Electronic Colloquium on Computational Complexity, Revision 1 of Report No. 103 (2025)

2D Minimal Graph Rigidity is in NC for One-Crossing-Minor-Free Graphs *

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October 10, 2025

Abstract

Minimally rigid graphs can be decided and embedded in the plane efficiently, i.e. in polynomial time. There is also an efficient randomized *parallel* algorithm, i.e. in RNC. We present an NC-algorithm to decide whether one-crossing-minor-free graphs are minimally rigid. In the special case of $K_{3,3}$-free graphs, we also compute an infinitesimally rigid embedding in NC.

1 Introduction

Graph rigidity is the combinatorial study of rigidity or flexibility of bar-and-joint frameworks, a set of solid bars connected via hinge joints. Historically, the problem goes back to Euler in 1776, who asked about the rigidity of polyhedrons in 3D. Rigidity of bar-and-joint frameworks, particularly in 2D, has been studied extensively for applications in mechanical engineering like for collision-free robot arm motion planning, or molecular conformations, or network topologies with distance and angle constraints.

Graph rigidity is also a central problem in theoretical computer science. It yields a combinatorial interpretation of certain *polynomial identity testing* (PIT) problems that can be solved efficiently for 2D-rigidity, but the complexity for larger dimensions is wide open, see for example the exposition of Raz and Wigderson [RW19] or the book of Lovász [Lov19, Chapter 15]. Other interesting aspects are that rigidity defines a matroid and that minimal rigidity reduces to bipartite perfect matching. The latter point motivates the question for the parallel complexity of rigidity.

Describing the problem. A bar-and-joint framework can be seen as a graph G, where vertices are the joints and edges are the bars, together with an embedding $p : V \to \mathbb{R}^2$, for the case of 2D. The vertices of G are allowed to move continuously subject to the constraint that the distance between adjacent vertices does not change, i.e., the length of the bars are fixed. A framework

*Partially supported by DFG grant TH 472/5-2 and SERB MATRICS grant MTR2022/001009

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ISSN 1433-8092

(G, p) is *rigid* if every such continuous motion of the framework preserves the distance of *all* pairs of vertices, i.e., also of non-adjacent vertices. Otherwise the framework is *flexible*.

A natural question is whether rigidity/flexibility is just the property of the underlying graph G or whether it also depends on the specific embedding p. That is, whether rigidity is just determined by the combinatorial structure of the bars, i.e. which tuples of bars are connected at one joint, and does not depend on the specific lengths of the bars. The first answer is that there are examples that show that rigidity depends on the embedding p. However, for any graph, either almost all of its embeddings are rigid or almost all of them are flexible [Glu75, AR79]. Therefore one defines a graph G to be *rigid* if almost all embeddings p result in a rigid framework (G, p), otherwise G is *flexible*. Graph rigidity can be equivalently defined based on the notion of *infinitesimal rigidity*, see Section 2.1 and Definition 2.1.

Characterizations of minimal rigidity. A graph is *minimally rigid* if the removal of any edge makes it flexible. Minimally rigid graphs have a combinatorial characterization that is often attributed to Laman, but was already known to Geiringer.

Definition 1.1 (Laman graph) *A graph* $G = (V, E)$ *is a* Laman graph *if it has* $m = 2n - 3$ *edges and for all* $S \subseteq V$ *with* $|S| \geq 2$*, the subgraph of* G *induced by* S *has at most* $2|S|-3$ *edges, i.e.*

$$\forall S \subseteq V \ (|S| \geq 2) \quad |E(S)| \leq 2|S| - 3. \tag{1}$$

Theorem 1.2 ([Pol27], [Lam70]) *A graph* G *is minimally rigid iff* G *is a Laman graph.*

The Laman condition (1) gives rise to a matroid, the *rigidity matroid* for a graph $G = (V, E)$. The ground set is the set of edges E. A set $E' \subseteq E$ is *independent*, if graph (V, E') fulfills (1). The base sets of this matroid are the minimally rigid subgraphs of G (on V).

There is another very interesting relationship. Let $G = (V, E)$ be a graph with $m = 2n - 2$ edges. Nash-Williams [NW61] and Tutte [Tut61] independently showed that G is the disjoint union of two spanning trees iff for all $S \subseteq V$ with $|S| \geq 2$, we have $|E(S)| \leq 2|S| - 2$, see also Lovász and Yemini [LY82]. Note again the similarity with Laman's condition (1). This yields again a combinatorial characterization.

Theorem 1.3 ([LY82]) *A graph* $G = (V, E)$ *is minimally rigid iff* $\forall e \in E$ *multigraph* $G + e$ *is the union of two edge-disjoint spanning trees.*

There is also an iterative way to construct Laman graphs, known as the Henneberg construction [Hen11], where we start with an edge, and then add a node to the graph constructed so far in one of two ways (Theorem 2.11). This is explained in more detail in Section 2.4.

Complexity and motivation. In 2D, rigidity can be solved in polynomial time, namely in time $\mathcal{O}(n^2)$ [Hen92, GW92]. For minimal rigidity, this follows from Theorem 1.3, because whether a graph is the disjoint union of two spanning trees can be decided in polynomial time [RT85]. There is a $\mathcal{O}(n\sqrt{n \log n} + m)$ algorithm for minimal rigidity [GW92]. We are interested in the *parallel* complexity of minimal rigidity.

Note that the characterization from Theorem 1.3 yields a reduction from minimal rigidity to the problem of deciding whether a graph consists of two edge-disjoint spanning trees. Moreover, the characterization of minimal rigidity in Theorem 1.2 is similar in nature to Hall's Theorem [Hal35] that characterizes the existence of perfect matchings in bipartite graphs. This can actually be formalized to a reduction from the minimal rigidity problem to the bipartite perfect matching problem [Hen92]. Since the reductions to the two edge-disjoint spanning trees problem and bipartite perfect matching are efficient even in *parallel*, i.e. in NC, the parallel complexity of these two problems carries over to minimal rigidity. Note that the problems are in randomized NC, RNC [MVV87, NSV94]. The derandomization question, whether one can avoid the randomness without much loss in efficiency, is of major interest in complexity theory. A few years back, the randomized algorithm has been derandomized to a quasi-NC bound for bipartite perfect matching [FGT21]. Later, the two edge-disjoint spanning tree problem was also shown to be in quasi-NC via *Linear Matroid Intersection* [GT20]. These results imply two different quasi-NC algorithms for minimal rigidity.

- One of the major motivations for us to study minimal graph rigidity is that we might have better chances to show that it is in NC than for bipartite perfect matching. In that sense, minimal rigidity is the litmus test for bipartite perfect matching.

Note that a reduction in the other direction is not known. Hence, minimal rigidity might be easier than bipartite perfect matching.

Interestingly, also the complexity questions around general rigidity (i.e., not necessarily minimal), have a status quite similar to those around the bipartite matching problem. Using the matroid property, one can get a polynomial-time algorithm to test if a graph is rigid: Start from the empty set, and keep adding edges from the graph while maintaining independence. If this algorithm ends with $2|V|-3$ edges, then the graph is indeed rigid [Hen92].

Both problems, matching and rigidity, reduce to the polynomial identity testing (PIT) problem [MVV87, RW19], and thus they have RNC algorithms. For any given embedding of the graph, one can form the *rigidity matrix* of the graph, a $|E| \times 2|V|$ matrix, where the entries are functions of the vertex coordinates. We define the matrix in Section 2.1. The rank of the rigidity matrix tells us about the rigidity of the particular embedding. For a rigid graph, a random embedding would be rigid and the corresponding rigidity matrix would have rank $2|V|-3$, with high probability [AR78]. Thus, to decide the rigidity of a graph one simply has to take a random embedding and find the rank of the rigidity matrix. Note that computing the rank of a matrix is highly parallelizable [Mul87] and so is the above algorithm, which puts the rigidity problem in the complexity class RNC. Its connection with matching and polynomial identity testing (PIT) makes graph rigidity a very interesting candidate for studying derandomization.

Our results. For certain graph classes we are able to derandomize and give an NC-algorithm for minimal rigidity. Streinu and Haas et al. [Str00, HOR+05] developed a geometric characterization of *planar* Laman graphs that yields a more efficient algorithm than for general graphs. This was improved further by Rollin, Schlipf, Schulz [RSS19]. Our first observation is that by combining the subroutines in [Str00, HOR+05, RSS19] appropriately, we obtain a parallel algorithm and get

planar minimal rigidity in NC^2. We can even compute a rigid embedding of a planar Laman graph in NC^2.

Then we extend these parallel complexity results to some minor-free graph classes. Our first main result is that for $K_{3,3}$-free graphs, we can decide whether a graph is Laman in NC^2 (Theorem 4.4) and also compute an embedding (Theorem 4.6). Also whether a K_5-free graph is Laman can be decided in NC^3. Actually, our technique works for the more general class of *one-crossing-minor-free graphs*. For this class, we can decide Laman graphs in NC^3 (Theorem 5.4). However, an NC-algorithm for computing a rigid embedding remains an open problem.

Note that extending results from planar graphs to such minor-free graphs is a technically non-trivial step that has also been done in other contexts, for example, famously by Vazirani [Vaz89]: Kasteleyn [Kas67] showed that the number of perfect matchings in planar graphs can be efficiently computed. Then, based on a result of Little [Lit74], Vazirani [Vaz89] extended the result to $K_{3,3}$-free graphs in NC^2. Later, the result was further extended to K_5-free graphs in NC^3 [STW16]. There are many more results in a similar flavor, see for example [DNTW09, TW14, Khu90b, Khu90a, AGGT16, CDGS24, CE10, EV21]. Although the literature gives a meta strategy, a result for planar graphs does not automatically imply a result for other minor-free graph classes and new ideas are required.

Following the literature, our idea is to decompose the given one-crossing-minor-free graph (or $K_{3,3}$-free graph) into planar components or components of bounded treewidth. We observe that there are parallel algorithms to decide rigidity of graphs of bounded treewidth, see Section 2.2. Hence, for planar and bounded treewidth components we can check whether they are Laman. Our main technical tools which allow us to lift the results from planar/bounded treewidth case to $K_{3,3}$-free/one-crossing-minor free graphs are as follows.

- Characterization of Laman graphs in terms of Lamanness of certain split graphs obtained by splitting the graph along separating pairs/triples (Lemmas 4.3, 5.1).
- For all families of 3-vertex graphs, we construct gadget graphs such that adding the gadget to a graph makes it Laman if and only if adding some member of the family makes it Laman (Lemma 5.3).
- Combining rigid embeddings of a set of Laman graphs to find a rigid embedding of a Laman graph obtained by joining them using 2-sum operations (Lemma 4.5).

Note that obtaining a parallel algorithm for minimal rigidity does not give the same for rigidity, as there is no parallel reduction known. In fact, finding an NC (or quasi-NC) algorithm for rigidity is open, even for planar graphs.

Organization of the paper. In the preliminaries, we give a self-contained derivation of the rigidity matrix in Section 2.1. We show that for graphs of bounded treewidth, rigidity can be decided efficiently in parallel in Section 2.2. In Section 2.4, we describe the characterization of Laman graphs via Henneberg extensions.

In Section 3, we explain our basic observation that the sequential algorithms to decide whether a *planar* graph is Laman and compute infinitesimally rigid embeddings can be efficiently parallelized,

it is in NC. The reader can safely skip this section in first reading, since we use the planar case only as a blackbox subroutine in the rest of the paper.

Our main results about $K_{3,3}$-free and one-crossing-minor-free Laman graphs are in Section 4 and 5, respectively.

2 Preliminaries

For $n \in \mathbb{N}$, we denote $[n] = \{1, 2, \ldots, n\}$. We use standard complexity classes, in particular, the classes NC^k that consist of uniform boolean circuit families with bounded fan-in, polynomial size, and depth $\mathcal{O}(\log^k n)$. The corresponding randomized classes are denoted by RNC^k. A slight extension is quasi-NC^k that is defined similarly to NC^k, but with circuits of quasi-polynomial size $2^{\log^{\mathcal{O}(1)} n}$. Many problems from Linear Algebra are in known to be efficiently solvable in parallel. For example the determinant and the rank of matrices can be computed in NC^2, as well as the solution of linear systems of equations [Mul87].

For a graph $G = (V, E)$, we denote $n = n(G) = |V|$ and $m = m(E) = |E|$ throughout the paper. For a set $S \subseteq V$, the edges of G that are within S are denoted by $E(S)$. A graph G is *planar* if it can be drawn in the Euclidean plane such that the edges intersect only at the endpoints. Such a drawing is called a *planar (topological) embedding* of the graph. The *faces of a planar embedding* are the regions of the embedding. The *outer face of a planar embedding* is the unbounded face. It is known that every planar embedding can be turned into a planar embedding with the same face structure such that every edge is a straight line [Fá48, Wag36]. A planar embedding can be turned into a *combinatorial embedding*, a cyclic order of the edges around each vertex in counter clockwise order of the planar embedding. One can check if a graph is planar and compute topological and combinatorial planar embeddings in NC^2 [RR94].

The *crossing number* of a graph G is the minimum number c such that G has an embedding in the plane with c edge crossings. For example planar graphs have crossing number 0 and $K_{3,3}$ and K_5 have crossing number 1. Planar graphs are minor-closed. However, this does not hold in general for graphs with crossing number ≥ 1: There are examples of graphs G with crossing number 1 that have minors H with crossing number 2. To capture such graphs H as well, Robertson and Seymour [RS93] defined a graph H to be *single-crossing* or *one-crossing*, if H is the minor of a graph G with crossing number ≤ 1. With this definition, one-crossing graphs are closed under minors. However, in the literature, often the simpler definition is used that one-crossing graphs are those with crossing number ≤ 1. Note that this is a subset of the one-crossing graphs as defined by Robertson and Seymour. Hence, our results hold for the simplified definition as well.

For graphs G, H, we say that G is H-*free*, if H is not a minor of G. When H is one-crossing, we call G *one-crossing-minor-free*. Hence, planar, $K_{3,3}$-free, and K_5-free graphs are all one-crossing-minor-free.

We also need to solve a *flow problem* on planar graphs. Given a planar flow graph with multiple sources and sinks, capacities and fixed demands at every source and sink, the objective is to compute a feasible flow function. Miller and Naor [MN95] showed that the problem can be solved in NC^2.

2.1 Infinitesimal rigidity of frameworks

In a *bar-and-joint framework* we are given a graph $G = (V, E)$ and an embedding $p : V \to \mathbb{R}^2$ in the plane. The edges of the graph are considered as bars and the vertices as joints that are free to move continuously. However, the length of the bars does not change. The framework is *flexible*, if there is a continuous motion such that the distance of some vertices changes. Otherwise, the framework is *rigid*. Figure 1 shows examples.

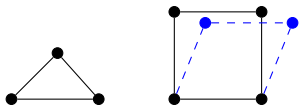


Figure 1: The triangle is rigid whereas the rectangle is flexible with the motion indicated in blue. The distance of the diagonally opposite vertices changes with the motion.

The problem whether a given framework (G, p) is flexible is NP-hard [Abb08]. It is complete for the existential theory of reals [ADD+16]. We consider a restricted version of the problem called *infinitesimal rigidity* that can be solved efficiently.

Let $G = (V, E)$ where $V = [n]$ and $|E| = m$. Let $p(i) = (x_i, y_i) \in \mathbb{R}^2$ be the coordinates of vertex $i \in V$. For an edge $e = (i, j) \in E$, the square of its length is

$$|e|^2 = (x_i - x_j)^2 + (y_i - y_j)^2. \tag{2}$$

Consider a smooth motion by treating the coordinates as functions $x_i(t)$ and $y_i(t)$ in time t, such that $(x_i(0), y_i(0)) = (x_i, y_i)$. The condition for the motion is that $|e|^2$ does not change, i.e. is constant. Hence, when we look at the derivative of (2) w.r.t. t, we get

$$2(x_i - x_j)(x_i' - x_j') + 2(y_i - y_j)(y_i' - y_j') = 0. \tag{3}$$

We get m such equations, one for every edge in G. We combine them in matrix-vector form. That is, we define the *rigidity matrix* $R = R(G, p)$ as a $m \times 2n$ matrix, with n columns for the x-part of the nodes and n columns for the y-part. Let R_e be the row for edge $e = (i, j) \in E$ in R. We define the k-th entry $R_{e,k}$ as

$$R_{e,k} = \begin{cases} x_i - x_j, & \text{for } k = i, \\ x_j - x_i, & \text{for } k = j, \\ y_i - y_j, & \text{for } k = n + i, \\ y_j - y_i, & \text{for } k = n + j, \\ 0, & \text{otherwise.} \end{cases}$$

The derivatives we put in the *velocity vector* v,

$$v = (x_1', x_2', \ldots, x_n', \; y_1', y_2', \ldots, y_n')^T.$$

Then (3) becomes

$$Rv = 0. \tag{4}$$

For example, consider the triangle graph on three nodes $1, 2, 3$. Then R is a 3×6 matrix,

$$R = \begin{pmatrix} x_1 - x_2 & x_2 - x_1 & 0 & y_1 - y_2 & y_2 - y_1 & 0 \\ 0 & x_2 - x_3 & x_3 - x_2 & 0 & y_2 - y_3 & y_3 - y_2 \\ x_1 - x_3 & 0 & x_3 - x_1 & y_1 - y_3 & 0 & y_3 - y_1 \end{pmatrix}$$

Any nonzero vector v that fulfills (4) corresponds to an *infinitesimal motion*. However, there are three trivial motions that are always possible: a shift s_x of all vertices in x-direction or s_y in y-direction, and a rotation r around the origin,

$$\begin{aligned} s_x &= (1, 1, \ldots, 1,\ 0, 0, \ldots, 0)^T, \\ s_y &= (0, 0, \ldots, 0,\ 1, 1, \ldots, 1)^T, \\ r &= (-y_1, -y_2, \cdots - y_n,\ x_1, x_2 \ldots, x_n)^T. \end{aligned}$$

Hence, the nullspace of R has dimension at least 3, and therefore, the rank of R can be at most $2n-3$.

Definition 2.1 (Infinitesimal rigidity) *A framework* (G, p) *is* infinitesimally rigid *if*

$$\operatorname{rank}(R(G, p)) = 2n - 3.$$

A graph G *is* (infinitesimally) rigid[1] *if there exists an embedding* p *such that the framework* (G, p) *is infinitesimally rigid.*

The definition for a graph G to be rigid given here is equivalent to the definition given in the Introduction [Glu75, AR79].

Clearly, the condition for the rank can only be achieved when we have $m \geq 2n - 3$ edges. The rank of a matrix can be computed in NC^2 [Mul87], thus, infinitesimal rigidity for a given embedding p can be tested in NC^2. If p is *not* given, the rigidity problem becomes a PIT-question. In 2D, it can be solved in polynomial time.

We remark that the notions of rigidity and infinitesimal rigidity of a framework (G, p) are *not* equivalent: If (G, p) is infinitesimal rigid, it is also rigid. But when $\operatorname{rank}(R(G, p)) < 2n - 3$, the framework might still be rigid. This is because a nonzero velocity vector v with $Rv = 0$ only guarantees that the edges of G maintain their lengths in direction v. It does not insure that two non-adjacent nodes change their distance. Examples where this happens are the trivial motions, but there are also examples with non-trivial motions.

2.2 Rigidity of graphs with bounded treewidth

A *tree decomposition* of a graph $G = (V, E)$ is a tree $\mathcal{T}$ with a set of nodes $\mathcal{B}$, the *bags*, where each $B \in \mathcal{B}$ is a subset of V, such that the following conditions hold.

1. $\bigcup_{B \in \mathcal{B}} B = V$,
2. $\forall (u, v) \in E\ \exists B \in \mathcal{B}\ \ u, v \in B$,

[1] In the literature it is common to skip the adjective *infinitesimally* when considering a graph G.

3. $\forall v \in V \;\; \mathcal{B}_v = \{ B \in \mathcal{B} \mid v \in B \}$ forms a subtree of $\mathcal{T}$.

The *width* of the tree decomposition is $w = \max_{B \in \mathcal{B}} |B| - 1$. The *treewidth of* G is the minimum width over all tree decompositions of G. A class of graphs $\mathcal{G}$ has *bounded treewidth*, if there is a constant c, such that all graphs in $\mathcal{G}$ have treewidth bounded by c.

Courcelle [Cou90] showed that when a graph property is expressible in monadic second-order logic (MSO-logic), then it can be decided in linear time when the input graph has bounded treewidth. Elberfeld, Jakoby and Tantau [EJT10] showed a logspace-version of Courcelle's Theorem.

Theorem 2.2 ([EJT10]) *All graph properties expressible in MSO-logic can be solved in* L *on graphs of bounded treewidth.*

The characterization in Theorem 1.3 can be used to express rigidity in MSO-logic. For a simple undirected graph $G = (V, E)$, there is a MSO-predicate $\mathsf{Tree}(V, E)$ that is true iff G is a tree, see [CE12, Section 1.3]. Then the following predicate $\mathsf{minRigid}(V, E)$ is true iff G is minimally rigid,

$$\begin{aligned} \mathsf{minRigid}(V, E) \;\; = \;\; \forall e \in E \; \exists T_1, T_2 \subseteq E \; (\; & \mathsf{Tree}(V, T_1) \wedge \mathsf{Tree}(V, T_2) \wedge \\ & e \in T_1 \wedge e \in T_2 \wedge \\ & \forall f \in E - e \; (\, f \in T_1 \oplus f \in T_2 \,) \;). \end{aligned}$$

Since a graph $G = (V, E)$ is rigid iff it has a spanning minimally rigid subgraph, we get a MSO-predicate for G being rigid,

$$\mathsf{Rigid}(V, E) = \exists F \subseteq E \;\; \mathsf{minRigid}(V, F)$$

It follows that whether a graph is Laman or a rigid graph in general is in NC^2, for graphs with bounded treewidth.

Corollary 2.3 *Rigidity of graphs with bounded treewidth can be decided in* L *and thus in* NC^2.

2.3 Structural Decompositions

Let $G = (V, E)$ be a graph. A set $S \subseteq V$ with $|S| = k$ is called a k-*separating set*, if $G - S$ is not connected. Let $G_1, \ldots, G_l$ be the connected components of $G - S$. The *split graphs with respect to* S are the subgraphs of G induced by $G_i \cup S$, where we add *virtual edges* between every pair of vertices in S. A graph G is called k-connected if there is no $(k-1)$-separating set in G.

A k-separating set is called *articulation point* for $k = 1$, *separating pair* for $k = 2$, and *separating triple* for $k = 3$.

Laman graphs are clearly connected, actually even 2-connected.

Lemma 2.4 ([JJ05, Lemma 2.6]) *Laman graphs are 2-connected.*

In particular, every node of a Laman graph has degree at least two.

3-connected components. Let $G = (V, E)$ a 2-connected graph. A separating pair $\{u, v\}$ in G is called 3-*connected*, if there are 3 vertex disjoint paths between u and v in G.

The *triconnected components* of G are the split graphs we obtain from G when we successively split G along all 3-connected separating pairs, in any order. If a separating pair $\{u, v\}$ is connected by an edge in G, we also define a 3-*bond* for $\{u, v\}$ as a triconnected component. This is the multigraph with two vertices u, v with 3 edges between them. The 3-bond components are there to be able to reconstruct the original graph from the components.

It is known that the triconnected components of G are uniquely determined, i.e. independent of the order of the separating pairs in which we do the splitting.

Lemma 2.5 ([Mac37, HT72]) *The triconnected components of a 2-connected graph are unique.*

The decomposition leads to the *triconnected component tree*: There is a node for each triconnected component and each 3-connected separating pair of G. There is an edge between triconnected component node C and separating pair node $\{u, v\}$, if $u, v \in C$.

4-connected components. We also need to further decompose 3-connected graphs along separating triples into 4-connected components. The split components of two separating triples might overlap and thus we cannot simply split along all separating triples. For example, in a $K_{3,3}$-graph both sides form separating triples and we cannot split along both. For an efficient splitting procedure with respect to parallel computation see [TW14] or [EV21].

The decomposition again leads to a tree, the 4-*connected component tree*. In this tree we have vertices for the separating triples and for the 4-connected components. In addition, there is a vertex representing a 3-bond component for every edge (u, v) from G, where u, v are part of a separating triple. Two vertices in the 4-connected component tree are adjacent if one of them corresponds to a separating triple and the other one to a 4-connected component or a 3-bond sharing vertices with the triple.

Complexity. Graph reachability problems can be solved in nondeterministic logspace, NL. In undirected graphs, even in deterministic logspace, L [Rei08]. Problems in NL like directed s-t-reachability can be reduced to matrix powering which yields efficient parallel algorithms. We have

$$\mathsf{NC}^1 \subseteq \mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{NC}^2.$$

In the following lemma, we list some known results along these lines that we will need later on.

Lemma 2.6 (Complexity summary for connectivity) *Let G be an undirected graph. The following problems can all be solved in* NC^2.

1. *Compute the articulation points and the connected components of* G.
2. *When G is 2-connected, compute the 3-connected separating pairs, the triconnected components, and the triconnected component tree of* G.
3. *When G is 3-connected, compute the separating triples, the fourconnected components, and the four connected component tree of* G [TW14].

The component trees can have large depth. As we want to process them in a bottom-up fashion in logarithmic time, we need to identify long paths and treat them separately. Let T be a rooted tree and let v be a vertex in T. Then $T(v)$ is the subtree of T rooted at v. A child u of v is a *large child* if $|T(u)| > |T(v)|/2$. A *large child path in* T is a maximal path such that every vertex along the path is a large child of its parent.

Lemma 2.7 ([STW16]) *Let T be a tree with n nodes and p be a simple path in T. Then*

- *the number of large child paths on p is $\leq \log n$,*
- *the number of nodes on p that are* not *large children is $\leq \log n$,*
- *all large child paths in T can be computed in NC^2.*

By the first two items in Lemma 2.7, the number of large child paths in T is polynomially bounded in n. Then the last item follows because we can compute the number of nodes in subtrees of T in L and hence, also in NC^2.

Structure theorems. The reason why we are considering the above decompositions is that for $K_{3,3}$-free, K_5-free, and one-crossing-minor-free graphs, we end up in components that are planar or of bounded treewidth. In more detail, for $K_{3,3}$-free graphs, the decomposition leads to planar components or K_5-components.

Theorem 2.8 ([Asa85]) *Every 3-connected $K_{3,3}$-free graph is either planar or K_5.*

For K_5-free Laman graphs, we end up in planar components or a special constant size graph.

Theorem 2.9 ([Wag37]) *Every 4-connected component of a K_5-free graph is either planar or the Möbius ladder on 8 vertices, also known as* Wagner graph.

For one-crossing-minor-free graphs, we get planar or bounded treewidth components.

Theorem 2.10 ([RS93]) *Every 4-connected component of a one-crossing-minor-free graph is either planar or of bounded treewidth.*

2.4 Henneberg sequences

Laman graphs can be constructed iteratively via *Henneberg extensions* [Hen11]. The starting point in a sequence of Henneberg extensions is a graph with two nodes connected by an edge. Let G be the graph constructed so far. There are two ways to add a new node v to G:

- A *Henneberg extension of type* 1 connects v with two arbitrary vertices of G.
- A *Henneberg extension of type* 2 takes an existing edge (u, w) in G and replaces it with edges (u, v) and (v, w) instead. Additionally v is connected to an arbitrary third vertex in G.

Theorem 2.11 ([TW85]) *A graph $G = (V, E)$ is Laman iff it can be constructed by a sequence of Henneberg extensions that starts with an arbitrary edge $e \in E$.*

Henneberg sequences can also be reversed in the following way.

Lemma 2.12 ([JJ05, Lemma 2.8]) *Let $G = (V, E)$ be a Laman graph with $|V| \geq 3$ and $v \in V$.*

1. *If $\deg(v) = 2$, then $G - v$ is Laman.*
2. *If $\deg(v) = 3$, then $G - v + (u, w)$ is Laman for some pair u, w of neighbors of v.*

Using reversed Henneberg operations one can generalize Theorem 2.11: One can start the Henneberg sequence for a Laman graph with *any* two nodes u and v and edge (u, v), even if (u, v) is not present in G.

Lemma 2.13 ([HOR+05, Lemma 2]) *A Laman graph $G = (V, E)$ has a Henneberg construction that starts from edge (u, v), for any two nodes $u, v \in V$.*

The proof for Lemma 2.13 roughly works by applying reversed Henneberg steps to the graph until we end up with the edge (u, v). We further generalize Lemma 2.13 by showing the existence of a Henneberg sequence that starts with a triangle on any three vertices $T = \{u, v, w\}$ that contain at least one edge in G. To do so, we first make a technical observation for the case that all vertices not in T have large degree.

Lemma 2.14 *Let $G = (V, E)$ be a Laman graph, $|V| \geq 4$, and $T = \{u, v, w\} \subseteq V$ such that every $x \in V' = V - T$ has $\deg(x) \geq 4$. Then we have*

- $\deg(u) = \deg(v) = \deg(w) = 2$,
- $|E(T)| = 0$,
- $G - T$ *is a connected graph.*

Proof. The sum of all node degrees of G is at least $\deg(u) + \deg(v) + \deg(w) + 4(n - 3)$. By the degree sum formula, we therefore have

$$\deg(u) + \deg(v) + \deg(w) + 4(n - 3) \leq 2m.$$

Since $m = 2n - 3$, we conclude that

$$\deg(u) + \deg(v) + \deg(w) \leq 6.$$

Since G is Laman, we also have $\deg(u), \deg(v), \deg(w) \geq 2$. This implies the first item.

To show the second item, let $n' = |V'|$ and $m' = |E(V')|$. Let $E(V', T)$ be the edges between V' and T. Since the nodes in T have degree 2, we have $|E(V', T)| \leq 6$. We argue that $|E(V', T)| = 6$, and hence $|E(T)| = 0$.

The sum of all node degrees in V' is at least $4n' - |E(V', T)|$. By the degree sum formula, we therefore have

$$4n' - |E(V', T)| \leq 2m'.$$

Since G is Laman, we also have $m' \leq 2n' - 3$, and therefore

$$|E(V', T)| \geq 6.$$

To argue that $G - T$ is connected, assume that there are two components in $G - T$, say on nodes V_1 and V_2, respectively. For each component we can make the same estimates as for V' above to show the second item. Hence, we would get that $|E(V_1, T)|$, $|E(V_2, T)| \geq 6$. But this contradicts the first item. □

We use Lemma 2.14 to show that a Henneberg sequence for a Laman graph G can start with a triangle on any three nodes, as long as there is at least one edge between the nodes in G. This generalizes a result by Haas et al. [HOR+05, Lemma 3].

Lemma 2.15 *Let $G = (V, E)$ be a Laman graph and $T = \{u, v, w\} \subseteq V$ such that $|E(T)| \geq 1$. Then there is a Henneberg sequence for G that starts with triangle (u, v, w).*

Proof. We prove the claim by induction on $n = |V|$. For $n = 3$ the claim is trivially true since the only possibility for G is to be exactly triangle (u, v, w). Let $n \geq 4$. Let vertex $x \in V' = V - T$ such that $\deg(x) \leq 3$. Note that x must exist because otherwise, when $\deg(x) \geq 4$, for all $x \in V'$, then $|E(T)| = 0$ by Lemma 2.14, but we have $|E(T)| \geq 1$.

We remove x by a reversed Henneberg step from Lemma 2.12. Let $H = G - x$. The removal operation can only increase the number of edges within T in H. Thus, by the induction hypothesis, there is a Henneberg sequence for H that starts with triangle (u, v, w). We extend the sequence by adding x back to H. Then the sequence produces G. □

The assumption in Lemma 2.15 is necessary: There are examples for a graph G where $|E(T)| = 0$ and it is *not* possible to construct G from triangle (u, v, w). In Section 5, we will consider the case where G is 3-connected and T is a separating triple. Then we can still get a useful statement about the starting point of a Henneberg sequence from Lemma 2.14 and 2.15.

Corollary 2.16 *Let $G = (V, E)$ be a 3-connected Laman graph with a separating triple $T = \{u, v, w\}$ and corresponding split graphs $G_1, G_2, \ldots, G_\ell$, where we have removed all virtual edges from the split graphs. Then there is a Henneberg sequence for G that starts*

- *either with triangle (u, v, w),*
- *or there is a split component, say $G_1 = (V_1, E_1)$, that is Laman and $|E_1(T)| = 0$, and the sequence initially constructs G_1.*

Proof. The claim follows from Lemma 2.15 when $|E(T)| \geq 1$. So assume that $|E(T)| = 0$. Like in the proof of Lemma 2.15, we remove vertices $x \in V' = V - T$ with $\deg(x) \leq 3$ by reversed Henneberg steps, as long as there are such vertices. In case this process introduces an edge within T at some point, then it will stop with triangle (u, v, w) as in the proof of Lemma 2.15. Otherwise, it will stop with a Laman graph $H = (V_H, E_H)$ with $|E_H(T)| = 0$. By Lemma 2.14, graph $H - T$ is connected. Hence, H can be obtained by reverse Henneberg steps from one of the split components $G_1, G_2, \ldots, G_\ell$, say G_1. Now a Henneberg sequence for H can be extended to a sequence for G_1

by adding the vertices of G_1 back that we removed above. Note that $T \subseteq V_H$. Hence, adding the remaining vertices of G_1 will not introduce an edge within T. Now we can add the rest of the vertices back that we removed above. This gives the desired Henneberg sequence for G. □

3 Rigid embeddings for planar graphs in NC^2

Streinu [Str00] and Haas et al. [HOR+05] showed that planar Laman graphs can be characterized by planar embeddings in the plane with certain geometric properties. We observe that this characterization can be verified efficiently in parallel. Moreover, the geometric properties can be used to compute an infinitesimally rigid embedding in NC^2.

Consider a straight-line embedding of a planar graph $G = (V, E)$. For an angle between two adjacent edges, we call the angle *convex*, if it is strictly less than 180°, and *reflex*, if it is strictly larger than 180°. We say that vertex $v \in V$ in the embedding of G is *pointed*, if some consecutive pair of edges in the circular order of edges around v span a reflex angle. We call the embedding of G *pointed*, if every $v \in V$ is pointed.

The faces of a straight-line embedding of G are simple polygons. A vertex of a polygon is *convex*, if the interior angle between its two adjacent edges is convex. The vertex is *reflex*, if the interior angle is reflex. A polygon is a *pseudo-triangle*, if it has exactly three convex vertices, and all other vertices being reflex. Note that a triangle is also a pseudo-triangle.

We say that a planar graph admits a *pointed pseudo-triangulation (PPT)*, if it can be embedded in the plane such that every vertex is pointed, every interior face is a pseudo-triangle, and the outer face is the complement of the convex hull of all vertices. Streinu [Str00] first showed that the underlying graph of a pointed pseudo-triangulation is always Laman. Then Haas et al. [HOR+05] showed the reverse direction.

Theorem 3.1 ([**Str00**, **HOR+05**]) *A planar graph* G *is Laman iff* G *admits a pointed pseudo-triangulation.*

Furthermore, a pointed pseudo-triangulation embedding of a Laman graph simultaneously provides an infinitesimally rigid embedding.

Theorem 3.2 ([**Str00**, **Str05**]) *Any pointed pseudo-triangulation embedding of a planar Laman graph is an infinitesimally rigid embedding.*

The difficulty in finding a PPT embedding of a graph are the conflicting conditions that around each vertex all angles except one should be convex, but every face must have exactly three convex angles. For a planar embedding of a graph, Haas et al. [HOR+05] defined a *combinatorial pointed pseudo-triangulation (CPPT)* that just assigns label C (for *convex*) or R (for *reflex*) to every angle such that the assignment combinatorially corresponds to a PPT embedding. That is:

1. Every vertex has exactly one angle labeled R,
2. every interior face has exactly three angles labeled C,
3. every angle incident to the outer face is labeled R.

Clearly, the angles of a PPT embedding of a graph G yield a CPPT, and hence, a CPPT is a necessary condition for a PPT. However, the converse might not be true. Nevertheless, by Euler's formula for planar graphs, one can show that a graph with a CPPT has exactly $m = 2n - 3$ edges [HOR+05, Lemma 5].

Finding a CPPT for a planar graph G is the first step to find a PPT embedding. Haas et al. [HOR+05] reduce the CPPT problem to a perfect matching problem on a bipartite graph H. However, H is no longer planar in general, and NC-algorithms for perfect matching are only known for planar bipartite graphs. Instead, Rollin, Schlipf, and Schultz [RSS19] reduce the CPPT problem to a maximum flow problem with multiple sources and sinks on a bipartite graph H' that is very similar to H, but is *planar*. Below we describe H' so that one can see that it can be constructed efficiently in parallel. Then the flow algorithm by Miller and Naor [MN95] can be used to compute a CPPT assignment efficiently in parallel.

Lemma 3.3 ([RSS19, Lemma 9]) *For a planar graph* G*, finding a CPPT can be reduced to the problem of finding a flow in a planar graph with multiple sources and sinks with given demands.*

Proof. Let $G = (V, E)$ have $m = 2n - 3$ edges and let F be the set of faces in a planar embedding. We want to determine which angles in a face should be assigned to be reflex, and the one reflex angle of every node.

For each face $f \in F$, let d_f be the number of nodes around f. Recall that $d_f - 3$ nodes should be reflex nodes in f, for the interior faces, and d_f when f is the outer face. Correspondingly, we set up the flow network H' such that for every face $f \in F$ there are two nodes f_1, f_2 connected by a directed edge (f_1, f_2) with capacity $d_f - 3$, for every inner face, and capacity d_f, for the outer face, where nodes f_1 are source nodes.

Then we connect the f_2-nodes with the vertices of G. That is, for every $v \in V$ there are two nodes v_1, v_2 in H' connected by a directed edge (v_1, v_2) with capacity 1, where the nodes v_2 are sink nodes. For the connection between faces and nodes, we put a directed edge (f_2, v_1) with capacity 1, whenever node v belongs to face f. Note that H' is planar.

Clearly, the sum of the sink-capacities is n. It follows from Euler's formula that also the sum of the source-capacities is n. Moreover, an integer flow of value n corresponds to a CPPT-assignment for G: the edges (f_2, v_1) with flow 1 indicate where the reflex angle of node v is. All other angles are set to convex. □

Since the network H' in the proof of Lemma 3.3 can be constructed efficiently in parallel from G, we can combine it with the flow algorithm from Miller and Naor [MN95].

Corollary 3.4 *For a planar graph* G*, we can find a CPPT in* NC^2*, if one exists.*

It is not immediately clear that a CPPT-assignment can also be realized geometrically as a corresponding PPT-embedding.

Definition 3.5 (Stretchable CPPT) *A CPPT of a graph* G *is* stretchable*, if* G *admits a compatible PPT, i.e. a PPT-embedding, where angles labeled* C *are convex and angles labeled* R *are reflex.*

Observe that the above flow network H' might have many maximum flows, and correspondingly, there can be many CPPT assignments for G. For Laman graphs, all these assignments are stretchable.

Theorem 3.6 ([HOR+05, Theorem 9]) *For a planar Laman graph, every CPPT assignment is stretchable.*

Haas et al. [HOR+05] gave a characterization of when a CPPT assignment of graph G is stretchable via an associated planar graph G^*. Graph G^* contains G, but some of the edges are turned into directed edges. There are also additional directed edges such that the inner faces of G become triangulated, but still are planar. Further conditions are

1. the boundary vertices of G have no outgoing edges in G^*,
2. every interior vertex v in G has 3 outgoing edges in G^*, where two of them are incident to the reflex angle of v and the third edge lies in the face containing the reflex angle.

The above conditions do not specify G^* uniquely. However, any graph with the above properties is fine for our purpose. Haas et al. [HOR+05] describe a recursive algorithm to construct G^*. Rollin, Schlipf, and Schulz [RSS19] gave a construction that works in linear time. Their construction can also be accomplished efficiently in parallel.

Lemma 3.7 ([HOR+05, RSS19]) *Given a CPPT assignment of a planar graph* G*, the graph* G^* *can be constructed in* NC^1.

Proof. First, for each inner vertex v, the two edges incident to the reflex angle of v are oriented away from v. Then we triangulate all inner faces that are not already triangles. Consider such a face and let $a, b, c \in V$ be the three nodes marked as convex. Let $V_{a,b}, V_{a,c}, V_{b,c}$ be the vertices between a, b, respectively a, c and b, c. At least one of the sets is non-empty, say $V_{a,b} \neq \emptyset$. For every $v \in V_{a,b}$, we put a directed edge (v, c). Let $x \in V_{a,b}$ be the neighbor of a and $y \in V_{a,b}$ be the neighbor of b. Then we add directed edges (v, x), for all $v \in V_{a,c}$, and (w, y), for all $w \in V_{b,c}$.

Both steps can be done in parallel for every inner vertex and every inner face. □

Whether a CPPT assignment for G is stretchable can now be characterized by a connectivity property of G^*. In a directed graph, a set of vertices S is 3-*connected to a disjoint set of vertices* T, if for every $v \in S$ there are 3 vertex disjoint paths to 3 distinct vertices in T.

Theorem 3.8 ([HOR+05]) *Let* G *be a planar graph with a CPPT assignment and* G^* *be the graph associated with* G*. Then the CPPT* G *is stretchable if and only if the set of interior vertices is* 3*-connected to the set of boundary vertices in* G^*.

It follows that checking stretchability can be done efficiently in parallel.

Corollary 3.9 *Given a planar graph* G *with a CPPT assignment, we can check if* G *is stretchable in* NC^2.

Proof. Let $G = (V, E)$. First, we construct graph G^* by Lemma 3.7. Let S be the interior nodes of G^* and T be the nodes on the boundary. We add a new vertex t in the outer face of G^* and connect all nodes $v \in T$ with t with a directed edge (v, t). By Menger's Theorem S is 3-connected to T if and only if for every $s \in S$ and for every $u, v \in V - \{s, t\}$ there is a path from s to t in $G^* - \{u, v\}$. The latter condition can be checked in NC^2 as explained in Section 2.3. Now the claim follows from Theorem 3.8. □

In summary, we can decide whether a planar graph is minimally rigid efficiently in parallel.

Theorem 3.10 *Deciding whether a planar graph is Laman is in* NC^2.

Proof. Let G be planar. We first compute a planar combinatorial embedding for G, and then a CPPT assignment for the embedding. Finally, we check if the CPPT is stretchable.

By Lemma 3.4 and Corollary 3.9 we know that G is Laman if and only if all subroutines work out positively. Moreover, all subroutines are in NC^2, and hence the overall algorithm as well. □

Remark 3.11 *In terms of logarithmic space classes, observe that all the subroutines in the decision algorithm for planar Laman graphs are actually in* NL*, inclusively the Miller-Naor flow problem [DGKT12]. Hence, a stronger form of Theorem 3.10 is that planar Laman decision is in* NL.

Haas et al. [HOR+05] also show that one can compute a PPT *embedding* for planar Laman graphs. By Theorem 3.2, this provides an embedding, where the rigidity matrix has full rank. Hence, this gives a derandomization of the rank problem for the rigidity matrix for planar graphs.

Theorem 3.12 ([HOR+05]) *Given a planar Laman graph* G *with* n *nodes, one can compute a PPT embedding for* G *in time* $O(n^{3/2})$.

The algorithm to compute the PPT embedding is based on Tutte's Theorem on barycentric embeddings of graphs. Essentially, we fix the coordinates of the outer face vertices $v_1, \ldots, v_k$ to be the vertices of a convex k-gon. The coordinates of the remaining vertices are computed by solving a system of linear equations which can be done in NC^2 [Mul87].

Corollary 3.13 *Given a planar Laman graph* G*, computing a PPT embedding for* G *is in* NC^2.

4 Rigid embeddings for $K_{3,3}$-free graphs in NC^2

Let G be a $K_{3,3}$-free graph. We want to check whether G is Laman efficiently in parallel. By Lemma 2.4, we may assume that G is 2-connected. By Theorem 2.8, when we decompose G into 3-connected components, these components are either planar or K_5. For planar components we can check if they are Laman by Theorem 3.10. Hence, what we need is a connection between the Laman properties of G and its 3-connected components. This is established by the following lemma.

Lemma 4.1 *Let* $G = (V, E)$ *be a 2-connected graph with a separating pair* $\{u, v\}$ *and corresponding split graphs* $G_1, G_2, \ldots, G_\ell$*, where we have removed all virtual edges from the split graphs.*

1. *If* $(u,v) \in E$, *then* G *is Laman iff* $G_1, \dots, G_\ell$ *are Laman.*

2. *If* $(u,v) \notin E$, *then* G *is Laman iff there exists one component, say* G_1, *that is Laman, and* $G_2 + (u,v), \dots, G_\ell + (u,v)$ *are Laman.*

Proof. Let G be Laman. By Lemma 2.13, there is a Henneberg construction for G that starts with edge (u,v). Recall that this is independent of whether (u,v) is an edge in G. When a new vertex is added in the Henneberg sequence, it belongs to exactly one of the split graphs G_i and the extension cannot interfere with some vertex from another split graph. Otherwise, the separating pair would not separate the split graphs from each other in G.

1. If $(u,v) \in E$, this gives us Henneberg sequences for all split graphs by subdividing the sequence for G in its parts for $G_1, \dots, G_\ell$, respectively. Hence, they are all Laman.

2. If $(u,v) \notin E$, it has been replaced by a type 2 extension adding a vertex in exactly one split graph, say G_1. Again we subdivide the Henneberg sequence for G, and get sequences for G_1 and for $G_2 + (u,v), \dots, G_\ell + (u,v)$. Therefore, they are all Laman.

For the backward direction, in the first case, we consider Henneberg constructions for $G_1, G_2, \dots, G_l$ that start with edge (u,v). Since the edge is present in all the components, it is never used in any of the Henneberg sequences. Hence, we can combine all the Henneberg sequences to a sequence for G. Hence, G is Laman.

The second case is similar. Since (u,v) is not present in G_1, the edge is used in the sequence for G_1, but not in any of the sequences for $G_2 + (u,v), \dots, G_\ell + (u,v)$. Hence, we can again combine all the Henneberg sequences to a sequence for G. Hence, G is Laman. □

The above lemma motivates us to define an operation on 2-connected graphs, which we call Laman-split.

Definition 4.2 (Laman-split) *For a 2-connected graph* $G = (V, E)$ *with a separating pair* $\{u,v\}$, *let* $G_1, G_2, \dots, G_\ell$ *be the split graphs obtained after splitting* G *along* $\{u,v\}$, *where we have removed all virtual edges. The* Laman-split *of* G *along* $\{u,v\}$ *are the graphs* $G'_1, G'_2, \dots, G'_\ell$, *where for each* $i \in [\ell]$,

$$G'_i = \begin{cases} G_i + (u,v), & \text{if } m(G_i) = 2n(G_i) - 4, \\ G_i, & \text{otherwise.} \end{cases}$$

For a Laman graph, all split graphs G_i in Definition 4.2 have either $2n(G_i) - 4$ or $2n(G_i) - 3$ edges by Lemma 4.1. Note that we define Laman-split also for graphs that are not Laman. In this case, the split graphs can also have other numbers of edges. In such a case, G_i, and hence G, are trivially detected as not being Laman.

Recall that by Lemma 2.5, the standard splitting of 2-connected graphs in triconnected components is unique, i.e. independent of the order of the separating pairs we do the splitting. The following lemma shows when we apply Laman-splits to the components on the way, the resulting Laman components are unique as well.

Lemma 4.3 *Let* $G = (V, E)$ *be a 2-connected graph. Then* G *is Laman iff there is a way to put the separating pair edges* $(u, v) \notin E$ *into the triconnected components of* G *such that* (u, v) *is in all but one of the components that contain* u, v *and that the resulting components are all Laman.*

Moreover, in case G *is Laman, this* Laman decomposition *is unique and can be computed in* NC^2.

Proof. Consider the standard recursive splitting procedure to compute the triconnected component tree of G. When we have split along a separating pair $\{u, v\}$, we can also compute the Laman-split that says in which split components edge (u, v) should be put. Note that the recursive splitting is always done on the components computed by the standard splitting procedure. The Laman-split is a post-computation on these components that does not affect the recursive splitting. By the characterization given in Lemma 4.1, we conclude that G is Laman iff all the components computed by Laman-splits are Laman.

It remains to argue about the uniqueness of the Laman decomposition. This property is crucial for our parallel algorithm to compute the decomposition. So assume that G is a Laman graph. By Lemma 2.5, the triconnected components are unique. We argue that also the corresponding components we get from Laman-splits are uniquely determined. That is, whether a separating pair edge is present or not in a component is irrespective of the order of the decomposition.

This is trivial in case a separating pair $\{u, v\}$ is connected by an edge (u, v) in G. Then all components will have the edge (u, v) as well by Lemma 4.1. We only have to argue for the case that $\{u, v\}$ is *not* connected by an edge in G. In this case, the edge (u, v) will be present in all but one of the components by Lemma 4.1. We argue that the component without edge (u, v) is uniquely determined.

Consider the triconnected component tree $\mathcal{T}$. We argue via induction on the number of component nodes in $\mathcal{T}$. If $\mathcal{T}$ has just one component node, then there is no separating pair and the claim is trivial.

In the inductive step, let $\mathcal{T}$ have more than one component node. Let C be a component node with a separating pair $\{u, v\}$ such that *all* other split components at $\{u, v\}$ are leafs in $\mathcal{T}$. In the leaf components $\{u, v\}$ is the only separating pair. Hence, it is uniquely defined whether the separating pair edge should be present in a leaf component or not, so that it has the right number of edges to be Laman. Therefore the same holds for the parent component C by Lemma 4.1. Note also that the presence or absence of a separating pair edge (u, v) in C is not affected when C is further split along a different separating pair.

Now we can prune the leaf components considered above from $\mathcal{T}$ and get a tree with a smaller number of component nodes where we can apply the induction hypothesis.

For the complexity bound, we describe a parallel procedure to obtain the Laman components. First we compute all triconnected components in NC^2 (Lemma 2.6). To determine where to put the separating pair edges, we do a Laman-split of G, for every separating pair $\{u, v\}$ in parallel. That is, we treat each separating pair as the starting point of a Laman decomposition of G. Thereby we will put the edges correctly in the respective components by the uniqueness property: For any triconnected component H that contains $\{u, v\}$, we add the separating pair edge (u, v) to H, if after Laman-split in G the component H' that contains H has edge (u, v). □

Now we have all the tools to decide efficiently in parallel whether a given $K_{3,3}$-free graph is Laman.

Theorem 4.4 *Given a $K_{3,3}$-free graph G, we can decide whether G is Laman in NC^2.*

Proof. Given a 2-connected $K_{3,3}$-free graph G, we apply the algorithm from Lemma 4.3 to compute its Laman components. Here we might already detect that G is not Laman when the Laman-split yields some component where the number of edges does not match the number according to Lemma 4.1. Otherwise, we have that G is Laman iff all the components are Laman. Note that the components are planar graphs or subgraphs of K_5 by Theorem 2.8, because separating pair edges only replace virtual edges, and hence do not affect planarity. Thus, we can apply Theorem 3.10 for all components in parallel to check if they are all Laman. All the subroutines used are in NC^2. □

The decision algorithm splits the graph G along separating pairs until all components are planar and then checks that these components are Laman. We observed in Section 3 that we can also compute rigid embeddings for the planar components. To find a rigid embedding of G, we now want to reassemble the embeddings of the components appropriately.

In Lemma 4.5, we make the assumption that the two nodes of a separating pair are mapped to the same pair of points in all the components, respectively. We show that then we directly have a rigid embedding for the whole graph G.

Lemma 4.5 *Let $G = (V, E)$ be a Laman graph with a separating pair $\{u, v\}$. Let $G_1, G_2, \ldots, G_\ell$ be the Laman components obtained after a Laman-split of G along $\{u, v\}$. Let $p_1, p_2, \ldots, p_\ell$ be infinitesimal rigid embeddings of the components such that*

$$\begin{aligned} p_1(u) = p_2(u) &= \cdots = p_\ell(u), \\ p_1(v) = p_2(v) &= \cdots = p_\ell(v), \\ p_1(u) &\neq p_1(v), \end{aligned}$$

so that the common embedding $p = \cup_{i=1}^{\ell} p_i$ is well defined. Then p is an infinitesimally rigid embedding of G.

Proof. We prove the claim for $\ell = 2$. For larger ℓ, we can iterate the argument, combining two graphs in every round. By Lemma 4.1, edge (u, v) is *not* contained in at most one of the components. Hence, we may assume that (u, v) is contained in G_2.

When we combine the rigidity matrices $R_1 = R(G_1, p_1)$ and $R_2 = R(G_2, p_2)$ as shown in Figure 2, we essentially get the rigidity matrix $R = R(G, p)$.

By our assumption, edge (u, v) is in G_2 and hence, there is a row (u, v) in R_2. Since R_2 has full rank, row (u, v) is linearly independent from the other rows of R_2.

- If $(u, v) \in E$, then also R_1 and R have a row (u, v), and the row is linearly independent of the other rows of R_1 as well.
- If $(u, v) \notin E$, then row (u, v) is present only in R_2, but not in R_1 and R. Now, row (u, v) might be linearly dependent on the rows of R_1.

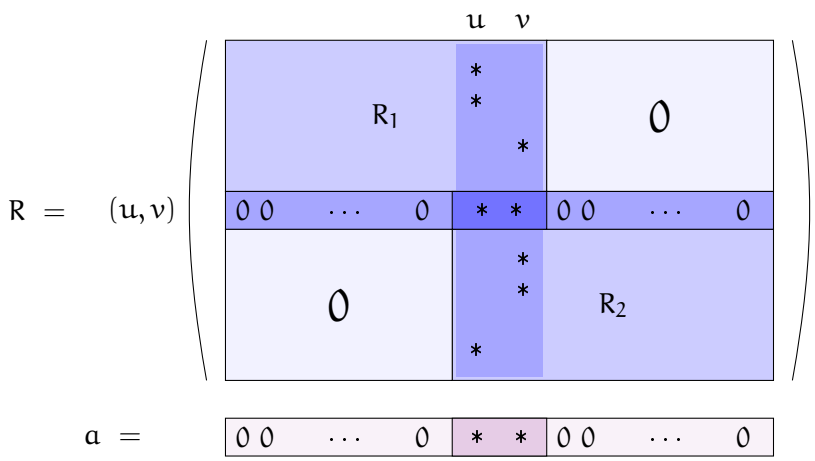


Figure 2: The rigidity matrix $R = R(G, p)$ in case $(u, v) \in E$, up to a permutation of the columns. If $(u, v) \notin E$, we have to remove row (u, v). The $*$-entries represent possible non-zero entries. Each $*$ stands for two values, an x- and a y-value.
Vector a is in the row-span of each, the upper part from R_1 and the lower part from $R_2' = R_2 - (u, v)$. Therefore the only non-zero entries of a can be at positions u and v.

We have to show that the rows of matrix R are linearly independent. Let R_2' denote the matrix consisting of all rows of R_2 except row (u, v). Recall that if row (u, v) belongs to R then it also belongs to R_1. Hence, the rows of R can be partitioned into R_1 and R_2'. Since R_1 and R_2' have full rank, the only way we can have a dependency in R is that there is a non-zero vector a that is in the row-span of R_1 and of R_2' as illustrated in Figure 2. Then $a - a = 0$ would give a non-trivial linear combination of the rows of R that produces the zero vector.

By the structure of R as shown in Figure 2, the only non-zero entries of a can be at positions u and v. We restrict our attention to these positions. Let $a_{u,v}$ and $b_{u,v}$ be the the part of a and row (u, v) at positions u and v, respectively,

$$a_{u,v} = (a_{u_x}, a_{v_x}, \ a_{u_y}, a_{v_y}),$$
$$b_{u,v} = (x_u - x_v, x_v - x_u, \ y_u - y_v, y_v - y_u).$$

By assumption, we have $b_{u,v} \neq 0$. Let M be the matrix consisting of the vectors for the three trivial motions, $M = (v_x, v_y, v_r)$, on these four positions. That is

$$M = \begin{pmatrix} 1 & 0 & -y_u \\ 1 & 0 & -y_v \\ 0 & 1 & x_u \\ 0 & 1 & x_v \end{pmatrix}.$$

Because a, and hence $a_{u,v}$ are a linear combination of the rows of the rigidity matrix, we have

$$a_{u,v} M = b_{u,v} M = 0.$$

Note that M is 4×3 matrix and $\operatorname{rank}(M) = 3$ since we assume that $(x_u, y_u) \neq (x_v, y_v)$. Hence, the codimension of M is 1, or, in other words, the kernel of M^T has dimension 1. It follows that $a_{u,v}$ must be a multiple of $b_{u,v}$. Hence, vector a is a multiple of the row-vector (u, v) of R. But recall that a is in the row-span of R'_2 and row-vector (u, v) is linearly independent of R'_2. Hence, we must have $a = 0$. Therefore, the rigidity matrix R has full rank. □

To get a rigid embedding of a $K_{3,3}$-free Laman graph, it now remains to show how to achieve the assumption of Lemma 4.5.

Theorem 4.6 *Given a $K_{3,3}$-free Laman graph G, we can compute an infinitesimally rigid embedding in NC^2.*

Proof. We follow the algorithm from Theorem 4.4 and apply Lemma 4.3 to decompose G into planar Laman components $C_1, C_2, \ldots, C_k$. Then we compute infinitesimally rigid embeddings $p_1, p_2, \ldots, p_k$ for the components in parallel by Corollary 3.13.

The vertices that belong to some separating pair occur in several components. We will construct new rigid embeddings $q_1, q_2, \ldots, q_k$ that will map different copies of any such vertex to the same point, and leave all other vertices unchanged. Then $q_1, q_2, \ldots, q_k$ will fulfill the assumption of Lemma 4.5, and $q = \bigcup_{i=1}^{k} q_i$ will be a rigid embedding for G.

For $v \in V$, let

$$S_v = \{\, i \in [k] \mid v \in V(C_i) \,\}.$$

For every $v \in V$ where $|S_v| > 1$, we construct a pair of univariate polynomials $(a_v(t), b_v(t))$ that interpolates the points $\{\, p_i(v) \mid i \in S_v \,\}$. That is, we compute the interpolation polynomials such that $(a_v(i), b_v(i)) = p_i(v)$, for every $i \in S_v$. Then we replace the coordinates of such a vertex v by $(a_v(t), b_v(t))$ in each component. That is, we define embeddings $p_{i,t}(v)$, for $i \in [k]$ and $v \in V(C_i)$,

$$p_{i,t}(v) = \begin{cases} (a_v(t), b_v(t)), & \text{if } |S_v| > 1, \\ p_i(v), & \text{otherwise.} \end{cases}$$

Note that a component C_i can have several separating pair nodes and we replace their coordinates by different polynomials, respectively, but all in the same variable t. The interpolation guarantees that $p_{i,t}(v)$ agrees with $p_i(v)$ for $t = i$, for every $v \in V(C_i)$,

$$p_{i,i}(v) = p_i(v). \tag{5}$$

Consider the rigidity matrices $R(C_i, p_{i,t})$, where some of the entries are polynomials in t. Our goal is to find a value for t such that all matrices $R(C_i, p_{i,t})$ have full rank. Let R_i be a non-singular $(2n_i - 3)$-square submatrix of $R(C_i, p_i)$ and define $R_{i,t}$ as the corresponding submatrix of $R(C_i, p_{i,t})$. Since $\det(R_i) \neq 0$ and $R_i = R_{i,i}$ by (5), we have that $\det(R_{i,t})$ is a non-zero polynomial. Hence, the product

$$A(t) = \det(R_{1,t}) \det(R_{2,t}) \cdots \det(R_{k,t})$$

is a non-zero polynomial too. For the degree of $A(t)$ note that $\deg(a_v), \deg(b_v) = |S_v| - 1 \leq n$. Therefore

$$\deg(\det(R_{i,t})) \leq n(2n_i - 3) < 2n^2.$$

Hence, for $A(t)$ we get

$$\deg(A(t)) < k2n^2 \leq 2n^3.$$

It follows that we can find a $t_0 \in [2n^3 + 1]$ such that $A(t_0) \neq 0$. Now we define $q_i = p_{i,t_0}$, for all $i \in [k]$. By construction, q_i is still a rigid embedding of C_i. Then $q = \bigcup_{i=1}^{k} q_i$ is a rigid embedding for G by Lemma 4.5.

For the complexity, note that polynomial interpolation and evaluation is in NC^2 [EGK90].

□

5 Deciding Lamanness of one-crossing-minor-free graphs in NC^3

In this section, we give an NC^3 algorithm for deciding whether a K_5-free graph is Laman, or even more general, whether a one-crossing-minor-free graph is Laman. We use Theorem 2.9, respectively Theorem 2.10, and further decompose the graph at separating triples into 4-connected components. We first show how the Laman property is preserved in the components. This can be seen as a generalization of Lemma 4.1 for separating pairs to separating triples.

Lemma 5.1 *Let $G = (V, E)$ be a 3-connected graph with a separating triple $T = \{u, v, w\}$ and corresponding split graphs $G_1, G_2, \ldots, G_\ell$, where we have removed all virtual edges from the split graphs. Let $\Delta_T = \{(u, v),\ (u, w),\ (v, w)\}$ be the triangle edges and $E(T)$ the actual edges of G in T.*

Then G is Laman iff there is a way to put each $e \in \Delta_T - E(T)$ in all but one of $G_1, G_2, \ldots, G_\ell$, such that the resulting components are all Laman.

Proof. The argument goes along the lines of the proof of Lemma 4.1, extended to triples.

Let G be Laman. By Corollary 2.16, there is a Henneberg construction for G that starts either with triangle (u, v, w), or with one component, say $G_1 = (V_1, E_1)$, where $|E_1(T)| = 0$.

- If the sequence starts with triangle (u, v, w), each triangle edge $e \in \Delta_T - E(T)$ will be subdivided by a type 2 step, adding a vertex in one split component, say G_i. Hence, to get a Henneberg sequence for all the components, we have to add e to all of them except G_i.
- If the sequence starts by constructing G_1, the rest of the sequence constructs $G_2, \ldots, G_\ell$ by extending from T, but without using any triangle edges, because $|E_1(T)| = 0$. Hence, we get Henneberg sequences for $G_2 + \Delta_T, \ldots, G_\ell + \Delta_T$.

For the reverse direction, we have Henneberg sequences for all the components, where we have added the edges from $\Delta_T - E(T)$ to $G_1, G_2, \ldots, G_\ell$, as described in the lemma. If all components have a Henneberg sequence that starts with triangle (u, v, w), then we can combine them to one sequence for G. If there is component, say $G_1 = (V_1, E_1)$ with $|E_1(T)| = 0$, that cannot be constructed from triangle (u, v, w), we start with the sequence for G_1. By definition, the other components are $G_2 + \Delta_T, \ldots, G_\ell + \Delta_T$, that have Henneberg sequences starting with triangle (u, v, w). These sequences will not use any triangle edge, and hence, we can attach them to the sequence for G_1. This yields a sequence for G. □

While using Lemma 5.1, we will be considering different choices of triangle edges which can be added to make a component Laman. The following lemma states that when two of the choices of a pair of triangle edges work, then so does the third one.

Lemma 5.2 *Let* $G = (V, E)$ *be a graph and* $u, v, w \in V$ *be three nodes in* G *with no edge between them. If* $G + \{(u, v), (u, w)\}$ *and* $G + \{(u, v), (v, w)\}$ *are Laman then* $G + \{(u, w), (v, w)\}$ *is also Laman.*

Proof. Assume that $G + \{(u, w), (v, w)\}$ is not Laman. Then there must be a subset S of vertices including at least two triple vertices such that

$$|E[S]| > 2|S| - 3. \tag{6}$$

Then S will also satisfy (6) in $G + \{(u, v), (u, w)\}$ or $G + \{(u, v), (v, w)\}$, a contradiction. □

Consider the 4-connected component tree of a 3-connected graph. Let components $H_1, \ldots, H_\ell$ be leaf nodes in the tree that are attached via a common separating triple to parent component H_0. The following lemma shows how to prune the leafs in the tree and replace them by a constant size gadget in H_0 such that the Laman property is maintained.

Lemma 5.3 *Let* $H = (V, E)$ *be a graph with a separating triple* $T = \{u, v, w\}$ *and corresponding split graphs* $H_0, H_1, \ldots, H_\ell$*, where we have removed all virtual edges, such that* $H_1, \ldots, H_\ell$ *are planar or of bounded treewidth (even with the virtual edges). Then there is an* NC^2*-algorithm that*

- *either computes a constant-size gadget graph* Γ *on* T *such that*

$$H \text{ is Laman} \iff H_0 \cup \Gamma \text{ is Laman}$$

- *or determines directly that* H *is not Laman.*

Moreover, let H_0' *be graph* H_0 *plus the edges missing from triangle* T*. If* H_0' *is planar or of bounded treewidth* $w \geq 5$*, then* $H_0 \cup \Gamma$ *is planar or of bounded treewidth* w*, respectively.*

Also, the choice of the gadget in the first item depends only on $E(T)$ *and* $H_1, \ldots, H_\ell$*, and* not *on* H_0*.*

Proof. Let $\Delta_T = \{(u, v), (v, w), (u, w)\}$ be the triangle edges on T. Lemma 5.1 describes how to put triangle edges in split components for graph H to be Laman. However, this does not uniquely determine the placement of the edges. Therefore we consider all distribution of the edges that are consistent with Lemma 5.1.

Let $\mathcal{F}_0$ be the family of those sets $F_0 \subseteq \Delta_T - E(T)$ for which there exist sets $F_1, F_2, \ldots, F_\ell \subseteq \Delta_T - E(T)$ such that

1. each edge in $\Delta_T - E(T)$ appears in all but one of $F_0, F_1, \ldots, F_\ell$ and
2. $H_1 + F_1, H_2 + F_2, \ldots, H_\ell + F_\ell$ are all Laman.

From Lemma 5.1, we have

$$H \text{ is Laman} \iff \exists\, F_0 \in \mathcal{F}_0 \;\; H_0 + F_0 \text{ is Laman.} \tag{7}$$

We claim that the family $\mathcal{F}_0$ can be computed in NC^2. To see this observe that the number of possible tuples $(F_0, F_1, \ldots, F_\ell)$ which satisfy item 1 above is $(\ell+1)^{|\Delta_T - E(T)|} \leq (\ell+1)^3$. For all such tuples, we can check the Lamanness of $H_j + F_j$ for all $j \in [\ell]$ in parallel. Since $H_1, \ldots, H_\ell$ are planar or of bounded treewidth (with virtual edges), we can invoke the NC^2-algorithm from Theorem 3.10 or Theorem 2.2 to check whether $H_j + F_j$ is Laman.

If $\mathcal{F}_0 = \emptyset$, then we can say that H cannot be Laman. When $\mathcal{F}_0 \neq \emptyset$, we construct an appropriate gadget. By (7), it suffices to construct a gadget Γ such that

$$\exists\, F_0 \in \mathcal{F}_0 \;\; H_0 + F_0 \text{ is Laman} \iff H_0 + \Gamma \text{ is Laman.}$$

Recall that we need the construction of Γ to depend only on $E(T)$, but to be independent of H_0. For each family $\mathcal{F}_0$ and $E(T)$, we construct a gadget graph Γ such that for any $F_0 \subseteq \Delta_T - E(T)$, we have

$$F_0 \in \mathcal{F}_0 \iff \Gamma \text{ can be obtained from } (T, F_0) \text{ via a sequence of Henneberg steps.} \tag{8}$$

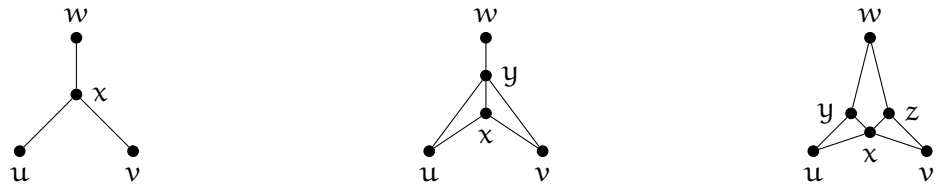
If $\mathcal{F}_0$ has a unique set $\mathcal{F}_0 = \{F_0\}$, then we take $\Gamma = (T, F_0)$. Note that by construction, each $F_0 \in \mathcal{F}_0$ has the same cardinality. Thus, F_0 is always unique when $|E(T)| = 3$ or $|E(T)| = 2$. But also when $|E(T)| = 1$ or $|E(T)| = 0$, $\mathcal{F}_0$ can have a unique set. In the following, we consider *all* cases where $\mathcal{F}_0$ contains at least two sets. For these, we construct gadgets Γ shown in Table 1 that we put into H_0. We use the notation $e_1 = (u, v), e_2 = (u, w), e_3 = (v, w)$. Clearly, the definition of the gadgets is up to vertex relabeling.

Below we explain why property (8) holds for each of the gadgets given in the table. The implication from left-to-right in (8) is given in the description of the gadgets in Figure 3. We argue for the reverse direction. Note that in both types of Henneberg steps, the quantity $2|V| - |E|$ remains constant. Hence, all graphs (T, F_0) which lead the same gadget via Henneberg steps must have the same number of edges.

Case	$E(T)$	$\mathcal{F}_0$	Γ
1	$\{e_1\}$	$\{\{e_2\}, \{e_3\}\}$	Figure 3a
2	$\emptyset$	$\{\{e_1, e_2\}, \{e_1, e_3\}\}$	Figure 3b
3	$\emptyset$	$\{\{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\}$	Figure 3c
4	$\emptyset$	$\{\{e_1\}, \{e_2\}, \{e_3\}\}$	Figure 3a
5	$\emptyset$	$\{\{e_1\}, \{e_2\}\}$	Not a valid possibility

Table 1: Gadgets for all possibilities of $\mathcal{F}_0$ and $E(T)$ where $|\mathcal{F}_0| \geq 2$.

Case 1. The only possible reverse Henneberg steps from the gadget in Figure 3a is to remove the degree 3 node x and add a triangle edge. Since F_0 is restricted to be a subset of $\Delta_T - E(T) = \{e_2, e_3\}$, the only possibility for the resulting graph is either $(T, \{e_2\})$ or $(T, \{e_3\})$.



(a) (b) (c)

Figure 3: Gadgets from Lemma 5.3. (a) The *or-gadget* that can be obtained from any one of the triangle edges via a Henneberg type 2 step. (b) The uv*-or-gadget* that can be obtained from edge (u, v) plus any one of the other two triangle edges via two Henneberg type 2 steps. (c) The *2-or-gadget* that can be obtained from any two triangle edges via a Henneberg type 1 step and two type 2 steps.

Case 2. Starting from the gadget in Figure 3b, the first reverse Henneberg step has to remove x because it has degree 3. Then the edge we have to add can only be $e_1 = (u, v)$. Then the second reverse Henneberg step can only be to remove y. The edge we can add has to be either e_2 or e_3. Thus, the resulting graph can only be either $(T, \{e_1, e_2\})$ or $(T, \{e_1, e_3\})$.

Case 3. From any two triangle edges we can construct the gadget in Figure 3c as explained in the caption. Hence, starting from the gadget, we can reverse these steps and and end up in any two triangle edges.

Case 4. The only possible reverse Henneberg steps from the gadget in Figure 3a is to remove the degree 3 node x and add one of the triangle edges.

Case 5. We show that $\mathcal{F}_0 = \{\{e_1\}, \{e_2\}\}$ is not a valid possibility. From the definition of $\mathcal{F}_0$, recall that we can have $F_0 = \{e_1\}$ only when, say, $F_1 = \{e_2, e_3\}$ and $F_j = \{e_1, e_2, e_3\}$ for $j \geq 2$. Similarly, we can have $F_0 = \{e_2\}$ only when $F_1 = \{e_1, e_3\}$. Then from Lemma 5.2, $F_1 = \{e_1, e_2\}$ is also a valid choice for H_1 to be Laman. Hence, $F_0 = \{e_3\}$ should also be present in $\mathcal{F}_0$.

Finally, we argue the last part of the lemma about planarity and bounded treewidth. If component H_0' is planar, it can be embedded such that the triangle is one face of the embedding. Then we can put any of the gadgets from Figure 3 inside the triangle so that $H_0 + \Gamma$ is planar.

If the component H_0' has bounded treewidth w, consider a tree decomposition $(\mathcal{T}, \mathcal{B})$ of width w. In a tree decomposition, every clique must be contained in a common bag [Bod89, Lemma 2.1]. Thus, there must exist a bag $B \in \mathcal{B}$ that contains the triangle nodes, i.e., $T \subseteq B$.

To get a tree decomposition of $H_0' + \Gamma$, we put an additional bag $B' = V(\Gamma)$ in $\mathcal{T}$ that is adjacent to B. Note that $|B'| \leq 6$. Hence, the treewidth of $H_0' + \Gamma$ is bounded by $\max\{w, 5\}$. The same holds for $H_0 + \Gamma$ because treewidth does not increase when edges are removed. □

Now we can prove our main Theorem.

Theorem 5.4 *Given a one-crossing-minor-free graph* G*, we can decide whether* G *is Laman in* NC^3*.*

Proof. We first decompose the input graph G into triconnected components in NC^2 by Lemma 2.6. Then, in parallel we further decompose each triconnected component into 4-connected components in NC^2 by Lemma 2.6 and we identify the large child paths in the 4-connected component trees by Lemma 2.7. As G is one-crossing-minor-free, the 4-connected components are planar or of bounded treewidth by Theorem 2.10.

By Lemma 4.3, we can decompose G into components resulting from the triconnected components, such that G is Laman iff all these components are Laman. Then for every such component C, in parallel, we decide whether it is Laman as follows. We apply Lemma 5.1 in a bottom up fashion along the 4-connected component tree of C. The leaf components in a 4-connected component tree contain a single separating triple and we can decide for what choices of triple edges the component is Laman by Theorem 3.10 or by Theorem 2.2. Then we put gadgets into the parent components according to Lemma 5.3. The gadgets we put into each separating triple in a parent component are only defined by the children components that are attached to the triple. In particular, we can put the gadgets into the parent components by working in parallel for every triple. Note that in case of overlapping separating triples, multi-edges could emerge in a parent component after we have added the gadgets. In this case, we detected that a parent component is not Laman. We continue this in a bottom-up fashion until we reach the root. Note that if we run this procedure as it is, the parallel complexity would be proportional to the depth of 4-connected component tree, which could be large.

Instead, when we reach a large child path along the way in component C, we deviate from the bottom-up evaluation. Let the large child path consists of components $P_1, P_2, \ldots, P_k$, where P_i is the parent component of P_{i+1} in the 4-connected component tree. Let T_i be the separating triple between components P_i and P_{i+1}, for $i = 1, 2, \ldots, k-1$, and T_0 be separating triple between component P_1 and its parent in the tree.

If the above procedure reaches some component of the large child path at a separating triple $T \neq T_i$, for all $i = 0, 1, \ldots, k-1$, then we put a gadget as described above. Then each path component P_i is planar or of bounded treewidth and has at most two triples that have not been replaced by a gadget. Therefore, for each path component in parallel we can apply Theorem 3.10 or Theorem 2.2 to check for what choices of edges in the two triples the component is Laman in NC^2. We describe how to merge the components $P_1, P_2, \ldots, P_k$ into one component.

Merging two components: Let H be a graph with a separating triple T. Let $A, B, C_1, C_2, \ldots, C_h$ be the components obtained when we split H at T. Let T_A and T_B be two other triples that are present in A and B, respectively. Suppose for each component C_i, we have already computed for which choices of edges in triple T it is Laman. Similarly, suppose we have computed for which choices of edges in triples T_A and T, component A is Laman, and analogously for the edges in triples T and T_B, w.r.t. component B. Then using the conditions in Lemma 5.1, we can find out for what edge choices in triples T_A and T_B, graph H is Laman. This can be done in NC^1 because there is only a polynomial number of possibilities of putting the edges of triple T in components $A, B, C_1, \ldots, C_h$ by the condition in Lemma 5.1, which can be checked in parallel. Moreover, the number of edge choices in T_A and T_B is constant.

Merging a path: We apply the process of merging two components recursively in a binary tree fashion on $P_1, P_2, \ldots, P_k$. At the bottom layer, we start with applying the above merge procedure on pairs of neighboring path components at their common separating triple T_i, in parallel. After merging two path components, we get a new component that again has two triples at each end and we have computed the edge choices in these two triples that make the component Laman.

When we have merged all path components into a single component, we find the choices of edges in the triple T_0 for which the split graph in $C - T_0$ that contains P_1 is Laman. Then we put the corresponding gadget in T_0 and carry on with the bottom up evaluation. Clearly, the above procedure for a large child path is in NC^2, as the merge step is in NC^1.

If it happens during the bottom up evaluation that a component or a large child path is not Laman for any choice of edges we can conclude that the graph is not Laman and stop the bottom-up evaluation.

Regarding the complexity, note that the NC^2-algorithms that we run as subroutines in the bottom up evaluation are the ones from Theorem 3.10 and 2.2. By Lemma 2.7, there are at most $\log n$ many large child paths on a path from a leaf node to the root in the 4-connected component tree Thus, the algorithm sequentially runs at most $\log n$ many NC^2 algorithms as subroutines and therefore we end up in NC^3. □

6 Open problems

For $K_{3,3}$-free Laman graphs we can compute an infinitesimally rigid embedding efficiently in parallel. This is open for the case of one-crossing-minor-free graphs. In fact, it is open even for graphs of bounded treewidth. It is also open for K_5-free Laman graphs, even though the 4-connected components are all planar. A problem there is that we do not have the analog of Lemma 4.5 for separating triples. For example, an embedding of a $K_{3,3}$ on a conic section as in Figure 4 is infinitesimally flexible [Whi84]. However, the split graphs G_1, G_2, G_3 corresponding to the separating triple $\{u, v, w\}$ are infinitesimally rigid in the same embedding, where one has to add two edges to each component to make it Laman.

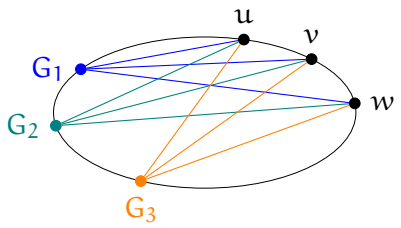


Figure 4: $K_{3,3}$ on an ellipsoid

The main open problem is still to show rigidity or minimal rigidity in 2D for arbitrary graphs in NC. Note that even for planar graphs, rigidity is not known to be in NC.

A seemingly even more challenging open problem is to consider infinitesimal rigidity in *higher dimensions*. The $m \times 2n$ rigidity matrix $R = R_2$ in 2D can clearly be generalized to the $m \times dn$ rigidity matrix R_d, for dimensions $d \geq 3$. The PIT problem for R_2 is in polynomial time because of the various characterizations we have for rigidity, like Theorems 1.2, 1.3, and 2.11. However, we do not have such characterizations even for $d = 3$. A derandomization in polynomial time of the PIT for R_3 is an open problem for decades. See also the exposition of Raz and Wigderson [RW19] on this topic.

Acknowledgments

The authors thank the organizers of Dagstuhl Seminar 24381 on *Algebraic and Analytic Methods in Computational Complexity* for inviting them. Part of this research was done during the seminar. We thank Dhara Thakkar for helpful discussions and Samir Datta for pointing out the NL-bound for planar Laman graph decision.

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ECCC ISSN 1433-8092

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