a B_j -join tree of $\langle\langle\rangle\rangle$ -depth $d - 1$ for graphs $B_1 \cup \dots \cup B_\ell = \text{Path}_k$. (Note: $\{B_1, \dots, B_\ell\}$ is not necessarily the T -branch covering of Path_k that we will eventually produce.)

Applying our Main Lemma 3.5(II) to graphs B_1, \dots, B_ℓ , we obtain a permutation $\pi : [\ell] \xrightarrow{\cong} [\ell]$ satisfying

$$(8) \quad \vec{\lambda} \Delta(B_{\pi(1)}, \dots, B_{\pi(\ell)}) \geq \frac{k}{30}.$$

For $j \in [\ell]$, it will be convenient to write

$$B^j := B_j \ominus (B_{\pi(1)} \cup \dots \cup B_{\pi(\pi^{-1}(j)-1)}).$$

(Note: B^j is not to be confused with $B_j \ominus (B_1 \cup \dots \cup B_{j-1})$.) Under this reindexing, inequality (8) is equivalent to

$$(9) \quad \sum_{j \in [\ell]} \lambda(B^j) \Delta(B^j) \geq \frac{k}{30}.$$

Graph sequences $G_1^j, \dots, G_{m_j}^j$

For each $j \in [\ell]$, let T^j be the B^j -join tree obtained from T_j by relabeling to \emptyset (the empty graph) any leaf originally labeled by an edge of $B_j \setminus B^j$. Note that T^j has $\langle\langle\rangle\rangle$ -depth $d - 1$, since relabeling does not increase $\langle\langle\rangle\rangle$ -depth.

Applying the induction hypothesis to each T^j , we obtain a T^j -branch covering \mathcal{H}^j of B^j and an enumeration $H_1^j, \dots, H_{n_j}^j$ of \mathcal{H}^j such that

$$(10) \quad \vec{\Delta}(H_1^j, \dots, H_{n_j}^j) \geq \frac{1}{30e} (d-1) \lambda(B^j)^{1/(d-1)} + \Delta(B^j) - d + 1.$$

We now define a sequence of graph $G_1^j, \dots, G_{m_j}^j$ where $m_j = n_j + j - 1$ via the three bullet points below. (See the diagram following the definition for a helpful example.)

- Note that $\pi^{-1}(1), \dots, \pi^{-1}(j)$ are j distinct numbers in $[m]$.

Let $\tau_j : [j] \xrightarrow{\cong} [j]$ be the permutation defined by $\tau_j^{-1}(a) < \tau_j^{-1}(b)$ iff $\pi^{-1}(a) < \pi^{-1}(b)$ for all $a, b \in [j]$.

For example, if $j = 4$ and $\pi^{-1}(1, 2, 3, 4) = (4, 7, 2, 6)$, then $\tau_j^{-1}(1, 2, 3, 4) = (2, 4, 1, 3)$ and $\tau_j(1, 2, 3, 4) = (3, 1, 4, 2)$.

- Let $j^* = \tau_j^{-1}(j) \in [j]$.

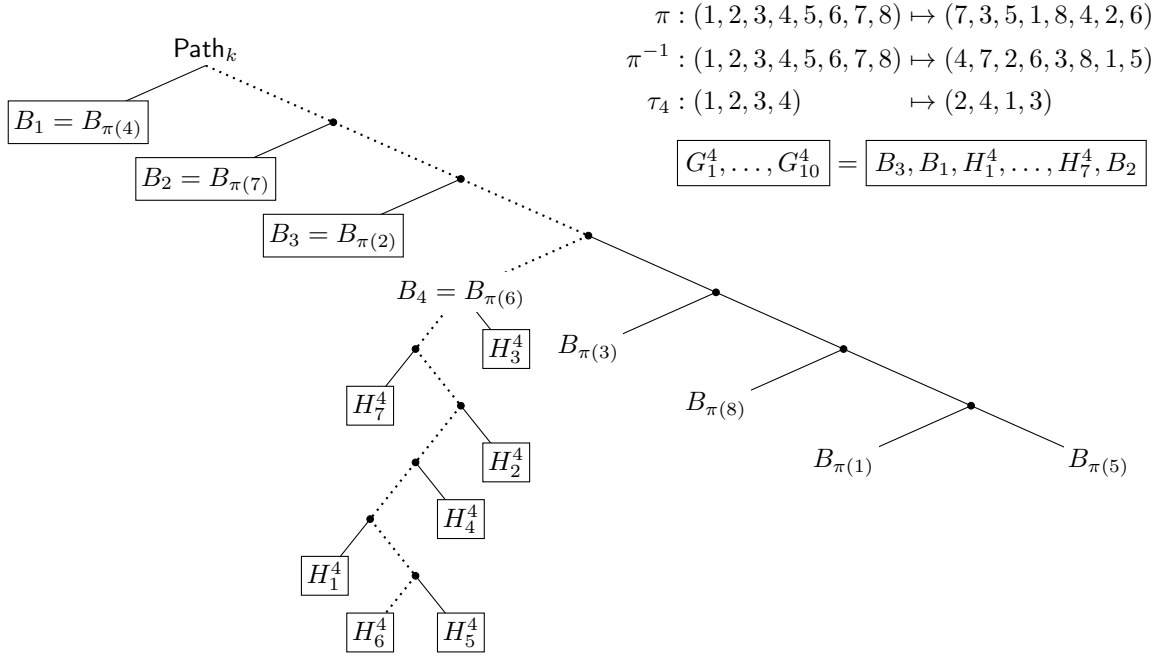
In our example, $j^* = 3$.

- Let $G_1^j, \dots, G_{m_j}^j$ be the sequence

$$B_{\tau_j(1)}, \dots, B_{\tau_j(j^*-1)}, H_1^j, \dots, H_{n_j}^j, B_{\tau_j(j^*+1)}, \dots, B_{\tau_j(j)}.$$

Said differently: $G_1^j, \dots, G_{m_j}^j$ is obtained from the sequence $B_{\tau_j(1)}, \dots, B_{\tau_j(j)}$ by removing the j^* -th entry (i.e., the graph $B_{\tau_j(j^*)} = B_j$) and inserting the subsequence $H_1^j, \dots, H_{n_j}^j$ in its place.

In our example (with $j = 4$ and $j^* = 3$), $G_1^j, \dots, G_{m_j}^j$ is the sequence $B_3, B_1, H_1^4, \dots, H_{n_4}^4, B_2$.



The T -branch covering $\mathcal{G}_j = \{G_1, \dots, G_{m_j+1}\}$

For each $j \in [\ell]$ and $s \in [m_j]$, let $H_{j,s}$ be the label in T_j of the node labeled by H_s^j in T^j . Note that $H_s^j = H_{j,s} \ominus (B_{\pi(1)} \cup \dots \cup B_{\pi(\pi^{-1}(j)-1)})$.

We now define a T -branch covering $\mathcal{G}_j = \{G_{j,1}, \dots, G_{j,m_j+1}\}$ of Path_k as follows:

$$\begin{aligned}
 (G_{j,1}, \dots, G_{j,m_j}) &:= (B_{\tau_j(1)}, \dots, B_{\tau_j(j^*-1)}, H_{j,1}, \dots, H_{j,n_j}, B_{\tau_j(j^*+1)}, \dots, B_{\tau_j(j)}), \\
 G_{j,m_j+1} &:= B_{j+1} \cup \dots \cup B_\ell.
 \end{aligned}$$

That is, G_{j,m_j+1} is the label in T of the j th right descendant of the root; we will ignore the contribution of this graph when bounding $\vec{\Delta}(G_{j,1}, \dots, G_{j,m_j+1})$. Note that \mathcal{G}_j is indeed a T -branch covering of Path_k .

Lower bound on $\vec{\Delta}(G_{j,1}, \dots, G_{j,m_j})$

By the ‘‘chain rule’’ for Δ (Lemma 2.5),

$$\begin{aligned}
 \vec{\Delta}(G_{j,1}, \dots, G_{j,m_j}) &= \vec{\Delta}(B_{\tau_j(1)}, \dots, B_{\tau_j(j^*-1)}) \\
 &\quad + \vec{\Delta}(H_{j,1}, \dots, H_{j,n_j} \mid B_{\tau_j(1)} \cup \dots \cup B_{\tau_j(j^*-1)}) \quad = B_j = B_{\tau_j(j^*)} \\
 &\quad + \vec{\Delta}(B_{\tau_j(j^*+1)}, \dots, B_{\tau_j(j)} \mid B_{\tau_j(1)} \cup \dots \cup B_{\tau_j(j^*-1)} \cup \overbrace{H_{j,1} \cup \dots \cup H_{j,n_j}}) \\
 &= \vec{\Delta}(H_{j,1}, \dots, H_{j,n_j} \mid B_{\tau_j(1)} \cup \dots \cup B_{\tau_j(j^*-1)}) \\
 &\quad + \sum_{h \in [j] \setminus \{j^*\}} \Delta(B_{\tau_j(h)} \ominus (B_{\tau_j(1)} \cup \dots \cup B_{\tau_j(h-1)})).
 \end{aligned}$$

Since $B_{\tau_j(1)} \cup \dots \cup B_{\tau_j(j^*-1)} \subseteq B_{\pi(1)} \cup \dots \cup B_{\pi(\pi^{-1}(j)-1)}$, we have

$$\begin{aligned}
 \vec{\Delta}(H_{j,1}, \dots, H_{j,n_j} \mid B_{\tau_j(1)} \cup \dots \cup B_{\tau_j(j^*-1)}) &\geq \vec{\Delta}(H_{j,1}, \dots, H_{j,n_j} \mid B_{\pi(1)} \cup \dots \cup B_{\pi(\pi^{-1}(j)-1)}) \\
 &= \vec{\Delta}(H_1^j, \dots, H_{n_j}^j).
 \end{aligned}$$

Next, we have

$$\begin{aligned} \sum_{h \in [j] \setminus \{j^*\}} \Delta(B_{\tau_j(h)} \ominus (B_{\tau_j(1)} \cup \dots \cup B_{\tau_j(h-1)})) &= \sum_{i \in [j-1]} \Delta(B_i \ominus \overbrace{(B_{\tau_j(1)} \cup \dots \cup B_{\tau_j(\tau_j^{-1}(i)-1)})}^{\subseteq B_{\pi(1)} \cup \dots \cup B_{\pi(\pi^{-1}(i)-1)}})) \\ &\geq \sum_{i \in [j-1]} \Delta(B_i \ominus \underbrace{(B_{\pi(1)} \cup \dots \cup B_{\pi(\pi^{-1}(i)-1)})}_{= B^i}). \end{aligned}$$

The above inequalities combine with bound (10) on $\vec{\Delta}(H_1^j, \dots, H_{n_j}^j)$ to yield:

$$\begin{aligned} \vec{\Delta}(G_{j,1}, \dots, G_{j,m_j}) &\geq \vec{\Delta}(H_{j,1}, \dots, H_{j,n_j}) + \sum_{i \in [j-1]} \Delta(B^i) \\ (11) \quad &\geq \frac{1}{30e} (d-1) \lambda(B^j)^{1/(d-1)} + \sum_{i \in [j]} \Delta(B^i) - d + 1. \end{aligned}$$

Completing the proof via the numerical inequality Lemma 3.9

Combining bounds (9) and (11) with the numerical inequality Lemma 3.9, we get

$$\begin{aligned} \max_{j \in [\ell]} \vec{\Delta}(G_1^j, \dots, G_{m_j+1}^j) &\geq \max_{j \in [\ell]} \vec{\Delta}(G_1^j, \dots, G_{m_j}^j) && \text{(ignoring } G_{m_j+1}^j) \\ &\geq \max_{j \in [\ell]} \left((d-1) \left(\frac{\lambda(B^j)}{(30e)^{d-1}} \right)^{1/(d-1)} + \sum_{i \in [j]} \Delta(B^i) \right) - d + 1 && \text{(by (11))} \\ &\geq d \left(\frac{1}{30^{d-1} e^d} \sum_{j \in [\ell]} \lambda(B^j) \Delta(B^j) \right)^{1/d} - d + 1 && \text{(by Lemma 3.9)} \\ &\geq \frac{1}{30e} dk^{1/d} - d + 1 && \text{(by (9)).} \end{aligned}$$

The proof is completed by picking the optimal $j \in [\ell]$, as inequality (7) is satisfied by the T -branch covering $\mathcal{G} = \mathcal{G}_j$ and enumeration $(G_1, \dots, G_m) = (G_{j,1}, \dots, G_{j,m_j+1})$. \square

5 Join tree tradeoff (II)

In this section we prove inequalities (II) of Lemmas 3.4–3.5 and Theorem 3.6, which we eventually use to prove tradeoffs (II)⁺ and (II)⁻ of Theorem 1.6 for monotone and non-monotone AC^0 circuits.

5.1 Lemma relating the $\llbracket \rrbracket$ operation and shift permutations

Our proof of Theorem 3.6(I) was based on the following inequality, which relates $\Psi(\langle\langle T_1, \dots, T_m \rangle\rangle)$ with permutations of $[j]$ where $j \leq m$. (We omit the proof, which is implicit in the argument of §4.)

Lemma 5.1. *Suppose $T = \langle\langle T_1, \dots, T_m \rangle\rangle$ where T_j are G_j -join trees with $G_j \subset \text{Path}_{\mathbb{Z}}$. Then for every $j \in [m]$ and permutation $\tau : [j] \xrightarrow{\cong} [j]$ and $j^* := \tau^{-1}(j)$,*

$$\Psi(T) \geq \Psi(T_j \ominus (G_{\tau(1)} \cup \dots \cup G_{\tau(j^*-1)})) - \Delta(T_j \ominus (G_{\tau(1)} \cup \dots \cup G_{\tau(j^*-1)})) + \vec{\Delta}(G_{\tau(1)}, \dots, G_{\tau(j)}).$$

In this section, we prove an analogous result (Lemma 5.4) for join trees $T = \llbracket T_1, \dots, T_m \rrbracket$, which will be the backbone of our proof of Theorem 3.6(II) in §5. In order to state this lemma, we require one definition.

Definition 5.2 (Induced shift permutations $\tilde{\sigma}_1, \dots, \tilde{\sigma}_m$). Consider a shift permutation $\sigma = \sigma_I : [m] \xrightarrow{\cong} [m]$ where $I = \{i_1, \dots, i_p\}$ with $0 =: i_0 < i_1 < \dots < i_p = m$. For each $j \in [m]$, we define $\tilde{\sigma}_j = \sigma_{\tilde{I}_j} : [m] \xrightarrow{\cong} [m]$ where

$$\tilde{I}_j := \begin{cases} I \cup [i_{h-1}] & \text{if } j = i_h \in I, \\ I \cup [j-1] & \text{if } j \notin I. \end{cases}$$

For example, if $\sigma = \sigma_{\{3,5\}} : (1, 2, 3, 4, 5) \mapsto (3, 1, 2, 5, 4)$, then

$$\begin{aligned} \tilde{\sigma}_1 &= \tilde{\sigma}_3 = \sigma_{\{3,5\}} : (1, 2, 3, 4, 5) \mapsto (3, 1, 2, 5, 4), \\ \tilde{\sigma}_2 &= \sigma_{\{1,3,5\}} : (1, 2, 3, 4, 5) \mapsto (1, 3, 2, 5, 4), \\ \tilde{\sigma}_4 &= \tilde{\sigma}_5 = \sigma_{\{1,2,3,5\}} : (1, 2, 3, 4, 5) \mapsto (1, 2, 3, 5, 4). \end{aligned}$$

For future reference (in the proof of Lemma 5.6 in the next subsection), we record a simple property of shift permutations $\tilde{\sigma}_j$.

Lemma 5.3. For all $\sigma = \sigma_I : [m] \xrightarrow{\cong} [m]$ and $j \in [m]$, we have

$$\vec{\Delta}(G_{\tilde{\sigma}_j(1)}, \dots, G_{\tilde{\sigma}_j(m)}) \geq \sum_{i \in I} \Delta(G_i \ominus (G_1 \cup \dots \cup G_{i-1})).$$

Proof. Consider any $\{m\} \subseteq I \subseteq [m]$ and $j \in [m]$. For each $i \in I$, we have $i \in \tilde{I}_j$. Let i^{\leftarrow} denote the predecessor of i in the set $\{0\} \cup \tilde{I}_j$, that is, $i^{\leftarrow} := \max(\{h \in \{0\} \cup \tilde{I}_j : h < i\})$. Note that

$$\tilde{\sigma}_j(i^{\leftarrow} + 1) = i \quad \text{and} \quad \{\tilde{\sigma}_j(1), \tilde{\sigma}_j(2), \dots, \tilde{\sigma}_j(i^{\leftarrow})\} = \{1, 2, \dots, i^{\leftarrow}\}.$$

We now obtain the desired inequality as follows:

$$\begin{aligned} \vec{\Delta}(G_{\tilde{\sigma}_j(1)}, \dots, G_{\tilde{\sigma}_j(m)}) &= \sum_{l=1}^m \Delta(G_{\tilde{\sigma}_j(l)} \ominus (G_{\tilde{\sigma}_j(1)} \cup \dots \cup G_{\tilde{\sigma}_j(l-1)})) \\ &\geq \sum_{i \in I} \Delta(G_{\tilde{\sigma}_j(i^{\leftarrow} + 1)} \ominus (G_{\tilde{\sigma}_j(1)} \cup \dots \cup G_{\tilde{\sigma}_j(i^{\leftarrow})})) \\ &= \sum_{i \in I} \Delta(G_i \ominus (G_1 \cup \dots \cup G_{i^{\leftarrow}})) \\ &\geq \sum_{i \in I} \Delta(G_i \ominus (G_1 \cup \dots \cup G_{i-1})). \quad \square \end{aligned}$$

The next lemma relates the $\llbracket \cdot \rrbracket$ operation and shift permutations.

Lemma 5.4. Suppose $T = \llbracket T_1, \dots, T_m \rrbracket$ where T_j are G_j -join trees with $G_j \subset \text{Path}_{\mathbb{Z}}$. Then for every shift permutation $\sigma : [m] \xrightarrow{\cong} [m]$ and $h \in [m]$, we have

- (i) $\Psi(T) \geq \Psi(T_{\sigma(1)}) + \vec{\Delta}(G_{\sigma(2)}, \dots, G_{\sigma(m)} \mid G_{\sigma(1)})$,
- (ii) $\Psi(T) \geq \Psi(T_{\sigma(h)}) + \vec{\Delta}(G_{\sigma(h+1)}, \dots, G_{\sigma(m)} \mid G_{\sigma(1)} \cup \dots \cup G_{\sigma(h)})$,
- (iii) $\Psi(T) \geq \Psi(T_j \ominus F_j) - \Delta(G_j \ominus F_j) + \vec{\Delta}(G_{\tilde{\sigma}_j(1)}, \dots, G_{\tilde{\sigma}_j(m)})$ where

$$j := \sigma(h) \quad \text{and} \quad F_j := G_{\sigma(1)} \cup \dots \cup G_{\sigma(h-1)}.$$

Proof. We illustrate bounds (i), (ii), (iii) by considering a specific example $\sigma : [5] \xrightarrow{\cong} [5]$ with $m = 5$:

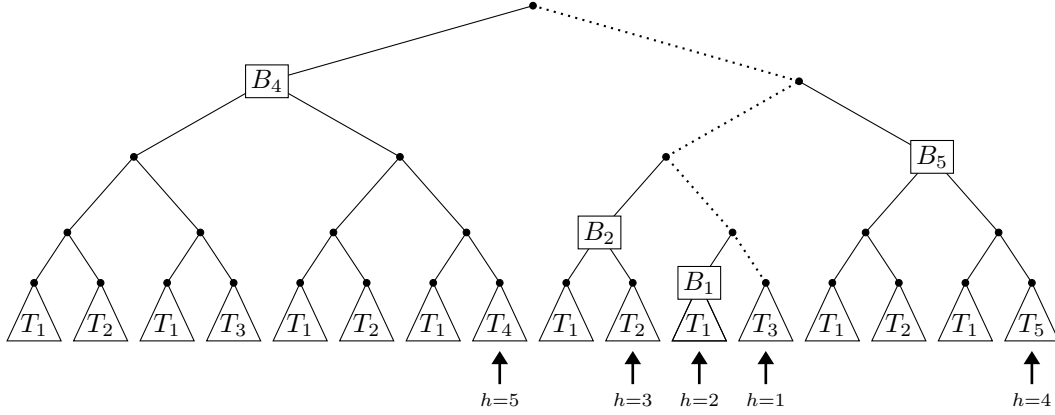
$$\sigma := \sigma_{\{3,5\}} : (1, 2, 3, 4, 5) \mapsto (3, 1, 2, 5, 4).$$

Following the proof of each bound (i), (ii), (iii) for this specific σ , we will comment on the generalization to arbitrary shift permutations.

In this case, bound (i) states

$$\Psi(T) \geq \Psi(T_{\sigma(1)}) + \vec{\Delta}(G_{\sigma(2)}, G_{\sigma(3)}, G_{\sigma(4)}, G_{\sigma(5)} \mid G_{\sigma(1)}) = \Psi(T_3) + \vec{\Delta}(G_1, G_2, G_5, G_4 \mid G_3),$$

To obtain this bound, we consider the dotted branch in T depicted below (ignoring for now arrows pointing to the sub-trees labeled $h = 1, \dots, 5$).



This branch descends right-left-right-right from the root before continuing as the optimal branch inside the T_3 -subtree (i.e., the branch that witnesses the value of $\Psi(T_3)$). Observe that the four branch-sibling graphs above this T_3 -subtree are, in order from the bottom up:

$$B_1 := G_1, \quad B_2 := G_1 \cup G_2, \quad B_5 := G_1 \cup G_2 \cup G_5, \quad B_4 := G_1 \cup G_2 \cup G_3 \cup G_4.$$

Note that in general we have

$$(12) \quad G_{\sigma(1)} \cup B_{\sigma(2)} \cup \dots \cup B_{\sigma(h)} = G_{\sigma(1)} \cup G_{\sigma(2)} \cup \dots \cup G_{\sigma(h)} \text{ for all } h \in [m].$$

Now consider the T -branch covering of $G_1 \cup \dots \cup G_5$ that consists of an optimal enumeration of the T_3 -branch covering of G_3 , followed by graphs B_1, B_2, B_5, B_4 . This shows that

$$\Psi(T) \geq \Psi(T_3) + \vec{\Delta}(B_1, B_2, B_5, B_4 \mid G_3) = \Psi(T_3) + \vec{\Delta}(G_1, G_2, G_5, G_4 \mid G_3),$$

that is, the required bound (i).

For general shift permutations $\sigma : [m] \xrightarrow{\cong} [m]$, we have

$$\Psi(T) \geq \Psi(T_{\sigma(1)}) + \vec{\Delta}(B_{\sigma(2)}, \dots, B_{\sigma(m)} \mid G_3) = \Psi(T_{\sigma(1)}) + \vec{\Delta}(G_{\sigma(2)}, \dots, G_{\sigma(m)} \mid G_3)$$

by fact (12) and Lemma 2.5(e).

To obtain bound (ii), for each $h \in [5]$, consider the $T_{\sigma(h)}$ -subtree indicated by arrows in the above diagram. Observe that the branch-sibling graphs above this $T_{\sigma(h)}$ -subtree include $B_{\sigma(h+1)}, \dots, B_{\sigma(5)}$. Now consider a T -branch covering of $G_1 \cup \dots \cup G_5$ that begins with an optimal enumeration of an optimal $T_{\sigma(h)}$ -branch covering

of $G_{\sigma(h)}$ (i.e. witnessing the value of $\Psi(T_{\sigma(h)})$), followed by graphs $B_{\sigma(h+1)}, \dots, B_{\sigma(5)}$. This yields the required bound (ii), respectively for $h = 1, \dots, 5$:

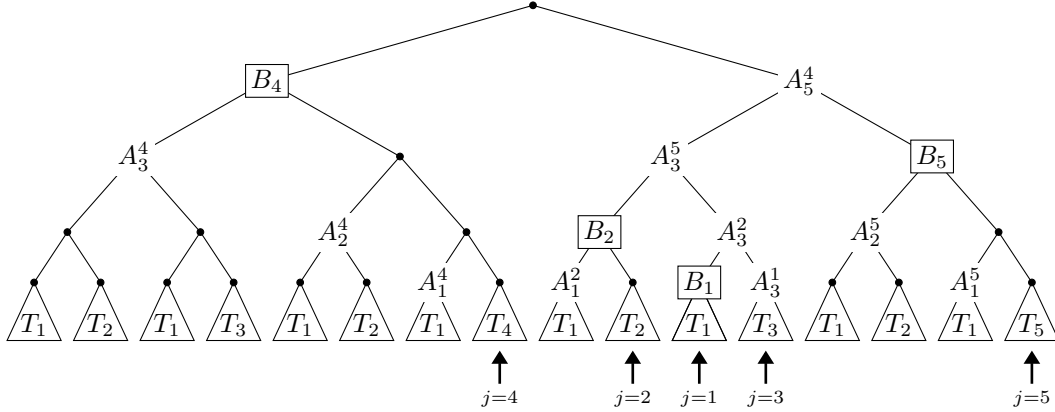
$$\begin{aligned} \Psi(T) &\geq \Psi(T_3) + \vec{\Delta}(B_1, B_2, B_5, B_4 \mid G_3) = \Psi(T_3) + \vec{\Delta}(G_1, G_2, G_5, G_4 \mid G_3), \\ \Psi(T) &\geq \Psi(T_1) + \vec{\Delta}(B_2, B_5, B_4 \mid G_1) \geq \Psi(T_1) + \vec{\Delta}(G_2, G_5, G_4 \mid G_3 \cup G_1), \\ \Psi(T) &\geq \Psi(T_2) + \vec{\Delta}(B_5, B_4 \mid G_2) \geq \Psi(T_2) + \vec{\Delta}(G_5, G_4 \mid G_3 \cup G_1 \cup G_2), \\ \Psi(T) &\geq \Psi(T_5) + \vec{\Delta}(B_4 \mid G_5) \geq \Psi(T_5) + \vec{\Delta}(G_4 \mid G_3 \cup G_1 \cup G_2 \cup G_5), \\ \Psi(T) &\geq \Psi(T_4) + \vec{\Delta}(\mid G_4) \geq \Psi(T_4) + \vec{\Delta}(\mid G_3 \cup G_1 \cup G_2 \cup G_5 \cup G_4). \end{aligned}$$

In general, we have

$$\begin{aligned} \Psi(T) &\geq \Psi(T_{\sigma(h)}) + \vec{\Delta}(B_{\sigma(h+1)}, \dots, B_{\sigma(m)} \mid G_{\sigma(1)} \cup \dots \cup G_{\sigma(h)}) \\ &= \Psi(T_{\sigma(h)}) + \vec{\Delta}(G_{\sigma(h+1)}, \dots, G_{\sigma(m)} \mid G_{\sigma(1)} \cup \dots \cup G_{\sigma(h)}) \end{aligned}$$

again by fact (12) and Lemma 2.5.

For the final bound (iii), for $j = 1, \dots, 5$, consider the T_j -subtrees indicated by arrows in the diagram below.



Letting $h := \sigma^{-1}(j)$, the branch-sibling graphs above this T_j -subtree are, in order from the bottom up:

$$A_{\tilde{\sigma}_j(1)}^j, \dots, A_{\tilde{\sigma}_j(h-1)}^j, B_{\sigma(h+1)}, \dots, B_{\sigma(5)}.$$

Note that, since $\{\sigma(1), \dots, \sigma(h-1)\} = \{\tilde{\sigma}_j(1), \dots, \tilde{\sigma}_j(h-1)\}$, we have

$$F_j := G_{\sigma(1)} \cup \dots \cup G_{\sigma(h-1)} = G_{\tilde{\sigma}_j(1)} \cup \dots \cup G_{\tilde{\sigma}_j(h-1)}.$$

Now consider a T -branch covering of $G_1 \cup \dots \cup G_m$ that consists of

- graphs $A_{\tilde{\sigma}_j(1)}^j, \dots, A_{\tilde{\sigma}_j(h-1)}^j$, followed by
- an optimal enumeration of an optimal $T_j \ominus F_j$ -branch covering of $G_j \ominus F_j$ (i.e. witnessing the value of $\Psi(T_j \ominus F_j)$), followed by
- graphs $B_{\sigma(h+1)}, \dots, B_{\sigma(m)}$.

Cases $j = 2, 5$ ($h = 3, 4$) of bound (iii) are obtained as follows:

$$\begin{aligned}
\Psi(T) &\geq \Psi(T_2 \ominus (A_1^2 \cup A_3^2)) + \vec{\Delta}(A_1^2, A_3^2) + \vec{\Delta}(B_5, B_4 \mid A_1^2 \cup A_3^2 \cup G_2) \\
&= \Psi(T_2 \ominus (G_1 \cup G_3)) + \vec{\Delta}(G_1, G_3) + \vec{\Delta}(G_5, G_4 \mid G_1 \cup G_3 \cup G_2) \\
&= \Psi(T_2 \ominus (G_1 \cup G_3)) + \vec{\Delta}(G_1, G_3, G_2, G_5, G_4) - \Delta(G_2 \ominus (G_1 \cup G_3)) \\
&= \Psi(T_2 \ominus F_2) + \vec{\Delta}(G_{\tilde{\sigma}_2(1)}, G_{\tilde{\sigma}_2(2)}, G_{\tilde{\sigma}_2(3)}, G_{\tilde{\sigma}_2(4)}, G_{\tilde{\sigma}_2(5)}) - \Delta(G_2 \ominus F_2), \\
\\
\Psi(T) &\geq \Psi(T_5 \ominus (A_1^5 \cup A_2^5 \cup A_3^5)) + \vec{\Delta}(A_1^5, A_2^5, A_3^5) + \vec{\Delta}(B_4 \mid A_1^5 \cup A_2^5 \cup A_3^5 \cup G_5) \\
&= \Psi(T_5 \ominus (G_1 \cup G_2 \cup G_3)) + \vec{\Delta}(G_1, G_2, G_3) + \vec{\Delta}(G_4 \mid G_1 \cup G_2 \cup G_3 \cup G_5) \\
&= \Psi(T_5 \ominus (G_1 \cup G_2 \cup G_3)) + \vec{\Delta}(G_1, G_2, G_3, G_5, G_4) - \Delta(G_5 \ominus (G_1 \cup G_2 \cup G_3)) \\
&= \Psi(T_5 \ominus F_5) + \vec{\Delta}(G_{\tilde{\sigma}_5(1)}, G_{\tilde{\sigma}_5(2)}, G_{\tilde{\sigma}_5(3)}, G_{\tilde{\sigma}_5(4)}, G_{\tilde{\sigma}_5(5)}) - \Delta(G_5 \ominus F_5).
\end{aligned}$$

These calculations (and those for $j = 1, 3, 4$) are easily verified.

For a general shift permutation $\sigma : [m] \xrightarrow{\cong} [m]$ and $j \in [m]$ and $h := \sigma^{-1}(j)$, we have

$$\begin{aligned}
\Psi(T) &\geq \vec{\Delta}(A_{\tilde{\sigma}_j(1)}^j, \dots, A_{\tilde{\sigma}_j(h-1)}^j) \\
&\quad + \Psi(T_j \ominus (A_{\tilde{\sigma}_j(1)}^j \cup \dots \cup A_{\tilde{\sigma}_j(h-1)}^j)) \\
&\quad + \vec{\Delta}(B_{\sigma(h+1)}, \dots, B_{\sigma(m)} \mid A_{\tilde{\sigma}_j(1)}^j \cup \dots \cup A_{\tilde{\sigma}_j(h-1)}^j \cup G_j) \\
&= \vec{\Delta}(G_{\tilde{\sigma}_j(1)}, \dots, G_{\tilde{\sigma}_j(h-1)}) \quad (\text{since } A_{\tilde{\sigma}_j(1)}^j \cup \dots \cup A_{\tilde{\sigma}_j(i)}^j = G_{\tilde{\sigma}_j(1)} \cup \dots \cup G_{\tilde{\sigma}_j(i)} \text{ for all } i \in [h-1]) \\
&\quad + \Psi(T_j \ominus F_j) \quad (\text{since } A_{\tilde{\sigma}_j(1)}^j \cup \dots \cup A_{\tilde{\sigma}_j(h-1)}^j = F_j) \\
&\quad + \vec{\Delta}(B_{\sigma(h+1)}, \dots, B_{\sigma(m)} \mid F_j \cup G_j) \\
&= \Psi(T_j \ominus F_j) - \Delta(G_j \ominus F_j) + \underbrace{\vec{\Delta}(G_{\tilde{\sigma}_j(1)}, \dots, G_{\tilde{\sigma}_j(h-1)}) + \Delta(G_j \ominus F_j) + \vec{\Delta}(B_{\sigma(h+1)}, \dots, B_{\sigma(m)} \mid F_j \cup G_j)}_{= \vec{\Delta}(G_{\tilde{\sigma}_j(1)}, \dots, G_{\tilde{\sigma}_j(h-1)}, G_j, B_{\sigma(h+1)}, \dots, B_{\sigma(m)})} \\
&= \vec{\Delta}(G_{\tilde{\sigma}_j(1)}, \dots, G_{\tilde{\sigma}_j(m)}).
\end{aligned}$$

The last equality above is justified by the observation that, for all $i \in \{h, h+1, \dots, m\}$,

$$\begin{aligned}
G_{\tilde{\sigma}_j(1)} \cup \dots \cup G_{\tilde{\sigma}_j(h-1)} \cup G_j \cup B_{\sigma(h+1)} \cup \dots \cup B_{\sigma(i)} &= G_{\sigma(1)} \cup \dots \cup G_{\sigma(h)} \cup B_{\sigma(h+1)} \cup \dots \cup B_{\sigma(i)} \\
&= G_{\sigma(1)} \cup \dots \cup G_{\sigma(i)} \\
&= G_{\tilde{\sigma}_j(1)} \cup \dots \cup G_{\tilde{\sigma}_j(i)}
\end{aligned}$$

by (12) and the fact that $\{\tilde{\sigma}_j(1), \dots, \tilde{\sigma}_j(i)\} = \{\sigma(1), \dots, \sigma(i)\}$. \square

Corollary 5.5. *Suppose $T = \llbracket T_1, \dots, T_m \rrbracket$ where T_j are G_j -join trees with $G_j \subset \text{Path}_{\mathbb{Z}}$. Then for every $j \in [m]$, we have*

$$\Psi(T) \geq \Psi(T_j) + \vec{\Delta}(G_1, \dots, G_m \mid G_j).$$

Proof. Applying Lemma 5.4(i) in the case $\sigma = \sigma_{[m] \setminus [j-1]} : (1, \dots, m) \mapsto (j, 1, \dots, j-1, j+1, \dots, m)$, we have

$$\Psi(T) \geq \Psi(T_{\sigma(1)}) + \vec{\Delta}(G_{\sigma(2)}, \dots, G_{\sigma(m)} \mid G_{\sigma(1)}) = \Psi(T_j) + \vec{\Delta}(G_1, \dots, G_{j-1}, G_{j+1}, \dots, G_m \mid G_j). \quad \square$$

5.2 Proof of Pre-Main Lemma 3.4(II)

We now prove bound (II) of our Pre-Main Lemma 3.4. The proof is significantly simpler than bound (I).

Pre-Main Lemma 3.4(II) (restated). *Suppose $G_1 \cup \dots \cup G_m = \text{Path}_k$ and $\vec{\Delta}(G_1, \dots, G_m) = 1$. Then there exists a shift permutation $\sigma : [m] \xrightarrow{\cong} [m]$ such that $\vec{\lambda}(G_{\sigma(1)}, \dots, G_{\sigma(m)}) \geq k/4$.*

Proof. Without loss of generality, assume that $G_1 = \text{Path}_{s_1, t_1}$ where $0 \leq s_1 < t_1 \leq k$ and $s \leq k/2$. For each $j \in \{2, \dots, m\}$, let Path_{s_j, t_j} be the rightmost connected component of G_j (i.e., with maximal t_j).

Define the set $J \subseteq [m]$ by

$$J := \{j \in [m] : t_j > \max\{0, t_1, \dots, t_{j-1}\}\}.$$

Note that $1 \in J$ and

$$\text{Path}_{s_1, k} \subseteq \bigcup_{j \in J} G_j \quad \text{and} \quad G_1 \cup \dots \cup G_j \subseteq \text{Path}_{0, t_j} \quad \text{for all } j \in J.$$

Now let $j_1, \dots, j_r \in J$, $1 \leq j_1 < \dots < j_r \leq m$, be any minimal increasing sequence of indices in J such that

$$\text{Path}_{s_1, k} \subseteq G_{j_1} \cup \dots \cup G_{j_r}.$$

Minimality of the sequence j_1, \dots, j_r implies that we have interleaving endpoints

$$s_{j_1} < s_{j_2} \leq t_{j_1} < s_{j_3} \leq t_{j_2} < s_{j_4} \leq t_{j_3} < s_{j_5} \leq t_{j_4} < s_{j_6} \leq \dots \leq t_{j_{r-2}} < s_{j_r} \leq t_{j_{r-1}} < t_{j_r}.$$

In particular,

$$s_{j_1} < t_{j_1} < s_{j_3} < t_{j_3} < s_{j_5} < t_{j_5} < \dots \quad \text{and} \quad s_{j_2} < t_{j_2} < s_{j_4} < t_{j_4} < s_{j_6} < t_{j_6} < \dots.$$

It follows that for all $h \in [r-2]$,

$$\text{Path}_{s_{j_{h+2}}, t_{j_{h+2}}} \subseteq G_{j_{h+2}} \ominus (G_1 \cup \dots \cup G_{j_h}).$$

Also note that $s_{j_1} \leq k/2$ and $t_{j_r} = k$, so that

$$\frac{k}{2} \leq \|\text{Path}_{s_1, k}\| \leq \|G_{j_1} \cup \dots \cup G_{j_r}\| \leq \sum_{h \in [r]} (t_{j_h} - s_{j_h}).$$

We next define a sequence (i_1, \dots, i_p) to be either (j_1, j_3, j_5, \dots) or (j_2, j_4, j_6, \dots) according to two cases.

- If $\sum_{\text{odd } h \in [r]} (t_{j_h} - s_{j_h}) \geq \frac{k}{4}$, then let $p := \lceil \frac{r}{2} \rceil$ and $(i_1, \dots, i_p) := (j_1, j_3, j_5, \dots, j_{2p-1})$.
- Otherwise, we have $\sum_{\text{even } h \in [r]} (t_{j_h} - s_{j_h}) \geq \frac{k}{4}$ and let $p := \lfloor \frac{r}{2} \rfloor$ and $(i_1, \dots, i_p) := (j_2, j_4, j_6, \dots, j_{2p})$.

Let $i_0 := 0$ and let

$$I := [m] \setminus \bigcup_{h \in [p]} \{l \in [m] : i_{h-1} < l < i_h\} \quad \left(= \{i_1, \dots, i_p\} \cup \{l \in [m] : i_p < l \leq m\} \right).$$

Recall that $\sigma := \sigma_I : [m] \xrightarrow{\cong} [m]$ is a shift permutation satisfying

$$\sigma(i_{h-1} + 1) = i_h \quad \text{and} \quad \{\sigma(1), \sigma(2), \dots, \sigma(i_{h-1})\} = \{1, 2, \dots, i_{h-1}\} \quad \text{for all } h \in [p].$$

We conclude the proof as follows:

$$\begin{aligned}
\vec{\lambda}(G_{\sigma(1)}, \dots, G_{\sigma(m)}) &= \sum_{j=1}^m \lambda(G_{\sigma(j)} \ominus (G_{\sigma(1)} \cup \dots \cup G_{\sigma(j-1)})) \\
&\geq \sum_{j \in \{i_0+1, i_1+1, \dots, i_{p-1}+1\}} \lambda(G_{\sigma(j)} \ominus (G_{\sigma(1)} \cup \dots \cup G_{\sigma(j-1)})) \\
&= \sum_{h=1}^p \lambda(G_{\sigma(i_{h-1}+1)} \ominus (G_{\sigma(1)} \cup \dots \cup G_{\sigma(i_{h-1})})) \\
&= \sum_{h=1}^p \lambda(G_{i_h} \ominus (G_1 \cup \dots \cup G_{i_{h-1}})) \\
&\geq \sum_{h=1}^p \lambda(\text{Path}_{s_{i_h}, t_{i_h}}) = \sum_{h=1}^p (t_{i_h} - s_{i_h}) \geq \frac{k}{4}. \quad \square
\end{aligned}$$

Under the same hypothesis where $G_1 \cup \dots \cup G_m = \text{Path}_k$ and $\vec{\Delta}(G_1, \dots, G_m) = 1$, the next lemma modifies this construction of a shift permutation $\sigma : [m] \xrightarrow{\cong} [m]$ to extract an additional useful property at a small cost to $\vec{\lambda}(G_{\sigma(1)}, \dots, G_{\sigma(m)})$.

Lemma 5.6 (Stronger Pre-Main Lemma (II)). *Suppose $G_1 \cup \dots \cup G_m = \text{Path}_k$ and $\vec{\Delta}(G_1, \dots, G_m) = 1$. Then there exists a shift permutation $\sigma : [m] \xrightarrow{\cong} [m]$ such that*

$$\begin{aligned}
\vec{\lambda}(G_{\sigma(1)}, \dots, G_{\sigma(m)}) &\geq \frac{k}{8} - \frac{\max\{\lambda(G_1), \dots, \lambda(G_m)\}}{2}, \\
\vec{\Delta}(G_{\sigma_j(1)}, \dots, G_{\sigma_j(m)}) &\geq \frac{\vec{\Delta}(G_1, \dots, G_m)}{2} \quad \text{for all } j \in [m].
\end{aligned}$$

Proof. Construct sequences $1 \leq j_1 < \dots < j_r \leq m$ and $0 = i_0 < i_1 < \dots < i_p = m$ exactly as in the previous proof of Lemma 3.5(II). For concreteness, let us assume that $(i_1, \dots, i_p) = (j_2, j_4, j_6, \dots, j_{2p})$; the argument is similar when $(i_1, \dots, i_p) = (j_1, j_3, j_5, \dots, j_{2p-1})$.

Let

$$\ell := \max\{\lambda(G_1), \dots, \lambda(G_m)\}$$

Since $t_{j_{2h}} - s_{j_{2h}} \leq \ell$ for all $h \in [p]$, there exists a partition $[p] = Q_1 \sqcup Q_2$ such that, for both $b \in \{1, 2\}$,

$$\sum_{h \in Q_b} (t_{j_{2h}} - s_{j_{2h}}) \geq \frac{k}{8} - \frac{\ell}{2}.$$

We consider two shift permutations $\sigma_{I_1}, \sigma_{I_2} : [m] \xrightarrow{\cong} [m]$ where sets $\{m\} \in I_1, I_2 \subseteq [m]$ are defined by

$$I_b := [m] \setminus \bigcup_{h \in Q_b} \{j_{2h-2} + 1, \dots, j_{2h} - 1\}.$$

Note that $I_1 \cup I_2 = [m]$ and $\sigma_{I_b}(j_{2h-2} + 1) = j_{2h}$ for all $h \in Q_b$.

For both $b \in \{1, 2\}$, we have

$$\begin{aligned}
\vec{\lambda}(G_{\sigma_{I_b}(1)}, \dots, G_{\sigma_{I_b}(m)}) &= \sum_{j=1}^m \lambda(G_{\sigma_{I_b}(j)} \ominus (F \cup G_{\sigma_{I_b}(1)} \cup \dots \cup G_{\sigma_{I_b}(j-1)})) \\
&\geq \sum_{h \in Q_b} \lambda(G_{\sigma_{I_b}(j_{2h-2}+1)} \ominus (F \cup G_{\sigma_{I_b}(1)} \cup \dots \cup G_{\sigma_{I_b}(j_{2h-2})})) \\
&= \sum_{h \in Q_b} \lambda(G_{\sigma_{I_b}(j_{2h})} \ominus (F \cup G_{\sigma_{I_b}(1)} \cup \dots \cup G_{j_{2h-2}})) \\
&\geq \sum_{h \in Q_b} (t_{j_{2h}} - s_{j_{2h}}) \geq \frac{k}{8} - \frac{\ell}{2}.
\end{aligned}$$

Next, since $I_1 \cup I_2 = [m]$, it follows that for at least one $b \in \{1, 2\}$, we have

$$\sum_{l \in I_b} \Delta(G_l \ominus (G_1 \cup \dots \cup G_{l-1})) \geq \frac{1}{2} \sum_{l=1}^m \Delta(G_l \ominus (G_1 \cup \dots \cup G_{l-1})) = \frac{\vec{\Delta}(G_1, \dots, G_m)}{2}.$$

Letting $\sigma := \sigma_{I_b}$ for one such $b \in \{1, 2\}$, the proof is completed noting that by Lemma 5.3,

$$\vec{\Delta}(G_{\tilde{\sigma}_j(1)}, \dots, G_{\tilde{\sigma}_j(m)}) \geq \sum_{l \in I_b} \Delta(G_l \ominus (G_1 \cup \dots \cup G_{l-1})). \quad \square$$

5.3 Proof of Main Lemma 3.5(II)

In this subsection we prove Main Lemma 3.5(II). In fact, we prove a stronger result (Lemma 5.10), which is what we actually use in our proof of Theorem 3.6(II). For graphs $G_1 \cup \dots \cup G_m = \text{Path}_k$, this result generalizes Lemma 5.6 in terms of a parameter, $\text{gap}(G_1, \dots, G_m)$, which measures the density within the real interval $[0, k]$ of the set of midpoints of connected components of graphs $G_j \ominus (G_1 \cup \dots \cup G_m)$.

Definition 5.7 (The parameter $\text{gap}(G_1, \dots, G_m)$). Suppose $G_1 \cup \dots \cup G_m = \text{Path}_k$. Let $c := \vec{\Delta}(G_1, \dots, G_m)$ and let $0 \leq s_1 < t_1 < \dots < s_c < t_c \leq k$ be the unique integers such that

$$\bigcup_{j=1}^m G_j \ominus (G_1 \cup \dots \cup G_m) = \bigcup_{i=1}^c \text{Path}_{s_i, t_i}.$$

Note that $0 < \frac{s_1+t_1}{2} < \frac{s_2+t_2}{2} < \dots < \frac{s_c+t_c}{2} < k$.

Let $[0, k]$ denote the closed interval of real numbers between 0 and k . We define $\text{gap}(G_1, \dots, G_m) \in [0, k]$ by

$$\text{gap}(G_1, \dots, G_m) := \max_{y \in [0, k]} \min_{i \in \{1, \dots, c\}} \left| \frac{s_i + t_i}{k} - y \right|.$$

Lemma 5.8. Let $G_1 \cup \dots \cup G_m = \text{Path}_k$ and let $g := \text{gap}(G_1, \dots, G_m)$.

- (i) $\vec{\Delta}(G_1, \dots, G_m) \geq \frac{k}{2g}$.
- (ii) $\vec{\Delta}(G_1, \dots, G_m \mid F) \geq \frac{k - \lambda(F)\Delta(F)}{2g} - \Delta(F)$ for all $F \subseteq \text{Path}_k$.
- (iii) $\vec{\Delta}(G_j, G_1, \dots, G_{j-1}, G_{j+1}, \dots, G_m) \geq \frac{k - \lambda(G_j)\Delta(G_j)}{4g}$ for all $j \in [m]$.

Proof. For $0 \leq s_1 < t_1 < \dots < s_c < t_c \leq k$ as in Definition 5.7, let $p_i := \frac{s_i + t_i}{2}$ for $i \in \{1, \dots, c\}$.

Inequality (i) follows from the observation that $\max\{p_1, p_2 - p_1, p_3 - p_2, \dots, p_c - p_{c-1}, 1 - p_c\} \geq \frac{k}{2c}$ with equality iff $(p_1, \dots, p_c) = (\frac{k}{2c}, \frac{3k}{2c}, \frac{5k}{2c}, \dots, \frac{(2c-1)k}{2c})$.

For inequality (ii), let $r := \lambda(F)/2$ and $b = \Delta(F)$. Assume $b \geq 1$, since otherwise (iii) reduces to (i). Let $0 < q_1 < \dots < q_b < k$ be the midpoints of components of F . Note that

$$\begin{aligned} \vec{\Delta}(G_1, \dots, G_m \mid F) &\geq \max\left\{0, \#\left\{i \in [c] : p_i \leq q_1 - r\right\} - 1\right\} + \max\left\{0, \#\left\{i \in [c] : p_i \geq q_b + r\right\} - 1\right\} \\ &\quad + \sum_{a=1}^{b-1} \max\left\{0, \#\left\{i \in [c] : q_a + r \leq p_i \leq q_b - r\right\} - 2\right\}. \end{aligned}$$

It follows that

$$\begin{aligned} \vec{\Delta}(G_1, \dots, G_m \mid F) &\geq \max\left\{0, \left\lceil \frac{q_1 - r}{2g} \right\rceil - 1\right\} + \max\left\{0, \left\lceil \frac{k - q_b - r}{2g} \right\rceil - 1\right\} + \sum_{a=1}^{b-1} \max\left\{0, \left\lceil \frac{q_{a+1} - q_a - 2r}{2g} \right\rceil - 2\right\} \\ &\geq -2b + \left\lceil \frac{q_1 - r}{2g} \right\rceil + \left\lceil \frac{k - q_b - r}{2g} \right\rceil + \sum_{a=1}^{b-1} \left\lceil \frac{q_{a+1} - q_a - 2r}{2g} \right\rceil \\ &\geq -2b + \frac{1}{2g} \left((q_1 - r) + (k - q_b - r) + \sum_{a=1}^{b-1} (q_{a+1} - q_a - 2r) \right) \\ &= -2b + \frac{k - 2rb}{2g} = \frac{k - \lambda(F)\Delta(F)}{2g} - 2\Delta(F). \end{aligned}$$

For inequality (iii), note that $\vec{\Delta}(G_j, G_1, \dots, G_{j-1}, G_{j+1}, \dots, G_m) \geq \Delta(G_j)$ and

$$\begin{aligned} \vec{\Delta}(G_j, G_1, \dots, G_{j-1}, G_{j+1}, \dots, G_m) &= \Delta(G_j) + \vec{\Delta}(G_1, \dots, G_{j-1}, G_{j+1}, \dots, G_m \mid G_j) \\ &= \Delta(G_j) + \vec{\Delta}(G_1, \dots, G_m \mid G_j) \\ &\geq \frac{k - \lambda(G_j)\Delta(G_j)}{2g} - \Delta(G_j) \quad (\text{by (ii) with } F = G_j). \end{aligned}$$

Inequality (iii) follows by the convex combination of these bounds. \square

The next lemma generalizes the proof of Pre-Main Lemma 3.4(II).

Lemma 5.9. *Suppose $G_1 \cup \dots \cup G_m = \text{Path}_k$ and let $g := \text{gap}(G_1, \dots, G_m)$. Then there exists a shift permutation $\sigma : [m] \xrightarrow{\cong} [m]$ such that*

$$\vec{\lambda}(G_{\sigma(1)}, \dots, G_{\sigma(m)}) \geq \frac{g - \max\{\lambda(G_1), \dots, \lambda(G_m)\}}{2}.$$

Proof. Let $\ell := \max\{\lambda(G_1), \dots, \lambda(G_m)\}$ and $U_j := G_1 \cup \dots \cup G_j$ and $H := \bigcup_{j=1}^m (G_j \ominus U_{j-1})$. Assume that $\ell \leq g/2$, since the lemma is trivial otherwise.

For $0 \leq s_1 < t_1 < \dots < s_c < t_c \leq k$ as in Definition 5.7, let $p_i := \frac{s_i + t_i}{2}$ for $i \in \{1, \dots, c\}$. Then either $p_1 \geq g$ or $k - p_c \geq g$ or $p_{i+1} - p_i \geq 2g$ for some $i \in \{1, \dots, c-1\}$.

Consider first the case that $p_{i+1} - p_i \geq 2g$ for some $i \in \{1, \dots, c-1\}$. Note that $s_{i+1} - t_i \leq 2g - \ell$. Let $l \in [m]$ be the minimum index such that $\text{Path}_{s_{i+1}, t_i} \subseteq U_l$. For each $j \in [l-1]$, note that

$$\text{Path}_{s_{i+1}, t_i} \cap U_j = \text{Path}_{s_{i+1}, a_j} \cup \text{Path}_{b_j, t_i}$$

for some unique $s_{i+1} \leq a_j < b_j \leq t_i$. Next, observe that $b_{l-1} - a_{l-1} \leq \ell$ since $\text{Path}_{a_{l-1}, b_{l-1}} \subseteq G_l$ and $\lambda(G_l) \leq \ell$. Therefore, $(a_{l-1} - s_{i+1}) + (t_i - b_{l-1}) \geq 2(g - \ell)$. It follows that $a_{l-1} - s_{i+1} \geq g - \ell$ or $t_i - b_{l-1} \geq g - \ell$.

Without loss of generality, assume that $a_{l-1} - s_{i+1} \geq g - \ell$. We now apply precisely the argument of Lemma 3.4(II), but focused on the interval $\text{Path}_{a_{l-1}, s_{i+1}}$ in place of $\text{Path}_{s_1, k}$ (of length $\geq k/2$) in the proof of Lemma 3.4(II). This produces a shift permutation σ with $\vec{\lambda}(G_{\sigma(1)}, \dots, G_{\sigma(m)}) \geq (g - \ell)/2$ (instead of $k/4$).

The case where $p_1 \geq g$ or $k - p_c \geq g$ follow from applying the argument of Lemma 3.4(II) with respect to intervals Path_{0, t_1} and $\text{Path}_{s_c, k}$, respectively. Here we get a shift permutation σ with an even stronger bound $\vec{\lambda}(G_{\sigma(1)}, \dots, G_{\sigma(m)}) \geq g/2$. \square

As a corollary of Lemma 5.9, we get a proof of:

Main Lemma 3.5(II) (restated). *Suppose $G_1 \cup \dots \cup G_m = \text{Path}_k$. Then there exists a shift permutation $\sigma : [m] \xrightarrow{\cong} [m]$ such that $\vec{\lambda}\vec{\Delta}(G_{\sigma(1)}, \dots, G_{\sigma(m)}) \geq \sqrt{k/8}$.*

Proof. Letting

$$\ell := \max\{\lambda(G_1), \dots, \lambda(G_m)\}, \quad g := \text{gap}(G_1, \dots, G_m),$$

we have

$$\begin{aligned} \max_{\sigma} \vec{\lambda}\vec{\Delta}(G_{\sigma(1)}, \dots, G_{\sigma(m)}) &\geq \max\left\{\ell, \max_{\sigma} \vec{\lambda}(G_{\sigma(1)}, \dots, G_{\sigma(m)}), \vec{\Delta}(G_1, \dots, G_m)\right\} \\ &\geq \max\left\{\ell, \frac{g}{2} - \ell, \frac{k}{2g}\right\} \geq \max\left\{\frac{g}{4}, \frac{k}{2g}\right\} \geq \sqrt{\frac{k}{8}}. \end{aligned} \quad \square$$

Our proof of Theorem 3.6(II) in the next subsection does not actually use Lemma 3.5(II), but rather the following stronger version which combines of Lemmas 5.6 and 5.9.

Lemma 5.10 (Stronger Main Lemma (II)). *Suppose $G_1 \cup \dots \cup G_m = \text{Path}_k$ and let $g := \text{gap}(G_1, \dots, G_m)$. Then there exists a shift permutation $\sigma : [m] \xrightarrow{\cong} [m]$ such that*

$$\begin{aligned} \vec{\lambda}(G_{\sigma(1)}, \dots, G_{\sigma(m)}) &\geq \frac{g - 3 \max\{\lambda(G_1), \dots, \lambda(G_m)\}}{4}, \\ \vec{\Delta}(G_{\tilde{\sigma}_j(1)}, \dots, G_{\tilde{\sigma}_j(m)}) &\geq \frac{k}{4g} \quad \text{for all } j \in [m]. \end{aligned}$$

Proof. Again let $\ell := \max\{\lambda(G_1), \dots, \lambda(G_m)\}$. In the case of $\vec{\Delta}(G_1, \dots, G_m) = 1$, given the shift permutation σ_I of Lemma 3.4(II) which satisfies $\vec{\lambda}(G_{\sigma_I(1)}, \dots, G_{\sigma_I(m)}) \geq k/4$, recall that the argument of Lemma 5.6 produces two shift permutations, σ_{I_1} and σ_{I_2} , both of which satisfy

$$\vec{\lambda}(G_{\sigma_{I_b}(1)}, \dots, G_{\sigma_{I_b}(m)}) \geq \frac{\vec{\lambda}(G_{\sigma_I(1)}, \dots, G_{\sigma_I(m)}) - \ell}{2} \geq \frac{k}{8} - \frac{\ell}{2},$$

and at least one of which additionally satisfies

$$\vec{\Delta}(G_{\tilde{\sigma}_{I_b j}(1)}, \dots, G_{\tilde{\sigma}_{I_b j}(m)}) \geq \frac{\vec{\Delta}(G_1, \dots, G_m)}{2} \quad \text{for all } j \in [m].$$

Applying the argument of Lemma 5.6 instead to the shift permutation σ_I of Lemma 5.9 which satisfies $\vec{\lambda}(G_{\sigma_I(1)}, \dots, G_{\sigma_I(m)}) \geq (g - \ell)/2$, we get

$$\begin{aligned} \vec{\lambda}(G_{\sigma_{I_b}(1)}, \dots, G_{\sigma_{I_b}(m)}) &\geq \frac{\vec{\lambda}(G_{\sigma_I(1)}, \dots, G_{\sigma_I(m)}) - \ell}{2} \geq \frac{g - 3\ell}{4}, \\ \vec{\Delta}(G_{\tilde{\sigma}_{I_b j}(1)}, \dots, G_{\tilde{\sigma}_{I_b j}(m)}) &\geq \frac{\vec{\Delta}(G_1, \dots, G_m)}{2} \geq \frac{k}{4g}, \end{aligned}$$

where the last inequality is by Lemma 5.8(i). \square

5.4 Proof of Theorem 3.6(II)

As in §4.3, we consider a more general restatement of Theorem 3.6(II) that is better suitable for induction on d .

Main Theorem 3.6(II) (slightly more general restatement). *Let P be a finite subgraph of $\text{Path}_{\mathbb{Z}}$ and let T be a P -join tree of $\llbracket \rrbracket$ -depth d . Then*

$$\Psi(T) \geq \frac{1}{\sqrt{32e}} d \lambda(P)^{1/2d} - d + \Delta(P).$$

Proof. We argue by induction on d . Here we treat $d = 0$ as the base case, where P is a single edge (or empty) and the bound $\Psi(T) \geq \Delta(P)$ is trivial.

For the induction step where $d \geq 1$, we reduce to the case $\Delta(P) = 1$ exactly as in our proof of Theorem 3.6(I). We therefore assume that $T = \llbracket T_1, \dots, T_m \rrbracket$ is a Path_k -join tree of $\llbracket \rrbracket$ -depth m and aim to show

$$(13) \quad \Psi(T) \geq \varepsilon d k^{1/2d} - d + 1 \quad \text{where} \quad \varepsilon := \frac{1}{\sqrt{32e}}.$$

Let $G_1 \cup \dots \cup G_m = \text{Path}_k$ where G_j is the root graph of T_j , and let $g := \text{gap}(G_1, \dots, G_m)$. We prove inequality (13) by considering three cases, according to whether $\max_{j \in [m]} \lambda(G_j)$ is:

$$\text{greater than } \frac{\varepsilon^2 k}{\Delta(G_j)}, \quad \text{between } \frac{g}{8} \text{ and } \frac{\varepsilon^2 k}{\Delta(G_j)}, \quad \text{or less than } \frac{g}{8}.$$

The most interesting is the third case, which relies on Lemma 5.10 (the stronger version of Main Lemma 3.5(II)).

Case 1: Assume there exists $j \in [m]$ such that $\lambda(G_j) \geq \frac{\varepsilon^2 k}{\Delta(G_j)}$.

By the induction hypothesis applied to T_j , we have

$$(14) \quad \begin{aligned} \Psi(T_j) &\geq \varepsilon(d-1)\lambda(G_j)^{1/(2d-2)} + \Delta(G_j) - d + 1 \\ &= (2d-2) \left(\left(\frac{\varepsilon}{2} \right)^{2d-2} \lambda(G_j) \right)^{1/(2d-2)} + \Delta(G_j) - d + 1. \end{aligned}$$

Therefore,

$$\begin{aligned} \Psi(T) + d - 1 &\geq \Psi(T_j) + d - 1 && \text{(since clearly } \Psi(T) \geq \Psi(T_j)\text{)} \\ &\geq (2d-2) \left(\left(\frac{\varepsilon}{2} \right)^{2d-2} \lambda(G_j) \right)^{1/(2d-2)} + \Delta(G_j) && \text{(by induction hypothesis (14))} \\ &\geq (2d-1) \left(\left(\frac{\varepsilon}{2} \right)^{2d-2} \lambda(G_j) \frac{\Delta(G_j)}{2} \right)^{1/(2d-1)} + \frac{\Delta(G_j)}{2} && \text{(by Lemma 3.10)} \\ &\geq 2d \left(\left(\frac{\varepsilon}{2} \right)^{2d-2} \lambda(G_j) \frac{\Delta(G_j)^2}{4} \right)^{1/2d} && \text{(by Lemma 3.10 again)} \\ &\geq 2d \left(\left(\frac{\varepsilon}{2} \right)^{2d-2} \frac{\varepsilon^2}{4} k \right)^{1/2d} && \text{(since } \lambda(G_j) \geq \frac{\varepsilon^2 k}{\Delta(G_j)} \geq \frac{\varepsilon^2 k}{\Delta(G_j)^2}\text{)} \\ &= \varepsilon d k^{1/2d}. \end{aligned}$$

Case 2: Assume there exists $j \in [m]$ such that $\frac{g}{8} \leq \lambda(G_j) \leq \frac{\varepsilon^2 k}{\Delta(G_j)}$.

We have

$$\begin{aligned}
\Psi(T) + d - 1 &\geq \Psi(T_j) + \vec{\Delta}(G_1, \dots, G_{j-1}, G_{j+1}, \dots, G_m \mid G_j) + d - 1 && \text{(by Corollary 5.5)} \\
&\geq (2d - 2) \left(\left(\frac{\varepsilon}{2} \right)^{2d-2} \lambda(G_j) \right)^{1/(2d-2)} + \vec{\Delta}(G_j, G_1, \dots, G_{j-1}, G_{j+1}, \dots, G_m) && \\
&&& \text{(by induction hypothesis (14))} \\
&\geq (2d - 2) \left(\left(\frac{\varepsilon}{2} \right)^{2d-2} \lambda(G_j) \right)^{1/(2d-2)} + \frac{(1 - \varepsilon^2)k}{4g} && \text{(by Lemma 5.8(iii))} \\
&\geq 2d \left(\left(\frac{\varepsilon}{2} \right)^{2d-2} \lambda(G_j) \left(\frac{(1 - \varepsilon^2)k}{8g} \right)^2 \right)^{1/2d} && \text{(by Lemma 3.10 twice)} \\
&\geq 2d \left(\left(\frac{\varepsilon}{2} \right)^{2d-2} \frac{(1 - \varepsilon^2)^2}{64} k \right)^{1/2d} && \text{(since } \lambda(G_j) \geq \frac{g}{8} \text{ and } k \geq g) \\
&= \varepsilon d \left(\frac{(1 - \varepsilon^2)^2}{16\varepsilon^2} k \right)^{1/2d} \\
&\geq \varepsilon d k^{1/2d} && \text{(since } \varepsilon = \frac{1}{\sqrt{32e}}).
\end{aligned}$$

Case 3: Assume $\lambda(G_j) \leq \frac{g}{8}$ for all $j \in [m]$.

By Lemma 5.10 (the stronger version of Main Lemma 3.5(II)), there exists a shift permutation $\sigma : [m] \xrightarrow{\cong} [m]$ such that

$$(15) \quad \vec{\lambda}(G_{\sigma(1)}, \dots, G_{\sigma(m)}) \geq \frac{g - 3 \max_{j \in [m]} \lambda(G_j)}{4} > \frac{g}{8},$$

$$(16) \quad \vec{\Delta}(G_{\tilde{\sigma}_j(1)}, \dots, G_{\tilde{\sigma}_j(m)}) \geq \frac{k}{4g} \quad \text{for all } j \in [m].$$

For each $j \in [m]$, let

$$\begin{aligned}
j^* &:= \sigma^{-1}(j), \\
F_j &:= G_{\sigma(1)} \cup \dots \cup G_{\sigma(j^*-1)} \quad (= G_{\tilde{\sigma}_j(1)} \cup \dots \cup G_{\tilde{\sigma}_j(j^*-1)}), \\
G^j &:= G_j \ominus F_j, \\
T^j &:= T_j \ominus F_j \quad (= T_j \text{ restricted to } G^j).
\end{aligned}$$

Note the inequality

$$\begin{aligned}
(17) \quad \vec{\Delta}(G_{\tilde{\sigma}_j(1)}, \dots, G_{\tilde{\sigma}_j(m)}) &\geq \vec{\Delta}(G_{\tilde{\sigma}_j(j^*)}, \dots, G_{\tilde{\sigma}_j(m)} \mid G_{\tilde{\sigma}_j(1)} \cup \dots \cup G_{\tilde{\sigma}_j(j^*-1)}) \\
&= \vec{\Delta}(G_{\sigma(j^*)}, \dots, G_{\sigma(m)} \mid G_{\sigma(1)} \cup \dots \cup G_{\sigma(j-1)}) \\
&= \sum_{i=j^*}^m \Delta(G^{\sigma(i)}).
\end{aligned}$$

Also, applying the induction hypothesis to each T^j , we have

$$(18) \quad \Psi(T^j) \geq \varepsilon(d-1)\lambda(G^j)^{1/(2d-2)} + \Delta(G^j) - d + 1.$$

For each $j \in [m]$, we have

$$\begin{aligned}
\Psi(T) + d - 1 &\geq \Psi(T^j) - \Delta(G^j) + \vec{\Delta}(G_{\tilde{\sigma}_j(1)}, \dots, G_{\tilde{\sigma}_j(m)}) + d - 1 && \text{(by Lemma 5.4(iii))} \\
&\geq (2d - 2) \left(\left(\frac{\varepsilon}{2} \right)^{2d-2} \lambda(G^j) \right)^{1/(2d-2)} + \vec{\Delta}(G_{\tilde{\sigma}_j(1)}, \dots, G_{\tilde{\sigma}_j(m)}) && \text{(by (18))} \\
&\geq (2d - 2) \left(\left(\frac{\varepsilon}{2} \right)^{2d-2} \lambda(G^j) \right)^{1/(2d-2)} + \frac{k}{8g} + \sum_{i=j^*}^m \frac{\Delta(G^{\sigma(i)})}{2} && \text{(by } \frac{1}{2} \text{(16) + } \frac{1}{2} \text{(17))} \\
&\geq (2d - 1) \left(\left(\frac{\varepsilon}{2} \right)^{2d-2} \frac{k}{8g} \lambda(G^j) \right)^{1/(2d-1)} + \sum_{i=j^*}^m \frac{\Delta(G^{\sigma(i)})}{2} && \text{(by Lemma 3.10).}
\end{aligned}$$

Maximizing over $j \in [m]$, we finally obtain the desired bound 13 using Lemma 3.9 (the key numerical inequality) as follows:

$$\begin{aligned}
\Psi(T) + d - 1 &\geq \max_{j \in [m]} (2d - 1) \left(\left(\frac{\varepsilon}{2} \right)^{2d-2} \frac{k}{8g} \lambda(G^j) \right)^{1/(2d-1)} + \sum_{i=j^*}^m \frac{\Delta(G^{\sigma(i)})}{2} \\
&= \max_{h \in [m]} (2d - 1) \left(\left(\frac{\varepsilon}{2} \right)^{2d-2} \frac{k}{8g} \lambda(G^{\sigma(h)}) \right)^{1/(2d-1)} + \sum_{i=h}^m \frac{\Delta(G^{\sigma(i)})}{2} \\
&\geq 2d \left(\frac{1}{e} \sum_{h=1}^m \left(\frac{\varepsilon}{2} \right)^{2d-2} \frac{k}{8g} \lambda(G^{\sigma(h)}) \frac{\Delta(G^{\sigma(h)})}{2} \right)^{1/2d} && \text{(by Lemma 3.9)} \\
&= \varepsilon d \left(\frac{k}{4e\varepsilon^2 g} \vec{\lambda}(G_{\sigma(1)}, \dots, G_{\sigma(m)}) \right)^{1/2d} \\
&\geq \varepsilon d \left(\frac{k}{4e\varepsilon^2 g} \vec{\lambda}(G_{\sigma(1)}, \dots, G_{\sigma(m)}) \right)^{1/2d} \\
&\geq \varepsilon d \left(\frac{k}{32e\varepsilon^2} \right)^{1/2d} && \text{(by (15))} \\
&= \varepsilon d k^{1/2d}. \quad \square
\end{aligned}$$

6 Tradeoffs for pathsets

In this section, we review the Pathset Framework of papers [15, 21, 23, 24] and prove new tradeoffs for pathset complexity (Theorem 6.11) that follow from our tradeoffs for join trees (Theorem 3.6).

6.1 Relations and joins

Throughout this section, we fix arbitrary positive integers k and n . We additionally fix an arbitrary parameter $\tilde{n} \leq n$. In our application, we will set $\tilde{n} := n^{(k-1)/k}$.

Definition 6.1 (G -relations). For a graph $G \subseteq \text{Path}_k$, we refer to sets $\mathcal{A} \subseteq [n]^{V(G)}$ as G -relations. We denote the set of all G -relations by \mathcal{R}_G . (That is, \mathcal{R}_G is the set of “ $V(G)$ -ary” relations on $[n]$.)

The join operation \bowtie combines a G -relation and an H -relation into a $G \cup H$ -relation.

Definition 6.2 (Join). For graphs $G, H \subseteq \text{Path}_k$ and relations $\mathcal{A} \in \mathcal{R}_G$ and $\mathcal{B} \in \mathcal{R}_H$, the *join* of \mathcal{A} and \mathcal{B} is the relation $\mathcal{A} \bowtie \mathcal{B} \in \mathcal{R}_{G \cup H}$ defined by

$$\mathcal{A} \bowtie \mathcal{B} := \{ \gamma \in [n]^{V(G) \cup V(H)} : \gamma_{V(G)} \in \mathcal{A} \text{ and } \gamma_{V(H)} \in \mathcal{B} \}.$$

Note that the join behaves as direct product $\mathcal{A} \times \mathcal{B}$ when $V(G) \cap V(H) = \emptyset$, and as intersection $\mathcal{A} \cap \mathcal{B}$ when $V(G) = V(H)$.

6.2 Density and pathsets

Definition 6.3 (Density). Let $G \subseteq \text{Path}_k$. The *density* of a relation $\mathcal{A} \in \mathcal{R}_G$ is defined by

$$\mu(\mathcal{A}) := \frac{|\mathcal{A}|}{n^{|V(G)|}} = \mathbb{P}_{\alpha \in [n]^{V(G)}} [\alpha \in \mathcal{A}].$$

For a graph $F \subseteq \text{Path}_k$ and $I = V(F) \cap V(G)$, the *maximum density of \mathcal{A} conditioned on F* is defined by

$$\mu(\mathcal{A} \mid F) := \max_{\beta \in [n]^{V(F)}} \frac{|\{\alpha \in \mathcal{A} : \alpha_I = \beta_I\}|}{n^{|V(G) \setminus V(F)|}} = \max_{\beta \in [n]^{V(F)}} \mathbb{P}_{\alpha \in [n]^{V(G)}} [\alpha \in \mathcal{A} \mid \alpha_I = \beta_I].$$

(Note that $0 \leq \mu(\mathcal{A}) = \mu(\mathcal{A} \mid \emptyset) \leq \mu(\mathcal{A} \mid F) \leq 1$.)

The next lemma and corollary relate the density of a join to the maximum conditional densities of the constituent relations. The proof is another simple exercise in relational algebra.

Lemma 6.4 (Chain rule for density of a join). *For all graphs $F, G, H \subseteq \text{Path}_k$ and relations $\mathcal{A} \in \mathcal{R}_G$ and $\mathcal{B} \in \mathcal{R}_H$, we have*

$$\mu(\mathcal{A} \bowtie \mathcal{B} \mid F) \leq \mu(\mathcal{A} \mid F) \cdot \mu(\mathcal{B} \mid F \cup G).$$

Corollary 6.5 (m -ary version of Lemma 6.4). *For all graphs $G_1, \dots, G_m \subseteq \text{Path}_k$ and relations $\mathcal{A}_j \in \mathcal{R}_{G_j}$ ($j \in [m]$), we have*

$$\mu(\mathcal{A}_1 \bowtie \dots \bowtie \mathcal{A}_m) \leq \prod_{j=1}^m \mu(\mathcal{A}_j \mid G_1 \cup \dots \cup G_{j-1}).$$

Furthermore, since the density of the join does not depend on the ordering of relations $\mathcal{A}_1, \dots, \mathcal{A}_m$, we have

$$\mu(\mathcal{A}_1 \bowtie \dots \bowtie \mathcal{A}_m) \leq \prod_{j=1}^m \mu(\mathcal{A}_{\pi(j)} \mid G_{\pi(1)} \cup \dots \cup G_{\pi(j-1)})$$

for every permutation $\pi : [m] \xrightarrow{\cong} [m]$.

The lifting technique of papers [21, 23] focuses on a special class of G -relations that are subject to a family of density constraints in terms of $\Delta(\cdot \mid \cdot)$.

Definition 6.6 (G -pathsets). For a graph $G \subseteq \text{Path}_k$, a G -*pathset* (with respect to parameters n and \tilde{n}) is a relation $\mathcal{A} \in \mathcal{R}_G$ such that for all graphs $F \subseteq \text{Path}_k$,

$$\mu(\mathcal{A} \mid F) \leq (1/\tilde{n})^{\Delta(G \mid F)}.$$

The set of G -pathsets is denoted by \mathcal{P}_G . (Note that \mathcal{P}_G is a proper subset of \mathcal{R}_G when G is nonempty.)

The next result (Corollary 6.5 applied to pathsets) bounds the density of a join of pathsets in terms of the operation $\vec{\Delta}(\cdot)$.

Corollary 6.7. *For all graphs $G_1, \dots, G_m \subseteq \text{Path}_k$ and pathsets $\mathcal{A}_j \in \mathcal{P}_{G_j}$ ($j \in [m]$), we have*

$$\mu(\mathcal{A}_1 \bowtie \dots \bowtie \mathcal{A}_m) \leq \min_{\pi : [m] \xrightarrow{\cong} [m]} (1/\tilde{n})^{\vec{\Delta}(G_{\pi(1)}, \dots, G_{\pi(m)})}.$$

6.3 Pathset complexity

For a graph $G \subseteq \text{Path}_k$, we define a family of complexity measures $\mu_T : \mathcal{P}_G \rightarrow \mathbb{N}$ for each G -join tree T .

Definition 6.8 (Pathset complexity). Let $G \subseteq \text{Path}_k$, let $\mathcal{A} \in \mathcal{P}_G$, and let T be a G -join tree. The T -pathset complexity of \mathcal{A} , denoted by $\chi_T(\mathcal{A})$, is defined inductively as follows:

- If T is a single node labeled by G (in which case $\|G\| \leq 1$), then

$$\chi_T(\mathcal{A}) := \begin{cases} 0 & \text{if } \|G\| = 0 \text{ or } \mathcal{A} \text{ is empty,} \\ 1 & \text{if } \|G\| = 1 \text{ and } \mathcal{A} \text{ is nonempty.} \end{cases}$$

- If $T = \langle T_1, T_2 \rangle$ where T_1, T_2 are G_1, G_2 -join trees, then

$$\chi_T(\mathcal{A}) := \min_{\substack{\text{sequences } \{(\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i)\}_i : \\ (\mathcal{A}_i, \mathcal{B}_i, \mathcal{C}_i) \in \mathcal{P}_G \times \mathcal{P}_{G_1} \times \mathcal{P}_{G_2}, \\ \mathcal{A}_i \subseteq \mathcal{B}_i \bowtie \mathcal{C}_i, \mathcal{A} \subseteq \bigcup_i \mathcal{A}_i}} \sum_i \max\{\chi_{T_1}(\mathcal{B}_i), \chi_{T_2}(\mathcal{C}_i)\}.$$

Here i ranges over an arbitrary finite index set.

The following lower bound on pathset complexity was first proved in [23], then again in [15] with an improved big- Ω constant.

Theorem 6.9 ([15, 23]). For every Path_k -join tree T and Path_k -pathset \mathcal{A} ,

$$\chi_T(\mathcal{A}) \geq \tilde{n}^{\Omega(\log k)} \cdot \mu(\mathcal{A}).$$

6.4 Lower bounds on pathset complexity

The main results of this section (Theorem 6.11 and Corollary 6.12) give tradeoffs for pathset complexity $\chi_T(\mathcal{A})$ with respect to the $\langle \rangle$ - and $\llbracket \rrbracket$ -depth of the join tree T . These lower bounds are based on the following lemma.

Lemma 6.10. For every Path_k -join tree T and Path_k -pathset \mathcal{A} ,

$$\chi_T(\mathcal{A}) \geq \tilde{n}^{\Psi(T)} \cdot \mu(\mathcal{A}).$$

Proof. Fix an enumerate G_1, \dots, G_m of a T -branch covering of Path_k such that $\Psi(T) = \vec{\Delta}(G_1, \dots, G_m)$. We first observe that pathset complexity $\chi_T(\mathcal{A})$ is invariant under *rotations* of T , that is, exchanging the left and right children of any subtree. Noting that the theorem statement has nothing to do with $\langle \rangle$ -depth (which is obviously not invariant under rotations), we may assume without loss of generality that $\{G_1, \dots, G_m\}$ is the T -branch covering associated with the right spine of T . That is, assume that $T = \langle \langle T_1, \dots, T_m \rangle \rangle$ where each T_j is a B_j -join tree and $(G_1, \dots, G_m) = (B_{\pi(1)}, \dots, B_{\pi(m)})$ for some permutation $\pi : [m] \xrightarrow{\cong} [m]$.

For each $j \in [m]$, let

$$A_j := B_j \cup \dots \cup B_m, \quad S_j := \langle \langle T_j, \dots, T_m \rangle \rangle.$$

Note that S_j is an A_j -join tree. In the case $j = 1$, we have $S_1 = T$ and $A_1 = \text{Path}_k$.

By definition of pathset complexity, for all $j \in [m-1]$ and indices i_1, \dots, i_{j-1} , there exist nonempty pathsets $\mathcal{A}_{i_1, \dots, i_{j-1}} \in \mathcal{P}_{A_j}$ and $\mathcal{B}_{i_1, \dots, i_j} \in \mathcal{P}_{B_j}$ such that, letting $\mathcal{A}_j := \mathcal{A}$, for all $j \in [m-1]$, we have

$$\begin{aligned} \mathcal{A}_{i_1, \dots, i_{j-1}} &\subseteq \bigcup_{i_j} (\mathcal{B}_{i_1, \dots, i_j} \bowtie \mathcal{A}_{i_1, \dots, i_j}), \\ \chi_{S_j}(\mathcal{A}_{i_1, \dots, i_{j-1}}) &= \sum_{i_j} \max\{\chi_{T_j}(\mathcal{B}_{i_1, \dots, i_j}), \chi_{S_{j+1}}(\mathcal{A}_{i_1, \dots, i_j})\}. \end{aligned}$$

It follows that

$$\chi_T(\mathcal{A}) = \chi_{S_1}(\mathcal{A}_0) \geq \sum_{i_1} \chi_{S_2}(\mathcal{A}_{i_1}) \geq \sum_{i_1, i_2} \chi_{S_3}(\mathcal{A}_{i_1, i_2}) \geq \cdots \geq \sum_{i_1, \dots, i_{m-1}} \chi_{S_m}(\mathcal{A}_{i_1, \dots, i_{j-1}}) = \sum_{i_1, \dots, i_{m-1}} 1.$$

Similarly, letting $\mathcal{B}_{i_1, \dots, i_{m-1}, 1} := \mathcal{A}_{i_1, \dots, i_{m-1}}$, we have

$$\begin{aligned} \mathcal{A} = \mathcal{A}_0 &\subseteq \bigcup_{i_1} \mathcal{B}_{i_1} \bowtie \mathcal{A}_{i_1} \subseteq \bigcup_{i_1} \mathcal{B}_{i_1} \bowtie \bigcup_{i_2} \mathcal{B}_{i_1, i_2} \bowtie \mathcal{A}_{i_1, i_2} \subseteq \cdots \\ &\cdots \subseteq \bigcup_{i_1} \mathcal{B}_{i_1} \bowtie \bigcup_{i_2} \mathcal{B}_{i_1, i_2} \bowtie \bigcup_{i_3} \mathcal{B}_{i_1, i_2, i_3} \bowtie \cdots \bowtie \bigcup_{i_{m-1}} \mathcal{B}_{i_1, \dots, i_{m-1}} \bowtie \mathcal{A}_{i_1, \dots, i_{m-1}} \\ &= \bigcup_{\substack{i_1, \dots, i_{m-1} \\ i_m=1}} \mathcal{B}_{i_1} \bowtie \mathcal{B}_{i_1, i_2} \bowtie \mathcal{B}_{i_1, i_2, i_3} \bowtie \cdots \bowtie \mathcal{B}_{i_1, \dots, i_{m-1}} \bowtie \mathcal{B}_{i_1, \dots, i_m}. \end{aligned}$$

We may now bound $\mu(\mathcal{A})$ as follows.

$$\begin{aligned} \mu(\mathcal{A}) &\leq \sum_{\substack{i_1, \dots, i_{m-1} \\ i_m=1}} \mu(\mathcal{B}_{i_1} \bowtie \mathcal{B}_{i_1, i_2} \bowtie \mathcal{B}_{i_1, i_2, i_3} \bowtie \cdots \bowtie \mathcal{B}_{i_1, \dots, i_m}) \\ &\leq \sum_{\substack{i_1, \dots, i_{m-1} \\ i_m=1}} \prod_{j=1}^m \mu(\mathcal{B}_{i_1, i_2, \dots, i_{\pi(j)}} \mid B_{\pi(1)} \cup B_{\pi(2)} \cup \cdots \cup B_{\pi(j-1)}) \quad (\text{by Corollary 6.5}) \\ &\leq \sum_{\substack{i_1, \dots, i_{m-1} \\ i_m=1}} \prod_{j=1}^m (1/\tilde{n})^{\Delta(B_{\pi(j)} \mid B_{\pi(1)} \cup B_{\pi(2)} \cup \cdots \cup B_{\pi(j-1)})} \quad (\text{since each } \mathcal{B}_{i_1, i_2, \dots, i_{\pi(j)}} \text{ is a } B_{\pi(j)}\text{-pathset}) \\ &= (1/\tilde{n})^{\vec{\Delta}(B_{\pi(1)}, \dots, B_{\pi(m)})} \cdot \sum_{\substack{i_1, \dots, i_{m-1} \\ i_m=1}} 1 \leq (1/\tilde{n})^{\vec{\Delta}(B_{\pi(1)}, \dots, B_{\pi(m)})} \cdot \chi_T(\mathcal{A}). \end{aligned}$$

Since $(B_{\pi(1)}, \dots, B_{\pi(m)}) = (G_1, \dots, G_m)$, we get the desired bound

$$\chi_T(\mathcal{A}) \geq \tilde{n}^{\vec{\Delta}(G_1, \dots, G_m)} \cdot \mu(\mathcal{A}) = \tilde{n}^{\Psi(T)} \cdot \mu(\mathcal{A}). \quad \square$$

As an immediate corollary of Lemma 6.10 and Theorem 3.6, we get the following tradeoff for pathset complexity.

Theorem 6.11 (Pathset complexity tradeoffs). *For every Path_k-join tree T and Path_k-pathset \mathcal{A} ,*

- (I) $\chi_T(\mathcal{A}) \geq \tilde{n}^{\frac{1}{30e} dk^{1/d} - d} \cdot \mu(\mathcal{A})$ *where d is the $\langle\langle\rangle\rangle$ -depth of T ,*
- (II) $\chi_T(\mathcal{A}) \geq \tilde{n}^{\frac{1}{\sqrt{32e}} dk^{1/2d} - d} \cdot \mu(\mathcal{A})$ *where d is the $\langle\rangle\rangle$ -depth of T .*

We remark that Theorem 6.11 does not imply Theorem 6.9, since the bounds of Theorem 6.11 are trivial for $d = \Omega(k)$ while Theorem 6.9 applies to join trees of arbitrary $\langle\langle\rangle\rangle$ - and $\langle\rangle\rangle$ -depth. The ‘‘maximally overlapping’’ Path_k-join tree provides an example which has $\langle\langle\rangle\rangle$ - and $\langle\rangle\rangle$ -depth $k - 1$ and Ψ -value 1. Together these bounds imply:

Corollary 6.12. *Suppose that $\tilde{n} = n^{\Omega(1)}$ (for example, $\tilde{n} = n^{1 - \frac{1}{k}}$ in our application in this paper). Then for every Path_k-join tree T and Path_k-pathset \mathcal{A} ,*

- (I) $\chi_T(\mathcal{A}) \geq n^{\Omega(d(k^{1/d} - 1))} \cdot \mu(\mathcal{A})$ *where d is the $\langle\langle\rangle\rangle$ -depth of T ,*
- (II) $\chi_T(\mathcal{A}) \geq n^{\Omega(d(k^{1/2d} - 1))} \cdot \mu(\mathcal{A})$ *where d is the $\langle\rangle\rangle$ -depth of T .*

The lower bounds of Corollary 6.12 are asymptotically tight for every fixed d and k , and both bounds converge to $n^{\Omega(\log k)}$ since $d(k^{1/d} - 1)$ converges to $\ln k$ as $d \rightarrow \infty$.

7 Tradeoffs for AC^0 and SAC^0 formulas

In this section we prove Theorem 1.6, our size-depth tradeoffs for SAC^0 and AC^0 formulas in both the monotone and non-monotone settings. In §7.1 we discuss two different ways of converting SAC^0 and AC^0 formula to DeMorgan formulas with binary \wedge and \vee gates. We then prove tradeoff (I)⁺ over the course of §7.2–7.5. At a high-level, the proof is a reduction that starts out with a monotone SAC^0 formula of \wedge -depth d and size s computing $\text{SUB-PMM}_{n,k}$. The reduction extracts a Path_k -join tree T of $\langle\langle\rangle\rangle$ -depth d and a Path_k -pathset $\mathcal{A} \subseteq [n]^{\{0,\dots,k\}}$ of density $n^{-O(1)}$ such that $\chi_T(\mathcal{A}) \leq n^{O(1)} \cdot s$. On the other hand, the pathset complexity lower bound of §6 (Corollary 6.12) implies $\chi_T(\mathcal{A}) = n^{\Omega(dk^{1/d})} \cdot \mu(\mathcal{A})$. Combining these inequalities, we get the desired size lower bound $s = n^{\Omega(dk^{1/d}) - O(1)}$ ($= n^{\Omega(dk^{1/d})}$ since $dk^{1/d} \geq \log k$).

We next prove tradeoff (II)⁻ for non-monotone AC^0 formulas in §7.6–7.8 via a novel reduction that makes a surprising connection between AC^0 formula depth to $\langle\langle\rangle\rangle$ -depth of join trees. The remaining two tradeoffs (I)⁻ and (II)⁺ of Theorem 1.6 (for non-monotone SAC^0 formulas and monotone AC^0 formulas) are proved by a “convex combination” of arguments in the proofs of (I)⁺ of (II)⁻. We omit the redundant details.

Proviso 7.1. Throughout this section, we assume that $d \leq \log k$ and either $k \leq \log \log n$ or $k \leq \log^* n$ in the monotone and non-monotone settings, respectively. In all statements involving pathsets and pathset complexity, the density parameter \tilde{n} is fixed to $n^{(k-1)/k}$.

7.1 Converting AC^0 formulas to DeMorgan formulas

Definition 7.2 (DeMorgan formulas). A *DeMorgan formula* is a rooted binary tree with non-leaves (“gates”) labeled by \wedge or \vee , and leaves (“inputs”) labeled by constants or literals. As with AC^0 formulas, we measure *size* by the number of inputs labeled by literals and *depth* (resp. \wedge -*depth*) by the maximum number of gates (resp. \wedge -gates) on a root-to-leaf branch. Additionally, we define *left-depth* (resp. \wedge -*left-depth*) of a DeMorgan formula as the maximum number of left descents (resp. left descents at \wedge -gates) on a root-to-leaf branch.

Our tradeoffs for SAC^0 and AC^0 formulas involve two different ways of converting an $(S)AC^0$ formula F to an equivalent DeMorgan formula f .

Definition 7.3 (DeMorgan conversions of AC^0 formulas).

- The **right-deep DeMorgan conversion** of an AC^0 formula F is the DeMorgan formula f obtained by replacing each m -ary each fan-in $m \vee / \wedge$ gate with a right-deep tree of $m - 1$ binary \vee / \wedge gates.

For example, if $F = \bigwedge_{i=1}^8 F_i$, then its right-deep DeMorgan conversion is the formula

$$f = f_1 \wedge (f_2 \wedge (f_3 \wedge (f_4 \wedge (f_5 \wedge (f_6 \wedge (f_7 \wedge f_8))))))$$

where each f_i is the right-deep DeMorgan conversion of F_i .

- The **balanced DeMorgan conversion** of an AC^0 formula F is the DeMorgan formula f obtained by replacing each m -ary each fan-in $m \vee / \wedge$ gate with a balanced tree of $m - 1$ binary \vee / \wedge gates.

For example, if $F = \bigwedge_{i=1}^8 F_i$, then its balanced DeMorgan conversion is the formula

$$f = ((f_1 \vee f_2) \vee (f_3 \vee f_4)) \vee ((f_5 \vee f_6) \vee (f_7 \vee f_8))$$

where each f_i is the balanced DeMorgan conversion of F_i .

Note that if F has depth d and fan-in m , then its balanced DeMorgan conversion has depth at most $d \lceil \log m \rceil$. If F has \wedge -fan-in d and \wedge -fan-in m , then its right-deep DeMorgan conversion has \wedge -left-depth d and \wedge -depth at most dm (though possibly much greater depth). Both conversions preserve size.

Our first tradeoff for SAC^0 formulas (§7.2–7.5) uses the right-deep DeMorgan conversion to establish a connection between \wedge -fan-in, \wedge -left-depth, and $\langle\langle\rangle\rangle$ -depth of join trees. Our second tradeoff for AC^0 formulas (§7.6–7.8) relies on a randomized version of the balanced DeMorgan conversion to establish a connection between AC^0 depth and $\langle\langle\rangle\rangle$ -depth of join trees. We state the definition here, but postpone analysis to §7.6.

Definition 7.4 (Randomized balanced DeMorgan conversion). Let $t \in \mathbb{N}$ be an arbitrary parameter. For every AC^0 formula F , we define a random DeMorgan formula \mathbf{f} with distribution $\mathcal{D}_t(F)$ as follows:

- If F has depth 0 (i.e., is a constant or literal), then $\mathbf{f} = F$ (with probability 1).
- Suppose that F is $\bigwedge_{i=1}^m F_i$ (resp. $\bigvee_{i=1}^m F_i$). Sample independent random DeMorgan formulas $\mathbf{f}_i \sim \mathcal{D}_t(F_i)$ and independent uniform random indices $i_1, \dots, i_{tm} \in [m]$. Now let \mathbf{f} be the DeMorgan formula consisting of a balanced tree of $tm - 1$ binary \wedge -gates (resp. \vee -gates) with subformulas $\mathbf{f}_{i_1}, \dots, \mathbf{f}_{i_{tm}}$ feeding in.

Note that if F has depth d and size s , then every \mathbf{f} in the support of $\mathcal{D}_t(F)$ has depth at most $d \lceil \log(ts) \rceil$ and size at most $t^d s$. In particular, if $t = (\log s)^{O(1)}$ and $d = o(\frac{\log s}{\log \log s})$, then \mathbf{f} has size $s^{1+o(1)}$. Also note that \mathbf{f} does not necessarily compute the same function as F . However, for $t \geq (\log s)^2$, we at least ensure that $\mathbf{f}(x) = F(x)$ with high probability for any given input x .

We conclude this subsection with a few remarks on notation.

Notation 7.5. Throughout this section, we speak of both AC^0 and DeMorgan formulas on both kn^2 variables and k variables. To keep these objects straight, we consistently write

- F for an AC^0 formula on kn^2 variables,
- \mathbf{f} for a DeMorgan formula on kn^2 variables,
- G for an AC^0 formula on k variables,
- \mathbf{g} for a DeMorgan formula on k variables.

We use typewriter font F, \mathbf{f} and G, \mathbf{g} to distinguish these formulas from the Boolean functions of kn^2 and k variables that they compute, denoted by italic f and g .

In the context of tradeoffs $(I)^\pm$, we refer to F as an “ SAC^0 formula” whenever we assume that it has \wedge -fan-in at most $n^{1/k}$.

Notation 7.6. Note the distinction between syntactic *equality* of formulas (notated with $=$) and semantic *equivalence* of formulas (notated with \equiv). For example, $\mathbf{f}_1 = \mathbf{f}_2$ expresses that \mathbf{f}_1 and \mathbf{f}_2 are isomorphic as labeled binary trees, whereas $\mathbf{g}_1 \equiv \mathbf{g}_2$ expresses that \mathbf{g}_1 and \mathbf{g}_2 compute the same Boolean function $\{0, 1\}^k \rightarrow \{0, 1\}$. We sometimes write \equiv between an AC^0 formula and an equivalent DeMorgan formula (e.g., $F \equiv \mathbf{f}$ or $G \equiv \mathbf{g}$).

7.2 Monotone functions on kn^2 variables

In the context of problems $\text{BMM}_{n,k}$, $\text{SUB-PMM}_{n,k}$, $\text{PMM}_{n,k}$, we identify elements of $\{0, 1\}^{kn^2}$ with k -tuples $\vec{M} = (M^{(1)}, \dots, M^{(k)})$ of Boolean matrices $M^{(i)} \in \{0, 1\}^{n \times n}$. However, for purposes of our lower bound method, it is convenient to instead identify $\{0, 1\}^{kn^2}$ with the set of subgraphs of the n -blow-up of Path_k , defined below.

Definition 7.7 (The n -blow-up of a graph). The n -blow-up of a graph G , denoted $G^{\uparrow n}$, is the graph with vertex and edge sets

$$\begin{aligned} V(G^{\uparrow n}) &:= \{v^{(a)} : v \in V(G), a \in [n]\} \quad (\text{a way of writing } V(G) \times [n]), \\ E(G^{\uparrow n}) &:= \{\{v^{(a)}, w^{(b)}\} : \{v, w\} \in E(G), a, b \in [n]\}. \end{aligned}$$

That is, for each vertex v of G , the blow-up contains n distinct vertices named $v^{(1)}, \dots, v^{(n)}$; and for each edge $\{v, w\}$ of G , the blow-up contains the complete bipartite graph between sets $\{v^{(1)}, \dots, v^{(n)}\}$ and $\{w^{(1)}, \dots, w^{(n)}\}$.

For $\alpha \in [n]^{V(G)}$, we write $G^{(\alpha)}$ for the isomorphic copy of G with

$$\begin{aligned} V(G^{(\alpha)}) &:= \{v^{(\alpha_v)} : v \in V(G)\}, \\ E(G^{(\alpha)}) &:= \{\{v^{(\alpha_v)}, w^{(\alpha_w)}\} : \{v, w\} \in E(G)\}. \end{aligned}$$

Graphs $G^{(\alpha)}$ are called *sections* of the blow-up $G^{\uparrow n}$.

To simplify notation: given a section $G^{(\alpha)}$ of $G^{\uparrow n}$ and a subgraph $H \subseteq G$, we write $H^{(\alpha)}$ instead of $H^{(\alpha_V(H))}$ for the corresponding section of $H^{\uparrow n}$.

Definition 7.8 (Identifying $\{0, 1\}^{kn^2}$ with the set of subgraphs of $\text{Path}_k^{\uparrow n}$).

- The blow-up $\text{Path}_k^{\uparrow n}$ has kn^2 edges and 2^{kn^2} subgraphs. (Recall that *graphs* have no isolated vertices according to our definition.)
- For $G \subseteq \text{Path}_k$ and $\alpha \in [n]^{V(G)}$, we refer to graphs $G^{(\alpha)}$ as *G-sections* of $\text{Path}_k^{\uparrow n}$.
- We identify elements of $\{0, 1\}^{kn^2}$ with subgraphs of $\text{Path}_k^{\uparrow n}$, as well as k -tuples of n -by- n Boolean matrices.

Definition 7.9 (*G*-minterms). For a monotone function $f : \{0, 1\}^{kn^2} \rightarrow \{0, 1\}$, we write $\mathcal{M}_G(f)$ for the *G*-relation

$$\mathcal{M}_G(f) := \{\alpha \in [n]^{V(G)} : G^{(\alpha)} \text{ is a minterm of } f\}.$$

That is, $\mathcal{M}_G(f)$ is the set of $\alpha \in [n]^{V(G)}$ such that $f(G^{(\alpha)}) = 1$ and $f(H^{(\alpha)}) = 0$ for all $H \subsetneq G$.

For example, we have $\mathcal{M}_{\text{Path}_k}(\text{BMM}_{n,k}) = \{\alpha \in [n]^{\{0, \dots, k\}} : \alpha_0 = \alpha_k = 1\}$. More generally:

Observation 7.10. A monotone function $f : \{0, 1\}^{kn^2} \rightarrow \{0, 1\}$ computes $\text{SUB-PMM}_{n,k}$ (i.e., agrees with $\text{BMM}_{n,k}$ whenever the input is a k -tuple of sub-permutation matrices) if, and only if, $\mathcal{M}_{\text{Path}_k}(f) = \{\alpha \in [n]^{\{0, \dots, k\}} : \alpha_0 = \alpha_k = 1\}$. In this case, we have $\mu(\mathcal{M}_{\text{Path}_k}(f)) = n^{-2}$.

The next lemma relates the *G*-minterm relations of $f_1 \vee f_2$ and $f_1 \wedge f_2$ to those of f_1 and f_2 .

Lemma 7.11. For every graph $G \subseteq \text{Path}_k$ and monotone functions $f_1, f_2 : \{0, 1\}^{kn^2} \rightarrow \{0, 1\}$, we have

- $\mathcal{M}_G(f_1 \vee f_2) \subseteq \mathcal{M}_G(f_1) \cup \mathcal{M}_G(f_2)$,
- $\mathcal{M}_G(f_1 \wedge f_2) \subseteq \bigcup_{\substack{G_1, G_2 \subseteq G: \\ G_1 \cup G_2 = G}} \mathcal{M}_{G_1}(f_1) \bowtie \mathcal{M}_{G_2}(f_2) \subseteq \mathcal{M}_G(f_1) \cup \mathcal{M}_G(f_2) \cup \bigcup_{\substack{G_1, G_2 \subsetneq G: \\ G_1 \cup G_2 = G}} \mathcal{M}_{G_1}(f_1) \bowtie \mathcal{M}_{G_2}(f_2)$.

Proof. These containments follow from the elementary observation: for any monotone Boolean functions h_1 and h_2 , every minterm of $h_1 \vee h_2$ is a minterm of h_1 or a minterm of h_2 , and every minterm of $h_1 \wedge h_2$ is the union of a minterm of h_1 and a minterm of h_2 . \square

The following corollary extends Lemma 7.11 to m -ary disjunctions and conjunctions. We do not directly use Corollary 7.12 in what follows, but similar reasoning shows up in the proof of Lemma 7.17.

Corollary 7.12 (m -ary version of Lemma 7.11). For every nonempty graph $\emptyset \neq G \subseteq \text{Path}_k$, monotone functions $f_1, \dots, f_m : \{0, 1\}^{kn^2} \rightarrow \{0, 1\}$, we have

- $\mathcal{M}_G(\bigvee_{j=1}^m f_j) \subseteq \bigcup_{j=1}^m \mathcal{M}_G(f_j)$,
- $\mathcal{M}_G(\bigwedge_{j=1}^m f_j) \subseteq \bigcup_{G_1, \dots, G_m \subseteq G: G_1 \cup \dots \cup G_m = G} \bigbowtie_{j=1}^m \mathcal{M}_{G_j}(f_j) \subseteq \bigcup_{t=1}^{\|G\|} \bigcup_{\substack{G_1, \dots, G_t \subseteq G: \\ G_1 \cup \dots \cup G_t = G, \\ \forall s \in [t], G_s \not\subseteq G_1 \cup \dots \cup G_{s-1}}} \bigcup_{1 \leq j_1 < \dots < j_t \leq m} \bigbowtie_{s=1}^t \mathcal{M}_{G_{j_s}}(f_{j_s})$.

In the bottom inclusions $\mathcal{M}_G(\bigwedge_{j=1}^m f_j) \subseteq \bigcup \bigbowtie \dots \subseteq \bigcup \bigcup \bigcup \bigbowtie \dots$, the lefthand union is indexed over $(2^{\|G\|} - 1)^m$ ($\leq 2^{km}$) possibilities for G_1, \dots, G_m , while the righthand unions are indexed over at most $2^{\|G\|^2} m^{\|G\|}$ ($\leq 2^{k^2} m^k$) possibilities for $t, j_1, \dots, j_t, G_1, \dots, G_t$. In particular, when $k \leq \log \log n$ and $m \leq n^{1/k}$, there are only $2^{k^2} m^k = n^{O(1)}$ indices in the righthand unions. This is essentially the reason why our SAC^0 formula lower bounds extend to \bigwedge -fan-in $n^{1/k}$.

7.3 Strict join trees

Definition 7.13 (Strict join trees). For $G \subseteq \text{Path}_k$, a G -join tree T is *strict* if either $\|G\| \leq 1$ and T is a single node labeled G , or $\|G\| \geq 2$ and $T = \langle T_1, T_2 \rangle$ where T_1, T_2 are G_1, G_2 -join trees with $G_1, G_2 \subsetneq G$. We denote the set of strict G -join trees by \mathcal{T}_G .

A simple counting argument shows that $|\mathcal{T}_G| \leq 2^{2^{\|G\|}}$. For bounded $\langle\langle\rangle\rangle$ - or $\llbracket\rrbracket$ -depth, we get an even better bound:

Lemma 7.14. *For all $G \subseteq \text{Path}_k$, there are at most $2^{\|G\|^{d+1}}$ distinct $T \in \mathcal{T}_G$ with $\langle\langle\rangle\rangle$ -depth (or $\llbracket\rrbracket$ -depth) at most d .*

Proof. We argue by induction on d . Let $\ell := \|G\|$ and assume $\ell \geq 1$, since otherwise \mathcal{T}_G is empty. In the case $d = 0$, we have $|\mathcal{T}_G| \leq 1$. In the case $d = 1$, we have $|\mathcal{T}_G| = \ell! < 2^{\ell^2}$.

Suppose $d \geq 2$ and consider any $T = \langle\langle T_1, \dots, T_m \rangle\rangle \in \mathcal{T}_G$ where $m \geq 2$ and each T_j is a G_j -join tree of $\langle\langle\rangle\rangle$ -depth at most $d - 1$. Note that $G_1 \cup \dots \cup G_m = G$ and $G_j \not\subseteq G_1 \cup \dots \cup G_{j-1}$ for all $j \in [m]$. It follows that $m \leq \ell$, and there are at most 2^{ℓ^2} possible sequences (G_1, \dots, G_m) .

For each $j \in [m]$, since $\|G_j\| \leq \ell - 1$, there are at most $2^{(\ell-1)^d}$ possibilities for T_j by the induction hypothesis. Therefore, the number of possible T is at most

$$2^{\ell^2} \cdot (2^{(\ell-1)^d})^\ell = 2^{\ell^2 + \ell(\ell-1)^d} \leq 2^{\ell^{d+1}},$$

since $\ell^2 + \ell(\ell-1)^d \leq \ell^{d+1}$ for all $\ell \geq 1$ and $d \geq 2$.

The exact same argument applies to $\llbracket\rrbracket$ -depth, since $\llbracket T_1, \dots, T_m \rrbracket \in \mathcal{T}_G$ also implies that $G_j \not\subseteq G_1 \cup \dots \cup G_{j-1}$ for all $j \in [m]$. \square

Definition 7.15. Let \mathbf{f} be a monotone DeMorgan formula on kn^2 variables. For each graph $G \subseteq \text{Path}_k$ and strict join tree $T \in \mathcal{T}_G$, we define a subset $\mathcal{M}_G^{(T)}(\mathbf{f}) \subseteq \mathcal{M}_G(\mathbf{f})$ inductively as follows. If $\|G\| \leq 1$, then $\mathcal{M}_G^{(T)}(\mathbf{f}) := \mathcal{M}_G(\mathbf{f})$. If $\|G\| \geq 2$ and $T = \langle T_1, T_2 \rangle$, then

$$\mathcal{M}_G^{(T)}(\mathbf{f}) := \begin{cases} \emptyset & \text{if } \mathbf{f} \text{ has depth } 0, \\ \mathcal{M}_G(\mathbf{f}) \cap \left(\mathcal{M}_G^{(T)}(\mathbf{f}_1) \cup \mathcal{M}_G^{(T)}(\mathbf{f}_2) \right) & \text{if } \mathbf{f} = \mathbf{f}_1 \vee \mathbf{f}_2, \\ \mathcal{M}_G(\mathbf{f}) \cap \left(\mathcal{M}_G^{(T)}(\mathbf{f}_1) \cup \mathcal{M}_G^{(T)}(\mathbf{f}_2) \cup \left(\mathcal{M}_{G_1}^{(T_1)}(\mathbf{f}_1) \bowtie \mathcal{M}_{G_2}^{(T_2)}(\mathbf{f}_2) \right) \right) & \text{if } \mathbf{f} = \mathbf{f}_1 \wedge \mathbf{f}_2. \end{cases}$$

The next lemma follows directly from Lemma 7.11 and the definition of subsets $\mathcal{M}_G^{(T)}(\mathbf{f})$.

Lemma 7.16. *For every monotone DeMorgan formula \mathbf{f} on kn^2 variables and graph $G \subseteq \text{Path}_k$, we have*

$$\bigcup_{T \in \mathcal{T}_G} \mathcal{M}_G^{(T)}(\mathbf{f}) = \mathcal{M}_G(\mathbf{f}).$$

Moreover, if \mathbf{f} has \wedge -left-depth d , then $\mathcal{M}_G^{(T)}(\mathbf{f})$ is nonempty only for $T \in \mathcal{T}_G$ with $\langle\langle\rangle\rangle$ -depth at most d . It follows that there exists a strict join tree $T \in \mathcal{T}_G$ with $\langle\langle\rangle\rangle$ -depth at most d such that

$$\mu(\mathcal{M}_G^{(T)}(\mathbf{f})) \geq \frac{\mu(\mathcal{M}_G(\mathbf{f}))}{|\{T \in \mathcal{T}_G : T \text{ has } \langle\langle\rangle\rangle\text{-depth at most } d\}|} \geq \frac{\mu(\mathcal{M}_G(\mathbf{f}))}{2^{\|G\|^{d+1}}}.$$

In the event that $\mathcal{M}_G(\mathbf{f}_0)$ is a G -pathset for every graph $G \subseteq \text{Path}_k$ and subformula \mathbf{f}_0 of \mathbf{f} , we are able to bound on the size of \mathbf{f} in terms of the pathset complexity of subsets $\mathcal{M}_G^{(T)}(\mathbf{f}) \subseteq \mathcal{M}_G(\mathbf{f}_0)$.

Lemma 7.17 (Lemma 6.6 of [23]). *Suppose that \mathbf{f} is a monotone DeMorgan formula on kn^2 variables with \wedge -depth D such that $\mathcal{M}_G(\mathbf{f}_0)$ is a G -pathset for every graph $G \subseteq \text{Path}_k$ and subformula \mathbf{f}_0 of \mathbf{f} . Then for every $G \subseteq \text{Path}_k$ and $T \in \mathcal{T}_G$, we have*

$$\chi_T(\mathcal{M}_G^{(T)}(\mathbf{f})) \leq \binom{D + \|G\| - 1}{\|G\| - 1} \cdot \text{size}(\mathbf{f}) \leq (D + 1)^{\|G\|} \cdot \text{size}(\mathbf{f}).$$

Proof. We prove the inequality by induction on \mathbf{f} . In the base case where \mathbf{f} is a constant, or \mathbf{f} is a positive literal and $\|G\| \geq 2$, then $\mathcal{M}_G^{(T)}(\mathbf{f})$ is empty, hence $\chi_T(\mathcal{M}_G^{(T)}(\mathbf{f})) = 0$ and $\text{size}(\mathbf{f}) = 0$. In the remaining base case that \mathbf{f} is a positive literal and $\|G\| = 1$, we $\chi_T(\cdot) \leq 1$ and $\text{size}(\mathbf{f}) = 1$.

The induction step is straightforward when \mathbf{f} is a disjunction $\mathbf{f}_1 \vee \mathbf{f}_2$. We are left with the case that \mathbf{f} is a conjunction $\mathbf{f}_1 \wedge \mathbf{f}_2$. Here the induction step is trivial when $\|G\| = 1$. So we assume that $\|G\| \geq 2$ and $T = \langle T_1, T_2 \rangle$.

For $i \in \{1, 2\}$, note that $\|G_i\| \leq \|G\| - 1$ and $\text{depth}_\wedge(\mathbf{f}_i) \leq D - 1$ and $\text{size}(\mathbf{f}) = \text{size}(\mathbf{f}_1) + \text{size}(\mathbf{f}_2)$. By definition of pathset complexity $\chi_T(\cdot)$ and the induction hypothesis applied to \mathbf{f}_1 and \mathbf{f}_2 ,

$$\begin{aligned}
\chi_T(\mathcal{M}_G^{(T)}(\mathbf{f})) &= \chi_T\left(\mathcal{M}_G(\mathbf{f}) \cap \left(\mathcal{M}_G^{(T)}(\mathbf{f}_1) \cup \mathcal{M}_G^{(T)}(\mathbf{f}_2) \cup \left(\mathcal{M}_{G_1}^{(T_1)}(\mathbf{f}_1) \bowtie \mathcal{M}_{G_2}^{(T_2)}(\mathbf{f}_2)\right)\right)\right) \\
&\leq \chi_T(\mathcal{M}_G^{(T)}(\mathbf{f}_1)) + \chi_T(\mathcal{M}_G^{(T)}(\mathbf{f}_2)) + \max\{\chi_{T_1}(\mathcal{M}_{G_1}^{(T_1)}(\mathbf{f}_1)), \chi_{T_2}(\mathcal{M}_{G_2}^{(T_2)}(\mathbf{f}_2))\} \\
&\leq \sum_{i \in \{1, 2\}} \left(\binom{\text{depth}_\wedge(\mathbf{f}_i) + \|G\| - 1}{\|G\| - 1} + \binom{\text{depth}_\wedge(\mathbf{f}_i) + \|G_i\| - 1}{\|G_i\| - 1} \right) \cdot \text{size}(\mathbf{f}_i) \\
&\leq \sum_{i \in \{1, 2\}} \left(\binom{D + \|G\| - 2}{\|G\| - 1} + \binom{D + \|G\| - 2}{\|G\| - 2} \right) \cdot \text{size}(\mathbf{f}_i) \\
&= \binom{D + \|G\| - 1}{\|G\| - 1} \cdot \text{size}(\mathbf{f}). \quad \square
\end{aligned}$$

Corollary 7.18. *Suppose that \mathbf{f} is a monotone DeMorgan formula on kn^2 variables with \wedge -left-depth d and \wedge -depth D such that $\mathcal{M}_G(\mathbf{f}_0)$ is a G -pathset for every graph $G \subseteq \text{Path}_k$ and subformula \mathbf{f}_0 of \mathbf{f} . Then*

$$\text{size}(\mathbf{f}) \geq \frac{n^{\Omega(d(k^{1/d}-1))}}{2^{k^{d+1}} \cdot (D+1)^k} \cdot \mu(\mathcal{M}_{\text{Path}_k}(\mathbf{f})).$$

Proof. By Lemma 7.16, there exists a strict join tree $T \in \mathcal{T}_{\text{Path}_k}$ such that

$$\mu(\mathcal{M}_{\text{Path}_k}^{(T)}(\mathbf{f})) \geq \frac{1}{2^{k^{d+1}}} \cdot \mu(\mathcal{M}_{\text{Path}_k}(\mathbf{f})).$$

We now obtain the desired bound as follows:

$$\begin{aligned}
\text{size}(\mathbf{f}) &\geq \frac{1}{(D+1)^k} \cdot \chi_T(\mathcal{M}_{\text{Path}_k}^{(T)}(\mathbf{f})) && \text{(by Lemma 7.17)} \\
&\geq \frac{n^{\Omega(d(k^{1/d}-1))}}{(D+1)^k} \cdot \mu(\mathcal{M}_{\text{Path}_k}^{(T)}(\mathbf{f})) && \text{(by Corollary 6.12(I))} \\
&\geq \frac{n^{\Omega(d(k^{1/d}-1))}}{2^{k^{d+1}} \cdot (D+1)^k} \cdot \mu(\mathcal{M}_{\text{Path}_k}(\mathbf{f})). \quad \square
\end{aligned}$$

The hypothesis of Corollary 7.18 requires that G -minterm relations $\mathcal{M}_G(\mathbf{f}_0)$ are pathsets for all subformulas \mathbf{f}_0 of \mathbf{f} . We point out that this condition may fail to hold even for relatively simple DeMorgan formulas \mathbf{f} . For example, suppose \mathbf{f} is the balanced DeMorgan conversion of the read-once CNF formula $\bigwedge_{i \in [k]} \bigvee_{(a,b) \in [n]^2} M_{a,b}^{(i)}$. In this case, $\mathcal{M}_{\text{Path}_k}(\mathbf{f})$ is the complete Path_k -relation $[n]^{\{0, \dots, k\}}$, which is not a Path_k -pathset since it has density 1 ($> \tilde{n} = \tilde{n}^{\Delta(\text{Path}_k)}$).

7.4 Random restriction lemma

In order to make use of Corollary 7.18, we require a result (Lemma 7.24) showing that whenever a monotone function $f : \{0, 1\}^{kn} \rightarrow \{0, 1\}$ is computable by a reasonably small AC^0 formula of reasonably bounded depth, the G -minterm relation of a certain *random restriction* of f is a G -pathset with very high probability.

Definition 7.19 (Random matrices $\xi \leq \zeta \in \{0, 1\}^{n \times n}$ and graphs $\Xi \subseteq \text{Path}_k^{\uparrow n}$).

- Let $\zeta \in \{0, 1\}^{n \times n}$ be a random Boolean matrix with i.i.d. Bernoulli($n^{-1-\frac{1}{2k}}$) entries.
- Let $\xi \in \{0, 1\}^{n \times n}$ be the sub-permutation matrix induced by ζ as follows:

$$\xi_{a,b} := \begin{cases} 1 & \text{if } \zeta_{a,b} = 1 \text{ and } \zeta_{a',b} = \zeta_{a,b'} = 0 \text{ for all } a' \neq a \text{ and } b' \neq b, \\ 0 & \text{otherwise.} \end{cases}$$

- Let $\zeta^{(1)}, \dots, \zeta^{(k)} \in \{0, 1\}^{n \times n}$ be independent random matrices with distribution ζ , let $\xi^{(1)}, \dots, \xi^{(k)} \in \{0, 1\}^{n \times n}$ be the induced sub-permutation matrices, and let $\Xi := (\xi^{(1)}, \dots, \xi^{(k)})$ which we regard as an element of $\{0, 1\}^{kn^2}$ as well as a subgraph $\Xi \subseteq \text{Path}_k^{\uparrow n}$.

Definition 7.20 (The restricted function $f^{\cup \Xi}$).

- For a Boolean function $f : \{0, 1\}^{kn^2} \rightarrow \{0, 1\}$, let $f^{\cup \Xi} : \{0, 1\}^{kn^2} \rightarrow \{0, 1\}$ denote the random function defined by $f^{\cup \Xi}(X) := f(X \cup \Xi)$ for all $X \subseteq \text{Path}_k^{\uparrow n}$. (Note that if f is monotone, then so is $f^{\cup \Xi}$.)
- For a kn^2 -variable DeMorgan formula \mathfrak{f} , let $\mathfrak{f}^{\cup \Xi}$ denote the formula obtained from \mathfrak{f} by substituting the constant 1 (resp. 0) at each positive (resp. negative) literal corresponding to an edge of Ξ . (Note that if \mathfrak{f} computes f , then $\mathfrak{f}^{\cup \Xi}$ computes $f^{\cup \Xi}$.)

The next lemma extends Observation 7.10 concerning the Path_k -minterms of monotone functions that compute $\text{SUB-PMM}_{n,k}$.

Lemma 7.21. *If $f : \{0, 1\}^{kn^2} \rightarrow \{0, 1\}$ is a monotone function which agrees with $\text{SUB-PMM}_{n,k}$ on every k -tuple of sub-permutation matrices, then*

$$\mathbb{P} \left[\mu(\mathcal{M}_{\text{Path}_k}(f^{\cup \Xi})) \geq \frac{1}{2n^2} \right] = 1 - o(1).$$

Proof. Let $\zeta^{(1)}, \dots, \zeta^{(k)} \in \{0, 1\}^{n \times n}$ be the random matrices which generate the $\xi^{(1)}, \dots, \xi^{(k)} \in \{0, 1\}^{n \times n}$ comprising Ξ . As noted in Observation 7.10, we have $\mathcal{M}_{\text{Path}_k}(f) = \{\alpha \in [n]^{\{0, \dots, k\}} : \alpha_0 = \alpha_k = 1\}$. For each α in this set, a sufficient condition for $\alpha \in \mu(\mathcal{M}_{\text{Path}_k}(f^{\cup \Xi}))$ is that $\zeta_{a, \alpha_i}^{(i)} = \zeta_{\alpha_{i-1}, b}^{(i)} = 0$ for all $a, b \in [n]$. (In this case, $\Xi \cup G^{(\alpha)}$ is a k -tuple of sub-permutation matrices for every $G \subseteq \text{Path}_k$, from which it follows that $\alpha \in \mu(\mathcal{M}_{\text{Path}_k}(f^{\cup \Xi}))$.) This sufficient condition occurs with probability $(1 - n^{-1-\frac{1}{2k}})^{k(2n-1)} = 1 - O(kn^{-\frac{1}{2k}}) = 1 - o(1)$. Therefore, by linearity of expectation,

$$\mathbb{E} \left[\mu(\mathcal{M}_{\text{Path}_k}(f^{\cup \Xi})) \right] \geq (1 - o(1)) \cdot \mu(\mathcal{M}_{\text{Path}_k}(f)) = (1 - o(1)) \cdot \frac{1}{n^2}.$$

A straightforward application of Janson's inequality [14] shows that $\mu(\mathcal{M}_{\text{Path}_k}(f^{\cup \Xi}))$ is at least half its expectation with very high probability. \square

The next definition gives a version of the relation $\mathcal{M}_G(f)$ for non-monotone functions f .

Definition 7.22 (The relation $\mathcal{N}_G(f)$).

- For a (not necessarily monotone) function $f : \{0, 1\}^{kn^2} \rightarrow \{0, 1\}$ and a graph $G \subseteq \text{Path}_k$ and $\alpha \in [n]^{V(G)}$, let $f \upharpoonright G^{(\alpha)} : \{0, 1\}^{\|G\|} \rightarrow \{0, 1\}$ denote restricted subfunction $f \upharpoonright G^{(\alpha)}(H) := H^{(\alpha)}$ where we identify each input $H \in \{0, 1\}^{\|G\|}$ with the corresponding subgraph of G .
- The relation $\mathcal{N}_G(f) \subseteq [n]^{V(G)}$ by

$$\mathcal{N}_G(f) := \{\alpha \in [n]^{V(G)} : f \upharpoonright G^{(\alpha)} \text{ depends on all } \|G\| \text{ coordinates}\}.$$

For monotone functions f , note that $\mathcal{M}_G(f) \subseteq \mathcal{N}_G(f)$ (hence if $\mathcal{N}_G(f)$ is a G -pathset, then so is $\mathcal{M}_G(f)$).

Since it will be useful later, we state here a variant of Lemma 7.17 with $\mathcal{N}_G(\cdot)$ in place of $\mathcal{M}_G(\cdot)$. (The proof is identical.)

Lemma 7.23. *Suppose that \mathbf{f} is a DeMorgan formula on kn^2 variables with depth D such that $\mathcal{N}_G(\mathbf{f}_0)$ is a G -pathset for every graph $G \subseteq \text{Path}_k$ and subformula \mathbf{f}_0 of \mathbf{f} . Then for every $G \subseteq \text{Path}_k$ and $T \in \mathcal{T}_G$, we have*

$$\chi_T(\mathcal{N}_G^{(T)}(\mathbf{f})) \leq (D+1)^{\|G\|} \cdot \text{size}(\mathbf{f}).$$

We conclude this subsection by stating a key lemma from [23], which implies that relations $\mathcal{M}_G(f^{\cup \Xi})$ are likely to be G -pathsets whenever f is computable by monotone AC^0 formulas of reasonable size and depth.

Lemma 7.24 (Lemma 6.5 of [23]). *Suppose that $f : \{0, 1\}^{kn^2} \rightarrow \{0, 1\}$ is computable by a (not necessarily monotone) AC^0 formula of size at most n^k and depth at most $\frac{\log n}{(\log \log n)^6}$. Then for every $G \subseteq \text{Path}_k$, we have*

$$\mathbb{P} [\mathcal{N}_G(f^{\cup \Xi}) \text{ is \underline{not} a } G\text{-pathset}] \leq O(n^{-2k}).$$

7.5 Tradeoff (I)⁺ for monotone SAC^0 formulas

We are ready to prove our size-depth tradeoff for monotone SAC^0 formulas.

Theorem 7.25 (Tradeoff (I)⁺ of Theorem 1.6). *Suppose that F is a monotone SAC^0 formula of \wedge -depth d and \wedge -fan-in $n^{1/k}$ which computes $\text{SUB-PMM}_{n,k}$ where $k \leq \log \log n$ and $d \leq \log k$. Then F has size $n^{\Omega(dk^{1/d})}$.*

Proof. Without loss of generality, assume that F has size at most n^k , since this is greater than the lower bound we wish to show. Let \mathbf{f} be the right-deep DeMorgan conversion of F . Note that even though \mathbf{f} has very large depth, it has \wedge -left-depth d and \wedge -depth at most $dn^{1/k}$. Moreover, each subformula \mathbf{f}_0 of \mathbf{f} is equivalent to a monotone AC^0 formula of \wedge -depth at most d (and depth at most $2d+1 = O(\log \log n)$) and size at most n^k . In particular, \mathbf{f}_0 satisfies the hypothesis of Lemma 7.24.

We now observe that each of the following statements hold with probability $1 - o(1)$:

- $\mathcal{M}_G(\mathbf{f}_0^{\cup \Xi})$ is a G -pathset for every subformula \mathbf{f}_0 of \mathbf{f} and $G \subseteq \text{Path}_k$
(by Lemma 7.24 and a union bound over at most $O(n)^k$ choices of \mathbf{f}_0 and G),
- $\text{size}(\mathbf{f}^{\cup \Xi}) \geq \frac{n^{\Omega(d(k^{1/d}-1))}}{2^{k^{d+1}} \cdot (dn^{1/k} + 1)^k} \cdot \mu(\mathcal{M}_{\text{Path}_k}(\mathbf{f}^{\cup \Xi}))$ (by Corollary 7.18),
- $\text{size}(\mathbf{f}^{\cup \Xi}) \geq \frac{n^{\Omega(d(k^{1/d}-1))}}{2^{k^{d+1}} \cdot (dn^{1/k} + 1)^k \cdot 2n^2}$ (by Lemma 7.21).

Since $\text{size}(F) = \text{size}(\mathbf{f}) \geq \text{size}(\mathbf{f}^{\cup \Xi})$, it follows that

$$\text{size}(F) \geq \frac{n^{\Omega(d(k^{1/d}-1))}}{2^{k^{d+1}} \cdot (dn^{1/k} + 1)^k \cdot 2n^2}.$$

Since $k \leq \log \log n$ and $d \leq \log k$, we have $2^{k^{d+1}} \leq n^{o(1)}$ and $(dn^{1/k} + 1)^k = n^{1+o(1)}$, so the denominator above is $n^{O(1)}$, whereas the numerator is at least $n^{\Omega(\log k)}$. We conclude that F has size $n^{\Omega(d(k^{1/d}-1))}$ as required. \square

Our proof of tradeoff (I)⁺ shows that we can slightly extend the range of k from $\log \log n$ to $2\sqrt{\log \log n}$ and \wedge -fan-in from $n^{1/k}$ to $n^{(\log k)/k}$. While the range of k can probably be pushed further, the proof breaks down for larger \wedge -fan-in $D = n^{\omega((\log k)/k)}$; this is ultimately due to the D^k factor in the bound of Lemma 7.17. Also notice that our $n^{\Omega(dk^{1/d})}$ lower bound for monotone SAC^0 formulas of \wedge -depth d cannot possibly extend to unbounded fan-in AC^0 formulas of \wedge -depth d , in light of the $n^{O(dk^{1/(2d+1)})}$ upper bound given by the monotone Σ_{2d+1} formulas of Proposition 1.3(II).

7.6 Strict support tree of a k -variable DeMorgan formula

We next define notions of *strictness* and $\llbracket \cdot \rrbracket$ -*depth* for DeMorgan formulas, which are analogous to the corresponding notions for join trees. Even though we only apply these notions to k -variable DeMorgan formulas, we state the definitions more generally without reference to k .

Definition 7.26 (Strict DeMorgan formulas and operations $\text{strict}(g)$ and $\text{strict}(T)$).

- A DeMorgan formula g is *strict* if either
 - it has depth 0 (i.e., g is a constant or literal), or
 - it is $g_1 \wedge g_2$ or $g_1 \vee g_2$ where g_1, g_2 are strict and compute distinct functions from g (i.e., $g_1 \not\equiv g$ and $g_2 \not\equiv g$).
- For every DeMorgan formula g , we define an equivalent strict DeMorgan formula $\text{strict}(g)$ inductively as follows:
 - if g has depth 0, then $\text{strict}(g) := g$,
 - if g is $g_1 \wedge g_2$ (resp. $g_1 \vee g_2$), then

$$\text{strict}(g) := \begin{cases} \text{strict}(g_1) & \text{if } g_1 \equiv g, \\ \text{strict}(g_2) & \text{if } g_1 \not\equiv g \text{ and } g_2 \equiv g, \\ \text{strict}(g_1) \wedge \text{strict}(g_2) \text{ (resp. } \text{strict}(g_1) \vee \text{strict}(g_2)) & \text{otherwise.} \end{cases}$$

(Note that g is strict if, and only if, $g = \text{strict}(g)$. On the other hand, $g \equiv \text{strict}(g)$ for all g .)

- For every G -join tree T , we define a strict G -join tree $\text{strict}(T)$ in a similar fashion:
 - if T has $\langle \cdot \rangle$ -depth 0, then $\text{strict}(T) := T$,
 - if $T = \langle T_1, T_2 \rangle$ where T_1, T_2 are G_1, G_2 -join trees, then

$$\text{strict}(T) := \begin{cases} \text{strict}(T_1) & \text{if } G_1 = G, \\ \text{strict}(T_2) & \text{if } G_1 \neq G \text{ and } G_2 = G, \\ \langle \text{strict}(T_1), \text{strict}(T_2) \rangle & \text{otherwise.} \end{cases}$$

To help the reader keep track of the types of objects, we use typewriter font for the strict operator on DeMorgan formulas and italics for the *strict* operator on join trees. (Later on in Definition 7.29, we use sans-serif $\text{Supp}(g)$ for a subgraph of Path_k and italic $S(g)$ for a join tree associated with g .)

Definition 7.27 ($\llbracket \cdot \rrbracket$ -depth of DeMorgan formulas).

- For any DeMorgan formulas g_1, \dots, g_m , we define DeMorgan formulas $\llbracket g_1, \dots, g_m \rrbracket_\wedge$ and $\llbracket g_1, \dots, g_m \rrbracket_\vee$ inductively by

$$\begin{aligned} \llbracket g_1 \rrbracket_\wedge &= \llbracket g_1 \rrbracket_\vee := g_1, & \text{for } m = 1, \\ \llbracket g_1, \dots, g_m \rrbracket_\wedge &:= \llbracket g_1, \dots, g_{m-1} \rrbracket_\wedge \wedge \llbracket g_1, \dots, g_{m-2}, g_m \rrbracket_\wedge & \text{for } m \geq 2, \\ \llbracket g_1, \dots, g_m \rrbracket_\vee &:= \llbracket g_1, \dots, g_{m-1} \rrbracket_\vee \vee \llbracket g_1, \dots, g_{m-2}, g_m \rrbracket_\vee & \text{for } m \geq 2. \end{aligned}$$

- The $\llbracket \cdot \rrbracket$ -*depth* of a DeMorgan formula g is defined inductively as follows:
 - if g has depth 0, then $\text{depth}_{\llbracket \cdot \rrbracket}(g) := 0$,
 - if g has depth ≥ 1 , then

$$\text{depth}_{\llbracket \cdot \rrbracket}(g) := 1 + \min_{\substack{m \geq 2, g_1, \dots, g_m : \\ g = \llbracket g_1, \dots, g_m \rrbracket_\wedge \text{ or } \llbracket g_1, \dots, g_m \rrbracket_\vee}} \max_{i \in [m]} \text{depth}_{\llbracket \cdot \rrbracket}(g_i).$$

We now restrict attention to k -variable DeMorgan formulas, whose variables we identify with edges of Path_k and whose inputs in $\{0, 1\}^k$ we identify with subgraphs of Path_k .

Lemma 7.28. *There are at most $2^{2^{(d+1)(k+1)}}$ distinct strict k -variable DeMorgan formulas of $\llbracket \rrbracket$ -depth at most d .*

Proof. Consider any strict k -variable DeMorgan formula g of $\llbracket \rrbracket$ -depth d . Without loss of generality,

$$g = \llbracket g_1, \dots, g_m \rrbracket \wedge$$

for some g_1, \dots, g_m of $\llbracket \rrbracket$ -depth at most $d - 1$. Note that g_1, \dots, g_m are necessarily also strict.

Let $g_1, \dots, g_m : \{0, 1\}^k \rightarrow \{0, 1\}$ be the functions computed by g_1, \dots, g_m . Strictness of g implies that

$$g_1, \quad g_1 \wedge g_2, \quad g_1 \wedge g_2 \wedge g_3, \quad \dots, \quad g_1 \wedge \dots \wedge g_m$$

is a strictly decreasing sequence of Boolean functions. Moreover, $g_1 \wedge \dots \wedge g_m \neq 0$ assuming $m \geq 1$. It follows that $m \leq 2^k$. (Whereas there are 2^{2^k} distinct Boolean functions $\{0, 1\}^k \rightarrow \{0, 1\}$, the longest decreasing sequence of non-zero functions has length 2^k .)

We conclude that

$$\begin{aligned} & |\{\text{strict } k\text{-variable DeMorgan formulas of } \llbracket \rrbracket\text{-depth at most } d\}| \\ & \leq 2 + 2k + 2 |\{\text{strict } k\text{-variable DeMorgan formulas of } \llbracket \rrbracket\text{-depth at most } d - 1\}|^{2^k}. \end{aligned}$$

(Here the additive $2 + 2k$ counts the depth-0 formulas, and the other factor 2 arises from the choice of top gate type.) The bound of the lemma follows since $2 + 2k + 2^{2^{d(k+1)}} \leq 2^{2^{(d+1)(k+1)}}$ for all $d, k \geq 0$. \square

The next definition associates a join tree $S(g)$ with each k -variable DeMorgan formula g .

Definition 7.29 (Support, restriction, and support tree).

- The *support* of a Boolean function $g : \{0, 1\}^k \rightarrow \{0, 1\}$ is the set $\text{Supp}(g) \subseteq [k]$ of coordinates on which g depends, that is,

$$\text{Supp}(g) := \{i \in [k] : \exists y \in \{0, 1\}^k, g(y) \neq g(y_1, \dots, y_{i-1}, 1 - y_i, y_{i+1}, \dots, y_k)\}.$$

(We identify $\text{Supp}(g)$ with the corresponding subgraph of Path_k .)

The *support* of a k -variable DeMorgan formula g , denoted $\text{Supp}(g)$, is the support of the Boolean function it computes.

- For a k -variable DeMorgan formula g and a subgraph $H \subseteq \text{Path}_k$, let $g \upharpoonright H$ denote the DeMorgan formula obtained from g by syntactically relabeling to 0 (resp. 1) each leaf labeled by a positive (resp. negative) literal whose coordinate lies in $E(\text{Path}_k) \setminus E(H)$.

(Observe that $\text{Supp}(g \upharpoonright H) \subseteq H$ and $g \equiv g \upharpoonright \text{Supp}(g)$, even though g and $g \upharpoonright \text{Supp}(g)$ are not necessarily equal.)

- For every k -variable DeMorgan formula g , we define a join tree $S(g)$, called the *support tree* of g , inductively as follows:
 - if g is a constant (i.e., 0 or 1), then $S(g)$ is the empty join tree;
 - if g is the i th positive or negative literal, then $S(g)$ is the single-node join tree labeled by E_i (i.e., the i th single-edge subgraph of Path_k);
 - if g is $g_1 \wedge g_2$ or $g_1 \vee g_2$, then $S(g) := S(g_1 \upharpoonright \text{Supp}(g)) \cup S(g_2 \upharpoonright \text{Supp}(g))$.

Note that $S(g)$ is a $\text{Supp}(g)$ -join tree. (Had we instead defined $S(g) := S(g_1) \cup S(g_2)$, then $S(g)$ would be a G -join tree where G is the supergraph of $\text{Supp}(g)$ that contain the edges of Path_k corresponding to all literals that appear in g .)

- We refer to $\text{strict}(S(\mathbf{g}))$ as the *strict support tree of \mathbf{g}* . Note that $\text{strict}(S(\mathbf{g}))$ is a strict $\text{Supp}(\mathbf{g})$ -join tree (i.e., an element of $\mathcal{T}_{\text{Supp}(\mathbf{g})}$).

The next lemma follows directly from definitions.

Lemma 7.30. *For every k -variable DeMorgan formula \mathbf{g} , the $\llbracket \cdot \rrbracket$ -depth of the join tree $S(\mathbf{g})$ (and hence also $\text{strict}(S(\mathbf{g}))$) is at most the $\llbracket \cdot \rrbracket$ -depth of \mathbf{g} . \square*

7.7 The distribution of $\text{strict}(\mathcal{D}_t(\mathbf{G}))$ for a k -variable AC^0 formula \mathbf{G}

In this subsection, we prove a key result (Lemma 7.33) on the distribution of $\text{strict}(\mathbf{g})$ for the randomized balanced DeMorgan conversion $\mathbf{g} \sim \mathcal{D}_t(\mathbf{G})$ (Def. 7.4) of a k -variable AC^0 formula \mathbf{G} . We begin with a preliminary lemma on the distribution of $\text{strict}(\mathbf{T}_t)$ for a certain family of random join trees \mathbf{T}_t .

Lemma 7.31. *Fix any real numbers $p_1 \geq \dots \geq p_k > 0$ with $p_1 + \dots + p_k = 1$. For each $t \in \mathbb{N}$, let \mathbf{T}_t be a random join tree of $\langle \cdot \rangle$ -depth t with 2^t leaves independently labeled by single-edge graphs E_1, \dots, E_k with probability p_1, \dots, p_k respectively. Then for all $t \geq \log(\frac{1}{p_k}) + k$, we have*

$$\mathbb{P} \left[\text{strict}(\mathbf{T}_t) = \llbracket E_1, \dots, E_k \rrbracket \right] \geq \exp(-O(2^k)).$$

A key point in Lemma 7.31 is that the bound $\exp(-O(2^k))$ is independent of t and the distribution p_1, \dots, p_k .

Proof. First, consider the case of the uniform distribution $p_1 = \dots = p_k = \frac{1}{k}$. Here we have

$$\mathbb{P} \left[\text{strict}(\mathbf{T}_k) = \llbracket E_1, \dots, E_k \rrbracket \right] = \mathbb{P} \left[\mathbf{T}_k = \llbracket E_1, \dots, E_k \rrbracket \right] = k^{-k}.$$

Note that for any $t \geq k$, we have

$$\mathbb{P} \left[\text{strict}(\mathbf{T}_t) = \llbracket E_1, \dots, E_k \rrbracket \right] \geq \mathbb{P} \left[\text{the leftmost depth-}k \text{ sub-join tree of } \mathbf{T}_t \text{ equals } \llbracket E_1, \dots, E_k \rrbracket \right] = k^{-k}.$$

To illustrate the general argument, consider the following (essentially geometric) distribution with $k = 4$:

$$p_1 \approx 0.99, \quad p_2 = 2^{-10}, \quad p_3 = 2^{-100}, \quad p_4 = 2^{-1000}.$$

First, note that

$$\mathbb{P} \left[\mathbf{T}_{1000} \text{ contains a single } E_4 \right] = \mathbb{P} \left[\text{Binomial}(2^{1000}, 2^{-1000}) = 1 \right] \approx \frac{1}{e}.$$

Letting $\mathbf{T}_{999}^{(L)}$ and $\mathbf{T}_{999}^{(R)}$ be the left and right children of the root of \mathbf{T}_{1000} , we next observe that

$$\mathbb{P} \left[\underbrace{\mathbf{T}_{999}^{(L)} \text{ contains no } E_4 \text{ and } \mathbf{T}_{999}^{(R)} \text{ contains a single } E_4}_{\text{call this event } \mathcal{E}} \right] \approx \frac{1}{e^2}.$$

Conditioned on event \mathcal{E} , let $\mathbf{T}_{100}^{(L)}$ be the leftmost depth- k sub-join tree of $\mathbf{T}_{999}^{(L)}$, and let $\mathbf{T}_{100}^{(R)}$ be the unique depth- k sub-join tree of $\mathbf{T}_{999}^{(R)}$ that contains E_4 . Now observe that a sufficient condition for $\text{strict}(\mathbf{T}_{1000}) = \llbracket E_1, E_2, E_3, E_4 \rrbracket$ is that

- event \mathcal{E} holds,
- $\text{strict}(\mathbf{T}_{100}^{(L)}) = \llbracket E_1, E_2, E_3 \rrbracket$, and
- $\text{strict}(\mathbf{T}_{100}^{(R)}) = \llbracket E_1, E_2, E_4 \rrbracket$.

(Given \mathcal{E} , condition (b) implies that $\text{strict}(\mathbf{T}_{999}^{(L)}) = \llbracket E_1, E_2, E_3 \rrbracket$ and (c) implies that $\text{strict}(\mathbf{T}_{999}^{(R)}) = \llbracket E_1, E_2, E_4 \rrbracket$. Conditions (a),(b),(c) therefore together imply that $\text{strict}(\mathbf{T}_{1000}) = \llbracket E_1, E_2, E_3, E_4 \rrbracket$.) We now have

$$\begin{aligned} \mathbb{P} \left[\text{strict}(\mathbf{T}_{1000}) = \llbracket E_1, E_2, E_3, E_4 \rrbracket \right] \\ \geq \mathbb{P} \left[\mathcal{E} \text{ and } \text{strict}(\mathbf{T}_{100}^{(L)}) = \llbracket E_1, E_2, E_3 \rrbracket \text{ and } \text{strict}(\mathbf{T}_{100}^{(R)}) = \llbracket E_1, E_2, E_4 \rrbracket \right] \\ \approx \frac{1}{e^2} \cdot \mathbb{P} \left[\text{strict}(\mathbf{T}_{100}^{(L)}) = \llbracket E_1, E_2, E_3 \rrbracket \mid \mathcal{E} \right] \cdot \mathbb{P} \left[\text{strict}(\mathbf{T}_{100}^{(R)}) = \llbracket E_1, E_2, E_4 \rrbracket \mid \mathcal{E} \right]. \end{aligned}$$

Splitting $\mathbf{T}_{100}^{(L)}$ (resp. $\mathbf{T}_{100}^{(R)}$) into left and right subtrees $\mathbf{T}_{99}^{(LL)}$ and $\mathbf{T}_{99}^{(LR)}$ (resp. $\mathbf{T}_{99}^{(RL)}$ and $\mathbf{T}_{99}^{(RR)}$), we continue this analysis:

$$\begin{aligned} \mathbb{P} \left[\underbrace{\mathbf{T}_{99}^{(LL)} \text{ contains no } E_3 \text{ and } \mathbf{T}_{99}^{(LR)} \text{ contains a single } E_3}_{\text{call this event } \mathcal{E}^{(L)}} \mid \mathcal{E} \right] &\approx \frac{1}{e^2}, \\ \mathbb{P} \left[\underbrace{\mathbf{T}_{99}^{(RL)} \text{ contains no } E_4 \text{ and } \mathbf{T}_{99}^{(RR)} \text{ contains a single } E_4}_{\text{call this event } \mathcal{E}^{(R)}} \mid \mathcal{E} \right] &\approx \frac{1}{e^2}. \end{aligned}$$

Next, for appropriate definitions of sub-join trees $\mathbf{T}_{10}^{(LL)}, \mathbf{T}_{10}^{(LR)}$ of $\mathbf{T}_{99}^{(L)}$ and $\mathbf{T}_{10}^{(RL)}, \mathbf{T}_{10}^{(RR)}$ of $\mathbf{T}_{99}^{(R)}$, we have

$$\begin{aligned} \mathbb{P} \left[\text{strict}(\mathbf{T}_{100}^{(L)}) = \llbracket E_1, E_2, E_3 \rrbracket \mid \mathcal{E} \right] \\ \approx \frac{1}{e^2} \cdot \mathbb{P} \left[\text{strict}(\mathbf{T}_{10}^{(LL)}) = \llbracket E_1, E_2 \rrbracket \mid \mathcal{E} \wedge \mathcal{E}^{(L)} \right] \cdot \mathbb{P} \left[\text{strict}(\mathbf{T}_{10}^{(LR)}) = \llbracket E_1, E_3 \rrbracket \mid \mathcal{E} \wedge \mathcal{E}^{(L)} \right], \\ \mathbb{P} \left[\text{strict}(\mathbf{T}_{100}^{(R)}) = \llbracket E_1, E_2, E_4 \rrbracket \mid \mathcal{E} \right] \\ \approx \frac{1}{e^2} \cdot \mathbb{P} \left[\text{strict}(\mathbf{T}_{10}^{(RL)}) = \llbracket E_1, E_2 \rrbracket \mid \mathcal{E} \wedge \mathcal{E}^{(R)} \right] \cdot \mathbb{P} \left[\text{strict}(\mathbf{T}_{10}^{(RR)}) = \llbracket E_1, E_4 \rrbracket \mid \mathcal{E} \wedge \mathcal{E}^{(R)} \right]. \end{aligned}$$

This analysis continues down one more layer. In the end we get a bound

$$\mathbb{P} \left[\text{strict}(\mathbf{T}_{1000}) = \llbracket E_1, E_2, E_3, E_4 \rrbracket \right] \gtrsim \frac{1}{e^2} \left(\frac{1}{e^2} \left(\frac{1}{e^2} \right) \left(\frac{1}{e^2} \right) \right) \left(\frac{1}{e^2} \left(\frac{1}{e^2} \right) \left(\frac{1}{e^2} \right) \right) = \frac{1}{e^{14}}.$$

Generalizing this example to larger k , we get a lower bound

$$\mathbb{P} \left[\text{strict}(\mathbf{T}_{10^k}) = \llbracket E_1, \dots, E_k \rrbracket \right] \gtrsim \exp(-2(2^k - 1)),$$

which is essentially the worst case for the lemma.

To generalize this argument for an arbitrary distribution $p_1 \geq \dots \geq p_k > 0$, in place of depths 1, 10, 100, 1000 in the example above, we consider the sequence $1 = t_1 < t_2 < \dots < t_k$ defined by

$$t_i := \left\lceil \log \frac{1}{p_i + \dots + p_k} \right\rceil + i.$$

The same analysis yield the desired $\exp(-O(2^k))$ lower bound. \square

Simply rephrasing Lemma 7.31 in terms of depth-1 AC^0 formulas, we get the following:

Corollary 7.32. *For every k -variable Π_1 or Σ_1 formula G (i.e., an m -ary \wedge or \vee of literals, where m may be arbitrarily large relative to k), there exists an equivalent strict DeMorgan formula G^* of $\llbracket \rrbracket$ -depth at most 1 such that for all $t \geq 2^k$, we have*

$$\mathbb{P}_{\mathbf{g} \sim \mathcal{D}_t(G)} \left[\text{strict}(\mathbf{g}) = G^* \right] \geq \exp(-O(2^k)).$$

Proof. Without loss of generality, it suffices to consider the case that G is a monotone Π_1 formula of size $m = m_1 + \dots + m_k$ with m_i leaves labeled by the positive literal X_i , where $m_1 \geq \dots \geq m_k \geq 1$. Further assume w.l.o.g. that m is a power of 2. Letting $p_i := \frac{m_i}{m}$ for $i \in [k]$, note that $\mathbf{g} \sim \mathcal{D}_t(G)$ is identical (after renaming \wedge to \cup and X_i to E_i) to the random join tree $T_{\log(tm)}$ of Lemma 7.31. Since $\log(tm) \geq k + \log m \geq k + \log \frac{1}{p_k}$, Lemma 7.31 implies

$$\mathbb{P}_{\mathbf{g} \sim \mathcal{D}_t(G)} [\text{strict}(\mathbf{g}) = \llbracket X_1, \dots, X_k \rrbracket_{\wedge}] \geq \exp(-O(2^k)).$$

The desired statement is proved by letting $G^* := \llbracket X_1, \dots, X_k \rrbracket_{\wedge}$, which is clearly strict, equivalent to G , and has $\llbracket \cdot \rrbracket$ -depth 1. \square

The next lemma extends Corollary 7.32 to AC^0 formulas G beyond depth 1. For the analysis to work out, we consider the distribution $\mathcal{D}_t(G)$ for larger t , which allows us to condition on the event that $\mathbf{g}_0 \sim \mathcal{D}_t(G_0)$ is equivalent to G_0 for every sub-formula G_0 of G .

Lemma 7.33. *For every k -variable AC^0 formula G of depth d , there exists an equivalent strict DeMorgan formula G^* of $\llbracket \cdot \rrbracket$ -depth at most d such that for all $t \geq 2^{2^{2^k}} \cdot (\log \text{size}(G))^3$, we have*

$$\mathbb{P}_{\mathbf{g} \sim \mathcal{D}_t(G)} [\text{strict}(\mathbf{g}) = G^*] \geq 2^{-\exp(O(d2^{2^k}))}.$$

Proof. The case $d = 0$ is trivial and Corollary 7.32 proves the case $d = 1$.

For the induction step when $d \geq 2$, assume that $G = \bigwedge_{i=1}^m G_i$. (The argument for $G = \bigvee_{i=1}^m G_i$ is identical after swapping \vee for \wedge .) By the induction hypothesis, there exist k -variable DeMorgan formulas G_1^*, \dots, G_m^* which satisfy the lemma with respect to subformulas G_1, \dots, G_m .

Let $h_1, \dots, h_s : \{0, 1\}^k \rightarrow \{0, 1\}$ enumerate the distinct functions computed by sub-formulas G_1, \dots, G_m , ordered such that $m_1 \geq \dots \geq m_s$ where

$$m_j := |\{i \in [m] : G_i \text{ computes } h_j\}|.$$

Next, for each $j \in [s]$, consider the set

$$\{G_i^* : i \in [m] \text{ such that } G_i \text{ computes } h_j\}.$$

Each element of this set is a strict k -variable DeMorgan formula of $\llbracket \cdot \rrbracket$ -depth at most $d-1$, hence the size of this set is at most $2^{2^{d(k+1)}}$ by Lemma 7.28. It follows that there exist strict k -variable DeMorgan formulas $\mathbf{h}_1, \dots, \mathbf{h}_s$ of $\llbracket \cdot \rrbracket$ -depth at most $d-1$ such that for all $j \in [s]$,

$$|\{i \in [m] : G_i^* = \mathbf{h}_j\}| \geq \frac{m_j}{2^{2^{d(k+1)}}}.$$

Let $J = \{j_1 < \dots < j_r\} \subseteq [s]$ be the subset of indices defined by

$$J := \{j \in [s] : h_1 \wedge \dots \wedge h_{j-1} \not\equiv h_1 \wedge \dots \wedge h_j\}.$$

Note that whereas $s \leq 2^{2^k}$, we have $r \leq 2^k$ since this is the maximum length of a strictly decreasing sequence of nonzero functions $\{0, 1\}^k \rightarrow \{0, 1\}$ (assuming w.l.o.g. that $G \not\equiv 0$).

At last, we choose G^* to be the formula $\llbracket \mathbf{h}_{j_1}, \dots, \mathbf{h}_{j_r} \rrbracket_{\wedge}$. Observe that G^* is equivalent to G and has $\llbracket \cdot \rrbracket$ -depth at most d . Moreover, it is strict and satisfies $\text{strict}(\llbracket \mathbf{h}_1, \dots, \mathbf{h}_s \rrbracket_{\wedge}) = G^*$.

Onto the probabilistic analysis. We generate $\mathbf{g} \sim \mathcal{D}_t(G)$ a balanced tree of binary \wedge -gates with subformulas $\mathbf{g}_{\iota_1}, \dots, \mathbf{g}_{\iota_{tm}}$ feeding in for independent uniform random $\iota_1, \dots, \iota_{tm} \in [m]$ and $\mathbf{g}_i \sim \mathcal{D}_t(G_i)$ ($i \in [m]$). Note that our choice of t ensures that $\mathbf{g}_i \equiv G_i$ for all $i \in [m]$ with probability $1 - o(1)$. Let us henceforth automatically condition on this event in all instances of $\mathbb{P}[\cdot]$ below.

By the induction hypothesis and our choice of formulas h_1, \dots, h_s , for all indices $j \in [s]$ and $q \in [tm]$, we have

$$\begin{aligned} \mathbb{P} \left[\text{strict}(\mathbf{g}_{\iota_q}) = h_j \mid \mathbf{G}_{\iota_q} \text{ computes } h_j \right] &\geq 2^{-2^{d(k+1)}} \cdot \mathbb{P} \left[\text{strict}(\mathbf{g}_{\iota_q}) = G_{\iota_q}^* \right] \\ &\geq 2^{-2^{d(k+1)}} \cdot 2^{-\exp(O((d-1)2^{2^k}))}. \end{aligned}$$

Now consider the top $\log(tm)$ layers of \mathbf{g} , which we view as a random tree of binary \wedge gates with inputs labeled by the functions computed at subformulas $\mathbf{g}_{\iota_1}, \dots, \mathbf{g}_{\iota_{tm}}$ (elements of the set $\{h_1, \dots, h_s\}$). Call this tree $T(\mathbf{g})$. Viewing $T(\mathbf{g})$ as a join tree over single-edge graphs E_1, \dots, E_s (or, alternatively, viewing $T(\mathbf{g})$ as a DeMorgan formula over literals corresponding to functions h_1, \dots, h_s), Lemma 7.31 (or from a different perspective Corollary 7.32) tells us

$$\mathbb{P} \left[\underbrace{\text{strict}(T(\mathbf{g})) = \llbracket h_1, \dots, h_s \rrbracket_{\wedge}}_{\text{call this event } \mathcal{E}} \right] \geq 2^{-c2^s}$$

for some absolute constant c determined by Lemma 7.31.

Conditioned on event \mathcal{E} , there exist 2^s indices — name them $1 \leq q_1 < \dots < q_{2^s} < tm$ — which embed the leaves of $\llbracket h_1, \dots, h_s \rrbracket_{\wedge}$ among the leaves of $T(\mathbf{g})$. By definition of this sequence, formulas $\mathbf{g}_{\iota_{q_1}}, \dots, \mathbf{g}_{\iota_{q_{2^s}}}$ compute functions $h_1, h_2, h_1, h_3, \dots, h_1, h_2, h_1, h_s$. Now observe that

$$\begin{aligned} &\mathbb{P} \left[\text{strict}(\mathbf{g}) = G^* \mid \mathcal{E} \right] \\ &= \mathbb{P} \left[\text{strict}(\mathbf{g}) = \text{strict}(\llbracket h_1, \dots, h_s \rrbracket_{\wedge}) \mid \mathcal{E} \right] \\ &\geq \mathbb{P} \left[(\text{strict}(\mathbf{g}_{\iota_{q_1}}), \dots, \text{strict}(\mathbf{g}_{\iota_{q_{2^s}}})) = (h_1, h_2, h_1, h_3, \dots, h_1, h_2, h_1, h_s) \mid \mathcal{E} \right] \\ &= \mathbb{P} \left[\text{strict}(\mathbf{g}_{\iota_{q_1}}) = h_1 \mid \mathbf{G}_{\iota_{q_1}} \text{ computes } h_1 \right] \cdot \mathbb{P} \left[\text{strict}(\mathbf{g}_{\iota_{q_2}}) = h_2 \mid \mathbf{G}_{\iota_{q_2}} \text{ computes } h_2 \right] \cdot \dots \\ &\quad \dots \cdot \mathbb{P} \left[\text{strict}(\mathbf{g}_{\iota_{q_{2^s-1}}}) = h_1 \mid \mathbf{G}_{\iota_{q_{2^s-1}}} \text{ computes } h_1 \right] \cdot \mathbb{P} \left[\text{strict}(\mathbf{g}_{\iota_{q_{2^s}}}) = h_s \mid \mathbf{G}_{\iota_{q_{2^s}}} \text{ computes } h_s \right] \\ &\geq \left(2^{-2^{d(k+1)}} \cdot 2^{-\exp(O((d-1)2^{2^k}))} \right)^{2^s}. \end{aligned}$$

Finally, we obtain the desired bound

$$\begin{aligned} \mathbb{P} \left[\text{strict}(\mathbf{g}) = G^* \right] &\geq \mathbb{P} \left[\mathcal{E} \right] \cdot \mathbb{P} \left[\text{strict}(\mathbf{g}) = G^* \mid \mathcal{E} \right] \geq 2^{-c2^s} \cdot \left(2^{-2^{d(k+1)}} \cdot 2^{-\exp(O((d-1)2^{2^k}))} \right)^{2^s} \\ &\geq \left(2^{-c} \cdot 2^{-2^{d(k+1)}} \cdot 2^{-\exp(O((d-1)2^{2^k}))} \right)^{2^{2^k}} \\ &\geq 2^{-\exp(O(d2^{2^k}))} \end{aligned}$$

for an appropriate big- O constant depending on the constant c . □

7.8 Tradeoff (II)⁻ for non-monotone AC^0 formulas

We are ready to prove size-depth tradeoff (II)⁻ for non-monotone AC^0 formulas. The proof has an overall similar structure to the proof of tradeoff (I)⁺ in §7.5.

Theorem 7.34 (Tradeoff (II)⁻ of Theorem 1.6). *Suppose that F is a depth- d AC^0 formula which computes $\text{SUB-PMM}_{n,k}$ where $k \leq \log^* n$ and $d \leq \log k$. Then F has size $n^{\Omega(dk^{1/2^d})}$.*

By a simple additional argument, the same $n^{\Omega(dk^{1/2^d})}$ lower bound may actually be obtained for AC^0 formulas of depth $d+1$ (precisely matching the statement of tradeoff (II)⁻ in Theorem 1.6). We present the lower bound for depth- d AC^0 formulas for simplicity sake, since the factor 2 in the second exponent is anyway probably not optimal.

Proof. Without loss of generality, assume that F has size at most n^k , since this is greater than the lower bound we wish to show. Let

$$t := 2^{2^{2^k}} \cdot (\log n)^4$$

and consider the randomized balanced DeMorgan conversion $f \sim \mathcal{D}_t(F)$. Lemma 7.24 implies that, with probability $1 - o(1)$, the G -relations $\mathcal{N}_G(\mathbf{f}_0^{\cup \Xi})$ are pathsets for all subformulas \mathbf{f}_0 of f and subgraphs $G \subseteq \text{Path}_k$. We proceed with our analysis conditioned on this almost sure event.

For each \mathbf{f}_0 and G , we define a covering $\bigcup_{T \in \mathcal{T}_G} \mathcal{N}_G^{(T)}(\mathbf{f}_0^{\cup \Xi}) = \mathcal{N}_G(\mathbf{f}_0^{\cup \Xi})$ by essentially Definition 7.15, except that in the induction step where $T = \langle T_1, T_2 \rangle$, we are now forced to treat the case $\mathbf{f}_0 = \mathbf{f}_1 \vee \mathbf{f}_2$ the same as $\mathbf{f}_0 = \mathbf{f}_1 \wedge \mathbf{f}_2$, in both cases defining

$$\mathcal{N}_G^{(T)}(\mathbf{f}_0^{\cup \Xi}) := \mathcal{N}_G(\mathbf{f}_0^{\cup \Xi}) \cap \left(\mathcal{N}_G^{(T)}(\mathbf{f}_1^{\cup \Xi}) \cup \mathcal{N}_G^{(T)}(\mathbf{f}_2^{\cup \Xi}) \cup \left(\mathcal{N}_{G_1}^{(T_1)}(\mathbf{f}_1^{\cup \Xi}) \bowtie \mathcal{N}_{G_2}^{(T_2)}(\mathbf{f}_2^{\cup \Xi}) \right) \right).$$

By Lemma 7.21, $\mathcal{N}_{\text{Path}_k}(\mathbf{F}^{\cup \Xi})$ almost surely has density at least $\frac{1}{2n^2}$. For each $\alpha \in \mathcal{N}_{\text{Path}_k}(\mathbf{F}^{\cup \Xi})$, we consider the restricted k -variable depth- d AC^0 formula

$$G_\alpha := \mathbf{F}^{\cup \Xi} \upharpoonright \text{Path}_k^{(\alpha)}$$

obtained by F by fixing variables corresponding to Ξ to 1, and all other variables except the k edges of $\text{Path}_k^{(\alpha)}$ to 0.

We now invoke the key Lemma 7.33 to obtain, for each $\alpha \in \mathcal{N}_{\text{Path}_k}(\mathbf{F}^{\cup \Xi})$, a strict DeMorgan formula G_α^* of $\llbracket \rrbracket$ -depth at most d , which is equivalent to G_α (hence depends on all k variables) and satisfies

$$\mathbb{P}_{\mathbf{g}_\alpha \sim \mathcal{D}_t(G_\alpha)} \left[\text{strict}(\mathbf{g}_\alpha) = G_\alpha^* \right] \geq 2^{-\exp(O(d2^{2^k}))}.$$

Let T_α^* denote the strict support tree of G_α^* :

$$T_\alpha^* := \text{strict}(S(G_\alpha^*)).$$

Note that T_α^* is a strict Path_k -join tree since G_α ($= \mathbf{F}^{\cup \Xi} \upharpoonright \text{Path}_k^{(\alpha)}$) depends on all variables corresponding to edges of $\text{Path}_k^{(\alpha)}$. Moreover, T_α^* has $\llbracket \rrbracket$ -depth at most d (i.e., at most the $\llbracket \rrbracket$ -depth of G_α^*) by Lemma 7.30.

Next, we observe that the distribution of $\mathbf{g}_\alpha \sim \mathcal{D}_t(G_\alpha)$ — that is, $\mathbf{g}_\alpha \sim \mathcal{D}_t(\mathbf{F}^{\cup \Xi} \upharpoonright \text{Path}_k^{(\alpha)})$ — is *identical* to distribution of $\mathbf{f} \upharpoonright \text{Path}_k^{(\alpha)}$ where $\mathbf{f} \sim \mathcal{D}_t(\mathbf{F}^{\cup \Xi})$. Generating \mathbf{g}_α as $\mathbf{f} \upharpoonright \text{Path}_k^{(\alpha)}$, we have the following implication:

$$\text{strict}(\mathbf{g}_\alpha) = G_\alpha^* \implies \alpha \in \mathcal{N}_{\text{Path}_k}^{(T_\alpha^*)}(\mathbf{f}).$$

Since there are at most $2^{k^{d+1}}$ strict Path_k -join trees of $\llbracket \rrbracket$ -depth at most d (by Lemma 7.14), we may fix one such join tree T^* such that

$$\mu(\{\alpha \in \mathcal{N}_{\text{Path}_k}(\mathbf{F}^{\cup \Xi}) : T_\alpha^* = T^*\}) \geq \frac{1}{2^{k^{d+1}}} \cdot \mu(\mathcal{N}_{\text{Path}_k}(\mathbf{F}^{\cup \Xi})) \geq \frac{1}{2^{k^{d+1}} \cdot 2n^2}.$$

It follows that

$$\mu(\mathcal{N}_{\text{Path}_k}^{(T^*)}(\mathbf{F}^{\cup \Xi})) \geq \frac{1}{2^{\exp(O(d2^{2^k}))} \cdot 2^{k^{d+1}} \cdot 2n^2} = \frac{1}{n^{2+o(1)}}.$$

Recall that the random DeMorgan formula $\mathbf{f} \sim \mathcal{D}_t(F)$ has depth at most $d \cdot \lceil \log(t \cdot \text{size}(F)) \rceil$ and size at most $t^d \cdot \text{size}(F)$. Moreover, with probability $1 - o(1)$, functions $\mathbf{F}^{\cup \Xi}$ and $\mathbf{f}^{\cup \Xi}$ agree on all inputs of Hamming weight $\leq k$ (by a union bound), hence $\mathcal{N}_{\text{Path}_k}^{(T^*)}(\mathbf{F}^{\cup \Xi}) = \mathcal{N}_{\text{Path}_k}^{(T^*)}(\mathbf{f}^{\cup \Xi})$.

Since $\mathcal{N}_G(\mathbf{f}_0^{\cup \Xi})$ are pathsets for all \mathbf{f}_0 of \mathbf{f} and $G \subseteq \text{Path}_k$, Lemma 7.23 yields an upper bound on pathset complexity:

$$\begin{aligned}
\chi_{T^*}(\mathcal{N}_{\text{Path}_k}^{(T^*)}(\mathbf{F}^{\cup \Xi})) &= \chi_{T^*}(\mathcal{N}_{\text{Path}_k}^{(T^*)}(\mathbf{f}^{\cup \Xi})) \\
&\leq (\text{depth}(\mathbf{f}^{\cup \Xi}) + 1)^k \cdot \text{size}(\mathbf{f}^{\cup \Xi}) \\
&= O(d \cdot \log(t \cdot \text{size}(\mathbf{F})))^k \cdot t^d \cdot \text{size}(\mathbf{F}) \\
&\leq O(d \cdot \log(2^{2^{2^k}} \cdot (\log n)^4 \cdot n^k))^k \cdot 2^{d2^{2^k}} \cdot (\log n)^{4d} \cdot \text{size}(\mathbf{F}) \\
&= n^{o(1)} \cdot \text{size}(\mathbf{F}).
\end{aligned}$$

On the other hand, since T^* has $\llbracket \rrbracket$ -depth at most d , Corollary 6.12(II) provides the lower bound

$$\chi_{T^*}(\mathcal{N}_{\text{Path}_k}^{(T^*)}(\mathbf{F}^{\cup \Xi})) \geq n^{\Omega(dk^{1/2d})} \cdot \mu(\mathcal{N}_{\text{Path}_k}^{(T^*)}(\mathbf{F}^{\cup \Xi})) \geq \frac{n^{\Omega(dk^{1/2d})}}{n^{2+o(1)}}.$$

Combining these inequalities, we conclude that \mathbf{F} has size $n^{\Omega(dk^{1/2d})}$ as required. \square

Remark 7.35. This proof shows that the range $k \leq \log^* n$ can be extended to $k \leq 0.99 \log \log \log \log n$, small enough to ensure that $2^{\exp(O(d2^{2^k}))} = n^{o(1)}$. We have made little effort to optimize the range of k and expect it can be extended further, potentially to $\log \log n$.

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