Communication complexity of pointer chasing via the fixed-set lemma

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The input to the k-pointer-chasing function are two arrays $a, b \in [n]^n$ of pointers and the goal is to output (say) the first bit of the pointer reached after following k pointers, starting at $a[0]$. For example, the output of 3-pointer chasing is the first bit of $b[a[b[a[0]]]]$. The communication complexity of this fundamental problem and its variants has a long, rich, and developing history, starting with [PS84]. For background see [KN97, RY19, Vio23]. Here I consider deterministic protocols with k rounds and 2 players: Alice, who sees b but not a, and Bob who sees a but not b. In a 0-round protocol Alice outputs the answer with no communication. In a 1-round, Alice sends a message then Bob outputs the answer, and so on. Alice always goes first.

[NW93] prove a $c(n - k \log n)$ lower bound on the communication complexity. They then write that getting rid of the $-k \log n$ term "requires a more delicate argument" which they sketch. The textbook [RY19] states an $n/16 - k$ lower bound and omits the proof because it is "too technical."

I give a very simple, apparently new proof of an $n/8$ lower bound using the *fixed-set Lemma* 3.14 from [GSV18]. This follows from the next theorem for $A = B = [n]^n$ and $F_A = F_B = \emptyset$. The bound holds for any k. In particular, for $k = n/8$ the bound holds regardless of the number of rounds. I write $[n]$ for $\{0, 1, \ldots, n-1\}$ and for a set A I write A for its size as well, following notation in [Vio23].

Theorem 1. *There is no k-round protocol with communication s and sets $A, B \subseteq [n]^n$ and $F_A, F_B \subseteq [n]$ such that:*

(0) The protocol computes k-pointer-chasing on every input in $A \times B$,

(1) The F_A coordinates in A are fixed, and the same for B,

(2) The unfixed density of A, defined as A/n^{n-F_A}, is $\geq 2^{2s-n/4+F_A}$, and the same for B,

(3) $A[0]$ is alive, defined as $\mathbb{P}_{a \in A}[a[0] = v] < 2/n$ for every $v \in [n]$.

Proof. Proceed by induction on k, a.k.a. round elimination. For the base case $k = 0$, we can fix B and hence Alice's output. But since $A[0]$ is alive, the probability that this is correct is $< (2/n) \cdot n/2 < 1$.

The induction step is by contrapositive. Assuming there is such a protocol and there are such sets, we construct a $(k-1)$-round protocol and sets violating the inductive assumption. Suppose Alice sends t bits as her first message. Fix the most likely message, and let $B_0 \subseteq B$ be the set of $\geq 2^{-t}B$ corresponding strings. If there is a pointer $i \in [n] - F_B$ which is not alive in B_0, fix it to its most likely value, call $B_1 \subseteq B_0$ the corresponding subset, and let $F_{B_1} := F_B \cup \{i\}$. Note that the unfixed density increases by a factor 2 since

$$\frac{B_1}{n^{n-F_{B_1}}} \geq \frac{2}{n}\frac{B_0}{n^{n-F_B-1}} = 2\frac{B_0}{n^{n-F_B}}.$$

Continue fixing until every unfixed pointer is alive, and call B', $F_{B'}$ the resulting sets. The unfixed density of B' is then

$$\frac{B'}{n^{n-F_{B'}}} \geq \frac{2^{-t}B}{n^{n-F_B}}2^{F_{B'}-F_B} \geq 2^{2s-n/4+F_B-t+F_{B'}-F_B} = 2^{2(s-t)-n/4+F_{B'}}.$$

Now note that $F_{B'} \leq n/4$ because the density cannot be larger than 1. We use this to analyze Alice's side. Because $A[0]$ is alive, $\mathbb{P}_{a\in A}[a[0] \in F_{B'}] \leq 2F_{B'}/n \leq 1/2$. So there is an alive pointer $B'[i]$ such that $\mathbb{P}[a[0] = i] \geq 1/2n$. Let $A' \subseteq A$ be the corresponding subset, and let $F_{A'} := F_A \cup \{0\}$. The unfixed density of A' is

$$\frac{A'}{n^{n-F_{A'}}} \geq \frac{A/2n}{n^{n-F_A-1}} = \frac{A/2}{n^{n-F_A}} \geq 2^{2s-n/4+F_A-1} = 2^{2s-n/4+F_{A'}-2} \geq 2^{2(s-t)-n/4+F_{A'}}$$

since $t \geq 1$.

This gives a $(k-1)$-round protocol where Bob goes first that computes $(k-1)$-pointer-chasing where the first pointer is $B'[i]$. We can swap players to let Alice go first and permute coordinates to let $A'[0]$ be the first pointer. □

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References

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