Electronic Colloquium on Computational Complexity, Report No. 201 (2024)

On one-way functions and the average time complexity of almost-optimal compression

Marius Zimand*

Abstract

We show that one-way functions exist if and only there exists an efficient distribution relative to which almost-optimal compression is hard on average. The result is obtained by combining a theorem of Ilango, Ren, and Santhanam [IRS21, IRS22] and one by Bauwens and Zimand [BZ23].

1 Introduction

Several recent papers show that the existence of one-way functions (OWF) is equivalent to the hardness of certain problems in meta-complexity [LP20, LP21, RS21, IRS21, IRS22, LP23a, LP23b, HLO24, LS24]. The motivation for this research line comes primarily from cryptography, where one-way functions play a central role[1]. Ilango, Ren and Santhanam [IRS21, IRS22] have obtained a result of this type involving standard (unbounded) Kolmogorov complexity. Informally speaking, they have shown that one-way functions exist if and only if "finding good approximations of Kolmogorov complexity" is hard on average with respect to some polynomial-time samplable distribution. Bauwens and Zimand [BZ23] have shown that given a good approximation of the Kolmogorov complexity of a string x, one can compress x in probabilistic polynomial time to a string of length close to its complexity (so, x is almost-optimally compressed). The combination of these 2 results yields the following theorem.

Theorem 1 (Informal statement)**.** *The following two assertions are equivalent:*

1. *There exists an one-way function.*
2. *Almost optimal compression is hard on average with respect to some polynomial-time samplable distribution.*

The result of Ilango et. al. [IRS21, IRS22] is not exactly stated in the form that we mentioned above. For this reason, we prefer to give a proof which does not directly invoke [IRS21, IRS22], but which closely follows their method. In one direction, it is based on results of Impagliazzo, Levin and Luby[IL90, IL89] connecting the existence of OWFs to the hardness of approximating poly-time samplable distributions, and, in the other direction, it is based on the connection between OWFs and pseudo-random generators established by Håstad, Impagliazzo, Levin and Luby [HILL99].

2 Definitions, and technical tools

Kolmogorov complexity. We fix an optimal universal Turing machine U with prefix-free domain. A program for string x is a string p such that $U(p) = x$. The prefix-free Kolmogorov complexity $K(x)$ of the string x is the length of a shortest program for x.[2]

Distributions. We consider ensembles of distributions. An ensemble has the form $D = (D_n)_{n \in \mathbb{N}}$, where each D_n is a distribution on $\{0,1\}^n$. The ensemble D is *samplable* if there exists a probabilistic algorithm Samp, such that for every n and every $x \in \{0,1\}^n$,

$$\text{Prob}\,[\mathsf{Samp}(1^n) = x] = D_n(x)$$

*Department of Computer and Information Sciences, Towson University, Baltimore, MD. Partially supported by a grant from the School of Emerging Technologies at Towson University.

[1] See [HLO24] and [LS24] for a discussion of some of these and related works.

[2] The prefix-free Kolmogorov complexity $K(x)$ is a little more convenient for the proof than the plain complexity $C(x)$. The difference $K(x) - C(x)$ is bounded by $2 \log |x|$ and, therefore, the result is valid for $C(x)$ as well.

ISSN 1433-8092

(the probability is over the randomness of Samp).

D is said to be *P-samplable*, in case Samp runs in polynomial time.

Some notation: For every x, we denote $D(x) = D_{|x|}(x)$. For every m, U_m denotes the uniform distribution over m-bit strings.

Lemma 1. *If D is samplable, then for every x in its support,*

$$K(x) \leq \log \frac{1}{D(x)} + 3\log(|x|) + O(1).$$

Proof. Fix a binary string x and let n be its length. Given n and the code of Samp, one can compute $D_n(y)$ for all strings y of length n and then list all these strings in descending order of their $D_n(\cdot)$ probability (with ties broken, say, lexicographically). The string x is described by its rank t in this list. Since the D_n-probability of the first t strings in the order is at most 1 and at least $t \cdot D_n(x)$, it follows that $t \leq \lceil 1/D_n(x) \rceil$. An overhead of $2\log(|x|) + O(1)$ bits is added to obtain a self-delimited description in the standard way. □

Lemma 2. *For every distribution D, and every $\Delta \geq 0$,*

$$\text{Prob}_{x \leftarrow D} \ [K(x) \geq \log \frac{1}{D(x)} - \Delta] \geq 1 - 2^{-\Delta}.$$

Proof. The complement of the event in the probability is $E = \{x \mid D(x) \leq 2^{-\Delta} \cdot 2^{-K(x)}\}$. We have

$$D(E) = \sum_{x \in E} D(x) \leq \sum_{x \in E} 2^{-\Delta} \cdot 2^{-K(x)} \leq 2^{-\Delta} \sum_{x \in \{0,1\}^*} 2^{-K(x)} \leq 2^{-\Delta} \cdot 1 = 2^{-\Delta}.$$

In the penultimate transition, we have used the Kraft inequality, which is legitimate because $K(\cdot)$ represents the lengths of a prefix-free code. □

Formal statement of Theorem 1. The following 2 assertions are equivalent:

Assertion (1): The hypothesis "∃ OWF": There exists a polynomial-time computable $f : \{0,1\}^* \to \{0,1\}^*$ with the following property: For every probabilistic polynomial-time algorithm $\mathsf{Inverter}$, every $q \in \mathbb{N}$ and every length $n \in \mathbb{N}$,

$$\text{Prob}_{x \leftarrow U_n, \mathsf{Inverter}}[\mathsf{Inverter}(1^n, f(x)) \in f^{-1}(f(x))] \leq 1/n^q.$$

(The notation $\text{Prob}_{x \leftarrow U_n, \mathsf{Inverter}}$ means that the probability is over $U_n \times$ randomness of $\mathsf{Inverter}$.)

Assertion (2): The hypothesis "almost optimal compression is hard on average" : There exists a P-samplable distribution D and a constant c with the following property: For every probabilistic polynomial-time algorithm $\mathsf{Compress}$, at every length n,

$$\text{Prob}_{x \leftarrow D_n, \mathsf{Compress}}[\mathsf{Compress}\,(x) \text{ outputs a program of } x \text{ of length } \leq K(x) + c\log^2 n] \leq 1/n.$$

Remark. The "infinitely often" version of Theorem 1 is also true, with essentially the same proof. More precisely, if we modify Assertions 1 and 2 by replacing "every length n" with "infinitely many lengths n," the modified assertions are also equivalent.

Results from the literature that we use.

Theorem 2 ([IL89, IL90]; this variant is stated and proved in [IRS21])**.** *Assume the hypothesis "∃ OWF" is not true. Let $D = (D_n)_{n \in \mathbb{N}}$ be a P-samplable ensemble of distributions, and $q \in \mathbb{N}$. There exists a probabilistic polynomial-time algorithm A and a constant $c > 1$ such that for infinitely many n,*

$$\text{Prob}_{x \leftarrow D_n, A} \ [D_n(x)/c \leq A(x) \leq D_n(x)] \geq 1 - \frac{1}{n^q}.$$

In other words: If there are no one-way functions, then P-samplable distributions can be approximated efficiently in the average sense.

Theorem 3 ([BZ23]). *There exists a probabilistic polynomial-time algorithm* Compress *that for every input triple* ($x \in \{0,1\}^*, m \in \mathbb{N}$, *rational* $\epsilon > 0$) *outputs with probability 1 a string* z *of length* $m + O(\log m \cdot \log |x|/\epsilon)$ *and if* $m \geq K(x)$ *then*

$$\text{Prob}[z \textit{ is a program for } x] \geq 1 - \epsilon.$$

In other words: Given a good approximation of the Kolmogorov complexity of a string x, one can efficiently compress x almost optimally (where efficiently means in probabilistic polynomial time).

3 Proof of Theorem 1

Proof of assertion (2) → assertion (1).

We actually prove the contrapositive: $\nexists$ OWF ⇒ almost optimal compression is easy on average.

Let $D = (D_n)_{n\in\mathbb{N}}$ be a P-samplable ensemble and $q \in \mathbb{N}$. By Lemma 2 and Lemma 1, for some constant c, for every n

$$\text{Prob}_{x\leftarrow D_n}[\log \frac{1}{D_n(x)} - c\log n \leq K(x) \leq \log \frac{1}{D_n(x)} + c\log n] \geq 1 - 1/n^q.$$

Under our assumption "$\nexists$ OWF ," Theorem 2 states that there exists an algorithm A that approximates $K(x)$. By rescaling we get that, for every n,

$$\text{Prob}_{x\leftarrow D_n}[K(x) \leq A(x) \leq K(x) + c\log n] \geq 1 - 1/n^q.$$

Using algorithm Compress from Theorem 3 with $m = A(x)$ and $\epsilon = 1/n^{2q}$, and after more rescaling we obtain the converse of assertion (2).

Proof of assertion (1) → assertion (2).

($\exists$ OWF ⇒ almost optimal compression is hard on average.)

The idea is that an efficient good compressor would break the security of any candidate pseudorandom generator (p.r.g.), because the output of the generator can be compressed to a much shorter string, whereas a genuinely random string cannot. Therefore, pseudorandom generators would not exist and hence there would be no OWF, contradicting assertion (1). Now, the details.

Suppose "$\exists$ OWF" is true. Then, by [HILL99] combined with the methods to obtain ensembles of p.r.g.'s with every possible output length [Gol01, Section 3.3.3], there exists an ensemble of p.r.g.'s $G = (G_n)_{n\in\mathbb{N}}$, with $G_n : \{0,1\}^{s(n)} \to \{0,1\}^n$, where the seed length $s(n)$ is bounded by $n^{1/3}$, that satisfies the following security guarantee: For every probabilistic polynomial-time algorithm T (the hypothetical distinguisher) and every $q \in \mathbb{N}$, for every $n \in \mathbb{N}$, the probabilities that (a) T accepts $G_n(U_{s(n)})$ and (b) T accepts U_n, differ by at most $1/n^q$.

Consider the following P-samplable distribution D_n:

with probability 1/2, output $G(U_{s(n)})$ and with probability 1/2, output U_n.

Clearly, if assertion (2) is false, then there exists a probabilistic polynomial-time algorithm A that, at infinitely many lengths n, with D_n-probability $\geq 1/n$ approximates $K(x)$ with slack at most $c\log^2 n$. Then, for infinitely many n, by Markov's inequality,

$$\text{Prob}_{x\leftarrow D_n}[\mathcal{E}_n] \geq 5/6,$$

where

$$\mathcal{E}_n = \{x \in \{0,1\}^n \mid \text{Prob}[\,|A(x) - K(x)| \leq c\log^2 n\,] \geq 6/(5n)\}.$$

For infinitely many n, the complement of $\mathcal{E}_n$ (an event that we henceforth denote by BAD and which says that A fails to approximate K), has D_n-probability at most 1/6. By inspecting the sampling procedure, we see that each element x in BAD has D_n-probability mass at least $(1/2)\cdot 2^{-n}$ and thus $1/6 \geq D_n(\text{BAD}) \geq (\#BAD)\cdot(1/2\cdot 2^{-n}) = 1/2\cdot \text{Prob}_{U_n}[BAD]$, and so

$$\text{Prob}_{U_n}[BAD] \leq 1/3.$$

Also, each element in BAD $\cap$ Im($G(U_{s(n)})$ has D_n-probability mass at least $(1/2) \cdot 2^{-s(n)}$, which, similarly to the above, implies that

$$\text{Prob}_{U_{s(n)}}[BAD \cap \text{Im}(G(U_{s(n)}))] \leq 1/3.$$

We now define the probabilistic polynomial-time distinguisher T: T on input z of length n outputs 1, if $A(z) \leq 2s(n)$, and 0 otherwise. Note that $G(U_{s(n)})$ with probability 1 has prefix-free complexity at most $s(n)+2\log s(n)+O(1) \leq n^{1/3}+O(\log n)$, and U_n, with probability at least 1-1/n, has complexity at least $n - \log n$.

Therefore, for infinitely many n,

$$\text{Prob}_{U_{s(n)},T}[T(G(U_{s(n)})) = 1] \geq (1 - 1/3) \cdot 6/(5n)$$

(the probability that $G(U_{s(n)})$ is not in BAD is at least $1 - 1/3$ and the probability that T uses good randomness is at least $6/(5n)$) and, by similar arguments,

$$\text{Prob}_{U_n,T}[T(U_{4s(n)}) = 1] \leq (1/n + 1/3) \cdot 6/(5n),$$

contradicting the security of G.

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ECCC ISSN 1433-8092
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