Electronic Colloquium on Computational Complexity, Revision 1 of Report No. 37 (2025)

Uniform Bounds on Product Sylvester-Gallai Configurations

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Abstract

In this work, we explore a non-linear extension of the classical Sylvester-Gallai configuration. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, and let $\mathcal{F} = \{F_1, \ldots, F_m\} \subset \mathbb{K}[x_1, \ldots, x_N]$ denote a collection of irreducible homogeneous polynomials of degree at most d, where each F_i is not a scalar multiple of any other F_j for $i \neq j$. We define $\mathcal{F}$ to be a *product Sylvester-Gallai configuration* if, for any two distinct polynomials $F_i, F_j \in \mathcal{F}$, the following condition is satisfied:

$$\prod_{\substack{k \in [m], \\ k \neq i,j}} F_k \in \operatorname{rad}(F_i, F_j).$$

We prove that product Sylvester-Gallai configurations are inherently low dimensional. Specifically, we show that there exists a function $\lambda : \mathbb{N} \to \mathbb{N}$, independent of $\mathbb{K}$, N, and m, such that any product Sylvester-Gallai configuration must satisfy:

$$\dim(\operatorname{span}_{\mathbb{K}}(\mathcal{F})) \leqslant \lambda(d).$$

This result generalizes the main theorems from [Shp20, PS20, OS24], and gets us one step closer to a full derandomization of the polynomial identity testing problem for the class of depth 4 circuits with bounded top and bottom fan-in.

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ISSN 1433-8092

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1 Introduction

In 1893, Sylvester posed the following basic problem in extremal combinatorial geometry.

Problem 1.1 ([Syl93])**.** *Suppose a finite set of real points are arranged such that any line through two points of the set contains a third point in the set. Prove that the set of points lies on a straight line.*

The first complete affirmative answer to Problem 1.1 was given by Melchior [Mel40]. The same problem was independently posed in 1943 by Erdős [EBW+43], and was answered by Gallai [Gal44]. The above statement is now known as the Sylvester-Gallai theorem.

It is important to note that the answer to Sylvester's problem depends on the base field. For instance, it was known (quite possibly even to Sylvester) that the answer to Problem 1.1 is negative when the base field is $\mathbb{C}$, as the nine inflection points of any non-singular planar cubic form a 2-dimensional complex configuration satisfying Sylvester's conditions. In 1966, Serre asked in [Ser66] whether any complex configurations satisfying Sylvester's conditions must be 2-dimensional – that is, whether the configurations arising from non-singular planar cubics are extremal. An affirmative answer to Serre's question was given in 1983 by Hirzebruch [Hir83], using highly non-elementary algebro-geometric tools.[1]

Ever since the solution of Sylvester's problem, several generalizations and variations have been studied in combinatorial geometry, theoretical computer science (TCS) and coding theory [EK66, Han65, PS09, EPS06, BDYW11, DSW14, PS20, PS21] – see [BM90, Dvi12] for brief surveys on these generalizations. The usual setup of such generalizations is as follows: given a *finite* collection of geometric objects (points, in the case of Sylvester's problem) satisfying *enough* **local conditions** (collinearity of certain triples of points, in the case of Problem 1.1), one wants to know if such collection of geometric objects must be "*low dimensional*" (all points must be in one line, in the case of Problem 1.1). As is usual in the literature, any configuration satisfying the proposed local conditions are called *Sylvester-Gallai configurations*, and the result stating that such configurations are low dimensional is referred to as a *Sylvester-Gallai* type theorem.

One line of generalizations of Problem 1.1 arises from projective duality, which we now discuss. By projective duality, any point $P = (p_1, \dots, p_N)$ gives rise to a dual hyperplane, defined by the zero set of the linear form $\ell_P := p_1x_1 + \cdots + p_Nx_N$, which we denote by $V(\ell_P)$.[2] Given three points P, Q, R, the condition that R is in the line defined by P, Q is equivalent to $V(\ell_P, \ell_Q) \subset V(\ell_R)$ in the dual space. Lastly, given points $P_1, \dots, P_m$, the condition that they are collinear is equivalent to $\dim \operatorname{span}_{\mathbb{R}} \{\ell_{P_1}, \dots, \ell_{P_m}\} = 2$. Thus, we can recast Sylvester's problem as follows: given a finite set of distinct hyperplanes defined by the set of linear forms $\mathcal{L} := \{\ell_1, \cdots, \ell_m\} \subset \mathbb{R}[x_1, \dots, x_N]$ such that for any $i \neq j \in [m]$, there is $k \neq i, j$ such that $V(\ell_i, \ell_j) \subset V(\ell_k)$, then it must be the case that $\dim \operatorname{span}_{\mathbb{R}} \{\mathcal{L}\} = 2$. For simplicity, if one applies projective duality in two dimensions, the foregoing statement becomes quite natural: given a finite set of distinct lines (the set $\mathcal{L}$) such that for any two lines $\ell_i, \ell_j \in \mathcal{L}$, there must be a third line in $\mathcal{L}$ which passes through the intersection of ℓ_1, ℓ_2 (i.e., $V(\ell_1, \ell_2)$), then it must be the case that the set of lines forms a pencil (i.e., all lines intersect at a common point). In fact, this dual formulation was precisely the setting treated in [Hir83].

Motivated by questions in algebraic complexity theory, Gupta [Gup14] proposed non-linear, algebro-geometric generalizations to the above (dual) formulation of Sylvester's problem (and their set variants, such as [EK66]). In this work, we make progress on Gupta's program and fully resolve one such generalization of Sylvester's problem. Before we describe our main result, in the following subsection we describe the connection between the Sylvester-Gallai configurations that

[1]By the Lefschetz principle (alternatively by [BDYW11, Theorem 8.3]), one can extend the same result to any algebraically closed field of characteristic zero.

[2]More generally, for any set of polynomials I, we denote the set of common zeroes of all polynomials in I by $V(I)$.

we study and the Polynomial Identity Testing (PIT) problem, a fundamental problem in algebraic complexity theory.

1.1 Polynomial Identity Testing (PIT) and Sylvester-Gallai configurations

The Polynomial Identity Testing (PIT) problem is the task of determining whether a given algebraic circuit computes the zero polynomial. While there are simple and efficient randomized algorithms for the PIT problem, developing an efficient deterministic solution remains a significant open problem in theoretical computer science. The PIT problem is closely tied to fundamental topics such as lower bounds for algebraic circuits [HS80, Agr05, KI04] and derandomizing key problems in mathematics and computer science [AKS04, FS13, KSS15, Mul17, GT17, FGT19]. For a deeper exploration of PIT and its applications, see [Sax09, SY10, Sax14].

The recent works [AV08, GKKS16, Tav15] have shown that the PIT problem for general circuits can be reduced to solving it for low-depth circuits, such as unrestricted depth-3 or homogeneous depth-4 circuits, which has made these circuit classes a focus of recent study. Notably, progress has been made towards deterministic, polynomial-time PIT for depth-3 circuits of constant top fanin via the connection between this problem and linear Sylvester-Gallai configurations arising from depth-3 identities [DS07, KS09, SS13].

Let us now explain how Sylvester-Gallai configurations arise in the context of depth-3 and depth-4 PIT. Let $\mathbf{x} := (x_1, \ldots, x_N)$ be a tuple of N variables and $S := \mathbb{C}[\mathbf{x}]$ be the polynomial ring in N variables. Suppose we are given a depth three circuit with top fan-in three (denoted $\Sigma^3\Pi\Sigma$) that computes a polynomial P. Such a polynomial has the form

$$P = \prod_{i=1}^{m} \ell_i(\mathbf{x}) + \prod_{j=1}^{m} g_j(\mathbf{x}) + \prod_{k=1}^{m} h_k(\mathbf{x}), \tag{1.1}$$

where ℓ_i, g_j, h_k are linear forms.[3] When $P \equiv 0$, that is, when we have an identity, and moreover this representation of the identity is "efficient,"[4] we may wonder whether the linear forms in the identity must be "low dimensional" (i.e., all such linear forms must "depend on few variables"). To capture the dimensionality of the circuit given by Eq. (1.1), [DS07] defined the *rank* of the circuit as being the linear span of the forms that appear in the circuit. In case of Eq. (1.1), we have that the rank is $\dim \operatorname{span}_{\mathbb{C}} \{\ell_i, g_j, h_k\}_{i,j,k \in [m]}$.

Consider a linear form ℓ_i from the first gate and a second linear form g_j from the second gate. Since $P \equiv 0$, we have that for any $\mathbf{a} \in \mathbb{C}^N$ such that $\ell_i(\mathbf{a}) = g_j(\mathbf{a}) = 0$, it must be the case that $\prod h_k(\mathbf{a}) = 0$. In other words, we have that $V(\ell_i, g_j) \subset V(\prod h_k) = \bigcup_k V(h_k)$. Since the algebraic set $V(\ell_i, g_j)$ is irreducible, we must have $V(\ell_i, g_j) \subset V(h_a)$ for some $a \in [m]$. Note that this last condition (combined with the symmetry among the gates) is exactly the local constraint arising from the dual formulation of Sylvester's problem. In [DS07], the authors asked whether sets $\{\ell_i\}_{i\in[m]}, \{g_j\}_{j\in[m]}, \{h_k\}_{k\in[m]}$ arising from Eq. (1.1) which satisfy the above conditions must be low-dimensional – more precisely, whether $\dim \operatorname{span}_{\mathbb{C}} \{\ell_i, g_j, h_k\}_{i,j,k} = O(1)$.

As it turns out, the Sylvester-Gallai like condition in the previous paragraph can be seen as a set-version of Sylvester's problem, which was studied by [EK66]. This work showed that such configurations must have constant rank. Thus, we deduce that $\operatorname{span}_{\mathbb{C}} \{\ell_i, g_j, h_k\} = O(1)$. This

[3] Note that, since we are only given the circuit computing P, we do not "know" the polynomial P in the usual way – that is, we do not know its coefficients and monomials.

[4] We mean "efficient" in the sense that no subset of the three summands are themselves 0, and all three summands do not have a common factor. Formally, we want the circuit to be simple and minimal. Definitions of simple and minimal can be found in [KS09].

shows that any $\Sigma^3\Pi\Sigma$ identity essentially depends on constantly many variables. Combined with the results of [KS08], this gives a black box deterministic PIT algorithm for such circuits. While significantly extra complexity and subtleties arise when one works with $\Sigma^k\Pi\Sigma$ circuits for $k > 3$, the linear forms in "efficient" identities arising from such circuits satisfy enough local relationships to deduce that they have low rank. This approach was carried out by [KS09, SS13], who generalized the above approach and showed an intrinsic and elegant connection between $\Sigma^k\Pi\Sigma$ identities and Sylvester-Gallai configurations (see [SS13, Theorem 1.4]). As a result of this connection, Sylvester-Gallai theorems imply deterministic, polynomial-time PIT algorithms for $\Sigma^k\Pi\Sigma$ circuits, when $k = O(1)$.

Motivated by these results, Gupta [Gup14] studied depth four circuits with constant top and bottom fan-ins. This circuit family is denoted $\Sigma^k\Pi\Sigma\Pi^d$, and it computes polynomials which can be written as a sum of k products of polynomials of degree at most d.

Consider a polynomial Q computed by a $\Sigma^3\Pi\Sigma\Pi^d$ circuit. It has the form

$$Q = \prod_{i=1}^{m_1} A_i(\mathbf{x}) + \prod_{j=1}^{m_2} B_j(\mathbf{x}) + \prod_{k=1}^{m_3} C_k(\mathbf{x}),$$

where A_i, B_j, C_k are polynomials of degree at most d. If $Q \equiv 0$ and the representation is efficient, as in the previous case, we have $V(A_i, B_j) \subseteq V(\prod C_k)$.[5] However, as the forms are not necessarily linear, we have that $V\left(A_i, B_j\right)$ is not necessarily irreducible, and thus we no longer have the same Sylvester-Gallai type configuration, as we cannot guarantee the existence of $k \in [m]$ such that $V\left(A_i, B_j\right) \subset V\left(C_k\right)$. Nevertheless, as there are many constraints of the form $V(A_i, B_j) \subseteq V(\prod C_k)$, [Gup14] asked whether such Sylvester-Gallai type conditions would be enough to prove that $\dim \operatorname{span}_{\mathbb{C}} \left\{A_i, B_j, C_k\right\}_{i,j,k}$ is small enough.

Given any "efficient" $\Sigma^3\Pi\Sigma\Pi^d$ circuit (not necessarily computing the zero polynomial), [Gup14] called it a Sylvester-Gallai circuit if the above constraints on the zerosets (i.e. $V(A_i, B_j) \subseteq V(\prod C_k)$) hold. The above discussion shows that all such circuits that compute 0 are Sylvester-Gallai circuits, but the converse is not true: for instance, multiplying a single form by a constant can change whether or not the circuit computes 0 without changing the above condition.

Non Sylvester-Gallai circuits are relatively simple compared to Sylvester-Gallai circuits. Non Sylvester-Gallai circuits do not have too many cancellations (the above discussion shows that they can never compute 0), and the circuit structure can be preserved by certain linear variable reduction maps, which is the main tool used in the works [Gup14, Guo21] to handle these circuits. This allows these works to certify the non-zeroness of non-Sylvester-Gallai circuits.

The cancellations in Sylvester-Gallai circuits on the other hand are a lot more subtle, and it is not clear if they are preserved by the linear maps of [Gup14, Guo21]. Similar to how linear Sylvester-Gallai configurations have constant rank, Gupta conjectured that sets of higher degree forms satisfying the local condition $V(A_i, B_j) \subset V(\prod_k C_k)$ must "depend on constantly many variables." If this is true, then we can efficiently test if Sylvester-Gallai circuits are nonzero using the methods of [BMS13]. A simpler form of Gupta's main conjecture is the following generalization of the configurations studied in [EK66]:

Conjecture 1.2 (Gupta's main conjecture - simple form)**.** *Let $\{A_i\}_{i\in[m_1]} \cup \left\{B_j\right\}_{j\in[m_2]} \cup \{C_k\}_{k\in[m_3]}$ be a set of irreducible polynomials of degree $\leqslant d$ such that for any $i \in [m_1], j \in [m_2]$, we have that $V\left(A_i, B_j\right) \subseteq V\left(\prod_k C_k\right)$ (and the same relations hold when exchanging the roles of the polynomials A, B, C). Then there exists a function $\lambda : \mathbb{N} \to \mathbb{N}$ such that*

$$\dim \operatorname{span}_{\mathbb{C}} \left\{A_i, B_j, C_k\right\}_{i,j,k} \leqslant \lambda(d).$$

[5] By symmetry among the gates, we also have $V(A_i, C_k) \subseteq V\left(\prod B_j\right)$ and $V(B_j, C_k) \subseteq V\left(\prod A_i\right)$.

A first step towards establishing the above conjecture is to establish the Sylvester-Gallai version of the above conjecture. This motivates us to consider the Sylvester-Gallai configurations we study in this article, which we now define.[6] By way of the algebraic-geometric dictionary, we will henceforth replace the geometric condition $V(A,B) \subset V(C)$ by the equivalent algebraic condition $\mathrm{rad}(C) \subset \mathrm{rad}(A,B)$.[7]

Definition 1.3 (Product Sylvester–Gallai configurations)**.** A finite set $\mathcal{F} \subset \mathbb{C}[x_1,\ldots,x_N]$ of irreducible forms of degree at most d is a d-product-SG-configuration if the following hold.

- $F_i \notin (F_j)$ for any $F_i, F_j \in \mathcal{F}$. (each form encodes a different hypersurface)
- For each $F_i, F_j \in \mathcal{F}$, we have (Sylvester-Gallai condition)

$$\prod_{F \in \mathcal{F}\setminus\{F_i,F_j\}} F \in \mathrm{rad}(F_i, F_j).$$

As stated above, the reader may find it strange that the product constraint no longer seems local, since the condition for each F_i, F_j involves every other form in the configuration. However, by a standard Bézout type argument it follows that the condition $\prod_{F\in\mathcal{F}\setminus\{F_i,F_j\}} F \in \mathrm{rad}(F_i,F_j)$ is equivalent to the (local) condition that there are indices $k_1,\ldots,k_{d^2}$ different from i,j such that

$$\prod_{a=1}^{d^2} F_{k_a} \in \mathrm{rad}(F_i, F_j).$$

Since d is a constant, each (local) condition now only depends on constantly many forms in the configuration as expected. As this equivalent condition is more cumbersome to state, we prefer to work with the above definition.

In a similar fashion to [Gup14], we can also conjecture that such "product" configurations are "low dimensional." This is the content of the following conjecture:

Conjecture 1.4 (Product Sylvester-Gallai Conjecture)**.** *There is a function* $\lambda : \mathbb{N} \to \mathbb{N}$ *such that if* $\mathcal{F}$ *is a* d-product-SG-configuration *then* $\dim \mathrm{span}_{\mathbb{C}}\{\mathcal{F}\} = \lambda(d)$.

The above conjecture was first considered by [PS20], where the authors proved Conjecture 1.4 for $d = 2$. Our main theorem, which we now state, generalizes their main result to forms of every degree, thus fully settling Conjecture 1.4.

Theorem 1.5 (Product Sylvester-Gallai Theorem)**.** *There is a function* $\lambda : \mathbb{N} \to \mathbb{N}$ *such that if* $\mathcal{F}$ *is a* d-product-SG-configuration *then* $\dim \mathrm{span}_{\mathbb{C}}\{\mathcal{F}\} = \lambda(d)$.

Our proof techniques build upon the techniques introduced in [Shp20, OS24] to study radical Sylvester-Gallai configurations, which we define in Section 1.3. In Section 1.3 we also discuss related and previous works on non-linear Sylvester-Gallai configurations and the PIT problem for depth-4 circuits. In Section 1.4, we give a high level outline of our proof, highlighting the technical difficulties that need to be overcome, as well as the new conceptual contributions of this work. We will now discuss the main contributions of this work, where we state our main technical result, which is of independent interest.

[6]We will later see that our proof strategy will require us to further generalise the definition given above, and we do this in Definition 3.3.

[7]Here we recall that the radical ideal generated by a set of polynomials $\mathcal{I}$, denoted by $\mathrm{rad}(\mathcal{I})$, is the set of polynomials H such that some power of H is in the ideal generated by $\mathcal{I}$.

1.2 Contributions of this paper

This paper has two main contributions. The first is the proof of Theorem 1.5, which states that product Sylvester-Gallai configurations of degree bounded by d have dimension uniformly upper bounded by a function which depends only on d. This result simultaneously generalizes [Shp20, PS20, OS24], putting us one step closer towards a proof of Conjecture 1.2, and also to the main conjecture of Gupta [Gup14]. To achieve this, we adapt the inductive approach from [OS24] to work with product configurations. This adaptation, as we discuss in Section 1.4, requires us to deal with several subtleties, one of them being to establish a useful property of the general quotients of [Shp20, OS24]: we prove that such general quotients behave well with respect to (a generalized notion of) absolute irreducibility. More precisely, we prove that that these quotients send *absolutely reducible* forms to *reducible* elements, and that forms which are *absolutely irreducible* over the vector space being quotiented out will *remain absolutely irreducible* after a general quotient.

The second contribution is our main technical result, Theorem 4.16, which relates absolute irreducibility of a polynomial P with an *effective and uniform bound* on the non-primality of certain ideals containing P. In a simplified form, it can be stated as:

Theorem 1.6 (Prime bound - informal)**.** *Let $S := \mathbb{C}[x_1, \ldots, x_N, y_1, \ldots, y_n]$ and $P \in S \setminus (x_1, \ldots, x_N)$ be a form of degree d that is irreducible over $\overline{\mathbb{C}(x_1, \ldots, x_N)}[y_1, \ldots, y_n]$. The number of non-associate irreducible forms $Q \in \mathbb{C}[x_1, \ldots, x_N]$ such that (P, Q) is not prime is bounded above by a function of d.*

We believe that the above theorem, and its more general version (Theorem 4.16), together with its proof, are of independent interest. We now briefly discuss the ingredients needed in the proof of the above theorem. If one only combines results from Gröbner basis theory and elimination theory with techniques from algebraic geometry, one can obtain an effective bound for the above theorem which depends on N, n and d. Since the goal of a Sylvester-Gallai theorem is to obtain bounds *independent of the number of variables*, such a bound will not be good enough for our purposes. By combining the Stillman's uniformity bounds of [AH20a] with results from elimination theory and algebraic geometry, we are able to obtain an effective bound (depending only on n, d) on the number of non-associate irreducible forms Q such that (P, Q) is not prime.

Once the dependence on N has been removed and a finite bound on the number of (non-associate) bad forms Q has been established, we can then use Bertini's theorem and the Ananyan-Hochster principle to further drop the dependency on n, finally obtaining a bound which simply depends on the degree d. To achieve this, we give very effective bounds on the degrees of generators of Gröbner bases for ideals generated by a small number of variables of low degree, and also for certain elimination ideals of such ideals. These bounds could be of independent interest.

An important point to note is that [OS24] builds on the Ananyan-Hochster principle to show their transfer theorems, which we also use. However, to prove the above theorem, we also need to apply the Ananyan-Hochster principle in a different manner: combined with results from Gröbner basis theory, we use it to find low degree polynomials in elimination ideals of ideals generated by a small number of low degree forms. The details of the above discussion can be found in Section 4.

1.3 Related Work

1.3.1 Sylvester-Gallai configurations

In [Gup14], as a first step of his plan, Gupta proposed the study of a direct generalization of the linear Sylvester-Gallai configurations, which are captured by the following definition.

Definition 1.7 (Radical Sylvester–Gallai configuration)**.** Let $\mathcal{F} \subset \mathbb{C}[x_1, \ldots, x_N]$ be a finite set of irreducible forms of degree at most d. We say that $\mathcal{F}$ is a d-radical Sylvester-Gallai configuration if the following hold.

- $F_i \notin (F_j)$ for any $F_i, F_j \in \mathcal{F}$.
- For every $F_i, F_j \in \mathcal{F}$, we have $F_k \in \text{rad}(F_i, F_j)$ for some $F_k \in \mathcal{F} \setminus \{F_i, F_j\}$.

The following conjecture is a strengthening of [Gup14, Conjecture 2].

Conjecture 1.8 (Radical Sylvester-Gallai conjecture)**.** *There is a function $\lambda : \mathbb{N} \to \mathbb{N}$ such that if $\mathcal{F}$ is a d-radical Sylvester-Gallai configuration then* $\dim \text{span}_{\mathbb{C}} \{\mathcal{F}\} = \lambda(d)$.

Conjecture 1.8 was proved by [Shp20] in the special case when $d = 2$. Subsequently, [OS22] proved Conjecture 1.8 for $d = 3$, and [OS24] fully resolved the above conjecture in the affirmative. Note that the condition $F_k \in \text{rad}(F_i, F_j)$ for forms of degree d is a natural generalization of the condition $\ell_k \in (\ell_i, \ell_j)$: by the Nullstellensatz, these conditions are equivalent to $V(F_k) \supset V(F_i, F_j)$ and $V(\ell_k) \supset V(\ell_i, \ell_j)$ respectively. Therefore Conjecture 1.8 is a direct generalisation of the usual Sylvester-Gallai theorem for forms of higher degrees.

The first work to consider product Sylvester-Gallai configurations was the work of [PS20], where the authors settled Conjecture 1.4 for quadratic forms. In follow up work [PS21], the same authors settled Conjecture 1.2 for the case when $d = 2$, therefore proving that Gupta's PIT algorithm runs in deterministic, poly-time for $\Sigma^3\Pi\Sigma\Pi^2$ circuits. Note that the above path in the proofs of Sylvester-Gallai type results is a natural path towards resolving the PIT problem for $\Sigma^3\Pi\Sigma\Pi^d$ circuits, since Conjecture 1.8 follows from Conjecture 1.4, and it can be shown that Conjecture 1.2 would follow from a robust version of Conjecture 1.4.

Robust and higher dimensional generalizations of the radical Sylvester-Gallai theorems have also been studied by the works [PS22, GOS22, GOPS23]. Aside from settling such interesting questions in extremal combinatorial geometry, such variations are also motivated by the PIT problem for depth-4 circuits, in the hope that these more general versions may be useful towards obtaining a (potentially simpler) proof of Gupta's conjectures.

1.3.2 PIT for depth four circuits

Depth-4 circuits with bounded top fan-in are among the "easiest" classes for which we do not have deterministic, poly-time PIT algorithms. There has thus been some work on deterministic PIT for these circuits using methods other than the Sylvester-Gallai based methods discussed above.

In [DDS21], the authors give a quasipolynomial time PIT algorithm for depth 4 circuits of bounded top and bottom fanins, via the Jacobian method of [ASSS16]. By using the logarithmic derivative and its power series, they are able to modify the top sum gate in the circuit to a powering gate. Even though this breaks the bounded top fan-in assumption, models of circuits with powering gates are well understood, and efficient PIT algorithms for them exist. This allowed them to harness the known algorithms for such models and obtain their result.

The breakthrough work of [LST22] on lower bounds for bounded depth circuits gives another approach to PIT for this model. Hardness-randomness tradeoffs have been well studied in algebraic complexity [Agr05, KI04], and the work of [CKS19] showed that these tradeoffs also hold in the bounded depth setting. Combining these results gives a subexponential time PIT algorithm for depth four circuits. Based on the lower bounds of [LST22], a hitting set generator for bounded depth circuits was constructed by [AF22], which gives another subexponential time PIT algorithm for depth four circuits, with improved parameters. This generator relies on the fact that low depth circuits cannot detect low rank matrices since determinant is hard for them.

It is important to note that neither of the above methods seem to be able to give a truly polynomial time algorithm for the PIT problem for $\Sigma^k\Pi\Sigma\Pi^d$ circuits. The only currently known method that might do so is the approach of [Gup14].

1.4 High-level proof overview

In this subsection we provide an overview of the proof of our main result, Theorem 1.5. We begin by outlining the high-level approach used in previous works to bound the dimension of the span of radical Sylvester-Gallai configurations and discuss the challenges of applying this method in our context. Next, we describe the tools that we employ to address these challenges. Finally, we provide a concise summary of how these components are combined to establish our main theorem.

To illustrate the ideas in this overview, we use a special case of product Sylvester-Gallai configurations with a simplifying assumption as an example. In the final part of this subsection, we explain how this assumption can be eliminated to extend the argument to the general case.

1.4.1 High-level approach

At a high level, our proof uses a similar approach as the previous works on higher degree Sylvester-Gallai configurations [Shp20, PS20, PS21, PS22, GOPS23, OS24], which we now summarize.

Let $R := \mathbb{C}[x_1, \ldots, x_N]$, and $S := R[y_1, \ldots, y_n]$ be two polynomial rings and $\mathcal{F} \subset S$ be a radical (or product) Sylvester-Gallai configuration, where the forms in $\mathcal{F}$ have degree at most d. Thus, we can write $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_d$, where $\mathcal{F}_e$ are the forms in $\mathcal{F}$ of degree e. We start with the assumption that each form only depends on constantly many variables in S.[8] A number of key ideas can already be highlighted in this easier setting.

We want to control the highest degree forms $\mathcal{F}_d$ in our configuration, with the goal of reducing to the case when the highest degree is $d-1$, where we can proceed inductively.

If many ideals of the form (F_i, F_j), where $F_i, F_j \in \mathcal{F}_d$, are radical (or prime), then the set $\mathcal{F}_d$ is essentially a (robust) linear Sylvester-Gallai configuration. In this case, the linear Sylvester-Gallai theorems imply that constantly many forms $F_1, \ldots, F_a$ are a basis for $\text{span}_{\mathbb{C}}\{\mathcal{F}_d\}$, and if $x_1, \ldots, x_r$ is the union of the set of variables of $F_1, \ldots, F_a$, then $\mathcal{F}_d \subset \mathbb{C}[x_1, \ldots, x_r]$. This is sufficient to control the forms of $\mathcal{F}_d$ and apply our inductive step. The interesting case is thus when there are many ideals generated by forms in $\mathcal{F}_d$ that are not radical (or prime). In this case, the goal is to show that the forms in $\mathcal{F}_d$ must share many variables in common. The steps listed below show how previous works have dealt with this case.

1. First, one devises a structure theorem on ideals that are not radical (or prime). In principle, such structure theorems show that if an ideal generated by two polynomials is not radical (or not prime), then the variables appearing in the polynomials must have some special "dependencies".

2. Next, combine the structure theorem with the local conditions to show the existence of constantly many "good variables" $x_1, \ldots, x_b$ such that $\mathcal{F}_d \subset (x_1, \ldots, x_b)$.

   In this step, our approach differs from previous works, as in a product configuration, it may not be possible to directly achieve this high level of control on the forms in $\mathcal{F}_d$. We discuss this in more detail in Step 2 below.

[8] The set of variables may be different across the forms, otherwise the main theorem is trivially true. Also, note that the variables of S are both the x and y variables.

3. Finally, we "randomly project" (i.e., apply a general quotient) the special variables $x_1, \ldots, x_b$ to a new variable z in order to reduce the degree of the forms in $\mathcal{F}_d$.

   Also, in this step our approach will differ from previous works. Instead of reducing degree directly, we will make the forms in $\mathcal{F}$ "*factor more*."

We now elaborate on this approach, pointing out the new difficulties we face in this work, and sketch how we deal with these issues.

Step 1 - Structure theorems: Earlier works ([Shp20, OS22]) gave fairly strong classification of ideals that are not radical in the first step. In particular, they consider the possible minimal primes that a non-radical ideal of the form (P, Q) can have, and possible multiplicities that these minimal primes can have, giving a structural result for each of these cases. However this strategy is hard to generalise beyond the case of ideals generated by two cubics, since these ideals have very high degree and their primary decomposition becomes much more complex as the degree grows.

Subsequent work by [OS24] used a significantly weaker (albeit more general) structure theorem, and showed that this suffices to carry out the strategy above. The structure theorem in [OS24] is the following. If $P \in S$ is an irreducible form of degree d that is not in the ideal $(x_1, \ldots, x_N)$, then there are at most $3d^3$ square-free forms $Q_i \in R$ such that (P, Q_i) is not radical. Informally, this statement says that if a form P depends non-trivially on some variable(s) y_i, then (P, Q) must be radical for almost every polynomial Q that only depends on $x_1, \ldots, x_N$.

For product Sylvester-Gallai configurations, the ideals of interest are those that are not prime. One could hope to generalize the structural result from [OS24] to this setting. However, a statement of the above form is no longer true here. Take the form $P = y_1^4 - x_1x_2y_2^2$. This form depends on the variables y_1, y_2. However, for every irreducible linear form $Q \in R$, the ideal $\left(P, x_1x_2 - Q^2\right)$ is not prime, even though P and $x_1x_2 - Q^2$ are irreducible. This is our first technical difficulty.

We overcome this by showing that the above only happens because P is reducible as a form in $\overline{\mathbb{C}(x_1, \ldots, x_N)}[y_1, y_2]$, since it factors as $P = \left(y_1^2 - \sqrt{x_1x_2}y_2\right)\left(y_1^2 + \sqrt{x_1x_2}y_2\right)$. This is where our first key conceptual and technical contribution comes in. We show that for any degree d form in $R[y_1, \ldots, y_m]$ that is irreducible in $\overline{\mathbb{C}(x_1, \ldots, x_N)}[y_1, \ldots, y_n]$ (such forms are called absolutely irreducible over $R[y_1, \ldots, y_m]$), there are only finitely many $Q_i \in R$ such that (P, Q_i) is not prime.

A fairly subtle issue remains in the above statement. In order to solve the product Sylvester-Gallai problem, we need quantitative bounds on the number of "bad forms" Q_i. Moreover, we need a bound which is solely a function of $\deg(P)$. If we use standard techniques from commutative algebra and algebraic geometry, we obtain bounds which depend on the number of variables (that is: N, n) and on $\deg(P)$. This is not sufficient for us, as we want to show that $\dim \operatorname{span}_{\mathbb{C}}\{\mathcal{F}\}$ does not depend on the number of variables in the polynomial ring. To prove that the number of bad forms is in fact *independent* of N and n, we need to work a bit harder, and we need to make use of two additional tools: Bertini-type theorems (to remove the dependence on n), and the Ananyan-Hochster construction of small strong subalgebras (this will remove the dependence on N). This yields, in its basic form, Theorem 1.6, which we discussed in Section 1.2.

Step 2 - Finding a small set of "good common variables": Suppose $\mathcal{F}$ is a radical Sylvester-Gallai configuration. By the above, we can assume that $\mathcal{F}_d$ is not a robust linear Sylvester-Gallai configuration. Our assumption implies that for many pairs F_i, F_j, the ideal $\left(F_i, F_j\right)$ is not radical. In particular, for a small $0 < \varepsilon < 1$, we can find $3d^3$ forms $F_1, \ldots, F_{3d^3} \in \mathcal{F}_d$ such that $\left(\mathcal{F}_i, \mathcal{F}_j\right)$ is not radical for $i \leqslant 3d^3$ and $j \leqslant (1-\varepsilon)|\mathcal{F}_d|$. The radical bound mentioned in Step 1 shows that there is $a \in O(1)$ and an ideal generated by linear forms $(\ell_1, \ldots, \ell_a)$ that contains every such F_j. Repeating this argument, we can find an ideal $(\ell_1, \ldots, \ell_b)$ with b constant such that $\mathcal{F}_d \subset (\ell_1, \ldots, \ell_b)$.

When $\mathcal{F}$ is a product Sylvester-Gallai configuration a number of issues arise: first, our structure theorem deals with prime ideals (instead of radical) and our assumptions are that the forms are absolutely irreducible. Hence, we will only be able to draw the following weaker conclusion: there are constantly many variables $x_1, \ldots, x_b$ such that every $F \in \mathcal{F}_d$ is either in $(x_1, \ldots, x_b)$, or absolutely reducible in the ring $S' := \mathbb{C}(x_1, \ldots, x_b)[x_{b+1}, \ldots, x_N, y_1, \ldots, y_n]$. Therefore, when we apply the general quotient (i.e. a random projection), we can no longer guarantee that the degree of our forms reduce. We deal with this issue by changing the way we "decompose" the configuration $\mathcal{F} = \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_d$. This is a key difference between our approach and previous approaches, since this issue does not arise when studying radical Sylvester-Gallai configurations. This new decomposition also allows us to solve an issue that arises when we apply the general quotient. We will now state this issue, and then discuss the solution.

Step 3 - General quotients & degree reduction/lowering degree: The image of an irreducible form under a projection map can be reducible. In order to apply induction, the definition of product Sylvester-Gallai configurations has to be generalised to allow the forms in $\mathcal{F}$ to be reducible. Hence, we update our definition of product Sylvester-Gallai configurations to the following.

Definition 1.9 (Product Sylvester–Gallai configurations)**.** A finite set $\mathcal{F} \subset \mathbb{C}[x_1, \ldots, x_N]$ of forms of degree at most d is a d-product-SG-configuration if the following hold.

- $\gcd(F_i, F_j) = 1$ for all $i \neq j$.
- For every $F_i, F_j \in \mathcal{F}$, we have $\prod_{F \in \mathcal{F} \setminus \{F_i, F_j\}} F \in \operatorname{rad}(F_i, F_j)$.

This generalisation already appears in [OS24], where reducibility is not an issue when dealing with radical ideals, since the radical bound also applies to reducible forms. The prime bound however can no longer apply, irrespective of any assumption: if P is reducible then (P, Q) can never be prime. In particular, if every form in $\mathcal{F}_d$ is reducible, then we can draw no conclusion about $\mathcal{F}_d$ using the above ideas.

We tackle both these issues by turning this reducibility into an advantage: instead of trying to reduce the degree as in the inductive approaches of [Shp20, OS24, PS20], we will try and make the forms in our configuration "factor more".

In order to formalise what we mean by "factor more", we introduce the notion of factor sets of a product configuration. Given a (generalized) product Sylvester-Gallai configuration $\mathcal{F}$, we define the factor set $\mathcal{I}$ of $\mathcal{F}$ to be the set of irreducible factors of all forms in $\mathcal{F}$. As before, we write $\mathcal{I} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_c$, where $\mathcal{I}_j \subset S_j$.

If every form in $\mathcal{F}$ factors a lot, then all the forms in $\mathcal{I}$ will have low degree. We will induct on the highest integer c such that $\mathcal{I}_c$ is nonzero. Now, the set of "common variables" we look for will not be variables common to $\mathcal{F}_d$, but variables common to $\mathcal{I}_c$. The absolute irreducibility condition remains, and we end up with variables $x_1, \ldots, x_b$ such that every form in $\mathcal{I}_c$ is either in the ideal $(x_1, \ldots, x_b)$ or absolutely reducible. We postpone the discussion of how we do this step to the next subsection. We now show how these common variables are used in the rest of the induction step.

Step 2 lets us find the "common variables" $x_1, \ldots, x_b$. Define a projection map φ that maps each x_i for $1 \leqslant i \leqslant b$ to a random multiple of a new variable z, and acts as identity on the other variables. If $F \in (x_1, \ldots, x_b)$ then $\varphi(F) = zF'$, where $\deg(F') < \deg(F)$. We define $\varphi(\mathcal{F})$ to be the image of $\mathcal{F}$ under φ, after factoring out powers of z. As we show in Sections 6 and 8 such a projection map preserves radical and product Sylvester-Gallai structure, and also allows bounds on the dimension of $\varphi(\mathcal{F})$ to be lifted back to bounds on the dimension of $\mathcal{F}$.

If we have $\mathcal{F}_d \subset (x_1, \ldots, x_b)$ (as in the case of radical Sylvester-Gallai configurations) then $\varphi(\mathcal{F})$ is a Sylvester-Gallai configuration with forms of degree at most $d-1$, and induction can be applied. If on the other hand $\mathcal{F}$ is a product Sylvester-Gallai configuration, then we have the weaker condition that every $F \in \mathcal{I}_c$ is either in $(x_1, \ldots, x_b)$ or absolutely reducible in S'. We show that the image of any F that is absolutely reducible in S' under the map φ is reducible. Thus, in either case we conclude that $\mathcal{I}'_c = \emptyset$, where $\mathcal{I}'$ is a factor set of $\varphi(\mathcal{F})$, and we can apply induction.

In the next subsection we show how we find these "common variables" in our setting. Note that we are still under the assumption that every form in $\mathcal{F}$ (and thus in $\mathcal{I}$) depends on only constantly many variables. We will also show how this assumption is removed.

1.4.2 Putting everything together

Let $\mathcal{F}$ be a product Sylvester-Gallai configuration of forms of degree at most d. We now allow forms in $\mathcal{F}$ to be reducible, but still require that they are square-free and are pairwise relatively prime. Let $\mathcal{I} := \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_d$ be the set of irreducible factors of forms in $\mathcal{F}$. We will induct on the largest integer c such that $\mathcal{I}_c \neq \emptyset$. We first go over the base case in our induction.

Our base case is when every form $F_i \in \mathcal{F}$ is a product of linear forms, say $F_i = \prod_a \ell_{ia}$. In this case, $\mathcal{I} = \mathcal{I}_1 = \{\ell_{ia}\}_{i,a}$. Now suppose we pick ℓ_{ia} and ℓ_{jb} with $i \neq j$. The ideal (ℓ_{ia}, ℓ_{jb}) is a minimal prime of (F_i, F_j). Therefore, the product Sylvester-Gallai condition implies that $F_k \in (\ell_{ia}, \ell_{jb})$ for some k, which in turn implies that $\ell_{ke} \in (\ell_{ia}, \ell_{jb})$ for some choice of index e. This is a linear Sylvester-Gallai relationship among the elements of $\mathcal{I}_1$. The choice of ℓ_{ia}, ℓ_{jb} was arbitrary, except the condition that $i \neq j$. Therefore, the set $\mathcal{I}_1$ is a robust linear Sylvester-Gallai configuration, and therefore has bounded rank. This in turn implies that the forms in $\mathcal{F}$ have bounded rank, since they are spanned by monomials of degree at most d in a basis for $\mathcal{I} = \mathcal{I}_1$.

We now sketch the induction step. Suppose c is the largest integer such that $\mathcal{I}_c \neq \emptyset$. Our induction step is further split into two steps, as discussed in the previous subsection.

1. First we show that there are variables $x_1, \ldots, x_b$ such that every form in $\mathcal{I}_c$ is either in the ideal $(x_1, \ldots, x_b)$ or absolutely reducible in $S' := \mathbb{C}(x_1, \ldots, x_b)[x_{b+1}, \ldots, x_N, y_1, \ldots, y_n]$. This is a relaxed version of step 2 in the previous approach, applied to $\mathcal{I}_c$ instead of $\mathcal{F}_d$.

2. We then show that projecting $x_1, \ldots, x_b$ to a new variable z results in a product Sylvester-Gallai configuration with $\mathcal{I}_e = \emptyset$ for any $e \geqslant c$.

We elaborate on these in the special case when $c = 2$, since this already captures all the main ideas. As before, if $\mathcal{I}_2$ is a robust linear Sylvester-Gallai configuration, then step 2 follows easily by picking a basis for $\mathcal{I}_2$. Consider $A, B \in \mathcal{I}_2$, and suppose $A | F_i$ and $B | F_j$ for some $F_i, F_j \in \mathcal{F}$. Suppose (A, B) is prime. Then (A, B) is a minimal prime of (F_i, F_j). The product Sylvester-Gallai configuration implies $F_k \in (A, B)$ for some k, and therefore $C \in (A, B)$ for some $C | F_k$. Further, by homogeneity, it must be that $C \in \mathcal{I}_2$. This gives us a linear Sylvester-Gallai relationship between the elements $A, B, C \in \mathcal{I}_2$. The assumption that $\mathcal{I}_2$ is not a robust linear Sylvester-Gallai configuration will therefore imply that (A, B) is not prime for many pairs A, B. Since the forms in $\mathcal{I}_2$ are irreducible (even though the forms in $\mathcal{F}$ might not be), by a combinatorial argument and Theorem 1.6, we deduce the existence of variables $x_1, \ldots, x_b$, completing step 1 above.

We now move on to second step. Suppose φ is a projection map that sends $x_1, \ldots, x_b$ to random multiples of a new variable z. The map φ sends elements in $(x_1, \ldots, x_b)$ to multiples of z. We also prove (Proposition 4.5) that φ sends elements of $\mathcal{I}_2 \setminus (x_1, \ldots, x_b)$ that are absolutely reducible in S' to reducible forms. Since $c = 2$, the degree of each of these forms has to be 1. Therefore, if we apply φ to some $F \in \mathcal{F}$, every factor of F will be linear, even though it is possible that

$\deg(F) = \deg(\varphi(F))$. This reduces the problem to the base case, and the properties of φ allow us to lift the resulting bound on the dimension of $\varphi(\mathcal{F})$ back to a bound on the dimension of $\mathcal{I}$.

Removing the initial assumption: We started with the assumption that every form in $\mathcal{F}$ depended only on constantly many variables. In general, it is of course possible that the forms depend on many or even all the variables. This issue was overcome in [OS24], building on the seminal work of [AH20a]. They show that forms $H_1, \ldots, H_a$ of high enough strength (a notion we define in Section 5) behave essentially like variables, in the sense that $\mathbb{C}[H_1, \ldots, H_a]$ is isomorphic to a polynomial ring. Further, the extension $\mathbb{C}[H_1, \ldots, H_a] \subset S$ has many of the properties that the extension $\mathbb{C}[x_1, \ldots, x_a] \subset S$ has, the most useful of them being that this extension preserves arbitrary intersection of ideals.

Motivated by this, [OS24] define the notion of strong vector spaces and algebras, which are vector spaces of small dimension spanned by forms of high rank, and the algebras they generate. They show that in the general case of the radical Sylvester-Gallai problem (when the forms depend on more than constantly many variables), while there might not be variables $x_1, \ldots, x_b$ such that $\mathcal{F}_d \subset (x_1, \ldots, x_b)$, we can find a strong vector space V such that $\mathcal{F}_d \subset (V)$.

The above change complicates the projection step, since random restriction of higher degree forms is no longer well defined. To fix this, [OS24] defined generalised projection maps, and again proved that such maps have all the properties that the linear projection maps have. In particular, they preserve the radical Sylvester-Gallai structure, and they allow bounds on the dimension of the image of $\mathcal{F}$ to be lifted back to bounds on $\mathcal{F}$. With these tools, they set up an inductive framework to show radical Sylvester-Gallai configurations have bounded dimension, with linear Sylvester-Gallai configurations acting as the base case.

Extending our results to the setting of strong algebras in order to apply this framework requires two technical results. First we define the notion of absolute reducibility with respect to strong vector spaces (Section 5.5). This allows us to extend the prime bound to strong algebras. We then show that general projection maps send forms that are absolutely reducible with respect to strong vector spaces to reducible forms. These two extensions to the above framework allow us to prove our main theorem in the general case.

1.5 Organisation

In Section 2, we give some notation that we will use in the rest of the paper. In Section 3, we formally define the notions of linear Sylvester-Gallai configurations that we need, and we formally define product Sylvester-Gallai configurations and the related notion of factor sets. In Section 4, we gather and establish the necessary results from commutative algebra and algebraic geometry that we need, and we establish our primality structure theorem. In Section 5 and Section 6, we collect the necessary facts about strong algebras and general quotients that will be required. Most proofs for the statements in these sections can be found in [OS24], and we only prove the new statements that we need. In Section 7, we use the results from previous sections to establish the structure of factor sets of product Sylvester-Gallai configurations. Finally, in Section 8, we combine all the above ingredients to prove our main theorem, Theorem 1.5.

Acknowledgments

The authors would like to thank Amir Shpilka and Shir Peleg for several helpful conversations in the course of this work.

2 Preliminaries

Throughout this paper, $\mathbb{K}$ will denote an algebraically closed field of characteristic zero, and $S = \mathbb{K}[x_1, \ldots, x_N]$ will denote the polynomial ring, graded by degree $S = \bigoplus_{i \geqslant 0} S_i$, where $S_0 = \mathbb{K}$. Given a ring R and elements $F_1, \cdots, F_n \in R$ we denote the ideal generated by $F_1, \cdots, F_n$ as $(F_1, \cdots, F_n)$. Given an ideal $I \subset R$ we denotes its radical as $\text{rad}(I)$, which is the ideal consisting of all elements F such that $F^r \in I$ for some positive integer r.

We use *form* to refer to a homogeneous polynomial. Given a graded vector space $V \subset S$ spanned by forms of S, we use V_i to denote the degree i piece, that is, $V_i = V \cap S_i$. Note that we have $V = \bigoplus_{i \geqslant 0} V_i$. Given two forms A, B we say that A, B are associate if there exists a non-zero scalar $c \in \mathbb{K}$ such that $A = cB$. Otherwise, we say that A, B are non-associate forms. For any integral domain A, we let $K(A)$ denote the fraction field of A, and $\overline{K(A)}$ its algebraic closure.

3 Sylvester-Gallai Configurations

We now define linear Sylvester-Gallai configurations and state known bounds on their dimensions. We then formally define our main object of study: product Sylvester-Gallai configurations.

Definition 3.1 (Robust linear Sylvester-Gallai configurations)**.** Let $c \in \mathbb{N}$, $0 < \delta \leqslant 1$ and V be a $\mathbb{K}$-vector space. Let $\mathcal{F} := \{v_1, \ldots, v_m\} \subset V$ be a finite set of *pairwise linearly independent* vectors. We say that $\mathcal{F}$ is a (r, δ)-linear-SG configuration over $\mathbb{K}$ if there exists a $\mathbb{K}$-vector subspace $U \subset V$ of dimension at most r such that the following condition holds:

- for any $v_i \in \mathcal{F} \setminus U$, there exist at least $\delta(m-1)$ indices $j \in [m] \setminus \{i\}$ such that $v_j \notin U$ and

$$|\text{span}_{\mathbb{K}}\{v_i, v_j\} \cap \mathcal{F}| \geqslant 3 \quad \text{or} \quad \text{span}_{\mathbb{K}}\{v_i, v_j\} \cap U \neq (0).$$

We will say that $\mathcal{F}$ is a (r, δ)-linear-SG configuration over the vector space U.

The following bound on such configurations is proved in [OS24, Proposition 3.5], which is a generalization of the result [Shp20, Corollary 16], using the sharper bounds from [DGOS18].

Proposition 3.2. *If $\mathcal{F}$ is a (r, δ)-linear-SG then $\dim \text{span}_{\mathbb{K}}\{\mathcal{F}\} \leqslant r + 1 + 8/\delta$.*

We now formally define product Sylvester-Gallai configurations.

Definition 3.3 (Product Sylvester-Gallai configurations)**.** Let $U \subset S$ be a graded finitely generated vector space such that $R := S/(U)$ is a UFD, and let $z \in R_1$. Let $\mathcal{F} := \{z, F_1, \ldots, F_m\} \subset R$ be a finite set of square-free forms of degree at most d. We say that $\mathcal{F}$ is a (d, c, z, R)-product Sylvester-Gallai configuration if the following conditions hold:

1. $\gcd(F_i, F_j) = 1$ for all $i \neq j$, and $\gcd(z, F_i) = 1$ for all $i \in [m]$ (non-associate forms)
2. every F_i is a product of irreducible forms of degree at most c. (factor bound)
3. for every $i \neq j$ (Sylvester-Gallai condition)

$$z \cdot \prod_{k \neq i,j} F_k \in \text{rad}(F_i, F_j).$$

The above definition generalises Definition 1.3 in a number of ways. The underlying ring is allowed to be any finitely generated UFD over $\mathbb{K}$ as opposed to just a polynomial ring. The forms are allowed to be reducible as long as they are square free and relatively prime. Further, the definition keeps track of the highest degree of any factor of any form in the configuration.

Each of these requirements are needed to allow our inductive framework to apply, following the same approach as was done in [OS24]. As we mentioned in Section 1, the change between our inductive approach and the one in [OS24] is that we will induct on the maximum degree of any irreducible factor of the forms F_i. Since these irreducible factors are our main object of focus in our inductive approach, we are naturally brought to the following definition.

Definition 3.4 (Factor set)**.** Let $\mathcal{F}$ be a (d, c, z, R)-product Sylvester-Gallai configuration. A set $\mathcal{I} \subset R$ consisting of every irreducible factor of every form $F \in \mathcal{F}$ is called a factor set of $\mathcal{F}$. [9]

We write $\mathcal{I}(\mathcal{F})$ if the underlying configuration $\mathcal{F}$ is not clear from context. Since factors of forms are forms, we can write $\mathcal{I} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_c$, where $\mathcal{I}_j \subset R_j$. The degree of $\mathcal{I}$ is the largest e such that $\mathcal{I}_e \neq \emptyset$. If $\mathcal{F}$ is a (d, c, z, R)-configuration then the factor set $\mathcal{I}$ has degree at most c.

The following lemma captures a Sylvester-Gallai like structural result that $\mathcal{I}$ satisfies.

Lemma 3.5. *Suppose $U \subset S$ is a graded finitely generated vector space such that $R := S/(U)$ is a UFD. Suppose $\mathcal{F}$ is a (d, c, z, R)-product Sylvester-Gallai configuration, and suppose $\mathcal{I}$ is a factor set of $\mathcal{F}$. For every $G_1, G_2 \in \mathcal{I} \setminus (z)$, at least one of the following is true.*

- *There is some $F \in \mathcal{F}$ such that $G_1 | F$ and $G_2 | F$.*
- *(G_1, G_2) is not prime.*
- *$|(G_1, G_2) \cap \mathcal{I}| \geqslant 3$.*

Proof. Let $G_1, G_2 \in \mathcal{I} \setminus (z)$. Suppose (G_1, G_2) is prime, and suppose there is no $F \in \mathcal{F}$ such that $G_1 | F$ and $G_2 | F$. We must show that the third item holds. If $z \in (G_1, G_2)$ then we are done, since $z \in \mathcal{I}$. We are therefore left with the case when $z \notin (G_1, G_2)$.

We have $\text{rad}(F_1, F_2) \subset (G_1, G_2)$. By the Sylvester-Gallai condition on $\mathcal{F}$, and the fact that (G_1, G_2) is prime, we have $F_3 \in (G_1, G_2)$ for some form $F_3 \in \mathcal{F}$ such that $F_3 \neq F_1, F_2$. This implies that $G_3 \in (G_1, G_2)$ for some irreducible factor G_3 of F_3. Since F_1, F_3 are relatively prime and similarly F_2, F_3 are relatively prime we have $G_3 \notin (G_1), (G_2)$. ☐

The following result shows that in order to bound the dimension of a product SG-configuration $\mathcal{F}$, it is enough to bound the dimension of a factor set $\mathcal{I}$.

Proposition 3.6. *Suppose $U \subset S$ is a graded finitely generated vector space such that $R := S/(U)$ is a UFD. Suppose $\mathcal{F}$ is a (d, c, z, R)-product Sylvester-Gallai configuration, and suppose $\mathcal{I}$ is a factor set of $\mathcal{F}$. If $\dim \text{span}_{\mathbb{K}} \{\mathcal{I}\} = s$ then $\dim \text{span}_{\mathbb{K}} \{\mathcal{F}\} \leqslant \binom{s+d}{d}$.*

Proof. Let $P_1, \ldots, P_s \in \mathcal{I}$ be such that every form in $\mathcal{I}$ is a linear combination of forms $P_1, \ldots, P_s$. Consider the subalgebra $\mathbb{K}[\mathcal{I}] \subset R$ generated by $\mathcal{I}$. We know that $\mathcal{F} \subset \mathbb{K}[\mathcal{I}]_{\leqslant d}$. Now, $\mathbb{K}[\mathcal{I}] = \mathbb{K}[P_1, \cdots, P_s]$. Thus, every form in $\mathcal{F}$ is a linear combination of monomials in $P_1, \ldots, P_s$ of degree at most d. Hence $\dim \text{span}_{\mathbb{K}} \{\mathcal{F}\} \leqslant \binom{s+d}{d}$. ☐

[9]The factor set is only defined up to units in R, which are just the nonzero scalars. The properties we are interested in, such as the dimension of the span of the factor set, and ideals generated by pairs of polynomials in $\mathcal{I}$ are independent of such scaling, therefore this ambiguity in the definition does not matter to us.

4 Algebraic-Geometric Toolkit

4.1 Commutative Algebraic preliminaries

Suppose that A is an integral domain and M a finitely generated A-algebra. If $M \otimes_A K(A) = 0$, then there exists a non-zero $f \in A$ such that the localization $M_f = 0$. We will be interested in a special case of this situation when A is a polynomial subalgebra of a polynomial ring S. More precisely, let $S = \mathbb{K}[x_1, \cdots, x_n, z_1, \cdots, z_\ell]$, $A = \mathbb{K}[x_1, \cdots, x_n]$ and $M = S/(g_1, \cdots, g_s)$, where $g_1, \cdots, g_s \in S$. Suppose that $M \otimes K(A) = 0$. In the following results, we will show that there exists a $f \in A$ with $M_f = 0$ such that $\deg(f)$ is uniformly bounded. We refer to [Eis95, CLO07] for commutative algebraic definitions and the background on Gröbner basis.

Lemma 4.1. *Let $e \geqslant 1$. Let $S = \mathbb{K}[x_1, \cdots, x_n, z_1, \cdots, z_\ell]$ and $A = \mathbb{K}[x_1, \cdots, x_n]$. Let $g_1, \cdots, g_s \in S$ with $\deg(g_i) \leqslant e$, and $B = S/(g_1, \cdots, g_s)$. If $B \otimes_A K(A) = 0$, then there exists a non-zero polynomial $f \in A \cap (g_1, \cdots, g_s) \cdot S$ with*

$$\deg(f) \leqslant 2(\frac{e^2}{2} + e)^{2^{\ell+n-1}}.$$

In particular, $B_f = 0$.

Proof. Note that A is an integral domain and (0) is a prime ideal in A. Since $S = A[z_1, \cdots, z_\ell]$, we have that $B = A[z_1, \cdots, z_\ell]/(g_1, \cdots, g_s)$ is a finitely generated A-algebra. Let $B_{(0)}$ be the localization of B at (0). Then we have $B_{(0)} \simeq B \otimes_A K(A) = 0$ as A-modules. Therefore there exists a non-zero polynomial $g \in A$ such that $g \cdot 1 = 0$ in B and hence $g \in (g_1, \cdots, g_s)$ in S. In particular $B_g = 0$ and $(g_1, \cdots, g_s) \cdot S \cap A \neq (0)$ in S. Although we do not have control over the degree of an arbitrary g above, it is enough to show that there exists a non-zero polynomial $f \in (g_1, \cdots, g_s) \cdot S \cap A$ such that $\deg(f) \leqslant 2(\frac{e^2}{2} + e)^{2^{\ell+n-1}}$.

Consider the lex monomial ordering on S where $z_\ell > \cdots > z_1 > x_n > \cdots > x_1$. Let $I = (g_1, \cdots, g_s) \cdot S$. Recall that $\deg(g_i) \leqslant e$ for all i. Therefore, by applying [Dub90] to S, there exists a Gröbner basis $f_1, \cdots, f_k$ of I such that $\deg(f_j) \leqslant 2(\frac{e^2}{2} + e)^{2^{\ell+n-1}}$ for $j \in [k]$. Now, by [CLO07, Section 3.1, Theorem 2], we know that $\{f_1, \cdots, f_k\} \cap A$ is a Gröbner basis for the elimination ideal $I \cap A$. In particular, there exists a non-zero element $f_j \in I \cap A$ for some $j \in [k]$. Therefore, we may take $f = f_j$. □

Lemma 4.2. *Fix $d, e \geqslant 1$. Let $S = \mathbb{K}[x_1, \cdots, x_n, z_1, \cdots, z_\ell]$ and $A = \mathbb{K}[x_1, \cdots, x_n]$. Suppose that $t_1, \cdots, t_r \in A$ is a regular sequence in S where $\deg_S(t_i) \leqslant d$. Let $A' := \mathbb{K}[t_1, \cdots, t_r] \subset S$ and $S' := A'[z_1, \cdots, z_\ell]$. Let $g_1, \cdots, g_s \in S'$ with $\deg_S(g_i) \leqslant e$. Let $B := S/(g_1, \cdots, g_s)$ and $B' := S'/(g_1, \cdots, g_s)$.*

If $B' \otimes_{A'} K(A') = 0$, then there exists a non-zero polynomial $f \in A' \subset A$ with

$$\deg_S(f) \leqslant 2d(\frac{e^2}{2} + e)^{2^{\ell+r-1}}$$

such that the localization $B_f = 0$.

Proof. Since $t_1, \cdots, t_r$ is a regular sequence in S, they are algebraically independent. Moreover, as $t_1, \cdots, t_r \in A$, we note that $t_1, \cdots, t_r, z_1, \cdots, z_\ell$ are algebraically independent. Therefore $S' = \mathbb{K}[t_1, \cdots, t_r, z_1, \cdots, z_m]$ is isomorphic to a polynomial ring and the elements $t_1, \cdots, t_r, z_1, \cdots, z_\ell$ can be treated as the variables of the polynomial ring.

Note that $\deg_{S'}(g_i) \leqslant \deg_S(g_i) \leqslant e$. By applying Lemma 4.1 to S', there exists $f \in A' \cap (g_1, \cdots, g_s) \cdot S'$ such that $\deg_{S'}(f) \leqslant 2(\frac{e^2}{2} + e)^{2^{\ell+r-1}}$. Since $\deg_S(t_i) \leqslant d$, we have that $\deg_S(f) \leqslant$

$d \cdot \deg_{S'}(f)$. Therefore we have $\deg_S(f) \leqslant 2d(\frac{e^2}{2}+e)^{2^{\ell+r-1}}$. Furthermore, since $f \in (g_1, \cdots, g_s) \cdot S$, we have that $B_f = 0$. □

4.2 Absolute irreducibility

In this subsection, we recall the notions of Cohen-Macaulay rings, absolute and geometric irreducibility.

Regular sequences, Cohen-Macaulay rings and homogeneous system of parameters. Let R be ring and M an R-module. A sequence of elements $F_1, F_2, \cdots F_n \in R$ is called an M-regular sequence if $(F_1, F_2, \cdots, F_n)M \neq M$, and for $i = 1, \cdots, n$, F_i is a non-zerodivisor on $M/(F_1, \cdots, F_{i-1})M$. If $M = R$, then we simply call it a regular sequence. A local ring $(R, \mathfrak{m})$ with $\dim(R) = n$, is Cohen-Macaulay (CM) if there exists a regular sequence $F_1, \cdots, F_n \in \mathfrak{m}$ of length n. A Noetherian ring R is Cohen-Macaulay if $R_\mathfrak{p}$ is Cohen-Macaulay for all maximal (equivalently prime) ideals $\mathfrak{p}$ in R.

Let $R = \oplus_{i\in\mathbb{N}} R_i$ be a finitely generated $\mathbb{N}$-graded $\mathbb{K}$-algebra such that $R_0 = \mathbb{K}$. A sequence of homogeneous elements $F_1, \cdots, F_n \in R$ is a homogeneous system of parameters, abbreviated as h.s.o.p., if we have $n = \dim(R)$ and $\dim(R/(F_1, \cdots, F_n)) = 0$.

Definition 4.3. A polynomial $P \in \mathbb{F}[x_1, \cdots, x_N]$ is called *absolutely irreducible* if P is irreducible in the polynomial ring $\overline{\mathbb{F}}[x_1, \cdots, x_n]$, over an algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$.

A scheme X over a field $\mathbb{F}$ is called *geometrically irreducible* if the base change $X_{\mathbb{F}'}$ is irreducible for any field extension $\mathbb{F}'$ over $\mathbb{F}$. A scheme X over a field $\mathbb{F}$ is called *geometrically reduced* if the base change $X_{\overline{\mathbb{F}}}$ is reduced for an algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$.

Proposition 4.4.
1. *A scheme X over a field $\mathbb{F}$ is geometrically irreducible iff the base change $X_{\overline{\mathbb{F}}}$ is irreducible for an algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$.*
2. *A polynomial $P \in S = \mathbb{F}[x_1, \cdots, x_N]$ is absolutely irreducible iff the scheme $X = \mathrm{Spec}(S/(P))$ is geometrically irreducible and geometrically reduced.*

Proof. The first part follows from [Sta18, Tag 038I]. For the second part, we note that P is irreducible over $\overline{\mathbb{F}}$ iff (P) is prime in $\overline{\mathbb{F}}[x_1, \cdots, x_N]$. Now we are done since primality of (P) over $\overline{\mathbb{F}}$ is equivalent to the scheme $X = \mathrm{Spec}(S/(P))$ being irreducible and reduced over $\overline{\mathbb{F}}$. □

Proposition 4.5. *Let $\mathbb{K}[z, y_1, \ldots, y_m]$ be a graded polynomial ring with* $\deg(z) = 1$. *Suppose* $P \in \mathbb{K}[z, y_1, \ldots, y_m]$ *is a homogeneous polynomial. Then the following holds.*

1. *If* $\gcd(P, z) = 1$ *and P is reducible in $\mathbb{K}[z, y_1, \cdots, y_m]$, then P is reducible in $\overline{\mathbb{K}(z)}[y_1, \ldots, y_m]$.*
2. *If P is reducible in $\overline{\mathbb{K}(z)}[y_1, \ldots, y_m]$, then P is reducible in $\mathbb{K}[z, y_1, \ldots, y_m]$.*

Proof. (2) We can assume that $\gcd(P, z) = 1$, otherwise P is clearly reducible in $\mathbb{K}[z, y_1, \ldots, y_m]$. Since P is homogeneous and $\gcd(z, P) = 1$, [Ful89, Section 2.6, Corollary 5] shows that P reducible in $\mathbb{K}[z, y_1, \ldots, y_m]$ iff $P' := P(1, y_1, \ldots, y_m)$ is reducible in $\mathbb{K}[y_1, \ldots, y_m]$. Suppose P factors in $\overline{\mathbb{K}(z)}[y_1, \ldots, y_m]$ as $P = AB$. Let $A = \sum_\alpha p_\alpha(z) y^\alpha$ and $B = \sum_\beta p_\beta(z) y^\beta$, where y^α, y^β are monomials, and $p_\alpha(z), p_\beta(z) \in \overline{\mathbb{K}(z)} \neq 0$. Consider the universal polynomials $C = \sum_\alpha c_\alpha y^\alpha$ and $D = \sum_\beta d_\beta y^\beta$ where c_α, d_β are formal variables. We can write the product $CD = \sum_\gamma q_\gamma(c, d) y^\gamma$ where q_γ are polynomials with coefficients in $\mathbb{K}$. That $P = AB$ is equivalent to the fact that when we set $c_\alpha = p_\alpha, d_\beta = p_\beta$, each q_γ is equal to the corresponding coefficient of P. Suppose P has degree d as a polynomial in $\overline{\mathbb{K}(z)}[y_1, \ldots, y_m]$, and suppose the space of monomials of degree d in $y_1, \ldots, y_m$ has dimension N.

The coefficients of CD define a map to the space $\mathbb{P}^{N-1}_{\mathbb{K}[z]}$, whose image is closed, since projective maps are closed. The image of this map is exactly those polynomials in $\overline{\mathbb{K}(z)}[y_1, \dots, y_m]$ that factor into two polynomials of the same multidegree as C, D respectively. We construct the defining equations of this image $F_1, \dots, F_r$. Each F_i has coefficients in $\mathbb{K}$, since the coefficients of CD themselves are in $\mathbb{K}$. That P is reducible implies that each F_i vanishes on the coefficients of P. In particular, each F_i vanishes when the coefficients of P are specialised to $z = 1$.

Similar to this construction, we can construct the defining equations for the space of polynomials in $\mathbb{K}[y_1, \dots, y_m]$ of degree d that factor into two polynomials of the same multidegree as C, D. These defining equations are also exactly $F_1, \dots, F_r$, since the only coefficients that occur in the construction in both cases are those from $\mathbb{K}$. By the above, the coefficients of P', which are exactly the coefficients of P specialised to $z = 1$, satisfy these equations. This shows that P' is reducible, whence P is reducible. □

We also remark that both the assumption that P is homogeneous and that $\deg(z) = 1$ are necessary conditions: for example consider $P = y_1^2 - x_1 y_2^2$. This form is not homogeneous with the standard grading, but is homogeneous if we set $\deg(z) = \deg(y_2) = 2$ and $\deg(y_1) = 3$. Further, $\varphi(P)$ is irreducible in $\mathbb{C}[z, y_1, y_2]$ but P factors as $P = \left(y_1 - \sqrt{z} y_2\right)\left(y_1 + \sqrt{z} y_2\right)$.

The following two results deal with polynomial rings with possibly non-standard grading.

Lemma 4.6. *Let* $B = \mathbb{K}[z, x_1, \cdots, x_n]$*, where* $\deg(z) = 1$ *and* $\deg(x_i) = d_i$ *for some* $d_i \geqslant 1$*. For* $\alpha \in \mathbb{K}^n$*, let* $\mathfrak{p}_\alpha$ *be the prime ideal* $(x_1 - \alpha_1 z^{d_1}, \cdots, x_n - \alpha_1 z^{d_n}) \subseteq \mathbb{K}[z, x_1, \cdots, x_n]$*. If* $\mathcal{T} \subseteq \mathbb{K}^n$ *is a dense subset, then the set* $\mathcal{S} := \{\mathfrak{p}_\alpha \mid \alpha \in \mathcal{T}\}$ *is dense in* $\operatorname{Spec}(B)$*.*

Proof. Suppose that $\mathcal{S}$ is not dense in $\operatorname{Spec}(B)$. Then $\mathcal{S}$ is contained in a proper closed subset $V(I)$ of $\operatorname{Spec}(B)$, where $I \subseteq B$ is a non-zero ideal. Let $V(I) = \cup_j V(\mathfrak{q}_j)$ be the decomposition into irreducible components where $\mathfrak{q}_1, \cdots, \mathfrak{q}_r \subseteq B$ are non-zero prime ideals. Since $\mathcal{S} \subseteq \cup_j V(\mathfrak{q}_j)$, each prime $\mathfrak{p}_\alpha \in \mathcal{S}$ contains at least one of the prime ideals $\mathfrak{q}_1, \cdots, \mathfrak{q}_r$. Let $\mathcal{T}_j := \{\alpha \in \mathcal{T} \mid \mathfrak{q}_j \subseteq \mathfrak{p}_\alpha\}$. Since $\cup_{j=1}^r \mathcal{T}_j = \mathcal{T}$, at least one of the sets $\mathcal{T}_j$ must be dense in $\mathbb{K}^n$. Suppose $\mathcal{T}_k$ is dense. Then we may replace $\mathcal{T}$ with $\mathcal{T}_k$. Let $\mathfrak{q} = \mathfrak{q}_k$. Then we have that $\mathfrak{q} \subseteq \mathfrak{p}_\alpha$ for all $\alpha \in \mathcal{T}$. We will see that this leads to a contradiction.

Let $f \in \mathfrak{q} \subseteq \mathbb{K}[z, x_1, \cdots, x_n]$ be a non-zero polynomial. Let $\deg(f) = e$ and $f = f_0 + \cdots + f_e$ be the decomposition of f into its homogeneous parts with the grading of B. We may write $f_e = g_0 + g_1 z + \cdots + g_d z^d$ where $g_1, \cdots, g_d \in \mathbb{K}[x_1, \cdots, x_n]$ are homogeneous with $\deg(g_i) = e - i$. Consider the map $\varphi_\alpha : B \to B/\mathfrak{p}_\alpha = \mathbb{K}[z]$. Then $\varphi_\alpha(f_e) = (\sum_{i=0}^d g_i(\alpha)) z^e$. As $\mathfrak{p}_\alpha \subseteq B$ is a homogeneous ideal and $f \in \mathfrak{p}_\alpha$, we must have $f_e \in \mathfrak{p}_\alpha$. Hence $\varphi_\alpha(f_e) = 0$ in $\mathbb{K}[z]$. Therefore, $(\sum_i g_i)(\alpha) = 0$ for all $\alpha \in \mathcal{U}$. Hence $\sum_i g_i = 0$ in $\mathbb{K}[x_1, \cdots, x_n]$, as $\mathcal{U} \subseteq \mathbb{K}^n$ is a non-empty open subset. Now $g_i \in \mathbb{K}[x_1, \cdots, x_n]$ is homogeneous with $\deg(g_i) = e - i$. Hence we must have $g_i = 0$ for all i. Thus $f_e = 0$, which is a contradiction. □

Proposition 4.7. *Let* $S := \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_m]$ *be a graded polynomial ring with* $\deg(x_i) = d_i \geqslant 1$ *and* $\deg(y_j) = e_j \geqslant 1$*. Let* $A = \mathbb{K}[x_1, \cdots, x_n]$ *and* $K(A)$ *the fraction field of* A*. Let* $d \geqslant 1$ *and* $P \in S_d \setminus (x_1, \dots, x_n)$*. Suppose* $\varphi_\alpha : S \to \mathbb{K}[z, y_1, \dots, y_m]$ *is a map defined by* $\varphi_\alpha(x_i) = \alpha_i z^{d_i}$ *with* $\alpha \in \mathbb{K}^n$ *and* $\varphi_\alpha(y_j) = y_j$*.*

1. *If* P *is absolutely irreducible over* $K(A)$*, then* $\varphi_\alpha(P)$ *is absolutely irreducible over* $\mathbb{K}(z)$ *for a general choice of* α*.*

2. *Suppose that* $\varphi_\alpha(P)$ *is absolutely irreducible over* $\mathbb{K}(z)$ *for all* $\alpha \in \mathcal{T}$*, where* $\mathcal{T} \subseteq \mathbb{K}^n$ *is a dense subset. Then* P *is absolutely irreducible over* $K(A)$*.*

Proof. (1) Suppose that P is absolutely irreducible over $K(A)$. Then, by [Sta18, Tag 0557], there exists $f \in A$ such that $\varphi(P)$ is irreducible over $\overline{\mathbb{K}(z)}[y_1, \cdots, y_m]$ for any ring homomorphism $\varphi : A_f \to \overline{\mathbb{K}(z)}$. Let $\varphi_\alpha : A \to \mathbb{K}[z] \hookrightarrow \mathbb{K}(z) \hookrightarrow \overline{\mathbb{K}(z)}$ be defined by $x_i \mapsto \alpha_i z^{d_i}$. Note that $\varphi_\alpha(f) \in \mathbb{K}[z]$. Let $\deg(f) = e$ and $f = f_0 + \cdots + f_e$ be the decomposition of f into its homogeneous parts. Then $\varphi_\alpha(f) = \sum_{i=0}^{e} f_i(\alpha) z^i$. Let $\mathcal{U} := \mathbb{K}^n \setminus V(f_e)$. Then $\mathcal{U} \subseteq \mathbb{K}^n$ is a non-empty open subset as f_e is non-zero. Moreover, $\varphi_\alpha(f) \neq 0$ for all $\alpha \in \mathcal{U}$. Therefore, for all $\alpha \in \mathcal{U}$, the map φ_α descends to a ring homomorphism $A_f \to \overline{\mathbb{K}(z)}$ by the universal property of localization. Hence $\varphi_\alpha(P)$ is irreducible in $\overline{\mathbb{K}(z)}[y_1, \cdots, y_m]$for a general choice of α, i.e. for all $\alpha \in \mathcal{U}$.

(2) Let $\mathcal{T} \subseteq \mathbb{K}^n$ be a dense subset such that $\varphi_\alpha(P)$ is irreducible for all $\alpha \in \mathcal{T}$. Suppose that P is reducible in $\overline{K(A)}[y_1, \cdots, y_m]$. Then it is also reducible in $\overline{K(B)}[y_1, \cdots, y_m]$, where $B := A[z] = \mathbb{K}[z, x_1, \cdots, x_n]$. We will show that this leads to a contradiction.

Consider the affine schemes $X := \operatorname{Spec}(B[y_1, \cdots, y_m]/(P))$ and $Y := \operatorname{Spec}(B)$. The ring homomorphism $B \to B[y_1, \cdots, y_m]/(P)$ gives rise to a morphism of finite type $f : X \to Y$. Let $\eta \in Y$ be the generic point of Y, i.e. η is the prime ideal (0). By applying [Sta18, Tag 055A], we know that there exists a non-empty open subset $\mathcal{V} \subseteq Y$ such that the number of irreducible components of the geometric fiber $X_{\overline{\mathfrak{p}}}$ is constant for all $\mathfrak{p} \in \mathcal{V}$.

Note that by definition $X_{\overline{\mathfrak{p}}} = \operatorname{Spec}(\frac{B[y_1, \cdots, y_m]}{(P)} \otimes_B \overline{k(\mathfrak{p})})$, where $k(\mathfrak{p}) = B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ is the residue field at $\mathfrak{p}$. Since $k(\mathfrak{p}) = K(B/\mathfrak{p})$, we know that $X_{\overline{\mathfrak{p}}} = \operatorname{Spec}(\overline{K(B/\mathfrak{p})}[y_1, \cdots, y_m]/(P))$. Therefore the number of irreducible components of $X_{\overline{\mathfrak{p}}}$ is the number of irreducible factors of the image of P in $\overline{K(B/\mathfrak{p})}[y_1, \cdots, y_m]$. Note that the geometric generic fiber $X_{\overline{\eta}}$ has at least two irreducible components as P is reducible in $\overline{K(B)}[y_1, \cdots, y_m]$ and $\eta = (0)$. Therefore, $X_{\overline{\mathfrak{p}}}$ is reducible for all $\mathfrak{p} \in \mathcal{V}$. Hence the image of P is reducible in $\overline{K(B/\mathfrak{p})}[y_1, \cdots, y_m]$ for all $\mathfrak{p} \in \mathcal{V}$.

For $\alpha \in \mathbb{K}^n$, let $\mathfrak{p}_\alpha$ be the prime ideal $(x_1 - \alpha_1 z^{d_1}, \cdots, x_n - \alpha_1 z^{d_n}) \subseteq B = \mathbb{K}[z, x_1, \cdots, x_n]$. Note that $B/\mathfrak{p}_\alpha = \mathbb{K}[z]$ and $K(B/\mathfrak{p}_\alpha) = \mathbb{K}(z)$. By assumption we know that the image of P, namely $\varphi_\alpha(P)$, is irreducible in $\overline{\mathbb{K}(z)}[y_1, \cdots, y_m]$ for all $\alpha \in \mathcal{T}$. By Lemma 4.6, the set of primes $\mathcal{S} := \{\mathfrak{p}_\alpha \mid \alpha \in \mathcal{T}\}$ is dense in Y. Hence $\mathcal{S} \cap \mathcal{V} \neq \emptyset$. This is a contradiction. □

4.3 Bertini-type results

The classical Bertini theorem says that if a projective variety X is non-singular, then a general hyperplane section is also non-singular and it is irreducible if $\dim(X) \geqslant 2$. Furthermore, several useful properties of algebraic schemes such as reducedness, Cohen-Macaulayness and normality are preserved under general hyperplane sections [FOV99, Section 3.4]. In this section, we prove versions of the Bertini theorems relevant to our applications.

General points. Let $\mathbb{F}$ be any field. We say that a property $\mathcal{P}$ holds for a *general* $\alpha \in \mathbb{F}^m$ (or $\mathbb{A}^m_\mathbb{F}$), if there exists a non-empty open subset $\mathcal{U} \subset \mathbb{F}^m$ such that the property $\mathcal{P}$ holds for all $\alpha \in \mathcal{U}$. Here $\mathcal{U} \subset \mathbb{F}^m$ is open with respect to the Zariski topology. Hence $\mathcal{U}$ is the complement of the zero set of finitely many polynomial functions on $\mathbb{F}^m$. Note that, equivalently a property $\mathcal{P}$ holds for a general $\alpha \in \mathbb{F}^m$, if there is a closed subset $\mathcal{Z} \subset \mathbb{F}^m$ such that the $\mathcal{P}$ holds for all $\alpha \notin \mathcal{Z}$.

Hyperplane sections. We fix $n, m \in \mathbb{N}$ such that $m \geqslant 4$. Let $S = \mathbb{K}[x_1, \cdots, x_n, y_1, \cdots, y_m]$ and $A = \mathbb{K}[x_1, \cdots, x_n]$. For $\alpha \in \mathbb{K}^m$, let φ_α be the quotient homomorphism

$$\varphi_\alpha : S \to R_\alpha := S/(\alpha_1 y_1 + \cdots + \alpha_m y_m)$$

Note that R_α is also a polynomial ring. In fact, if $\alpha_1 \neq 0$, then $R_\alpha \simeq \mathbb{K}[x_1, \cdots, x_n, y_2, \cdots, y_m]$. Furthermore, $\varphi_\alpha(x_i) = x_i$ for all $i \in [n]$ and hence φ_α is an A-algebra homomorphism. Therefore φ_α extends to a $\overline{K(A)}$-algebra homomorphism

$$\overline{K(A)}[y_1, \cdots, y_m] \to \overline{K(A)}[y_1, \cdots, y_m]/(\alpha_1 y_1 + \cdots + \alpha_m y_m),$$

which we continue to denote by φ_α. For the remainder of this section we fix the notations φ_α and R_α defined as above.

Affine schemes. Recall that an ideal ideal I in the polynomial ring S is prime iff the quotient ring S/I is an integral domain. Given such an ideal I, the corresponding geometric object is the affine scheme $\mathrm{Spec}(A/I)$. In order to study primality of such ideals I we will study the geometric properties the affine scheme $\mathrm{Spec}(A/I)$. In particular, I is prime iff the affine scheme $\mathrm{Spec}(A/I)$ is irreducible and reduced. Affine schemes are a generalization of the classically studied affine varieties defined by ideals in polynomial rings. Working in this generalized setting is essential for us since the classical setting of varieties is not well-equipped to study nilpotents in the ring S/I, which are an obstruction to primality of I. In our arguments we will use basic definitions and properties of affine schemes, and we refer to [Har77] for the necessary background.

The following result is a version of the Bertini theorem. It shows that absolute irreducibility over $K(A)$ is preserved under suitable hyperplane sections defined over $\mathbb{K}$.

Lemma 4.8. *Let $n, m \in \mathbb{N}$ such that $m \geqslant 4$. Let $S = \mathbb{K}[x_1, \cdots, x_n, y_1, \cdots, y_m]$ and $A = \mathbb{K}[x_1, \cdots, x_n]$. Suppose that $P \in S$ is absolutely irreducible over $K(A)$. Then there exists an open dense subset $\mathcal{T} \subset \mathbb{K}^m$ such that $\varphi_\alpha(P) \in R_\alpha$ is absolutely irreducible over $K(A)$ for all $\alpha \in \mathcal{T}$.*

Proof. Let $\mathbb{F} := \overline{K(A)}$ and $S' = \mathbb{F}[y_1, \cdots, y_m]$. For $\beta \in \mathbb{F}^m$, we let $R'_\beta = S'/(\beta_1 y_1 + \cdots + \beta_m y_m)$ and φ_β be the quotient homomorphism $S' \to R'_\beta$. Consider the affine hypersurface $X = \mathrm{Spec}(S'/(P)) \subset \mathbb{A}^m_{\mathbb{F}}$. Since P is an irreducible polynomial over $\mathbb{F}$, we know that X is an irreducible and reduced scheme over $\mathbb{F}$. Note that $\dim(X) \geqslant 3$ as $m \geqslant 4$. For any $\beta \in \mathbb{F}^m$, let X_β denote the hyperplane section $X \cap V(\beta_1 y_1 + \cdots + \beta_m y_m)$. We will apply the Bertini theorems [FOV99, Corollary 3.4.9, Theorem 3.4.10] to show that a general hyperplane section X_β is also irreducible and reduced.

We consider the linear system on X spanned by $y_1, \cdots, y_m$. Then reducedness of X_β follows from [FOV99, 3.4.8, Corollary 3.4.9]. For irreducibility, we verify that the conditions in [FOV99, Theorem 3.4.10] hold in our setting. We note that the base locus of the linear system $y_1, \cdots, y_m$ is, by definition, the common zero set of $P, y_1, \cdots, y_m$ in $\mathbb{F}^m$. Therefore the base locus is contained in $\{(0, \cdots, 0)\}$, and hence at most 0-dimensional. Since $\dim(X) \geqslant 3$, we have that the codimension of the base locus of is at least 3. Moreover, consider the rational map $\phi : X \dashrightarrow \mathbb{P}^{m-1}_{\mathbb{F}}$ defined by $x \to [y_1(x) : \cdots : y_m(x)]$. Since ϕ factors through the natural map $\mathbb{A}^m_{\mathbb{F}} \dashrightarrow \mathbb{P}^{m-1}_{\mathbb{F}}$, we have that $\dim(\phi^{-1}(p)) \leqslant 1$ for any $p \in \mathbb{P}^{m-1}_{\mathbb{F}}$. Since $\dim(X) \geqslant 3$, we conclude that $\dim(\phi(X)) \geqslant 2$. Hence the linear system is not composed with a pencil. Therefore, by [FOV99, Theorem 3.4.10], we conclude that X_β is irreducible for a general $\beta \in \mathbb{F}^m$.

Since $X_\beta = \mathrm{Spec}(R'_\beta/(\varphi_\beta(P)))$, we conclude that $\varphi_\beta(P)$ is irreducible over $\mathbb{F}$ for a general $\beta \in \mathbb{F}^m$. Therefore there exists an open subset $\mathcal{U} \subset \mathbb{F}^m$ such that for all $\beta \in \mathcal{U}$ we have that $\varphi_\beta(P)$ is irreducible over $\mathbb{F}$. By [Ras99, Proposition 3.3], we know that $\mathcal{U} \cap \mathbb{K}^m$ is an open dense subset of $\mathbb{K}^m$, and we conclude by defining $\mathcal{T} := \mathcal{U} \cap \mathbb{K}^m$. ☐

The following lemma is a modification of [OS24, Lemma 4.21], which provides a criterion to deduce irreducibility and reducedness of schemes using properties of the fibers of a morphism.

Lemma 4.9. *Let $\phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of finitely generated $\mathbb{K}$-algebras. Let $Y = \mathrm{Spec}(\mathcal{B})$, $X = \mathrm{Spec}(\mathcal{A})$ and $\pi : Y \to X$ be the corresponding morphism of affine schemes. Suppose that X is irreducible and reduced, i.e. $\mathcal{A}$ is an integral domain. Furthermore, suppose that every irreducible component of Y dominates X.*

1. *If $\pi^{-1}(x)$ is irreducible for a general closed point $x \in X$, then Y is irreducible.*

2. *Suppose* Y *satisfies Serre's property* $\mathcal{S}_1$, *i.e.* Y *does not have embedded primes. If for every irreducible component* W *of* X, *we have that* $\pi^{-1}(x)$ *is reduced for a dense set of closed points* $x \in W$, *then* Y *is reduced.*

Proof. We note that (1) follows directly from [OS24, Lemma 4.21]. Furthermore, in [OS24, Lemma 4.21], statement (2) was proved under the assumption that $\pi^{-1}(x)$ is reduced for a general closed point $x \in W$. However, the same argument works if we have that a dense set of fibers $\pi^{-1}(x)$ is reduced. □

We prove the following Bertini-type result which shows that non-primality is preserved by general hyperplane sections of codimension 2 complete intersections.

Lemma 4.10. *Let* $n, m \in \mathbb{N}$ *such that* $m \geqslant 5$ *and* $S = \mathbb{K}[x_1, \cdots, x_n, y_1, \cdots, y_m]$. *Let* $P, Q \in S$ *be a regular sequence. Suppose that* (P, Q) *is not prime. Then the ideal* $(\varphi_\alpha(P), \varphi_\alpha(Q))$ *is not prime in* R_α *for a general* $\alpha \in \mathbb{K}^m$.

Proof. Let $X = \mathrm{Spec}(S/(P, Q))$ be the affine scheme defined by P, Q. Since (P, Q) is not prime, we have that X is not irreducible or not reduced. Since P, Q is a regular sequence, we know that X is Cohen-Macaulay. In particular, X does not have any embedded components and all the irreducible components of X have the same dimension. Let $X = X_1 \cup \cdots X_k$ be the irreducible decomposition of X. For $\alpha \in \mathbb{K}^m$, let $H_\alpha \subset \mathbb{A}^{n+m}$ be the hyperplane defined by $\ell_\alpha := \alpha_1 y_1 + \cdots + \alpha_m y_m$. We let X_α and $X_{i,\alpha}$ denote the corresponding hyperplane sections $X \cap H_\alpha$ and $X_i \cap H_\alpha$ respectively. Note that $X_\alpha = \mathrm{Spec}(S/(P, Q, \ell_\alpha)) = \mathrm{Spec}(R_\alpha/(\varphi_\alpha(P), \varphi_\alpha(Q)))$.

Case 1. Non-irreducible X. We will show that if X is not irreducible, then a general hyperplane section X_α is not irreducible. Suppose that X is not irreducible, i.e. $k > 1$.

First, we will apply the Bertini theorem [FOV99, Theorem 3.4.10] to the schemes X_i and show that $X_{i,\alpha}$ is irreducible for general α. Fix $i \in [k]$. Consider the linear system on X_i spanned by $y_1, \cdots, y_m$. Let $\phi : X_i \dashrightarrow \mathbb{P}^{m-1}_{\mathbb{K}}$ be the rational map defined as $x \to [y_1(x) : \cdots : y_m(x)]$. Note that $\dim(X_i) = \dim(X) = n + m - 2 \geqslant n + 3$. Since $\dim(\phi^{-1}(p)) \leqslant n + 1$ for all $p \in \mathbb{P}^{m-1}_{\mathbb{K}}$, we conclude that $\dim(\phi(X_i)) > 1$. Therefore the linear system spanned by $y_1, \cdots, y_m$ is not composed of a pencil. We note that the base locus or the common zero set of $y_1, \cdots, y_m$ on X_i is contained in the affine subspace $V(y_1, \cdots, y_m) \subset \mathbb{A}^{n+m}$. Hence the dimension of the base locus is at most n. Since $\dim(X_i) \geqslant n + 3$, we see that the codimension of the base locus is at least 3. By [FOV99, Theorem 3.4.10], we conclude that $X_{i,\alpha}$ is irreducible for a general α. Note that there are finitely many X_i. Hence, for a general α, we have that $X_{i,\alpha}$ are irreducible for all $i \in [k]$.

Now, we have that $X_\alpha = X_{1,\alpha} \cup \cdots \cup X_{k,\alpha}$. We will show that $X_{i,\alpha} \not\subset X_{j,\alpha}$ for $i \neq j$. Then $X_\alpha = X_{1,\alpha} \cup \cdots \cup X_{k,\alpha}$ will be the irreducible decomposition of X_α. Hence X_α will be non-irreducible, since $k > 1$.

We have $X_i \neq X_j$ for $i \neq j$ and we know that $\dim(X_i) = \dim(X_j)$. Therefore $U := X_i \setminus X_j$ is a non-empty open subset of X_i. Let $Z = X_i \cap X_j$. Note that $\dim(Z) \leqslant \dim(X_i) - 1 \leqslant n + m - 3$. We will show that for a general α, we must have $X_{i,\alpha} \cap U \neq \emptyset$. Otherwise, we have $X_{i,\alpha} \subset Z$ for a dense set of α in $\mathbb{K}^m$. Now $\dim(X_{i,\alpha}) = \dim(X_i \cap H_\alpha) \geqslant n + m - 3$ by [Har77, Propsition 7.1]. Therefore, we must have $\dim(Z) = n + m - 3$ and $X_{i,\alpha}$ is an irreducible component of Z for a dense set of α in $\mathbb{K}^m$. For any irreducible component W of Z, the set of hyperplanes containing W is a vector subspace of $\mathrm{span}_{\mathbb{K}}(y_1, \cdots, y_m)$. In particular, the set of $\alpha \in \mathbb{K}^m$ such that $W \subset H_\alpha$ is an linear subspace of $\mathbb{K}^m$. Since there are only finitely many possibilities for W, we must have that there is an irreducible component W of Z such that $H_\alpha \supset W$ for a dense set of $\alpha \in \mathbb{K}^m$. This is a contradiction since $\dim(V(y_1, \cdots, y_m)) \leqslant n$. Therefore, $X_{i,\alpha} \cap U \neq \emptyset$, and hence $X_{i,\alpha} \not\subset X_{j,\alpha}$ for a general α, if $i \neq j$.

Case 2. Non-reduced X. By the previous case, we may assume that X is irreducible. We will show that if X is not reduced, then a general hyperplane section X_α is also non-reduced.

Since P, Q is a regular is sequence, the ideal (P, Q) does not have any embedded primes. Since $X = \mathrm{Spec}(S/(P, Q))$ is irreducible, we know that (P, Q) has a unique minimal prime $\mathfrak{p}$. In particular, rad $(P, Q) = \mathfrak{p}$. Since X is Cohen-Macaulay, it is reduced iff it is generically reduced. Therefore any non-empty open subset $U \subset X$ is a non-reduced scheme. Suppose that there is a dense set of $\alpha \in \mathbb{K}^m$, such that X_α is reduced. We let $U := X \setminus V(y_1, \cdots, y_m)$ and in the following we will show that U is a reduced scheme, and obtain a contradiction.

Let $t_1 \cdots, t_m$ be new variables and let $\mathbb{A}^m$ be the corresponding affine space. We define $S' = \mathbb{K}[x_1, \cdots, x_n, y_1, \cdots, y_m, t_1, \cdots, t_m]$. We consider the following affine scheme $\mathcal{X} \subset X \times \mathbb{A}^m \subset \mathbb{A}^{2m+n}$ defined by the ideal $(P, Q, \sum t_i y_i) \subset S'$. Let $q : \mathcal{X} \to \mathbb{A}^m$ and $p : \mathcal{X} \to X$ denote the projection morphisms. For any $\alpha \in \mathbb{K}^m$, we know that the fiber $q^{-1}(\alpha)$ is the hyperplane section X_α. By [FOV99, Lemma 3.4.4], we know that the morphism $p : p^{-1}(U) \to U$ is a locally trivial fibration with fibers $\mathbb{A}^{m-1}$. Therefore, it is enough to show that $p^{-1}(U)$ is reduced.

We note that $P, Q, \sum t_i y_i$ is a regular sequence in S'. Otherwise, $\sum t_i y_i \in \mathfrak{p} \cdot S'$, as $\mathfrak{p} \cdot S'$ is the minimal prime of (P, Q) in S'. This is a contradiction, since we may substitute $t_i = 1$ and $t_j = 0$ for $j \neq i$, to obtain that $y_i \in \mathfrak{p}$ for all i. This is a contradiction since $\mathrm{ht}(\mathfrak{p}) = 2$ and $m \geqslant 5$. Therefore $\mathcal{X}$ is Cohen-Macaulay. Let $\mathcal{X}' \subset \mathcal{X}$ be the union of the irreducible components of $\mathcal{X}$ which dominate $\mathbb{A}^m$ under the morphism $q : \mathcal{X}' \to \mathbb{A}^m$. Note that $\mathcal{X}'$ does not have any embedded primes, as $\mathcal{X}$ is Cohen-Macaulay. Since X is irreducible, we know that X_α is irreducible for a general α (by the Bertini theorem as proved in Case 1.). Thus a general fiber $q^{-1}(\alpha) = X_\alpha$ is irreducible, and we conclude that $\mathcal{X}'$ is irreducible by Lemma 4.9. Moreover, we know that the fibers X_α are non-reduced for a dense set of α, by assumption. Therefore, $\mathcal{X}'$ is reduced by Lemma 4.9.

Now $p^{-1}(U)$ is an irreducible open subset of $\mathcal{X}$, as U is irreducible. Therefore $p^{-1}(U)$ is contained in a unique irreducible component of $\mathcal{X}$. Now $p^{-1}(U)$ dominates $\mathbb{A}^m$ under the morphism q, since a general point of X is contained in a general hyperplane H_α parametrized by $\alpha \in \mathbb{A}^m$. Therefore, $p^{-1}(U)$ must be contained in $\mathcal{X}'$. Since $\mathcal{X}'$ is reduced, we obtain that $p^{-1}(U)$ is reduced, as desired. $\square$

4.4 Irreducibility of specializations

Let $S = \mathbb{K}[x_1, \cdots, x_n, y_1, \cdots, y_m]$ be a polynomial ring and $A = \mathbb{K}[x_1, \cdots, x_n]$. Let $K(A)$ denote the fraction field of A. For any polynomial $P \in S$ and $\alpha \in \mathbb{K}^n$, we let $P_\alpha(y_1, \cdots, y_m) = P(\alpha_1, \cdots, \alpha_n, y_1, \cdots, y_m) \in \mathbb{K}[y_1, \cdots, y_m]$. Suppose that $P \notin A$, and P is absolutely irreducible over $K(A)$, then it follows from [Sta18, Tag 0559], that P_α is irreducible in $\mathbb{K}[y_1, \cdots, y_m]$ for a general $\alpha \in \mathbb{K}^n$. Following [Sta18, Tag 0557], we show that we can find a polynomial equation for the locus of α for which the specialization P_α is reducible. Furthermore, we show that we can also control the degree of the polynomial defining the equation.

For any $d, m, r \geqslant 1$ we let

$$C(m, r, d) = md^4(d^2 + 2d)^{2^{(2(d+1)m^d + r - 1)}}.$$

Lemma 4.11. *Suppose that* $t_1, \cdots, t_r \in A$ *is a regular sequence in* S *where* $\deg(t_i) \leqslant d$ *and* $A' := \mathbb{K}[t_1, \cdots, t_r] \subset S$. *Let* $P \in S$ *be a polynomial of degree* d. *Suppose that*

1. P *is absolutely irreducible over* $K(A)$.

2. $P \in A'[y_1, \cdots, y_m]$.

Then there exists a polynomial $H \in A$ *of degree* $\deg(H) \leqslant C(m, r, d)$ *such that* $P_\alpha \in \mathbb{K}[y_1, \cdots, y_m]$ *is irreducible for any* $\alpha \in \mathbb{K}^n$ *with* $H(\alpha) \neq 0$.

Proof. We let d_i be the degree of P in y_i for $i \in [m]$. We may write P as a polynomial in y_m as follows:

$$P = (\sum h_L y^L) y_m^{d_m} + P_1$$

where $h_L \in A$ and L varies over tuples $L = (\ell_1, \cdots, \ell_{m-1})$ with $0 \leqslant \ell_j \leqslant d_j$, and $P_1 \in S$ consists of monomials of P with degree of y_m strictly less than d_m. By assumption (2), we know that $h_L \in A'$ for all L. We will show that there exists a non-zero polynomial $f_m \in A$ such that, if for some $\alpha \in \mathbb{K}^n \setminus V(\{h_L\})$, the specialization P_α factors into a product of two polynomials of strictly smaller degree in y_m , then we must have $f_m(\alpha) = 0$.

For any two integers $e_1, e_2 > 0$ with $e_1 + e_2 = d_m$, we define the universal polynomials Q_1, Q_2 as follows.

$$Q_1 = \sum_{0 \leqslant \ell \leqslant e_1} (\sum_L a_{1,\ell,L} y^L) y_m^\ell$$

and

$$Q_2 = \sum_{0 \leqslant \ell \leqslant e_2} (\sum_L a_{2,\ell,L} y^L) y_m^\ell$$

where the coefficients $a_{1,\ell,L}, a_{2,\ell,L}$ are indeterminates and L varies over the set of multi-indices $L = (\ell_1, \cdots, \ell_{m-1})$ with $\ell_i \leqslant d_i$. Note that the total number of indeterminates $a_{i,\ell,L}$ appearing as coefficients in Q_1, Q_2 is at most $2(d+1)m^d$.

Let $T := A[\{a_{1,\ell,L}\}, \{a_{2,\ell,L}\}]$ and $T' := A'[\{a_{1,\ell,L}\}, \{a_{2,\ell,L}\}]$. Let $I_{e_1,e_2} \subset T$ be the ideal generated by the coefficients of $P - Q_1 Q_2$ in $T[y_1, \cdots, y_m]$. Since $P - Q_1 Q_2 \in T'[y_1, \cdots, y_m]$, these coefficients belong to T' and we similarly let $I'_{e_1,e_2} \subset T'$ be the ideal generated by these coefficients. Let B'_{e_1,e_2} be the finitely generated A'-algebra $T'/I'_{e_1,e_2}$ and similarly $B = T/I_{e_1,e_2}$.

Note that $P = Q_1 Q_2$ in $B'_{e_1,e_2}[y_1, \cdots, y_m]$. Since P is also irreducible over $\overline{K(A')}$, we conclude that there does not exist any non-trivial A'-algebra homomorphism $B'_{e_1,e_2} \to \overline{K(A')}$. By Hilbert's Nullstellensatz, we conclude that $B'_{e_1,e_2} \otimes K(A') = 0$. Therefore, by Lemma 4.2, we know that there exists a non-zero polynomial $f_{e_1,e_2} \in A$ with

$$\deg(f_{e_1,e_2}) \leqslant d(d^2 + 2d)^{2^{(2(d+1)m^d + r - 1)}}$$

such that $(B_{e_1,e_2})_{f_{e_1,e_2}} = 0$. Let $\alpha \in \mathbb{K}^n \setminus V(h_L f_{e_1 e_2})$ for some L. Then P_α is a non-zero polynomial of degree d_m in y_m. If P_α factors into a product of two polynomials of smaller degree e_1, e_2, then there exist specializations $(Q_1)_\beta, (Q_2)_\gamma$ of the polynomials Q_1, Q_2 given by $a_{1,\ell,L} = \beta_{1,\ell,L} \in \mathbb{K}$ and $a_{2,\ell,L} = \gamma_{2,\ell,L} \in \mathbb{K}$ such that $P_\alpha = (Q_1)_\beta (Q_2)_\gamma$. Therefore the evaluation $x_i \mapsto \alpha_i$, $a_{1,\ell,L} \mapsto \beta_{1,\ell,L}$ and $a_{2,\ell,L} \mapsto \gamma_{2,\ell,L}$ gives a non-zero homomorphism $B_{e_1,e_2} \to \mathbb{K}$. Now, by the universal property of localization, this homomorphism must factor through $B_{e_1,e_2} \to (B_{e_1,e_2})_{f_{e_1,e_2}}$, as $f_{e_1,e_2}(\alpha) \neq 0$. This is a contradiction as $(B_{e_1,e_2})_{f_{e_1,e_2}} = 0$. Hence P_α does not have a factorization into polynomials of degree e_1, e_2 if $h_L f_{e_1,e_2}(\alpha) \neq 0$.

We let $f_m = \prod_{e_1,e_2>0, e_1+e_2=d_m} f_{e_1,e_2}$ and $H_m = h_L f$ for some L. Similarly, we may define the polynomials H_j corresponding to the variables y_j for $j \in [m]$. Therefore, we have that if $H_j(\alpha) \neq 0$, then P_α is an irreducible polynomial of degree d_j in y_j. Therefore we are done by taking $H = \Pi_j H_j$. Since $\deg(h_L) \leqslant d$, $d_j \leqslant d$ and the number of pairs $e_1, e_2 > 0, e_1 + e_2 \leqslant d$ is at most d^2, the bound on $\deg(f_{e_1,e_2})$ above implies that $\deg(H) \leqslant C(m, n, d)$. □

Corollary 4.12 (Locus of reducible specializations)**.** *Let* $S = \mathbb{K}[x_1, \cdots, x_n, y_1, \cdots, y_m]$ *be a polynomial ring and* $A = \mathbb{K}[x_1, \cdots, x_n]$*. Let* $P \in S$ *be a polynomial of degree* d*. Suppose that* P *is absolutely irreducible over* $K(A)$*. Then there exists a polynomial* $H \in A$ *of degree* $\deg(H) \leqslant C(m, n, d)$ *such that* $P_\alpha \in \mathbb{K}[y_1, \cdots, y_m]$ *is irreducible for any* $\alpha \in \mathbb{K}^n$ *with* $H(\alpha) \neq 0$*.*

Proof. Since $x_1, \cdots, x_n$ is a regular sequence in S, we apply Lemma 4.11 with $A' = A$ to obtain the desired conclusion. ☐

Now we show that we can actually get rid of the dependence on n in the bound for the degree of H in Corollary 4.12.

Proposition 4.13 (Improved Locus of reducible specializations)**.** *There exists a function* $\widetilde{C} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *such that the following holds. Let* $S = \mathbb{K}[x_1, \cdots, x_n, y_1, \cdots, y_m]$ *be a polynomial ring and* $A = \mathbb{K}[x_1, \cdots, x_n]$*. Let* $P \in S$ *be a polynomial of degree* d*. Suppose that* P *is absolutely irreducible over* $K(A)$*. Then there exists a polynomial* $H \in A$ *of degree* $\deg(H) \leqslant \widetilde{C}(m, d)$ *such that* $P_\alpha \in \mathbb{K}[y_1, \cdots, y_m]$ *is irreducible for any* $\alpha \in \mathbb{K}^n$ *with* $H(\alpha) \neq 0$*.*

Proof. Let us consider P as an element of $A[y_1, \cdots, y_m]$ and write $P = \sum_{\boldsymbol{e}} a_{\boldsymbol{e}} y^{\boldsymbol{e}}$, where $\boldsymbol{e} = (e_1, \cdots, e_m) \in \mathbb{N}^m$ with $e_1 + \cdots + e_m \leqslant d$ and $y^{\boldsymbol{e}}$ denotes the monomial $y_1^{e_1} \cdots y_m^{e_m}$. Note that the number of such coefficients is $\binom{m+d}{d}$ and moreover $\deg_S(a_{\boldsymbol{e}}) \leqslant d$ for all $\boldsymbol{e}$. Consider the graded vector space $V = \oplus_{i=1}^{d} V_i \subset S$, spanned by all the coefficient polynomials $a_{\boldsymbol{e}}$. Now we have $\dim(V_i) \leqslant \binom{m+d}{d}$ and $\dim(V) \leqslant d\binom{m+d}{d}$. By [AH20a, Corollary B], there exists a regular sequence $t_1, \cdots, t_r$ in S such that all the coefficients $a_{\boldsymbol{e}}$ are contained in the algebra $A' := \mathbb{K}[t_1, \cdots, t_r]$ and we also have $r \leqslant R(m, d) := {}^3B(d\binom{m+d}{d}, d)$, where ${}^{\eta}B : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is the function defined in [AH20a]. In particular, r is upper bounded by a function of m, d only, independent of n. Since $P \in A'[y_1, \cdots, y_m]$, we apply Lemma 4.11 to conclude that there exists H with the desired properties and we also have $\deg(H) \leqslant C(m, R(m, d), d)$. Hence, we define $\widetilde{C}(m, d) = C(m, R(m, d), d)$. ☐

4.5 Primality criterion and Effective bounds

In this section we prove effective bounds on the number of non-prime ideals of the form (P, Q). The following lemma proves a primality criterion for certain ideals of the form (P, Q).

Lemma 4.14. *Let* $S := \mathbb{K}[x_1, \dots, x_n, y_1, \dots, y_m]$ *be a graded polynomial ring with* $\deg(x_i) = d_i \geqslant 1$ *and* $\deg(y_j) = e_j \geqslant 1$*. Let* $A = \mathbb{K}[x_1, \cdots, x_n]$ *and* $K(A)$ *the fraction field of* A*. Let* $d \geqslant 1$ *and* $P \in S_d \setminus (x_1, \dots, x_n)$ *such that* P *is absolutely irreducible over* $K(A)$*. Then there exists a polynomial* $H \in \mathbb{K}[x_1, \cdots, x_n]$ *with* $\deg(H) \leqslant \widetilde{C}(m, d)$ *such that the following holds. For any irreducible homogeneous* $Q \in \mathbb{K}[x_1, \cdots, x_n]$ *such that* $H \notin (Q)$*, we have that* (P, Q) *is prime.*

Proof. Let H be the polynomial given by Proposition 4.13. Recall that $\deg(H) \leqslant \widetilde{C}(m, d)$. Let $Q \in \mathbb{K}[x_1, \cdots, x_n]$ be an irreducible homogeneous element such that $H \notin (Q)$. We will show that (P, Q) is prime.

By [OS24, Proposition 4.22], we have that P is a non-zero divisor in $S/(Q)$. In particular $S/(P, Q)$ is Cohen-Macaulay. Furthermore, we also have that for any minimal prime $\mathfrak{p}$ over (P, Q) in S, the ideal $\mathfrak{p} \cap A$ is a minimal prime of (Q) in A. In particular, $\mathfrak{p} \cap A = (Q)$, since Q is irreducible. Let $\pi : \mathbb{A}^{m+n} \to \mathbb{A}^n$ be the morphism corresponding to $A \subset S$. Note that $\pi(x_1, \cdots, z_n, y_1, \cdots, x_m) = (x_1, \cdots, x_n)$ for closed points. Consider the affine schemes given by $Y = \mathrm{Spec}(S/(P, Q))$ and $X = \mathrm{Spec}(A/(Q))$. Note that we have a homomorphism of finitely generated $\mathbb{K}$-algebras $A/(Q) \to S/(P, Q)$, as $(Q) \subset (P, Q) \cap A$. The corresponding morphism of affine schemes is given by $\pi|_Y : Y \to X$ and we have commutative diagram.

$$\begin{array}{ccc} Y & \longrightarrow & \mathbb{A}^{m+n} \\ \big\downarrow{\scriptstyle \pi|_Y} & & \big\downarrow{\scriptstyle \pi} \\ X & \longrightarrow & \mathbb{A}^{n} \end{array}$$

Since Q is irreducible, we have that the scheme X is irreducible and reduced. Since $\mathfrak{p} \cap A = (Q)$ for any minimal prime $\mathfrak{p}$ over (P, Q) in S, we have that any irreducible component of Y dominates X. Therefore, by Lemma 4.9, it is enough to show that $\pi|_Y^{-1}(x)$ is irreducible and reduced for a general closed point $x \in X$. Let $x = (c_1, \cdots, c_n) \in X$ be a closed point. Let $P_x \in \mathbb{K}[y_1 \cdots, y_m]$ denote the polynomial $P(c_1, \cdots, c_n, y_1, \cdots, y_m)$. Note that $\pi|_Y^{-1}(x) = \mathrm{Spec}(\mathbb{K}[y_1, \cdots, y_m]/(P_x))$, as $x \in X$. Since $H \notin (Q)$, we have that $H(x) \neq 0$ for a general closed point $x \in X$. Therefore, by Proposition 4.13, we conclude that P_x is irreducible in $\mathbb{K}[y_1, \cdots, y_m]$ for a general closed point $x \in X$. Therefore, $\pi|_Y^{-1}(x) = \mathrm{Spec}(\mathbb{K}[y_1, \cdots, y_m]/(P_x))$ is irreducible and reduced for a general closed point $x \in X$. Hence (P, Q) is prime. □

Proposition 4.15. *Let $S = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be a graded polynomial ring with $\deg(x_i) = d_i \geqslant 1$ and $\deg(y_j) = e_j \geqslant 1$. Let $A = \mathbb{K}[x_1, \cdots, x_n]$ and $K(A)$ the fraction field of A. Let $d \geqslant 1$ and $P \in S_d \setminus (x_1, \ldots, x_n)$ such that P is absolutely irreducible over $K(A)$. Then there are at most $\widetilde{C}(m, d)$ pairwise non-associate irreducible homogeneous elements $Q_i \in A$ such that (P, Q_i) is not prime.*

Proof. Let $H \in \mathbb{K}[x_1, \cdots, x_n]$ be the polynomial given by Lemma 4.14 . Therefore for any $Q_i \in A$ such that (P, Q_i) is not prime, we must have that $H \in (Q_i)$, by Lemma 4.14 . Since, $\deg(H) \leqslant \widetilde{C}(m, d)$, we conclude that there exist at most $\widetilde{C}(m, d)$ pairwise non-associate irreducible homogeneous elements $Q_i \in \mathbb{K}[x_1, \ldots, x_n]$ such that (P, Q_i) is not prime. □

By applying our Bertini-type results we can improve the bound in Proposition 4.15. In fact, we show that the number of non-prime pairs (P, Q_i) can be effectively bounded independent of m.

Theorem 4.16. *There exists a function $\mathcal{C} : \mathbb{N} \to \mathbb{N}$ such that the following holds.*

Let $S = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be a graded polynomial ring with $\deg(x_i) = d_i \geqslant 1$ and $\deg(y_j) = e_j \geqslant 1$. Let $A = \mathbb{K}[x_1, \cdots, x_n]$ and $K(A)$ the fraction field of A. Let $d \geqslant 1$ and $P \in S_d \setminus (x_1, \ldots, x_n)$ such that P is absolutely irreducible over $K(A)$. Then there are at most $\mathcal{C}(d)$ pairwise non-associate irreducible homogeneous elements $Q_i \in A$ such that (P, Q_i) is not prime.

Proof. By Proposition 4.15, we know that there exist at most $\widetilde{C}(m, d)$ pairwise non-associate irreducible homogeneous elements $Q_i \in A$ such that (P, Q_i) is not prime. Since there are finitely many such Q_i, we may apply Lemma 4.10 to each pair (P, Q_i) to assume that $(\varphi_\alpha(P), \varphi_\alpha(Q_i))$ is not prime for a general $\alpha \in \mathbb{K}^m$. Note that $\varphi_\alpha(Q_i) = Q_i$. We may apply $(m-5)$ number of successive hyperplane sections corresponding to $\boldsymbol{\alpha}^{(1)}, \cdots, \boldsymbol{\alpha}^{(m-5)} \in \mathbb{K}^m$ to obtain a quotient ring $S \to R_\alpha$. Note that R_α is isomorphic to $A[y_1, \cdots, y_5]$. Let $\widetilde{P}$ be the image of P in R_α under the successive quotient morphism. By choosing $\boldsymbol{\alpha} = (\boldsymbol{\alpha}^{(1)}, \cdots, \boldsymbol{\alpha}^{(m-5)})$ generally, we may assume that $\widetilde{P}$ is absolutely irreducible over $K(A)$ by Lemma 4.8. Since $\boldsymbol{\alpha}^{(1)}, \cdots, \boldsymbol{\alpha}^{(m-5)} \in \mathbb{K}^m$ are chosen generally and there are finitely many Q_i, we conclude that $(\widetilde{P}, Q_i)$ is not prime in R_α for all i. Then by applying Proposition 4.15 to R_α, we know that there are at most $\widetilde{C}(5, d)$ number of $Q_i \in A$ such that $(\widetilde{P}, Q_i)$ is not prime in R_α. Therefore we are done by defining $\mathcal{C}(d) = \widetilde{C}(5, d)$. □

5 Strong Algebras

In this section we recall the definitions and results that we will need about strong algebras. Most of the definitions and results from this section are taken from [OS24, Section 5], and we refer the reader to this paper for motivation and further discussions about these definitions.

5.1 Strength

Let $R = \oplus_{d \geqslant 0} R_d$ be a finitely generated graded $\mathbb{K}$-algebra, generated by R_1. In [AH20a] the notions of collapse and strength were defined for a polynomial ring. We will extend those definitions to finitely generated graded $\mathbb{K}$-algebras and prove the necessary properties below. Henceforth, we refer to a homogeneous element of R as a *form*, adopting the same notation for polynomial rings.

Definition 5.1 (Collapse)**.** Given a non-zero form $F \in R_d$, we say that F has a k-*collapse* if there exist k forms $G_1, \dots, G_k$ such that $1 \leqslant \deg(G_i) < d$ and $F \in (G_1, \dots, G_k)$.

Definition 5.2 (Strength)**.** Given a non-zero form $F \in R_d$, the *strength* of F, denoted by $s(F)$, is the least positive integer such that F has a $(s(F)+1)$-collapse but it has no $s(F)$-collapse. We say that $s(F) \geqslant t$ whenever F does not have a t-collapse.

Remark 5.3. By the definitions above, a form $x \in R_1$ does not have a k-collapse for any $k \in \mathbb{N}$. Thus, we say that for any $x \in R_1$, $s(x) = \infty$. In particular, linear forms in the polynomial ring S have infinite strength. We will make the convention that $s(0) = -1$.

Definition 5.4 (Minimum collapse)**.** Given a non-zero form $F \in R_d$ and $s \in \mathbb{N}^*$ such that $s(F) = s - 1$, a *minimum collapse* of F is any identity of the form $F = G_1 H_1 + \cdots + G_s H_s$, where G_i, H_i are forms of degree in $[d-1]$.

It is useful to define the min and max strength of a linear system of forms of the same degree.

Definition 5.5 (Min and max strength)**.** Given a set of forms $F_1, \dots, F_r \in R_d$, define $s_{\min}(F_1, \dots, F_r)$ as the *minimum* strength of a *non-zero* form in $\operatorname{span}_{\mathbb{K}}\{F_1, \dots, F_r\}$ and $s_{\max}(F_1, \dots, F_r)$ as the *maximum* strength of a form in $\operatorname{span}_{\mathbb{K}}\{F_1, \dots, F_r\}$.

In particular, given any non-zero finite dimensional vector space $V \subset R_d$, define $s_{\min}(V)$ ($s_{\max}(V)$) as the minimum (maximum) strength of any non-zero form in V. If $V = (0)$, then there are no non-zero forms in V. In this case, by convention we define $s_{\min}((0)) = s_{\max}((0)) = \infty$. We will say that a vector space V is k-strong if $s_{\min}(V) \geqslant k$. Note that the zero vector space is infinitely strong.

5.2 Strong Ananyan-Hochster Vector Spaces

Let $R = \oplus_{d \geqslant 0} R_d$ be a finitely generated graded $\mathbb{K}$-algebra, generated by R_1. Given a graded $\mathbb{K}$-vector space $V = \bigoplus_{i=1}^{d} V_i \subset R$, where $\delta_i := \dim V_i$, we denote its dimension sequence by $\delta := (\delta_1, \dots, \delta_d)$.

Definition 5.6 (Strong Ananyan-Hochster vector spaces)**.** Let $R = \oplus_{d \geqslant 0} R_d$ be a finitely generated graded $\mathbb{K}$-algebra, generated by R_1. For any function $B = (B_1, \cdots, B_d) : \mathbb{N}^d \to \mathbb{N}^d$, we say that a non-zero graded vector subspace $V = \oplus_{i=1}^{d} V_i \subset R$, with dimension sequence δ, is a B-strong AH vector space if V_i is $B_i(\delta)$-strong for all i, i.e. $s_{\min}(V_i) \geqslant B_i(\delta)$. The subalgebra $\mathbb{K}[V] \subset R$ generated by a B-strong AH vector space V is called a B-strong AH algebra.

Note that if $V = (0)$, then V is B-strong for any function B, since $s_{\min}((0)) = \infty$. The following result is a corollary of [AH20b, Theorem A], and a proof can be found in [OS24, Corollary 5.9]. In the following lemma and the rest of this article, the function $A(\eta, d) : \mathbb{N}^2 \to \mathbb{N}$ is the function defined in [AH20b, Theorem A].

Corollary 5.7. *Let $V = \oplus_{i=1}^{d} V_i \subset S$ be a B-strong AH vector space for some $B : \mathbb{N}^d \to \mathbb{N}^d$. Suppose $B_i(\delta) \geqslant A(\eta, i) + 3(\sum_i \delta_i - 1)$ for some $\eta \in \mathbb{N}$. Then any sequence of $\mathbb{K}$-linearly independent forms in V is an $\mathcal{R}_\eta$-sequence. If $\eta \geqslant 3$, then $S/(V)$ is a Cohen-Macaulay, unique factorization domain.*

5.3 Lifted strength

Definition 5.8 (Lifted strength)**.** Let $U \subset S$ be a graded vector space and $R = S/(U)$. Let $F \in R_d$ be a non-zero form. We define the lifted strength of F with respect to U as

$$\widetilde{s}_{\min}(U, F) := \min\{s_{\min}(U_d + \operatorname{span}_{\mathbb{K}}\left\{\widetilde{F}\right\})\}$$

where $\widetilde{F}$ varies over all forms in S_d such that the image of $\widetilde{F}$ in R is F. Given a set of forms $F_1, \cdots, F_m \in R_d$, we define

$$\widetilde{s}_{\min}(U, F_1, \cdots, F_m) = \min\{s_{\min}(U_d + \operatorname{span}_{\mathbb{K}}\left\{\widetilde{F}_1, \cdots, \widetilde{F}_m\right\})\},$$

where $\widetilde{F}_i$ varies over all forms in S_d such that the image of $\widetilde{F}_i$ in R is F_i. Given a non-zero vector space $V \subset R_d$, we define

$$\widetilde{s}_{\min}(U, V) = \min\{\widetilde{s}_{\min}(U_d, F_1, \cdots, F_m)\},$$

where $F_1, \cdots, F_m$ vary over all possible bases of V. We say that $V \subset R_d$ is k-lifted strong with respect to U if $\widetilde{s}_{\min}(U, V) \geqslant k$. For simplicity, we omit U from the notation and write $\widetilde{s}_{\min}(V)$ when U is clear from the context.

Let $V = \oplus_{i=1}^{d} V_i \subset R$ be a graded vector space. For any function, $B : \mathbb{N}^d \to \mathbb{N}^d$ we will say that V is B-lifted strong with respect to U, if V_i is $B_i(\dim(U_i) + \dim(V_i))$-lifted strong, i.e. $\widetilde{s}_{\min}(U, V_i) \geqslant B_i(\dim(U_i) + \dim(V_i))$ for all $i \in [d]$. In other words, V is B-lifted strong with respect to U, if the vector space $U + \operatorname{span}_{\mathbb{K}}\left\{\widetilde{F}_1, \cdots, \widetilde{F}_m\right\}$ is B-strong in S, for any homogeneous basis $F_1, \cdots, F_m \in R$ of V and any set of homogeneous lifts $\widetilde{F}_1, \cdots, \widetilde{F}_m \in S$.

5.4 Strengthening and Robustness

For any $\mu \in \mathbb{N}^d$, we define the translation function $t_\mu : \mathbb{N}^d \to \mathbb{N}^d$ as $t_\mu = (t_{\mu,1}, \cdots, t_{\mu,d})$ where the i-th component is defined by $t_{\mu,i}(\delta) = \delta_i + \mu_i$. In other words, for all $i \in [d]$ we add μ_i to the i-th component of δ. For any $n \in \mathbb{N}$, we let $t_n := t_{(n,\cdots,n)}$.

The following lemma is proved in [OS24, Lemma 5.15].

Lemma 5.9 (Strengthening of Algebras)**.** *For any $d \in \mathbb{N}$ and a function $B : \mathbb{N}^d \to \mathbb{N}^d$, there exist functions $C_B : \mathbb{N}^d \to \mathbb{N}^d$ and $h_B : \mathbb{N}^d \to \mathbb{N}^d$, depending on B, such that the following holds:*

Given a graded vector space $U = \oplus_{i=1}^{d} U_i \subset S$ with dimension sequence $\delta \in \mathbb{N}^d$, there exists a B-strong AH vector space $V = \oplus_{i=1}^{d} V_i$ such that

1. $\mathbb{K}[U] \subset \mathbb{K}[V]$,

2. *for all* $i \in [d]$, *we have* $\dim(V_i) \leqslant C_{B,i}(\delta)$, *where* $C_{B,i}$ *denotes the* i*-th component of* $C_B = (C_{B,1}, \cdots, C_{B,d}) : \mathbb{N}^d \to \mathbb{N}^d$.

Furthermore, suppose $H = \oplus_{i=1}^d H_i \subset U$ *is a graded subspace such that* $s_{\min}(H_i) \geqslant h_{B,i}(\delta)$ *for all* $i \in [d]$. *Then there exists a* B*-strong AH vector space* V *satisfying (1) and (2) above such that* $H \subset V$.

The following corollary is from [OS24, Corollary 5.16].

Corollary 5.10 (Robustness of strong algebras)**.** *Let* $B, G : \mathbb{N}^d \to \mathbb{N}^d$ *and* $\mu \in \mathbb{N}^d$. *Suppose that* $B_i(\delta) \geqslant h_{G,i}(\delta + \mu)$ *for all* $\delta \in \mathbb{N}^d$ *and* $i \in [d]$, *where* $h_G : \mathbb{N}^d \to \mathbb{N}^d$ *is the function defined in Lemma 5.9. Let* $U \subset S$ *be a* B*-strong AH vector space and* $W \subset S$ *is a graded vector space with dimension sequences* δ *and* μ *respectively. Then there exists a* G*-strong AH vector space* V *such that*

1. $\mathbb{K}[U + W] \subset \mathbb{K}[V]$,
2. $U \subset V$,
3. *for all* $i \in [d]$, $\dim(V_i) \leqslant C_{G,i}(\delta + \mu)$, *where* $C_G : \mathbb{N}^d \to \mathbb{N}^d$ *is the function defined in Lemma 5.9.*

The following corollary corresponds to [OS24, Corollary 5.17].

Corollary 5.11. *Let* $B : \mathbb{N}^d \to \mathbb{N}^d$. *Let* $U \subset S$ *be a graded vector space with dimension sequence* $\delta_U \in \mathbb{N}^d$ *and let* $R = S/(U)$. *Let* $V \subset R$ *is a graded vector space with dimension sequence* $\delta_V \in \mathbb{N}^d$. *Suppose* V *is* $h_{2B} \circ t_k$*-lifted strong with respect to* U. *Let* $P_1, \cdots, P_k \in R_{\leqslant d}$ *be homogeneous elements. Then there exists a graded vector space* $V' \subset R_{\leqslant d}$ *such that:*

1. V' *is* B*-lifted strong with respect to* U.
2. $P_1, \cdots, P_k \in \mathbb{K}[V']$.
3. $V \subset V'$.
4. *for all* $i \in [d]$, *we have* $\dim(V'_i) \leqslant C_{2B,i}(t_k(\delta_U + \delta_V)) - \delta_{U,i}$.

5.5 Absolute irreducibility with respect to strong vector spaces

Definition 5.12. Let $B : \mathbb{N}^d \to \mathbb{N}^d$. Suppose $B_i(\delta) \geqslant A(\eta, i) + 3(\sum_i \delta_i - 1)$ for some $\eta \in \mathbb{N}$. Suppose $R = S/(U)$. Suppose $V \subset R$ is a graded vector space that is $h_{2B} \circ t_1$-lifted strong with respect to U. Suppose $P \in R$ is a form. Let V' be the vector space obtained by applying Corollary 5.11 to V and P. Let $y_1, \ldots, y_a$ be a basis of homogeneous forms of V, and $y_{a+1}, \ldots, y_b$ extend this to a basis of V'. We say P is absolutely reducible over V if P is absolutely reducible as a polynomial in $\mathbb{K}(y_1, \ldots, y_a)[y_{a+1}, \ldots, y_b]$.

Remark 5.13. Corollary 5.11 guarantees the existence of the vector space V' with the stated properties by constructing such a vector space iteratively, however the vector space V' with the stated properties is not unique. Our definition of absolute reducibility depends on the choice of vector space V'. We say that P is absolutely reducible if the property in Definition 5.12 is satisfied for some choice of V', and we say that P is absolutely irreducible if P is not absolutely reducible.

Absolute reducibility and irreducibility with respect to vector spaces allows us to apply Theorem 4.16 in more general settings. The function $\mathcal{C} : \mathbb{N} \to \mathbb{N}$ referred to in the following lemma is the same function whose existence is guaranteed by Theorem 4.16.

Corollary 5.14 (Corollary of Theorem 4.16)**.** *Let* $B : \mathbb{N}^d \to \mathbb{N}^d$. *Suppose* $B_i(\delta) \geqslant A(\eta, i)+3(\sum_i \delta_i - 1)$ *for some* $\eta \in \mathbb{N}$. *Suppose* $R = S/(U)$. *Suppose* $V \subset R$ *is a graded vector space that is* $h_{2B} \circ t_1$*-lifted strong with respect to* U. *Suppose* $P \in R_d$ *is absolutely irreducible with respect to* V. *There are at most* $\mathcal{C}(d)$ *irreducible non-associate forms* $Q_i \in \mathbb{K}[V]$ *such that* (Q_i, P) *is not prime.*

Proof. Let V' be the vector space obtained by applying Corollary 5.11 to V and P. Let $y_1, \dots, y_a$ be a basis of homogeneous forms of V, and $y_{a+1}, \dots, y_b$ extend this to a basis of V'. The ring $\mathbb{K}[V']$ is isomorphic to a polynomial ring in the variables y_i. Further, P is irreducible as a form in $\mathbb{K}[V']$ by definition of absolute irreducibility with respect to V. Therefore we can apply Theorem 4.16 to deduce that there are at most $\mathcal{C}(d)$ forms $Q_i \in \mathbb{K}[V]$ such that (P, Q_i) is not prime as an ideal of $\mathbb{K}[V']$. Further, since $y_1, \dots, y_b$ form a prime sequence, prime ideals of $\mathbb{K}[V']$ extend to prime ideals in R. This completes the proof. □

6 General Quotients

In the introduction, we outlined our inductive approach for proving Theorem 1.5, which involves reducing the degree of the factor set in a product SG-configuration through a series of quotient homomorphisms $R \to R[z]/(W)$, where W is a sufficiently strong vector space. This section defines such general quotients and the essential properties needed for this degree reduction. Most of the definitions and results in this section are taken from [OS24, Section 6], which we restate here to make the paper self-contained. We also provide proofs of the new facts that we will need which did not appear in [OS24].

Throughout this section, we fix positive integers $d, \eta \in \mathbb{N}$ with $\eta \geqslant 3$, and $B : \mathbb{N}^d \to \mathbb{N}^d$ denotes an ascending function such that $B_i(\delta) \geqslant A(\eta, i) + 3(\sum_i \delta_i - 1)$ for all $i \in [d]$.

The following definition corresponds to [OS24, Definition 6.1].

Definition 6.1 (Graded Quotients)**.** Let $U = \oplus_{i=1}^d U_i \subset S$ be a graded vector space of dimension sequence δ in S and $R := S/(U)$ be the quotient ring. Let $V = \oplus_{i=1}^d V_i \subset R$ be a graded subspace of dimension sequence μ.

Let $F_1, \dots, F_n$ be a homogeneous basis for V and z be a new variable. For $\alpha \in \mathbb{K}^n$, let $V_\alpha := \operatorname{span}_{\mathbb{K}} \{F_1 - \alpha_1 z^{\deg(F_1)}, \dots, F_n - \alpha_n z^{\deg(F_n)}\}$ and $I_\alpha \subset R[z]$ be the homogeneous ideal $I_\alpha = (V_\alpha)$. We define the graded quotient map $\varphi_{V,\alpha}$ as the quotient homomorphism of finitely generated graded $\mathbb{K}$-algebras given by

$$\varphi_{V,\alpha} : R[z] \to R[z]/I_\alpha.$$

For simplicity we will often drop the subscripts V or α, and write φ_α or φ for our quotient map when there is no ambiguity about the vector space V or the vector α.

Remark 6.2. In the above definition we abuse notation and define V_α with a fixed basis in mind. This abuse of notation is only to simplify our definition of the quotients we will use, and the choice of basis is not very important, since [AH20a] shows that any basis of V_α forms an $\mathcal{R}_\eta$ sequence.

The next proposition and lemma correspond to [OS24, Proposition 6.3] and [OS24, Lemma 6.4].

Proposition 6.3. *Suppose* $V \subset R$ *is* B*-lifted strong with respect to* U. *Then* $R[z]$ *and* $R[z]/I_\alpha$ *are quotients of* $S[z]$ *by* $\mathcal{R}_\eta$*-sequences, for any choice of* $\alpha \in \mathbb{K}^n$. *In particular, they are Cohen-Macaulay UFDs.*

Lemma 6.4. *Let* $S = \mathbb{K}[x_1, \cdots, x_N]$ *and* z *be a new variable. Fix positive integers* $d_1, \cdots, d_n \in \mathbb{N}$. *For* $\alpha \in \mathbb{K}^n$, *let* $I_\alpha = (x_1 - \alpha_1 z^{d_1}, \cdots, x_n - \alpha_n z^{d_n})$. *Let* $\varphi_\alpha : S[z] \to S[z]/I_\alpha$ *be the quotient ring homomorphism.*

1. *The ideal* I_α *is prime in* $S[z]$*, and the composition morphism* $\mathbb{K}[z] \hookrightarrow S[z] \to S[z]/I_\alpha$ *is injective.*

2. *If* $F \in \mathbb{K}[x_1, \cdots, x_n]$ *is a non-zero polynomial, then* $\varphi_\alpha(F) \neq 0$ *in* $S[z]/I_\alpha$ *for a general* $\alpha \in \mathbb{K}^n$*.*

3. *Let* $F \in S \setminus \mathbb{K}[x_1, \cdots, x_n]$*, then* $\varphi_\alpha(F) \notin \mathbb{K}[z]$ *in* $S[z]/I_\alpha$*, for a general* $\alpha \in \mathbb{K}^n$*.*

4. *If* $F \in S$ *is a non-zero polynomial, then* $\varphi_\alpha(F) \neq 0$ *in* $S[z]/I_\alpha$ *for a general* $\alpha \in \mathbb{K}^n$*.*

5. *If* $F, G \in S$ *have no common factor, then* $\gcd(\varphi_\alpha(F), \varphi_\alpha(G)) \in \mathbb{K}[z]$ *for a general* $\alpha \in \mathbb{K}^n$*.*

6. *If* $F \in S$ *is square-free. Then, for a general* $\alpha \in \mathbb{K}^n$*, the multiple factors of* $\varphi_\alpha(F)$ *must be in* $\mathbb{K}[z]$*.*

Proposition 6.5. *Let* $V \subset R$ *be a* B*-lifted strong vector space and* $\varphi_\alpha : R[z] \to R[z]/I_\alpha$ *be a graded quotient as defined in Definition 6.1.*

1. *The ideal* I_α *is a prime ideal in* $R[z]$ *and the composition* $\mathbb{K}[z] \hookrightarrow R[z] \to R[z]/I_\alpha$ *is injective.*

2. *If* $F \in R \setminus \{0\}$*, then* $\varphi_\alpha(F) \neq 0$ *for a general* $\alpha \in \mathbb{K}^n$*.*

3. *If* $F \notin \mathbb{K}[V] \subset R$*, then* $\varphi_\alpha(F) \notin \mathbb{K}[z]$ *for a general* $\alpha \in \mathbb{K}^n$*.*

4. *If* F *is absolutely reducible with respect to* V *then* $\varphi_\alpha(F)$ *is a product of forms of positive degree.*

Proof. The proof of items (1)-(3) can be found in [OS24, Proposition 6.5]. So we are only left with item (4), which we now prove.

Let V' be the vector space from the definition of F being absolutely reducible with respect to V: in particular, V' is obtained by applying Corollary 5.11 to V, F. Suppose $x_1, \ldots, x_a$ is a basis for V, and $y_1, \ldots, y_b$ extends this basis to a basis of V'. By definition, F is absolutely reducible in the ring $A := \mathbb{K}(x_1, \ldots, x_a)[y_1, \ldots, y_b]$.

Let $J_\alpha := I_\alpha \cap A[z]$. The proof of [OS24, Proposition 6.5] shows the following facts: The ideal J_α is generated by $x_i - \alpha_i z^{\deg x_i}$, and that $A[z]/J_\alpha \to R[z]/I_\alpha$ is an injective map. Further, $A[z]/J_\alpha$ is identified by a subalgebra of $R[z]/I_\alpha$ that is generated by a prime sequence.

Since $J_\alpha = \left(x_i - \alpha_i z^{\deg x_i}\right)$, we can apply Proposition 4.7 to deduce that F is absolutely reducible in $A(z)/J_\alpha \cong A(z)[y_1, \ldots, y_b]$. Therefore, we can apply Proposition 4.5 to deduce that F is reducible in $A[z]/J_\alpha$. Since this a subalgebra of $R[z]/I_\alpha$ generated by a prime sequence, the image of F under this inclusion, which is the same as $\varphi_\alpha(F)$, is also reducible in $R[z]/I_\alpha$. □

The next proposition corresponds to [OS24, Proposition 6.6].

Proposition 6.6. *Let* $G : \mathbb{N}^d \to \mathbb{N}^d$ *be a function such that* $G_i(\delta) \geqslant h_{B,i} \circ t_2(\delta)$ *for all* $\delta \in \mathbb{N}^d$*. Let* $V \subset R_{\leqslant d}$ *be a* G*-lifted strong vector space and* $\varphi_\alpha : R[z] \to R[z]/I_\alpha$ *be a general quotient as defined in Definition 6.1. Let* $F, G \in R_{\leqslant d}$ *be such that they have no common factor. Then,*

1. $\gcd(\varphi_\alpha(F), \varphi_\alpha(G)) \in \mathbb{K}[z]$

2. *If* F, G *are homogeneous, then* $\gcd(\varphi_\alpha(F), \varphi_\alpha(G)) = z^k$ *for some* $k \in \mathbb{N}$*. In particular, we have* $\gcd(\varphi_\alpha(zF), \varphi_\alpha(zG)) = z^{k+1}$ *for some* $k \in \mathbb{N}$*. Furthermore, if* $F, G \notin K[V] \subset R$ *then* $\varphi_\alpha(F)$*,* $\varphi_\alpha(G)$ *are linearly independent.*

3. *If* $F \in R$ *is a square-free form, then* $\varphi_\alpha(F)$ *does not have multiple factors other than* z^k *for some* $k \in \mathbb{N}$*.*

The next proposition, corresponding to [OS24, Proposition 6.9], tells us that if the product SG-configuration resulting from a general quotient has small vector space dimension, then it must be the case that the original configuration must have small vector space dimension. We refer to this result as "lifting from general quotients," as we are lifting our upper bounds for the quotiented configurations to the original configurations. One important aspect to point out is that the lemma below works for any finite set $\mathcal{F}$ (not necessarily coming from any Sylvester-Gallai configuration).

Proposition 6.7 (Lifting from general quotients)**.** *Let $d, e \in \mathbb{N}$ such that $1 \leqslant d \leqslant e$. Let $U \subset S_{\leqslant e}$ be a graded vector space generated by forms $H_1, \cdots, H_t$. Let $R = S/(U)$. Let $V \subset R_{\leqslant e}$ be a B-lifted strong vector space with basis $F_1, \cdots, F_n \in R$. Let $\varphi_\alpha : R[z] \to R[z]/I_\alpha$ be a graded quotient as defined in Definition 6.1. Let $\mathcal{F} \subset R_{\leqslant d}$ be a finite set of homogeneous elements. Suppose that there exists $D \in \mathbb{N}$ such that $\dim \operatorname{span}_{\mathbb{K}} \{\varphi_\alpha(\mathcal{F})\} \leqslant D$ for a general $\alpha \in \mathbb{K}^n$. Then*

$$\dim \operatorname{span}_{\mathbb{K}} \{\mathcal{F}\} \leqslant d^2(1+d)^{2n+2} D \cdot \Pi_{i=1}^{t} \deg(H_i) \cdot \Pi_{j=1}^{n} \deg(F_j).$$

We are now ready to show that product Sylvester-Gallai configurations are preserved under general quotients. But before we can prove this, we have to properly define the new configuration that will arise from such quotients.

Definition 6.8. Suppose $U \subset S$ is a graded finitely generated vector space such that $R := S/(U)$ is a Cohen-Macaulay UFD. Let $\mathcal{F}$ be a (d, c, z, R)-product Sylvester-Gallai configuration. Let $V \subset R_{\leqslant d}$ be a $h_B \circ t_2$-lifted strong vector space such that $z \in V$, and $\varphi_\alpha : R[y] \to R[y]/I_\alpha$ be a graded quotient, where $I_\alpha := (V_\alpha)$.

We define a configuration $\varphi_\alpha(\mathcal{F})$ as

$$\varphi_\alpha(\mathcal{F}) := \left\{F' \mid F \in \mathcal{F}, \varphi_\alpha(F) = y^r F', F' \neq 1, (F', y) = 1\right\} \cup \{y\}.$$

Informally, we apply the map φ_α to every element of $\mathcal{F}$, and then factor out all powers of y from the images, and add the result to our set as long as it is different from 1. Further, we also add y to the set.

The next proposition shows that this new set is also a product Sylvester-Gallai configuration.

Proposition 6.9. *Suppose $U \subset S$ is a graded finitely generated vector space such that $R := S/(U)$ is CM and a UFD. Suppose $\mathcal{F}$ is a (d, c, z, R)-product Sylvester-Gallai configuration. Suppose $V \subset R_{\leqslant d}$ is a $h_B \circ t_2$-lifted strong vector space such that $z \in V$, and $\varphi_\alpha : R[y] \to R[y]/I_\alpha$ is a graded quotient. Then $\varphi_\alpha(\mathcal{F})$ is a $(d, c, y, R[y]/I_\alpha)$-product Sylvester-Gallai configuration.*

Proof. We have $y \in \varphi_\alpha(\mathcal{F})$ by construction. By item 3 of Proposition 6.6, for any F and F' such that $\varphi_\alpha(F) = y^r F'$, we have that F' is squarefree.

Suppose $F', G' \in \varphi_\alpha(\mathcal{F}) \setminus \{y\}$. Suppose further that $F, G \in \mathcal{F}$ are such that $\varphi_\alpha(F) = y^r F'$ and $\varphi_\alpha(G) = y^s G'$. By item 2 of Proposition 6.6 we have $(\varphi_\alpha(F), \varphi_\alpha(G)) = y^t$ for some t. Since $(F', G') | (\varphi_\alpha(F), \varphi_\alpha(G))$, and since $(y, F') = (y, G') = 1$, we deduce that $(F', G') = 1$. Finally, since the map φ_α is degree preserving, the set $\varphi_\alpha(\mathcal{F})$ is a collection of squarefree and pairwise relatively prime forms of degree at most d, each of which factor into forms of degree at most c.

By the definition of $\mathcal{F}$, for any $F, G \in \mathcal{F}$ different from z we have

$$z \cdot \prod_{H \in \mathcal{F} \setminus \{F, G, z\}} H \in \operatorname{rad}(F, G).$$

Applying the map φ_α, we obtain

$$y \cdot \prod_{H \in \mathcal{F} \setminus \{F, G, z\}} y^{t_H} \varphi_\alpha(H) \in \operatorname{rad}\left(y^r F', y^s G'\right) \subset \operatorname{rad}\left(F', G'\right).$$

For each such H, we either have $\varphi_\alpha(H) = y^{t_H}$ for some t_H, or $\varphi_\alpha(H) = y^{t_H}H'$ for some $H' \in \varphi_\alpha(\mathcal{F})$. Therefore we have

$$y \cdot \prod_{H' \in \varphi_\alpha(\mathcal{F}) \setminus \{F', G', y\}} H' \in \operatorname{rad}\left(F', G'\right).$$

This completes the proof. □

7 Controlling Factor Sets

In this section, we state key lemmas describing the structure of factor sets in product Sylvester-Gallai configurations. The function $\mathcal{C}$ used in this section is the same function whose existence is guaranteed by Theorem 4.16.

The first lemma states that forms of maximum degree in a factor set cannot generate a prime ideal with many forms of lower degree in the factor set.

Lemma 7.1. *Let U be a graded finitely generated vector space such that $R := S/(U)$ is a CM UFD. Let $1 < c \leqslant d$ be integers and $\mathcal{F}$ be a (d, c, z, R)-product Sylvester-Gallai configuration with factor set $\mathcal{I}$ of degree c. For any $G \in \mathcal{I}_c$, there are at most $d + c^2 \cdot |\mathcal{I}_c|$ forms $H \in \mathcal{I}_{<c}$ such that (G, H) is prime.*

Proof. Suppose $H_1, \ldots, H_t \in \mathcal{I}_{<c}$ are such that (G, H_i) is prime for all i, and $t > d + c^2 \cdot |\mathcal{I}_c|$. Among these forms, there are at most d forms H_i such that $G|F$ and $H_i|F$ for some $F \in \mathcal{F}$. For any other H_i, the product SG-condition implies $|\mathcal{I} \cap (G, H_i)| \geqslant 3$, and since $G \in \mathcal{I}_c$ and $H_i \in \mathcal{I}_{<c}$ we have $|\mathcal{I}_c \cap (G, H_i)| \geqslant 3$.

If we assume $t > d + c^2 \cdot |\mathcal{I}_c|$, by the pigeonhole principle, we have w.l.o.g. that there is some $G' \in \mathcal{I}_c \setminus \{G\}$ such that $G' \in (G, H_i)$ for $1 \leqslant i \leqslant c^2 + 1$. We have $\operatorname{codim}((G, G')) = 2$, and $\deg(G, G') \leqslant c^2$. Since $(G, G') \subset (G, H_i)$ for each $1 \leqslant i \leqslant c^2 + 1$, each such (G, H_i) is a minimal prime of (G, G'). By the associativity formula, (G, G') has at most c^2 distinct minimal primes, therefore without loss of generality we have $(G, H_1) = (G, H_2)$. Since $\deg H_1, \deg H_2 < c$, this implies that $H_1 \in (H_2)$, which contradicts the definition of $\mathcal{I}$. This shows that $t \leqslant d + c^2 \cdot |\mathcal{I}_c|$. □

The next lemma states that if we can find a vector space W such that $\mathbb{K}[W]$ contains a large fraction of a factor set, then a general projection simplifies the product Sylvester-Gallai configuration.

Lemma 7.2. *Let U be a graded finitely generated vector space such that $R := S/(U)$ is a CM UFD. Let $\mathcal{F}$ be a (d, c, z, R)-product Sylvester-Gallai configuration with factor set $\mathcal{I}$ where $|\mathcal{I}_c| \geqslant 12(c+1)\mathcal{C}(c) + d$. Suppose $W \subset S_{\leqslant d}$ is a $h_{2B} \circ t_1$-lifted strong vector space with respect to U with $z \in W$, such that W satisfies one of the following conditions:*

$$|\mathcal{I}_c \cap \mathbb{K}[W]| \geqslant 3\,|\mathcal{I}_c|/4 > 0 \quad \text{or} \quad |\mathcal{I} \cap \mathbb{K}[W]| \geqslant 2(c+1)\,|\mathcal{I}_c|$$

If $G \in \mathcal{I}_c$ is absolutely irreducible with respect to W, then $G \in (W)$.

Proof. Let $\{z, H_1, \ldots, H_t\} = \mathcal{I} \cap \mathbb{K}[W]$. Let $G \in \mathcal{I}_c$ be absolutely irreducible with respect to W such that $G \notin (W)$. We will show that the existence of such a G results in a contradiction.

Let V be the B-lifted strong vector space obtained from W and G by applying Corollary 5.11. By Corollary 5.14 there are at most $\mathcal{C}(c)$ many H_i such that (G, H_i) is not prime. There are at most d forms H_i such that G, H_i divide the same form in $\mathcal{F}$. By Lemma 3.5 there are at least $s := t - \mathcal{C}(c) - d$ forms H_i such that (G, H_i) is prime. W.l.o.g. we assume that these are $H_1, \ldots, H_s$. We now consider the two cases, and derive a contradiction from each of them.

Suppose $|\mathcal{I}_c \cap \mathbb{K}[W]| \geqslant 3|\mathcal{I}_c|/4$. By the assumption on the size of $|\mathcal{I}_c|$ we have $|\mathcal{I}_c \cap \mathbb{K}[W]| \geqslant 2|\mathcal{I}_c|/3 + \mathcal{C}(c) + d$. We can assume $H_1, \dots, H_r \in \mathcal{I}_c$ for $r = 2|\mathcal{I}_c|/3$ and $r \leqslant s$. For each $1 \leqslant i \leqslant r$ we have (G, H_i) is prime, therefore by Lemma 3.5 we have $G_i \in (G, H_i) \cap \mathcal{I}_c$. If $G_i \in (W)$ for any i, then $G \in (W)$, which is contrary to assumption. Therefore $G_i \notin (W)$ for all i, and in particular we have $G_i \neq H_j$ for any i, j. Since $r = 2|\mathcal{I}_c|/3$, by the pigeonhole principle we must have $G_j = G_{j'}$ for some j, j', which in turn implies $G \in (H_j, H_{j'}) \subset (W)$, contradicting our assumption.

Suppose now that the second condition holds, that is $|\mathcal{I} \cap \mathbb{K}[W]| \geqslant 2(c+1)|\mathcal{I}_c|$. By the assumption on the size of $\mathcal{I}_c$ we have $|\mathcal{I} \cap \mathbb{K}[W]| \geqslant (c+1)|\mathcal{I}_c| + \mathcal{C}(c) + d$. Let $r = (c+1)|\mathcal{I}_c|$ so $r \leqslant s$. Since (G, H_i) is prime, by Lemma 3.5 we have $G_i \in (G, H_i)$ for each $1 \leqslant i \leqslant r$. Further, we must have $G_i \in \mathcal{I}_c$, as $(G, H_i) \cap R_{<c} = (H_i) \cap R_{<c}$. By the pigeonhole principle, we can assume w.l.o.g. that $G_1 = \dots = G_{c+1}$. Therefore for each H_j with $1 \leqslant j \leqslant c+1$ we have $G_1 = \alpha_j G + R_j H_j$. If $\alpha_j \neq \alpha_{j'}$ for some $1 \leqslant j, j' \leqslant c+1$ then $G \in (H_j, H_{j'}) \subset (W)$, contradicting assumption. Therefore we can assume that $\alpha_j = \alpha_1$ for all j, and $G_1 - \alpha_1 G = R_j H_j$. Since the forms H_i are non associate, we have $c+1$ distinct factors of $G - G_1$, which contradicts the fact that this form has degree c. □

Finally, the next lemma gives us a way of constructing vector spaces W for which $\mathbb{K}[W]$ intersects a large fraction of the factor set.

Lemma 7.3. *Let $\mu \in (0,1)$, $c \leqslant d \in \mathbb{N}$, $r := 2 \cdot \mathcal{C}(c) + d$ and $k \geqslant 20 \cdot dr/\mu$. Let U be a finitely generated graded vector space with dimension sequence δ_U such that $R := S/(U)$ is a CM UFD. Moreover, suppose $\tilde{B} \geqslant h_{2B} \circ t_1$ and U is $h_{2\tilde{B}} \circ t_k$ strong in S. Let $\mathcal{F}$ be a (d, c, z, R)-product Sylvester-Gallai configuration, with factor set $\mathcal{I}$ satisfying $|\mathcal{I}_c| \geqslant r$. Then there exists a $\tilde{B}$-lifted strong vector space W such that*

1. $\dim W \leqslant C_{2\tilde{B},i}(t_k(\delta_U)) - \delta_U$.
2. $z \in W$.
3. $|\mathcal{I}_c \cap \mathbb{K}[W]| \geqslant 2 \cdot \mathcal{C}(c) + d$.
4. *There are at most $\mu|\mathcal{I}_c|$ forms $G \in \mathcal{I}_c$ s.t. $G \notin (W)$ and G is absolutely irreducible relative to W.*
5. *If further $|\mathcal{I}_c| < \varepsilon|\mathcal{I}|$ for some $0 < \varepsilon < (4 \cdot r \cdot c^2)^{-1}$ then there are at most $|\mathcal{I}|/2$ forms $G' \in \mathcal{I}_{<c}$ such that $G' \notin (W)$ and G' is absolutely irreducible with respect to W.*

Proof. Set $\nu := \mu/2r$. For any $H \in \mathcal{I}_c$, define

$$\mathcal{F}_{\text{span}}(H) := \{G \mid G \in \mathcal{I}_c, |(G, H) \cap \mathcal{I}_c| \geqslant 3\}.$$

If $\mathcal{I}_c$ is a (r, ν)-linear SG configuration, then $\mathcal{I}_c$ has a basis of size at most $r + 1 + 8/\nu < k$ by Proposition 3.2. Let W be the vector space obtained by applying Corollary 5.11 to (0), and the set consisting of z and a basis for $\mathcal{I}_c$. This W has the first four required properties.

Suppose $\mathcal{I}_c$ is not a (r, ν)-linear SG configuration. Let $H_1, \dots, H_{r+1}$ be a set of witnesses to this fact, that is, each H_i is such that $|\mathcal{F}_{\text{span}}(H_i)| < 2\nu|\mathcal{I}_c|$. For each $i \in [r+1]$, let $\{H_{ij}\}_j$ be the set of forms such that $H_i | F$ and $H_{ij} | F$ for some $F \in \mathcal{F}$. For each i, there are at most d such forms H_{ij}. Let W be the vector space obtained by applying Corollary 5.11 to (0), and the set consisting of z, H_{ij}, H_i for all $i \in [r+1], j \in [d]$. This set has size at most k, therefore W is $\tilde{B}$-lifted strong. So far we have shown that W satisfies the first three properties.

Let $\mathcal{B} := \mathcal{I}_c \setminus \bigcup_{i \leqslant r+1} \mathcal{F}_{\text{span}}(H_i)$. We have $|\mathcal{B}| \geqslant (1 - r \cdot 2\nu)|\mathcal{I}_c| = (1-\mu)|\mathcal{I}_c|$. Let $G \in \mathcal{B}$ be such that G is absolutely irreducible with respect to W, we will show that $G \in (W)$. If $G = H_{ij}$ for some i, j then $G \in \mathbb{K}[W]$ by construction, therefore we can assume that this does not hold. If (G, H_i) is prime for some i, then $|\mathcal{I} \cap (G, H_i)| \geqslant 3$. Since H_i, G both have degree c, it must be that

$|\mathcal{I}_c \cap (G, H_i)| \geqslant 3$, contradicting the definition of $\mathcal{B}$. Therefore we deduce that (G, H_i) is not prime for every $1 \leqslant i \leqslant r$. Let V be the B-lifted strong vector space obtained by applying Corollary 5.11 to (W), and G. If $G \notin (W)$ then there at most $\mathcal{C}(c)$ forms $H \in \mathbb{K}[W]$ such that (G, H) is not prime. Since $r > \mathcal{C}(c)$, and since (G, H_i) is not prime for all i, we reach a contradiction to Corollary 5.14. Therefore $G \in (W)$. This shows that W has the first four properties.

We now show the fifth property for both constructions of W above. We have $|\mathcal{I}_c \cap \mathbb{K}[W]| \geqslant r$. Let $H_1, \ldots, H_r \in \mathcal{I}_c \cap \mathbb{K}[W]$. Let $\mathcal{B} \subset \mathcal{I}_{<c}$ be the set of forms G such that (G, H_i) is prime for some $1 \leqslant i \leqslant r$. By Lemma 7.1 we have $|\mathcal{B}| \leqslant r \cdot \left(c^2 \cdot |\mathcal{I}_c| + d\right) \leqslant 2 \cdot r \cdot c^2 |\mathcal{I}_c| \leqslant 2 \cdot r \cdot c^2 \cdot \varepsilon \cdot |\mathcal{I}|$. Suppose $G' \notin \mathcal{B}$ is such that G' is absolutely irreducible with respect to W. Let V be the B-lifted strong vector space obtained by applying Corollary 5.11 to (W), and G'. If $G' \notin (W)$ then there at most $\mathcal{C}(c)$ forms $H \in \mathbb{K}[W]$ such that (G', H) is not prime. Since $r > \mathcal{C}(c)$, and since (G', H_i) is not prime for all $1 \leqslant i \leqslant r$, we reach a contradiction. Therefore $G' \in (W)$. The desired bound now holds by the assumption on ε. $\square$

8 Product Sylvester-Gallai Theorem

We are now ready to prove our main result, Theorem 1.5.

At a high level, as we mentioned in the introduction, the proof proceeds in 3 steps. In the first step, we repeatedly apply Lemma 7.3 to construct a strong algebra $\mathbb{K}[V]$ which intersects a large fraction of the factor set, and apply the general projection to our configuration. After constantly many such steps, the space we obtain will satisfy the conditions of Lemma 7.2, and the general projection will result in a configuration with factor sets of smaller degree.

Parameter setting: let $d \in \mathbb{N}$ be a fixed constant. We define a number of functions and constants based on d. Let $B : \mathbb{N}^d \to \mathbb{N}^d$ be the ascending function given by $B_i(\delta) := A(\eta, i) + 3(\sum_i \delta_i - 1)$ for all $i \in [d]$, where $A(\eta, i)$ is the function provided by [AH20b, Theorem A].

Let $c \in \mathbb{N}$ be such that $1 \leqslant c \leqslant d$. Define the following parameters:

$$r_c := 12(c+1)(\mathcal{C}(c) + d), \quad \varepsilon_c := \left(12 \cdot r_c \cdot c^2\right)^{-1}, \quad \mu_c := \varepsilon_c/3, \quad k_c := (40 \cdot r_c \cdot d)/\mu_c, \quad g_c = 9c^2.$$

We inductively define functions $\Lambda_{i,j}$ for each $1 \leqslant i \leqslant d$ and $0 \leqslant j \leqslant g_i$. Let $\Lambda_{1,0} := 2B \circ t_2$. Given $\Lambda_{i,j}$ with $j < g_i$, define $\Lambda_{i,j+1} = h_{2\Lambda_{i,j}} \circ t_{k_i}$. Given Λ_{i,g_i}, define $\Lambda_{i+1,0} = h_{2\Lambda_{i,g_i}} \circ t_{k_{i+1}}$.

For each $1 \leqslant i \leqslant d$ and $0 \leqslant j \leqslant g_i$, let $\Gamma_{i,j} : \mathbb{N} \to \mathbb{N}$, be the following function:

$$\Gamma_{i,j}(n) := \max_{\delta \in \mathbb{N}^d, \|\delta\|_1 \leqslant n} \sum_{\ell=1}^{d} C_{2\Lambda_{i,j},\ell}(t_{k_i}(\delta)),$$

where $\|\delta\|_1$ is the ℓ_1-norm of the integer vector δ.

Define $\eta_c : \mathbb{N} \to \mathbb{N}$ as

$$\eta_c(n) = (d+3)^{4g_c^2 + 4n + \sum_{i=0}^{g_c} 4\Gamma_{c,i}(n)}.$$

Finally, define $\xi_c : \mathbb{N} \to \mathbb{N}$ as $\xi_1(n) = (51d)^d$, and $\xi_c(n) = \eta_c(n) \cdot \xi_{c-1}(\Gamma_{c,0}(n))$.

Lemma 8.1. *Let $U \subset S_{\leqslant d}$ be a Λ_{c,g_c}-strong vector space and $R := S/(U)$ (thus R is a CM UFD). Let $\mathcal{F}$ be a (d, c, z, R)-product Sylvester-Gallai configuration.*

For each $0 \leqslant i < g_c$, there exist vector spaces $V^{(i)} \subset S[z_1, \ldots, z_i]$ and graded quotient maps

$$\varphi^{(i)}_{\alpha_i} : (S[z_1, \ldots, z_i]/V^{(i)})[z_{i+1}] \to S[z_1, \ldots, z_{i+1}]/V^{(i+1)}$$

such that the following hold.

1. *Each* $V^{(i)}$ *is* Λ_{c,g_c-i}*-strong.*
2. $\dim V^{(i)} \leqslant \Gamma_{c,g_c-i}(\dim U)$*.*
3. $\varphi^{(g_c)}_{\alpha_{g_c}} \circ \cdots \circ \varphi^{(0)}_{\alpha_0}(\mathcal{F})$ *is a* $(d, c-1, z_{g_c}, S[z_1, \dots, z_{g_c}]/V^{(g_c)})$*-product Sylvester-Gallai configuration.*
4. $\dim \operatorname{span}_{\mathbb{K}}\{\mathcal{F}\} \leqslant \eta_c(\dim U) \cdot \dim \operatorname{span}_{\mathbb{K}}\left\{\varphi^{(g_c)}_{\alpha_{g_c}} \circ \cdots \circ \varphi^{(0)}_{\alpha_0}(\mathcal{F})\right\}$*.*

Proof. We construct $V^{(i)}$ via the following iterative process.

Set $R^{(0)} \leftarrow R$, $z_0 = z$, $V^{(0)} \leftarrow U$, $\mathcal{F}^{(0)} \leftarrow \mathcal{F}$, and let $\mathcal{I}^{(0)}$ be the factor set of $\mathcal{F}^{(0)}$
for $0 \leqslant i \leqslant g_c - 1$ **do**
  if $\left|\mathcal{I}^{(i)}_c\right| \geqslant r_c$ **then**
    Apply Lemma 7.3 to $\mathcal{F}^{(i)}$ with $\mu = \mu_c$, and $k = 20 \cdot d \cdot r_c/\mu_c$ to obtain $W^{(i)}$
  else
    Let $W^{(i)}$ be the vector space obtained by applying Corollary 5.11 to $\mathcal{I}_c$
  end if
  Let z_{i+1} be a new variable, and let $\varphi^{(i)}_{\alpha_i} : R^{(i)} \to R^{(i)}[z]/W^{(i)}_{\alpha_i}$ be a general quotient, where $\alpha_i \in \mathbb{K}^{\dim W^{(i)}}$ is chosen to be general with respect to all the conditions in Section 6.
  Set $R^{(i+1)} \leftarrow R^{(i)}[z_{i+1}]/W^{(i)}_{\alpha_i}$, $V^{(i+1)} \leftarrow \ker\left(S[z_1, \dots, z_{i+1}] \to R^{(i+1)}\right)$.
  Set $\mathcal{F}^{(i+1)} \leftarrow \varphi^{(i)}_{\alpha_i}(\mathcal{F}^{(i)})$ according to Definition 6.8, and $\mathcal{I}^{(i+1)}$ to be the factor set of $\mathcal{F}^{(i+1)}$.
end for

For $i \in [g_c]$, let $m_i := \left|\mathcal{I}^{(i)}_c\right|$ and $M_i := \left|\mathcal{I}^{(i)}\right|$. We establish the following facts by induction on i.

Fact 1: The space $V^{(i)}$ is Λ_{c,g_c-i}-strong for every i, and satisfies $\dim V^{(i)} \leqslant \Gamma_{c,g_c-i}(\dim U)$.

Proof of fact 1. The base case holds by assumption for $V^{(0)} = U$. Now, if $V^{(i)}$ is Λ_{c,g_c-i}-strong, then $W^{(i)}$ is Λ_{c,g_c-i-1}-lifted strong in both cases of the construction: by Lemma 7.3 in the if branch, and by Corollary 5.11 in the else branch. Therefore $V^{(i+1)}$ is also Λ_{c,g_c-i-1}-lifted strong. Further, by induction and Lemma 7.3, Corollary 5.11 the vector space $W^{(i)}$ has dimension sequence bounded by $C_{2\Lambda_{c,g_c-i-1}}(t_k(\delta_{V^{(i-1)}})) - \delta_{V^{(i-1)}}$. Since $V^{(i)} = V^{(i-1)} + W^{(i)}$, we deduce that $V^{(i)}$ has dimension sequence bounded by $C_{2\Lambda_{c,g_c-i-1}}(t_k(\delta_{V^{(i-1)}}))$. By definition of Γ_{c,g_c-i}, we have $\dim V^{(i)} \leqslant \Gamma_{c,g_c-i}(\dim U)$. □

Fact 2: Suppose $m_i \neq 0$ for every $1 \leqslant i \leqslant g_c$. Then $m_i \leqslant \mu_c^i m_0$ and

$$M_i \geqslant M_1 - 2\mu_c(c+1)m_0/(1-\mu_c) \geqslant (1/4 - \mu_c)m_0 - 2\mu_c(c+1)m_0/(1-\mu_c).$$

Proof of fact 2. Consider the space $W^{(i-1)}$ for $i \geqslant 1$. If $W^{(i-1)}$ is constructed by following the else branch above, then $\mathcal{I}^{(i)}_c \subset \mathbb{C}\left[W^{(i-1)}\right]$. If $W^{(i-1)}$ satisfies either one of the conditions in Lemma 7.2, then every form in $\mathcal{I}^{(i)}_c$ is either in the ideal $\left(W^{(i-1)}\right)$, or absolutely reducible with respect to $W^{(i-1)}$, and hence the images of every such under the map $\varphi^{(i-1)}_{\alpha_{i-1}}$ has degree less than c by item 4 of Proposition 6.5. In all of these cases, we get $m_i = 0$. Therefore, we can assume that $W^{(i-1)}$ is constructed by applying Lemma 7.3.

Every form in $\mathcal{I}^{(i)}_c$ that is either in the ideal $\left(W^{(i-1)}\right)$, or absolutely reducible with respect to $W^{(i-1)}$ is mapped to a form of degree less than c by the map $\varphi^{(i-1)}_{\alpha_{i-1}}$. By Lemma 7.3 there are at most $\mu_c \cdot m_{i-1}$ forms that do not satisfy either of these properties, therefore we get that $m_i \leqslant \mu_c m_{i-1} \leqslant \mu_c^i m_0$ by induction. We further have $\left|\mathcal{I}^{(i-1)} \cap \mathbb{C}\left[W^{(i-1)}\right]\right| < 2(c+1)m_{i-1}$. By item 2 of Proposition 6.5, every element in $\mathcal{I}^{(i-1)} \setminus \mathbb{C}\left[W^{(i-1)}\right]$ is mapped to a unique element in $\mathcal{F}^{(i)}$ by the map $\varphi_{\alpha_{i-1}}^{(i-1)}$, therefore we have $M_i \geqslant M_{i-1} - 2(c+1)m_{i-1}$.

By Lemma 7.3, we have $\left|\mathcal{I}_c^{(0)} \cap \mathbb{C}\left[W^{(0)}\right]\right| < 3m_0/4$. Further, there are at most $\mu_c m_0$ forms in $\mathcal{I}_c^{(0)} \setminus \left(W^{(0)}\right)$ which are absolutely irreducible with respect to $W^{(0)}$. Thus, at least $(1/4-\mu_c)m_0$ forms are either absolutely reducible with respect to $\left(W^{(0)}\right)$, or in $\left(W^{(0)}\right) \setminus \mathbb{C}\left[W^{(0)}\right]$. By item 2 of Proposition 6.5, each of these forms is mapped to a unique element in $\mathcal{F}^{(1)}$ under $\varphi_{\alpha_0}^{(0)}$, therefore $M_1 \geqslant (1/4-\mu_c)m_0$. Combined with the above derived equation $M_i \geqslant M_{i-1} - 2(c+1)m_{i-1}$, and the fact that $m_i \leqslant \mu_c^i m_0$ we get the claimed bound on M_i. □

Note that Fact 1 establishes items 1 and 2 of our lemma.

Establishing item 3: Note that, if $m_i = 0$ for some $1 \leqslant i \leqslant g_c$, then $V^{(i)}, \varphi_{\alpha_i}^{(i)}$ satisfies item 3. Thus, all we need to show is that the iterative process above will reach $m_i = 0$ for some $i \leqslant g_c$.

Assume, for the sake of contradiction, that $m_i \neq 0$ for $1 \leqslant i \leqslant g_c$. Facts 1 and 2, combined with $\mu_c \leqslant (12(c+1))^{-1}$, and the fact that $m_i \neq 0$ for $1 \leqslant i \leqslant g_c$ imply the following inequalities:

- $m_i/M_i \leqslant \varepsilon_c$ for every $i \geqslant 2$.
- $M_i \geqslant M_1/2$ for every $1 \leqslant i \leqslant g_c$.

Now, define potentials Φ_i for each $1 \leqslant i \leqslant g_c$ by

$$\Phi_i := \sum_{G \in \mathcal{I}^{(i)}} (\deg G)^2 .$$

We have $\Phi_1 \leqslant c^2 M_1$ by definition. Consider a form $G \in \mathcal{I}^{(i)}$ such that $\deg G = e$. This form contributes e^2 to Φ_i. We now consider the contribution of the image of $\varphi_{\alpha_i}^{(i)}(G)$ in $\mathcal{F}^{(i+1)}$. There are 3 cases to analyze:

Case 1: $G \in \left(W^{(i)}\right)$ and G is absolutely irreducible with respect to $W^{(i)}$.

In this case, $\varphi_{\alpha_i}^{(i)}(G) = z^\beta \cdot G'$, for some $\beta \in \mathbb{N}^*$ and G' irreducible. Therefore, $\varphi_{\alpha_i}^{(i)}(G)$ only contributes G' to the factor set $\mathcal{I}^{(i+1)}$. Since $\deg(G') \leqslant e-1$, its contribution to Φ_{i+1} is at most $(e-1)^2$, and we have $(e-1)^2 < e^2$. So the potential decreases in this case.

Case 2: G is absolutely reducible with respect to $W^{(i)}$.

In this case, $\varphi_{\alpha_i}^{(i)}(G) = z_{i+1}^{e'} \prod G_j$, with $\sum \deg G_j \leqslant e$. Therefore t factors G_i of G contribute $\sum \left(\deg G_j\right)^2$ to Φ_{i+1}. Since $\sum \left(\deg G_j\right) < e$ we have $\sum \left(\deg G_j\right)^2 < e^2$. So the potential also decreases in this case.

Case 3: $G \notin (W^{(i)})$ and it remains absolutely irreducible with respect to $W^{(i)}$.

In this case, $\varphi_{\alpha_i}^{(i)}(G) \in \mathcal{I}^{(i+1)}$ and $\deg(\varphi_{\alpha_i}^{(i)}(G)) = e$, so the contribution of G to the potential doesn't change.

By Lemma 7.3, and the fact that $m_i/M_i \leqslant \varepsilon_c$ for $i \geqslant 2$ proved above, at least $(M_i - m_i)/2$ forms in $\mathcal{I}^{(i)}$ fall under cases 1 and 2. The bounds on M_i, m_i imply that $(M_i - m_i)/2 \geqslant M_i/4$. Therefore, combined with the above case analysis, we have $\Phi_{i+1} \leqslant \Phi_i - M_i/4 \leqslant \Phi_i - M_1/8$. After $8c^2$ steps, we have $\Phi_i = 0$, which implies that the factor set is empty. Therefore, in iteration $8c^2 + 1 < 9c^2$, we must have $m_i = 0$, which contradicts our assumption.

Proving item 4 (dimension bounds): Since $\dim V^{(i)} \leqslant \Gamma_{c,g_c-i}(\dim U)$, and since $V^{(i)}$ is generated in degree at most d, we can bound the product of the degrees of a basis of $V^{(i)}$ by $e_i := (d+3)^{\Gamma_{c,g(c)-i}(\dim U)}$. Let $e_0 := (d+3)^{\dim U}$ be a bound on the product of the degrees of a basis of U. By Proposition 6.7 we have

$$\operatorname{span}_{\mathbb{K}}\left\{\varphi^{(g_i)}_{\alpha_{g_i}} \circ \cdots \circ \varphi^{(0)}_{\alpha_0}(\mathcal{F})\right\} \leqslant (d+3)^{4+\dim V^{(i-1)}} e_i e_{i-1} \operatorname{span}_{\mathbb{K}}\left\{\varphi^{(g_{i-1})}_{\alpha_{g_{i-1}}} \circ \cdots \circ \varphi^{(0)}_{\alpha_0}(\mathcal{F})\right\}.$$

By induction, we can deduce that that

$$\dim \operatorname{span}_{\mathbb{K}}\{\mathcal{F}\} \leqslant (d+3)^{4g_c^2+4(\dim U)+\sum_{i=0}^{g_c} 4\Gamma_{c,i}(\dim U)} \cdot \dim \operatorname{span}_{\mathbb{K}}\left\{\varphi^{(g_c)}_{\alpha_{g_c}} \circ \cdots \circ \varphi^{(0)}_{\alpha_0}(\mathcal{F})\right\}.$$

Substituting the definition of $\eta_c(n)$ proves item 4. □

We can now prove our main theorem, with Lemma 8.1 acting as our inductive step.

Theorem 8.2. *Let $U \subset S_{\leqslant d}$ be a Λ_{d,g_d}-strong vector space and $R = S/U$ (hence R is a CM UFD). If $\mathcal{F}$ is a (d, c, z, R)-product Sylvester-Gallai configuration then $\dim \operatorname{span}_{\mathbb{C}}\{\mathcal{F}\} \leqslant \xi_c(\dim U)$. In particular, if $\mathcal{F}$ is a $(d, c, z, S[z])$-product Sylvester-Gallai configuration then $\dim \operatorname{span}_{\mathbb{C}}\{\mathcal{F}\} \leqslant \xi_c(0)$.*

Proof. The second statement follows from the first since the zero vector space is Λ_{d,g_d}-strong. We prove the theorem by induction on c. Let $\mathcal{I}$ be the factor set of $\mathcal{F}$.

Base case: $c = 1$. That is, when every $G \in \mathcal{I}$ is a linear form.

By Lemma 3.5, for every $G_1, G_2 \in \mathcal{I} \setminus \{z\}$, either $G_1|F$ and $G_2|F$ for some form $F \in \mathcal{F}$, or $|(G_1, G_2) \cap \mathcal{I}| \geqslant 3$. Consequently, as long as $|\mathcal{I}| \geqslant 3d$, the set $\mathcal{I}$ is a $(1, 1/2)$-linear SG configuration, and $\dim \operatorname{span}_{\mathbb{C}}\{\mathcal{I}\} \leqslant 51 \leqslant 51d$. If $|\mathcal{I}| \leqslant 3d$ then we trivially have $|\mathcal{I}| \leqslant 51d$, and Proposition 3.6 implies $\dim \operatorname{span}_{\mathbb{C}}\{\mathcal{F}\} \leqslant (51d)^d = \xi_1(\dim U)$.

Inductive step: suppose now that $c > 1$.

In this case, U satisfies the hypothesis of Lemma 8.1. Thus, there is a vector space V and a sequence of graded quotient maps $\varphi^{(i)}_{\alpha_i}$ such that $\varphi(\mathcal{F})$ is a $(d, c-1, z_{g_c}, S[z_1, \ldots, z_{g_c}]/V)$-product Sylvester-Gallai configuration, where $\varphi(\mathcal{F})$ is the composition of the graded quotients $\varphi^{(i)}_{\alpha_i}$. The lemma also guarantees that $\dim V \leqslant \Gamma_{c,0}(\dim U)$, and that V is $\Lambda_{c,0}$-strong, and hence $\Lambda_{c-1,g_{c-1}}$-strong. Therefore we can apply the inductive hypothesis, and we have $\dim \operatorname{span}_{\mathbb{C}}\{\varphi(\mathcal{F})\} \leqslant \xi_{c-1}(\dim V) \leqslant \xi_{c-1}(\Gamma_{c,0}(\dim U))$. By item 4 of Lemma 8.1 we have

$$\dim \operatorname{span}_{\mathbb{C}}\{\mathcal{F}\} \leqslant \eta_c(\dim U) \cdot \xi_{c-1}(\Gamma_{c,0}(\dim U)) \leqslant \xi_c(\dim U).$$ □

9 Conclusion

In this work, we prove that product Sylvester-Gallai configurations of forms have bounded dimension, generalising the result of [OS24], and getting one step closer towards a PIT algorithm for $\Sigma^k\Pi\Sigma\Pi^d$ circuits.

To achieve this, we give a novel, and very effective, sufficient condition for the primality of ideals of the form (P, Q), where P "depends on more variables" than Q, a notion that we make explicit in Section 4 and Section 5, via the transfer principle from [OS24]. This turns out to be far more subtle, and to require more tools from algebraic geometry, than giving a sufficient condition for the ideal to be radical.

Our work leaves two important questions, in order to bridge the gap between our results and a complete analysis for a polynomial-time, deterministic PIT algorithm for $\Sigma^k\Pi\Sigma\Pi^d$ circuits. The first question is to extend this work and prove a higher codimension product Sylvester-Gallai theorem for all degrees, simultaneously generalising the main theorems of [GOPS23] and this work. The second question is to provide a tight connection between such Sylvester-Gallai problems and rank bounds for identities in $\Sigma^k\Pi\Sigma\Pi^d$, thereby generalizing the beautiful result in [SS13, Theorem 1.4] to circuits of depth four.

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ECCC ISSN 1433-8092
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