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# Equality is Far Weaker than Constant-Cost Communication

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**Abstract**

We exhibit an $n$-bit communication problem with a constant-cost randomized protocol but which requires $n^{\Omega(1)}$ deterministic (or even non-deterministic) queries to an Equality oracle. Therefore, even *constant-cost* randomized protocols cannot be efficiently "derandomized" using Equality oracles. This improves on several recent results and answers a question from the survey of Hatami and Hatami (SIGACT News 2024). It also gives a significantly simpler and quantitatively superior proof of the main result of Fang, Göös, Harms, and Hatami (STOC 2025), that constant-cost communication does not reduce to the $k$-Hamming Distance hierarchy.

## 1 Introduction

In this paper we prove strong limits on the power of "simple hashing" for the purpose of communication. In communication complexity, simple hashing, i.e. the Equality communication problem, is the most dramatic example of the power of randomness. Alice and Bob each have binary strings $x, y \in \{0,1\}^n$ and their goal is to decide whether $x = y$. With public randomness, they can generate a random 2-bit hash $\boldsymbol{h}(z)$ for every $z$ and check if $\boldsymbol{h}(x) = \boldsymbol{h}(y)$. This succeeds with probability at least $3/4$ and requires only 2 bits of communication regardless of the size of the input $n$ [KN96, RY20].

Randomized communication can often be "derandomized" by allowing Alice and Bob access to an oracle which (deterministically) computes Equality. Write $\mathsf{D}^{\mathrm{EQ}}(F)$ for the minimum cost of a deterministic protocol computing $F$ with access to this oracle, and $\mathsf{R}(F)$ for the public-coin randomized communication cost (with error probability $1/3$). Here are three examples covering a range of complexities:

**Example 1** (Greater Than)**.** Alice and Bob are given $n$-bit integers $x, y$ and their goal is to decide whether $x > y$. They can perform binary search to find the highest-order bit where $x$ and $y$ differ, by querying the Equality oracle (check if the first half of their bits are equal, etc.), so $\mathsf{D}^{\mathrm{EQ}}(\mathrm{GT}_n) = O(\log n)$, where $\mathrm{GT}_n$ denotes the Greater-Than problem on $n$ bits. Randomized communication satisfies $\mathsf{R}(F) = O(\mathsf{D}^{\mathrm{EQ}}(F))$ [Nis93, HR24], which in this case gives the optimal randomized protocol: $\mathsf{R}(\mathrm{GT}_n) = \Theta(\log n)$ [Nis93, BW16, Vio15, SY23].

**Example 2** (Planar Adjacency [Har20, HWZ22])**.** Alice and Bob are given vertices $x, y$ in a (shared) planar graph $P$ and wish to decide if they are adjacent. A planar graph is the edge union of 3 forests, so they can check adjacency in $P$ by checking adjacency in each forest: $x$ is equal to the parent of $y$ or vice versa. So $\mathsf{D}^{\mathrm{EQ}} \leq 6$ for this problem and therefore its randomized cost is *constant* (independent of the input size), which is clearly optimal.



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**Example 3** (1-Hamming Distance)**.** Alice and Bob are given $x, y \in \{0,1\}^n$ and their goal is to decide if $x, y$ differ on at most 1 bit. Using binary search with the Equality oracle, we get $\mathsf{D}^{\mathrm{EQ}}(\mathrm{HD}_1^n) = O(\log n)$, where $\mathrm{HD}_1^n$ denotes the 1-Hamming Distance problem on $n$ bits. This protocol is *not* optimal, since $\mathsf{R}(\mathrm{HD}_1^n) = O(1)$, but it is still *efficient*, i.e. $\mathsf{D}^{\mathrm{EQ}}(\mathrm{HD}_1^n) = \operatorname{poly}\log n$.

Can *every* efficient randomized protocol be replaced with an Equality oracle protocol? This is especially interesting for problems like Planar Adjacency and 1-Hamming Distance that have *constant* randomized cost, because one may reasonably expect the answer to be *yes*, i.e. any constant-cost randomized protocol can be replaced with $\operatorname{poly}\log n$ Equality queries, as in Example 3. On the contrary, we show:

**Theorem 1.** *There exists a communication problem* $F\colon \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ *with*

$$\mathsf{R}(F) = O(1) \qquad \text{and} \qquad \mathsf{D}^{\mathrm{EQ}}(F) = \Theta(\sqrt{n}).$$

In terms of complexity classes, this implies $\mathsf{BPP}_0 \not\subseteq \mathsf{P}^{\mathrm{EQ}}$, where $\mathsf{BPP}_0$ is the class of randomized constant-cost problems and $\mathsf{P}^{\mathrm{EQ}}$ is the class of problems with Equality oracle cost $\operatorname{poly}\log n$.

Communication oracles including Equality, $\mathrm{HD}_1$, and others, are well studied [CLV19, HHH23, HWZ22, EHK22, PSS23, CHHS23, CHZZ24, HZ24, FHHH24, FGHH25, CHH+25]. Theorem 1 improves on results from several of these works and our proof answers a number of open questions.

**Separation of $\mathsf{R}(F)$ and $\mathsf{D}^{\mathrm{EQ}}(F)$.** Whether Equality oracles can efficiently derandomize communication was first asked in [CLV19], who showed that there exists a problem $F$ with

$$\mathsf{R}(F) = O(\log n) \qquad \text{and} \qquad \mathsf{D}^{\mathrm{EQ}}(F) = \Theta(n).$$

This established that $\mathsf{BPP} \not\subseteq \mathsf{P}^{\mathrm{EQ}}$, where $\mathsf{BPP}$ is the class of $n$-bit communication problems with randomized communication cost $\operatorname{poly}\log n$. This separation was improved in [CHH+25] to hold for XOR functions (see also [CHHS23, PSS23, Tom25]).

Recently, there has been significant effort to understand the most extreme examples of the power of randomness in communication, i.e. the problems with constant cost. The question of how well Equality oracles can derandomize constant cost communication was posed in [HHH23, HWZ22], who showed that Equality is not "complete" for the class of constant-cost problems, i.e. $\mathsf{BPP}_0 \not\subseteq \mathsf{P}_0^{\mathrm{EQ}}$, where $\mathsf{P}_0^{\mathrm{EQ}}$ is the class of problems $F$ with $\mathsf{D}^{\mathrm{EQ}}(F) = O(1)$. But the best known lower bound on $\mathsf{D}^{\mathrm{EQ}}$ for a constant-cost problem is from [HHH23] (see also [HR24]), who proved a $\mathsf{D}^{\mathrm{EQ}}$ lower bound matching Example 3:

$$\mathsf{R}(\mathrm{HD}_1) = O(1) \qquad \text{and} \qquad \mathsf{D}^{\mathrm{EQ}}(\mathrm{HD}_1) = \Theta(\log n).$$

This leads to the question, implicit in [CHHS23] and explicit in the survey [HH24], of whether constant-cost protocols can be replaced with $\operatorname{poly}\log n$ Equality queries. We answer *no*, improving the separation $\mathsf{BPP} \not\subseteq \mathsf{P}^{\mathrm{EQ}}$ to $\mathsf{BPP}_0 \not\subseteq \mathsf{P}^{\mathrm{EQ}}$. Our example $F$ is an XOR function, so we also improve on the separation of [CHH+25].

**Separation of $\mathsf{BPP}_0$ and $k$-Hamming Distance.** Following the result that Equality is not complete for $\mathsf{BPP}_0$, [FGHH25] showed that the infinite hierarchy of $k$-Hamming Distance problems $\mathrm{HD}_k$ (decide if $x, y$ differ on at most $k$ bits) is also not complete: there exists $F$ with $\mathsf{R}(F) = O(1)$ and yet $\mathsf{D}^{\mathrm{HD}_k}(F) = \omega(1)$ for every constant $k$, where $\mathsf{D}^{\mathrm{HD}_k}(F)$ is the minimum

number of queries to an $\mathrm{HD}_k$ oracle required to compute $F$. We give a significantly simpler and quantitatively superior proof (for a different $F$), with a lower bound of $\tilde{\Omega}(\sqrt{n})$ instead of $\omega(1)$:

**Corollary 2.** *There exists an $n$-bit function $F$ such that for any constant $k$,*

$$\mathsf{R}(F) = O(1) \qquad \textit{and} \qquad \mathsf{D}^{\mathrm{HD}_k}(F) \geq \tilde{\Omega}(\sqrt{n}).$$

### Two Proofs with Five Corollaries

We give two incomparable proofs of Theorem 1: one analytic, one combinatorial, each with different consequences beyond the ones above, which we elaborate in Section 5.

**Analytic proof: $\gamma_2$-norm and decision trees.** We show in Theorem 6 that the $\gamma_2$-*norm* of our problem $F$ (equivalently, the spectral norm of $f$, see Section 5.2) is $2^{\Omega(\sqrt{n})}$. As a consequence:

- Our function $F$ is an XOR function, meaning $F(x, y) = f(x \oplus y)$ for some function $f$. Since XOR functions $F(x, y) = f(x \oplus y)$ satisfy the property that the $\gamma_2$-norm of $F$ is equal to the Fourier spectral norm of $f$ [HHH23], we exhibit a function $f$ with *approximate* spectral norm $O(1)$ yet *exact* spectral norm $2^{\Omega(\sqrt{n})}$ (Corollary 15).
- We answer [CHH+25, Question 8], asking whether there exists a function $f\colon \{0,1\}^n \to \{0,1\}$ with randomized parity decision tree size $O(1)$ but deterministic parity decision tree size $n^{\omega(1)}$. Our $f$ has deterministic parity decision tree size $2^{\Omega(\sqrt{n})}$ (Corollary 16).
- We make progress on [CHHS23, Question 5], asking whether any $n$-bit function with $\mathsf{R}(F) = O(1)$ has $\gamma_2$-norm $2^{\Omega(n)}$. Our function does not achieve this maximal value but it improves exponentially on the best known value of $\Theta(\sqrt{n})$, for the $\mathrm{HD}_1$ function [HHH23].

**Combinatorial proof: non-deterministic Equality protocols.** Using the *blocky cover number* of [PSS23], we show that $F$ has Equality oracle cost $\Omega(\sqrt{n})$ even for a *non-deterministic* protocol. As a consequence of our proof:

- We improve the complexity class separation in Theorem 1, from $\mathsf{BPP}_0 \not\subseteq \mathsf{P}^{\mathrm{EQ}}$ to $\mathsf{BPP}_0 \not\subseteq \mathsf{NP}^{\mathrm{EQ}}$. This improves on a result of [PSS23] who showed $\mathsf{BPP} \not\subseteq \mathsf{NP}^{\mathrm{EQ}}$.
- We tighten the theorem of [PSS23] relating the *blocky cover number* and the non-deterministic Equality oracle cost $\mathsf{ND}^{\mathrm{EQ}}(\cdot)$. This improves the lower bounds on $\mathsf{ND}^{\mathrm{EQ}}(\cdot)$ given in [PSS23, CHH+25] and answers the question posed after [CHH+25, Theorem 5], asking if $\mathsf{ND}^{\mathrm{EQ}}(\mathrm{IIP}_3^n) = \Omega(n)$, where $\mathrm{IIP}_3^n$ is the Integer Inner Product function defined in [CLV19].

## 2 The Problem

The problem is a special case of those studied recently by Sherstov & Storozhenko [SS24]. Alice and Bob are given matrices $A, B \in \mathbb{F}_2^{n\times n}$ and their goal is to decide if the $\mathbb{F}_2$-rank of $A + B$ is at most 1; i.e. $\textsc{RankOne}_n\colon \{0,1\}^{n\times n} \times \{0,1\}^{n\times n} \to \{0,1\}$ is defined by

$$\textsc{RankOne}_n(A, B) := \begin{cases} 1 & \text{if } A \oplus B \text{ has rank} \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that a matrix $C \in \{0,1\}^{n\times n}$ has rank $\leq 1$ if and only if its 1-entries form a combinatorial rectangle, i.e. there are sets $X, Y \subseteq [n]$ such that $C(x,y) = 1$ iff $(x,y) \in X \times Y$. This problem is an XOR problem, i.e. $\textsc{RankOne}_n(A,B) = f(A \oplus B)$ for the boolean function $f\colon \{0,1\}^{n\times n} \to \{0,1\}$ defined as $f(M) = 1$ if and only if $M$ has rank at most 1.

Sherstov & Storozhenko [SS24] prove tight bounds for the randomized communication complexity of computing rank $r$ over any finite field. For the sake of completeness, we include a simple proof that $\mathsf{R}(\textsc{RankOne}_n) = O(1)$. More specifically, we prove that the randomized parity decision tree depth of $f$ is $O(1)$ (which immediately implies the same for randomized communication).

A function $g\colon \{0,1\}^n \to \{0,1\}$ is computed by a depth-$d$ parity decision tree if, for any $x \in \{0,1\}^n$, $g(x)$ can be computed with at most $d$ adaptive queries of the form $\oplus_{i\in S} x_i$ for chosen $S \subseteq [n]$. A function $g$ is computed by a *randomized* parity decision tree if there exists a distribution $\mu$ over parity decision trees such that for every input $x \in \{0,1\}^n$, $\Pr_{\boldsymbol{T}\sim\mu}[\boldsymbol{T}(x) = g(x)] \geq 2/3$.

**Theorem 3.** *Let $f\colon \{0,1\}^{n\times n} \to \{0,1\}$ be as above: $f(M) = 1$ iff $\mathrm{rk}_{\mathbb{F}_2}(M) \leq 1$. Then it is computed by a randomized parity decision tree of constant depth (i.e. independent of $n$).*

*Proof.* Given input $M$, the tree chooses sets $\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{B}_1, \boldsymbol{B}_2 \subseteq [n]$ uniformly at random. For each $\alpha, \beta \in [2]$, the tree queries $\boldsymbol{C}_{\alpha,\beta} := \bigoplus_{(i,j)\in \boldsymbol{A}_\alpha \times \boldsymbol{B}_\beta} M_{ij}$, to form the $2\times 2$ matrix $\boldsymbol{C}$. The tree outputs 1 iff $\mathrm{rk}(\boldsymbol{C}) \leq 1$.

The transformation from $M$ to $\boldsymbol{C}$ can be seen as a two-step process: first, we generate a $n \times 2$ matrix $\boldsymbol{C}'$ where column $i \in [2]$ is the sum of columns $j \in \boldsymbol{B}_i$ of $M$. Then we generate the $2 \times 2$ matrix $\boldsymbol{C}$ where row $j$ is the sum of rows $i \in \boldsymbol{A}_j$ of $\boldsymbol{C}'$. Neither step may increase the rank of the matrix, so if $\mathrm{rk}(M) \leq 1$, then $\Pr[\mathrm{rk}(\boldsymbol{C}) \leq 1] = 1$.

Now suppose $\mathrm{rk}(M) \geq 2$. Each column of $\boldsymbol{C}'$ is chosen uniformly from the column space of $M$. The size of the column space is at least 4, so the probability that both columns of $\boldsymbol{C}'$ are nonzero and distinct is at least $\frac{3}{4}\cdot\frac{1}{2} = \frac{3}{8}$. Conditional on this event, by the same argument, the probability that both rows of $\boldsymbol{C}$ are nonzero and distinct is at least $\frac{3}{8}$. Therefore, the tree will output 0 with probability at least $9/64$. We can boost this to $2/3$ by repetition. ∎

To clarify notation, observe that the inputs to this function are naturally expressed with $n^2$ variables, whereas Theorem 1 is stated for a function on $n$ variables. So to prove Theorem 1 we must show

$$\mathsf{D}^{\mathrm{EQ}}(\textsc{RankOne}_n) = \Theta(n). \tag{1}$$

The upper bound is simple:

**Proposition 4.** $\mathsf{D}^{\mathrm{EQ}}(\textsc{RankOne}_n) = O(n)$.

*Proof.* Suppose Alice receives a matrix $A$ with rows $a_1, \ldots, a_n \in \{0,1\}^n$, and Bob receives a matrix $B$ with rows $b_1, \ldots, b_n \in \{0,1\}^n$. Then $\mathrm{rk}(A \oplus B) \leq 1$ if and only if each pair $i, j \in [n]$ with $a_i \neq b_i$ and $a_j \neq b_j$ satisfy $a_i \oplus b_i = a_j \oplus b_j$. Using $n$ queries $\textsc{Eq}(a_1, b_1), \textsc{Eq}(a_2, b_2), \ldots, \textsc{Eq}(a_n, b_n)$, Alice and Bob can find $i$ such that $a_i \neq b_i$ if such an $i$ exists (otherwise $\mathrm{rk}(A \oplus B) = 0$). Then for every $j \in [n]$ such that $a_j \neq b_j$ they check that $a_j \oplus b_j = a_i \oplus b_i$ by making the query $\textsc{Eq}(a_j \oplus a_i, b_j \oplus b_i)$. If any Equality-query comes out negative, Alice and Bob output 0, otherwise they output 1. ∎

## 3 Analytic Proof

We analyze the $\gamma_2$-norm of $\textsc{RankOne}_n$. The $\gamma_2$-norm of a matrix $M \in \mathbb{R}^{N\times N}$ is defined as

$$\gamma_2(M) := \min_{M=UV} \|U\|_{\mathrm{row}} \|V\|_{\mathrm{col}},$$

where the minimum is over matrices $U, V \in \mathbb{R}^d$ (in any dimension $d$) satisfying $M = UV$, $\|U\|_{\text{row}}$ is the maximum $\ell_2$-norm of any row of $U$, and $\|V\|_{\text{col}}$ is the maximum $\ell_2$-norm of any column of $V$. The $\gamma_2$-norm is related to the EQUALITY oracle cost by

$$\forall M \in \{0,1\}^{N \times N} : \qquad \frac{1}{2} \log \gamma_2(M) \leq \mathsf{D}^{\text{EQ}}(M) \qquad ([\text{HHH23}]). \tag{2}$$

To give a lower bound on $\gamma_2$, we use the Hölder's inequality framework introduced in [CHH+25]. Write $\|M\|_F := (\sum_{i,j \in [N]} M_{i,j}^2)^{1/2}$ for the Frobenius norm of a matrix $M \in \mathbb{R}^{N \times N}$. Then:

**Lemma 5** (Corollary of Hölder's inequality, [CHH+25])**.** *For any matrix $M \in \mathbb{R}^{N \times N}$,*

$$\gamma_2(M) \geq \frac{1}{N} \cdot \frac{\|M\|_F^3}{\sqrt{\text{tr}((M^T M)^2)}}.$$

In the next theorem, note that the number of bits in the input of $\text{RANKONE}_n$ is $n^2$, not $n$; by renaming the number of bits and applying Equation (2), we get the bound in Theorem 1.

**Theorem 6.** *$\gamma_2(\text{RANKONE}_n) = 2^{\Theta(n)}$. As a consequence, $\mathsf{D}^{\text{EQ}}(\text{RANKONE}_n) = \Omega(n)$.*

*Proof.* The upper bound follows from Proposition 4 and Equation (2). For the lower bound, we shall apply Lemma 5 to the matrix $\text{RANKONE}_n \in \{0,1\}^{N \times N}$ where $N = 2^{n^2}$. For convenience we write $M := \text{RANKONE}_n$. Since this matrix is boolean, the Frobenius norm is $\|M\|_F = (\sum_{i,j \in [N]} M_{i,j}^2)^{1/2} = (\sum_{i,j \in [N]} M_{i,j})^{1/2}$, so it suffices to count the number of 1-valued entries. Rows of $M$ are identified with matrices $A \in \{0,1\}^{n \times n}$. There are $2^{2n}$ rectangles $R \subseteq [n] \times [n]$ and they are in 1-to-1 correspondence with rank-1 matrices $Z \in \{0,1\}^{n \times n}$; here and in what follows, let us abuse language and consider the all-0 matrix (corresponding to the empty rectangle) to also be "rank-1". For fixed row $A$, each rank-1 matrix $Z$ has a unique $B \in \{0,1\}^{n \times n}$ such that $A \oplus B = Z$. Therefore the number of 1-valued entries in each row $A$ is $2^{2n}$. So

$$\|M\|_F = \sqrt{N \cdot 2^{2n}} = 2^n \cdot \sqrt{N}. \tag{3}$$

Now we bound the denominator in Lemma 5,

$$\text{tr}((M^T M)^2) = \sum_{x,y,z,w \in [N]} M_{x,z} M_{x,w} M_{y,z} M_{y,w}.$$

Rows and columns $x, y, z, w \in [N]$ are identified with matrices $X, Y, Z, W \in \{0,1\}^{n \times n}$. The product $M_{x,z} M_{x,t} M_{y,z} M_{y,w}$ equals 1 iff $R_1 := X \oplus Z$, $R_2 := X \oplus W$, $R_3 := Y \oplus Z$, and $R_4 := Y \oplus W$ are all rank-1 matrices in $\{0,1\}^{n \times n}$. Then the quadruple $(X, Y, Z, W)$ is uniquely determined by $(X, R_1, \ldots, R_4)$ which in turn is determined by $(X, R_1, R_2, R_3)$ since $R_4 = R_1 \oplus R_2 \oplus R_3$. So:

$$\sum_{x,y,z,w} M_{x,z} M_{x,t} M_{y,z} M_{y,w} = N \cdot \left| \{(R_1, R_2, R_3) \in (\{0,1\}^{n \times n})^3 \mid \text{rk}(R_i) \leq 1;\ \text{rk}(R_1 \oplus R_2 \oplus R_3) \leq 1\} \right|.$$

Now it remains to bound the number of triples $(R_1, R_2, R_3)$ of rank-1 matrices that sum to a rank-1 matrix. Recall that rank-1 matrices over $\mathbb{F}_2$ are precisely the matrices whose 1-entries form a combinatorial rectangle, let $A_i \times B_i \subseteq [n]^2$ be this rectangle for $R_i$ for $i \in [3]$.

We say that two sets $A, B$ are in *general position* if they are not disjoint and neither set is a subset of another. In other words $A \setminus B$, $A \cap B$, and $B \setminus A$ are all non-empty. We first show

that there are at most $6 \cdot 3^n \cdot 2^{4n}$ triples $(R_1, R_2, R_3)$ where the sides of $R_1, R_2$ are *not* in general position. Then we show that, if the sides of the first two rectangles $R_1, R_2$ *are* in general position, there is only a constant number of choices for $R_3$, meaning that there are at most $9 \cdot 2^{4n}$ of these triples. Therefore

$$\operatorname{tr}((M^T M)^2) = \sum_{x,y,z,w} M_{x,z} M_{x,t} M_{y,z} M_{y,w} \leq N \left(6 \cdot 3^n \cdot 2^{4n} + 9 \cdot 2^{4n}\right) \leq 15 \cdot N \cdot 3^n \cdot 2^{4n}.$$

Then Lemma 5 implies

$$\gamma_2(\textsc{RankOne}_n) \geq \frac{1}{N} \cdot \frac{N^{3/2} 2^{3n}}{\sqrt{15} \cdot \sqrt{N} \cdot (3 \cdot 2^4)^{n/2}} = \frac{1}{\sqrt{15}} \cdot \left(\frac{8}{\sqrt{3} \cdot 4}\right)^n = 2^{\Omega(n)},$$

which concludes the proof, once we establish the claimed bounds on the number of triples.

**Structured pairs.** We count the number of triples $(R_1, R_2, R_3)$ where $R_1 = A_1 \times B_1$ and $R_2 = A_2 \times B_2$ have either $A_1, A_2$ or $B_1, B_2$ not in general position. First we count the number of choices for $A_1, A_2$ where $A_1 \subseteq A_2$. This is at most $3^n$ because each $i \in [n]$ can be included in $A_2 \setminus A_1$, $A_1$, or neither. Therefore the number of triples $(R_1, R_2, R_3)$ where $A_1 \subseteq A_2$ is at most $3^n \cdot (2^n)^4$, accounting for the choices of $A_3, B_1, B_2, B_3$. Similar arguments hold for the other cases of non-general position of $A_1, A_2$ or $B_1, B_2$. There are 6 cases in total, so the number of triples $(R_1, R_2, R_3)$ is at most $6 \cdot 3^n \cdot 2^{4n}$.

**General position.** We count the number of triples $(R_1, R_2, R_3)$ which sum to a rank-1 matrix, and where $A_1, A_2$ and $B_1, B_2$ are both in general position. We claim that there is only a constant number options for $R_3 = A_3 \times B_3$ such that $\operatorname{rk}(R_1 \oplus R_2 \oplus R_3) = 1$. Let us first rule out the case where $A_3 \supseteq A_1 \cup A_2$ or $B_3 \supseteq B_1 \cup B_2$. Let us take representatives $i, j, k$ from $A_1 \setminus A_2$, $A_1 \cap A_2$, $A_2 \setminus A_1$ respectively, and $i', j', k'$ from $B_1 \setminus B_2$, $B_1 \cap B_2$, $B_2 \setminus B_1$ respectively. The submatrix of $R_1 \oplus R_2$ on rows $\{i, j, k\}$ and columns $\{i', j', k'\}$ is

$$P := (R_1 \oplus R_2)|_{\{i,j,k\} \times \{i',j',k'\}} = J_3 \oplus I_3,$$

where $J_3$ is $3 \times 3$ all-1 matrix and $I_3$ is the $3 \times 3$ identity matrix. Then if $A_3 \supseteq \{i, j, k\}$ then $P \oplus R_3|_{\{i,j,k\} \times \{i',j',k'\}}$ has either two distinct columns from $J_3 \oplus I_3$ or two distinct columns from $I_3$, in both cases $\operatorname{rk}(R_1 \oplus R_2 \oplus R_3) \geq 2$. The same argument shows that $B_3 \not\supseteq \{i', j', k'\}$.

Then there are a row and a column that are nonzero in $R_1 \oplus R_2$ and zero in $R_3$. Suppose that their values in $R_1 \oplus R_2$ are $a \in \{0,1\}^n$ and $b \in \{0,1\}^n$, respectively. Since $\operatorname{rk}(R_1 \oplus R_2 \oplus R_3) = 1$, any two nonzero rows of $R_1 \oplus R_2 \oplus R_3$ must be equal, and any two nonzero columns must also be equal. Since $a, b$ are nonzero, this requires $R_1 \oplus R_2 \oplus R_3 = a \cdot b^T$. For fixed $R_1, R_2$, there are only three possible values of $a$ and $b$, so there are at most 9 options for the value of $R_3$. Therefore there are at most $9 \cdot (4^n)^2$ triples $(R_1, R_2, R_3)$ in this case. ■

# 4 Combinatorial Proof

We prove a lower bound on the non-deterministic Equality oracle cost of $\textsc{RankOne}_n$. The non-deterministic Equality oracle cost of any matrix $M \in \{0,1\}^{N \times N}$ is denoted $\mathsf{ND}^{\mathrm{EQ}}(M)$. It is defined as the minimum value $t$ such that there are numbers $m, d$ with $m + d = t$, where there exist matrices $M_1, \ldots, M_{2^m} \in \{0,1\}^{N \times N}$ satisfying:

1. $\mathsf{D}^{\mathrm{EQ}}(M_i) \leq d$ for all $i \in [2^m]$;

2. If $M(x,y) = 1$ then $\exists i \in [2^m]$, $M_i(x,y) = 1$; and
3. If $M(x,y) = 0$ then $\forall i \in [2^m]$, $M_i(x,y) = 0$.

By definition, $\mathsf{ND}^{\mathrm{EQ}}(M) \le \mathsf{D}^{\mathrm{EQ}}(M)$. To prove a lower bound on $\mathsf{ND}^{\mathrm{EQ}}$, we require the *blocky cover number* defined in [PSS23].

**Definition 7** (Blocky Cover)**.** A matrix $B \in \{0,1\}^{N\times N}$ is called *blocky* if it is obtained from an identity matrix $I_{m,m}$ with $m \le N$ by duplicating rows or columns, permuting rows or columns, or adding all-0 rows or columns. Let $M \in \{0,1\}^{N\times N}$ be a boolean matrix. The *blocky cover number* $\mathrm{bc}(M)$ is the minimum number $r$ such that there exist $r$ blocky matrices $B_1, \dots, B_r \in \{0,1\}^{N\times N}$ with $M = \bigvee_{i=1}^{r} B_i$, where $\bigvee$ denotes entrywise OR.

Blocky covers are related to non-deterministic Equality oracle cost by the following inequality from [PSS23]. For all $A \in \{0,1\}^{N\times N}$,

$$\mathsf{ND}^{\mathrm{EQ}}(A) \le \log \mathrm{bc}(A) \le O(\mathsf{ND}^{\mathrm{EQ}}(A) \cdot \log\log N). \tag{4}$$

In Appendix A, we improve the upper bound in this inequality:

**Lemma 8.** *For any $A \in \{0,1\}^{N\times N}$,*

$$\mathsf{ND}^{\mathrm{EQ}}(A) \le \log \mathrm{bc}(A) \le \mathsf{ND}^{\mathrm{EQ}}(A) + O(\log\log N).$$

We may now improve Theorem 1 to $\mathsf{ND}^{\mathrm{EQ}}(\textsc{RankOne}_n) = \Omega(n)$ with the following bound on blocky cover number.

**Theorem 9.** $\mathrm{bc}(\textsc{RankOne}_n) = 2^{\Omega(n)}$. *As a consequence,* $\mathsf{ND}^{\mathrm{EQ}}(\textsc{RankOne}_n) = \Omega(n)$.

*Proof.* In this proof, write $M^{-1}(1)$ for the set of 1-valued entries in a boolean matrix $M$, and write $|M| = |M^{-1}(1)|$ for the number of 1-valued entries. We use standard notation $\mathrm{H}(\boldsymbol{Z})$ for the binary entropy of a random variable $\boldsymbol{Z}$, so $\mathrm{H}(\boldsymbol{Z}) := -\sum_{z\in\mathcal{Z}} \Pr[\boldsymbol{Z} = z] \log \Pr[\boldsymbol{Z} = z]$.

Recall from Equation (3) that $|\textsc{RankOne}_n| = 2^{n^2} \cdot 2^{2n}$. A rectangle $R = U \times V$ is *1-chromatic* if, for all $(u,v) \in U \times V$, $\textsc{RankOne}_n(u,v) = 1$. Let $\mathrm{bc}(\textsc{RankOne}_n) = K$, so $\textsc{RankOne}_n = \bigvee_{i\in[K]} B_i$, where the $B_i$ are blocky matrices, which can be written as a disjoint union of 1-chromatic rectangles $B_i = \bigvee_{j\in[b_j]} R_{i,j}$ where $R_{i,j} = U_{i,j} \times V_{i,j}$ is a 1-chromatic rectangle in $B_i$.

**Claim 10.** *There exists a 1-chromatic rectangle $R = U \times V$ with $|U|, |V| \ge 2^{2n}/(2K)$.*

*Proof of claim.* For the sake of contradiction, assume each 1-chromatic rectangle $R = U \times V$ has $|U| < 2^{2n}/(2K)$, in which case we call the rectangle *short*, or $|V| < 2^{2n}/(2K)$, in which case we call

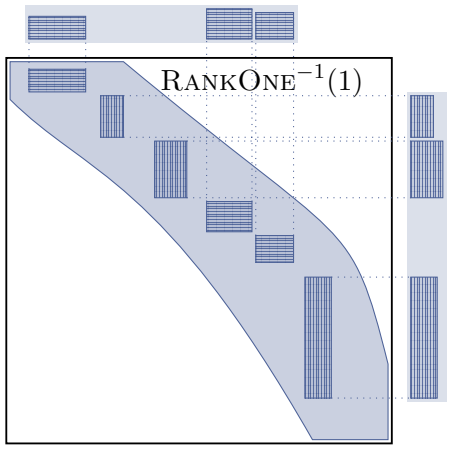


Figure 1: Illustration of Claim 10. The rectangles represent the 1-entries of a blocky matrix $B$ with $B^{-1}(1) \subseteq \textsc{RankOne}^{-1}(1)$. Short rectangles are filled with horizontal lines and narrow ones are filled with the vertical lines. We then can conclude that the total area of each type of the rectangles is small.

it *narrow*. Then by definition of $K$, we obtain a contradiction as follows:

$$\begin{aligned}
2^{n^2} \cdot 2^{2n} = |\textsc{RankOne}_n| &\le \sum_{i=1}^{K} |B_i| = \sum_{i=1}^{K} \sum_{j=1}^{b_i} |U_{i,j}| \cdot |V_{i,j}| \\
&< \sum_{i=1}^{K} \left( \sum_{j \in [b_i], R_{i,j} \text{ short}} |V_{i,j}| \cdot 2^{2n}/(2K) + \sum_{j \in [b_i], R_{i,j} \text{ narrow}} |U_{i,j}| \cdot 2^{2n}/(2K) \right) \\
&\le \tfrac{1}{2} 2^{2n} \max_{i \in [K]} \left( \sum_{j \in [b_i]} |V_{i,j}| + \sum_{j \in [b_i]} |U_{i,j}| \right) \\
&\le \tfrac{1}{2} 2^{2n} \max_{i \in [K]} \left( 2^{n^2} + 2^{n^2} \right) = 2^{2n} \cdot 2^{n^2}.
\end{aligned}$$

■

Suppose that $K < 2^{n/1000}$. Then Claim 10 implies that there exists a 1-chromatic rectangle $R = U \times V$ with $|U|, |V| \ge 2^{n(2-1/100)}$. This contradicts the next claim:

**Claim 11.** *Let $R = U \times V$ be a 1-chromatic rectangle. Then $\min(|U|, |V|) < 2^{\alpha n}$ for $\alpha = 2 - 10^{-2}$.*

*Proof of claim.* Suppose for the sake of contradiction that $|U|, |V| \ge 2^{\alpha n}$. Let $u \in U$ and $v \in V$ be arbitrary, and let $\boldsymbol{A} \sim U$, $\boldsymbol{B} \sim V$ be uniformly random, so that $\boldsymbol{A}, \boldsymbol{B} \in \{0,1\}^{n \times n}$. Since $R$ is 1-chromatic, $\boldsymbol{A} \oplus v$ and $\boldsymbol{B} \oplus u$ are always rank-1 matrices, so we may write $\boldsymbol{A} \oplus v$ as an outer product $\boldsymbol{\ell}^A \cdot (\boldsymbol{r}^A)^T$ where $\boldsymbol{\ell}^A, \boldsymbol{r}^A \in \{0,1\}^n$, and similarly $\boldsymbol{B} \oplus u := \boldsymbol{\ell}^B (\boldsymbol{r}^B)^T$. Then $\mathrm{H}(\boldsymbol{\ell}^A, \boldsymbol{r}^A), \mathrm{H}(\boldsymbol{\ell}^B, \boldsymbol{r}^B) \ge \alpha n$. By the chain rule for entropy, $\sum_{i \text{ odd}} \mathrm{H}(\boldsymbol{\ell}^A_i, \boldsymbol{r}^A_i, \boldsymbol{\ell}^A_{i+1}, \boldsymbol{r}^A_{i+1}) \ge n(2 - 10^{-2})$, therefore for at least 2/3-fraction of odd $i \in [n]$, $\mathrm{H}(\boldsymbol{\ell}^A_i, \boldsymbol{r}^A_i, \boldsymbol{\ell}^A_{i+1}, \boldsymbol{r}^A_{i+1}) \ge 2 - 3 \cdot 10^{-2}$. Similarly, for at least 2/3-fraction of odd $i \in [n]$, $\mathrm{H}(\boldsymbol{\ell}^B_i, \boldsymbol{r}^B_i, \boldsymbol{\ell}^B_{i+1}, \boldsymbol{r}^B_{i+1}) \ge 2 - 3 \cdot 10^{-2}$. Then for 1/3-fraction of odd coordinates $i$, both inequalities are satisfied. Suppose without loss of generality that this is the case for $i = 1$.

Then $\mathrm{H}(\boldsymbol{\ell}^A_1, \boldsymbol{r}^A_1, \boldsymbol{\ell}^A_2, \boldsymbol{r}^A_2) \ge 2(2 - 10^{-2}) > \log 15$, hence $|\operatorname{supp}(\boldsymbol{\ell}^A_1, \boldsymbol{r}^A_1, \boldsymbol{\ell}^A_2, \boldsymbol{r}^A_2)| = 16$, and similarly $|\operatorname{supp}(\boldsymbol{\ell}^B_1, \boldsymbol{r}^B_1, \boldsymbol{\ell}^B_2, \boldsymbol{r}^B_2)| = 16$. Thus $(\boldsymbol{A} \oplus v)|_{[2] \times [2]}$ and $(u \oplus \boldsymbol{B})|_{[2] \times [2]}$ can be arbitrary rank-1 matrices, so in particular there exists $(a, b) \in \operatorname{supp}(\boldsymbol{A}) \times \operatorname{supp}(\boldsymbol{B})$ such that $(a \oplus b)|_{[2] \times [2]}$ is the identity matrix: take $a$ and $b$ such that $(a \oplus v)|_{[2] \times [2]}$ and $(u \oplus b)|_{[2] \times [2]}$ form the decomposition of $(v \oplus u)|_{[2] \times [2]} \oplus I_2$ as a sum of rank-1 matrices. Hence $\mathrm{rk}(a \oplus b) \ge 2$, a contradiction. ■

This concludes the proof of the theorem. ■

# 5 Consequences

We conclude the paper with some corollaries of our proofs, as mentioned in the introduction.

## 5.1 Constant-Cost Communication vs. $k$-Hamming Distance

We will prove Corollary 2, which shows that constant-cost randomized communication does not reduce to $k$-Hamming Distance.

**Remark 12.** We must clarify a subtlety in the definition of $\mathsf{D}^{\mathcal{Q}}(P)$, the cost of a deterministic communication protocol with access to an oracle for problem $\mathcal{Q}$. For the purpose of studying the class $\mathsf{BPP}_0$, the natural definition of $\mathsf{D}^{\mathcal{Q}}$ is as follows (see e.g. [HWZ22, FHHH24, FGHH25]). $\mathcal{Q}$ is a family of boolean matrices, i.e. a communication problem, containing matrices $Q \in \{0,1\}^{N\times N}$ for a variety of input sizes $N$. $\mathsf{D}^{\mathcal{Q}}(P)$ is the minimum cost of a deterministic protocol computing $P \in \{0,1\}^{N\times N}$, where on input $x, y$, the two parties Alice and Bob in each round may construct inputs $a(x)$ and $b(y)$ of *arbitrary size*, and query the oracle on $Q(a(x), b(y))$, where $Q \in \mathcal{Q}$ is any instance of problem $\mathcal{Q}$. The difference between this definition and the standard definition of oracle protocols is that the query inputs $a(x), b(y)$ do not have bounded size. This is natural because for $\mathcal{Q} \in \mathsf{BPP}_0$, the randomized communication cost is independent of input size.

Recent work [FHHH24] showed that there is no complete problem for $\mathsf{BPP}_0$: there is no problem $\mathcal{Q} \in \mathsf{BPP}_0$ such that $\mathsf{D}^{\mathcal{Q}}(P) = O(1)$ for all problems $P \in \mathsf{BPP}_0$. The reason is that, for every $\mathcal{Q} \in \mathsf{BPP}_0$, there exists a constant $k$ such that $\mathsf{D}^{\mathcal{Q}}(\mathrm{HD}^n_k) = \omega(1)$. This raised the question of whether the infinite hierarchy of $k$-Hamming Distance problems $\mathrm{HD}_1, \mathrm{HD}_2, \cdots$ is complete, meaning that for every problem $\mathcal{P} \in \mathsf{BPP}_0$ there exists a constant $k$ such that $\mathsf{D}^{\mathrm{HD}_k}(P_N) = O(1)$. The question was answered in [FGHH25], whose main result was:

**Theorem 13** ([FGHH25])**.** *There exists a communication problem $F_n\colon \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ such that $\mathsf{R}(F_n) = O(1)$, yet for every constant $k$, $\mathsf{D}^{\mathrm{HD}_k}(F_n) = \omega(1)$.*

The $\omega(1)$ function is not given explicitly and is the result of a lengthy Ramsey-theoretic proof. We give a simpler proof (using a different function $F$) with a quantitative bound.

*Proof of Corollary 2.* We prove that, for every constant $k$, $\mathsf{D}^{\mathrm{HD}_k}(\textsc{RankOne}_n) = \Omega(n/\log n)$. Given a deterministic protocol computing $\textsc{RankOne}_n$ using $\mathrm{HD}_k$ queries, we may replace each query $Q$ with a protocol computing $Q$ using Equality queries. Then

$$\mathsf{D}^{\mathrm{EQ}}(\textsc{RankOne}_n) \le \mathsf{D}^{\mathrm{HD}_k}(\textsc{RankOne}_n) \cdot \max_Q \mathsf{D}^{\mathrm{EQ}}(Q),$$

where the maximum is over all $2^{n^2} \times 2^{n^2}$ matrices $Q$ which represent oracle queries to $\mathrm{HD}_k$; formally, these matrices $Q$ are the ones which are obtained by taking any submatrix of $\mathrm{HD}^{n'}_k$ for arbitrarily large $n'$ (due to Remark 12), and then duplicating rows or columns to ensure the matrix is the same size as $\textsc{RankOne}_n$. We now require a bound on $\mathsf{D}^{\mathrm{EQ}}(Q)$ for arbitrary $2^{n^2} \times 2^{n^2}$ submatrices of $\mathrm{HD}^{n'}_k$ where $n'$ can be arbitrarily large. This was proved in [FHHH24]:

**Theorem 14** ([FHHH24, Proposition 4.1])**.** *For any $k$ and any $n, N \in \mathbb{N}$ satisfying $N \le 2^n$, let $M$ be an $N \times N$ submatrix of $\mathrm{HD}^n_k \in \{0,1\}^{2^n \times 2^n}$. Then $\mathsf{D}^{\mathrm{EQ}}(M) \le O(k \log\log N)$.*

So, by Theorem 14,

$$\mathsf{D}^{\mathrm{EQ}}(\textsc{RankOne}_n) \le O\left(\mathsf{D}^{\mathrm{HD}_k}(\textsc{RankOne}_n) \cdot k \log\log(2^{n^2})\right).$$

By Theorem 1, we conclude that $\mathsf{D}^{\mathrm{HD}_k}(\textsc{RankOne}_n) = \Omega\left(\frac{n}{k \log n}\right)$. ■

### 5.2 Parity Decision Trees and Exact vs. Approximate Spectral Norm

We state some consequences for parity decision trees and exact vs. approximate spectral norms. The Fourier coefficients of a function $f\colon \{0,1\}^n \to \mathbb{R}$ are defined on subsets $S \subseteq [n]$ by

$$\hat{f}(S) \coloneqq \mathbb{E}_{\boldsymbol{x} \sim \{0,1\}^n}[f(\boldsymbol{x})(-1)^{\sum_{i \in S} \boldsymbol{x}_i}].$$

The Fourier norm is $\|\hat{f}\|_1 \coloneqq \sum_{S \subseteq [n]} |\hat{f}(S)|$. For an XOR problem $F(x,y) = f(x \oplus y)$, it is equivalent to the $\gamma_2$-norm:

$$\gamma_2(F) = \|\hat{f}\|_1 \qquad \text{([HHH23, Corollary 3.9])}$$

For $\epsilon \in (0,1)$, the approximate spectral norm is defined as

$$\|\widehat{f}\|_{1,\epsilon} \coloneqq \inf_g \|\widehat{g}\|_1$$

where the infimum is over functions $g\colon \{0,1\}^n \to \mathbb{R}$ which satisfy $\|f - g\|_\infty < \epsilon$.

We can now state the corollaries. The first is an exponential improvement over the separation given by [CHZZ24, Lemma 3].

**Corollary 15.** *There exists a function $f\colon \{0,1\}^{n \times n} \to \{0,1\}$ with approximate spectral norm $\|\widehat{f}\|_{1,1/3} = O(1)$ but exact spectral norm $\|\widehat{f}\|_1 = 2^{\Omega(n)}$.*

We remark that for some constants $\epsilon$ and $\ell$ the bound $\|\widehat{f}\|_{1,\epsilon} \le \ell$ [San19, Theorem 2.5] (known as quantitative Cohen idempotence theorem) implies that $\|\widehat{f}\|_1 = O(1)$, so this corollary (as well as [CHZZ24, Lemma 3]) gives a barrier for improving the dependency between $\ell$ and $\epsilon$ in this result.

The second corollary answers [CHH+25, Question 8].

**Corollary 16.** *There exists a function $f\colon \{0,1\}^{n \times n} \to \{0,1\}$ with randomized parity decision tree size $O(1)$ and deterministic parity decision tree size $2^{\Omega(n)}$.*

The separating function $f\colon \{0,1\}^{n \times n} \to \{0,1\}$ in both cases is defined by $f(A) = 1$ iff $A \in \{0,1\}^{n \times n}$ has $\mathrm{rk}(A) \le 1$, so $\textsc{RankOne}_n(x,y) = f(x \oplus y)$.

*Proof of Corollary 15.* By [HHH23, Proposition 4.3] we get that $\|\widehat{f}\|_{1,1/3} = O(1)$ since 1/3-error randomized communication cost of RankOne is $O(1)$ by Theorem 3. On the other hand, by [HHH23, Corollary 3.9] combined with Theorem 6 we get $\|\widehat{f}\|_1 = 2^{\Omega(n)}$ as required. ■

*Proof of Corollary 16.* The upper bound on randomized parity decision tree size of $f$ was proved in Theorem 3. If $f$ has a parity decision tree (PDT) of size $s$, then $f(x) = \sum_{i \in [s]} A_i(x)$ where $A_i$ are characteristic functions of the affine spaces corresponding to 1-labeled leaves of the PDT. It is well known that $\|\widehat{A_i}\|_1 \le 1$. Then $\|\hat{f}\|_1 \le \sum_{i \in [s]} \|\widehat{A_i}\|_1 \le s$. By Theorem 6,

$$2^{\Omega(n)} \le \gamma_2(\textsc{RankOne}_n) = \|\hat{f}\|_1 \le s.$$ ■

### 5.3 Non-Deterministic Lower Bounds

The paper [CLV19] showed that $\mathsf{BPP} \not\subseteq \mathsf{P}^{\mathrm{EQ}}$ using the Integer Inner Product function, defined as $\mathrm{IIP}_3^n \colon [-2^n, 2^n]^3 \times [-2^n, 2^n]^3 \to \{0,1\}$, where $\mathrm{IIP}_3^n(x,y) = 1$ iff $\sum_{i\in[3]} x_i y_i = 0$. Later, [PSS23, CHH+25] improved this separation to $\mathsf{BPP} \not\subseteq \mathsf{NP}^{\mathrm{EQ}}$ using $\mathrm{IIP}_3$ in the latter work (and the higher-dimensional $\mathrm{IIP}_6$ in the prior work). [CHH+25] prove a lower bound of $\mathsf{ND}^{\mathrm{EQ}}(\mathrm{IIP}_3^n) = \Omega(n/\log n)$ using Equation (5) and asking whether the $\log n$ factor could be removed. With our improved Lemma 8 we can remove this log factor.

We need the notion of maxrect$(\cdot)$ from [PSS23]. For a matrix $A \in \{0,1\}^{N\times N}$ let $\alpha(A)$ be the number of 1-entries in $A$ and let $\beta(A)$ be the area of the largest 1-chromatic rectangle in $A$. Then

$$\mathrm{maxrect}(A) := \frac{\alpha(A)}{N\sqrt{\beta(A)}}.$$

The following two theorems, together with the fact that $\mathrm{bc}(\cdot)$ does not increase when taking submatrices, imply $\mathrm{bc}(\mathrm{IIP}_3^n) = 2^{\Omega(n)}$:

**Theorem 17** (Theorem 5 in [CHH+25])**.** *$\mathrm{IIP}_3^n$ has a submatrix $A$ such that* $\mathrm{maxrect}(A) = 2^{\Omega(n)}$.

**Theorem 18** (Theorem 29 in [PSS23])**.** *For every $A \in \{0,1\}^{N\times N}$ we have* $\mathrm{bc}(A) \geq \Omega(\mathrm{maxrect}(A))$.

Then, by Lemma 8 (and the trivial bound $\mathsf{D}^{\mathrm{EQ}}(\mathrm{IIP}_3^n) = O(n)$):

**Corollary 19.** $\mathsf{ND}^{\mathrm{EQ}}(\mathrm{IIP}_3^n) = \Theta(n)$.

## A Appendix: Proof of Lemma 8

Let us now prove Lemma 8. Let $A \in \{0,1\}^{N\times N}$. Our goal is to show that

$$\log \mathrm{bc}(A) \leq \mathsf{ND}^{\mathrm{EQ}}(A) + O(\log\log N).$$

To prove this, we will use the fractional blocky cover instead of the blocky cover:

**Definition 20** (Fractional Blocky Cover)**.** For any $A \in \{0,1\}^{N\times N}$, a *fractional blocky cover* of $A$ is a choice of values $m \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_m \geq 0$, and blocky matrices $B_1, \ldots, B_m$ such that

$$A(x,y) = 0 \implies \sum_{i=1}^{m} \lambda_i B_i(x,y) = 0 \qquad A(x,y) = 1 \implies \sum_{i=1}^{m} \lambda_i B_i(x,y) \geq 1.$$

We write $\mathrm{fbc}(A)$ for the minimum value of $\sum_i \lambda_i$ over all fractional blocky covers of $A$.

We require the following properties of the fractional blocky cover:

**Proposition 21** (Properties of FBC)**.** *Let $P, Q \in \{0,1\}^{N\times N}$ be any boolean matrices and let $J$ be the all-1s matrix. Then*

1. *For any blocky matrix $B$,* $\mathrm{fbc}(J - B) \leq 4$.
2. $\mathrm{fbc}(P \wedge Q) \leq \mathrm{fbc}(P) \cdot \mathrm{fbc}(Q)$.
3. $\mathrm{fbc}(P \vee Q) \leq \mathrm{fbc}(P) + \mathrm{fbc}(Q)$.

4. $\mathrm{bc}(P) \le O(\mathrm{fbc}(P) \cdot \log N)$.

We prove these properties below. First let us complete the proof of Lemma 8.

*Proof of Lemma 8.* Let $A \in \{0,1\}^{N \times N}$. Due to property (4) of Proposition 21, it suffices to show $\log \mathrm{fbc}(A) = O(\mathsf{ND}^{\mathrm{EQ}}(A))$. Let $d, m \in \mathbb{N}$ be such that $d + m = \mathsf{ND}^{\mathrm{EQ}}(A)$ and there exist $2^m$ deterministic EQUALITY-oracle protocols $T_i$ of depth at most $d$ such that

$$\forall x, y \in [N] : \qquad A(x,y) = \bigvee_{i=1}^{2^m} T_i(x,y).$$

By property (3) of Proposition 21, $\mathrm{fbc}(A) \le \sum_{i=1}^{2^m} \mathrm{fbc}(M_i)$ where $M_i \in \{0,1\}^{N \times N}$ is the matrix computed by protocol $T_i$. It now suffices to show that $\mathrm{fbc}(M_i) \le 5^d$ for every $i$, so that $\log \mathrm{fbc}(A) \le \log(2^m 5^d) < 3 \cdot \mathsf{ND}^{\mathrm{EQ}}(A)$.

Fix any $T = T_i$, which is a decision tree with each inner node $v$ associated with a blocky matrix $B_v$. Without loss of generality we may assume the depth is exactly $d$ and that the tree is full. For each node $v$, let $M_v \in \{0,1\}^{N \times N}$ be the matrix computed by the subtree rooted at $v$, and write $\mathrm{depth}(v)$ for the depth of this subtree. We prove by induction that $\mathrm{fbc}(M_v) \le 5^{\mathrm{depth}(v)}$. If $\mathrm{depth}(v) = 1$, then $M_v$ is either a blocky matrix $B$ or its complement $J - B$, so by property (1) of Proposition 21, $\mathrm{fbc}(M_v) \le 4$. For depth $d' > 1$, let $\ell, r$ be the left and right child of $v$. Then

$$M_v = (B_v \wedge M_\ell) \vee ((J - B_v) \wedge M_r),$$

so by properties (1), (2), and (3) of Proposition 21,

$$\begin{aligned} \mathrm{fbc}(M_v) &\le \mathrm{fbc}(B_v)\mathrm{fbc}(M_\ell) + \mathrm{fbc}(J - B_v)\mathrm{fbc}(M_r) \le \mathrm{fbc}(M_\ell) + 4\mathrm{fbc}(M_r) \\ &\le 5^{d'-1} + 4 \cdot 5^{d'-1} = 5^{d'}, \end{aligned}$$

with the last inequality by induction. This completes the proof of the lemma. ∎

Finally, we establish the properties of fractional blocky covers that we have used above.

*Proof of Proposition 21.*

**Claim 22.** *Let $J$ be the all-1s matrix and let $B$ be any blocky matrix. Then $\mathrm{fbc}(J - B) \le 4$.*

*Proof.* It suffices to prove the claim for the identity matrix $B = I_{N \times N}$, since duplicating rows and columns does not change the value of fbc. The claim follows from the fact that the EQUALITY communication problem has a constant-cost one-sided error randomized protocol. That is, define the following probability distribution over rectangles $R \subseteq [N] \times [N]$. We choose $\boldsymbol{R}$ by including each $x \in [N]$ in a set $\boldsymbol{X} \subseteq [N]$ independently with probability $1/2$, and then take $\boldsymbol{R} = \boldsymbol{X} \times ([N] \setminus \boldsymbol{X})$. If $x = y$, then $\Pr[(x,y) \in \boldsymbol{R}] = 0$. If $x \neq y$ then $\Pr[(x,y) \in \boldsymbol{R}] = 1/4$. For each rectangle $R$, set $\lambda_R := 4 \cdot \Pr[\boldsymbol{R} = R]$. It follows that

$$\mathrm{fbc}(J - I_{N \times N}) \le \sum_R \lambda_R = 4.$$ ∎

**Claim 23.** *Let $P, Q \in \{0,1\}^{N \times N}$. Then $\mathrm{fbc}(P \wedge Q) \le \mathrm{fbc}(P) \cdot \mathrm{fbc}(Q)$.*

*Proof of claim.* Let $\alpha_1, \dots, \alpha_p$ and $B_1, \dots, B_p$ be a fractional blocky cover of $P$, and let $\beta_1, \dots, \beta_q$ and $B'_1, \dots, B'_q$ be a fractional blocky cover of $Q$. For any $x, y \in [N]$, consider

$$\sum_{i,j} \alpha_i \beta_j (B_i \wedge B'_j)(x,y) = \left( \sum_i \alpha_i B_i(x,y) \right) \cdot \left( \sum_j \beta_j B'_j(x,y) \right) \tag{5}$$

Note that $B_i \wedge B'_j$ is itself a blocky matrix. If $P \wedge Q(x,y) = 0$ then $P(x,y) = 0$ or $Q(x,y) = 0$ so Equation (5) is 0. If $P \wedge Q(x,y) = 1$ then $P(x,y) = Q(x,y) = 1$ so Equation (5) is at least 1. So

$$\mathrm{fbc}(P \wedge Q) \leq \sum_{i,j} \alpha_i \beta_j = \mathrm{fbc}(P) \cdot \mathrm{fbc}(Q). \qquad \blacksquare$$

**Claim 24.** *Let $P, Q \in \{0,1\}^{N \times N}$. Then $\mathrm{fbc}(P \vee Q) \leq \mathrm{fbc}(P) + \mathrm{fbc}(Q)$.*

*Proof of claim.* Let $\alpha_1, \dots, \alpha_p$ and $B_1, \dots, B_p$ be a fractional blocky cover of $P$, and let $\beta_1, \dots, \beta_q$ and $B'_1, \dots, B'_q$ be a fractional blocky cover of $Q$. It is straightforward to check that the union of these fractional blocky covers is itself a fractional blocky cover for $P \vee Q$, so that $\mathrm{fbc}(P \vee Q) \leq \sum_i \alpha_i + \sum_j \beta_j$. $\blacksquare$

**Claim 25.** *Let $P \in \{0,1\}^{N \times N}$. Then $\mathrm{bc}(P) \leq O(\mathrm{fbc}(P) \cdot \log N)$.*

*Proof of claim.* This is by standard randomized rounding. Let $\lambda_1, \dots, \lambda_m \geq 0$ and $B_1, \dots, B_m$ be a fractional blocky cover of $P$ with $\sum_\ell \lambda_\ell = \mathrm{fbc}(P)$. Let $\boldsymbol{M}_1, \dots, \boldsymbol{M}_t$ be random blocky matrices chosen independently from the distribution $\Pr[\boldsymbol{M}_i = B_j] = \frac{\lambda_j}{\mathrm{fbc}(P)}$. Fix any $x, y \in [N]$ and note that

$$P(x,y) = 0 \implies \Pr[\exists i : \boldsymbol{M}_i(x,y) = 1] = 0$$

since $B_\ell(x,y) = 0$ for all $\ell \in [m]$. Now suppose $P(x,y) = 1$. Then $\sum_\ell \lambda_\ell B_\ell(x,y) \geq 1$, so, by independence of each $\boldsymbol{M}_i$,

$$\Pr[\forall i : \boldsymbol{M}_i(x,y) = 0] = \Pr[\boldsymbol{M}_1(x,y) = 0]^t = \left(1 - \frac{1}{\mathrm{fbc}(P)}\right)^t \leq e^{-\frac{t}{\mathrm{fbc}(P)}}.$$

Setting $t = O(\mathrm{fbc}(P) \log N)$ and using the union bound over at most $N^2$ pairs $x, y \in [N]$ with $P(x,y) = 1$, we conclude that there exists a choice of $t = O(\mathrm{fbc}(P) \log N)$ blocky matrices which cover the 1-valued entries of $P$. $\blacksquare$

This completes the proof of all of the properties from Proposition 21. $\blacksquare$

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## References


[BW16] Mark Braverman and Omri Weinstein. A discrepancy lower bound for information complexity. *Algorithmica*, 76:846–864, 2016. doi:10.1007/s00453-015-0093-8.

[CHH+25] Tsun-Ming Cheung, Hamed Hatami, Kaave Hosseini, Aleksandar Nikolov, Toniann Pitassi, and Morgan Shirley. A lower bound on the trace norm of boolean matrices and its applications. In *Proceedings of the Innovations in Theoretical Computer Science Conference (ITCS)*, pages 37–1. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2025. doi:10.4230/LIPIcs.ITCS.2025.37.

[CHHS23] Tsun-Ming Cheung, Hamed Hatami, Kaave Hosseini, and Morgan Shirley. Separation of the factorization norm and randomized communication complexity. In *Proceedings of the Computational Complexity Conference (CCC)*, pages 1–1. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2023. doi:10.4230/LIPIcs.CCC.2023.1.

[CHZZ24] Tsun Ming Cheung, Hamed Hatami, Rosie Zhao, and Itai Zilberstein. Boolean functions with small approximate spectral norm. *Discrete Analysis*, 2024. doi:10.19086/da.122971.

[CLV19] Arkadev Chattopadhyay, Shachar Lovett, and Marc Vinyals. Equality alone does not simulate randomness. In *Proceedings of the Computational Complexity Conference (CCC)*, pages 14–1. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2019. doi:10.4230/LIPIcs.CCC.2019.14.

[EHK22] Louis Esperet, Nathaniel Harms, and Andrey Kupavskii. Sketching distances in monotone graph classes. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques (APPROX/RANDOM)*, pages 18–1. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2022. doi:10.4230/LIPIcs.APPROX/RANDOM.2022.18.

[FGHH25] Yuting Fang, Mika Göös, Nathaniel Harms, and Pooya Hatami. Constant-cost communication does not reduce to $k$-Hamming Distance. In *Proceedings of the ACM SIGACT Symposium on Theory of Computing (STOC)*, 2025. doi:10.48550/arXiv.2407.20204.

[FHHH24] Yuting Fang, Lianna Hambardzumyan, Nathaniel Harms, and Pooya Hatami. No complete problem for constant-cost randomized communication. In *Proceedings of the ACM SIGACT Symposium on Theory of Computing (STOC)*, 2024. doi:10.48550/arXiv.2404.00812.

[Har20] Nathaniel Harms. Universal communication, universal graphs, and graph labeling. In *Proceedings of the Innovations in Theoretical Computer Science Conference (ITCS)*, volume 151, page 33. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 2020. doi:10.4230/LIPIcs.ITCS.2020.33.

[HH24] Hamed Hatami and Pooya Hatami. Guest column: Structure in communication complexity and constant-cost complexity classes. *ACM SIGACT News*, 55(1):67–93, 2024. doi:10.1145/3654780.3654788.

[HHH23] Lianna Hambardzumyan, Hamed Hatami, and Pooya Hatami. Dimension-free bounds and structural results in communication complexity. *Israel Journal of Mathematics*, 253(2):555–616, 2023. doi:10.1007/s11856-022-2365-8.

[HR24] Nathaniel Harms and Artur Riazanov. Better boosting of communication oracles, or not. In *IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS)*, 2024. doi:10.4230/LIPIcs.FSTTCS.2024.25.

[HWZ22] Nathaniel Harms, Sebastian Wild, and Viktor Zamaraev. Randomized communication and implicit graph representations. In *Proceedings of the ACM SIGACT Symposium on Theory of Computing (STOC)*, 2022. doi:10.1145/3519935.3519978.

[HZ24] Nathaniel Harms and Viktor Zamaraev. Randomized communication and implicit representations for matrices and graphs of small sign-rank. In *Proceedings of the ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1810–1833. SIAM, 2024. doi:10.1137/1.9781611977912.72.

[KN96] Eyal Kushilevitz and Noam Nisan. *Communication complexity*. Cambridge University Press, 1996.

[Nis93] Noam Nisan. The communication complexity of threshold gates. *Combinatorics, Paul Erdos is Eighty*, 1(301-315):6, 1993.

[PSS23] Toniann Pitassi, Morgan Shirley, and Adi Shraibman. The strength of equality oracles in communication. In *Proceedings of the Innovations in Theoretical Computer Science Conference (ITCS)*, pages 89–1. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2023. doi:10.4230/LIPIcs.ITCS.2023.89.

[RY20] Anup Rao and Amir Yehudayoff. *Communication Complexity: and Applications*. Cambridge University Press, 2020.

[San19] Tom Sanders. Boolean functions with small spectral norm, revisited. *Mathematical Proceedings of the Cambridge Philosophical Society*, 167(2):335–344, 2019. doi:10.1017/S030500411800035X.

[SS24] Alexander Sherstov and Andrey Storozhenko. The communication complexity of approximating matrix rank. In *Proceedings of the IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 433–462. IEEE, 2024. doi:10.1109/FOCS61266.2024.00035.

[SY23] Srikanth Srinivasan and Amir Yehudayoff. The discrepancy of greater-than. *arXiv preprint arXiv:2309.08703*, 2023. doi:10.48550/ARXIV.2309.08703.

[Tom25] István Tomon. Factorization norms and Zarankiewicz problems. *arXiv preprint arXiv:2502.18429*, 2025. doi:10.48550/arXiv.2502.18429.

[Vio15] Emanuele Viola. The communication complexity of addition. *Combinatorica*, 35(6):703–747, 2015. doi:10.1007/s00493-014-3078-3.



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