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A high dimensional Cramer's rule connecting homogeneous multilinear equations to hyperdeterminants

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Abstract. We present a new algorithm for solving homogeneous multilinear equations, which are high dimensional generalisations of solving homogeneous linear equations. First, we present a linear time reduction from solving generic homogeneous multilinear equations to computing hyperdeterminants, via a high dimensional Cramer's rule. Hyperdeterminants are generalisations of determinants, associated with tensors of formats generalising square matrices. Second, we devise arithmetic circuits to compute hyperdeterminants of boundary format tensors. Boundary format tensors are those that generalise square matrices in the strictest sense. Consequently, we obtain arithmetic circuits for solving generic homogeneous boundary format multilinear equations. The complexity as a function of the input dimension varies across boundary format families, ranging from quasi-polynomial to sub exponential. Curiously, the quasi-polynomial complexity arises for families of increasing dimension, including the family of multipartite quantum systems made of d qubits and one qudit.

1. Introduction

Homogeneous multilinear systems. The familiar problem of solving homogeneous linear equations is to take a square matrix A and find a non zero vector x such that Ax is the zero vector. We devise algorithms for the natural high dimensional generalisation, which we call solving homogeneous multilinear equations. Let us rephrase what it means to solve homogeneous linear equations, to emphasise the motif that generalises. Given a square matrix A, find a pair of non zero vectors $\left(x^{(0)}, x^{(1)}\right)$ such that removing one of the vectors from the bilinear product $x^{(0)}Ax^{(1)}$ equals the zero vector. The solutions are merely pairs of non zero vectors $\left(x^{(0)}, x^{(1)}\right)$ with $x^{(0)}$ in the left kernel of A and $x^{(1)}$ in the right kernel of A. In homogeneous multilinear equations, a tensor A of dimension $r+1$ will be cast as the input in place of the square matrix. Multiplying A by the vectors $\left(x^{(0)}, x^{(1)}, \ldots, x^{(r)}\right)$ in the corresponding dimensions is a multilinear map taking this tuple of vectors to a scalar. If we remove one of the vectors from the multiplication, the result is a vector. The solution we demand is a tuple of vectors $\left(x^{(0)}, x^{(1)}, \ldots, x^{(r)}\right)$ such that removing one of the vectors from the multilinear product gives the zero vector, irrespective

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of which one was removed. See definition 3.1 for the formal statement. We address the problem over the field of complex numbers and describe algorithms in the formalism of arithmetic circuits. Key to our approach is the hyperdeterminant.

Hyperdeterminants. The determinant of a square matrix is a homogeneous integer polynomial in the matrix entries that vanishes precisely when the matrix is singular. The hyperdeterminant is a high dimensional analogue of the determinant conceived by Cayley [3]. The hyperdeterminant is a homogeneous integer polynomial in the coordinates of the tensor that vanishes precisely when the tensor is singular. This notion of singularity is geometric and defined through projective duality. Except for a foray by Schläfli, the subject remained dormant for nearly a century and a half [12]. It was revived in the comprehensive work of Gelfand, Kapranov and Zelevinsky [6, 7], which contains most of the mathematical ingredients required in this paper.

Tensor formats. Just as the determinant is only defined for square matrices, the hyperdeterminant is only defined for certain formats of tensors. An $(r+1)$-dimensional tensor product of complex vector spaces of dimensions $k_0+1, k_1+1, \ldots, k_r+1$ constitutes a $(k_0+1)\times(k_1+1)\times\ldots\times(k_r+1)$ format. Say $k_0 \geq k_1 \geq \ldots \geq k_r$. The hyperdeterminant is defined for formats where the largest vector space dimension k_0 satisfies the convexity constraint $k_0 \leq k_1+k_2+\ldots+k_r$. Such formats generalise square matrices. Boundary formats are those satisfying the convexity constraint with equality, that is, $k_0 = k_1+k_2+\ldots+k_r$. The special case $r=1$ corresponds to square matrices. Boundary formats generalise square matrices to higher dimensions in the strictest sense. To quote Gelfand, Kapranov and Zelevinsky [6], *"It is instructive to think of matrices of boundary format as proper high dimensional analogs of ordinary square matrices"*. Formats satisfying $k_0 < k_1+k_2+\ldots+k_r$ are called interior formats.

Hardness of Hyperdeterminants. We use hyperdeterminants as a means to solve homogeneous multilinear equations, but they are of intrinsic interest in complexity theory. The computational complexity of hyperdeterminants remains a mystery, either restricted to three dimensions or in general. Unlike the determinant (the two dimensional case), computing the hyperdeterminant (in three or more dimensions) is believed to be VNP-hard, yet a proof remains elusive. Testing if a given tensor is singular (has hyperdeterminant zero) is conjectured to be NP-hard [8]. Likewise, computing the hyperdeterminant is conjectured to be #P-hard in the counting model and VNP-hard in the arithmetic circuit model [8]. Several closely related three dimensional problems (such as zero testing singular values, defined for general formats in [9]) are proven to be NP, VNP or #P hard, but these instance are of formats where the hyperdeterminant is not defined. In particular, known hardness reductions to tensor problems seem to fall apart in formats satisfying the convexity constraint. Computing the combinatorial hyperdeterminant is known to be hard [8]. But the combinatorial hyperdeterminant more closely resembles the permanent and does not have the algebraic/geometric structure that underlies the hyperdeterminant. Another aspect to keep in mind is that the hyperdeterminant can have degree exponential in the size of the input, even in three dimensions. For instance, to write down a $(2n+1)\times(n+1)\times(n+1)$ boundary format tensor takes only cubic in n entries. However the degree of the hyperdeterminant is $(2n+1)!/n!^2 \approx 2^n$.

Hyperdeterminants and quantum information. Hyperdeterminants arise in quantum information when the amplitudes of quantum states are considered as normalised tensors in a projective space. The absolute value of the hyperdeterminant of three qubits ($r = 2, k_0 = k_1 = k_2 = 2$) is known as 3-tangle, an important entanglement measure generalising concurrence (the usual determinant) of bipartite systems [4]. A broader significance of hyperdeterminants to quantum information was identified by Miyake and Wadati [10], through projective duality between separability and singularity. In particular, the hyperdeterminant is invariant under stochastic local operations and classical communication (SLOCC).

Our Contribution.

Reducing homogeneous multilinear systems to hyperdeterminants. The geometric notion of singularity of a tensor (the hyperdeterminant vanishing) is equivalent to the algebraic notion of degeneracy that ensures the existence of a solution to homogeneous multilinear equations. Therefore, we may test if there is a solution to the homogeneous multilinear equation by checking if the hyperdeterminant of the tensor is zero. This correspondence begs the question as to if the solutions of the multilinear equation can be inferred through computation of the hyperdeterminant. It is important to consider the model of computation for the hyperdeterminant. The minimal computational assumption is a black-box that computes the hyperdeterminant of a given tensor. But, it is not obvious how useful black-box access is. The exponential degree of the hyperdeterminant and the lack of obvious structure (such as sparsity) make interpolating the hyperdeterminant as a polynomial using black-box evaluations difficult. We instead consider white-box computation: an arithmetic circuit that takes tensor entries as inputs and outputs the hyperdeterminant.

In § 3, we present a reduction. Given an arithmetic circuit that computes the hyperdeterminant (for a tensor format), we build an arithmetic circuit of asymptotically the same size that solves generic multilinear equations (of the same format). The key to the reduction is a theorem of Gelfand, Kapranov and Zelevinsky relating the Segre embedding of solutions of multilinear equations with partial derivatives of the hyperdeterminant. It may be thought of as a high dimensional generalisation of Cramer's rule. We invoke the Baur-Strassen algorithm to construct arithmetic circuits for all these partial derivatives at once from the arithmetic circuit computing the hyperdeterminant, thereby completing the reduction with only linear complexity.

The qualifier "generic" in generic homogeneous multilinear equations refers to there being at most one projective solution. Geometrically, this translates to the input tensor either being non-singular or a simple singularity. That is, the tensor cannot be a zero of the hyperdeterminant of multiplicity greater than one. Weyman and Zelevinsky proved that non-generic tensors (roots of the hyperdeterminant of multiplicity greater than one) form a co-dimension one projective subvariety of singular tensors [16]. Therefore non-generic tensors fall into a Zariski closed subspace, justifying the "generic" label. It remains an open problem if the non generic case can be handled by methods similar to our reduction.

Computing Hyperdeterminants of boundary formats. In § 4, we devise arithmetic circuits to compute hyperdeterminants of boundary format tensors. There is one circuit for each boundary format. The tensor entries are the inputs to the arithmetic circuit. The main ingredient in the construction is a correspondence between the hyperdeterminant of boundary format tensors and the determinant of a linear transformation connecting sections of vector bundles built from the tensor [6][Theorem 4.3](see also, [5]). Concretely, the linear transformation is between two spaces of multihomogeneous polynomials in the coordinate ring with prescribed degrees. The square matrix of this linear transformation is of dimension equal to the degree $(k_0+1)!/(k_1!k_2!\dots k_r!)$ of the hyperdeterminant, which could range from 2^{k_0} to $(k_0+1)!$. By choosing a monomial bases for the polynomial spaces, we ensure that the matrix entries are either zero or entries from the tensor. An arithmetic circuit for computing the determinant for this square matrix yields an arithmetic circuit for computing the hyperdeterminant. The circuit complexity is the degree of the hyperdeterminant $(k_0+1)!/(k_1!k_2!\dots k_r!)$ raised to the matrix multiplication exponent ω.

Complexity. The reduction and the algorithm for computing the hyperdeterminant in concert yield $O\left(\left(\frac{(k_0+1)!}{k_1!k_2!\dots k_r!}\right)^{\omega}\right)$ sized arithmetic circuits to solve generic homogeneous boundary format multilinear equations. To make sense of this complexity, consider the following two families of boundary format tensors. For the three dimensional family $(2n+1)\times(n+1)\times(n+1)$, the complexity is $O\left(\left((2n+1)!/n!^2\right)^{\omega}\right)$. This is roughly $O\left(2^{n\omega}\right)$, simply exponential in the dimension $4n+3$ of the output. This is sub-exponential in the dimension $(2n+1)(n+1)^2$ of the input. For the $d+1$ dimensional family $(d+1)\times\underbrace{2\times 2\times\dots\times 2}_{d}$, the circuit complexity $O\left((d+1)!\right)$ is quasi polynomial in the input dimension $(d+1)2^d$. It is remarkable that a natural tensor problem without structure has a quasi polynomial time algorithm time for a family of increasing dimension. Further, this family captures $(d+1)$-partite quantum system consisting of d qubits and a qudit. The hyperdeterminant vanishing is related to the existence of a partition of the $(d+1)$-partite system across which the quantum state splits into a product [10].

In terms of algorithms to compare with, Gröbner basis methods can be deployed to solve homogeneous multilinear equations. But applying them naively only guarantees a solution in double exponential time (over Global fields such as $\mathbb{C}$). The performance of Gröbner basis techniques tailored to this problem warrants further investigation. For instance, determinantal structure was exploited in Gröbner basis algorithms addressing similar problems by Spaenlehauer [14, 15] and M. Safey El Din, and Ê. Schost [11]. Our results hint that there are monomial orderings for which Gröbner methods tailored to solving homogeneous multilinear equations are fast.

Towards proving hardness of hyperdeterminants. The mystery surrounding the hardness of computing the hyperdeterminant drew us to the problems addressed in this work. A technical difficulty in proving the hardness of the hyperdeterminants using well known techniques (such as in [9]) is the following. When one tries to embed a hard computational problem into computing hyperdeterminants of three dimensional tensors, one of the dimensions of blows up and we land in a tensor format for which the hyperdeterminant does not exist. An important consequence of our reduction is that *if solving homogeneous multilinear equations is hard for some family boundary or interior formats, then so is computing the hyperdeterminant! Therefore, to prove the hardness of computing hyperdeterminants, it now suffices to prove the hardness of solving homogeneous multilinear equations for some family of boundary or interior formats.* Further, our work suggests it may be fruitful to consider boundary formats such as $(2n+1)\times(n+1)\times(n+1)$, for they may accommodate more geometric methods.

2. Preliminaries: Tensor singularity and hyperdeterminants

2.1. **Cayley's hyperdeterminants.** Let $V_0, V_1, \ldots, V_r$ be $r+1$ vector spaces over the complex numbers $\mathbb{C}$ of respective dimensions $k_0+1, k_1+1, \ldots, k_r+1$. Fix a coordinate system $x^{(j)} = (x_0^{(j)}, x_1^{(j)}, \ldots, x_{k_j}^{(j)})$ for the j^{th}-vector space V_j, or equivalently an ordered basis for the dual V_j^*. Identify an $(r+1)$ dimensional tensor $A \in V_0^* \otimes V_1^* \otimes \ldots \otimes V_r^*$ with an $r+1$-dimensional matrix

$$A = (a_{i_0,i_1,\ldots,i_r}, 0 \le i_0 \le k_0, 0 \le i_1 \le k_1, \ldots, 0 \le i_r \le k_r)$$

of format $(k_0+1)\times(k_1+1)\times\ldots\times(k_r+1)$.

Square matrices are a special case ($r=1$ and $k_0=k_1$) and come with the familiar determinant whose vanishing characterises singularity/degeneracy. The hyperdeterminant is a multidimensional generalisation of the determinant that characterises singularity/degeneracy for tensors formats that generalise square matrices. We start with a geometric definition, equivalent analytic (singularity) and algebraic (degeneracy) formulations follow thereafter.

Geometric definition. Let $\mathbb{P}(V_j) \cong \mathbb{P}^{k_j}$ be the projectivisation of V_j. We need the Cartesian product of these projective spaces, yet desire that the product itself is projective. Let X be the image of the Cartesian product (purely separable tensors) $\mathbb{P}^{k_0}\times\mathbb{P}^{k_1}\times\ldots\times\mathbb{P}^{k_r}$ under the Segre embedding

$$\mathbb{P}(V_0)\times\mathbb{P}(V_1)\times\ldots\times\mathbb{P}(V_r) \hookrightarrow \mathbb{P}\left(V_0\otimes V_1\otimes\ldots\otimes V_r\right) \cong \mathbb{P}^{(k_0+1)(k_1+1)\ldots(k_r+1)-1}$$

$$\left(\left(x_0^{(0)}:x_1^{(0)}:\ldots:x_{k_0}^{(0)}\right),\ldots,\left(x_0^{(r)}:x_1^{(r)}:\ldots:x_{k_r}^{(r)}\right)\right) \longmapsto \left(x_0^{(0)}x_0^{(1)}\ldots x_0^{(r)}:\ldots:x_{k_0}^{(0)}x_{k_1}^{(1)}\ldots x_{k_r}^{(r)}\right).$$

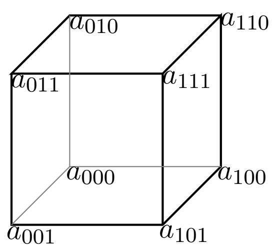
The image under the embedding is a smooth projective variety called the Segre variety, which we denote by X. Let $X\check{}$ denote the projectively dual variety of X consisting of all hyperplanes in the ambient projective space $\mathbb{P}\left(V_0\otimes V_1\otimes\ldots\otimes V_r\right)$ tangent to X at some point. By projective duality (hyperplanes $\leftrightarrow$ points), $X\check{}$ is a variety in the dual projective space $\mathbb{P}\left(V_0\otimes V_1\otimes\ldots\otimes V_r\right)^*$. Gelfand, Kapranov and Zelevinsky characterised precisely when $X\check{}$ is a hypersurface (that is, co-dimension one). It is when the convexity condition

$$\forall\ 0 \le j \le r, \quad k_j \le \sum_{\ell\ne j} k_\ell$$

holds, which we assume from here on. Being a hypersurface, the defining equation of $X\check{}$ is a homogeneous polynomial in the coefficients $a_{i_1,i_2,\ldots,i_d}$, defined to be the hyperdeterminant

Det(). It is an irreducible polynomial with integer coefficients. It can be made unique by insisting that the coefficients are co-prime and choosing a sign.

Example. The first example is the $r = 1$ case, of the usual 2 dimensional matrices. The convexity constraint simplifies to $k_0 = k_1$, confining to square matrices. For square matrices, the hyperdeterminant coincides with the classical determinant. The following first example in 3 dimensions goes back to Cayley [3] and the advent of hyperdeterminants. It is synonymous with the tripartite entanglement measure 3-tangle of three qubits. The hyperdeterminant of a $2 \times 2 \times 2$ format tensor A indexed by the vertices of a cube is



$$\begin{aligned} Det(A) = a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{011}^2 a_{100}^2 \\ -2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} + a_{000}a_{011}a_{100}a_{111} \\ +a_{001}a_{010}a_{101}a_{110} + a_{001}a_{011}a_{110}a_{100} + a_{010}a_{011}a_{101}a_{100}) \\ +4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111}). \end{aligned}$$

The first group of monomials correspond to the four opposing vertices across the main diagonals. The second group to the six pairs of opposing sides. The last group to the two tetrahedrons with edges on the diagonals of the faces.

Boundary and interior formats. Without loss of generality, assume from here on that $k_0 \geq k_1 \geq \ldots \geq k_r$. The convexity condition (which we remind, we always assume) simplifies to $k_0 \leq \sum_{\ell=1}^{r} k_\ell$. Boundary formats are those meeting the convexity constraint with equality, that is $k_0 = \sum_{\ell=1}^{r} k_\ell$. Interior formats are those satisfying the strict convexity constraint $k_0 < \sum_{\ell=1}^{r} k_\ell$.

3. Solving multilinear equations through hyperdeterminants

3.1. Hyperdeterminants, degeneracy of tensors and multilinear equations.

Definition 3.1. (Solving homogeneous multilinear equations) Given a tensor A, decide if there is a $w \in X$ such that in every dimension j,

$$\nabla_{A,j}(w) := \sum_{0 \leq i_j \leq k_j} \left(\sum_{\substack{0 \leq i_0 \leq k_0 \\ \cdots \\ 0 \leq i_r \leq k_r}} a_{i_0,i_1,\ldots,i_r} w_{i_0}^{(0)} w_{i_1}^{(1)} \ldots w_{i_{j-1}}^{(j-1)} w_{i_{j+1}}^{(j+1)} \ldots w_{i_r}^{(r)} \right) x_{i_j}^{(j)} = 0 \left(\in V_j^*\right).$$

The inner summation is over all dimensions except j. If such a solution w exists, find one. A tensor A is said to be degenerate if there is such a w.

Since the Segre variety X lives in the projective tensor space $\mathbb{P}(V_0 \otimes V_1 \otimes \ldots \otimes V_r)$, it is expensive to write down the solution $w \in X$ as a tensor. Instead, we may output a tuple of vectors in the Cartesian space $\mathbb{P}^{k_0} \times \mathbb{P}^{k_1} \times \ldots \times \mathbb{P}^{k_r}$ whose image under the Segre embedding is a solution $w \in X$.

Lemma 3.2. [7, Chap 14, Prop. 1.1] *A tensor A is degenerate if and only if $Det(A) = 0$.*

Proof. We sketch the proof from [7, Chap 14, Prop. 1.1] to give an impression of the connection between degeneracy and hyperdeterminants. To this end, consider the following analytic notion of singularity to complement the geometric definition of hyperdeterminants. To clarify the relation of hyperdeterminants to singularity of tensors, ask when the hyperplane $\{A = 0\}$ carved by A is in $X^{\vee}$. It is precisely when there is a point $x \in X$ at which the hyperplane $\{A = 0\}$ is tangent. This happens precisely when there is a point $x \in X$ such that the multilinear form (arising from the restriction of A on X)

$$f_A(x) := \sum_{\substack{0 \le i_0 \le k_0 \\ \cdots \\ 0 \le i_r \le k_r}} a_{i_0, i_1, \ldots, i_r} x^{(0)}_{i_0} x^{(1)}_{i_1} \ldots x^{(r)}_{i_r} \text{ and all its partial derivatives } \quad \frac{\partial f_A(x)}{\partial x^{(j)}_{i_j}}, \quad \forall j, i_j$$

vanish. In particular, such an x is a singular point of the hyperplane $\{A = 0\}$. We may thus identify the hyperdeterminantal variety $X^{\vee}$ with singular tensors. By inspection, we see that the condition for singularity and degeneracy are the same. □

Therefore the decision making part of solving homogeneous multilinear equations is equivalent to testing if $Det(A) = 0$. Can hyperdeterminants be used to find solutions? We prove that they can, for the generic case of the problem, which we next define.

Definition 3.3. (Solving generic homogeneous multilinear equations) Given a tensor A with a promise that A is a non singular point of $X^{\vee}$, solve the homogeneous multilinear equation with input A.

We next justify why this promise version captures generic instances of the problem. As we saw before, by projective duality, there is a point $w \in X$ in the Segre variety solving the homogeneous multilinear equation corresponding to A if and only if A is in the Hyperdeterminantal variety $X^{\vee}$. Further, A could either be a singular point (that is, a zero of multiplicity greater than one) or a non singular point on $X^{\vee}$. If A is a non singular point in $X^{\vee}$, then there is a unique solution w, which the problem demands that we find. If A is a singular point in $X^{\vee}$, then we have to detect that this is the case. But, we do not have to find a solution. Remarkably, for dimension at least three (that is, $r > 1$), excluding the interior format $2 \times 2 \times 2$, Weyman and Zelevinsky proved that the singular points of $X^{\vee}$ form a co-dimension one projective sub variety [16]. Therefore, by dimension considerations, a generic singular tensor is indeed a non singular point on $X^{\vee}$. The non-generic tensor inputs we abandon are in a Zariski closed subspace.

A high dimensional Cramer's rule. We reduce solving generic homogeneous multilinear equation to computing hyperdeterminants through the following characterisation of the unique solution by Gelfand, Kapranov and Zelevinsky [6, Proposition 1.2]. Let A be the input describing the homogeneous multilinear equation with the promise that A is a non singular point of $X^{\vee}$. If $Det(A) \neq 0$, output that there is no solution. If $Det(A) = 0$, then the promise ensures that there is a unique solution w. Let B be a tensor of the same format as the input A, but with with entries $b_{i_0, i_1, \ldots, i_r}$ that are commuting indeterminates. The hyperdeterminant $Det(B)$ is then an integer polynomial in the indeterminates $b_{i_0, i_1, \ldots, i_r}$.

Up to a normalisation factor, for all $i_0, i_1, \ldots, i_r$, the unique solution

$$w = \left(w_0^{(0)} w_0^{(1)} \ldots w_0^{(r)} : \ldots : w_{k_0}^{(0)} w_{k_1}^{(1)} \ldots w_{k_r}^{(r)}\right) \in X \subseteq \mathbb{P}\left(V_0 \otimes V_1 \otimes \ldots \otimes V_r\right)$$

in the Segre variety satisfies

$$w_{i_0}^{(0)} w_{i_1}^{(1)} \ldots w_{i_r}^{(r)} = \frac{\partial Det(B)}{\partial b_{i_0,i_1,\ldots,i_r}} \Big|_{B=A}. \tag{3.1}$$

It is too expensive to write out w as a point in the ambient tensor space $\mathbb{P}\left(V_0 \otimes V_1 \otimes \ldots \otimes V_r\right)$. Its pre-image

$$\left(\left(w_0^{(0)} : w_1^{(0)} : \ldots : w_{k_0}^{(0)}\right), \ldots, \left(w_0^{(r)} : w_1^{(r)} : \ldots : w_{k_r}^{(r)}\right)\right) \in \mathbb{P}(V_0) \times \mathbb{P}(V_1) \times \ldots \times \mathbb{P}(V_r)$$

in the Cartesian space is a succinct representation that we can explicitly write down. Returning a tuple of vectors as the solution is also in the spirit of the homogeneous linear equations that we are generalising. Without loss of generality, we may set the first coordinate of each vector as one, that is,

$$w_0^{(0)} = w_0^{(1)} = \ldots = w_0^{(r)} = 1.$$

With the first coordinates set, the i_j-th coordinate of the j-th vector is given by

$$w_{i_j}^{(j)} = \frac{\partial Det(B)}{\partial b_{0,\ldots,0,i_j,0,\ldots,0}} \Big|_{B=A}. \tag{3.2}$$

Given an arithmetic circuit to compute the hyperdeterminant (for the format of A), the Baur-Strassen algorithm [1] constructs an arithmetic circuit that computes all $k_0 k_1 \ldots k_r$ of the partial derivatives

$$\left(\frac{\partial Det(B)}{\partial b_{0,\ldots,0,i_j,0,\ldots,0}} \Big|_{B=A}, 0 \leq j \leq r, 1 \leq i_0 \leq k_0, 1 \leq i_1 \leq k_1, \ldots, 1 \leq i_r \leq k_r\right)$$

sought in equation 3.2 at once. Remarkably, the size of this arithmetic circuit is only a small constant times that of the circuit for computing the hyperdeterminant.

4. Hyperdeterminants of boundary format

In this section, we devise an algorithm that given a boundary format tensor A, computes its hyperdeterminant. The algorithm can be realised as an arithmetic circuit. Recall that for boundary formats, $k_0 = k_1 + k_2 + \ldots + k_r$. Hyperdeterminants of boundary formats have a simple interpretation as resultants of a system of multilinear forms following the "Cayley trick". Slices of A in the first dimension form a collection of $k_0 + 1$ multilinear forms

$$f_A^{(i_0)}(x) := \sum_{\substack{0 \leq i_1 \leq k_1 \\ \cdots \\ 0 \leq i_r \leq k_r}} a_{i_0,i_1,\ldots,i_r} x_{i_1}^{(1)} x_{i_2}^{(2)} \ldots x_{i_r}^{(r)}, \quad 0 \leq i_0 \leq k_0. \tag{4.1}$$

The hyperdeterminant $Det(A)$ vanishes precisely when the resultant of $f_A^{(i_0)}(x),\ 0 \leq i_0 \leq k_0$ does.

For positive integers $s_1, s_2, \ldots, s_r$, let

$$S(s_1, s_2, \ldots, s_r) \cong \mathrm{Sym}^{s_1}(V_1) \otimes \mathrm{Sym}^{s_2}(V_2) \otimes \ldots \otimes \mathrm{Sym}^{s_r}(V_r)$$

denote the space of polynomials that are for each $j \in 1, 2, \ldots, d$ homogeneous of degree s_j in the coordinates $x^{(j)}_{i_j}, 0 \le i_j \le k_j$. Set $m_1 := 0$ and for $j > 1$, set $m_j := k_1 + k_2 + \ldots + k_{j-1}$. The conception of our algorithms is primarily due to the following theorem relating boundary format hyperdeterminants to determinants of large matrices built with the above slices.

Theorem 4.1. *(Gelfand-Kapranov-Zelevinsky,* [6]*[Theorem 4.3]) The hyperdeterminant $Det(A)$ of a boundary format $A \in V_0^* \otimes V_1^* \otimes \ldots \otimes V_r^*$ equals (up to sign) the determinant of the linear operator*

$$\delta_A : S(m_1, m_2, \ldots, m_r)^{k_0+1} \longrightarrow S(m_1 + 1, m_2 + 1, \ldots, m_r + 1)$$

$$(g_0, g_1, \ldots, g_{k_0}) \longmapsto \sum_{i_0=0}^{k_0} f_A^{(i_0)} g_{i_0}.$$

Proof. See [6][Theorem 4.3] or [5] for proofs. □

The matrix of δ is indeed square. We may count monomials and verify that the dimensions

$$\dim\left(S(m_1, m_2, \ldots, m_r)^{k_0+1}\right) = \frac{(k_0+1)!}{k_1!k_2!\ldots k_r!}$$

and

$$\dim\left(S(m_1 + 1, m_2 + 1, \ldots, m_r + 1)\right) = (k_1 + k_2 + \ldots + k_r + 1)\binom{k_1 + k_2 + \ldots + k_r}{k_1, k_2, \ldots, k_r}$$

of the two spaces of polynomials are the same. By theorem 4.1, this count is also the degree of the hyperdeterminant

$$\deg(Det(A)) = \frac{(k_0+1)!}{k_1!k_2!\ldots k_r!} = (k_1 + k_2 + \ldots + k_r + 1)\binom{k_1 + k_2 + \ldots + k_r}{k_1, k_2, \ldots, k_r}. \tag{4.2}$$

Theorem 4.1 gives a determinantal identity for each choice of permutation of the vector spaces $V_1, V_2, \ldots, V_r$, which is implicit in the statement. It is not clear if there is a choice of permutation better suited to computation than the others. We now have all the ingredients to describe the hyperdeterminant computation.

Computation of the hyperdeterminant. Let A be the input tensor. Fix lexicographic ordered monomial bases P and Q respectively for $S(m_1, m_2, \ldots, m_r)$ and $S(m_1 + 1, m_2 + 1, \ldots, m_r + 1)$. With bases fixed, we will also denote the matrix of δ_A by δ_A. For $p \in P, q \in Q$, the (p, q)-th entry $\delta_A^{(p,q)}$ of δ_A is either 0 or an $a_{i'_0, i'_1, \ldots, i'_r}$ for some $i'_0, i'_1, \ldots, i'_r$. Since $\log(\deg(Det(A)))$ is polynomial in k_0, the encoding $(p, q) \longmapsto 0$ or $i'_0, i'_1, \ldots, i'_r$ is easy to compute in time polynomial in k_0. This encoding transforms an arithmetic circuit for computing the determinant of a $\frac{(k_0+1)!}{k_1!k_2!\dots k_r!} \times \frac{(k_0+1)!}{k_1!k_2!\dots k_r!}$ square matrix into an arithmetic circuit to compute the hyperdeterminant. The complexity of the circuit is $O\left(\left(\frac{(k_0+1)!}{k_1!k_2!\dots k_r!}\right)^{\omega}\right)$.

We conclude with an illustrative example of the determinantal identity underlying the hyperdeterminant computation.

Example. Let $A = (a_{i_0,i_1,i_2})$ be a tensor of the simplest three dimensional boundary format, namely $3\times 2\times 2$. Through lexicographic monomial orderings, fix ordered bases $\left(x_0^{(2)}, x_1^{(2)}\right)$ of $S(0,1)$ and $\left(x_0^{(1)}x_0^{(2)}x_0^{(2)}, x_0^{(1)}x_0^{(2)}x_1^{(2)}, x_0^{(1)}x_1^{(2)}x_1^{(2)}, x_1^{(1)}x_0^{(2)}x_0^{(2)}, x_1^{(1)}x_0^{(2)}x_1^{(2)}, x_1^{(1)}x_1^{(2)}x_1^{(2)}\right)$ of $S(1,2)$. Then

$$\delta_A : S(0,1)^3 \longrightarrow S(1,2)$$

corresponds to the matrix

$$\begin{pmatrix} a_{000} & 0 & a_{100} & 0 & a_{200} & 0 \\ a_{001} & 0 & a_{101} & 0 & a_{201} & 0 \\ a_{010} & a_{000} & a_{110} & a_{100} & a_{210} & a_{200} \\ a_{011} & a_{001} & a_{111} & a_{101} & a_{211} & a_{201} \\ 0 & a_{010} & 0 & a_{110} & 0 & a_{210} \\ 0 & a_{011} & 0 & a_{111} & 0 & a_{211} \end{pmatrix}.$$

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