Provability of the Circuit Size Hierarchy and Its Consequences

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Abstract

The *Circuit Size Hierarchy* (CSH^a_b) states that if $a > b \geq 1$ then the set of functions on n variables computed by Boolean circuits of size n^a is strictly larger than the set of functions computed by circuits of size n^b. This result, which is a cornerstone of circuit complexity theory, follows from the *non-constructive* proof of the existence of functions of large circuit complexity obtained by Shannon in 1949.

Are there more "constructive" proofs of the Circuit Size Hierarchy? Can we quantify this? Motivated by these questions, we investigate the provability of CSH^a_b in theories of bounded arithmetic. Among other contributions, we establish the following results:

(i) Given any $a > b > 1$, CSH^a_b is provable in Buss's theory T^2_2.

(ii) In contrast, if there are constants $a > b > 1$ such that CSH^a_b is provable in the theory T^1_2, then there is a constant $\varepsilon > 0$ such that P^{NP} requires non-uniform circuits of size $n^{1+\varepsilon}$.

In other words, an improved *upper bound* on the proof complexity of CSH^a_b would lead to new *lower bounds* in complexity theory.

We complement these results with a proof of the *Formula Size Hierarchy* (FSH^a_b) in PV_1 with parameters $a > 2$ and $b = 3/2$. This is in contrast with typical formalizations of complexity lower bounds in bounded arithmetic, which require APC_1 or stronger theories and are not known to hold even in T^1_2.

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1 Introduction

1.1 Context and Motivation

The existence of Boolean functions requiring large circuits can be shown by a non-constructive counting argument, as established by Shannon in 1949 [Sha49]. It follows from Shannon's seminal result and a simple padding argument that if $a > b \geq 1$ there are functions computable by circuits of size n^a that cannot be computed by circuits of size n^b. In other words, the classification of Boolean functions by their minimum circuit size forms a strict *hierarchy*.

Obtaining a "constructive" form of these results has been a holy grail in computational complexity theory for several decades due to its connections to derandomization and as an approach to separating P and NP. For instance, if there is a polynomial-time algorithm that given 1^n outputs the truth-table of a function $f \colon \{0,1\}^{\log n} \to \{0,1\}$ that requires circuits of size $n^{\Omega(1)}$, then $\mathsf{P} = \mathsf{BPP}$ [IW97]. In results of this form, a constructive form of the (non-constructive) proof of the existence of hard functions is interpreted *computationally* as the existence of an algorithm of bounded complexity that computes a hard function.

In this paper, rather than focusing on the existence of algorithms to capture the constructiveness of a statement, we explore this notion from the perspective of mathematical logic, specifically concerning its *provability* in certain mathematical theories. We are interested in identifying the weakest theory capable of establishing the aforementioned circuit size hierarchy for Boolean circuits and related results.

As one of our contributions, we present a tight connection between the computational and proof-theoretic perspectives. We demonstrate that proving the non-uniform circuit size hierarchy in a theory known as T^1_2 implies the existence of a function in P^{NP} that requires Boolean circuits of size at least $n^{1+\varepsilon}$. The latter is a frontier question in complexity theory (see, e.g., [CMMW19]). Thus, in a precise sense, developing more constructive proofs of the circuit size hierarchy would lead to significant progress on explicit circuit lower bounds.

We now proceed to describe this result and other contributions of this work in detail.

1.2 Results

We will be concerned with standard theories of bounded arithmetic. These theories are designed to capture proofs that manipulate and reason with concepts from a specified complexity class. Notable examples include Cook's theory PV_1 [Coo75], which formalizes polynomial-time reasoning; Jeřábek's theory APC_1 [Jeř04, Jeř05, Jeř07], which extends PV_1 by incorporating the dual weak pigeonhole principle for polynomial-time functions and formalizes probabilistic polynomial-time reasoning; and Buss's theories T^i_2 [Bus86], which incorporate induction principles corresponding to various levels of the polynomial-time hierarchy.

For an introduction to bounded arithmetic, we refer to [Bus97]. For its connections to computational complexity and a discussion on the formalization of complexity theory, we refer to [Oli24].[1] Here we only recall that theory PV_1 corresponds essentially to T^0_2 [Jeř06], and that $\mathsf{T}^0_2 \subseteq \mathsf{T}^1_2 \subseteq \mathsf{T}^2_2$ correspond to the first levels of Buss's hierarchy. A brief overview of the theories is provided in Section 2.

For a given $n \in \mathbb{N}$, we use $\mathsf{CIRCUIT}[s(n)]$ to denote the set of Boolean functions $f \colon \{0,1\}^n \to \{0,1\}$ computed by circuits of size at most $s(n)$. Similarly, we write $\mathsf{FORMULA}[s(n)]$ when referring to formula size. We use $\mathsf{SIZE}[s(n)]$ to denote the set of languages $L \subseteq \{0,1\}^*$ that admit a sequence of circuits of size at most $s(n)$.

[1] In particular, the reference [Oli24] contains a detailed discussion of some aspects of the formalization of the statements appearing below.

Circuit Size Hierarchy. For rationals $a > b \geq 1$ and n_0, we consider the following sentence:[2]

$$\begin{aligned}\mathsf{CSH}[a,b,n_0] \quad \equiv \quad & \forall n \geq n_0 \in \mathsf{Log},\ \exists \text{ circuit } D\colon \{0,1\}^n \to \{0,1\} \text{ of size} \leq n^a, \\ & \forall \text{ circuit } C\colon \{0,1\}^n \to \{0,1\} \text{ of size} \leq n^b,\ \exists x \in \{0,1\}^n \text{ such that } D(x) \neq C(x).\end{aligned}$$

In other words, $\mathsf{CSH}[a,b,n_0]$ states that $\mathsf{CIRCUIT}[n^a] \nsubseteq \mathsf{CIRCUIT}[n^b]$ whenever $n \geq n_0$.

Next, we state our first result.

Theorem 1. *The following results hold:*

(i) *For every choice of rationals a and b with $a > b > 1$, and for every large enough $n_0 \in \mathbb{N}$,*

$$\mathsf{T}^2_2 \vdash \mathsf{CSH}[a,b,n_0]\,.$$

(ii) *If there are rationals $a > b > 1$ and a constant $n_0 \in \mathbb{N}$ such that*

$$\mathsf{T}^1_2 \vdash \mathsf{CSH}[a,b,n_0]\,,$$

then there is a constant $\varepsilon > 0$ and a language $L \in \mathsf{P}^{\mathsf{NP}}$ such that $L \notin \mathsf{SIZE}[n^{1+\varepsilon}]$.

(iii) *Similarly to the previous item, if $\mathsf{PV}_1 \vdash \mathsf{CSH}[a,b,n_0]$, there is $L \in \mathsf{P}$ such that $L \notin \mathsf{SIZE}[n^{1+\varepsilon}]$.*

To put it another way, we can establish a circuit size hierarchy within the theory T^2_2. If this result could also be proven in the theory T^1_2, it would lead to a significant breakthrough in circuit lower bounds. Thus, by enhancing the proof complexity upper bound for the provability of the circuit size hierarchy, we can achieve new circuit lower bounds.

Note that in Theorem 1 Items (ii) and (iii) we obtain a lower bound against circuits of size $n^{1+\varepsilon}$, where the constant $\varepsilon > 0$ depends on the proof of $\mathsf{CSH}[a,b,n_0]$ in the corresponding theory. In other words, while the sentence claims the existence of hardness against circuits of size n^b, we are only able to extract a weaker lower bound for an explicit problem.

In our next result, we describe a setting where we can extract all the hardness from a proof of the corresponding sentence.

Succinct Circuit Size Hierarchy. For rationals $a > b \geq 1$ and n_0, we consider the following sentence:

$$\begin{aligned}\mathsf{SCSH}[a,b,n_0] \quad \equiv \quad & \forall n \geq n_0 \in \mathsf{Log},\ \exists \text{ collection } \{(x^1,b^1),\ldots,(x^\ell,b^\ell)\} \text{ of size } \ell \leq n^a \text{ with} \\ & |x^i| = n \wedge |b^i| = 1 \text{ for each } i \in [\ell] \text{ and } x^i \neq x^j \text{ for distinct } i,j \in [\ell]\,, \\ & \forall \text{ circuit } C\colon \{0,1\}^n \to \{0,1\} \text{ of size} \leq n^b,\ \exists i \in [\ell] \text{ such that } C(x^i) \neq b^i.\end{aligned}$$

In other words, $\mathsf{SCSH}[a,b,n_0]$ states that for every $n \geq n_0$ there is a collection of $\ell \leq n^a$ labelled examples such that every circuit of size at most n^b disagrees with at least one of its labels.

We obtain the following results on the proof complexity of the succinct circuit size hierarchy.

Theorem 2. *The following results hold:*

[2]The abbreviation $n \in \mathsf{Log}$ denotes that n is the length of a variable N (see, e.g., [Oli24] for more details).

(*i*) *For every choice of rationals $a > b > 1$ and for every large enough $n_0 \in \mathbb{N}$,*

$$\mathsf{T}^2_2 \vdash \mathsf{SCSH}[a, b, n_0]\,.$$

(*ii*) *If there are rationals $a > b > 1$ and a constant $n_0 \in \mathbb{N}$ such that*

$$\mathsf{T}^1_2 \vdash \mathsf{SCSH}[a, b, n_0]\,,$$

then there is a language $L \in \mathsf{P}^{\mathsf{NP}}$ such that $L \notin \mathsf{SIZE}[n^b]$.

In our final result, we investigate the provability of size hierarchies for more restricted computational models in T^1_2 and weaker theories.

Formula Size Hierarchy. For rationals $a > b \geq 1$ and n_0, we consider the following sentence:

$$\begin{aligned}\mathsf{FSH}[a, b, n_0] \quad \equiv \quad & \forall n \geq n_0 \in \mathsf{Log},\ \exists \text{ formula } F\colon \{0,1\}^n \to \{0,1\} \text{ of size } \leq n^a, \\ & \forall \text{ formula } G\colon \{0,1\}^n \to \{0,1\} \text{ of size } \leq n^b,\ \exists x \in \{0,1\}^n \text{ such that } F(x) \neq G(x).\end{aligned}$$

In other words, $\mathsf{FSH}(a, b, n_0)$ states that $\mathsf{FORMULA}[n^a] \nsubseteq \mathsf{FORMULA}[n^b]$ whenever $n \geq n_0$.

We establish that for some parameters a formula size hierarchy is provable already in PV_1.

Theorem 3. *Consider rationals $a > 2$ and $b = 3/2$, and let n_0 be a large enough positive integer. Then*

$$\mathsf{PV}_1 \vdash \mathsf{FSH}[a, b, n_0]\,.$$

While many lower bounds can be proven in APC_1 and stronger theories (see [MP20, Oli24, CLO24] and references therein), Theorem 3 provides an example of a non-trivial lower bound (under a "Log" formalization; see [Oli24, Section 4.1]) that can be established in PV_1, which might be of independent interest.

1.3 Techniques

The proofs of Items (*ii*) and (*iii*) in Theorem 1 are inspired by arguments from [KO17, Kra21] that rely on a combination of a witnessing theorem with a term elimination strategy. Recall that the witnessing theorem allows us to extract computational information from a proof of the sentence in the theory. Roughly speaking, in our context this implies that the first existential quantifier in the sentence $\mathsf{CSH}[a, b, n_0]$, which corresponds to a circuit computing a hard function, can be witnessed by a finite number of terms $t_1, \ldots, t_k$ of the corresponding theory. In PV_1, a term yields a polynomial-time function, while in T^1_2 a term yields a polynomial-time function with access to an NP oracle. The main difficulty is that (1) for a given input length n it is not clear which term among $t_1, \ldots, t_k$ succeeds in constructing a hard function, and (2) for a term to succeed we must provide counter-examples to the candidate witnesses provided by previous terms.

As in previous papers, we assume that the conclusion of the theorem does not hold, and use this assumption to rule out the correctness of each term. This leads to a contradiction, meaning that the original sentence is not provable in the corresponding theory. Implementing this plan requires a careful argument, and we are currently only able to carry it out under a complexity inclusion in $\mathsf{SIZE}[n^{1+\varepsilon}]$ as opposed to $\mathsf{SIZE}[n^b]$. The proof of the result is given in Section 3.1.

On the other hand, in the case of the succinct circuit size hierarchy, the argument for Item (*ii*) of Theorem 2 is simpler and allows us to start with the weaker assumption that $\mathsf{P}^{\mathsf{NP}} \subseteq \mathsf{SIZE}[n^b]$. Without getting into the technical details, the main reason for not losing hardness in this result is that given a labelled list of examples and access to an NP oracle, we can efficiently compute a minimum size circuit that agrees with this list of inputs. Consequently, we can check if a candidate labelled list provided by a term is indeed hard, or produce a counter-example when this is not the case. The same computation is not available in the case of Theorem 1, since it is not clear how to efficiently compute with access to an NP oracle if a given circuit admits a smaller equivalent circuit. The proof of Item (*ii*) of Theorem 2 appears in Section 3.2.

The proofs of Theorem 1 Item (*i*) and Theorem 2 Item (*i*) are given in Section 3.3. The formalization of these hierarchies in T^2_2 is easily done with access to the dual Weak Pigeonhole Principle for polynomial-time functions, a principle which is known to be available in T^2_2. In more detail, CSH follows from SCSH in PV_1, while SCSH can be established in theory APC_1, which is contained in T^2_2.

Finally, in the proof of Theorem 3 we formalize in PV_1 that the parity function on n bits can be computed by formulas of size $O(n^2)$ and require formulas of size $\Omega(n^{3/2})$. This yields in PV_1 a proof of $\mathsf{FSH}[a, b, n_0]$ for any choice of parameters $a > 2$, large enough n_0, and $b = 3/2$. The upper bound on the complexity of parity follows from a straightforward formalization of the correctness of the formula obtained via a divide-and-conquer procedure. On the other hand, in order to show the formula lower bound we formalize Subbotovskaya's argument [Sub61] based on the method of restrictions. To implement the proof in PV_1, we directly define an efficient refuter that given a small formula outputs an input string where it fails to compute the parity function. The correctness of the refuter is established by induction using an induction principle available in the theory S^1_2. We then rely on a conservation result showing that the proof can also be done in PV_1. A detailed exposition of the argument appears in Section 4.

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2 Preliminaries

2.1 Complexity Theory

We employ standard definitions from complexity theory, such as basic complexity classes, Boolean circuits, and Boolean formulas (see, e.g., [AB09]).

Let $\mathbb{N}$ represent the set of non-negative integers. For any $a \in \mathbb{N}$, let $|a|$ denote the length of its binary representation, defined as $|a| \triangleq \lceil \log_2(a+1) \rceil$. For a constant $k \geq 1$, a function $f \colon \mathbb{N}^k \to \mathbb{N}$ is said to be computable in polynomial time if $f(x_1, \ldots, x_k)$ can be computed in time polynomial in $|x_1|, \ldots, |x_k|$. For convenience, we might write $|\vec{x}| \triangleq |x_1|, \ldots, |x_k|$. The class FP denotes the set of polynomial-time computable functions. Although the definition of polynomial time typically refers to a machine model, FP can also be defined in a machine-independent manner as the closure of a set of base functions $\mathcal{F}$ (not described here) under *composition* and *limited recursion on notation*. A function $f(\vec{x}, y)$ is defined from functions $g(\vec{x})$, $h(\vec{x}, y, z)$, and $k(\vec{x}, y)$ by *limited recursion on notation* if

$$\begin{array}{rcl} f(\vec{x}, 0) & = & g(\vec{x}) \\ f(\vec{x}, y) & = & h(\vec{x}, y, f(\vec{x}, \lfloor y/2 \rfloor)) \\ f(\vec{x}, y) & \leq & k(\vec{x}, y) \end{array}$$

for every sequence $(\vec{x}, y)$ of natural numbers. Cobham [Cob65] established that FP is the smallest class of functions that contains the base functions $\mathcal{F}$ and is closed under composition and limited recursion on notation.

2.2 Bounded Arithmetic

2.2.1 Logical Theories

We recall the definitions of some standard theories of bounded arithmetic. For more details, the reader can consult [Kra95, CN10, Kra19].

Cook's Theory PV **[Coo75].** The theory PV_1 is designed to model the set $\mathbb{N}$ of natural numbers with the standard interpretations for constants and function symbols like $0, +, \times$, etc. The vocabulary (language) of PV, denoted $\mathcal{L}_{\mathsf{PV}}$, includes a function symbol for each polynomial-time algorithm $f \colon \mathbb{N}^k \to \mathbb{N}$, where k is any constant. These function symbols and their defining axioms are derived using Cobham's characterization of polynomial-time functions discussed above. While Cook's PV was an equational theory, it was later extended in [KPT91] to a first-order theory PV_1, which includes an induction axiom scheme that simulates binary search. It can be shown that PV_1 allows induction over quantifier-free formulas (i.e., polynomial-time predicates).

PV_1 can be formulated with all axioms as universal formulas (i.e., $\forall \vec{x}\, \phi(\vec{x})$, where ϕ is free of quantifiers). Thus, PV_1 is a *universal theory*. Although the definition of PV_1 is quite technical, the theory is fairly robust and the details of its definition are often unnecessary for practical purposes. In particular, PV_1 has an equivalent formalizations that does not rely on Cobham's result, e.g. [Jeř06].

Jeřábek's Theory APC_1 **[Jeř04, Jeř05, Jeř07].** APC_1 extends PV_1 with the *dual Weak Pigeonhole Principle* (dWPHP) for PV_1 functions:

$$\mathsf{APC}_1 \triangleq \mathsf{PV} \cup \{\mathsf{dWPHP}(f) \mid f \in \mathcal{L}_{\mathsf{PV}}\}.$$

Each sentence $\mathsf{dWPHP}(f)$ postulates that, for every length $n = |N|$ and for every choice of $\vec{z}$, there is $y < (1 + 1/n) \cdot 2^n$ such that $f(\vec{z}, x) \neq y$ for every $x < 2^n$. It is known that APC_1 is contained in T^2_2 [MPW02].

Buss's Theories S^i_2 **and** T^i_2 **[Bus86].** The language $\mathcal{L}_B$ for these theories includes predicate symbols $=$ and $\leq$, constant symbols 0 and 1, and function symbols S (successor), $+$, $\cdot$, $\lfloor x/2 \rfloor$, $|x|$ (interpreted as the length of x), and $\#$ (interpreted as $x \# y = 2^{|x| \cdot |y|}$, known as "smash").

Recall that a *bounded quantifier* is a quantifier of the form $Qy \leq t$, where $Q \in \{\exists, \forall\}$ and t is a term not involving y. Similarly, a *sharply bounded quantifier* is one of the form $Qy \leq |t|$. A formula where each quantifier appears bounded (or sharply bounded) is called a bounded (or sharply bounded) formula.

We can create a hierarchy of formulas by counting alternations of bounded quantifiers. The class $\Pi^b_0 = \Sigma^b_0$ contains the sharply bounded formulas. Recursively, for each $i \geq 0$, the classes Σ^b_i and Π^b_i are defined by the quantifier structure of the sentence, ignoring sharply bounded quantifiers. For instance, if $\varphi \in \Sigma_0^b$ and $\psi \triangleq \exists y \leq t(\vec{x})\ \varphi(y, \vec{x})$, then $\psi \in \Sigma_1^b$. For the general case of the definition, see [Kra95]. It is known that for each $i \geq 1$, a predicate $P(\vec{x})$ is in Σ_i^p (the i-th level of the polynomial hierarchy) if and only if there is a Σ_i^b-formula that agrees with it over $\mathbb{N}$.

These theories share a common set of finitely many axioms, BASIC, which postulate the expected arithmetic behavior of the constants, predicates, and function symbols. The only difference among the theories is the type of induction axiom scheme each one postulates.

T_2^i is a theory in the language $\mathcal{L}_B$ that extends BASIC by including the induction axiom IND:

$$\varphi(0) \wedge \forall x\, (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow\ \forall x\, \varphi(x)$$

for all Σ_i^b-formulas $\varphi(a)$. The formula $\varphi(a)$ may contain other free variables in addition to a.

S_2^i is a theory in the language $\mathcal{L}_B$ that extends BASIC by including the polynomial induction axiom PIND:

$$\varphi(0) \wedge \forall x\, (\varphi(\lfloor x/2 \rfloor) \rightarrow \varphi(x)) \rightarrow\ \forall x\, \varphi(x)$$

for all Σ_i^b-formulas $\varphi(a)$. The formula $\varphi(a)$ may contain other free variables in addition to a.

Theory $\mathsf{S}_2^1(\mathsf{PV})$. When proving some results in S_2^1, it is often convenient to use a more expressive vocabulary that easily describes any polynomial-time function. This can be done in a *conservative* manner, meaning the power of the theory is not increased. Specifically, let Γ be a set of $\mathcal{L}_B$-formulas. We say that a polynomial-time function $f \colon \mathbb{N}^k \rightarrow \mathbb{N}$ is Γ-*definable* in S_2^1 if there exists a formula $\psi(\vec{x}, y) \in \Gamma$ such that the following conditions are met:

(*i*) For every $\vec{a} \in \mathbb{N}^k$, $f(\vec{a}) = b$ if and only if $\mathbb{N} \models \varphi(\vec{a}, b)$.

(*ii*) $\mathsf{S}_2^1 \vdash \forall \vec{x}\, (\exists y\, (\varphi(\vec{x}, y) \wedge \forall z\, (\varphi(\vec{x}, z) \rightarrow y = z)))\,.$

Every function $f \in \mathsf{FP}$ is Σ_1^b-definable in S_2^1. By incorporating all functions in FP into the vocabulary of S_2^1 and extending the axioms of S_2^1 with their defining equations, we obtain a theory $\mathsf{S}_2^1(\mathsf{PV})$. This theory allows polynomial-time predicates to be referred to using quantifier-free formulas. $\mathsf{S}_2^1(\mathsf{PV})$ remains conservative over S_2^1, meaning any $\mathcal{L}_B$-sentence provable in $\mathsf{S}_2^1(\mathsf{PV})$ is also provable in S_2^1. Finally, it is known that $\mathsf{S}_2^1(\mathsf{PV})$ proves the polynomial induction scheme for both Σ_1^b-formulas and Π_1^b-formulas within the extended vocabulary.

2.2.2 The KPT Witnessing Theorem

The following witnessing theorem (a variant of Herbrand's theorem) is proved in [KPT91] (cf. also [Kra95, Theorem 7.4.1]) for universal theories (like the theory PV_1).

Theorem 4 (KPT Theorem for $\forall\exists\forall\exists$ sentences)**.** *Let* T *be a universal theory with vocabulary* $\mathcal{L}$. *Let* φ *be an open* $\mathcal{L}$*-formula, and suppose that*

$$\mathsf{T} \vdash \forall x\, \exists y\, \forall z\, \exists w\ \varphi(x, y, z, w).$$

Then there is a finite sequence $s_1, \ldots, s_k$ *of* $\mathcal{L}$*-terms such that*

$$\mathsf{T} \vdash \forall x, z_1, \ldots, z_k\, \big(\psi(x, s_1(x), z_1) \vee \psi(x, s_2(x, z_1), z_2) \vee \cdots \vee \psi(x, s_k(x, z_1, \ldots, z_{k-1}), z_k)\big),$$

where

$$\psi(x, y, z) \triangleq \exists w\ \varphi(x, y, z, w).$$

For completeness, we describe a proof of Theorem 4 in Appendix A.

We can also apply the KPT Theorem to each theory T^i_2 (for $i \geq 1$) using a conservative extension of the theory that admits a universal axiomatization. The corresponding theory is called PV_{i+1} [KPT91]. In PV_{i+1}, each term is equivalent to an $\mathsf{FP}^{\Sigma^p_i}$ function over the standard model. This leads to the following result.

Theorem 5 (Consequence of the KPT Theorem for Theory T^i_2)**.** *Let $i \geq 1$, $\varphi(x, y, w, z)$ be a Π^b_i-formula, and suppose that*

$$\mathsf{T}^i_2 \vdash \forall x\, \exists y\, \forall z\, \exists w\; \varphi(x, y, w, z).$$

Then there is a finite sequence $f_1, \dots, f_k$ of function symbols, each corresponding to an $\mathsf{FP}^{\Sigma^p_i}$ function, such that

$$\mathbb{N} \models \forall x, z_1, \dots, z_k \left(\psi(x, f_1(x), z_1) \vee \psi(x, f_2(x, z_1), z_2) \vee \dots \vee \psi(x, f_k(x, z_1, \dots, z_{k-1}), z_k)\right),$$

where

$$\psi(x, y, z) \triangleq \exists w\; \varphi(x, y, z, w).$$

3 Circuit Size Hierarchies in Bounded Arithmetic

3.1 Explicit Circuit Lower Bounds from Provability in PV_1 and T^1_2

In this section, we prove Theorem 1 Items (*ii*) and Items (*iii*).

Theorem 6 (Theorem 1 Item (*iii*))**.** *If there are rationals $a > b > 1$ and $n_0 \in \mathbb{N}$ such that*

$$\mathsf{PV}_1 \vdash \mathsf{CSH}[a, b, n_0]\,,$$

then there is a constant $\varepsilon > 0$ and a language $L \in \mathsf{P}$ such that $L \notin \mathsf{SIZE}[n^{1+\varepsilon}]$.

Proof. Towards a contradiction, suppose that $\mathsf{PV}_1 \vdash \mathsf{CSH}[a, b, n_0]$ for rationals $a > b > 1$ and some constant n_0 and that $P \subseteq \bigcap_{\varepsilon>0} \mathsf{SIZE}[n^{1+\varepsilon}]$. The sentence $\mathsf{CSH}[a, b, n_0]$ has the form $\forall\exists\forall\exists$:

$$\mathsf{CSH}[a, b, n_0] \triangleq \forall n \geq n_0 \in \mathsf{Log},\ \exists \text{ circuit } D\ \forall \text{ circuit } C\ \ \psi_{a,b}(n, D, C)\,,$$

where $\psi_{a,b}(n, D, C)$ is the existential formula:

$$\psi_{a,b}(n, D, C) \triangleq \exists x\ |x| \leq n \wedge \mathsf{SIZE}(D) \leq n^a \wedge (\mathsf{SIZE}(C) \leq n^b \rightarrow\ D(x) \neq C(x)).$$

Therefore, we can apply the KPT Theorem (Theorem 4), which provides PV_1-terms, equivalently FP functions, $s_1, \dots, s_k$, where k is a constant, such that

$$\mathbb{N} \models\ \psi_{a,b}(n, s_1(1^{(n)}), C_1) \vee \psi_{a,b}(n, s_2(1^{(n)}, C_1), C_2) \vee \dots \vee \psi_{a,b}(n, s_k(1^{(n)}, C_1, \dots, C_{k-1}), C_k). \quad (1)$$

In the relation above the circuits $C_1, \dots, C_k$ are universally quantified.

Next, we use $P \subseteq \bigcap_{\varepsilon>0} \mathsf{SIZE}[n^{1+\varepsilon}]$ to refute each of these disjuncts. We start by considering the following language, D-Eval:

```
Input: A string $x$ and a sequence $\langle C_1, C_2, \ldots, C_r \rangle$ of $r \leq k-1$ circuits
1 Define $n \triangleq |x|$;
2 Simulate $s_{r+1}(1^{(n)}, C_1, \ldots, C_r)$ and interpret the output as a Boolean circuit $D\colon \{0,1\}^n \to \{0,1\}$;
  // We assume w.l.o.g. that $D$ is a valid $n$-bit circuit of size
     $\leq n^a$, since otherwise the disjunct is trivially false.
3 Evaluate $D$ on input $x$ and output the result.
```
Algorithm 1: The pseudocode of an algorithm that decides the language D-Eval.

D-Eval is in P due to the fact that $s_1, \ldots, s_k \in \mathsf{FP}$ and circuit evaluation is in FP. By our assumption on the circuit complexity of the complexity class P, for every input length m and every $\varepsilon > 0$, D-Eval $\in \mathsf{SIZE}[m^{1+\varepsilon}]$, so we can choose

$$\varepsilon_0 \triangleq b^{1/(2k)} - 1 > 0$$

and have D-Eval $\in \mathsf{SIZE}[m^{b^{1/(2k)}}]$. We also define the constants

$$\epsilon_i \triangleq b^{i/k} \quad \text{and} \quad \delta_i \triangleq b^{(2i-1)/(2k)}$$

for $i = 1, \ldots, k$. Note that $\epsilon_i = (1+\varepsilon_0)\delta_i$ and $\delta_{i+1} > \epsilon_i$.

We start by refuting $\psi_{a,b}(n, s_1(1^{(n)}), C_1)$. We consider inputs of the form x, λ to D-Eval, where λ is the empty sequence. Then the input has length $n + c$, where $c = O(\log n)$ accounts for the overhead in the encoding of the input. We consider the circuit $C_1^* \in \mathsf{CIRCUIT}[(n+c)^{1+\varepsilon_0}]$, which evaluates as D-Eval on inputs of length $n + c$, and we fix the input variables not related to x to represent the empty sequence. The resulting circuit has as input an n-bit string x and computes according to $s_1(1^{(n)})$ by definition of the D-Eval algorithm. For sufficiently large n, we have that $n + c \leq n^{\delta_1} \Rightarrow (n+c)^{1+\varepsilon_0} \leq n^{(1+\varepsilon_0)\delta_1} = n^{\epsilon_1}$, therefore we have the circuit $C_1^* \in \mathsf{CIRCUIT}[n^{\epsilon_1}]$ which agrees with the circuit $s_1(1^{(n)})$ on all n-bit inputs. Since $\epsilon_1 \leq b$, we have that $\mathbb{N} \not\models \psi_{a,b}(n, s_1(1^{(n)}), C_1^*)$.

We can apply a similar argument to the next disjunct using the aforementioned circuit C_1^*. In more detail, we consider the input $(x, \langle C_1^* \rangle)$ on D-Eval, which has length $m = n + 9n^{\epsilon_1}\log(n^{\epsilon_1}) + c \leq n^{\delta_2}$ for sufficiently large n due to $\delta_2 > \epsilon_1$, and a corresponding circuit $C_2^* \in \mathsf{CIRCUIT}[m^{1+\varepsilon_0}]$ provided by the circuit upper bound hypothesis. Similarly, we can fix the $9n^{\epsilon_1}\log(n^{\epsilon_1}) + c$ variables not related to the input string x. This provides an n-bit circuit $C_2^* \in \mathsf{CIRCUIT}[n^{\epsilon_2}]$ that computes according to the circuit $s_2(1^{(n)}, C_1^*)$, due to the definition of the D-Eval algorithm. Since $\epsilon_2 < b$, we have that $\mathbb{N} \not\models \psi_{a,b}(n, s_2(1^{(n)}, C_1^*), C_2^*)$.

Inductively, if we have circuits $C_1^*, C_2^*, \ldots, C_i^*$ for some $i \leq k-1$ of sizes at most $n^{\epsilon_1}, n^{\epsilon_2}, \ldots, n^{\epsilon_i}$, respectively, we consider the input $(x, \langle C_1^*, \ldots, C_i^* \rangle)$ to D-Eval, which has length $m = n + 9n^{\epsilon_1}\log(n^{\epsilon_1}) + \cdots + 9n^{\epsilon_i}\log(n^{\epsilon_i}) + c \leq n^{\delta_{i+1}}$ for sufficiently large n. Therefore, by taking a corresponding $m^{1+\varepsilon_0}$-size circuit for D-Eval and fixing all the inputs except for x, we get the circuit $C_{i+1}^* \in \mathsf{CIRCUIT}[n^{\epsilon_{i+1}}] \subseteq \mathsf{CIRCUIT}[n^b]$ which agrees with the circuit $s_{i+1}(1^{(n)}, C_1^*, \ldots, C_i^*)$ on all n-bit inputs. Consequently, $\mathbb{N} \not\models \psi_{a,b}(n, s_{i+1}(1^{(n)}, C_1^*, \ldots, C_i^*), C_{i+1}^*)$.

Overall, we can refute all disjuncts in Equation (1), which gives us a contradiction. This completes the proof. □

Theorem 7 (Theorem 1 Item (*ii*)). *If there are rationals $a > b > 1$ and $n_0 \in \mathbb{N}$ such that*

$$\mathsf{T}_2^1 \vdash \mathsf{CSH}[a, b, n_0]\,,$$

then there is a constant $\varepsilon > 0$ and a language $L \in \mathsf{P}^{\mathsf{NP}}$ such that $L \notin \mathsf{SIZE}[n^{1+\varepsilon}]$.

Proof. In this case, provability in T^1_2 provides by the KPT Theorem (Theorem 5) functions $s_1, \ldots, s_k$ which are in $\mathsf{FP}^{\mathsf{NP}}$ instead of FP as in the previous proof. Therefore, the algorithm D-Eval is in P^{NP} and we use the upper bound $\mathsf{P}^{\mathsf{NP}} \subseteq \bigcap_{\varepsilon>0} \mathsf{SIZE}[n^{1+\varepsilon}]$ to get a contradiction in the same way as above. □

Note that in the arguments above we have no control over the constant $\varepsilon > 0$. It depends on the number of disjuncts obtained from the KPT Theorem, which depends on the supposed proof of the hierarchy sentence.

3.2 Extracting All the Hardness from Proofs of a Succinct Hierarchy Theorem

In this section, we prove Theorem 2 Item (*ii*).

Theorem 8 (Theorem 2 Item (*ii*))**.** *If there are rationals $a > b > 1$ and a constant $n_0 \in \mathbb{N}$ such that*

$$\mathsf{T}^1_2 \vdash \mathsf{SCSH}[a, b, n_0]\,,$$

then there is a language $L \in \mathsf{P}^{\mathsf{NP}}$ such that $L \notin \mathsf{SIZE}[n^b]$.

Proof. The main idea here is to use the proof of SCSH in order to define a Turing machine M which runs in polynomial time using an NP oracle and its language is hard against n^b-size circuits.

Starting from $\mathsf{T}^1_2 \vdash \mathsf{SCSH}[a, b, n_0]$, we see that the structure of the sentence is $\forall\exists\forall\exists$:

$$\mathsf{SCSH}[a, b, n_0] \triangleq \forall n \geq n_0 \in \mathsf{Log},\ \exists \text{ collection } \mathcal{F},\ \forall \text{ circuit } C\ \ \phi_{a,b}(n, \mathcal{F}, C),$$

where $\phi_{a,b}(n, \mathcal{F}, C)$ is the formula that states that $\mathcal{F}$ is a collection $\{(x^1, b^1), \ldots, (x^\ell, b^\ell)\}$ with $\ell \leq n^a$, where $|x^i| = n$ and $|b^i| = 1$, and that if C is a circuit on n variables and of size $\leq n^b$, then there is some $i \in [\ell]$ such that $C(x^i) \neq b^i$ (we can move the existential quantifier at the front of the formula).

Thus, by the KPT Theorem (Theorem 5), there are $\mathsf{FP}^{\mathsf{NP}}$ functions $f_1, \ldots, f_k$, where k is a fixed constant, such that

$$\mathbb{N} \models \phi_{a,b}(n, f_1(1^{(n)}), C_1) \vee \phi_{a,b}(n, f_2(1^{(n)}, C_1), C_2) \vee \cdots \vee \phi_{a,b}(n, f_k(1^{(n)}, C_1, \ldots, C_{k-1}), C_k). \quad (2)$$

From the relation above, we can see that one of the functions $f_1, \ldots, f_k$ will output a collection that refutes every circuit of size $\leq n^b$. If it is not f_1, then there is a counterexample circuit C_1, which is used as extra input in f_2 and so on. Since $f_1, \ldots, f_k$ are in $\mathsf{FP}^{\mathsf{NP}}$, we can simulate this procedure in a P^{NP} Turing machine M:

```
  Input: A bit-string $x$
1 Define $n \triangleq |x|$;
2 for $i = 1, \ldots, k$ do
3     Simulate $f_i$ with input $1^{(n)}$ and, if $i > 1$, $C_1, \ldots, C_{i-1}$. Interpret the output as a collection
        $\mathcal{F} = \{(x^1, b^1), \ldots, (x^\ell, b^\ell)\}$ with $\ell = n^a$;
4     Check with an NP oracle whether there exists a circuit $C$ of size $\leq n^b$, such that $C(x^i) = b^i$ for
        all $i \in [\ell]$;
5     If not or if $i = k$, exit the for-loop with the current $\mathcal{F}$;
6     If there is such a circuit, then use the NP oracle to find it and name it $C_i$.
7 end
8 If the pair $(x, 1)$ is in the collection $\mathcal{F}$, then accept. Else reject.
```

Algorithm 2: The Turing machine $M_{a,b}$, whose language is hard for n^b-size circuits.

It is easy to see that the language $L(M_{a,b})$ recognised by the Turing machine $M_{a,b}$, is in P^{NP}. It suffices to show that $L(M_{a,b}) \notin \mathsf{SIZE}[n^b]$.

Consider a circuit $C \in \mathsf{CIRCUIT}[n^b]$. We will show that it fails to recognise $L(M_{a,b})$. Assume that the for-loop in Algorithm 2 ends in the r-th iteration with $r \leq k$. We fix the circuits $C_1, C_2, \ldots, C_{r-1}$ found by the algorithm. Then the formula $\phi_{a,b}(n, f_r(1^{(n)}, C_1, \ldots, C_{r-1}), C)$ always holds. If $r < k$ and C did not satisfy it, then the NP oracle would find C as a counterexample and it would continue to the $(r+1)$-th iteration. If $r = k$, then by the construction of $C_1, C_2, \ldots, C_{k-1}$, the formulas $\phi_{a,b}(n, f_i(1^{(n)}, C_1, \ldots, C_{i-1}), C_i)$ for $i < k$ do not hold, which means by Equation (2) that $\phi_{a,b}(n, f_k(1^{(n)}, C_1, \ldots, C_{k-1}), C)$ is true.

Since $\mathcal{F} \equiv f_r(1^{(n)}, C_1, \ldots, C_{r-1})$, from $\phi_{a,b}(n, \mathcal{F}, C)$ we get that there is some $i \in [\ell]$, such that $C(x^i) \neq b^i$. However, if $b^i = 1$, then $x^i \in L(M_{a,b})$, and if $b^i = 0$, then $x^i \notin L(M_{a,b})$. In both cases, the circuit C fails to recognise the language $L(M_{a,b})$, and the proof is complete. □

3.3 Formalization in T^2_2

In this section, we prove Theorem 1 Item (*i*) and Theorem 2 Item (*i*). To achieve this, we show that the succinct circuit size hierarchy is provable in APC_1, which is contained in T^2_2. We then observe that the circuit size hierarchy is easily provable from the succinct circuit size hierarchy.

Theorem 9. *For every choice of rationals $a > b > 1$ and for every large enough $n_0 \in \mathbb{N}$,*

$$\mathsf{APC}_1 \vdash \mathsf{SCSH}[a, b, n_0]\,.$$

In particular, $\mathsf{SCSH}[a, b, n_0]$ is provable in T^2_2.

Proof. We define the polynomial-time function, f, which takes as input the description of a circuit, C, of size n^b, which means that the length of the description of C is $9n^b \log n^b$, and outputs a bit string y of length n^a with the property that for all $i = 0, 1, \ldots, n^a - 1$, $y_i = C(i)$.

The correctness of the polynomial-time algorithm f is provable in PV_1. In other words,

$$\mathsf{PV}_1 \vdash \forall n \in \mathsf{Log}\ (\,|x| \leq 9n^b \log n^b \wedge |y| \leq n^a\,) \rightarrow [\,|f(x)| \leq n^a \wedge (f(x) = y \leftrightarrow \forall i < n^a\ y_i = \mathsf{Eval}(x, i))]. \tag{3}$$

The quantifier $\forall i \leq n^a$ is sharply bounded, so this formula is provable in PV_1.

The theory APC_1 includes the dWPHP axiom for all PV functions with input length n and output length $n+1$, or equivalently input length n and output length m with $n < m$. From the first part of Equation (3), the input length of f is $9n^b \log n^b$, while the output length is n^a. Furthermore, it is provable in PV_1 that there is some constant n_0, such that $\forall n \geq n_0\ n^a > 9n^b \log n^b$. Therefore, we can use the axiom:

$$\mathsf{dWPHP}(f) \triangleq \forall n \geq n_0\ \exists y\ (|y| = n^a)\ \forall x\ (|x| = 9n^b \log n^b)\ f(x) \neq y \tag{4}$$

Every circuit of size n^b can be described by a string of size $9n^b \log n^b$, which means that

$$\forall C \in \mathsf{CIRCUIT}[n^b]\ |C| \leq 9n^b \log n^b.$$

Also, from the second part of Equation (3), using the notation for the circuit C, we get that

$$f(C) \neq y \leftrightarrow \exists i < n^a\ C(i) \neq y_i.$$

Substituting the last two relations to Equation (4), we get that

$$\mathsf{APC}_1 \vdash \forall n \geq n_0 \in \mathsf{Log}\ \exists y\ (|y| = n^a)\ \forall C \in \mathsf{CIRCUIT}[n^b]\ \exists i < n^a\ C(i) \neq y_i,$$

which is equivalent with $\mathsf{SCSH}[a, b, n_0]$, if we interpret y as the collection $\mathcal{F}_y \triangleq \{(0, y_0), (1, y_1), \ldots\}$. □

Corollary 10. *For every choice of rationals $a > b > 1$ and for every large enough $n_0 \in \mathbb{N}$,*

$$\mathsf{T}^2_2 \vdash \mathsf{CSH}[a, b, n_0]\,.$$

Proof. Since $a > b$, there is some rational $\epsilon > 0$, such that $a - \epsilon > b$. From Theorem 9, we have got a collection $\mathcal{F} = \{(x^1, b^1), \ldots, (x^\ell, b^\ell)\}$ of size $\ell \leq n^{a-\epsilon}$, such that for all circuits C of size less than n^b, there exists $i \in [\ell]$ such that $C(x^i) \neq b^i$. So, we only need to prove that

$$\mathsf{PV}_1 \vdash \exists \text{ circuit } D\colon \{0,1\}^n \to \{0,1\} \text{ of size} \leq n^a,\ \forall i \in [\ell]\ D(x^i) = b^i,$$

and then we can easily deduce that $\mathsf{APC}_1 \vdash \mathsf{CSH}[a, b, n_0]$. The same holds also for T^2_2.

It is sufficient to argue in PV_1 that there is a polynomial-time function $\mathsf{Circuit}(\mathcal{F})$ such that given the collection $\mathcal{F}$ from Theorem 9 outputs a circuit $D\colon \{0,1\}^n \to \{0,1\}$ of the required size such that $\forall i \in [\ell]\ D(x^i) = b^i$. In order to optimize the circuit size, we use that the obtained collection has a specific structure. More precisely, we have that for any $i \in [\ell]$, the strings x^i is the n-bit binary representation of the integer $i - 1$. Therefore, we can construct the circuit D in the following way: For every n-bit string x^i such that $(x^i, 1) \in \mathcal{F}$, we construct the term T^i, which is the conjunction of the first $|\ell|$ least significant bits of x^i (we put the literal z_j if the j-th bit of x^i is 1 and $\neg z_j$ if the j-th bit of x^i is 0, where $j \leq |\ell|$). Then we make the DNF

$$D \triangleq \bigvee_{(x^i,1)\in\mathcal{F}} T^i.$$

It is easy to see that D agrees with all the pairs of the collection $\mathcal{F}$. For an arbitrary pair (x^i, b^i), if $b^i = 1$, then the bits of x^i satisfy the term T^i, hence $D(x^i) = 1$. Otherwise, if $b^i = 0$, we know that the first $|\ell|$ least significant bits of x^i do not satisfy any term of the disjunction (since for all i, $x^i \leq \ell$), thus we get that $D(x^i) = 0$.

The DNF D can be viewed as a circuit and its correctness is easily provable in PV_1. This circuit has size at most $n^{a-\epsilon}|\ell|$ (derived by $|\ell| - 1$ $\wedge$-gates for each one of the at most $n^{a-\epsilon}$ terms and at most $n^{a-\epsilon}$ $\vee$-gates for the final disjunction), which is at most $n^{a-\epsilon}(\log n^{a-\epsilon} + 1)$. For large enough n_0, we can prove that $\forall n \geq n_0,\ n^{a-\epsilon}(\log n^{a-\epsilon} + 1) \leq n^a$, hence we have the desired result. ☐

3.4 On the Gap Between T^1_2 and T^2_2

We noticed above that it is possible to prove the circuit size hierarchy in the theory T^2_2. In contrast, it seems difficult to implement a similar proof in the theory T^1_2. The reason behind this difficulty is connected to the proof complexity of the dual Weak Pigeonhole Principle. If there is a proof of the circuit size hierarchy in T^1_2, either it uses an approach that relies on a principle that is not equivalent to $\mathsf{dWPHP(PV)}$, or $\mathsf{dWPHP(PV)}$ is also provable in T^1_2.

Paris, Wilkie, and Woods [PWW88] were the first to establish the provability of $\mathsf{dWPHP(PV)}$ in Buss's hierarchy. Subsequently, Maciel, Pitassi, and Woods [MPW02] provided an alternative proof with an explicit inclusion of the principle in T^2_2. In this section, we explain why the same argument is not available in T^1_2. (Their original proof is more general, and an exposition can be found in [Kra19].)

Assume that we have a PV-function $g'\colon \{0,1\}^n \to \{0,1\}^{n+1}$ with $n \in \mathsf{Log}$ or equivalently $g'\colon N \to 2N$, such that $\neg\mathsf{dWPHP}^{2N}_N(g')$ holds. It is easy even in $\mathsf{S}^1_2(\mathsf{PV})$ to extend this to a new function $g\colon N \to N^2$, such that $\neg\mathsf{dWPHP}^{N^2}_N(g) \triangleq \forall y < N^2\, \exists x < N\ g(x) = y$ holds.

For $\ell = 0, \ldots, |N|$, we consider all sequences $w \in [N]^\ell$. We extend a sequence by a new element using the operation $\frown$ (e.g., $(a_1, a_2, a_3) \frown a_4 = (a_1, a_2, a_3, a_4)$). For all sequences w, we define functions $g_w\colon N/2^\ell \to N^2$ recursively as follows:

- If $\ell = 0$, $g_\emptyset = g$.
- For $i < N$, $g_{w \frown i}(x) = y$ if $\exists z < N$ such that $g(z) = y \wedge g_w(x) = iN + z$, otherwise output $\emptyset$. (Here $\emptyset$ is just a fixed symbol that we use to denote "error" or that the function is undefined.)
- $g_{w \frown N}(x) = y$ if $\exists z < N\, \exists u < N$ such that $g(z) = y \wedge g_w(x + N/2^{\ell+1}) = zN + u$, otherwise output $\emptyset$.

Note that the formula $g_w(x) = y$ is Σ^b_1-definable and that $g_w(x)$ cannot have more than one value.

The key step of the proof is showing that

$$\mathsf{S}^3_2 \vdash \neg\mathsf{dWPHP}^{N^2}_N(g) \rightarrow \exists w \in [N]^\ell\ \neg\mathsf{dWPHP}^{N^2}_{N/2^l}(g_w). \tag{5}$$

The right-hand size can be also written as

$$\exists w \in [N]^\ell\ \forall y < N^2\ \exists x < N/2^\ell\ g_w(x) = y,$$

which is a Σ^b_3 formula. Therefore, for the proof of Equation (5), we use Σ^b_3-LIND, which is available in S^3_2. The intuition behind the inductive step is that if we split the domain into two equal intervals and the range into N intervals, from the surjectivity of g and g_w, either the first domain interval has all its values into the ith range interval, which gives us the new sequence $w \frown (i-1)$, or the second domain interval has value at each one of the range intervals, which gives us the new sequence $w \frown N$.

To complete the argument, plugging $\ell = |N|$ in Equation (5), we get a surjective function from 1 to N^2, which is a clear contradiction when $N > 1$. Therefore, $\mathsf{S}^3_2 \vdash \mathsf{dWPHP}(g)$, and since S^3_2 is $\forall\Sigma^b_3$-conservative over T^2_2, we also have $\mathsf{T}^2_2 \vdash \mathsf{dWPHP}(g)$.

The bottleneck to implement the proof in T^1_2 is the quantifier complexity of the inductive statement associated with Equation (5). Another barrier for such a proof in T^1_2 is the fact that for an arbitrary relation R, $\mathsf{dWPHP}(R)$ is not provable in $\mathsf{S}^2_2(R)$ [Kra92], so a proof of $\mathsf{dWPHP}(\mathsf{PV})$ has to use some properties of PV functions.

4 Provability of Formula Size Bounds in PV_1

In this section, we prove Theorem 3. To achieve this, we establish that:

1. The parity function on n bits requires formulas of size $\geq n^{3/2}$ (Section 4.1).
2. The parity function on n bits can be computed by formulas of size $O(n^2) \leq n^a$ for any fixed rational $a > 2$ and large enough n (Section 4.2).
3. Consequently, the formula size hierarchy holds with parameters $a > 2$ and $b = 3/2$, provided that n_0 is large enough (Section 4.3).

4.1 Subbotovskaya's Lower Bound

4.1.1 High-Level Details of the Formalization

In this section, we sketch a formalization in PV_1 of the proof that the parity function on n bits requires Boolean formulas of size $\geq n^{3/2}$ [Sub61].[3] We adapt the argument presented in [Juk12, Section 6.3], which

[3]For concreteness, we let the size of a Boolean formula F be the number of leaves of F labeled by an input literal. We allow leaves that are labeled by constants, but we do not charge for them. Consequently, a constant function has formula complexity 0, while a non-constant function has formula complexity at least 1.

proceeds as follows:

1. [Juk12, Lemma 6.8]: Given a Boolean formula F on n-bit inputs, it is possible to fix one of its variables so that the resulting formula F_1 satisfies

$$\mathsf{Size}(F_1) \leq (1 - 1/n)^{3/2} \cdot \mathsf{Size}(F).$$

   (In order to pick the variable to be restricted and its value, one first "normalizes" the formula F, as implicitly described in [Juk12, Claim 6.9].)

2. [Juk12, Theorem 6.10]: By applying this result $\ell \triangleq n - k$ times, it is possible to obtain a formula F_ℓ on k-bit inputs such that

$$\mathsf{Size}(F_\ell) \leq \mathsf{Size}(F) \cdot (1 - 1/n)^{3/2} \cdot (1 - 1/(n-1))^{3/2} \ldots (1 - 1/(k+1))^{3/2} = \mathsf{Size}(F) \cdot (k/n)^{3/2}.$$

3. [Juk12, Example 6.11]: If the initial formula F computes the parity function, by setting $\ell = n - 1$ we obtain

$$1 \leq \mathsf{Size}(F_\ell) \leq (1/n)^{3/2} \cdot \mathsf{Size}(F),$$

   and consequently $\mathsf{Size}(F) \geq n^{3/2}$.

We recommend reading this section with [Juk12, Section 6.3] at hand. We will slightly modify the argument when formalizing the lower bound in PV_1. In more detail, given a small formula F, we recursively construct (and establish correctness by induction) an n-bit input y witnessing that F does not compute the parity function. (Actually, for technical reasons related to the induction step, we will simultaneously construct an n-bit input y_n^0 witnessing that F does not compute the parity function and an n-bit input y_n^1 witnessing that F does not compute the negation of the parity function.)

Let $s(n)$ be a size bound and $\oplus(x)$ be a PV function that computes the parity of the binary string described by x, i.e., $\oplus(x) \triangleq x_1 \oplus x_2 \oplus \ldots \oplus x_n$, where x_i denotes the i-th bit of x. To simplify notation, we tacitly view x as a binary string. We assume that the formalization employs a well-behaved function symbol $\oplus$ such that PV_1 proves the basic properties of the parity function, e.g., $\mathsf{PV}_1 \vdash \oplus(x1) = 1 - \oplus(x)$ and $\mathsf{PV}_1 \vdash \oplus(x0) = \oplus(x)$.

We consider the following $\mathcal{L}_{\mathsf{PV}}$-sentence stating that the parity function requires formulas of size at least $s(n)$ for every input length $n \geq 1$:

$$\mathsf{FLB}_s \triangleq \forall N\, \forall n\, \forall F\, (n = |N| \geq 1 \wedge \mathsf{Size}(F) < s(n) \rightarrow \exists x\, (|x|_\ell = n \wedge \mathsf{Eval}(F, x) \neq \oplus(x)) \,, ^{4}$$

where for convenience of notation we use the function symbol $|w|_\ell$ to compute the bit-length of the string represented by w (under some reasonable encoding).

Theorem 11. *Let* $s(n) \triangleq n^{3/2}$. *Then* $\mathsf{PV}_1 \vdash \mathsf{FLB}_s$.

Proof. Given $b \in \{0, 1\}$, we introduce the function $\oplus^b(x) \triangleq \oplus(x) + b \pmod 2$. In order to prove FLB_s in PV_1, we explicitly consider a polynomial-time function $R(1^{(n)}, F, b)$ with the following properties:[5]

1. Let $b \in \{0, 1\}$.

[4]To simplify notation, we ommit from the sentence FLB_s and in other parts of the exposition certain straightforward conditions, such as checking that F represents a valid formula and that it computes over n-bit input strings.

[5]For convenience, we often write $1^{(n)}$ instead of explicitly considering parameters N and $n = |N|$. We might also write just $F(x)$ instead of $\mathsf{Eval}(F, x)$.

2. If $\mathsf{Size}(F) < s(n)$ then $R(1^{(n)}, F, b)$ outputs an n-bit string y^b_n such that $\mathsf{Eval}(F, y^b_n) \neq \oplus^b(y^b_n)$.

In other words, $R(1^{(n)}, F, b)$ witnesses that the formula F does not compute the function $\oplus^b$ over n-bit strings. Note that the correctness of R is captured by the bounded universal sentence:

$$\mathsf{Ref}_{R,s} \triangleq \forall 1^{(n)}\, \forall F\, (\mathsf{Size}(F) < s(n) \rightarrow |y^0_n|_\ell = |y^1_n|_\ell = n \wedge F(y^0_n) \neq \oplus^0(y^0_n) \wedge F(y^1_n) \neq \oplus^1(y^1_n))\,,$$

where we employed the abbreviations $y^0_n \triangleq R(1^{(n)}, F, 0)$ and $y^1_n \triangleq R(1^{(n)}, F, 1)$. Our plan is to define R and show that $\mathsf{PV}_1 \vdash \mathsf{Ref}_{R,s}$. Note that this implies FLB_s in PV_1. Jumping ahead, the correctness of $R(1^{(n)}, F, b)$ will be established by polynomial induction on N (equivalently, induction on $n = |N|$). Since $\mathsf{Ref}_{R,s}$ is a universal sentence and S^1_2 is $\forall\Sigma^b_1$-conservative over PV_1, polynomial induction for NP and coNP predicates (admissible in S^1_2; see, e.g., [Kra95, Section 5.2]) is available during the formalization. More details follow.

The procedure $R(1^{(n)}, F, b)$ makes use of a few polynomial-time sub-routines (discussed below) and is defined in the following way:

```
  Input: $1^{(n)}$ for some $n \geq 1$, formula $F$ over $n$-bit inputs, $b \in \{0,1\}$.
1 Let $s(n) \triangleq n^{3/2}$. If $\mathsf{Size}(F) \geq s(n)$ return "error";
2 If $\mathsf{Size}(F) = 0$, $F$ computes a constant function $b_F \in \{0,1\}$. In this case, return the $n$-bit string
    $y^b_n \triangleq y^b_1 0^{n-1}$ such that $\oplus^b(y^b_1 0^{n-1}) \neq b_F$;
3 Let $\widetilde{F} \triangleq \mathsf{Normalize}(1^{(n)}, F)$;
  // $\widetilde{F}$ satisfies [Juk12, Claim 6.9], $\mathsf{Size}(\widetilde{F}) \leq \mathsf{Size}(F)$,
      $\forall x \in \{0,1\}^n\ F(x) = \widetilde{F}(x)$.
4 Let $\rho \triangleq \mathsf{Find\text{-}Restriction}(1^{(n)}, \widetilde{F})$, where $\rho\colon [n] \to \{0,1,\star\}$ and $|\rho^{-1}(\star)| = n-1$;
  // $\rho$ restricts a suitable variable $x_i$ to a bit $c_i$, as in [Juk12,
      Lemma 6.8].
5 Let $F' \triangleq \mathsf{Apply\text{-}Restriction}(1^{(n)}, \widetilde{F}, \rho)$. Moreover, let $b' \triangleq b \oplus c_i$ and $n' \triangleq n-1$;
  // $F'$ is an $n'$-bit formula; $\forall z \in \{0,1\}^{\rho^{-1}(\star)}\ F'(z) = \widetilde{F}(z \cup x_i \mapsto c_i)$.
6 Let $y^{b'}_{n'} \triangleq R(1^{n'}, F', b')$ and return the $n$-bit string $y^b_n \triangleq y^{b'}_{n'} \cup y_i \mapsto c_i$;
```

Algorithm 3: Refuter Algorithm $R(1^{(n)}, F, b)$.

$\mathsf{Normalize}(1^{(n)}, F)$ **and its properties (in S^1_2).** We say that a subformula G of F is a *neighbor* of a leaf z if either $z \wedge G$ or $z \vee G$ is a subformula of F. We say that a formula F over variables $\{x_1, \ldots, x_n\}$ is in *normal form* if for every $i \in [n]$ and every literal $z \in \{x_i, \overline{x_i}\}$, if z is a leaf of F and G is a neighbor of z in F, then G does not contain the variable x_i.

Lemma 12. *There is a polynomial-time function* $\mathsf{Normalize}(1^{(n)}, F)$ *that given a Boolean formula F over n input variables, outputs a formula $\widetilde{F}$ over n input variables such that the following holds:*

(*i*) $\mathsf{Size}(\widetilde{F}) \leq \mathsf{Size}(F)$.

(*ii*) *For every input $x \in \{0,1\}^n$, $\widetilde{F}(x) = F(x)$.*

(*iii*) *$\widetilde{F}$ is in normal form.*

(*iv*) *$\widetilde{F}$ is either a constant 0 or 1, or $\widetilde{F}$ contains no leaves labeled by constants 0 and 1.*

Moreover, the correctness of $\mathsf{Normalize}(1^{(n)}, F)$ *is provable in* S^1_2.

Proof Sketch. It is enough to verify that the proof of [Juk12, Claim 6.9] provides such a polynomial-time function and that its correctness can be established in S^1_2. In more detail, if F is not in normal form, we can efficiently compute a literal $z \in \{x_i, \overline{x_i}\}$ and a neighbor G of z that violates the corresponding property. As shown in [Juk12, Claim 6.9], we can fix any leaf $z' \in \{x_i, \overline{x_i}\}$ in G by an appropriate constant c so that the resulting formula F_1 satisfies conditions (*i*) and (*ii*) of Lemma 12. After at most $\ell \triangleq \mathsf{Size}(F)$ iterations, we obtain a sequence $F_1, \ldots, F_\ell$ of formulas such that $\widetilde{F} \triangleq F_\ell$ satisfies conditions (*i*), (*ii*), and (*iii*) of the lemma. Moreover, condition (*iv*) can always be guaranteed by simplifying the final formula, i.e., by replacing subformulas $0 \vee G$ by G, $1 \vee G$ by 1, $0 \wedge G$ by 0, and $1 \wedge G$ by G. The correctness of $\widetilde{F} \triangleq \mathsf{Normalize}(1^{(n)}, F)$ can be established by polynomial induction for coNP predicates (i.e., Π^b_1 formulas), which is available in S^1_2. $\square$

$\mathsf{Find\text{-}Restriction}(1^{(n)}, \widetilde{F})$ **and its properties (in S^1_2).** We argue in S^1_2 and follow the argument from the proof of [Juk12, Lemma 6.8]. Let $\widetilde{F}$ be a formula over n input variables in normal form. We focus on the non-trivial case, and assume that $n \geq 2$, $\mathsf{Size}(\widetilde{F}) \geq 2$, and that $\widetilde{F}$ contains no leaves labeled by constants. Let $\mathsf{Count}(1^{(n)}, F, i)$ be a polynomial-time algorithm that outputs the number of leaves of F that contain the variable x_i (including its appearances as $\overline{x_i}$). Let $w = (w_1, \ldots, w_n)$ be the corresponding sequence of multiplicities, i.e., $w_i \triangleq \mathsf{Count}(1^{(n)}, F, i)$. Note that $\sum_i w_i = \widetilde{s}$, where $\widetilde{s} \triangleq \mathsf{Size}(\widetilde{F})$.

We claim that S^1_2 proves the existence of an index $i \in [n]$ such that $w_i \geq \widetilde{s}/n$. First, for each $j \in [n]$, we define the cumulative sum $v_j \triangleq \sum_{i \leq j} w_j$. Let $v \triangleq (v_0, v_1, \ldots, v_n)$ be the corresponding sequence, where we set $v_0 \triangleq 0$. Notice that $v_n = \widetilde{s}$. Since v contains $n+1$ elements, it can be efficiently computable from w. We now argue by induction on n that for some index $j \in [n]$ we have $v_j - v_{j-1} \geq v_n/n$. This implies that $w_j = v_j - v_{j-1} \geq v_n/n = \widetilde{s}/n$, as desired.

If $n = 1$, then $v_1 - v_0 = v_1 = v_1/1$ and the result holds for $j = 1$. Assume the result holds for $n-1$, and consider v_n. If $v_n - v_{n-1} \geq v_n/n$, we can pick $j = n$ and we are done. Otherwise, $v_{n-1} \geq v_n - v_n/n = v_n(n-1)/n$. By the induction hypothesis, there is an index $j \in [n-1]$ such that $v_j - v_{j-1} \geq v_{n-1}/(n-1)$. Using the lower bound on v_{n-1}, we get that $v_j - v_{j-1} \geq v_n/n$, which concludes the proof.

Consequently, S^1_2 proves the existence of a variable x_i which appears $t \geq \widetilde{s}/n$ times as a leaf of $\widetilde{F}$. Let $z_1, \ldots, z_t$ be the leaves of $\widetilde{F}$ labeled by either x_i or $\overline{x_i}$. Recall that we assume that $n \geq 2$, $\mathsf{Size}(\widetilde{F}) \geq 2$, and that $\widetilde{F}$ satisfies conditions (*iii*) and (*iv*) of Lemma 12. Therefore, each leaf z_j has a neighbor subformula G_j in $\widetilde{F}$ that contains some leaf labeled by a literal not in $\{x_i, \overline{x_i}\}$. For this reason, if we set x_i to an appropriate constant c_j, G_j will disappear from F, thereby erasing at least another leaf not among $z_1, \ldots, z_t$. As in the proof of [Juk12, Lemma 6.8], if we let $c \in \{0, 1\}$ be the constant that appears more often among $c_1, \ldots, c_t$ and set $x_i \mapsto c$ in the restriction ρ, all the leaves $z_1, \ldots, z_t$ will be eliminated from $\widetilde{F}$ together with at least $t/2$ additional leaves.[6] Thus the total number of eliminated leaves, which we specify using a polynomial-time function $\mathsf{NumRemoved}(1^{(n)}, \widetilde{F}, \rho)$, satisfies

$$\mathsf{NumRemoved}(1^{(n)}, \widetilde{F}, \rho) \geq t + \frac{t}{2} \geq \frac{3\widetilde{s}}{2n}.$$

Overall, it follows that

$$\mathsf{S}^1_2 \vdash \widetilde{F} = \mathsf{Normalize}(1^{(n)}, F) \wedge \rho = \mathsf{Find\text{-}Restriction}(1^{(n)}, \widetilde{F}) \rightarrow \mathsf{NumRemoved}(1^{(n)}, \widetilde{F}, \rho) \geq \frac{3}{2n} \cdot \mathsf{Size}(\widetilde{F})\,.$$

[6]The existence of such a constant c can be proved in S^1_2 in a way that is similar to the proof that some variable x_i appears in at least $\widetilde{s}/n$ leaves.

$\mathsf{Apply\text{-}Restriction}(1^{(n)}, \widetilde{F}, \rho)$ **and its properties (in S^1_2).** We only sketch the details. This is simply a polynomial-time algorithm that, given a formula $\widetilde{F}$ on n input variables and a restriction $\rho\colon [n] \to \{0, 1, *\}$ with $|\rho^{-1}(\star)| = n - 1$ (i.e., ρ restricts a single variable x_i to a constant $c_i \in \{0, 1\}$), outputs a formula F' over $n - 1$ input variables that sets every literal $z \in \{x_i, \overline{x_i}\}$ to the corresponding constant and simplifies the resulting formula, e.g., replaces subformulas $0 \vee G$ by G, $1 \vee G$ by 1, $0 \wedge G$ by 0, and $1 \wedge G$ by G. Additionally, for $F' = \mathsf{Apply\text{-}Restriction}(1^{(n)}, \widetilde{F}, \rho)$, we have

$$\mathsf{S}^1_2 \vdash \mathsf{Size}(F') \leq \mathsf{Size}(\widetilde{F}) - \mathsf{NumRemoved}(1^{(n)}, \widetilde{F}, \rho) \ \wedge\ \forall z \in \{0,1\}^{\rho^{-1}(\star)}\ F'(z) = \widetilde{F}(z \cup x_i \mapsto c_i)\,. \quad (6)$$

Using the previously computed bound on $\mathsf{NumRemoved}(1^{(n)}, \widetilde{F}, \rho)$ for $\rho = \mathsf{Find\text{-}Restriction}(1^{(n)}, \widetilde{F})$, we obtain that for $\widetilde{F}$ and F' defined as above (with $s' \triangleq \mathsf{Size}(F')$ and $\widetilde{s} \triangleq \mathsf{Size}(\widetilde{F})$), and assuming that $n \geq 2$,

$$\mathsf{S}^1_2 \vdash s' \leq \widetilde{s} - \frac{3}{2n} \cdot \widetilde{s} = \widetilde{s} \cdot \left(1 - \frac{3}{2n}\right) \leq \widetilde{s} \cdot \left(1 - \frac{1}{n}\right)^{3/2}. \quad (7)$$

The last inequality uses that $\mathsf{S}^1_2 \vdash \forall a,\ a \geq 2 \to (1 - 3/(2a))^2 \leq (1 - 1/a)^3$, which one can easily verify.

Note that $R(1^{(n)}, F, b)$ runs in time polynomial in $n + |F| + |b|$ and that it is definable in S^1_2. Next, we establish the correctness of $R(1^{(n)}, F, b)$ in S^1_2.

Lemma 13. *Let* $s(n) \triangleq n^{3/2}$. *Then* $\mathsf{S}^1_2 \vdash \mathsf{Ref}_{R,s}$.

Proof. We consider the formula $\varphi(N)$ defined as

$$\forall F\, \forall n\, (n = |N| \wedge n \geq 1 \wedge \mathsf{Size}(F) < s(n)) \to (|y^0_n|_\ell = |y^1_n|_\ell = n \wedge F(y^0_n) \neq \oplus^0(y^0_n) \wedge F(y^1_n) \neq \oplus^1(y^1_n))\,,$$

where as before we use $y^0_n \triangleq R(1^{(n)}, F, 0)$ and $y^1_n \triangleq R(1^{(n)}, F, 1)$. Note that $\varphi(N)$ is a Π^b_1 formula. Below, we argue that

$$\mathsf{S}^1_2 \vdash \varphi(1) \quad \text{and} \quad \mathsf{S}^1_2 \vdash \forall N\, \varphi(\lfloor N/2 \rfloor) \to \varphi(N)\,.$$

Then, by polynomial induction for Π^b_1 formulas (available in S^1_2) and using that $\varphi(0)$ trivially holds, it follows that $\mathsf{S}^1_2 \vdash \forall N\, \varphi(N)$. In turn, this yields $\mathsf{S}^1_2 \vdash \mathsf{Ref}_{R,s}$.

Base Case: $\mathsf{S}^1_2 \vdash \varphi(1)$. In this case, for a given formula F and length n, the hypothesis of $\varphi(1)$ is satisfied only if $n = 1$ and $\mathsf{Size}(F) = 0$. Let $y^0_1 \triangleq R(1, F, 0)$ and $y^1_1 \triangleq R(1, F, 1)$. We need to prove that

$$|y^0_1|_\ell = |y^1_1|_\ell = 1 \wedge F(y^0_1) \neq \oplus^0(y^0_1) \wedge F(y^1_1) \neq \oplus^1(y^1_1)\,.$$

Since $n = 1$ and $\mathsf{Size}(F) = 0$, F evaluates to a constant b_F on every input bit. The statement above is implied by Line 2 in the definition of $R(n, F, b)$.

(Polynomial) Induction Step: $\mathsf{S}^1_2 \vdash \forall N\, \varphi(\lfloor N/2 \rfloor) \to \varphi(N)$. Fix an arbitrary N, let $n \triangleq |N|$, and assume that $\varphi(\lfloor N/2 \rfloor)$ holds. By the induction hypothesis, for every formula F' with $\mathsf{Size}(F') < n'^{3/2}$, where $n' \triangleq n - 1$, we have

$$|y^0_{n'}|_\ell = |y^1_{n'}|_\ell = n' \ \wedge\ F'(y^0_{n'}) \neq \oplus^0(y^0_{n'}) \ \wedge\ F'(y^1_{n'}) \neq \oplus^1(y^1_{n'})\,, \quad (8)$$

where $y^0_{n'} \triangleq R(1^{n'}, F', 0)$ and $y^1_{n'} \triangleq R(1^{n'}, F', 1)$.

Now let $n \geq 2$, and let F be a formula over n-bit inputs of size $< n^{3/2}$. By the size bound on F, $R(1^{(n)}, F, b)$ ignores Line 1. If $\mathsf{Size}(F) = 0$, then similarly to the base case it is trivial to check that the conclusion of $\varphi(N)$ holds. Therefore, we assume that $\mathsf{Size}(F) \geq 1$ and $R(1^{(n)}, F, b)$ does not stop at Line 2. Let $\widetilde{F} \triangleq \mathsf{Normalize}(1^{(n)}, F)$ (Line 3), $\rho \triangleq \mathsf{Find\text{-}Restriction}(1^{(n)}, \widetilde{F})$ (Line 4), $F' \triangleq \mathsf{Apply\text{-}Restriction}(1^{(n)}, \widetilde{F}, \rho)$ (Line 5), $n' \triangleq n - 1$ (Line 5), and $b' \triangleq b \oplus c_i$ (Line 5), where ρ restricts the variable x_i to the bit c_i. Moreover, for convenience, let $s \triangleq \mathsf{Size}(F)$, $\widetilde{s} \triangleq \mathsf{Size}(\widetilde{F})$, and $s' \triangleq \mathsf{Size}(F')$. By Lemma 12 Item (*i*), Equation (7), and the bound $s < n^{3/2}$,

$$\mathsf{S}^1_2 \vdash s' \leq \widetilde{s} \cdot (1 - 1/n)^{3/2} \leq s \cdot (1 - 1/n)^{3/2} < n^{3/2} \cdot (1 - 1/n)^{3/2} = (n-1)^{3/2} .$$

Thus F' is a formula on n'-bit inputs of size $< n'^{3/2}$. Recall that for a given $b \in \{0,1\}$ we have $b' = b \oplus c_i$. Let $y^{b'}_{n'} \triangleq R(1^{n'}, F', b')$ (Line 6). By the first condition in the induction hypothesis (Equation (8)) and the definition of each $y^b_n \triangleq y^{b'}_{n'} \cup y_i \mapsto c_i$, we have $|y^0_n|_\ell = |y^1_n|_\ell = n$. Below, we also rely on the last two conditions in the induction hypothesis (Equation (8)), Lemma 12 Item (*ii*), and the last condition in Equation (6). We derive the following statements, where $b \in \{0,1\}$:

$$\begin{array}{rcl} F'(y^{b'}_{n'}) & \neq & \oplus^{b'}(y^{b'}_{n'}) \, , \\ F(y^b_n) & = & F'(y^{b'}_{n'}) \, , \\ F(y^b_n) & \neq & \oplus^{b'}(y^{b'}_{n'}) \, . \end{array}$$

Notice that

$$\oplus^{b'}(y^{b'}_{n'}) = \oplus^{b \oplus c_i}(y^{b'}_{n'}) = c_i \oplus (\oplus^b(y^{b'}_{n'})) = c_i \oplus (\oplus^b(y^b_n) \oplus c_i) = \oplus^b(y^b_n) \, .$$

These statements imply that, for each $b \in \{0,1\}$, $F(y^b_n) \neq \oplus^b(y^b_n)$. In other words, the conclusion of $\varphi(N)$ holds. This completes the proof of the induction step. $\square$

As explained above, the provability of $\mathsf{Ref}_{R,s}$ in S^1_2 implies its provability in PV_1. Since $\mathsf{PV}_1 \vdash \mathsf{Ref}_{R,s} \rightarrow \mathsf{FLB}_s$, this completes the proof of Theorem 11. $\square$

4.1.2 On the Low-Level Details of the Formalization

In order to make our presentation accessible to a broader audience, in this section we provide more details about the formalization of algorithms and about the proofs of their basic properties. For concreteness and convenience, we consider the theory $\mathsf{S}^1_2(\mathsf{PV})$, i.e., S^1_2 extended with function symbols and axioms for all polynomial-time functions as in Cobham's characterization of efficient computations. Since this theory is $\forall\Sigma^b_1$-conservative over PV_1 (see Section 2.2.1), the provability of FLB_s in $\mathsf{S}^1_2(\mathsf{PV})$ yields its provability in PV_1.

As a concrete example, we elaborate on a sub-routine employed by some algorithms discussed in Section 4.1. We consider a polynomial-time function $\mathsf{Fix}(1^{(n)}, F, i, b)$ that, given the description of a formula F over n input variables, a variable index $i \in [n]$, and a bit $b \in \{0,1\}$, replaces every leaf of F labeled by x_i with b and every leaf of F labeled by $\overline{x_i}$ with $1 - b$, then returns the corresponding restricted formula F' over $n - 1$ input variables (without the application of formula simplification rules). Next, we provide more details about the specification of the procedure Fix in $\mathsf{S}^1_2(\mathsf{PV})$ and about a proof of its correctness, i.e.,

$$\mathsf{S}^1_2(\mathsf{PV}) \vdash \forall 1^{(n)} \, \forall F \, \forall F' \, \forall x \, \forall z \, \forall i \qquad (9)$$

$$(n \geq 2 \wedge |x|_\ell = n \wedge |z|_\ell = n - 1 \wedge 1 \leq i \leq n \wedge F' = \mathsf{Fix}(1^{(n)}, F, i, b)) \rightarrow (\mathsf{Eval}(F', z) = \mathsf{Eval}(F, z \cup x_i \mapsto b)) \, ,$$

where $z \cup x_i \mapsto b$ denotes a function that takes (z, i, b), where z assigns bits to $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$, and outputs the n-bit string that agrees with z and sets x_i to b.

On the Specification of $\mathsf{Fix}(1^{(n)}, F, i, b)$ in $\mathsf{S}^1_2(\mathsf{PV})$. Theory $\mathsf{S}^1_2(\mathsf{PV})$ contains function symbols for all polynomial-time algorithms according to Cobham's characterization of polynomial-time computations. Consequently, to specify $\mathsf{Fix}(1^{(n)}, F, i, b)$ we employ a definition of this computation in Cobham's formalism, i.e., we define $\mathsf{Fix}(1^{(n)}, F, i, b)$ using simple base functions together with composition and recursion on notation. In order to be completely formal (a rather cumbersome task), one would first specify how formulas are represented by numbers and the polynomial-time functions that manipulate the corresponding representation. We could then interpret the binary representation of an integer as two sequence of tuples, one describing the edges in the binary tree representation of the formula, and another describing the labels of each node of the tree. Finally, $\mathsf{Fix}(1^{(n)}, F, i, b)$ would be a routine that iterates over each leaf of F labelled by the i-th variable or its negation and replaces it with the appropriate constant. Using previously defined routines and their corresponding function symbols, a sequential algorithm of this form can be described as a recursive procedure in Cobham's characterization of polynomial-time functions. Moreover, we need to argue in the theory that the output length of the function on a given input is bounded by a polynomial, similarly to the constraint in the limited recursion on notation from Cobham's theorem.

On the Proof of the Correctness of $\mathsf{Fix}(1^{(n)}, F, i, b)$ in $\mathsf{S}^1_2(\mathsf{PV})$ (Equation (9)). $\mathsf{S}^1_2(\mathsf{PV})$ also contains axioms describing how the function symbols (polynomial-time functions) are obtained from each other. For instance, $\mathsf{Fix}(1^{(n)}, F, i, b)$ might use in its specification a routine R that takes as input a tuple describing a formula G, a bit b, and a leaf of G and its label, replaces the label of this leaf by the constant b, and outputs the new formula G'. We can then reason in $\mathsf{S}^1_2(\mathsf{PV})$ about the correctness of $\mathsf{Fix}(1^{(n)}, F, i, b)$ (as in Equation (9)) using the provable properties of R and of the function symbol Eval. In more detail, Eval can be defined recursively based on the structure of the input formula, and the base case of the proof of correctness relies on the properties of R and the fact that the internal evaluations of $\mathsf{Eval}(F', z)$ for $F' = \mathsf{Fix}(1^{(n)}, F, i, b))$ and $\mathsf{Eval}(F, z \cup x_i \mapsto b))$ agree over all leaves. Crucially, the recursive nature of the specification of polynomial-time functions in Cobham's definition and in $\mathsf{S}^1_2(\mathsf{PV})$ is compatible with the polynomial induction axioms available in $\mathsf{S}^1_2(\mathsf{PV})$, in the sense that we can define recursive procedures while simultaneously proving their relevant properties by induction.

4.2 Upper Bound

In this section, we show that the parity function on n bits can be computed by formulas of size $O(n^2)$, provably in PV_1. We can formalize this upper bound in the language of PV, defining an $\mathcal{L}_{\mathsf{PV}}$-sentence stating that the parity function can be computed by a formula of size $s(n)$ for every input length $n \geq 1$:

$$\mathsf{FUB}_s \triangleq \forall N \, \forall n \, \exists F \, (n = |N| \geq 1 \land \mathsf{Size}(F) < s(n) \land \forall x \, (|x| \leq n \to \mathsf{Eval}(F, x) = \oplus^0_n(x)) \, .$$

Theorem 14. *Let $s(n) \triangleq 4n^2$. Then $\mathsf{PV}_1 \vdash \mathsf{FUB}_s$.*

Proof. FUB_s is a $\forall\Sigma^b_2$ sentence and our intended theory is PV_1. In order to implement some inductive proofs, it will be helpful to reduce the complexity of the formula. For this, we introduce a new polynomial-time function, $\mathsf{ParForm}(1^{(n)})$, which generates the desired formula that computes the parity function on n bits. Since it is a polynomial-time function, there is a symbol for it in PV and we can use it in the new formalization:

$$\mathsf{FUB}'_s \triangleq \forall N \, \forall n \, (n = |N| \geq 1 \land \mathsf{Size}(\mathsf{ParForm}(1^{(n)})) < s(n) \land \forall x \, (|x| \leq n \to \mathsf{Eval}(\mathsf{ParForm}(1^{(n)}), x) = \oplus^0_n(x)) \, .$$

It is immediate that $\mathsf{FUB}'_s \Rightarrow \mathsf{FUB}_s$, thus we focus on proving FUB'_s. We continue with the following steps:

1. We prove an upper bound of n^2 for the formulas calculating the parity function and its negation, when n is a power of 2.

2. We use this construction to derive the $4n^2$ upper bound for any n.

Next, we define a polynomial-time algorithm $\mathsf{Par}(1^{(n)})$ which computes a formula that calculates the parity function on n bits and a formula that calculates the negation of the parity function on n bits, if n is a power of 2.

```
   Input: $1^{(n)}$ for some $n \geq 1$.
1  Let $k \triangleq |n-1|$. If $n \neq 2^k$ ($n$ is not a power of 2), then return "error";
   // $F$ will compute the parity function, while $\overline{F}$ will compute its negation
2  if $k = 0$ then
3      Define $F$ to be the formula with one leaf $x_1$ and $\overline{F}$ to be the formula with one leaf $\neg x_1$.
4  else if $k \geq 1$ then
       // Construct a pair $(F, \overline{F})$ of formulas on input bits $x_1, \ldots, x_{2^k}$ as follows:
5      Let $(F_1, \overline{F_1}) \triangleq \mathsf{Par}(1^{n/2})$, and define a corresponding pair $(F_2, \overline{F_2})$:
6      In $F_2$ and $\overline{F}_2$, relabel the leaves by putting $x_{2^{k-1}+i}$ instead of $x_i$ for every $i = 1, \ldots, 2^{k-1}$;
7      Now let $F \triangleq (F_1 \vee F_2) \wedge (\overline{F}_1 \vee \overline{F}_2)$ and $\overline{F} \triangleq (F_1 \wedge F_2) \vee (\overline{F}_1 \wedge \overline{F}_2)$.
8  end
9  return $(F, \overline{F})$.
```

Algorithm 4: $\mathsf{Par}(1^{(n)})$ outputs Boolean formulas for $\oplus_n^0$ and $\oplus_n^1$ when n is a power of 2.

Lemma 15. *If n is a power of 2, the algorithm $\mathsf{Par}(1^{(n)})$ correctly outputs two formulas $(F, \overline{F})$ of size n^2 which calculate the parity function and its negation, provably in $\mathsf{S}_2^1(\mathsf{PV})$.*

Proof. We split the proof of the correctness for the algorithm $\mathsf{Par}(1^{(n)})$ into 3 properties:

1. $\phi_1(n) \triangleq F, \overline{F} \in \mathsf{VALIDFORM}(n)$, where $\mathsf{VALIDFORM}(n)$ is the set of formulas on n variables;

2. $\phi_2(n) \triangleq \mathsf{Size}(F) = \mathsf{Size}(\overline{F}) = n^2$;

3. $\phi_3(n) \triangleq \forall x\ |x| \leq n \to \mathsf{Eval}(F, x) = \oplus_n^0(x) \wedge \mathsf{Eval}(\overline{F}, x) = \oplus_n^1(x)$.

For now we only care about the case that n is a power of 2, so we prove these properties conditionally (equivalently we prove $(n = (n-1)\#1) \to \phi(n)$).[7] That is why it suffices to use polynomial induction on n, which is available in S_2^1, since our formulas are at most Π_1^b.

We skip the proof of ϕ_1, which is proven by simple induction as below, using the fact that if F_1, F_2 are formulas then $F_1 \wedge F_2$ and $F_1 \vee F_2$ are also formulas.

Property 2: $\mathsf{S}_2^1 \vdash \phi_2(n)$. For the base case, $\phi_2(1)$, we have $k = 0$, which means that the output $(F, \overline{F}) \triangleq \mathsf{Par}(1^1)$ will be two formulas with one leaf each, hence

$$\mathsf{Size}(F) = \mathsf{Size}(\overline{F}) = 1.$$

[7]It is easy to check that this is true if and only if n is a power of 2.

For the induction step, we need $\mathsf{S}^1_2 \vdash \forall n\, \phi_2(\lfloor n/2 \rfloor) \to \phi_2(n)$. If n is not a power of 2, then the statement is true by default. In the case of n being a power of 2, we fix $k = |n-1|$ and we want to prove equivalently:

$$\mathsf{S}^1_2 \vdash \phi_2(2^{k-1}) \to \phi_2(2^k).$$

Assume that $\phi_2(2^{k-1}) \equiv \phi_2(n/2)$ holds. From Line 8 we have that

$$F = (F_1 \vee F_2) \wedge (\overline{F}_1 \vee \overline{F}_2) \text{ and } \overline{F} = (F_1 \wedge F_2) \vee (\overline{F}_1 \wedge \overline{F}_2), \tag{10}$$

where $(F_1, \overline{F_1})$ and $(F_2, \overline{F_2})$ are copies of $\mathsf{Par}(1^{n/2})$. From the induction hypothesis, this means that $\mathsf{Size}(F_1) = \mathsf{Size}(\overline{F_1}) = \mathsf{Size}(F_2) = \mathsf{Size}(\overline{F_2}) = (n/2)^2 = 2^{2(k-1)}$. Therefore, from (Equation (10)) and the properties of the function Size, we get

$$\mathsf{Size}(F) = \mathsf{Size}(F_1) + \mathsf{Size}(\overline{F_1}) + \mathsf{Size}(F_2) + \mathsf{Size}(\overline{F_2}) = 4 \cdot 2^{2(k-1)} = 2^{2k} = n^2.$$

Similarly for $\overline{F}$, which means that $\phi_2(2^k) \equiv \phi_2(n)$ holds. This completes the proof of the induction for ϕ_2.

Property 3: $\mathsf{S}^1_2 \vdash \phi_3(n)$. Here the base case is trivial: for $F \triangleq x_1$ and $x \in \{0,1\}$, then $\mathsf{Eval}(F, x) = x = \oplus^0_1(x)$. Similarly for $\overline{F}$.

For the induction step, we assume as above that $n = 2^k$ and we want to prove:

$$\mathsf{S}^1_2 \vdash \phi_3(2^{k-1}) \to \phi_3(2^k).$$

We assume that $\phi_2(2^{k-1}) \equiv \phi_2(n/2)$ holds and we write F in the form

$$F = (F_1 \vee F_2) \wedge (\overline{F}_1 \vee \overline{F}_2) \text{ and } \overline{F} = (F_1 \wedge F_2) \vee (\overline{F}_1 \wedge \overline{F}_2),$$

where $(F_1, \overline{F_1})$ and $(F_2, \overline{F_2})$ are copies of $\mathsf{Par}(1^{n/2})$. Therefore, instead of $\mathsf{Eval}(F, x)$, we can calculate

$$\mathsf{Eval}((F_1 \vee F_2) \wedge (\overline{F}_1 \vee \overline{F}_2), x).$$

We need to prove that $\mathsf{Eval}(F, x) = \oplus^0_n(x)$ for all x with $|x| \leq n$. So, taking one such x we can split its binary representation into two parts x_1, x_2 with lengths $|x_1|, |x_2| \leq n/2$, such that $x = (x_2 x_1)_b = x_1 + 2^{n/2} x_2$.

The input to subformulas $F_2, \overline{F_2}$ from the definition are the bits $x_{2^{k-1}+i}$ for $i = 1, \ldots, 2^{k-1}$, which means that their input is x_2. Similarly, the input to subformulas $F_1, \overline{F_1}$ is x_1. Hence, we can define

$$\begin{aligned} b_1 &\triangleq \mathsf{Eval}(F_1, x_1) & b_3 &\triangleq \mathsf{Eval}(\overline{F_1}, x_1) \\ b_2 &\triangleq \mathsf{Eval}(F_2, x_2) & b_4 &\triangleq \mathsf{Eval}(\overline{F_2}, x_2) \end{aligned}$$

From the properties of the evaluation function and the form of F, we can prove in S^1_2 that $\mathsf{Eval}(F, x) = (b_1 \vee b_2) \wedge (b_3 \vee b_4)$, where the symbols $\vee, \wedge$ are used as Boolean symbols here.

However, since $|x_1|, |x_2| \leq n/2$ and $(F_1, \overline{F_1}) = (F_2, \overline{F_2}) = \mathsf{Par}(1^{n/2})$, from the induction hypothesis we get that

$$\begin{aligned} b_1 &= \oplus^0(x_1) & b_3 &= \oplus^1(x_1) = 1 - b_1 \\ b_2 &= \oplus^0(x_2) & b_4 &= \oplus^1(x_2) = 1 - b_2 \end{aligned}$$

Next, it is easy to prove by checking all the 4 cases that

$$\forall b_1, b_2 \in \{0,1\}\ (b_1 \vee b_2) \wedge ((1-b_1) \vee (1-b_2)) = b_1 \oplus b_2,$$

and as a result, we get

$$\mathsf{Eval}(F, x) = (\oplus^0(x_1)) \oplus (\oplus^0(x_2)) = \oplus^0(x_2 x_1) = \oplus^0(x)$$

by the properties of the parity function. Similarly, we can prove that $\mathsf{Eval}(\overline{F}, x) = \oplus^1_n(x)$, which concludes the induction. □

For the general case, we use a simple padding argument. For a number n, we can define the number

$$\tilde{n} \triangleq (n-1)\#1.$$

This number is the least power of 2 that is greater or equal to n. It is easy to see that

$$\mathsf{PV}_1 \vdash n \leq \tilde{n} < 2n.$$

If we replace $\mathsf{ParForm}(1^{(n)})$ by $\mathsf{Par}_1(1^{\tilde{n}})$ (the first coordinate of $\mathsf{Par}(1^{\tilde{n}})$), we have by the above lemma that

1. $\mathsf{Size}(\mathsf{ParForm}(1^{(n)})) = \mathsf{Size}(\mathsf{Par}_1(1^{\tilde{n}})) = \tilde{n}^2 < (2n)^2 = s(n)$.
2. For all x with $|x| \leq n$, we have $|x| \leq \tilde{n}$, which by the lemma gives us $\mathsf{Eval}(\mathsf{ParForm}(1^n), x) = \mathsf{Eval}(\mathsf{Par}_1(1^{\tilde{n}}), x) = \oplus^0_{\tilde{n}}(x)$. Since $|x| \leq n$, we also have $\oplus^0_{\tilde{n}}(x) = \oplus^0_n(x)$. Consequently, we have $\mathsf{Eval}(\mathsf{ParForm}(1^n), x) = \oplus^0_n(x)$.

These two together show that $\mathsf{PV}_1 \vdash \mathsf{FUB}'_s$ and the proof is complete. □

4.3 Formula Size Hierarchy

In this section, we provide the proof of Theorem 3.

Theorem 16 (Theorem 3)**.** *Consider rationals $a > 2$ and $b = 3/2$, and let n_0 be a large enough positive integer. Then*

$$\mathsf{PV}_1 \vdash \mathsf{FSH}[a, b, n_0]\,.$$

Proof. We combine the results of Section 4.1 and Section 4.2. We argue in PV_1. From Theorem 11, we get that

$$\forall n \in \mathsf{Log}\ \forall F \in \mathsf{FORMULA}[n^{3/2}]\ \exists x\ (|x| \leq n \wedge F(x) \neq \oplus_n(x)), \tag{11}$$

and from Theorem 14, we have that

$$\forall n \in \mathsf{Log}\ \exists G \in \mathsf{FORMULA}[4n^2]\ \forall x\ (|x| \leq n \rightarrow G(x) = \oplus_n(x)).$$

We can eliminate the constant 4 from the latter using that $a > 2$ and choosing a large enough n_0, such that for every $n \geq n_0$, $n^a \geq 4n^2$ (provably in PV_1). Consequently,

$$\forall n \geq n_0 \in \mathsf{Log}\ \exists G \in \mathsf{FORMULA}[n^a]\ \forall x\ (|x| \leq n \rightarrow G(x) = \oplus_n(x)). \tag{12}$$

Finally, combining Equation (11) and Equation (12), we get that

$$\forall n \geq n_0 \in \mathsf{Log}\ \exists G \in \mathsf{FORMULA}[n^a]\ \forall F \in \mathsf{FORMULA}[n^{3/2}]\ \exists x\ (|x| \leq n\ \wedge\ F(x) \neq G(x)),$$

which is exactly the formula size hierarchy, $\mathsf{FSH}[a, b, n_0]$, for our choice of parameters $a > 2$ and $b = 3/2$. □

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A Proof of the KPT Theorem for $\forall\exists\forall\exists$ Sentences

In order to make our results more accessible and the presentation self-contained, in this section we describe a standard model-theoretic proof of the KPT Witnessing Theorem. We restate the result below for convenience of the reader.

Theorem 17. *Let* T *be a universal theory with vocabulary* $\mathcal{L}$. *Let* φ *be an open* $\mathcal{L}$*-formula, and suppose that*

$$\mathsf{T} \vdash \forall x \, \exists y \, \forall z \, \exists w \; \varphi(x,y,z,w).$$

Then there is a finite sequence $s_1,\ldots,s_k$ *of* $\mathcal{L}$*-terms such that*

$$\mathsf{T} \vdash \forall x, z_1, \ldots, z_k \left(\psi(x, s_1(x), z_1) \vee \psi(x, s_2(x,z_1), z_2) \vee \cdots \vee \psi(x, s_k(x, z_1, \ldots, z_{k-1}), z_k)\right),$$

where

$$\psi(x,y,z) \triangleq \exists w \; \varphi(x,y,z,w).$$

Proof. Let $b, c_1, c_2, \ldots$ be a list of new constants, and let $u_1, u_2, \ldots$ be an enumeration of all terms built from the functions and constants in $\mathcal{L}$ together with $b, c_1, c_2, \ldots$, where the only new constants in u_k are among $b, c_1, \ldots, c_{k-1}$.

For convenience, let $\psi(x, y, z) \triangleq \exists w \varphi(x, y, z, w)$, as in the statement of the theorem. We will argue that there exists a constant $k \geq 1$ such that no model of T satisfies the sentence

$$\neg\psi(b, u_1, c_1) \wedge \neg\psi(b, u_2, c_2) \wedge \ldots \wedge \neg\psi(b, u_k, c_k) \ .$$

This implies that every model of T satisfies the negation of this sentence, and by the completeness theorem,

$$\mathsf{T} \ \vdash \ \psi(b, u_1, c_1) \vee \psi(b, u_2, c_2) \vee \ldots \vee \psi(b, u_k, c_k) \ .$$

Since $b, c_1, c_2, \ldots$ are new constants and each term u_k depends only on $b, c_1, \ldots, c_{k-1}$ (among the new constant symbols), the result follows.

To show the remaining claim, we argue by contradiction. Suppose that no finite k satisfies the claim. Then, by compactness, we get that

$$\mathsf{T} \cup \{\neg\psi(b, u_1, c_1), \neg\psi(b, u_2, c_2), \neg\psi(b, u_3, c_3), \ldots\}$$

admits a model $\mathcal{M}$. Consequently, using the definition of ψ,

$$\mathcal{M} \ \models \ \mathsf{T} \cup \{\forall w \neg\varphi(b, u_1, c_1, w), \forall w \neg\varphi(b, u_2, c_2, w), \ldots\}$$

Let $\mathsf{T}^+ \triangleq \mathsf{T} \cup \{\forall w \neg\varphi(b, u_1, c_1, w), \forall w \neg\varphi(b, u_2, c_2, w), \ldots\}$. Since T is a universal theory and φ is an open formula, it follows that T^+ is also a universal theory. For this reason, the substructure $\mathcal{M}'$ of $\mathcal{M}$ consisting of the denotations of the terms $u_1, u_2, \ldots$ is also a model of T^+. Now it is not hard to prove that

$$\mathcal{M}' \ \models \ \mathsf{T} \ + \ \exists x \, \forall y \, \exists z \, \forall w \ \neg\varphi(x, y, z, w) \ ,$$

which contradicts the hypothesis of the theorem and completes the proof. To see this, it is enough to show that $\mathcal{M}' \models \forall y \, \exists z \, \forall w \ \neg\varphi(b^{\mathcal{M}'}, y, z, w)$. Given an arbitrary element m in $\mathcal{M}'$, by construction of $\mathcal{M}'$, there is some term u_k such that $m = u_k^{\mathcal{M}'}(b^{\mathcal{M}'}, c_1^{\mathcal{M}'}, \ldots, c_{k-1}^{\mathcal{M}'})$. Since $\mathcal{M}'$ is a model of T^+, which includes the sentence $\forall w \neg\varphi(b, u_k, c_k, w)$, we get that $\mathcal{M}' \models \forall w \, \neg\varphi(b^{\mathcal{M}'}, m, c_k^{\mathcal{M}'}, w)$. This finishes the proof that $\mathcal{M}' \models \forall y \, \exists z \, \forall w \ \neg\varphi(b^{\mathcal{M}'}, y, z, w)$. $\square$