Electronic Colloquium on Computational Complexity, Report No. 7 (2025)

Improved Debordering of Waring Rank

Amir Shpilka *

Abstract

We prove that if a degree-d homogeneous polynomial f has border Waring rank $\underline{\mathrm{WR}}(f) = r$, then its Waring rank is bounded by

$$\mathrm{WR}(f) \leqslant d \cdot r^{O(\sqrt{r})}.$$

This result significantly improves upon the recent bound $\mathrm{WR}(f) \leqslant d \cdot 4^r$ established in [Dutta, Gesmundo, Ikenmeyer, Jindal, and Lysikov, STACS 2024], which itself was an improvement over the earlier bound $\mathrm{WR}(f) \leqslant d^r$.

1 Introduction

Given a circuit class $\mathcal{C}$, its closure, $\overline{\mathcal{C}}$, is defined as the closure of the set of polynomials computable in $\mathcal{C}$.Specifically, this includes all polynomials that are limits, in the Zariski topology, of a converging sequence of polynomials computable in $\mathcal{C}$. Over the complex or real fields, this is equivalent to the coefficient vectors converging in the usual sense (i.e., with respect to the Euclidean topology). However, this notion also applies to arbitrary fields and can be defined algebraically.

In this paper, we study the closure of depth-3 powering circuit, denoted by $\Sigma\wedge\Sigma$. The output of a $\Sigma^{[r]}\wedge^{[d]}\Sigma$ circuit is a degree-d homogeneous polynomial of the form

$$f(\mathbf{x}) = \sum_{i=1}^{r} \ell_i(\mathbf{x})^d.$$

where ℓ_i are linear forms (i.e., homogeneous linear polynomials).[1] The Waring rank of a homogeneous polynomial of degree d is defined as the minimal r such that f can be computed by a $\Sigma^{[r]}\wedge^{[d]}\Sigma$ circuit. It is also known as the *symmetric tensor rank* of f. The *border Waring rank* of a polynomial f, denoted $\underline{\mathrm{WR}}(f)$, is the minimal r such that f is in the closure of polynomials of Waring rank at most r.

Alder [Ald84] (see also [BCS97, Appendix 20.6]) showed that $\underline{\mathrm{WR}}(f) = r$ if there exist r linear functions $\ell_i \in \mathbb{C}(\varepsilon)[\mathbf{x}]$ such that

$$f = \lim_{\varepsilon\to 0} \sum_{i=1}^{r} \ell_i^d.$$

Equivalently, $\underline{\mathrm{WR}}(f) = r$ if there exist a degree-d polynomial $g \in \mathbb{C}[\varepsilon][\mathbf{x}]$, an integer q, and linear forms $\ell_i \in \mathbb{C}[\varepsilon][\mathbf{x}]$ such that

$$\varepsilon^q f + \varepsilon^{q+1} g = \sum_{i=1}^{r} \ell_i^d.$$

This alternative definition generalizes to arbitrary characteristic fields.

*This research was co-funded by the European Union by the European Union (ERC, EACTP, 101142020), the Israel Science Foundation (grant number 514/20) and the Len Blavatnik and the Blavatnik Family Foundation. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

[1] We can allow representations of the form $\sum_{i=1}^{r} c_i \cdot \ell_i^d$, for scalars c_i, but over algebraically closed fields this does not change the complexity.

ISSN 1433-8092

Understanding the relationship between $\underline{\mathrm{WR}}(f)$ and $\mathrm{WR}(f)$ is a longstanding open problem. Forbes [For16] conjectures that $\overline{\Sigma\wedge\Sigma} = \Sigma\wedge\Sigma$. In other words, if $\underline{\mathrm{WR}}(f) = \mathrm{poly}(n, d)$, then $\mathrm{WR}(f)$ is also polynomially bounded. Ballico and Bernardi [BB17] proposed a stronger conjecture, asserting that taking limits can save at most a factor of d. Specifically, they conjectured

$$\mathrm{WR}(f) \leqslant (\underline{\mathrm{WR}}(f) - 1) \cdot \deg(f).$$

This conjecture was verified for small values of r ($r = 3, 4$, and when $d \geqslant 9$, also for $r = 5$) [LT10, BB13, Bal18].

In [DGI+24], Dutta, Gesmundo, Ikenmeyer, Jindal, and Lysikov studied the general case and proved that if a polynomial f of degree d has $\underline{\mathrm{WR}}(f) = r$, then

$$\mathrm{WR}(f) \leqslant d \cdot 4^r.$$

We significantly improve upon the upper bound given in [DGI+24].

Theorem 1.1. *Let $f \in \mathbb{C}[\mathbf{x}]_d$ be a homogeneous polynomial. If $\underline{\mathrm{WR}}(f) = r$, then*

$$\mathrm{WR}(f) \leqslant d \cdot r^{10\sqrt{r}}.$$

For more on debordering, Waring rank, and related problems see [BCC+18, DDS22, DGI+24].

2 Preliminaries

In this section, we introduce some notation and formally define the Waring rank and border Waring rank. We work over the field $\mathbb{C}$ of complex numbers. The space of homogeneous polynomials of degree d in variables $x = (x_1, \ldots, x_n)$ is denoted by $\mathbb{C}[\mathbf{x}]_d$. We write $f \simeq g$ for $f, g \in \mathbb{C}(\varepsilon)[\mathbf{x}]$ if $\lim_{\varepsilon\to 0} f = \lim_{\varepsilon\to 0} g$ (in particular, both limits must exist).

The projective space $\mathbb{P}V$ is defined as the set of lines passing through the origin in V. For each nonzero $v \in V$, the corresponding line is denoted $[v] \in \mathbb{P}V$, where $[v] = [w]$ if and only if $v = c \cdot w$ for some scalar $c \in \mathbb{C}$.

For integers $j \leqslant d$, we denote

$$(d)_j = j! \cdot \binom{d}{j} = \prod_{i=0}^{j-1}(d - i),$$

where $(d)_j$ represents the falling factorial.

2.1 Facts from [DGI+24]

Definition 2.1 (Waring Rank)**.** Let $f \in \mathbb{C}[\mathbf{x}]$ be a degree-d homogeneous polynomial. The *Waring rank* of f, denoted $\mathrm{WR}(f)$, is the smallest integer r such that there exist homogeneous linear forms $\ell_1, \ldots, \ell_r$ satisfying:

$$f = \sum_{i=1}^{r} \ell_i^d.$$

Definition 2.2 (Border Waring Rank)**.** The *border Waring rank* of f, denoted $\underline{\mathrm{WR}}(f)$, is the smallest r such that f can be expressed as a limit of a sequence of polynomials with Waring rank at most r.

As shown in [Ald84], the next definition is equivalent to Definition 2.2.

Definition 2.3 (Border Waring Rank Decomposition)**.** A *border Waring rank decomposition* of a degree-d homogeneous polynomial $f \in \mathbb{C}[\mathbf{x}]_d$ is an expression of the form:

$$f = \lim_{\varepsilon\to 0} \sum_{i=1}^{r} \ell_i^d,$$

where $\ell_1, \ldots, \ell_r \in \mathbb{C}(\varepsilon)[\mathbf{x}]_1$ are linear forms with coefficients rationally dependent on ε. The border Waring rank $\underline{\mathrm{WR}}(f)$ is the smallest number r of summands in such a decomposition.

A rational family of linear forms $\ell \in \mathbb{C}(\varepsilon)[\mathbf{x}]_1$ always has a well-defined limit when viewed projectively. Specifically, if $\ell(\varepsilon)$ is expanded as a Laurent series:

$$\ell(\varepsilon) = \sum_{i=q}^{\infty} \varepsilon^i \ell_i, \quad \text{with } \ell_q \neq 0,$$

then:

$$\lim_{\varepsilon \to 0} [\ell(\varepsilon)] = \lim_{\varepsilon \to 0} \sum_{i=q}^{\infty} \varepsilon^i \ell_{q+i} = [\ell_q]. \tag{1}$$

A border Waring rank decomposition is called *local* if, for all summands in the decomposition, this limit is the same.

Definition 2.4 (Local Decomposition [DGI+24]). Let $f \in \mathbb{C}[\varepsilon]_d$ be a degree-d homogeneous polynomial. A border Waring rank decomposition:

$$f = \lim_{\varepsilon \to 0} \sum_{i=1}^{r} \ell_i^d$$

is called a *local border decomposition* if there exists a linear form $\ell \in \mathbb{C}[\mathbf{x}]_1$ such that:

$$\lim_{\varepsilon \to 0} [\ell_i(\varepsilon)] = [\ell] \quad \text{for all } i \in [r].$$

The point $[\ell] \in \mathbb{P}\mathbb{C}[\mathbf{x}]_1$ is called the *base* of the decomposition. A local decomposition is called *standard* if $\ell_1 = c \cdot \varepsilon^q \ell$ for some $q \in \mathbb{Z}$ and $c \in \mathbb{C}$.

The number of essential variables of a homogeneous polynomial f is the smallest integer m such that, after a linear change of coordinates, f can be expressed as a polynomial in m variables. Denote the number of essential variables of f by $N(f)$.

Lemma 2.5 (Lemma 4 of [DGI+24]). *For a homogeneous polynomial* $f \in \mathbb{C}[\mathbf{x}]_d$*, we have* $N(f) \leqslant \underline{\mathrm{WR}}(f)$.

Lemma 2.6 (Lemma 6 of [DGI+24]). *If f has a local border decomposition, then it has a standard local border decomposition with the same base and the same number of summands.*

The following lemma can be proved by induction on the degree. It also follows from Corollary 3.2, as discussed in Remark 3.4.

Lemma 2.7 (Lemma 7 of [DGI+24]). *Suppose* $f \in \mathbb{C}[\mathbf{x}]_d$ *has a local border decomposition with* r *summands based at* $[\ell]$*. If* $d \geqslant r-1$*, then:*

$$f = \ell^{d-r+1} \cdot g(\mathbf{x}),$$

where g is a homogeneous polynomial of degree $r-1$.

The following lemma shows that when the degree is greater than the rank, the decomposition must be local (or consist of local decompositions).

Lemma 2.8 (Lemma 10 of [DGI+24]). *Let* $f \in \mathbb{C}[\mathbf{x}]_d$ *be such that* $\underline{\mathrm{WR}}(f) = r$*. If* $d \geqslant r-1$*, then there exists a partition* $r = r_1 + \cdots + r_m$ *such that f has a decomposition:*

$$f = \sum_{k=1}^{m} \ell_k^{d-r_k+1} \cdot g_k,$$

where each $\ell_k^{d-r_k+1} g_k$ *has a local decomposition, with* $[\ell_k]$ *as the base of the decomposition, and:*

$$\underline{\mathrm{WR}}\left(\ell_k^{d-r_k+1} \cdot g_k\right) \leqslant r_k.$$

The following simple lemma is implicit in [DGI+24].

Lemma 2.9 (Perturbing a Variable by ε)**.** *Let* $f \in \mathbb{C}[\varepsilon][\mathbf{x}]$*. If* $x_1 \simeq g$*, where* $g \in \mathbb{C}(\varepsilon)[\mathbf{x}]$*, then:*

$$f(\mathbf{x}) \simeq f(g, x_2, \ldots, x_n).$$

Proof. Let $f_0 = \lim_{\varepsilon \to 0} f(\mathbf{x})$, where $f_0 \in \mathbb{C}[\mathbf{x}]$. For some $f_1 \in \mathbb{C}[\varepsilon][\mathbf{x}]$, we can write $f = f_0 + \varepsilon f_1$. Similarly, since $\lim_{\varepsilon \to 0} x_1 = \lim_{\varepsilon \to 0} g$, we have $g = x_1 + \varepsilon g'$ for some $g' \in \mathbb{C}[\varepsilon][\mathbf{x}]$. Expanding f_0 monomial-wise and substituting g for x_1, it is straightforward to verify that:

$$f_0(g, x_2, \ldots, x_n) = f_0(\mathbf{x}) + \varepsilon f_2(\mathbf{x}),$$

for some $f_2 \in \mathbb{C}[\varepsilon][\mathbf{x}]$. Thus,

$$f(g, x_2, \ldots, x_n) = f_0(g, x_2, \ldots, x_n) + \varepsilon f_1(g, x_2, \ldots, x_n) = f_0(\mathbf{x}) + \varepsilon f_2(\mathbf{x}) + \varepsilon f_1(g, x_2, \ldots, x_n) \simeq f_0,$$

as required. □

We will frequently use this lemma to simplify border Waring rank decompositions.

Additionally, if $\underline{\mathrm{WR}}(f) = r$ and $f = \lim_{\varepsilon \to 0} \sum_{i=1}^{r} \ell_i^d$, where each $\ell_i(\mathbf{x}) \in \mathbb{C}(\varepsilon)[\mathbf{x}]_1$, then for an integer q such that $\varepsilon^q \cdot \ell_i(\mathbf{x}) \in \mathbb{C}[\varepsilon][\mathbf{x}]_1$ for all i, it holds that

$$\varepsilon^{qd} f + \varepsilon^{qd+1} g = \sum_{i=1}^{r} (\varepsilon^q \ell_i)^d,$$

for some $g \in \mathbb{C}[\varepsilon][\mathbf{x}]_d$.

Conversely, if for some integer q, polynomial $g \in \mathbb{C}[\varepsilon][\mathbf{x}]_d$, and linear functions $\ell_i(\mathbf{x}) \in \mathbb{C}[\varepsilon][\mathbf{x}]_1$, we have

$$\varepsilon^q f + \varepsilon^{q+1} g = \sum_{i=1}^{r} \ell_i^d, \tag{2}$$

then $\underline{\mathrm{WR}}(f) \leqslant r$.

From this point onward, we will consider the representation in (2) for polynomials f with $\underline{\mathrm{WR}}(f) \leqslant r$.

3 Improved debordering

In this section, we provide the proof of Theorem 1.1. The proof begins by describing an ε-perturbed diagonalization process. We consider homogeneous polynomials $f \in \mathbb{C}[\mathbf{x}]_d$ and assume, without loss of generality, that $N(f) = n \leqslant \underline{\mathrm{WR}}(f)$ (see Lemma 2.5).

Lemma 3.1 (Perturbed Diagonalization)**.** *Let*

$$\varepsilon^q f + \varepsilon^{q+1} g = \sum_{i=1}^{r} \ell_i^d$$

be a decomposition of f*, where* $\ell_i(\mathbf{x}) \in \mathbb{C}[\varepsilon][\mathbf{x}]_1$*. Then, there exist:*

- *a matrix* $A = A_0 + \varepsilon A_1 \in \mathbb{C}[\varepsilon]^{r \times r}$*, where* $A_0 \in \mathbb{C}^{r \times r}$ *is invertible,*
- *integers* $0 = q_1 \leqslant q_2 \leqslant \ldots \leqslant q_n \leqslant q$*,*
- *a permutation* $\pi : [r] \to [r]$*, and*
- *linear functions* $L_1, \ldots, L_r$*,*

such that, for every $i \in [n]$ *and* $m \in [q_n]$*, defining*

$$k_{i,m} = \arg\max_{k \in [i-1]} \{q_k \leqslant m\},$$

the linear function $L_i(\mathbf{x})$ *satisfies the following:*

$$L_i(\mathbf{x}) := \ell_{\pi(i)}(A\mathbf{x}) = \begin{cases} \sum_{m=0}^{q_i-1} \varepsilon^m \sum_{k=1}^{k_{i,m}} c_{i,m,k} x_k + \varepsilon^{q_i} x_i & \textit{if } i \leqslant n, \\ \sum_{m=0}^{q} \varepsilon^m \sum_{k=1}^{n} c_{i,m,k} x_k & \textit{if } n < i \leqslant r. \end{cases} \tag{3}$$

Furthermore, for some polynomial $\tilde{g} \in \mathbb{C}[\varepsilon][\mathbf{x}]_d$*, we have*

$$\varepsilon^q f(A_0\mathbf{x}) + \varepsilon^{q+1}\tilde{g}(\mathbf{x}) = \sum_{i=1}^{r} L_i^d,$$

which is a border Waring rank decomposition of $f(A_0\mathbf{x})$*. Moreover, if the original decomposition of* f *was local, then the decomposition of* $f(A_0\mathbf{x})$ *is also local, based at* x_1*.*

To better understand this construction, consider the matrix Q representing the L_i's, where $Q_{i,j}$ is the linear form corresponding to the coefficient of ε^j in L_i. Explicitly:

$$Q_{i,j} = \begin{cases} \sum_{k=1}^{k_{i,j}} c_{i,j,k} x_k & \text{if } j < q_i, \\ x_i & \text{if } j = q_i, \\ 0 & \text{if } j > q_i. \end{cases}$$

Thus, the first n rows of the matrix Q are in lower triangular form. Importantly, a variable x_k does not appear in $L_1, \ldots, L_{k-1}$ and can only appear in columns $j \geqslant q_k$.

Proof of Lemma 2.9. First, observe that we can assume, without loss of generality, that no ℓ_i contains powers of ε larger than q. This simplification can be achieved by removing these higher-order terms from ℓ_i, which would only modify g without affecting the decomposition.

Let us denote $C_j[\ell_i] \in \mathbb{C}[\mathbf{x}]_1$ as the coefficient of ε^j in ℓ_i.

In Algorithm 1, we outline the process for constructing the matrix A. The algorithm begins by constructing a basis of linear functions. At each iteration, it identifies the smallest power of ε such that one of the remaining ℓ_i has a coefficient at that power which is a linear function linearly independent of all previously constructed basis elements. This ℓ_i is then removed from the set, and the identified linear function is added to the basis. This process repeats until all ℓ_i are processed.

After this step, each ℓ_i is associated with a basis element and a corresponding power of ε, representing the coefficient of that power in ℓ_i during the iteration. Additionally, the ℓ_is are re-indexed based on the order in which they were removed.

Next, an invertible linear transformation is applied to ensure that each basis element corresponds to a variable. Specifically, under the new indexing, x_i becomes the variable associated with the basis element derived from ℓ_i. After the transformation, each ℓ_i takes the form:

$$\ell_i = \tilde{\ell}_i(x_1, \ldots, x_{i-1}) + \varepsilon^{q_i} x_i + \varepsilon^{q_i+1} \ell_i',$$

where $\deg_\varepsilon(\tilde{\ell}_i) < q_i$. Finally, we adjust x_i by adding $\varepsilon \ell_i'$, ensuring that:

$$\ell_i = \tilde{\ell}_i(x_1, \ldots, x_{i-1}) + \varepsilon^{q_i} x_i.$$

Algorithm 1 Perturbed Diagonalization

Set $I_0 = [r]$ and $\mathcal{L} = \emptyset$.
for $k = 1 \dots n$ **do**
  Find the smallest power j such that $C_j[\ell_i]$ is linearly independent of all elements in $\mathcal{L}$, for some $i \in I_{k-1}$.
  Set $q_k = j$, and let i_k be the smallest index i such that $C_{q_k}[\ell_i]$ is linearly independent of all the linear forms in $\mathcal{L}$.
  Update $I_k = I_{k-1} \setminus \{i_k\}$ and set $\pi(k) = i_k$.
  Define $\tilde{L}_k = C_{q_k}[\ell_{i_k}]$ and $\tilde{L}_{k,\varepsilon} = \sum_{j=q_k+1}^{q} \varepsilon^{j-q_k} C_j[\ell_{i_k}]$.
  Update $\mathcal{L} = \mathcal{L} \cup \tilde{L}_k$.
end for
Complete π to be a permutation on $[r]$.
Define $A \in \mathbb{C}[\varepsilon]^{r \times r}$ such that for all $k \in [r]$, $(\tilde{L}_k + \tilde{L}_{k,\varepsilon})(A\mathbf{x}) = x_k$.
Define $L_k(\mathbf{x}) := \ell_{\pi(k)}(A\mathbf{x})$.

To verify the correctness of the constructed matrix A, observe that for each k, $\tilde{L}_{k,\varepsilon}|_{\varepsilon=0} = 0$, meaning that $\tilde{L}_k + \tilde{L}_{k,\varepsilon} = \tilde{L}_k + O(\varepsilon)$. By the definition of q_k, for every $m < q_k$, $C_m[\ell_i] \in \operatorname{span}\{\tilde{L}_j \mid j < k\}$ for all $i \in [r] \setminus I_{k-1}$. Thus, after the variable transformation defined by A, only variables $x_1, \dots, x_{k-1}$ appear in the coefficients of ε^m for $m < q_k$. Furthermore, the coefficient of ε^{q_k} in L_k is precisely x_k, ensuring that (3) holds.

Note that A can be written as $A = A_0 + \varepsilon A_1$, where $A_0 \in \mathbb{C}^{r \times r}$ is invertible because $\tilde{L}_k(A_0\mathbf{x}) = x_k$. Consequently, for some polynomial $\tilde{g} \in \mathbb{C}[\varepsilon][\mathbf{x}]_d$, we have:

$$f(A\mathbf{x}) + \varepsilon \cdot g(A\mathbf{x}) = f(A_0\mathbf{x}) + \varepsilon \cdot \tilde{g}(A_0\mathbf{x}).$$

Therefore:

$$\varepsilon^q f(A_0\mathbf{x}) + \varepsilon^{q+1}\tilde{g}(A_0\mathbf{x}) = \varepsilon^q f(A\mathbf{x}) + \varepsilon^{q+1} g(A\mathbf{x}) = \sum_{i=1}^{r} \ell_i(A\mathbf{x})^d = \sum_{i=1}^{r} L_i^d.$$

The fact that the new decomposition remains local, provided the original decomposition was local, follows directly from the invertibility of A_0. □

Corollary 3.2. *Let f, L_i, and $\tilde{g}$ be as in Lemma 2.9. Then, for every $k \in [n]$ and $j \in [r-1]$, it holds that*

$$\underline{\mathrm{WR}}\left(\frac{\partial^j f}{\partial x_k^j}\right) \leqslant r - k.$$

Proof. By the definition of the L_is, x_k does not appear in $L_1, \dots, L_{k-1}$ (see (3)). Therefore, using the notation of Lemma 2.9, we obtain:

$$\varepsilon^q \frac{\partial^j f(A_0\mathbf{x})}{\partial x_k^j} + \varepsilon^{q+1}\frac{\partial^j \tilde{g}(\mathbf{x})}{\partial x_k^j} = \frac{\partial^j}{\partial x_k^j}\left(\varepsilon^q f(A\mathbf{x}) + \varepsilon^{q+1} g(A\mathbf{x})\right) = \frac{\partial^j}{\partial x_k^j}\left(\sum_{i=1}^{r} L_i^d\right) = (d)_j \cdot \sum_{i=k+1}^{r} \left(\frac{\partial L_i}{\partial x_k}\right)^j \cdot L_i^{d-j}.$$

Since A_0 is invertible, it follows that

$$\underline{\mathrm{WR}}\left(\frac{\partial^j f}{\partial x_k^j}\right) \leqslant r - k.$$

□

Remark 3.3. The conclusion of Corollary 3.2 can be strengthened to

$$\underline{\mathrm{WR}}\left(\frac{\partial^j f}{\partial x_k^j}\right) \leqslant r - k - j + 1,$$

as after each derivative, we can re-diagonalize and conclude that each successive derivative, not taken with respect to x_1, reduces the rank further.

Remark 3.4. We note that Corollary 3.2 implies Lemma 2.7, as it shows that taking a derivative with respect to any variable other than x_1 reduces the Waring rank. Consequently, any monomial can contain at most $r-1$ other variables.

We now give the proof of Theorem 1.1.

Proof of Theorem 1.1. The proof proceeds by induction on r. For $1 < r \leqslant 100$, the result of [DGI+24] implies that $\mathrm{WR}(f) \leqslant 4^r < r^{10\sqrt{r}}$. Hence, we assume from now on that $r \geqslant 100$.

Let $\varepsilon^q f + \varepsilon^{q+1} g = \sum_{i=1}^{r} \ell_i^d$ be a border Waring rank decomposition of f, where $\ell_i(\mathbf{x}) \in \mathbb{C}[\varepsilon][\mathbf{x}]_1$. By applying Lemma 2.9, we can assume without loss of generality that the ℓ_is are in diagonal form, as described in the lemma.

We handle two separate cases. The first is when $d \geqslant r-1$, and the second is when $d < r-1$.

The case $d \geqslant r-1$. From Lemma 2.8, there exists a partition $r = r_1 + \ldots + r_m$ such that f has a decomposition

$$f = \sum_{k=1}^{m} \ell_k^{d-r_k+1} \cdot g_k,$$

where $\ell_k^{d-r_k+1} g_k$ has a local decomposition, with $[\ell_k]$ being the base of the decomposition, and $\underline{\mathrm{WR}}\left(\ell_k^{d-r_k+1} \cdot g_k\right) \leqslant r_k$.

Since $d \cdot \sum_{k=1}^{m} r_k^{10\sqrt{r_k}} \leqslant d \cdot r^{10\sqrt{r}}$, it suffices to prove Theorem 1.1 for local decompositions when $d \geqslant r-1$.

Assume that f has a local border Waring rank decomposition, based in x_1, as in the conclusion of Lemma 2.9.

Let $Y = \{x_1, \ldots, x_{\lfloor 10\sqrt{r} \rfloor}\}$ and $Z = \{x_{\lfloor 10\sqrt{r} \rfloor + 1}, \ldots, x_n\}$. For convenience, rename the variables in Z as $Z = \{z_1, \ldots, z_m\}$ for $m = n - \lfloor 10\sqrt{r} \rfloor$. By Lemma 2.7, we have the following representation of f:

$$f = x_1^{d-r+1} \left(f_0(Y) + \sum_{i=1}^{m} \sum_{k=1}^{r-1} z_i^k \cdot g_{i,k}(Y, z_{i+1}, \ldots, z_m) \right). \tag{4}$$

In other words, we first consider monomials involving only the Y variables. Then, each other monomial contains one or more variables from Z, and we group these monomials according to the minimal i and then the maximal k such that z_i^k divides them.

Clearly, f_0 is a polynomial of degree $r-1$ in $\lfloor 10\sqrt{r} \rfloor$ variables, and hence its Waring rank satisfies

$$\mathrm{WR}(f_0) \leqslant \binom{\lfloor 10\sqrt{r} \rfloor + r - 3}{r-2} = \binom{\lfloor 10\sqrt{r} \rfloor + r - 3}{\lfloor 10\sqrt{r} \rfloor - 1} \leqslant \left(\frac{e(r + \lfloor 10\sqrt{r} \rfloor - 3)}{10\sqrt{r} - 1} \right)^{\lfloor 10\sqrt{r} \rfloor - 1} < (5 \cdot r)^{5\sqrt{r}}.$$

Consequently,

$$\mathrm{WR}\left(x_1^{d-r+1} \cdot f_0\right) < d \cdot (5 \cdot r)^{5\sqrt{r}}.$$

Next, observe that $x_1^{d-r+1} \cdot g_{i,k}$ can be obtained by taking k derivatives of f with respect to z_i, setting $z_1 = \ldots = z_i = 0$, and multiplying the result by $k!$.

From Corollary 3.2, we conclude that $\underline{\mathrm{WR}}\left(x_1^{d-r+1} \cdot g_{i,k}\right) \leqslant r - \lfloor 10\sqrt{r} \rfloor - i$, and clearly it is a polynomial on at most $n-i$ variables. The induction hypothesis implies that

$$\mathrm{WR}\left(x_1^{d-r+1} \cdot g_{i,k}\right) \leqslant d \cdot (r - \lfloor 10\sqrt{r} \rfloor - i)^{10\sqrt{r - \lfloor 10\sqrt{r} \rfloor - i}}.$$

Hence,

$$\mathrm{WR}\left(x_1^{d-r+1} \cdot z_i^k \cdot g_{i,k}\right) \leqslant r \cdot d \cdot (r - \lfloor 10\sqrt{r} \rfloor - i)^{10\sqrt{r - \lfloor 10\sqrt{r} \rfloor - i}}.$$

It follows that

$$\begin{aligned}
\mathrm{WR}(f) &< d \cdot (5 \cdot r)^{5\sqrt{r}} + d \cdot \sum_{i=1}^{m} \sum_{k=1}^{r-1} r \cdot (r - \lfloor 10\sqrt{r} \rfloor - i)^{10\sqrt{r - \lfloor 10\sqrt{r} \rfloor - i}} \\
&< d \cdot (5 \cdot r)^{5\sqrt{r}} + d \cdot r^2 \cdot \sum_{i=1}^{m} (r - \lfloor 10\sqrt{r} \rfloor - i)^{10\sqrt{r - \lfloor 10\sqrt{r} \rfloor - i}} \\
&< d \cdot (5 \cdot r)^{5\sqrt{r}} + d \cdot r^3 \cdot r^{10(\sqrt{r}-5)} \\
&< d \cdot r^{10\sqrt{r}}.
\end{aligned}$$

The case $d < r - 1$**.** Assume that f has a border Waring rank decomposition as in the conclusion of Lemma 2.9. As before, set $Y = \{x_1, \ldots, x_{\lfloor 10\sqrt{r} \rfloor}\}$ and $Z = \{x_{\lfloor 10\sqrt{r} \rfloor + 1}, \ldots, x_n\}$, and rename the variables in Z as $Z = \{z_1, \ldots, z_m\}$ for $m = n - \lfloor 10\sqrt{r} \rfloor$. Using the same reasoning as before, we conclude that

$$\begin{aligned}
\mathrm{WR}(f) &< d \cdot (5 \cdot r)^{5\sqrt{r}} + d \cdot r \cdot \sum_{i=1}^{m} \sum_{k=1}^{r-1} (r - \lfloor 10\sqrt{r} \rfloor - i)^{10\sqrt{r - \lfloor 10\sqrt{r} \rfloor - i}} \\
&< d \cdot r^{10\sqrt{r}}.
\end{aligned}$$

□

Acknowledgement

I thank Prerona Chatterjee, Shir Peleg and Ben lee Volk for valuable discussions on the problem.

Bibliography

[Ald84] Alexander Alder. *Grenzrang und Grenzkomplexität aus algebraischer und topologischer Sicht*. Phd thesis, Universität Zürich, 1984. 1, 2

[Bal18] Edoardo Ballico. On the ranks of homogeneous polynomials of degree at least 9 and border rank 5. *Note Mat.*, 38(2):55–92, 2018. 2

[BB13] Edoardo Ballico and Alessandra Bernardi. Stratification of the fourth secant variety of Veronese varieties via the symmetric rank. *Adv. Pure Appl. Math.*, 4(2):215–250, 2013. 2

[BB17] Edoardo Ballico and Alessandra Bernardi. Curvilinear schemes and maximum rank of forms. *Matematiche (Catania)*, 72(1):137–144, 2017. 2

[BCC+18] Alessandra Bernardi, Enrico Carlini, Maria Virginia Catalisano, Alessandro Gimigliano, and Alessandro Oneto. The hitchhiker guide to: Secant varieties and tensor decomposition. *Mathematics*, 6(12):314, 2018. 2

[BCS97] Peter Bürgisser, Michael Clausen, and M. Amin Shokrollahi. *Algebraic complexity theory*, volume 315 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1997. With the collaboration of Thomas Lickteig. 1

[DDS22] Pranjal Dutta, Prateek Dwivedi, and Nitin Saxena. Demystifying the border of depth-3 algebraic circuits. In *2021 IEEE 62nd Annual Symposium on Foundations of Computer Science—FOCS 2021*, pages 92–103. IEEE Computer Soc., Los Alamitos, CA, [2022] ©2022. 2

[DGI+24] Pranjal Dutta, Fulvio Gesmundo, Christian Ikenmeyer, Gorav Jindal, and Vladimir Lysikov. Fixed-parameter debordering of waring rank. In Olaf Beyersdorff, Mamadou Moustapha Kanté, Orna Kupferman, and Daniel Lokshtanov, editors, *41st International Symposium on Theoretical Aspects of Computer Science, STACS 2024, March 12-14, 2024, Clermont-Ferrand, France*, volume 289 of *LIPIcs*, pages 30:1–30:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2024. 2, 3, 7

[For16] Michael Forbes. Some concrete questions on the border complexity of polynomials. Presentation given at the Workshop on Algebraic Complexity Theory (WACT), 2016. 2

[LT10] J. M. Landsberg and Zach Teitler. On the ranks and border ranks of symmetric tensors. *Found. Comput. Math.*, 10(3):339–366, 2010. 2

ECCC ISSN 1433-8092
https://eccc.weizmann.ac.il