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Sign-Rank of k-Hamming Distance is Constant

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Abstract

We prove that the sign-rank of the k-Hamming Distance matrix on n bits is $2^{O(k)}$, independent of the number of bits n. This strongly refutes the conjecture of Hatami, Hatami, Pires, Tao, and Zhao (RANDOM 2022), and Hatami, Hosseini, and Meng (STOC 2023), repeated in several other papers, that the sign-rank should depend on n. This conjecture would have qualitatively separated *margin* from *sign-rank* (or, equivalently, bounded-error from unbounded-error randomized communication). In fact, our technique gives constant sign-rank upper bounds for all matrices which reduce to k-Hamming Distance, as well as large-margin matrices recently shown to be *irreducible* to k-Hamming Distance.

Contents

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1 Introduction

A boolean matrix $M \in \{0,1\}^{N \times N}$ can always be represented as a point–halfspace arrangement. For example, in Figure 1, the identity and a lower triangular matrix are represented with rows assigned to points on the unit sphere and columns assigned to halfspaces with boundary through the origin, such that point x_i belongs to a halfspace h_j if and only if the entry i, j of the matrix is 1.

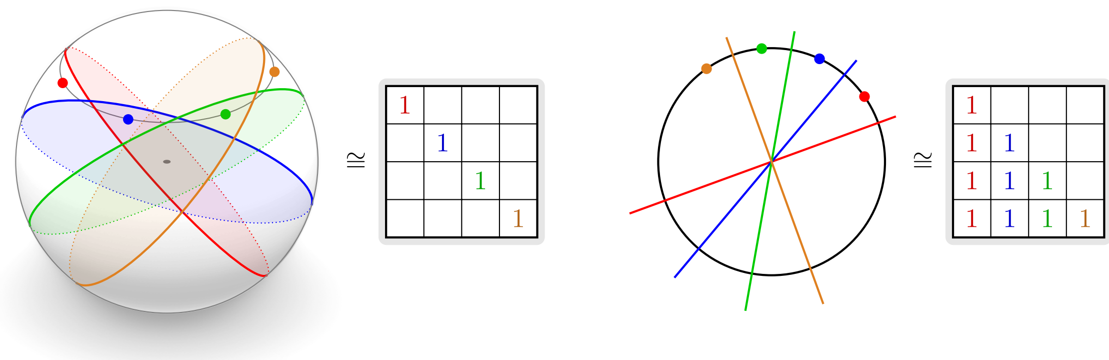


Figure 1: Sign-rank representations of the identity matrix and lower triangular matrix.

The smallest dimension d in which M can be represented in this way, with rows assigned to points and columns assigned to halfspaces through the origin, is called the *sign-rank* of M, denoted $\operatorname{rank}_\pm(M)$ (also called *dimension complexity* in learning theory). We may equivalently define sign-rank of a boolean matrix M as the smallest rank d of a matrix $A \in \mathbb{R}^{N \times N}$ which satisfies $M_\pm(i,j) = \operatorname{sign}(A(i,j))$ for all i, j, where $M_\pm(i,j) = 2M(i,j) - 1$ is a representation of M as a sign matrix $M_\pm \in \{\pm 1\}^{N \times N}$.

In this paper we prove that the k-Hamming Distance matrices have sign-rank at most $2^{O(k)}$. These are the matrices $\mathsf{HD}_k^n \colon \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ where $\mathsf{HD}_k^n(x,y) = 1$ if and only if the Hamming distance $\operatorname{dist}(x,y)$ between row index $x \in \{0,1\}^n$ and column index $y \in \{0,1\}^n$ is exactly k.

Theorem 1.1. *For all $n, k \in \mathbb{N}$, $\operatorname{rank}_\pm(\mathsf{HD}_k^n) = 2^{O(k)}$.*

For $k = 0$, HD_k^n is the identity matrix in Figure 1, but for $k \geq 1$ our theorem improves on the best known (and trivial) bound of $\operatorname{poly}(n)$, and refutes the conjecture that for some constant k the sign-rank must depend on n. This was a conjecture of Hatami, Hatami, Pires, Tao, and Zhao [HHP+22], Hatami, Hosseini, and Meng [HHM23], the basis for a question of Harms and Zamaraev [HZ24], and an open problem in [FHHH24, HH24, HR24, FGHH25]. The goal of this conjecture was to separate the class of matrices with *large margin*, i.e., those which can be represented as point–halfspace arrangements with a large (constant) margin between any hyperplane and any point; from those of *constant sign-rank*, i.e., the ones which can be represented as point–halfspace arrangements whose dimension is independent of matrix size. This is a question of Linial, Mendelson, Schechtman, and Shraibman [LMSS07] with consequences for communication complexity, learning theory, circuit complexity, distributed computing, privacy, and other areas of computer science [FKL+01, BES02, For02, LS09, FX14, BNS19, HWZ22, HHP+22, HZ24, AN25]. We explain this question in more detail in Section 1.1.

Generalization. Surprisingly, our simple technique applies not only to k-Hamming Distance but to all large-margin matrices obtained from it by *reductions* (i.e., boolean combinations), as well as large-margin matrices which are *irreducible* to k-Hamming Distance [FGHH25]. While new large-margin matrices can be created by reductions, it is a well-known open problem whether reductions preserve sign-rank [BMT21, HHP+22], so reductions to k-Hamming Distance are not handled *a priori* by Theorem 1.1. Furthermore, [FGHH25] proved that there exist large-margin matrices which are *irreducible* to k-Hamming Distance. These can be obtained from k-Hamming Distance by "distance-r compositions". [FGHH25] observed that all large-margin matrices known prior to their work could be obtained from k-Hamming Distance by compositions and reductions. We show that none of these examples can separate margin from sign-rank:

Theorem 1.2 (Generalization of Theorem 1.1, Informal)**.** *Any boolean matrix M that is obtained from k-*HAMMING DISTANCE *by reductions and "compositions" has* $\mathrm{rank}_\pm(M) = O(1)$.

Subsequent to [FGHH25] and concurrent with the present study, recent work of Sherstov and Storozhenko [SS24] introduces a new class of large-margin matrices, which are now the only remaining candidates that we are aware of for separating large margin from constant sign-rank.

The proofs of Theorems 1.1 and 1.2 involve new techniques (a focused study of *support-rank*) that we outline in Section 2. These techniques in turn lead us to define some useful new complexity classes, which we describe in Section 5 along with some open problems. For the remainder of this introduction, we discuss the implications of our main results.

1.1 Context and Consequences for Sign-Rank vs. Margin

Sign-rank and margin. Given a boolean matrix M, we can ask to minimize the *dimension* of its point–halfspace representation, which leads to the definition of sign-rank. But another way to optimize the point–halfspace representation is to ignore the dimension and maximize the *margin*, so that no point is too close to the boundary of any halfspace. For a boolean matrix $M \in \{0,1\}^{N\times N}$, we write

$$\mathrm{mar}(M) \coloneqq \sup_{u,v} \min_{i,j\in[N]} |\langle u_i, v_j\rangle|$$

where the supremum is over all assignments $u, v\colon [N] \to \mathbb{R}^d$ of the rows and columns to unit vectors in any dimension d such that the signed matrix is $M_\pm(i,j) = \mathrm{sign}(\langle u_i, v_j\rangle)$.

It is not well understood how these two types of representations relate to one another. Small sign-rank does not imply large margin: the triangular matrix in Figure 1 has sign-rank 2 but small (sub-constant) margin [BW16, Vio15, SY23], while matrices of sign-rank 3 can have margin (equivalently, discrepancy [LS09]) as small as $(\mathrm{poly}(N))^{-1}$ [HHL20, ACHS24]. A basic open question is the converse:

Open Problem 1.3 ([LMSS07, HHP$^+$22])**.** *Is there a function η such that any boolean matrix M satisfies $\mathrm{rank}_\pm(M) \le \eta(\mathrm{mar}(M)^{-1})$? That is, do matrices of large (constant) margin also have constant sign-rank?*

Contrary to conjectures in earlier work, our Theorem 1.2 shows that all large-margin matrices covered in [FGHH25] also have constant sign-rank. Sign-rank and margin are important in several areas of computer science, leading to several equivalent formulations of this question, for example:

- The margin determines the performance of the *perceptron* algorithm. Are there hypothesis classes of dimension (sign-rank) $\omega(1)$ that the perceptron algorithm can learn with only $O(1)$ mistakes?
- Via a relationship to randomized communication complexity, hypothesis classes $\mathcal{H}$ with constant margin are exactly those which are PAC learnable under pure differential privacy [FX14, BNS19]. Is any super-constant dimensional problem learnable under pure differential privacy?
- Is every hereditary graph class with constant size *adjacency sketches* (e.g., [FK09, Har20, HWZ22, EHK22, NP24, AN25]) a point–halfspace incidence graph in constant dimension?

We next focus on communication complexity, where, as we explain below, the equivalent question is:

Open Problem 1.4. *Is there any communication problem with* constant *bounded-error randomized cost, but* super-constant *unbounded-error cost?*

Sign-rank in communication. One of the main goals in communication complexity is to understand the power of randomness. When allowing randomness in a communication protocol, there are a few choices we can make about what to demand from our protocol:

1. Is the source of randomness *public-coin* (both parties share the source of randomness), or *private-coin* (each party has their own source of randomness that the other doesn't see)?
2. Should our protocol have *bounded error* (the probability of error is at most, say $1/4$) or are we satisfied with *unbounded error* (the probability of error is strictly less than $1/2$)?

The most interesting choices to compare are *bounded-error, public-coin* and *unbounded-error, private-coin*, because, unlike the other choices, these are not obviously weaker or stronger than the other. For a boolean matrix $M \in \{0,1\}^{N \times N}$, we write $\mathrm{R}(M)$ for the least cost of a *bounded error, public-coin* protocol computing M, and $\mathrm{U}(M)$ for the least cost of an *unbounded error, private-coin* protocol.

Moreover, Newman's theorem [New91] says that any bounded-error randomized protocol requires at most $O(\log\log N)$ bits of randomness. One player can privately generate these bits and send them to the other player, giving

$$\mathrm{U}(M) \leq \mathrm{R}(M) + O(\log\log N). \tag{1}$$

Is this the best we can do in general, or can we remove the dependence on N? In other words, if we fix $\mathrm{R}(M)$ to be *constant* (i.e., independent of the matrix size N), does this imply $U(M)$ is also constant? This is equivalent to Open Problem 1.3, by the following argument. Paturi and Simon [PS86] showed

$$\mathrm{U}(M) = \log(\mathrm{rank}_{\pm}(M)) \pm 2. \tag{2}$$

This means that constant $\mathrm{U}(M)$ is equivalent to constant sign-rank. Linial and Shraibman [LS09] showed that margin is equivalent to discrepancy, and therefore

$$\Omega(\log(\mathrm{mar}(M)^{-1})) \leq \mathrm{R}(M) \leq O(\mathrm{mar}(M)^{-2}), \tag{3}$$

so that constant $\mathrm{R}(M)$ is equivalent to constant margin. We may therefore rephrase Open Problem 1.3 as: *If* $\mathrm{R}(M) = O(1)$, *is* $\mathrm{U}(M) = O(1)$*?*

The classes of communication problems with $\mathrm{R}(M) = O(1)$ have been well studied [HHH23, HWZ22, HHH22, DHP$^+$22, EHK22, CHZZ24, FHHH24, HR24, FGHH25, Tom25], and several papers [HHP$^+$22, HHM23, HH24, HZ24] have conjectured a negative answer to Open Problem 1.3:

Conjecture 1.5. *There exists a communication problem M with $\mathrm{R}(M) = O(1)$ but $\mathrm{U}(M) = \omega(1)$.*

The most obvious candidates for this conjecture are the k-Hamming Distance problems, which have $\mathrm{R}(\mathsf{HD}_k^n) = \Theta(k \log k)$ for $k < \sqrt{n}$ [HSZZ06, Sağ18]. Any problem which reduces to HD_0 (i.e., Equality) has constant sign-rank [HHP$^+$22], but the question remained open for problems which do not reduce to Equality, including 1-Hamming Distance and its generalizations [HHH23, HWZ22, FHHH24, FGHH25]. Several papers [HHP$^+$22, HHM23, HZ24] worked towards the conjecture that for constant $k \geq 1$, these problems should satisfy Conjecture 1.5:

Conjecture 1.6 (Now false)**.** *For some constant $k \geq 1$, $\mathrm{rank}_{\pm}(\mathsf{HD}_k^n) = \omega(1)$.*

In particular, [HHP$^+$22] showed that all known lower bound techniques fail to prove this conjecture; [HZ24] suggested a (now false) characterization of the problems with both $\mathrm{R}(M) = O(1)$ and $\mathrm{U}(M) = O(1)$ as exactly those which reduce to Equality (which for XOR functions would follow from Conjecture 1.6 for $k = 1$, due to the result of [CHZZ24]); and [HHM23] settled Conjecture 1.5 for *partial* matrices by an elegant application of the Borsuk–Ulam theorem to the Gap Hamming Distance problem (where two parties given $x, y \in \{0,1\}^n$ must decide if the Hamming distance is either at most αn or at least $(1-\alpha)n$ for some constant $\alpha > 0$). Another approach to solving Conjecture 1.5 is to find a completion M of Gap Hamming Distance which has $\mathrm{R}(M) = O(1)$; this was proven impossible in concurrent (independent) work [BHH$^+$25].

Theorem 1.1 gives an upper bound of $\mathrm{U}(\mathsf{HD}_k^n) = O(k)$ on the unbounded-error communication cost, showing that it is always smaller than the bounded-error cost $\Theta(k \log k)$, whereas Conjecture 1.6 posits an arbitrarily large gap in the other direction. However, the conjecture was sensible for several reasons:

- It was not known whether any other fundamentally different candidates exist; [FGHH25] only recently showed that there are candidates which are irreducible to k-Hamming Distance.

- Earlier work on symmetric XOR problems, of which k-Hamming Distance is the simplest example, have not witnessed any upper bounds superior to $O(n)$ [HQ17] and this is tight for Gap Hamming Distance [HHM23].
- The Borsuk–Ulam technique can be used to give a lower bound of $\Omega(n)$ for a *continuous* version of the problem, 1-Hamming Distance on strings in $[0,1]^n$.
- Problems reducing to Equality satisfy many nice properties that 1-Hamming Distance does not, see e.g., the close relationship between Equality and the γ_2-norm (see Definition 5.3) [HHH22, CHHS23, PSS23, CHZZ24, CHH+25, Tom25], and four different proofs that 1-Hamming Distance does not reduce to Equality [HHH22, HWZ22, FHHH24, HR24]. Equality is "special" among the Hamming distance problems, so one may expect it to be special with respect to sign-rank as well.

Owing partly to the latter reason, [HHP+22, HHM23, HZ24] stated the strongest form of Conjecture 1.6, for $k = 1$ rather than an arbitrary constant k. The stronger conjecture is easier to refute than Conjecture 1.6 (see Section 2.3), but the easier refutation does not generalize to $k = 2$.

The main contribution of this paper is the more general technique allowing to go beyond $k = 1$, including the candidates for Conjecture 1.5 which [FGHH25] showed cannot be reduce to k-Hamming Distance.

2 Technique: Support-Rank

The main conceptual idea allowing for our upper bounds is to switch from *sign-rank* to *support-rank*.

Definition 2.1 (Support-Rank)**.** Let $M \in \{0,1\}^{N\times N}$ be a boolean matrix. Its *support-rank* $\mathrm{rank}_0(M)$ is the minimal r for which there exists some $A \in \mathbb{R}^{N\times N}$ with rank r, satisfying

$$\forall i,j \in [N]: \qquad M(i,j) = 0 \iff A(i,j) = 0.$$

That is, the matrices A and M have the same support.

Support-rank has been studied previously in the context of tensor rank [CU13, BCZ17, BCZ18]. In quantum communication complexity, it has been called *nondeterministic rank* [dW03], in circuit complexity, *equality rank* [HP10], and, in graph theory, *minimum rank* [FH07]. It is also closely related to *unit-distance graphs* [EHT65, AK14]; see our discussion in Section 2.3.

Why support-rank? A basic fact is that any boolean matrix of support-rank r has sign-rank $O(r^2)$; see Section 2.1. The converse is false: we have $\mathrm{rank}_0(I_N) = N$ but $\mathrm{rank}_\pm(I_N) = 3$ for the $N \times N$ identity matrix. Thus, proving upper bounds on support-rank is a more difficult task. Nevertheless, what is convenient about support-rank is that it behaves better than sign-rank under *boolean combinations* (or *reductions*). To explain this, recall that our goal is to give a sign-rank upper bound for

$$\mathsf{HD}^n_k = \mathsf{HD}^n_{\le k} \wedge \neg \mathsf{HD}^n_{\le k-1},$$

where $\wedge$ and $\neg$ are understood entry-wise and $\mathsf{HD}^n_{\le k}(x,y) = 1$ if and only if $\mathrm{dist}(x,y) \le k$. The challenge with sign-rank is that it is not known whether the sign-rank of $A \wedge B$ can be bounded in terms of the sign-ranks of A and B [BMT21, HHP+22]. So even if we can prove, say, $\mathrm{rank}_\pm(\mathsf{HD}_{\le k}) = O(1)$, this would not imply any bound on $\mathrm{rank}_\pm(\mathsf{HD}_k)$. By contrast, we show the following useful properties of support-rank:

- (Section 2.1): We show that any matrix reducible to matrices with bounded support-rank has bounded sign-rank. All of our arguments for sign-rank upper bounds will rely on this fact.
- (Section 2.2): We explain how to transform polynomial identities $P(x,y) = 0$ into linear identities $\langle x', y'\rangle = 0$. This is useful for giving upper bounds on support-rank via polynomials.

2.1 From Support-Rank to Sign-Rank via Reductions

Reductions for constant-cost randomized communication are defined similarly to standard oracle reductions in communication complexity (e.g., [BFS86, CLV19]) except that there is no bound on the size of the oracle query inputs. For any communication problem $\mathcal{Q}$ (i.e., a family of boolean matrices), and any boolean matrix P, we write $\mathsf{D}^{\mathcal{Q}}(P)$ for the minimum cost of a *deterministic* communication protocol computing P with access to a unit-cost oracle that computes $\mathcal{Q}$. We say problem $\mathcal{P}$ *reduces to* $\mathcal{Q}$ if there is a constant c such that for every $P \in \mathcal{P}$, $\mathsf{D}^{\mathcal{Q}}(P) \leq c$. More formally:

Definition 2.2 (Oracle Protocols)**.** Let $\mathcal{Q}$ be a communication problem. For any matrix $P \in \{0,1\}^{N \times N}$, we write $\mathsf{D}^{\mathcal{Q}}(P)$ for the smallest depth of a communication tree T, where each inner node v is labelled by a matrix $Q_v \in \mathcal{Q}$ and two functions a_v, b_v; and each leaf ℓ is labelled with an output value. On inputs $i, j \in [N]$ the protocol at node v proceeds by computing $Q_v(a_v(i), b_v(j))$ and descending to the left or right child depending on the result. At a leaf ℓ the protocol outputs the value of ℓ, which must be equal to $P(i,j)$.

Since we are concerned only with constant vs. non-constant costs, one may equivalently say that $\mathcal{P}$ reduces to $\mathcal{Q}$ if there is a constant c such that every $P \in \mathcal{P}$ can be written as

$$P = \Gamma(Q_1, \ldots, Q_c)$$

for some choice of $Q_i \in \mathcal{Q}$ and boolean function $\Gamma \colon \{0,1\}^c \to \{0,1\}$ which is applied entry-wise to $Q_1, \ldots, Q_c$ to produce P; see e.g., [CLV19, FHHH24, FGHH25] for more details on these reductions and [ABSZ24] for applications in graph theory. It is not hard to see that reductions preserve constant-cost randomized communication in the bounded-error model: if $\mathrm{R}(\mathcal{Q}) = O(1)$ and $\mathsf{D}^{\mathcal{Q}}(\mathcal{P}) = O(1)$ then $\mathrm{R}(\mathcal{P}) = O(1)$, because we may replace each query $Q \in \mathcal{Q}$ with a randomized subroutine computing Q using standard majority-vote error boosting to bring the total error down to $1/4$.

The most important property of support rank is that it allows to upper bound $\mathrm{rank}_{\pm}(M)$ in terms of the number of queries $\mathsf{D}^{\mathcal{Q}}(M)$ required to compute M with an oracle $\mathcal{Q}$ that has bounded support-rank. This lemma generalizes a theorem of [HHP+22] which held for queries to the Equality oracle.

Lemma 2.3. *Let $\mathcal{Q}$ be a family of boolean matrices with $\mathrm{rank}_0(\mathcal{Q}) \leq r$. Then, for any boolean P,*

$$\mathrm{rank}_{\pm}(P) \leq O(r^2)^{\mathsf{D}^{\mathcal{Q}}(P)}.$$

Proof. (Generalization of [HHP+22, Theorem 3.8].) Let T be a decision tree for P of depth $q = \mathsf{D}^{\mathcal{Q}}(M)$, which queries problems in $\mathcal{Q}$. We prove that $\mathrm{rank}_{\pm}(P) \leq \left(1 + r^2\right)^q$ by induction on q. The base case $q = 0$ is immediate. For $q \geq 1$, let $R \in \mathcal{Q}$ be the problem queried at the root of T. Let A_0 and A_1 be the sign matrices corresponding to the two sub-problems computed by T after R returns 0 and 1, respectively. Thus,

$$P_{\pm} = R \circ A_1 + (\neg R) \circ A_0,$$

where $A \circ B$ is the entry-wise (or *Hadamard*) product defined by $(A \circ B)_{ij} = A_{ij} B_{ij}$. By the inductive hypothesis, there are real matrices $\tilde{A}_0$ and $\tilde{A}_1$ with rank at most $\left(1 + r^2\right)^{q-1}$ and with the same sign pattern as A_0 and A_1, respectively. Similarly, let $\tilde{R}$ be a real matrix with the same support as R and rank at most r. Note that for a sufficiently large $\gamma > 0$, the real matrix $\tilde{A}_1 + \gamma\big(\tilde{R} \circ \tilde{R} \circ \tilde{A}_0\big)$ has the same sign pattern as $P_{\pm}$. This is because on the support of $\tilde{R}$, the second term will dominate, whilst the first term dictates the sign wherever $\tilde{R}$ is zero. Using the fact[1] that $\mathrm{rank}(A \circ B) \leq \mathrm{rank}(A)\,\mathrm{rank}(B)$, we conclude

$$\mathrm{rank}_{\pm}(P) \leq \mathrm{rank}\Big(\tilde{A}_1 + \lambda\big(\tilde{R} \circ \tilde{R} \circ \tilde{A}_0\big)\Big) \leq \mathrm{rank}\big(\tilde{A}_1\big) + \mathrm{rank}\big(\tilde{R}\big)^2\,\mathrm{rank}\big(\tilde{A}_0\big) \leq \left(1 + r^2\right)^q. \qquad \blacksquare$$

[1] Write $A = \sum_{i=1}^{r} a_i u_i^T$ and $B = \sum_{j=1}^{s} b_j v_j^T$. Then, we have $A \circ B = \left(\sum_i a_i u_i^T\right) \circ \left(\sum_j b_i v_j^T\right) = \sum_{i,j} (a_i u_i^T) \circ (b_j v_j^T) = \sum_{i,j} (a_i \circ b_j)(u_i \circ v_j)^T$. This shows that $A \circ B$ can be written as the sum of rs many rank-1 matrices.

2.2 From Polynomials to Support-Rank via Veronese Maps

Note that the support-rank of $M \in \{0,1\}^{N\times N}$ is at most r if and only if there are vectors $u_1, u_2, \ldots, u_N \in \mathbb{R}^r$ and $v_1, v_2, \ldots, v_N \in \mathbb{R}^r$ such that

$$\forall i,j \in [n]: \qquad M_{ij} = 0 \iff \langle u_i, v_j\rangle = 0. \tag{4}$$

It is more convenient to work with polynomial equations $P(u_i, v_j) = 0$. For example, suppose we have

$$\forall i,j \in [n]: \qquad M_{ij} = 0 \iff P(u_i, v_j) = 0 \tag{5}$$

where, say, $u_i, v_j \in \mathbb{R}^2$ and $P(a,b)$ is a polynomial on 4 variables, say $P(a,b) = a_1^2 + b_1^2 - a_1 b_1 + 3a_2 b_2$. Then we can write P as an inner product of two vectors, each depending only on a or only on b, by grouping each monomial into its own dimension:

$$P(a,b) = \left\langle (a_1^2, 1, -a_1, 3a_2),\ (1, b_1^2, b_1, b_2)\right\rangle.$$

In this way we transform equations like Equation (5) into the equations like Equation (4) required for the definition of support-rank, where the dimension is at most the number of monomials in P. In general we have the following proposition (whose proof simply generalizes the above discussion and is hence omitted).

Proposition 2.4 (Veronese Map)**.** *Let $M \in \{0,1\}^{n\times n}$ and let P be a real polynomial in $2m$ variables. Assume that there are functions $\alpha_t\colon [n] \to \mathbb{R}$ and $\beta_t\colon [n] \to \mathbb{R}$ for $t \in [m]$ that satisfy*

$$\forall i,j \in [n]: \qquad M_{ij} = 0 \iff P\big(\alpha_1(i),\ \ldots,\ \alpha_m(i),\ \beta_1(j),\ \ldots,\ \beta_m(j)\big) = 0.$$

Then, $\operatorname{rank}_0(M)$ is at most the number of monomials in P with non-zero coefficients.

2.3 Support-Rank and Unit-Distance Graphs

A *(faithful) unit-distance graph* [EHT65, AK14] in dimension d is a graph $G = (V,E)$ whose vertices $x \in V$ can be identified with points $u_x \in \mathbb{R}^d$ such that

$$\{x,y\} \in E \iff \|u_x - u_y\|_2 = 1.$$

We claim that the complement of the adjacency matrix of a unit-distance graph in dimension d has support-rank at most $O(d)$. Indeed, we have $\|u_x - u_y\|_2 = 1 \Leftrightarrow P(u_x, u_y) = 0$ for the $2d$-variate polynomial $P(a,b) \coloneqq \sum_{i=1}^d (a_i - b_i)^2 - 1$ with $O(d)$ monomials, and the claim follows from Proposition 2.4. Conversely, it is easy to show that that any boolean matrix M with $\operatorname{rank}_0(M) = d$ is the complement of the bi-adjacency matrix of a bipartite unit-distance graph in dimension d (indeed, normalize all u_i, v_j to have length $1/\sqrt{2}$).

It is a classic fact [EHT65] that the hypercube graph (the bipartite graph with bi-adjacency matrix HD_1^n) is a unit-distance graph in dimension 2. By the above discussion, we have $\operatorname{rank}_0(\neg\mathsf{HD}_1^n) \le O(1)$, which implies $\operatorname{rank}_\pm(\mathsf{HD}_1^n) \le O(1)$ via Lemma 2.3. This already proves Theorem 1.1 in the special case $k = 1$. However, this argument does not generalize to $k = 2$, because $\neg\mathsf{HD}_2^n$ (and also HD_2^n) contains an unbounded-size identity submatrix, which shows that its support-rank is unbounded.

3 Sign-Rank of k-Hamming Distance

In this section, we prove Theorem 1.1. We have $\mathsf{HD}_k^n = \mathsf{HD}_{\ge k}^n \wedge \neg\mathsf{HD}_{\ge k+1}^n$ and hence by Lemma 2.3 it suffices to prove an upper bound on the support-rank of $\mathsf{HD}_{\ge k}^n$.

Theorem 3.1. *We have $2^k \le \operatorname{rank}_0(\mathsf{HD}_{\ge k}^n) \le 4^k$ for all $k \le n$.*

It is important here to consider $\mathsf{HD}_{\ge k}^n$ rather than $\mathsf{HD}_{\le k}^n$, as the latter contains an identity submatrix of size $\Omega(2^n)$ and therefore its support-rank depends on n. We also note that this theorem, together with Lemma 2.3, implies more generally that any matrix reducible to HD_k^n has constant sign-rank.

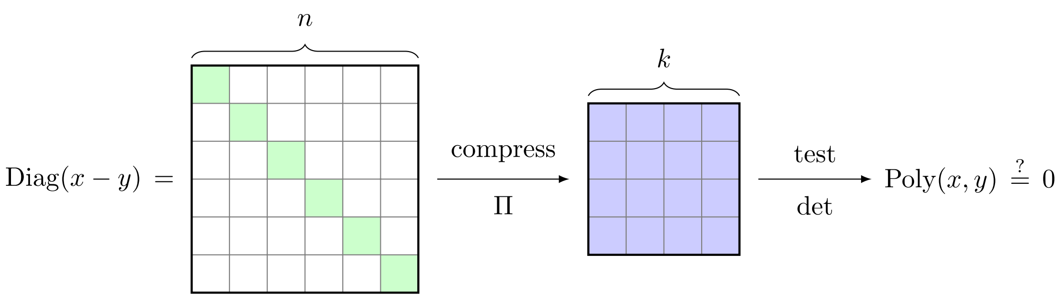
Corollary 3.2. *For all boolean matrices M, $\operatorname{rank}_{\pm}(M) = 2^{O(k \cdot \mathsf{D}^{\mathsf{HD}_k}(M))}$ and $\mathrm{U}(M) = O(k \cdot \mathsf{D}^{\mathsf{HD}_k}(M))$.*

The lower bound in Theorem 3.1 follows directly from the fact that $\mathsf{HD}_{\geq k}$ contains an identity submatrix of size 2^k, induced by the set of all strings ending in $n-k$ zeros.

The upper bound uses the following method (which we further generalize in Section 4). We show that there exists a map $A\colon \{0,1\}^n \to \mathbb{R}^{k\times k}$ assigning to each binary string $x \in \{0,1\}^n$ a $k \times k$ matrix $A(x)$ with the property that

$$\forall x, y \in \{0,1\}^n : \qquad \mathsf{HD}^n_{\geq k}(x,y) = 1 \iff \operatorname{rank}\big(A(x) - A(y)\big) = k. \tag{6}$$

In other words, the output of the communication problem depends only on whether the matrix $A(x) - A(y)$ has full rank. This can be verified by testing if its determinant is 0. Since the determinant is given by a polynomial in the entries of the matrix, we can then use a Veronese map to obtain a support-rank upper bound. Figure 2 illustrates the proof that follows.



$$\operatorname{dist}(x,y) \geq k \Leftrightarrow \operatorname{rank}\big(\mathrm{Diag}(x-y)\big) \geq k \Leftrightarrow \operatorname{rank}\big(\Pi\big(\mathrm{Diag}(x-y)\big)\big) = k \Leftrightarrow \det\big(\Pi\big(\mathrm{Diag}(x-y)\big)\big) \neq 0$$

Figure 2: A sketch of the argument: We reduce $\mathsf{HD}_{\geq k}$ to checking whether a polynomial vanishes.

As a first step, observe that if we view $\{0,1\}^n$ as a subset of $\mathbb{R}^n$, then

$$\forall x, y \in \{0,1\}^n : \quad \mathsf{HD}^n_{\geq k}(x,y) = 1 \iff x - y \text{ has } \geq k \text{ non-zero entries} \iff \operatorname{rank}(\mathrm{Diag}(x-y)) \geq k. \tag{7}$$

The matrices $\mathrm{Diag}(x-y)$ are of size $n \times n$ and the next step is to reduce their size to $k \times k$, without changing the rank, provided it is at most k. Informally, this can be accomplished by applying a random projection which has probability 1 of preserving ranks. Formally, we have the following lemma.

Lemma 3.3 (Rank Compression)**.** *Let $\mathcal{M}$ be a finite set of matrices in $\mathbb{R}^{a\times b}$. For any a' and b', there exists a linear map $\Pi\colon \mathbb{R}^{a\times b} \to \mathbb{R}^{a'\times b'}$ which satisfies*

$$\forall M \in \mathcal{M} : \qquad \operatorname{rank}\big(\Pi(M)\big) = \min\big(\operatorname{rank}(M),\ a',\ b'\big).$$

Proof. It suffices to prove the statement with $b = b'$, since the general result is recovered by applying this case twice, transposing in between. We can further assume $a' < a$, as otherwise we can just take any injective linear map for Π. For each $M \in \mathcal{M}$, pick a subspace $V_M \subseteq \operatorname{image}(M) \subseteq \mathbb{R}^a$ of dimension $\min(\operatorname{rank}(M), a')$. Since $\mathcal{M}$ is finite, there is a subspace $V \subseteq \mathbb{R}^a$ of dimension $a - a'$ such that

$$V \cap \bigcup_{M \in \mathcal{M}} V_M = \{0\}.$$

We now let $\Pi(x) = Px$ where $P \in \mathbb{R}^{a'\times a}$ is the projection with kernel V. Thus, for all $M \in \mathcal{M}$ we have

$$\operatorname{rank}\big(\Pi(M)\big) = \operatorname{rank}(PM) = a' - \dim\big(V \cap \operatorname{image}(M)\big) \geq \dim(V_M) = \min\big(\operatorname{rank}(M),\ a'\big).$$

The converse inequality clearly holds as well. ■

In particular, using $\mathcal{M} = \{\mathrm{Diag}(x-y) \mid x, y \in \{0,1\}^n\}$ and the fact that $k \leq n$, we obtain a linear map $\Pi\colon \mathbb{R}^{n\times n} \to \mathbb{R}^{k\times k}$ with the desired property

$$\forall x, y \in \{0,1\}^n: \qquad \mathrm{rank}\Big(\Pi\big(\mathrm{Diag}(x-y)\big)\Big) = k \iff \mathrm{rank}\big(\mathrm{Diag}(x-y)\big) \geq k \iff \mathrm{dist}(x,y) \geq k. \tag{8}$$

Thus, setting $A(x) = \Pi(\mathrm{Diag}(x))$ for all $x \in \{0,1\}^n$, we obtain the desired characterisation of Equation (6).

It remains to express the rank bound of this $k \times k$ matrix in terms of a polynomial suitable for the Veronese map. Writing S_k for the set of permutations on $[k]$, we further have

$$\forall x, y \in \{0,1\}^n: \qquad \det\big(A(x) - A(y)\big)\big) = \sum_{\pi \in S_k} \mathrm{sign}(\pi) \prod_{i=1}^{k} \Big(A(x)_{i\pi(i)} - A(y)_{i\pi(i)}\Big).$$

Expanding the right side of the above, we obtain a polynomial over $2k$ variables, corresponding to the entries of $A(x)$ and $A(y)$, with at most $2^k \cdot k!$ many monomials. Thus, using the Veronese map from Proposition 2.4, we conclude that

$$\mathrm{rank}_0(\mathsf{HD}^n_{\geq k}) \leq k! \cdot 2^k.$$

This bound is already independent of n, but for Theorem 3.1 we claim an even better upper bound of 4^k. To achieve this, we can be smarter when constructing the Veronese map. In the following section, we prove the following fact, which gives an improved Veronese map, concluding the proof of Theorem 3.1.

Proposition 3.4. *For matrices $A, B \in \mathbb{R}^{k\times k}$, we can write $\det(A-B)$ as a sum of at most 4^k many terms, each of the form $\pm\det(A')\det(B')$ where A' and B' denote some square submatrices of A and B, respectively.*

Remark 3.5. The above proof works for k-Hamming Distance over any finite alphabet, not only for $\{0,1\}$. One simply has to identify the alphabet with some subset of $\mathbb{R}$ to satisfy Equation (7).

3.1 Proof of Proposition 3.4

We use the following lemma from [Mar90], whose proof we include for completeness. Note that, in this lemma, the right side of the equation contains $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \leq 4^n$ many terms, which proves Proposition 3.4.

Lemma 3.6. *Let $n \in \mathbb{N}$ and let $A, B \in \mathbb{R}^{n\times n}$. Then,*

$$\det(A+B) = \sum_{\substack{\alpha,\beta \subseteq [n] \\ |\alpha| = |\beta|}} (-1)^{s(\alpha)+s(\beta)} \det(A_{|\alpha\times\beta}) \det(B_{|\bar{\alpha}\times\bar{\beta}}),$$

where $s(\cdot)$ denotes the sum of a set, and $A_{|\alpha\times\beta}$ denotes the submatrix of A indexed by sets α and β.

Proof. Let S_n denote the set of all permutations on the set $[n]$. We have

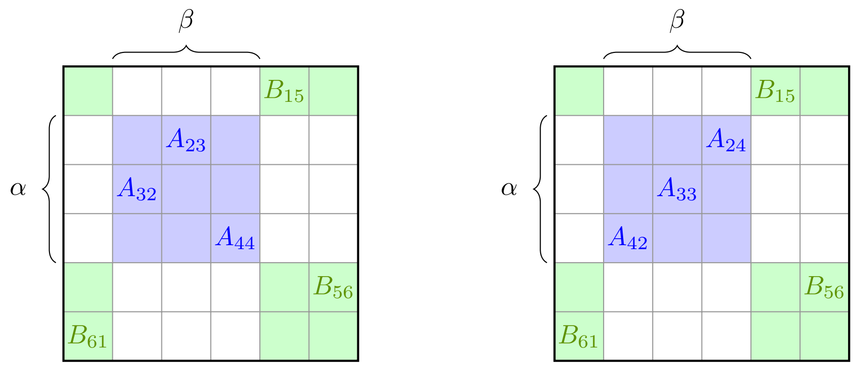
$$\det(A+B) = \sum_{\pi\in S_n} \mathrm{sign}(\pi) \prod_{i=1}^{n} (A+B)_{i\pi(i)} = \sum_{\alpha\subseteq[n]} \sum_{\pi \in S_n} \mathrm{sign}(\pi) \prod_{i\in\alpha} A_{i\pi(i)} \prod_{i\in\bar{\alpha}} B_{i\pi(i)}.$$

Given a choice of $\alpha \subseteq [n]$, any permutation $\pi \in S_n$ can be uniquely decomposed as $\pi = \tau \circ \pi_\alpha \circ \pi_{\bar{\alpha}}$ where

$$\pi_\alpha(x) = x \text{ for all } x \in \bar{\alpha} \qquad \text{and} \qquad \pi_{\bar{\alpha}}(x) = x \text{ for all } x \in \alpha,$$

and $\tau \in S_n$ is the unique permutation with $\tau(\alpha) = \pi(\alpha)$ that is order preserving on both α and $\bar{\alpha}$. Note that τ only depends on $\pi(\alpha)$, which we now call β. Moreover, after fixing β, the correspondence between pairs $(\pi_\alpha, \pi_{\bar{\alpha}})$ and π is bijective. Thus,

$$\det(A+B) = \sum_{\substack{\alpha,\beta\subseteq[n] \\ |\alpha|=|\beta|}} \sum_{\pi_\alpha} \sum_{\pi_{\bar{\alpha}}} \mathrm{sign}(\tau\circ\pi_\alpha\circ\pi_{\bar{\alpha}}) \prod_{i\in\alpha} A_{i(\tau\circ\pi_\alpha)(i)} \prod_{i\in\bar{\alpha}} B_{i(\tau\circ\pi_{\bar{\alpha}})(i)}$$



$$B_{15}A_{23}A_{32}A_{44}B_{56}B_{61} + B_{15}A_{24}A_{33}A_{42}B_{56}B_{61} = (A_{23}A_{32}A_{44} + A_{24}A_{33}A_{42})(B_{15}B_{56}B_{61})$$

Figure 3: Two terms in the expansion of $\det(A+B)$ that only differ on the square given by $\alpha\times\beta$ and can thus be factored. Doing this for all like terms of each square, we arrive at Lemma 3.6.

$$= \sum_{\substack{\alpha,\beta\subseteq[n]\\|\alpha|=|\beta|}} \operatorname{sign}(\tau)\left[\sum_{\pi_\alpha}\operatorname{sign}(\pi_\alpha)\prod_{i\in\alpha}A_{i(\tau\circ\pi_\alpha)(i)}\right]\left[\sum_{\pi_{\bar\alpha}}\operatorname{sign}(\pi_{\bar\alpha})\prod_{i\in\bar\alpha}B_{i(\tau\circ\pi_{\bar\alpha})(i)}\right]$$

$$= \sum_{\substack{\alpha,\beta\subseteq[n]\\|\alpha|=|\beta|}} \operatorname{sign}(\tau)\det(A_{|\alpha\times\beta})\det(B_{|\bar\alpha\times\bar\beta}).$$

Finally, note that the number of inversions of τ is given by $\sum_{i\in\alpha}|\pi(i)-i|$, which has the same parity as $\sum_{i\in\alpha}(\pi(i)+i) = s(\beta)+s(\alpha)$. Thus, $\operatorname{sign}(\tau) = (-1)^{s(\alpha)+s(\beta)}$, just as desired. ∎

4 Rank Problems and Generalizations of k-Hamming Distance

Corollary 3.2 showed that every problem that reduces in q queries to k-Hamming Distance has sign-rank $2^{O(qk)}$. But [FGHH25] recently showed that there exist problems with $\mathrm{R}(M) = O(1)$ that *do not* reduce to k-Hamming Distance. At first, this provides some hope of using these problems to separate constant margin from constant sign-rank, but we shall crush this hope, using another new idea.

The problems of [FGHH25] were constructed via *distance-r compositions*. The simplest example, which was the main focus of their paper, is the $\{4,4\}$-Hamming Distance problem. We will discuss only this example here and leave the formal definition of *distance-r compositions* for Section 4.3.

Example 4.1 ($\{4,4\}$-Hamming Distance)**.** Alice and Bob receive matrices $X, Y \in \{0,1\}^{n\times n}$ respectively. Write X_i for the i^{th} row of X, and similar for Y. The players should output 1 if and only if the following conditions are satisfied:

1. There are at most 2 rows $i, j \in [n]$ such that $X_i \neq Y_i$ and $X_j \neq Y_j$; and
2. For each row i where $X_i \neq Y_i$, it holds that $\mathrm{dist}(X_i, Y_i) \leq 4$.

In essence, the reason that $\{4,4\}$-Hamming Distance does not reduce to k-Hamming Distance for any constant k is that a k-Hamming Distance query is not capable of distinguishing between two rows each of distance 4 (where the correct output should be 1), and two rows of distance 6 and 2 (where the correct output should be 0). This poses a challenge for sign-rank as well, since a naïve application of our method for k-Hamming Distance encounters the same issue.

To handle this type of problem, we define a class of problems called *rank problems*. A rank problem is any problem which can be expressed as a function of the rank of the *difference* between matrices held by

Alice and Bob; in other words, any problem that can put in the form similar to Equation (6) in our upper bound of k-Hamming Distance. This notion of a rank problem resembles the problems defined in [SS24], but concerning matrices over $\mathbb{R}$ instead of a finite field $\mathbb{F}$. This difference is crucial in our argument.

Definition 4.2 (Rank Problem)**.** A boolean matrix $P \in \{0,1\}^{N\times N}$ is a *rank problem* of *order* k if for some a, b and every $x \in [N]$, there exist real matrices $A(x)$ and $B(x)$ in $\mathbb{R}^{a\times b}$, satisfying

$$\forall x, y \in [N]: \qquad P(x,y) = g\Big(\operatorname{rank}\big(A(x) + B(y)\big)\Big),$$

where $g\colon \{0,1,2,\dots\} \to \{0,1\}$ is a function which is constant for inputs $\geq k$. We say that a family $\mathcal{P}$ of boolean matrices is *family of rank problems* of order k if there is some function g such that each $P \in \mathcal{P}$ is a rank problem of order k with associated function g. Moreover, we call the rank problem *symmetric* if $A(x) + B(x) = 0$ for all $x \in [N]$.

Remark 4.3. In the above definition, we can equivalently take $a = b = k$. This is because we can always embed A and B in larger matrices, or compress them using Lemma 3.3 without changing the ranks pertinently. We also note that it suffices to define g on the domain $\{0, 1, \dots, k\}$.

Example 4.4. In the proof of Theorem 3.1, we showed in Equation (8) that $\mathsf{HD}_{\geq k}$ is a symmetric rank problem of order k, with associated function $g(t) = \mathbb{1}[t \geq k]$.

We prove the following three properties of rank problems.

Theorem 4.5. *Rank problems satisfy the following three properties:*

1. *Rank problems of constant order have constant sign rank.*
2. *Rank problems of constant order are closed under reductions.*
3. *Symmetric rank problems of constant order are closed under distance-r compositions.*

Since k-Hamming Distance is a rank problem of constant order, the above 3 properties guarantee that any problem obtained by reductions and compositions of it will have constant sign-rank. As explained in [FGHH25], this covers all known examples of problems with $\mathrm{R}(M) = O(1)$, apart from the remaining separation candidates from [SS24].

In the following three subsections, we will give proofs of each of the three parts in the above theorem, with quantitative bounds.

4.1 Rank Problems have Bounded Sign-Rank

Lemma 4.6 (Theorem 4.5, Part 1)**.** *If P is a rank problem of order k, then*

$$\operatorname{rank}_\pm(P) = 2^{O(k \log k)}.$$

We first deal with a special class of simple rank problems, and then reduce general rank problems to this case. Let $g\colon \{0,1,\dots\} \to \{0,1\}$ be some function as in the definition of rank problems. We write ∂g for the number of times g changes value. If $\partial g \leq 1$, then we say that the associated rank problem is *monotone*.

Lemma 4.7. *If P is a monotone rank problem of order k, then*

$$\min\Big(\operatorname{rank}_0(P),\ \operatorname{rank}_0(\neg P)\Big) \leq 4^k.$$

Proof. Let $P \in \{0,1\}^{N\times N}$ be a monotone rank problem of order k. Without loss of generality, replacing P

with $\neg P$ if necessary, there are functions $A, B\colon [N] \to \mathbb{R}^{a\times b}$ and some $r \le \min(k, a, b)$ that satisfy

$$\forall x, y \in [N]: \qquad P(x, y) = 1 \iff \operatorname{rank}\big(A(x) + B(y)\big) \ge r.$$

But now, we finish just like in the proof of Theorem 3.1 using rank compression and the Veronese map: By Lemma 3.3, there is a rank-preserving linear map $\Pi : \mathbb{R}^{a\times b} \to \mathbb{R}^{r\times r}$ such that

$$\begin{aligned}\forall x, y \in [N]: \qquad P(x, y) = 1 &\iff \operatorname{rank}\big(A(x) + B(y)\big) \ge r\\ &\iff \operatorname{rank}\Big(\Pi(A(x)) + \Pi(B(y))\Big) = r\\ &\iff \det\Big(\Pi(A(x)) + \Pi(B(y))\Big) \ne 0.\end{aligned}$$

The determinant can be expanded using Lemma 3.6 to obtain a polynomial. Using the Veronese map from Proposition 2.4 together with Proposition 3.4, we conclude that $\operatorname{rank}_0(P) \le 4^r \le 4^k$. ∎

We now show that any rank problem reduces to a bounded number of monotone rank problems.

Lemma 4.8. *If P is a rank problem of order k, then there exists a family $\mathcal{Q}$ of monotone rank problems of order k, such that*

$$\mathsf{D}^{\mathcal{Q}}(P) = O(\log k).$$

Proof. For $P \in \{0,1\}^{N\times N}$, let $A, B\colon [N] \to \mathbb{R}^{a\times b}$, and $g\colon \{0, 1, 2, \dots\} \to \{0, 1\}$ constant on $[k, \infty)$, such that

$$\forall x, y \in [N]: \qquad P(x, y) = g\Big(\operatorname{rank}\big(A(x) + B(y)\big)\Big).$$

Using binary search, $\operatorname{rank}(A(x) + B(y))$ can be determined exactly by a decision tree of depth $O(\log k)$ that makes queries to the family of monotone rank problems of order k, given by the same maps A and B. Since P only depends on this rank, we conclude that $\mathsf{D}^{\mathcal{Q}}(P) = O(\log k)$.

We note that this bound can be improved to $\mathsf{D}^{\mathcal{Q}}(P) = O(\log \partial g)$, since we do not need to determine $\operatorname{rank}(A(x) + B(y))$ exactly, but only need to determine whether or not it lies in the support of g. ∎

The proof of Lemma 4.6 is now a simple application of Lemma 4.7 and Lemma 2.3.

4.2 Rank Problems are Closed under Reductions

Lemma 4.9 (Theorem 4.5, Part 2)**.** *If the problem P is a boolean combination of q rank problems of order k each, then P is a rank problem of order $O(k)^q$.*

Proof. Let $P \in \{0,1\}^{N\times N}$ be a problem that is the combination of rank problem instances $Q_1, Q_2, \dots, Q_q$ of order k each. For each $i \in [q]$, if $Q_i \in \{0,1\}^{N_i\times N_i}$, then by Remark 4.3, there are $A_i, B_i\colon [N_i] \to \mathbb{R}^{k\times k}$ and $g_i\colon \{0, 1, \dots, k\} \to \{0, 1\}$, such that

$$\forall x, y \in [N_i]: \qquad Q_i(x, y) = g_i\Big(\operatorname{rank}\big(A_i(x) + B_i(y)\big)\Big).$$

For each $x \in [N]$, we now define the diagonal block matrix $A(x)$ with q blocks as follows. For each $i \in [q]$, let x_i denote the input to Q_i corresponding to input x of P. We now construct the i-th block of $A(x)$ by aligning $(k+1)^{i-1}$ many copies of $A_i(x_i)$ side by side, as illustrated in Figure 4:

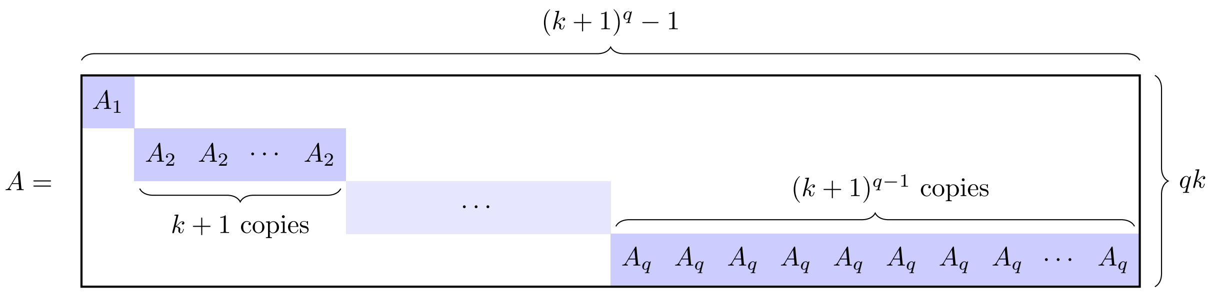


Figure 4: Construction of the matrix $A(x)$ from the matrices $A_i(x_i)$ with $i \in [q]$.

We similarly define the diagonal block matrix $B(x)$. Now, we have

$$\forall x, y \in [N]: \qquad \operatorname{rank}\Big(A(x) + B(y)\Big) = \sum_{i=1}^{q} (k+1)^{i-1} \operatorname{rank}\Big(A_i(x_i) + B_i(y_i)\Big).$$

Since $\operatorname{rank}(A_i(x) + B_i(y)) \leq k$ for all $i \in [q]$,

$$\forall x, y \in [N]: \qquad Q_i(x,y) = g_i\Big(\operatorname{rank}\Big(A_i(x_i) + B_i(y_i)\Big)\Big) = g_i\left(\left\lfloor \frac{\operatorname{rank}\big(A(x)+B(y)\big)}{(k+1)^{i-1}} \right\rfloor \text{ modulo } k+1\right).$$

Thus, $\operatorname{rank}(A(x)+B(y))$ encodes the answer to every query required to compute $P(x,y)$ as the digits in its base $k+1$ expansion. We conclude that there is some function $g : \{0,1,\dots\} \to \{0,1\}$ that is constant for inputs at least $(k+1)^q - 1$ and such that

$$\forall x, y \in [N]: \qquad P(x,y) = g\Big(\operatorname{rank}\Big(A(x) + B(y)\Big)\Big).$$

Thus, P is a rank problem of order $(k+1)^q - 1 = O(k)^q$. ■

Remark 4.10. Since the construction in the above proof preserves symmetry, we also find that symmetric rank problems of constant order are closed under reductions.

Example 4.11. Let $S \subseteq \mathbb{N}$ be a finite set. Consider the problem $P \in \{0,1\}^{2^n \times 2^n}$ of deciding whether two strings $x, y \in \{0,1\}^n$ satisfy $\operatorname{dist}(x,y) \in S$. Note that P can be computed by $O(|S|)$ many queries to $\mathsf{HD}_{\geq k}$ with $k \leq 1 + \max S$. By Example 4.4, all of these problems are rank problems of order $O(\max S)$. Thus, by Lemma 4.9, it follows that P is a rank problem of order $O(\max S)^{O(|S|)}$.

4.3 Rank Problems are Closed under Distance-r Compositions

We now define the distance-r compositions of [FGHH25]. For simplicity, we present a definition in which each inner problem P_i is boolean-valued, although [FGHH25] allows an arbitrary constant-size range of values. It is not difficult to show that the latter can be reduced to queries of the former.

Definition 4.12. Fix any r and an *outer* function $h\colon \{0,\dots,r\} \to \{0,1\}$. For boolean matrices $P_1,\dots,P_m \in \{0,1\}^{N\times N}$, we define their *distance-r composition* $h[\![P_1,\dots,P_m]\!]\colon [N]^m \times [N]^m \to \{0,1\}$ as follows. For any $x, y \in [N]^m$, write $\Delta(x,y) \coloneqq \{i \in [m] \mid x_i \neq y_i\}$. Then,

$$h[\![P_1,\dots,P_m]\!](x,y) = \begin{cases} 0 & \text{if } |\Delta(x,y)| > r, \\ h\left(\sum_{i\in\Delta(x,y)} P_i(x_i,y_i)\right) & \text{otherwise.} \end{cases}$$

Example 4.13. The problem HD_k^n is the distance-k composition where we take $h(t) = \mathbb{1}[t = k]$ and take each P_i to be $\neg I_{2,2}$, the negation of the 2×2 identity matrix.

Example 4.14. The $\{4,4\}$-HAMMING DISTANCE problem (Example 4.1) is the distance-2 composition where we take $h(t) = \mathbb{1}[t \le 2]$ and take each P_i to be $\mathsf{HD}^n_{\le 4}$.

Lemma 4.15 (Theorem 4.5, Part 3)**.** *Let $\mathcal{P}$ be a family of symmetric rank problems of order k and let $P_1, \dots, P_n \in \mathcal{P}$. Then, for any $h\colon \{0,\dots,r\} \to \{0,1\}$, the distance-$r$ composition*

$$P = h[\![P_1, \dots, P_n]\!]$$

is a symmetric rank problem of order $O(rk)^{O(rk^2)}$.

Proof. We may assume without loss of generality that each P_i is an $N \times N$ matrix. The goal is to write P as a boolean combination of $O(rk^2)$ many rank problems, each of order $O(rk)$; from there, the conclusion holds by Lemma 4.9. First consider the case where $h = \mathbb{1}$ is the constant 1 function. In this case

$$\forall x, y \in [N]^n : \qquad P(x,y) = \mathbb{1}[\![P_1,\dots,P_n]\!](x,y) = 1 \iff |\{i \in [n] \mid x_i \ne y_i\}| \le r.$$

This is just the Hamming distance problem $\neg\mathsf{HD}^n_{\ge r+1}$ over the alphabet $\Sigma = \{0,1\}^N$, which is a rank problem of order $k+1$ (see Remark 3.5). We now prove the statement for general h. Let $g\colon \{0,1,2,\dots\} \to \{0,1\}$ and $A_i(x) \in \mathbb{R}^{a_i \times b_i}$ for $i \in [M]$ be such that g is constant on inputs at least k, and

$$\forall x_i, y_i \in [N] : \qquad P_i(x_i, y_i) = g\Big(\operatorname{rank}\big(A_i(x_i) - A_i(y_i)\big)\Big).$$

We wish to recover the multiset $\big\{\operatorname{rank}\big(A_i(x_i) - A_i(y_i)\big) \;\big|\; i \in [n],\ x_i \ne y_i\big\}$. Due to the simple fact stated below, it suffices to determine the values

$$\forall t \in [k] : \qquad \sum_{i=1}^{n} \min\Big(\operatorname{rank}\big(A_i(x_i) - A_i(y_i)\big),\ t\Big). \tag{9}$$

Fact 4.16. *Let U and V be two multisets whose elements are integers from $\{0,\dots,s\}$. If for all $t \in [s]$ we have $\sum_{u \in U} \min(u,t) = \sum_{v \in V} \min(v,t)$, then necessarily $U = V$.*

Now fix any $t \in [k]$. We determine the value of Equation (9) by constructing a diagonal block matrix whose blocks are compressions of the matrices $A_i(x_i)$ for $i \in [n]$, capping their ranks at t, as follows.

For every $i \in [n]$, by Lemma 3.3, there is a linear map $\Pi_i^{(t)}\colon \mathbb{R}^{a_i \times b_i} \to \mathbb{R}^{t \times t}$ such that

$$\forall x_i, y_i \in [N] : \qquad \operatorname{rank}\Big(\Pi_i^{(t)}\big(A_i(x_i) - A_i(y_i)\big)\Big) = \min\Big(\operatorname{rank}\big(A_i(x_i) - A_i(y_i)\big),\ t\Big).$$

For every $x \in [N]^n$ and all $i \in [n]$, define $B_i^{(t)}(x_i) = \Pi_i^{(t)}(A_i(x_i)) \in \mathbb{R}^{t \times t}$ and construct the diagonal block matrix $B^{(t)}(x) \in \mathbb{R}^{nt \times nt}$ whose blocks are given by $B_i^{(t)}(x_i)$, illustrated in Figure 5. By Lemma 3.3, there is a linear map $\Pi^{(t)}\colon \mathbb{R}^{nt \times nt} \to \mathbb{R}^{rt \times rt}$ that satisfies

$$\forall x, y \in [N]^n : \qquad \operatorname{rank}\Big(\Pi^{(t)}\big(B^{(t)}(x) - B^{(t)}(y)\big)\Big) = \min\Big(\operatorname{rank}\big(B^{(t)}(x) - B^{(t)}(y)\big),\ rt\Big).$$

Now set $A^{(t)}(x) = \Pi^{(t)}(B^{(t)}(x))$ for all $x \in [N]^n$. We claim that $P(x,y)$ is a function of

$$\mathbb{1}[\![P_1,\dots,P_n]\!](x,y) \quad \text{and} \quad \operatorname{rank}\left(A^{(t)}(x) - A^{(t)}(y)\right) \quad \text{for} \quad 1 \le t \le k.$$

If $\mathbb{1}[\![P_1,\dots,P_n]\!](x,y) = 0$, then $P(x,y) = 0$. Else, we must have $|\{i \in [M] \mid x_i \ne y_i\}| \le r$ which implies that for all $t \in [k]$, $\operatorname{rank}(B^{(t)}(x) - B^{(t)}(y)) \le rt$ and thus

$$\forall x, y \in [N]^n : \quad \operatorname{rank}\Big(A^{(t)}(x) - A^{(t)}(y)\Big) = \operatorname{rank}\Big(B^{(t)}(x) - B^{(t)}(y)\Big) = \sum_{i=1}^{n} \min\Big(\operatorname{rank}\big(A_i(x) - A_i(y)\big),\ t\Big).$$

Now for fixed $x, y \in [N]^m$, the above expression is a function of all the $O(rk)$ many possible monotone rank problems that correspond to the map $A^{(t)}$, each of order rt. Doing this for all $t \in [k]$ and using

$$B^{(t)}(x) = \overbrace{\begin{pmatrix} B_1^{(t)}(x_1) & & & \\ & B_2^{(t)}(x_2) & & \\ & & \ddots & \\ & & & B_n^{(t)}(x_n) \end{pmatrix}}^{nt} \Bigg\} nt$$

Figure 5: Construction of $B^{(t)}(x)$ from the $B_i^{(t)}(x_i)$ for $i \in [n]$.

Fact 4.16, we deduce that the multiset $\{\text{rank}(A_i(x) - A_i(y)) \mid i \in [M]\}$ is a boolean combination of $O(rk^2)$ many rank problems of order rk each. But then the same is true for the value of

$$P(x, y) = h\left(\sum_{i \in \Delta(x,y)} P_i(x_i, y_i)\right) = h\left(\sum_{i \in \Delta(x,y)} g\Big(\text{rank}\big(A_i(x) - A_i(y)\big)\Big)\right).$$

Thus, we have expressed P as a boolean combination of the rank problem $\mathbb{1}[\![P_1, \ldots, P_n]\!]$ of order $k+1$, together with $O(rk^2)$ many symmetric rank problems of order rk each. By Lemma 4.9 and Remark 4.10, we conclude that P is a symmetric rank problem of order $O(rk)^{O(rk^2)}$. ∎

5 What's SUPP? Complexity Classes and their Relations

5.1 New Classes

One of the main ideas that allowed us to get general upper bounds on sign-rank was the use of support-rank as an intermediate step, and we have shown that all known communication problems $\mathcal{P}$ with $\text{R}(\mathcal{P}) = O(1)$ can be computed by queries to matrices of constant support-rank. This suggests the following new communication complexity classes:

SUPP: The class of communication problems $\mathcal{P}$ with constant support-rank, $\text{rank}_0(\mathcal{P}) = O(1)$.

coSUPP: The class of communication problems $\mathcal{P}$ whose negation is in SUPP.

$\mathsf{P}^{\mathsf{SUPP}}$: The class of communication problems $\mathcal{P}$ which can be computed by a constant-cost deterministic protocol with access to an oracle $\mathcal{Q} \in$ SUPP.

We note that several recent works [CLV19, CHHS23, CHH+25, Tom25] have implicitly studied problems in SUPP: the Integer Inner Product functions, which belong to SUPP by definition (the problem asks to decide if integer vectors $u, v \in [-M, M]^d$ are orthogonal in fixed dimension d). For example, [CLV19] proved that these problems form an infinite hierarchy and have efficient (but still super-constant) randomized protocols.

In terms of these complexity classes, we may state non-quantitative versions of our results as:

1. For all constant k, $\mathsf{HD}_{\geq k} \in$ SUPP (Theorem 3.1).

2. $\mathsf{P}^{\mathsf{SUPP}} \subseteq \mathsf{UPP}_0$, the class of problems with constant sign-rank, i.e., constant unbounded-error communication cost (Lemma 2.3)

3. Therefore for all constant k, $\mathsf{P}^{\mathsf{HD}_k} \in \mathsf{UPP}_0$.

4. Constant-order rank problems, including all BPP_0 (constant-cost bounded-error randomized communication) problems from [FGHH25], are in $\mathsf{P}^{\mathsf{SUPP}} \subseteq \mathsf{UPP}_0$ (Section 4).

These new classes might informally be described as "one-sided" versions of UPP_0. By analogy, one might consider similar "one-sided" versions of constant margin, which we call "support margin":

SMAR: The class of communication problems $\mathcal{P}$ for which there exists a constant $\gamma > 0$ such that for all $P \in \mathcal{P}$, there exist vectors $u_1, \ldots, u_N, v_1, \ldots, v_N$ (in arbitrary dimension) such that

$$\begin{aligned} P(i,j) = 0 &\implies \langle u_i, v_j \rangle = 0, \\ P(i,j) = 1 &\implies |\langle u_i, v_j \rangle| > \gamma. \end{aligned} \tag{10}$$

coSMAR: The class of communication problems $\mathcal{P}$ whose negation is in SMAR.

It is not immediately obvious whether these classes are as interesting as SUPP, but we will see below that they are non-trivial classes which contain interesting problems, in particular the problems of constant γ_2-norm.

5.2 Relations Between Classes

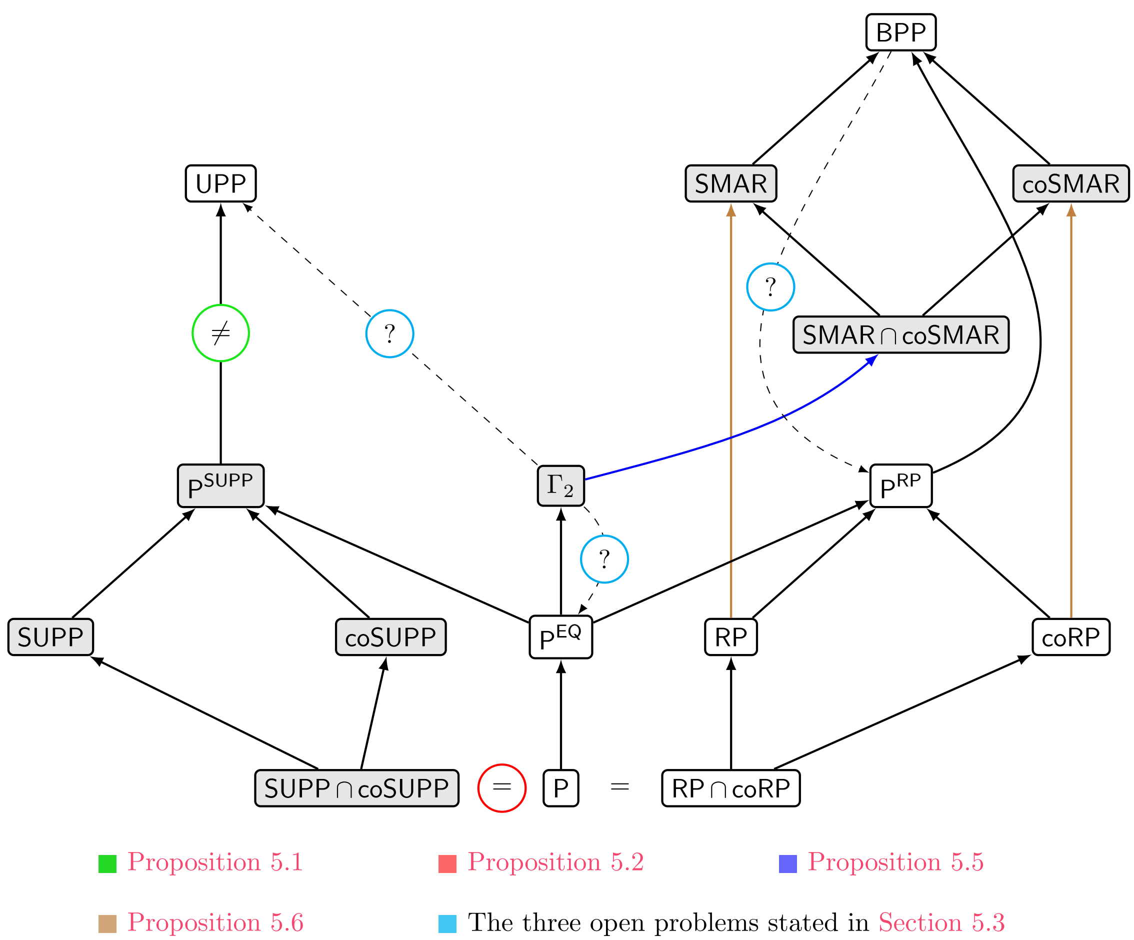


Figure 6: Hierarchy of constant-cost communication classes (with typical subscript 0 dropped). New classes are shaded, and the relations we discuss in this paper are highlighted in color.

Let us discuss how these classes fit within the relevant landscape of constant-cost communication complexity classes, as described in [HH24]. Communication complexity classes are sometimes written with the superscript *cc*, e.g., BPP^{cc}, but we drop this superscript for simplicity. The emerging convention is to write constant-cost communication classes with the subscript 0, e.g., BPP_0, by analogy to constant-depth circuit complexity classes like TC^0 or AC^0, but we will also drop this subscript since we are only concerned with the constant-cost classes. Some standard classes are as follows, and are shown with their relations in Figure 6.

BPP: The class of communication problems $\mathcal{P}$ with $\mathrm{R}(\mathcal{P}) = O(1)$, i.e., problems with constant-cost randomized public-coin bounded-error protocols, or equivalently families of boolean matrices with constant margin.

RP: The class of communication problems $\mathcal{P}$ with $\mathsf{R}_1(\mathcal{P}) = O(1)$, where $\mathsf{R}_1(P)$ denotes the optimal cost of a public-coin randomized protocol with bounded *one-sided error* (i.e., its output is correct with probability 1 on inputs x, y with $P(x, y) = 0$).

UPP: The class of communication problems $\mathcal{P}$ with $\mathrm{U}(\mathcal{P}) = O(1)$, i.e., problems with constant-cost randomized private-coin unbounded-error protocols, or equivalently families of boolean matrices with constant sign-rank.

P: The class of communication problems $\mathcal{P}$ with $\mathsf{D}(\mathcal{P}) = O(1)$, i.e., problems with constant-cost *deterministic* protocols.

P^{EQ}: The class of communication problems $\mathcal{P}$ with $\mathsf{D}^{\mathsf{EQ}}(\mathcal{P}) = O(1)$, i.e., problems with constant-cost deterministic protocols that have access to an Equality oracle.

P^{RP}: The class of communication problems $\mathcal{P}$ with $\mathsf{D}^{\mathcal{Q}}(\mathcal{P}) = O(1)$ for some $\mathcal{Q} \in \mathsf{RP}$.

Γ_2: The class of communication problems $\mathcal{P}$ with constant γ_2-norm (Definition 5.3).

Below, we establish some relationships between classes to fill in Figure 6.

5.2.1 The Classes SUPP, coSUPP, and UPP

Proposition 5.1. $\mathsf{P}^{\mathsf{SUPP}} \subsetneq \mathsf{UPP}$.

The proof follows a similar Ramsey-theoretic strategy as in [FHHH24].

Proof. We show that the Greater-Than problem (which has sign-rank 2 as in Figure 1) does not belong to the class $\mathsf{P}^{\mathsf{SUPP}}$. Suppose for the sake of contradiction that there exist constants q, s such that for all N, the $N \times N$ Greater-Than matrix $\mathsf{GT}_N \in \{0,1\}^{N \times N}$ can be written as

$$\mathsf{GT}_N = \Gamma(Q_1, \dots, Q_q)$$

where $\Gamma\colon \{0,1\}^q \to \{0,1\}$ is applied entry-wise to the matrices $Q_i \in \{0,1\}^{N\times N}$, and each Q_i satisfies $\mathrm{rank}_0(Q_i) \le s$. Consider an auxiliary complete graph G on vertices $[N]$ where each edge $\{x, y\} \in \binom{[N]}{2}$ with $x < y$ is assigned the color

$$\mathsf{col}(\{x, y\}) \coloneqq (Q_i(x, y), Q_i(y, x))_{i\in[q]}\,.$$

In other words, the color of $\{x, y\}$ is the vector in $\{0,1\}^{2q}$, made up of entries in Q_i for row x and column y, as well as row y and column x. Ramsey's theorem guarantees that for any $n \in \mathbb{N}$ there exists sufficiently large N such that there is a set $T \subseteq [N]$ of size $|T| = n$ such that all edges $\{x, y\} \in \binom{T}{2}$ have the same color. Therefore, all $x, y \in T$ with $x < y$ share the same matrix entries $Q_i(x, y)$ and $Q_i(y, x)$. In particular, for each $i \in [q]$ there are $b_i, b_i' \in \{0,1\}$ such that

$$\forall x, y \in T \text{ such that } x < y: \qquad Q_i(x, y) = b_i \text{ and } Q_i(y, x) = b_i'.$$

We argue that there must exist some $i \in [q]$ such that $b_i \neq b'_i$. If this were not the case, then for every $x, y \in T$ with $x < y$, we have

$$\mathsf{GT}_N(x, y) = \Gamma(Q_1(x, y), \ldots, Q_q(x, y)) = \Gamma(Q_1(y, x), \ldots, Q_q(y, x)) = \mathsf{GT}_N(y, x),$$

a contradiction. Therefore, we have Q_i such that

$$Q_i(x, y) = \begin{cases} b_i & \text{if } x < y \\ \neg b_i & \text{if } x > y. \end{cases}$$

Then, Q_i contains an $n \times n$ submatrix on the rows and columns in T, which has all 1s in the upper triangle and all 0s in the lower triangle, or vice versa. This submatrix therefore has support-rank at least $n - 1$. Since this can be found for all values of n, we reach a contradiction. ∎

Proposition 5.2. $\mathsf{SUPP} \cap \mathsf{coSUPP} = \mathsf{P}$.

Proof. It is well-known that the class P consists of exactly the communication problems with constant rank. Then, $\mathsf{P} \subseteq \mathsf{SUPP} \cap \mathsf{coSUPP}$, so we must only show that $\mathsf{SUPP} \cap \mathsf{coSUPP} \subseteq \mathsf{P}$. Let $\mathcal{P}$ be any communication problem in $\mathsf{SUPP} \cap \mathsf{coSUPP}$. There exists a constant s such that for all $P \in \mathcal{P}$, $\mathrm{rank}_0(P) \leq s$ and $\mathrm{rank}_0(\neg P) \leq s$. Lemma 3.6 of [HHH23] shows that for each N there is some r such that every boolean matrix $M \in \{0,1\}^{N \times N}$ with $\mathrm{rank}(M) \geq r$ contains an $N \times N$ submatrix isomorphic to one of

$$\mathsf{EQ}^N, \ \neg\mathsf{EQ}^N, \ \mathsf{GT}^N, \ \neg\mathsf{GT}^N. \tag{11}$$

Suppose for the sake of contradiction that $\mathcal{P} \notin \mathsf{P}$, so that for every r there exists $P \in \mathcal{P}$ with $\mathrm{rank}(P) \geq r$. Then for every $N > s$ there exists $P \in \mathcal{P}$ containing one of the matrices in Equation (11) as a submatrix. But for each of these matrices, either it or its complement has support-rank at least $N > s$, contradiction. ∎

5.2.2 The Classes SMAR, coSMAR, and Γ_2

It is now necessary to define the γ_2-norm and the class Γ_2.

Definition 5.3 (γ_2-Norm and Γ_2)**.** Let $M \in \mathbb{R}^{N \times N}$. The γ_2-norm is defined as

$$\gamma_2(M) \coloneqq \min_{UV=M} \|U\|_{\mathrm{row}} \|V\|_{\mathrm{col}},$$

where the minimum is over matrices U, V with $UV = M$, and $\|U\|_{\mathrm{row}}$ is the maximum ℓ_2-norm of any row of U, while $\|V\|_{\mathrm{col}}$ is the maximum ℓ_2-norm of any column of V. In other words, $\gamma_2(M)$ is the smallest $\lambda > 0$ for which there exist real vectors u_i, v_j satisfying $\|u_i\|_2, \|v_j\|_2 \leq \lambda$ and

$$\forall i, j \in [N] : \qquad M(i, j) = \langle u_i, v_j \rangle$$

We will write Γ_2 for the class of all communication problems $\mathcal{P}$ with $\gamma_2(P) \leq \lambda$ for all $P \in \mathcal{P}$, where λ is some constant only depending on $\mathcal{P}$.

In Figure 6 we have shown that $\mathsf{P}^{\mathsf{EQ}} \subseteq \Gamma_2$, which was proved by [HHH22]. We will require the fact that Γ_2 is closed under negation:

Fact 5.4. *If $\mathcal{P} \in \Gamma_2$ then $\neg\mathcal{P} \in \Gamma_2$.*

Proof. Fix a constant $\lambda > 0$ such that $\gamma_2(P) \leq \lambda$ for all $P \in \mathcal{P}$. Take any $P \in \mathcal{P}$ and write $P = UV$ where $\|U\|_{\mathrm{row}}, \|V\|_{\mathrm{col}} \leq \lambda$. Now, $\neg P = J - UV$, where J is the all-1s matrix. Writing u_i for the i^{th} row of U and v_j for the j^{th} column of V, we have

$$\forall i, j : \qquad \neg P(i, j) = 1 - \langle u_i, v_j \rangle = \langle (1, u_i), (1, -v_j) \rangle,$$

where $\|(1, u_i)\|_2^2, \|(1, -v_j\|_2^2 \leq 1 + \lambda^2$. We conclude that $\gamma_2(\neg P) \leq \sqrt{1 + \lambda^2}$ for all $P \in \mathcal{P}$. ∎

Proposition 5.5. $\Gamma_2 \subseteq \mathsf{SMAR} \cap \mathsf{coSMAR}$.

Proof. Since Γ_2 is closed under negation, it suffices to show $\Gamma_2 \subseteq \mathsf{SMAR}$. Let $\mathcal{P} \in \Gamma_2$ and let $\lambda > 0$ be a constant with $\gamma_2(P) \leq \lambda$ for all $P \in \mathcal{P}$. For any $P \in \mathcal{P}$, write $P = UV$ with $\|U\|_{\mathrm{row}}, \|V\|_{\mathrm{col}} \leq \lambda$, and write u_i and v_j for the i^{th} row of U and j^{th} column of V, respectively. Then, we have

$$P(i,j) = 0 \implies \frac{\langle u_i, v_j \rangle}{\|u_i\|_2 \|v_j\|_2} = 0 \qquad \text{and} \qquad P(i,j) = 1 \implies \frac{|\langle u_i, v_j \rangle|}{\|u_i\|_2 \|v_j\|_2} \geq \frac{1}{\lambda^2}.$$

Thus, the normalized vectors witness that $\mathcal{P} \in \mathsf{SMAR}$ with constant λ^{-2}. ∎

Proposition 5.6. $\mathsf{RP} \subseteq \mathsf{SMAR}$ *and* $\mathsf{coRP} \subseteq \mathsf{coSMAR}$.

Proof. It suffices to show that $\mathsf{RP} \subseteq \mathsf{SMAR}$. Let $\mathcal{P} \in \mathsf{RP}$, so that for some constant $c > 0$, every $P \in \mathcal{P}$ has a one-sided error randomized protocol with cost c. We may assume without loss of generality that the protocol succeeds with probability at least 1/2, and also that the protocol is one-way, i.e., Alice sends a single message to Bob, who then produces the output (this assumption holds because we are interested only in constant cost; see e.g., [HWZ22]).

For each random seed r and inputs x, y, let $a_r(x) \in \{0,1\}^c$ be the message which Alice would send given input x and random seed r and let $B_r(y) \subseteq \{0,1\}^c$ be the subset of messages on which Bob would output 1. Since the protocol has one-sided error, for random $\boldsymbol{r}$, we have

$$\begin{aligned} P(x,y) = 0 &\implies \mathbb{P}\left[a_{\boldsymbol{r}}(x) \in B_{\boldsymbol{r}}(y)\right] = 0, \\ P(x,y) = 1 &\implies \mathbb{P}\left[a_{\boldsymbol{r}}(x) \in B_{\boldsymbol{r}}(y)\right] \geq 1/2. \end{aligned}$$

We may assume that $\boldsymbol{r}$ is drawn uniformly from a finite universe $\{0,1\}^R$. For a subset $S \subseteq \{0,1\}^c$ of messages, let $\chi_S \in \{0,1\}^{2^c}$ be the vector indicating membership in S. Now for each x, y, we may construct vectors $u_x, v_y \in \{0,1\}^{2^{c+r}}$ as the concatenations

$$u_x \coloneqq \left(\chi_{\{a_r(x)\}}\right)_{r \in \{0,1\}^R} \qquad \text{and} \qquad v_y \coloneqq \left(\chi_{B_r(y)}\right)_{r \in \{0,1\}^R}.$$

They have the property that

$$\langle u_x, v_y \rangle = 2^R \cdot \mathbb{P}_{\boldsymbol{r}}\left[\mathbb{1}\left[a_{\boldsymbol{r}}(x) \in B_{\boldsymbol{r}}(y)\right]\right] \begin{cases} = 0 & \text{if } P(x,y) = 0 \\ \geq \frac{1}{2} 2^R & \text{if } P(x,y) = 1. \end{cases}$$

Moreover, the vectors have ℓ_2-norm at most $\sqrt{2^{R+c}}$, so normalizing these vectors gives a margin of at least 2^{-c-1} in the case $P(x,y) = 1$. ∎

Proposition 5.7. $\mathsf{P}^{\mathsf{SMAR} \cap \mathsf{coSMAR}} = \mathsf{SMAR} \cap \mathsf{coSMAR}$.

Proof. Since $\mathsf{SMAR} \cap \mathsf{coSMAR}$ is closed under negations, it suffices to show that SMAR is closed under OR, so that coSMAR is closed under AND and their intersection is closed under all boolean operations. We now let $\mathcal{P}, \mathcal{Q} \in \mathsf{SMAR}$ so that there exists a constant $\gamma > 0$ such that all $\mathcal{P}$ and $\mathcal{Q}$ satisfy Equation (10) with constant γ. For any $N \times N$ matrices $P \in \mathcal{P}_N$ and $Q \in \mathcal{Q}_N$ and each $i, j \in [N]$, let u_i, v_j be the unit vectors witnessing Equation (10) for P, and define u'_i, v'_j similarly for Q.

The normalized concatenations $\left(\frac{\sqrt{\gamma}}{\sqrt{2}} u_i, u'_i\right)$ and $\left(\frac{\sqrt{\gamma}}{\sqrt{2}} v_j, v'_j\right)$, both with ℓ_2-norm $\sqrt{1 + \gamma/2}$, satisfy

$$\begin{aligned} (P \vee Q)(i,j) = 0 &\implies \langle \left(\sqrt{\tfrac{\gamma}{2}} u_i, u'_i\right), \left(\sqrt{\tfrac{\gamma}{2}} v_j, v'_j\right) \rangle = 0, \\ (P \vee Q)(i,j) = 1 &\implies \left| \langle \left(\sqrt{\tfrac{\gamma}{2}} u_i, u'_i\right), \left(\sqrt{\tfrac{\gamma}{2}} v_j, v'_j\right) \rangle \right| = \left| \tfrac{\gamma}{2} \langle u_i, v_j \rangle + \langle u'_i, v'_j \rangle \right| \geq \min\left(\frac{\gamma^2}{2}, \frac{\gamma(1-\gamma)}{2}\right), \end{aligned}$$

where the last inequality holds because $|\langle u_i, v_j \rangle| \geq \gamma$ or $|\langle u'_i, v'_j \rangle| \geq \gamma$. Therefore, $\mathcal{P} \vee \mathcal{Q} \in \mathsf{SMAR}$. ∎

5.3 Open Problems

We have added some classes to the hierarchy in Figure 6, which suggests some new open problems. Recall that Conjecture 1.5 asks to separate BPP_0 from UPP_0 (i.e., show $\mathsf{BPP}_0 \setminus \mathsf{UPP}_0 \neq \emptyset$). The introduction of support-rank suggests an intermediate problem:

Open Problem 5.8. *Is* $\mathsf{BPP}_0 \setminus \mathsf{P}_0^{\mathsf{SUPP}} \neq \emptyset$*? (Is there a problem with constant bounded-error randomized cost, which cannot be reduced to any problem of constant support-rank?)*

Our results show that all examples in BPP_0 known up to [FGHH25] are also contained in $\mathsf{P}_0^{\mathsf{SUPP}}$. This leaves the recent examples of [SS24] as promising candidates for proving this separation.

Another question suggested by new classes in Figure 6 is whether $\mathsf{SMAR} \cap \mathsf{coSMAR} = \Gamma_2$. We refer to [HH24] for many other open problems regarding the complexity classes in Figure 6, and more. Let us repeat two of the most interesting, originally from [HHH22]:

Open Problem 5.9. *Is* $\mathsf{BPP}_0 = \mathsf{P}_0^{\mathsf{RP}}$*?*

Open Problem 5.10. *Is* $\mathsf{P}_0^{\mathsf{EQ}} = \Gamma_2$*?*

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