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Supercritical Tradeoff Between Size and Depth for Resolution over Parities

Dmitry Itsykson*[1,2] and Alexander Knop†[3]

[1]Ben-Gurion University of the Negev, Beer Sheva, Israel
[2]On leave from Steklov Institute of Mathematics at St. Petersburg
[3]Google Research

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Abstract

Alekseev and Itsykson (STOC 2025) proved the existence of an unsatisfiable CNF formula such that any resolution over parities (Res(⊕)) refutation must either have exponential size (in the formula size) or superlinear depth (in the number of variables). In this paper, we extend this result by constructing a formula with the same hardness properties, but which additionally admits a resolution refutation of quasi-polynomial size. This establishes a *supercritical tradeoff* between size and depth for resolution over parities.

The proof builds on the framework of Alekseev and Itsykson and relies on a lifting argument applied to the supercritical tradeoff between width and depth in resolution, proposed by Buss and Thapen (IPL 2026).

1 Introduction

Propositional proof complexity investigates proof systems used to demonstrate the unsatisfiability of Boolean formulas. A central goal — often referred to as Cook's program, motivated by the NP vs. coNP problem — is to establish *superpolynomial lower bounds* on the size of refutations within stronger and stronger proof systems.

Resolution is the most extensively studied such a system, valued for its conceptual simplicity and its close relationship with modern SAT solvers. A resolution refutation of a CNF formula φ is a sequence of clauses $C_1, C_2, \ldots, C_s$, concluding with the empty clause (representing a contradiction). Each clause in the sequence is either an original clause of φ or is derived from earlier clauses using the *resolution rule*: $\frac{A \vee x \quad B \vee \neg x}{A \vee B}$. Numerous techniques have been developed for proving lower bounds in Resolution, and exponential lower bounds are known for a wide range of formulas. Notably, Urquhart [Urq87] showed that certain unsatisfiable systems of linear equations over $\mathbb{F}_2$ require exponential-size resolution refutations when encoded naturally in CNF.

In this paper, we study the proof system resolution over parities (Res(⊕)), which extends classical resolution by integrating linear algebra over the finite field $\mathbb{F}_2$. Proof lines in this proof

*e-mail: dmitrits@gmail.com. Supported by European Research Council Grant No. 949707.
†e-mail: aaknop@gmail.com.

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system are disjunctions of linear equations over $\mathbb{F}_2$, called *linear clauses*. A Res($\oplus$) refutation of a CNF formula φ is a sequence of linear clauses $C_1, C_2, \ldots, C_s$ that ends with the empty clause (a contradiction), every clause of this sequence is either a clause of φ or is obtained from previous clauses by one of the two inference rules:

1. The *resolution rule*, which infers $C \vee D$ from premises $C \vee (f = 0)$ and $D \vee (f = 1)$ for some linear form f.

2. The *weakening rule*, which derives a linear clause D from C if C semantically implies D, i.e. if any assignment satisfying C also satisfies D.

The question of proving a superpolynomial lower bound on the size of Res($\oplus$) refutations remains open and appears to be very challenging. The study of lower bounds for Res($\oplus$) is motivated by the long-standing challenge of proving exponential lower bounds for Frege systems — a formalization of textbook propositional logic. Despite decades of effort, no such bounds are known, even for considerably weaker subsystems. The strongest Frege subsystem for which we currently have lower bounds is AC^0-Frege [Ajt94], which operates with constant-depth formulas using only $\wedge$, $\vee$, and $\neg$ gates. However, once parity gates are added — resulting in $\mathrm{AC}^0[2]$-Frege — existing lower bound techniques completely break down. This is in sharp contrast to circuit complexity, where exponential lower bounds for $\mathrm{AC}^0[2]$ circuits have been known for over 30 years [Smo87, Raz87]. Bridging this discrepancy requires a deeper understanding of proof systems that reflect the power of reasoning with parities. As a subsystem of $\mathrm{AC}^0[2]$-Frege, Res($\oplus$) offers a natural and tractable framework for exploring the power of reasoning with parity, making it a central object of study in this context.

1.1 Recent progress on lower bounds for subsystems of resolution over parities

There are numerous results establishing exponential lower bounds for tree-like Res($\oplus$) refutations of standard combinatorial formulas, using a variety of techniques [IS14, IS20, GK18, GOR24, IR21, Kra18, BI24].

Independently, Chattopadhyay, Mande, Sanyal, and Sherif [CMSS23] and Beame and Kroth [BK23] introduced a lifting approach for establishing lower bounds in tree-like Res($\oplus$).

Given a CNF formula $\varphi(y_1, y_2, \ldots, y_n)$ and a Boolean function $g : \{0,1\}^\ell \to \{0,1\}$ (called a *gadget*), we define the *lifted formula* $\varphi \circ g$ as the CNF encoding of the formula $\varphi(g(x_{1,1}, x_{1,2}, \ldots, x_{1,\ell}), \ldots, g(x_{n,1}, x_{n,2}, \ldots, x_{n,\ell}))$ where each variable y_i in φ is replaced by $g(x_{i,1}, \ldots, x_{i,\ell})$ for fresh variables $x_{i,1}, x_{i,2}, \ldots, x_{i,\ell}$.

Chattopadhyay, Mande, Sanyal, and Sherif [CMSS23] introduced the notion of k-stifling gadgets as follows. A Boolean function $g : \{0,1\}^\ell \to \{0,1\}$ is called a k-stifling gadget if, for every $a \in \{0,1\}$ and every choice of $\ell - k$ input variables, there exists a setting of those $\ell - k$ variables such that the output of g is always equal to a, regardless of the values of the remaining k variables. They further showed that if every resolution refutation of a formula φ has depth at least h, and g is a k-stifling gadget, then any tree-like Res($\oplus$) refutation of the lifted formula $\varphi \circ g$ must have size at least 2^{kh}.

Efremenko, Garlik, and Itsykson [EGI24] made the first significant progress beyond the tree-like setting by establishing exponential lower bounds for bottom-regular Res($\oplus$) refutations of the Binary Pigeonhole Principle formula BPHP_n^{n+1}. Their work introduced the notions of *closure* and a random-walk technique, both of which have proven instrumental in subsequent research. Building on these ideas, and combining them with lifting techniques, Bhattacharya, Chattopadhyay, and Dvořák [BCD24] showed that certain formulas require exponential-size refutations in bottom-regular Res($\oplus$), while still admitting polynomial-size refutations in Resolution.

Alekseev and Itsykson [AI25] showed that one can construct formulas hard for bottom-regular Res($\oplus$) based on any formula that requires large resolution depth. Specifically, they proved that if φ is an unsatisfiable CNF formula over n variables with resolution depth at least $\Omega(n)$, then any regular Res($\oplus$) refutation of the lifted and transformed formula $\mathrm{mix}(\varphi) \circ g$ must have size at least $2^{\Omega(n)}$, where g is a constant-size gadget and mix is a semantic-preserving transformation of φ.

Moreover, Alekseev and Itsykson [AI25] made progress beyond bottom-regular Res($\oplus$) by establishing a tradeoff between the size and depth of general Res($\oplus$) refutations. Specifically, they constructed a family of formulas — lifted Tseitin formulas — over n variables and of size $\mathrm{poly}(n)$ such that any Res($\oplus$) refutation must have either size at least $2^{\Omega(n/\log n)}$ or depth at least $\Omega(n \log\log n)$. Subsequently, Efremenko and Itsykson [EI25] improved the depth lower bound to $\Omega(n \log n)$. In particular, this result implies exponential lower bounds for regular Res($\oplus$) for all reasonable notions of regularity. However, a limitation of this result is that it applies to a specific formula. In this paper, we address this issue by developing a general lifting result.

1.2 Main question addressed

It is important to note that, for the lifted Tseitin formulas used in the size-or-depth lower bound of [AI25], it remains unclear whether they actually admit short Res($\oplus$) refutations. In other words, it is still unclear whether the observed phenomenon constitutes a genuine tradeoff or merely reflects the current limitations of our techniques for proving size lower bounds. In this paper, we address the following question: Does there exist a formula that admits a short Res($\oplus$) refutation, yet any such refutation must have either superlinear (in the number of variables) depth or exponential (in the size of the formula) size?

A negative answer to this question — combined with the result from [AI25] — would yield exponential lower bounds on the size of Res($\oplus$) refutations.

On the other hand, a positive answer would establish a *supercritical tradeoff* between size and depth in Res($\oplus$). Here, *supercritical* means that for refutations of small size, the required depth significantly exceeds the worst-case upper bound achievable in the unrestricted setting. A positive answer would also lend support to a possible explanation for why proving Res($\oplus$) lower bounds for seemingly simple formulas — such as the pigeonhole principle — remains so challenging. It may be that these formulas do admit short refutations, but all such refutations necessarily have large depth, making them difficult to construct.

1.3 Supercritical tradeoffs in proof complexity

A supercritical tradeoff between two proof complexity measures μ and ν for a formula φ occurs when φ has proofs with small μ and others with small ν, but any proof with μ below a certain threshold forces ν to significantly exceed the worst-case upper bound known for all formulas.

In the last few years, many supercritical tradeoffs in proof complexity have been established [Raz16, BBI16, Ber12, BNT13, Raz18, Raz17, BN20, FPR22, BT26, dRFJ$^+$25, GMRS25, CD24]. We briefly overview the most relevant results concerning Resolution and Res($\oplus$). Razborov [Raz16] established a supercritical tradeoff between width and size for tree-like Resolution. Fleming, Pitassi, and Robere [FPR22] proved supercritical tradeoffs between size/width and depth for Resolution.

More recently, Buss and Thapen [BT26] introduced a simple and highly flexible construction yielding a supercritical tradeoff between size/width and depth in Resolution. Finally, de Rezende et al. [dRFJ+25] and Göös et al. [GMRS25] showed that many of these tradeoffs can be made truly supercritical, meaning that the lower bounds are expressed in terms of the formula's size, rather than merely the number of variables.

Chattopadhyay and Dvořák [CD24] established a supercritical tradeoff between width and size for tree-like Res(⊕). Their proof builds on a corresponding lifting theorem, which directly lifts Razborov's result for tree-like Resolution [Raz16] to the Res(⊕) setting.

Note that all the tradeoffs mentioned above have been established for proof systems for which superpolynomial size lower bounds are already known.

1.4 Our contributions

Our main result is the following theorem.

Theorem 1.1 (Theorem 6.1)**.** *Let ψ be an unsatisfiable CNF formula such that ψ does not have a resolution refutation with width at most w and depth at most h. Assume that there is a natural number $s \geq 2$ such that $h \geq s^2 w - w$. Let $g\colon \{0,1\}^\ell \to \{0,1\}$ be a 2-stifling gadget (for example, g can be the 5-bit Majority function). Then any* Res(⊕) *refutation of $\psi \circ \oplus_s \circ g$ has either size at least 2^w or depth at least $\frac{s^2 w}{4\ell}$.*

By applying the lifting from Theorem 1.1 to the width-depth tradeoff established by Buss and Thapen [BT26], we obtain a supercritical tradeoff between the size and depth of Res(⊕) refutations.

Theorem 1.2 (Theorem 6.3)**.** *For every natural $K \geq 2$ and $n \geq 2$ such that $K \leq \sqrt{\frac{n}{10 \log^3 n}}$ there is a CNF formula $\varphi_{n,K}$ that contains $O(Kn \log n)$ variables, the formula $\varphi_{n,K}$ is in $O(K \log^2 n)$-CNF and of size $n^{O(K \log n)}$. The formula $\varphi_{n,K}$ has a resolution refutation of size $n^{O(K \log n)}$ and of width $O(K \log^2 n)$ but every its* Res(⊕) *refutation has either size at least $2^{\Omega(n/\log n)}$ or depth at least $\Omega(K^2 n \log n)$.*

To our knowledge, this is the first instance of a supercritical tradeoff demonstrated in a proof system lacking known superpolynomial lower bounds on proof size.

Corollary 1.3 (Corollary 6.4)**.** *For every $\delta > 0$, Depth-$\frac{n^{4/3}}{\log^{4/3+\delta} n}$* Res(⊕) *does not p-simulate Resolution.*

Another important specific case of Theorem 1.1 is a size-depth tradeoff for Res(⊕) obtained by lifting from resolution width.

Theorem 1.4 (Theorem 6.5)**.** *Let ψ_n be a family of unsatisfiable $O(1)$-CNF formulas such that ψ_n has n variables and the resolution width of ψ_n is $w(n)$. For every natural $K \geq 2$ consider a formula $\Psi_{n,K} := \psi_n \circ \oplus_K \circ Maj_5$; it has $5nK$ variables, $\Psi_{n,K}$ is an $O(K)$-CNF formula of size at most* $\mathrm{poly}(n)2^K$ *and any* Res(⊕) *refutations of Ψ_n has either depth at least $\Omega(w(n)K^2)$ or size at least $2^{\Omega(w(n))}$.*

Consider several interesting specific cases of Theorem 1.4.

- **Size-depth tradeoff for formula of size polynomial in number of variables.** Let $w(n) = \Omega(n)$. Then the formula $\Psi_{n,\lceil \log n \rceil}$ contains $m := \Theta(n \log n)$ variables, $\Psi_{n,\lceil \log n \rceil}$ is an $O(\log m)$-CNF formula of size $\mathrm{poly}(m)$ and any $\mathrm{Res}(\oplus)$ refutations of $\Psi_{n,\lceil \log n \rceil}$ has either depth at least $\Omega(m \log m)$ or size at least $2^{\Omega(m/\log m)}$.

  This result extends the result from [EI25] for a large number of formulas.

- **Maximal depth.** Let $w(n) = \Omega(n)$. Consider an arbitrary $1 > \delta > 0$. The formula $\Psi_{n,\lceil n^{1-\delta} \rceil}$ contains $m := \Theta(n^{2-\delta})$ variables, $\Psi_{n,\lceil n^{1-\delta} \rceil}$ is a CNF formula of size $poly(n)2^{n^{1-\delta}}$ and any $\mathrm{Res}(\oplus)$ refutations of $\Psi_{n,\lceil n^{1-\delta} \rceil}$ has either depth at least $\Omega(m^{3/2-\delta/2})$ or size at least $2^{\Omega(n)}$.

  So if we do not restrict ourselves to formulas of polynomial size in the number of variables, then we can get a superpolynomial size lower bound for depth less than $m^{3/2-\delta}$.

- **Minimal width.** Let $w(n) = \Omega\left(n^{1/2+\delta}\right)$, where $1/2 > \delta > 0$. The formula $\Psi_{n,\lceil n^{1/2} \rceil}$ contains $m := \Theta(n^{3/2})$ variables, $\Psi_{n,\lceil n^{1/2} \rceil}$ is a CNF formula of size $poly(n)2^{n^{1/2}}$ and any $\mathrm{Res}(\oplus)$ refutations of $\Psi_{n,\lceil n^{1/2} \rceil}$ has either depth at least $\Omega\left(n^{3/2+\delta}\right) = \Omega\left(m^{1+2\delta/3}\right)$ or size at least $2^{\Omega\left(n^{1/2+\delta}\right)}$.

  So we can get a non-trivial size-depth tradeoff starting from a formula with the resolution width $\Omega\left(n^{1/2+\delta}\right)$.

1.5 Technique

Proof of Theorem 1.1 builds on the size-depth tradeoff established by Alekseev and Itsykson [AI25], with subsequent improvements by Efremenko and Itsykson [EI25].

As a first step, we develop a more flexible size-depth tradeoff that applies to a broad class of formulas. Below, we outline the main ideas behind the tradeoff established by Alekseev and Itsykson [AI25].

Consider a $\mathrm{Res}(\oplus)$ refutation Π. We identify certain linear clauses within Π as *good* clauses. By definition, a good clause cannot be an axiom of the original formula. These clauses satisfy a crucial property we refer to as the *dichotomy property*. Specifically, for every good clause C of moderately small width, one of the following holds:

- The size of the refutation Π is exponential;
- There exists another good linear clause in Π that appears at a significantly greater depth than C and whose width exceeds that of C by only a small amount.

Assuming that the empty clause is good and the size of Π is small, the dichotomy property implies that one can iteratively find increasingly deeper good clauses within Π, ultimately yielding a lower bound on the depth.

The dichotomy property is established through a combination of a random walk argument and a bottleneck argument. We begin by defining a set Σ of good full assignments that falsify a given linear clause C_0 (for simplicity, one may think of Σ as the set of all assignments falsifying C_0). We then select a random assignment $\sigma \in \Sigma$ and perform a t-step random walk along the refutation graph. At each step, we move from a linear clause to one of its premises that is also falsified by σ, counting only applications of resolution rules (applications of weakening are ignored).

The random walk theorem asserts that, with probability at least p, such a walk ends at a good linear clause. Now, if all good linear clauses reachable from C_0 within t steps have width greater than $|C_0| + s$, then there must be at least $p \cdot 2^s$ such clauses. This is because a random assignment falsifying C_0 can falsify a clause of width at least $|C_0| + s$ with probability at most 2^{-s}.

Alekseev and Itsykson [AI25] proved a random walk theorem for formulas of the form $\varphi \circ g$, where g is a 2-stifling gadget and φ is an unsatisfiable CNF formula that admits a sufficiently strong strategy for Delayer in a specific Prover–Delayer game. This game is defined with respect to the formula φ and a set $\mathcal{A}$ of its partial assignments, which must satisfy two conditions: (1) no assignment in $\mathcal{A}$ falsifies any clause of φ, and (2) $\mathcal{A}$ is closed under restriction, i.e., any restriction of an assignment in $\mathcal{A}$ also belongs to $\mathcal{A}$.

The game proceeds as follows. It begins with some initial assignment $\rho_0 \in \mathcal{A}$. In each round, Prover selects a variable x and queries its value. Delayer then has two options:

1. Pay one black coin to choose a value $a \in \{0, 1\}$ and extend the current assignment by setting $x := a$; or

2. Reply with $*$, allowing Prover to choose the value a.

Regardless of the outcome, Delayer earns one white coin for every move. The game terminates as soon as the current assignment no longer belongs to $\mathcal{A}$.

The required property of Delayer's strategy is as follows: for every starting assignment $\rho_0 \in \mathcal{A}$ in the $(\varphi, \mathcal{A})$-game, there exists a strategy for Delayer that guarantees earning at least $t - |\rho_0|$ white coins while spending at most n black coins, where t is significantly larger than n. Alekseev and Itsykson [AI25] provide an example of such a strategy for Tseitin formulas, in which t is, roughly speaking, the number of edges and n is the number of vertices in the underlying graph.

Our first observation is that Delayer strategies for such games can be derived from strategies in the Atserias–Dalmau games [AD08], which characterize resolution width, via a lifting transformation using a parity gate. This simple but powerful idea allows us to establish Theorem 1.4, thereby completing the first step of our approach.

However, formulas with large resolution width are inherently hard for resolution and can therefore only be used to show that small-depth Res($\oplus$) refutations require large size. To establish a supercritical tradeoff, we also need to demonstrate the existence of short refutations with large depth. Our second step is to refine the lifting theorem so that it can be applied starting from formulas whose every resolution refutation must have either width at least w or depth at least h. This refinement precisely enables us to apply lifting to the known supercritical tradeoffs between resolution width and depth.

In Section 4, we introduce an analogue of the Atserias–Dalmau games tailored to formulas that require resolution width at least w for any resolution proof of depth at most h. The properties of winning positions in these games closely resemble those in the original Atserias–Dalmau games, provided we focus on positions within distance h from the empty position. We then apply the parity gadget to these games. Notably, in the proof of the size-versus-depth tradeoff for suitable parameters, it suffices to consider Delayer's strategy only on positions that remain close to the empty position. This insight enables us to establish the size-depth tradeoff starting from formulas that have no refutations of width at most h and depth at most w simultaneously, thereby proving Theorem 1.1.

2 Preliminaries

2.1 Resolution

Let φ be an unsatisfiable CNF formula. A resolution refutation of φ is a sequence of clauses $C_1, C_2, \ldots, C_s$ such that C_s is the empty clause (i.e., identically false) and for every $i \in [s]$ the clause C_i is either a clause of φ or is obtained from previous clauses by the *resolution rule* that allows us to derive a clause $C \vee D$ from clauses $C \vee x$ and $D \vee \neg x$.

The *size* of a resolution refutation is the number of clauses in it. The *depth* of a resolution refutation is the length of the longest path between the empty clause and the clause of the original formula. The *width* of a resolution refutation is the maximal size of a clause from the refutation. The resolution width of an unsatisfiable CNF formula φ is the minimal possible width over all resolution refutations of φ.

2.2 Resolution Over Parities

Here and after, all scalars are from the field $\mathbb{F}_2$. Let X be a set of variables taking values in $\mathbb{F}_2$. A linear form in variables from X is a homogeneous linear polynomial over $\mathbb{F}_2$ in variables from X or, in other words, a polynomial $\sum_i^n x_i a_i$, where $x_i \in X$ is a variable and $a_i \in \mathbb{F}_2$ for all $i \in [n]$. A linear equation is an equality $f = a$, where f is a linear form and $a \in \mathbb{F}_2$.

A *linear clause* is a disjunction of linear equations: $\bigvee_{i=1}^t (f_i = a_i)$. Note that over $\mathbb{F}_2$ a linear clause $\bigvee_{i=1}^t (f_i = a_i)$ may be represented as the negation of a linear system: $\neg \bigwedge_{i=1}^t (f_i = a_i + 1)$.

Now we define the proof system resolution over parities (Res($\oplus$)) [IS20].

Let φ be an unsatisfiable CNF formula. A Res($\oplus$) refutation of φ is a sequence of linear clauses $C_1, C_2, \ldots, C_s$ such that C_s is the empty clause (i.e., identically false) and for every $i \in [s]$ the clause C_i is either a clause of φ or is obtained from previous clauses by one of the following inference rules:

- *Resolution rule* allows us to derive a linear clause $C \vee D$ from linear clauses $C \vee (f = a)$ and $D \vee (f = a + 1)$.
- *Weakening rule* allows us to derive from a linear clause C any linear clause D in the variables of φ that semantically follows from C (i.e., any assignment satisfying C also satisfies D).

The *size* of a Res($\oplus$) refutation is the number of linear clauses in it. The *depth* of a Res($\oplus$) refutation is the maximal number of resolution rules applied on a path between a clause of the initial formula and the empty clause.

Remark 2.1. A resolution refutation of a formula φ is a special case of a Res($\oplus$) refutation, where all linear clauses are plain (i.e., disjunctions of literals).

For any function $f(n)$, we denote by Depth-$f(n)$ Res($\oplus$) the subsystem of Res($\oplus$) consisting of refutations with depth at most $f(n)$, where n is the number of variables in the formula being refuted.

For a linear clause C we denote by $L(C)$ the set of linear forms that appear in C; i.e. $L\left(\bigvee_{i=1}^t (f_i = a_i)\right) = \{f_1, f_2, \ldots, f_t\}$. The same notation we use for linear systems: if Ψ is a $\mathbb{F}_2$-linear system, $L(\Psi)$ denotes the set of all linear forms from Ψ.

2.3 Lifted formulas

For every CNF formula Φ over the variables $Y = \{y_1, y_2, \dots, y_m\}$ and every Boolean function $g\colon \{0,1\}^\ell \to \{0,1\}$ we define a CNF formula $\Phi \circ g$ with variables $X = \{x_{i,j} \mid i \in [m], j \in [\ell]\}$ representing in CNF $\Phi(g(x_{1,1}, x_{1,2}, \dots, x_{1,\ell}), g(x_{2,1}, x_{2,2}, \dots, x_{2,\ell}), \dots, g(x_{m,1}, x_{m,2}, \dots, x_{m,\ell}))$ (i.e. we substitute to every variable of Φ the function g applied to ℓ fresh variables). Let $\Phi = \bigwedge_{i \in I} C_i$, where C_i is a clause for all $i \in I$. For every $i \in [m]$ we denote by $y_i \circ g$ the canonical CNF formula representing $g(x_{i,1}, x_{i,2}, \dots, x_{i,\ell})$ which has ℓ variables in every clause and by $(\neg y_i) \circ g$ the canonical CNF formula representing $\neg g(x_{i,1}, x_{i,2}, \dots, x_{i,\ell})$ which has ℓ variables in every clause. Let $C_i = l_{i,1} \vee l_{i,2} \vee \dots \vee l_{i,n_i}$, where l_i is a literal. Then we denote by $C_i \circ g$ a CNF formula that represents $l_{i,1} \circ g \vee l_{i,2} \circ g \vee \dots \vee l_{i,n_i} \circ g$ as follows: $C_i \circ g$ is the conjunction of all clauses of the form $D_1 \vee D_2 \vee \dots \vee D_{n_i}$, where D_j is a clause of $l_{i,j} \circ g$ for all $j \in [m_i]$. And $\Phi \circ g := \bigwedge_{i \in I} C_i \circ g$.

We refer to $\Phi \circ g$ as a formula Φ *lifted with a gadget* g, to the set $Y = \{y_1, y_2, \dots, y_m\}$ as a set of *unlifted* variables, and to the set $X = \{x_{i,j} \mid i \in [m], j \in [\ell]\}$ as the set of *lifted* variables.

Lemma 2.2 (Folklore, see [IS20], for example)**.** *Let $g : \{0,1\}^\ell \to \{0,1\}$ be a gadget. If a CNF formula φ has a resolution refutation of size S and width w, then the formula $\varphi \circ g$ has a resolution refutation of size $S2^{O(w\ell)}$ and width $O(w\ell)$.*

2.4 Closure and Amortized Closure

We consider the set of propositional variables $X = \{x_{i,j} \mid i \in [m], j \in [\ell]\}$. The variables from X are divided into m blocks by the value of the first index. The variables $x_{i,1}, x_{i,2}, \dots, x_{i,\ell}$ form the i-th block, for $i \in [m]$. We may view the set X as the set of lifted variables with respect to a gadget of size ℓ.

Let $F = \{f_1, f_2, \dots, f_n\}$ be a set of $\mathbb{F}_2$-linear forms with variables from X. Consider a coefficient matrix M of F: its columns correspond to X, and for all $i \in [n]$, i-th row is the coefficient vector of f_i. For every $i \in [m]$, let M_i consist of matrix columns corresponding to the variables from the i-th block. Let $I \subseteq [m]$. We say that $\{M_i\}_{i \in I}$ is *safe* if there are distinct indices $i_1, i_2, \dots, i_t \in I$ and elements $v_{i_1} \in M_{i_1}, v_{i_2} \in M_{i_2}, \dots, v_{i_t} \in M_{i_t}$ such that $v_{i_1}, v_{i_2}, \dots, v_{i_t}$ is a basis of $\langle \cup_{i \in I} M_i \rangle$.

A *closure* [EGI24] of a set of linear forms F is any inclusion-wise minimal set $S \subseteq [m]$ such that $\{M_i\}_{i \in [m] \setminus S}$ is safe. Informally, the closure indicates the set of the most essential unlifted variables for the set of linear forms F.

Lemma 2.3 (Uniqueness [EGI24])**.** *For any F, its closure is unique.*

We denote the closure of F by $\mathrm{Cl}(F)$.

Lemma 2.4 ([EGI24])**.** *Closure has the following properties.*

1. *If $F \subseteq G$, then $\mathrm{Cl}(F) \subseteq \mathrm{Cl}(G)$;*
2. *$\mathrm{Cl}(F) = \mathrm{Cl}(\langle F \rangle)$;*
3. *$|\mathrm{Cl}(F)| \le \dim\langle F \rangle$.*

We also require the notion of amortized closure, introduced by Efremenko and Itsykson [EI25]. Unlike the plain closure, which can grow dramatically with the addition of a single linear form, the amortized closure is a superset of the plain closure designed to grow more gradually and smoothly.

We say that a subset $A \subseteq [m]$ is *coverable* w.r.t. $\{M_i \mid i \in [m]\}$ if for every $i \in A$ there is $v_i \in M_i$ such the set $\{v_i \mid i \in A\}$ is linearly independent. For subsets $A, B \subseteq [m]$, we say that A is less than B ($A \preceq B$) if the largest element in the symmetric difference $A \triangle B$ belongs to B.

An *amortized* closure of F [EI25], denoted by $\widetilde{\mathrm{Cl}}(F)$, is the $\preceq$-maximal subset of $[m]$ that is coverable w.r.t. $\{M_i \mid i \in [m]\}$. It is easy to see that $\widetilde{\mathrm{Cl}}(F)$ does not depend on the permutation of rows in the coefficient matrix of F.

Lemma 2.5 (Size bound [EI25])**.** $|\widetilde{\mathrm{Cl}}(F)| \leq \dim\langle F\rangle$.

Proof. $|\widetilde{\mathrm{Cl}}(F)|$ is at most the rank of a coefficient matrix of F that equals $\dim\langle F\rangle$. □

Lemma 2.6 ([EI25])**.** $\mathrm{Cl}(F) \subseteq \widetilde{\mathrm{Cl}}(F)$

Lemma 2.7 (Lipschitz continuity [EI25])**.** $\widetilde{\mathrm{Cl}}(F) \subseteq \widetilde{\mathrm{Cl}}(F \cup \{f\})$ *and* $|\widetilde{\mathrm{Cl}}(F \cup \{f\})| \leq |\widetilde{\mathrm{Cl}}(F)| + 1$.

Lemma 2.8 ([EI25])**.** *Let Φ and Ψ be two linear systems in variables $X = \{x_{i,j} \mid i \in [m], j \in [\ell]\}$. Let π be a partial assignment defined on $\{x_{i,j} \mid i \in \mathrm{Cl}(L(\Phi)), j \in [\ell]\}$. Let Σ consist of all solutions σ of Φ such that σ extends π. Assume that $\Sigma \neq \emptyset$. Let τ be a random element of Σ. Then* $\Pr[\tau$ *satisfies* $\Psi] \leq 2^{|\widetilde{\mathrm{Cl}}(L(\Phi))| - |\widetilde{\mathrm{Cl}}(L(\Psi))|}$.

2.5 Lifting via stifling gadgets

In the lifting settings, we will identify subsets of $[m]$ with corresponding subsets of the lifted variables Y. It is especially convenient to use such correspondence for closure and amortized closure. So, we will assume that the support and the (amortized) closure of the set of linear forms over lifted variables is the set of unlifted variables.

A partial assignment ρ to the set of variables X is called block-respectful if, for every i, ρ either assigns values to all variables with support i or does not assign values to any of them.

Suppose that ρ is a block-respectful partial assignment. Then we define by $\hat{\rho}$ the partial assignment on the set of variables Y such that $\hat{\rho}(y_i) = g(\rho(x_{i,1}, x_{i,2}, \ldots, x_{i,\ell}))$ (here we assume that if the right-hand side is undefined, then the left-hand side is also undefined).

Let $k < \ell$. A gadget (i.e. Boolean function) $g : \{0,1\}^\ell \to \{0,1\}$ is called *k-stifling* [CMSS23] if for every $A \subset [\ell]$ of size k for every $c \in \{0,1\}$ there exists $a \in \{0,1\}^\ell$ such that for every $b \in \{0,1\}^\ell$ if a and b agree on set of indices $[\ell] \setminus A$, then $g(b) = c$.

It is easy to see that the majority function $Maj_{2k+1} : \{0,1\}^{2k+1} \to \{0,1\}$ is a k-stifling for every k.

Lemma 2.9 ([AI25])**.** *Let Ψ be a satisfiable linear system in the lifted variables X. Let $g\colon \{0,1\}^\ell \to \{0,1\}$ be a 1-stifling gadget. Suppose there exists a full assignment σ to lifted variables X satisfying Ψ such that $\hat{\sigma}|_{\mathrm{Cl}(L(\Psi))}$ does not falsify any clause of φ. Then, Ψ does not contradict any clause of $\varphi \circ g$.*

2.6 Supercritical tradeoff between width and depth for resolution

Here, we state the supercritical tradeoff between width and depth in resolution, as established by Buss and Thapen [BT26].

Theorem 2.10 ([BT26])**.** *Let $b, c, d \geq 2$ be integers and b be a power of two. Then there is an explicit formula $\Phi_{b,c,d}$ that has the following properties:*

- $\Phi_{b,c,d}$ *has* $bcd + b\log b$ *variables;*
- $\Phi_{b,c,d}$ *has* $c + (d^c - 1)bc^2 + b \le d^c b^2 c$ *clauses, of width at most* $c + \log b + 1$*;*
- $\Phi_{b,c,d}$ *has a resolution refutation of size* $O(d^c b^2 c)$ *and width* $c + \log b + 1$*;*
- *Any resolution refutation of* $\Phi_{b,c,d}$ *of width strictly below* $b/2$ *must have depth at least* d^c*.*

Corollary 2.11. *There exists a family of unsatisfiable CNF formulas* $\{\Psi_n\}_{n=1}^{\infty}$ *such that*

- Ψ_n *contains* n *variables;*
- *the width of* Ψ_n *is* $O(\log n)$ *and, moreover,* Ψ_n *has a resolution refutation of size* $\mathrm{poly}(n)$ *and of width* $O(\log n)$*;*
- *any resolution refutation of* Ψ_n *of width at most* $n/40\log n$ *has depth greater than* $n^2/400\log^2 n$*.*

Proof. Let b be the maximal power of two such that $5b\log b \le n$. Then $n < 10b(\log b + 1)$. Hence, $b > \frac{n}{20\log n}$.

Let us fix $d = 2$, $c = 2\log b$.

Consider $\Phi_{b,c,d}$ from Theorem 2.10. $\Phi_{b,c,d}$ contains exactly $5b\log b$ variables. We define Ψ_n as $\Phi_{b,c,d}$ with $n - 5b\log b$ fictive fresh variables. Precisely we take $\Psi_n = \Phi_{b,c,d} \wedge y_1 \wedge y_2 \cdots \wedge y_{n-5b\log b}$, where variables y_i have no occurrences in $\Phi_{b,c,d}$.

It is easy to see that Ψ_n contains exactly n variables. The width of Ψ_n is $3\log b + 1 = O(\log n)$. Any resolution refutation of $\Phi_{b,c,d}$ is also a refutation of Φ_n. Hence, Ψ_n has a resolution refutation of size $\mathrm{poly}(n)$ and of width $O(\log n)$. Since $\Phi_{b,c,d}$ can be obtained form Ψ_n by substitution of all y_i to 1, we get that any resolution refutation of Ψ_n of width at most $n/40\log n$ (which is less than $b/2$) has depth at least $d^c = b^2 > n^2/400\log^2 n$. □

3 Prover-Adversary and Prover-Delayer games

In this section, we define two games based on an unsatisfiable CNF formula φ. Let $\mathcal{A}$ be a set of partial assignments for the variables of φ. We say that $\mathcal{A}$ is *proper* for φ if the following two properties hold:

- $\mathcal{A}$ is closed under restrictions: for every $\rho \in \mathcal{A}$ for every $\sigma \subseteq \rho$, $\sigma \in \mathcal{A}$.
- For every $\sigma \in \mathcal{A}$, σ does not falsify any clause of φ.

$(\varphi, \mathcal{A})$-game of Prover and Adversary with starting position $\rho_0 \in \mathcal{A}$. In this game, two players, Prover and Adversary, maintain a partial assignment ρ for variables of φ that initially equals ρ_0. On every move, Prover chooses a variable x, and Adversary earns a coin and chooses a Boolean value a of x. The current assignment ρ is updated: $\rho := \rho \cup \{x := a\}$. The game ends when $\rho \notin \mathcal{A}$. The goal of Adversary is to earn as many coins as he can.

Let $\mathcal{C}(\varphi)$ denote the set of all partial assignments that do not falsify any clause of φ. It is easy to see that $\mathcal{C}(\varphi)$ is proper and the maximal number of coins that Adversary can earn in the $(\varphi, \mathcal{C}(\varphi))$ with the empty starting position is precisely the resolution depth of φ. (See [Urq11] for details.)

Remark 3.1. In terms of Prover-Adversary games, one can also define the resolution width. Indeed, Atserias and Dalmau [AD08] showed that the resolution width of φ is at least w if and only if there exists a proper set of assignments $\mathcal{A}$ such that for every $\rho_0 \in \mathcal{A}$ if $|\rho_0| < w$, then in the $(\varphi, \mathcal{A})$-game with starting position ρ_0 Adversary has a strategy that guarantees him to earn at least $|w| - |\rho_0|$ coins.

Alekseev and Itsykson defined the following games [AI25].

$(\varphi, \mathcal{A})$-game of Prover and Delayer with starting position $\rho_0 \in \mathcal{A}$. In this game, two players, Prover and Delayer, maintain a partial assignment ρ for variables of φ that initially equals ρ_0. On every move, Prover chooses a variable x, and Delayer has two options:

- Delayer can earn a *white* coin and reports $*$. Then, Prover chooses a Boolean value a of x.
- Delayer can earn a *white* coin and pay a *black* coin to choose a Boolean value a of x by himself.

The current assignment ρ is updated: $\rho := \rho \cup \{x := a\}$. The game ends when $\rho \notin \mathcal{A}$.

Remark 3.2. It is easy to see that if, in the $(\varphi, \mathcal{C}(\varphi))$-game starting from the empty position, there exists a value t such that Delayer has a strategy ensuring that at some point he accumulates t more white coins than the total number of spent black coins, then any *tree-like* resolution refutation of φ must have size at least 2^t. See [PI00] for details.

Delayer's strategy is called *linearly described* [AI25] if there exists a map f that takes as input an ordered set of variables L and a variable x, and returns either $*$ or an $\mathbb{F}_2$-affine function h depending on the variables in L. The strategy is applied as follows: given a game history $x_1 = a_1, x_2 = a_2, \dots, x_k = a_k$ and a requested variable x, Delayer evaluates $f((x_1, x_2, \dots, x_k), x)$. If $f((x_1, x_2, \dots, x_k), x) = *$, then Delayer reports $*$. Otherwise, if $f((x_1, x_2, \dots, x_k), x) = h$ for some affine function h, Delayer reports $h(a_1, a_2, \dots, a_k)$.

3.1 Lifting by parity

Let $\mathcal{A}$ be a proper set of partial assignments for a CNF formula $\varphi(y_1, y_2, \dots, y_n)$.

We denote by $\oplus_k$ the parity gadget $\{0,1\}^k \to \{0,1\}$ that maps $(a_1, a_2, \dots, a_k)$ to $a_1 + a_2 + \dots + a_k \bmod 2$.

For every partial assignment ρ to the variables of the formula $\varphi \circ \oplus_k$ we define the partial assignment $\tilde{\rho}$ to the variables of φ as follows:

- $\tilde{\rho}$ is defined on y_i, if and only if ρ is defined on all $x_{i,1}, x_{i,2}, \dots, x_{i,k}$;
- $\tilde{\rho}(y_i) = \bigoplus_{j=1}^{k} \rho(x_{i,j})$.

Based on the formula $\varphi \circ \oplus_k$ we define a set $\mathcal{A}^{\oplus_k}$ that consists of partial assignments ρ to variables of $\varphi \circ \oplus_k$ such that $\tilde{\rho} \in \mathcal{A}$.

Proposition 3.3. *If $\mathcal{A}$ is a proper set for φ, then $\mathcal{A}^{\oplus_k}$ is a proper set for $\varphi \circ \oplus_k$.*

Proof. Consider $\rho \in \mathcal{A}^{\oplus_k}$ and let $\rho' \subseteq \rho$. By the definition, $\tilde{\rho}' \subseteq \tilde{\rho}$. Since $\tilde{\rho} \in \mathcal{A}$, then $\tilde{\rho}' \in \mathcal{A}$, then $\rho' \in \mathcal{A}^{\oplus_k}$.

Consider $\rho \in \mathcal{A}^{\oplus_k}$, every clause of $\varphi \circ \oplus_k$ is a clause of $C \circ \oplus_k$, where C is a clause of φ. Since $\tilde{\rho}$ doesn't falsify C, there is a variable y_j of C such that $\tilde{\rho}$ is not defined on y_j. Hence, there is $i \in [k]$ such that ρ is not defined on $x_{j,i}$, hence ρ doesn't falsify $C \circ \oplus_k$. ☐

Now we explore the simple idea of Urquhart [Urq11] that the strategy of Adversary can be lifted to the strategy of Delayer if we lift the formula by the parity gadget. The following lemma extends [AI25, Lemma 6.1].

Lemma 3.4. *Assume that Adversary has a strategy in a Prover-Adversary game* $(\varphi, \mathcal{A})$ *with starting position* $\rho_0 \in \mathcal{A}$ *that guarantees him to earn at least* t *coins. Consider a Prover-Delayer game* $(\varphi \circ \oplus_k, \mathcal{A}^{\oplus_k})$ *with starting position* σ_0*, where* $\tilde{\sigma}_0 = \rho_0$*. Then Delayer has a linearly described strategy that guarantees him to earn at least* $k(t + |\rho_0|) - |\sigma_0|$ *white coins while paying at most* t *black coins.*

Proof. We describe a strategy for Delayer in the Prover–Delayer game $(\varphi \circ \oplus_k, \mathcal{A}^{\oplus_k})$, obtained from the Adversary's strategy in the Prover–Adversary game $(\varphi, \mathcal{A})$. Let σ denote the current partial assignment in the first game. We maintain the invariant that the corresponding partial assignment in the second game is $\tilde{\sigma}$.

Initially, we set $\sigma = \sigma_0$ and, thus, $\tilde{\sigma} = \rho_0$. Suppose the requested variable in the first game is $x_{j,i}$. If there exists an index $i' \in [k] \setminus \{i\}$ such that σ is undefined on $x_{j,i'}$, then Delayer responds with $*$, and $\tilde{\sigma}$ remains unchanged.

Otherwise, if σ is defined on all $x_{j,i'}$ for $i' \in [k] \setminus \{i\}$, we simulate a Prover request for variable y_j in the second game. Let $a \in \{0, 1\}$ be the Adversary's response according to his strategy. Delayer then responds with $a \oplus \bigoplus_{i' \in [k] \setminus \{i\}} \sigma(x_{j,i'})$.

To show that this strategy is linearly described, it suffices to prove that the value of a is uniquely determined by the Adversary's strategy, the ordered list of queried variables, and the initial assignment σ_0.

Indeed, given the ordered list of queried variables in the first game and the initial assignment σ_0, we can compute both the initial assignment ρ_0 in the second game and the corresponding sequence of variable requests. Since the Adversary's strategy deterministically specifies the response to each query in the second game, we can compute all answers, in particular the last one, which is the value of a.

While $\tilde{\sigma} \in \mathcal{A}$, we have $\sigma \in \mathcal{A}^{\oplus_k}$. Since the Adversary in the first game earns at least t white coins, consider the first moment when $|\tilde{\sigma}| = |\rho_0| + t$. Each payment of a black coin corresponds to an increment of $\tilde{\sigma}$ by one, so by this point Delayer has paid exactly t black coins. The number of earned white coins is at least $|\sigma| - |\sigma_0| \geq k|\tilde{\sigma}| - |\sigma_0| = k(|\rho_0| + t) - |\sigma_0|$. □

4 Bounded-depth width games

In this section, we present a combinatorial characterization — an analogue of the Atserias–Dalmau games [AD08] — that captures when an unsatisfiable formula φ admits no resolution refutation of both width at most w and depth at most h simultaneously. A different game characterization for the same property, more closely related to depth-based games, has already appeared in the literature [Ber12, FPR22, EI25]. In contrast, our characterization is more aligned with width-based games.

Let φ be a CNF formula. Let $\mathcal{H}$ be a set of pairs (ρ, i) of a partial assignment ρ and an integer number i. We say that $\mathcal{H}$ is a (w, h)-winning strategy for φ if the following conditions hold:

- $(\varepsilon, 0) \in \mathcal{H}$, where ε is an empty assignment.
- If $(\rho, i) \in \mathcal{H}$, then $|\rho| \leq w$, $i \leq h$ and ρ doesn't falsify any clause of φ.

- If $(\rho, i) \in \mathcal{H}$ and $\rho' \subseteq \rho$, then $(\rho', i) \in \mathcal{H}$.
- If $(\rho, i) \in \mathcal{H}$, $|\rho| < w$, $i < h$, and $x \in \mathrm{Vars}(\varphi) \setminus \mathrm{Dom}(\rho)$, then there exists $a \in \{0, 1\}$ such that $(\rho \cup \{x := a\}, i + 1) \in \mathcal{H}$.

Theorem 4.1. *Let $w \geq 0$ and $h \geq 0$ be some integers; and let φ be an unsatisfiable CNF formula such that φ doesn't have a resolution refutation of width at most w and simultaneously with depth at most h. Then there exists a (w, h)-winning strategy for φ.*

Proof. Proof by induction on h. The base case is $h = 0$. In this case, we can take $\mathcal{H}$ consisting of the only element $(\varepsilon, 0)$, where ε is an empty clause.

We only have to verify that the formula does not contain an empty clause; this is true since the formula φ does not have refutation with depth zero and width zero.

Induction step. Let ϕ' be a CNF formula containing all clauses that can be derived from ϕ in at most one step with width at most w. It is easy to see that any resolution refutation of ϕ' has either width greater than w or depth greater than $h - 1$. We apply the induction hypothesis to ϕ'. Let $\mathcal{H}'$ be a $(w, h - 1)$ winning strategy for ϕ'.

- For every $(\rho, i) \in \mathcal{H}'$, ρ does not falsify clauses of φ' and hence clauses of φ of width at most w; $|\rho| \leq w$, hence ρ does not falsify any clause of φ.
- Consider $(\rho, h - 1) \in \mathcal{H}'$ such that $|\rho| \leq w - 1$. Let x be a variable from $\mathrm{Vars}(\varphi) \setminus \mathrm{Dom}(\rho)$. We claim that either $\rho \cup \{x := 0\}$ or $\rho \cup \{x := 1\}$ does not falsify any clause of φ. Suppose that $\rho \cup \{x := 0\}$ falsifies C_0 and $\rho \cup \{x := 0\}$ falsifies C_1, where C_0 and C_1 are clauses of φ. Since $|\rho \cup \{x := 0\}| \leq w$, width of C_0 and C_1 are at most w. Since ρ doesn't falsify neither C_0 nor C_1, $C_0 = D_0 \vee x$ and $C_1 = D_1 \vee \neg x$ and ρ falsifies D_0 and D_1. Therefore ρ falsifies $D_0 \vee D_1$. Note that $|D_0 \vee D_1| \leq |\rho| \leq w$ and $D_0 \vee D_1$ is the result of resolution rule applied to C_0 and C_1, hence, $D_0 \vee D_1$ is a clause of ϕ'. We get a contradiction since $(\rho, h - 1) \in \mathcal{H}'$.

Let us define Γ to be the set of all pairs of the form $(\rho \cup \{x := a\}, h)$ such that $(\rho, h - 1) \in \mathcal{H}'$, $|\rho| \leq w - 1$, and $a \in \{0, 1\}$, provided that $\rho \cup \{x := a\}$ does not falsify any clause of φ. Let Γ' be the set of all pairs (ρ', h) such that there exists an assignment ρ with $\rho' \subseteq \rho$ and $(\rho, h) \in \Gamma$.

We define $\mathcal{H} := \mathcal{H}' \cup \Gamma'$. Let us verify that $\mathcal{H}$ is a (w, h)-winning strategy for φ.

- Consider a pair $(\rho, i) \in \mathcal{H}$. If $i < h - 1$, then $(\rho, i) \in \mathcal{H}'$, so $|\rho| \leq w$ and ρ does not falsify any clause of φ. If $i = h$, then $(\rho, h) \in \Gamma'$, which implies that there exists an assignment σ such that $\rho \subseteq \sigma$ and $(\sigma, h) \in \Gamma$. Therefore, σ (and thus ρ) does not falsify any clause of φ, and $|\rho| \leq |\sigma| \leq w$.
- Consider a pair $(\rho, i) \in \mathcal{H}$ and let $\rho' \subseteq \rho$. If $i < h-1$, then $(\rho, i) \in \mathcal{H}'$ and thus $(\rho', i) \in \mathcal{H}' \subseteq \mathcal{H}$. If $i = h$, $(\rho, i) \in \Gamma'$, hence $(\rho', i) \in \Gamma' \subseteq \mathcal{H}$.
- Consider a pair $(\rho, i) \in \mathcal{H}$ such that $|\rho| < w$ and $i < h$, and let $x \in \mathrm{Vars}(\varphi) \setminus \mathrm{Dom}(\rho)$. If $i < h-1$, then $(\rho, i) \in \mathcal{H}'$. By the properties of $\mathcal{H}'$, there exists $a \in 0, 1$ such that $\rho \cup \{x := a\}$ does not falsify any clause of φ' (and hence none of φ), and $(\rho \cup \{x := a\}, i + 1) \in \mathcal{H}' \subseteq \mathcal{H}$. If $i = h - 1$, then as noted above, there exists $a \in 0, 1$ such that $\rho \cup \{x := a\}$ does not falsify any clause of φ. Thus, $(\rho \cup \{x := a\}, i + 1) \in \Gamma \subseteq \Gamma' \subseteq \mathcal{H}$.

$\square$

Proposition 4.2. *Let $\mathcal{H}$ be a (w,h)-winning strategy for the formula φ. Consider a set $\mathcal{B}$ that consists of all partial assignments ρ such that $(\rho,i) \in \mathcal{H}$ for some i. Then for every $(\rho_0,i_0) \in \mathcal{H}$ in the Prover-Adversary game $(\varphi,\mathcal{B})$ there is a strategy of Adversary with starting position ρ_0 that guarantees him to earn $\min\{w-|\rho_0|, h-i_0\}$ coins.*

Proof. Let ρ denote the current partial assignment in the Prover-Adversary game $(\varphi,\mathcal{B})$. Adversary will maintain the number i such that $(\rho,i) \in \mathcal{A}$. Initially $\rho := \rho_0$ and $i := i_0$. Let $|\rho| < w$ and $i < h$ and x be the requested variable. Then there exists such $a \in \{0,1\}$ such that $(\rho \cup \{x := a\}, i+1) \in \mathcal{H}$, hence $\rho \cup \{x := a\} \in \mathcal{B}$. The Adversary chooses any of such a and updates $\rho := \rho \cup \{x := a\}$ and $i := i+1$. After each step, the value $\min\{w-|\rho|, h-i\}$ decreases by one. And we can not make the next step if $\min\{w-|\rho|, h-i\} = 0$. Thus Adversary earns $\min\{w-|\rho_0|, h-i_0\}$ coins. ☐

Definition 4.3. Consider a $(\varphi,\mathcal{A})$-game between Prover and Delayer, and let H be a strategy for Delayer in this game. For any two assignments $\sigma,\sigma' \in \mathcal{A}$, we define the distance between them, denoted $\Delta_{\mathcal{A},H}(\sigma,\sigma')$, as the minimal integer K such that there exists a sequence of assignments $\sigma_0 = \sigma, \sigma_1, \ldots, \sigma_n = \sigma' \in \mathcal{A}$ satisfying the following conditions:

- For every $i \in [n]$, either σ_{i+1} is obtained from σ_i by one step of Delayer's strategy H, or $\sigma_{i+1} \subseteq \sigma_i$;
- The total number of steps of the first type (i.e., applications of H) is exactly K.

If no such sequence exists, we define $\Delta_{\mathcal{A},H}(\sigma,\sigma') = \infty$.

Proposition 4.4. *Let $\mathcal{H}$ be a (w,h)-winning strategy for a formula φ. Define $\mathcal{B}$ as the set of all partial assignments ρ such that $(\rho,i) \in \mathcal{H}$ for some i. Now consider a strategy H for Delayer in the game $(\varphi \circ \oplus_k, \mathcal{B}^{\oplus_k})$, obtained via Lemma 3.4 from the Adversary's strategy in the $(\varphi,\mathcal{B})$-game constructed in Proposition 4.2.*

1. *Let $(\rho_0,j) \in \mathcal{H}$, consider $\sigma_0 \in \mathcal{B}^{\oplus_k}$ such that $\tilde{\sigma}_0 = \rho_0$. Let $\sigma \in \mathcal{B}^{\oplus_k}$ such that $\Delta_{\mathcal{B}^{\oplus_k},H}(\sigma_0,\sigma) < \infty$. Then $(\tilde{\sigma}, j') \in \mathcal{H}$ for some $j' \leq j + \Delta_{\mathcal{B}^{\oplus_k},H}(\sigma_0,\sigma)$.*
2. *Let $\sigma \in \mathcal{B}^{\oplus_k}$ such that $\Delta_{\mathcal{B}^{\oplus_k},H}(\epsilon,\sigma) \leq h-w$, where ϵ is an empty assignment. Then, the strategy H with starting position σ guarantees that Delayer earns at least $kw - |\sigma|$ white coins while paying at most w black coins.*

Proof.
1. Let us denote $K = \Delta_{\mathcal{B}^{\oplus_k},H}(\sigma_0,\sigma)$. There are exist $\sigma_1, \sigma_2, \ldots, \sigma_m = \sigma$ such that for every $i \in [m]$ either $\sigma_i \subseteq \sigma_{i-1}$ or σ_i can be obtained from σ_{i-1} by one step according the strategy H. Let us define the sequence $j_0, j_1, \ldots, j_m$ as follows: $j_0 = j$, for every $i \in [m]$: if $\sigma_i \subseteq \sigma_{i-1}$, then $j_i = j_{i-1}$; if σ_i is obtained from σ_{i-1} by one step according the strategy H, then if $\tilde{\sigma}_i = \tilde{\sigma}_{i-1}$, then $j_i = j_{i-1}$ and if $\tilde{\sigma}_i \neq \tilde{\sigma}_{i-1}$, then $j_i = j_{i-1}+1$. Taking into account the definition of the game $(\varphi \circ \oplus_k, \mathcal{B}^{\oplus_k})$ and the construction of the strategy H, it is easy to verify that $(\tilde{\sigma}_i, j_i) \in \mathcal{H}$ for all $i \in [m]$. In particular, this implies that $(\tilde{\sigma}, j_m) \in \mathcal{H}$. Observe that $j_i > j_{i-1}$ only when s_i is obtained from s_{i-1} by a single step of the strategy H. Therefore, $j_m \leq j + K$.
2. By the previous item, there exists $j' \leq h-w$ such that $(\tilde{\sigma}, j') \in \mathcal{H}$. Proposition 4.2 gives a strategy of Adversary in the Prover-Adversary game $(\varphi,\mathcal{B})$ with starting position $\tilde{\sigma}$ that guarantees him to earn $w - |\tilde{\sigma}|$ coins. Then by Lemma 3.4 the strategy H in the Prover-Delayer

game with starting position σ guarantees Delayer to earn at least $k(w-|\tilde{\sigma}|+|\tilde{\sigma}|)-|\sigma| = kw-|\sigma|$ white coins while paying at most w black coins.

□

5 Random-walk theorem

In this section, we present the main tool developed by Itsykson and Alekseev [AI25] for proving size-depth tradeoffs in $\mathrm{Res}(\oplus)$.

Let Π be a $\mathrm{Res}(\oplus)$ refutation, C_0 be a linear clause from Π, Σ be a set of full assignments that falsify C_0, and $t \in \mathbb{N}$ be a natural number. A (Π, C_0, Σ, t)-*random walk* is defined as follows: sample an assignment σ uniformly at random from Σ, and perform a walk of weighted length t on the refutation graph of Π, starting at the node labeled by C_0. At each step, the walk proceeds from a linear clause to a premise falsified by σ. There are two cases: if the step uses the weakening rule, there is a single premise and the step has weight zero; if it uses the resolution rule, there are two premises and the step has weight one. The walk terminates either upon reaching a clause from the initial formula or when the total weight accumulated over all steps reaches t. If the walk terminates at a node labeled with a linear clause C, then C is the value of the random variable defined by the walk.

Theorem 5.1 (Theorem 4.3 from [AI25])**.** *Let φ be an unsatisfiable CNF formula and $g\colon \{0,1\}^\ell \to \{0,1\}$ be a 2-stifling gadget. Consider a $\mathrm{Res}(\oplus)$ refutation Π of $\varphi \circ g$ and a linear clause C_0 from Π. Let τ be a solution of $\neg C_0$ and let ρ_0 be the restriction of $\hat{\tau}$ to $\mathrm{Cl}(L(C_0))$. Let Σ be the set of all full assigmnets π such that π satisfies $\neg C_0$ and $\hat{\pi}$ extends ρ_0. Let t be integer number such that $t \le w - |\widetilde{\mathrm{Cl}}(L(C_0))| + |\rho_0|$. Let a linear clause C denote the result of the (Π, C_0, Σ, t)-random walk defined by a random variable σ distributed uniformly on Σ.*

Let $\mathcal{A}$ be a proper set of partial assignments for $\mathrm{Vars}(\varphi)$. Assume that in the $(\varphi, \mathcal{A})$-game with starting position $\rho_0 \in \mathcal{A}$, Delayer has a linearly described strategy H that guarantees him to earn w white coins while paying at most c black coins. Then $\hat{\sigma}|_{\mathrm{Cl}(L(C))} \in \mathcal{A}$ and $\Delta_{\mathcal{A},H}(\rho_0, \hat{\sigma}|_{\mathrm{Cl}(L(C))}) \le w$ with probability at least $2^{-c(\ell-1)}$.

Theorem 5.1 is a slightly modified version of [AI25, Theorem 4.3], with two minor adjustments to the statement that do not affect the validity of the original proof. The first modification, introduced in [EI25], concerns the inequality for t, namely the bound $t \le w - |\widetilde{\mathrm{Cl}}(L(C_0))| + |\rho_0|$ was originally stated in a stronger form as $t \le w - \mathrm{rk}(\neg C_0) + |\rho_0|$. We refer the reader to [EI25] for an explanation of why the proof remains valid under the weaker bound.

Here, we focus on the second modification: namely, we additionally assert that $\Delta_{\mathcal{A},H}(\rho_0, \hat{\sigma}|_{\mathrm{Cl}(L(C))}) \le w$.

First, observe that in the "lucky" execution of the random walk — that is, when $\hat{\sigma}|_{\mathrm{Cl}(L(C))} \in \mathcal{A}$ — Lemma 2.9 implies that C is not a clause of $\varphi \circ g$. Hence, the random walk successfully makes t steps (i.e., it does not terminate prematurely at a leaf). Let us denote the sequence of visited linear clauses by $C_0, C_1, \ldots, C_m = C$. There exist indices $0 \le i_1 < i_2 < \cdots < i_t \le m$ such that:

- For every $i \in \{i_1, i_2, \ldots, i_t\}$, the clause C_i is obtained from C_{i+1} and another premise by applying the resolution rule over the linear form f_i.
- For every $i \in \{0, 1, \ldots, m-1\} \setminus \{i_1, i_2, \ldots, i_t\}$, the clause C_i is obtained from C_{i+1} by the weakening rule.

By the definitions of the resolution and weakening rules, it follows that $L(C) \subseteq \langle L(C_0), f_1, f_2, \dots, f_t \rangle$. Hence, by Lemma 2.4, we have

$$\mathrm{Cl}(L(C)) \subseteq \mathrm{Cl}\left(L(C_0) \cup \{f_1, f_2, \dots, f_t\}\right).$$

The proof of [AI25, Theorem 4.3] shows that, with probability at least $2^{-c(\ell-1)}$, the restriction of $\hat{\sigma}$ to $\mathrm{Cl}(L(C_0) \cup \{f_1, f_2, \dots, f_t\})$ is consistent with the strategy H in the game $(\varphi, \mathcal{A})$ starting from position ρ_0. This means that there exists some sequence of moves by Prover for which Delayer's strategy H reaches the assignment $\hat{\sigma}|_{\mathrm{Cl}(L(C_0)\cup\{f_1,f_2,\dots,f_t\})}$. Since $\mathcal{A}$ is closed under restrictions, it follows that $\hat{\sigma}|_{\mathrm{Cl}(L(C))} \in \mathcal{A}$. The number of moves in the game does not exceed

$$|\,\mathrm{Cl}\left(L(C_0) \cup \{f_1, f_2, \dots, f_t\}\right)| \overset{\text{(Lem. 2.6)}}{\leq} |\,\widetilde{\mathrm{Cl}}\left(L(C_0) \cup \{f_1, f_2, \dots, f_t\}\right)| \overset{\text{(Lem. 2.7)}}{\leq} |\,\widetilde{\mathrm{Cl}}(L(C_0))| + t \leq w.$$

Hence, $\Delta_{\mathcal{A},H}(\rho_0, \hat{\sigma}|_{\mathrm{Cl}(L(C))}) \leq w$ by the definition of the distance Δ.

6 Size vs depth tradeoff

Theorem 6.1. *Let ψ be an unsatisfiable CNF formula such that ψ does not have a resolution refutation with width at most w and simultaneously depth at most h. Assume that there is a natural number $s \geq 2$ such that $h \geq s^2 w - w$. Let $\varphi := \psi \circ \oplus_s$.*

Let $g\colon \{0,1\}^\ell \to \{0,1\}$ be a 2-stifling gadget. Then any $\mathrm{Res}(\oplus)$ *refutation of $\varphi \circ g$ has either size at least 2^w or depth at least $\frac{s^2 w}{4\ell}$.*

Proof. By Theorem 4.1 for the formula ψ there is a (w,h)-winning strategy $\mathcal{H}$. Let $\mathcal{B}$ be the set of all partial assignments ρ such that $(\rho, i) \in \mathcal{H}$ for some i. By Proposition 4.2, for every $(\rho, i) \in \mathcal{H}$ in Prover-Adversary game $(\psi, \mathcal{B})$ there is a strategy of Adversary with starting position ρ that guarantees him to earn $\min\{w - |\rho|, h - i\}$ coins.

We define $\mathcal{A} = \mathcal{B}^{\oplus_k}$ and consider a linearly described strategy H for Delayer in the Prover-Delayer game $(\varphi, \mathcal{A})$ that exists by Lemma 3.4.

Let us denote $t := ws$. By Proposition 4.4, for every $\rho_0 \in \mathcal{A}$ such that $\Delta_{\mathcal{A},H}(\epsilon, \sigma) \leq h - w$, in the game with starting position ρ_0 the strategy H guaranties Delayer to earn at least $t - |\rho_0|$ white coins while paying at most w black coins.

In what follows, we use our lifting notations assuming that variables of φ are unlifted and variables of $\varphi \circ g$ are lifted.

Let C be a linear clause over lifted variables (i.e., variables of the formula $\varphi \circ g$), and let $\rho \in \mathcal{A}$ be a partial assignment over the original (unlifted) variables. We say that C *corresponds* to ρ if there exists an assignment τ satisfying $\neg C$ such that the restriction of $\hat{\tau}$ onto $\mathrm{Cl}(L(C))$ coincides with ρ, that is, $\hat{\tau}|_{\mathrm{Cl}(L(C))} = \rho$. We denote this relation by $C \sim \rho$. For convenience, we also define a measure μ of a clause C as $\mu(C) := \left|\widetilde{\mathrm{Cl}}(L(C))\right|$.

Consider a $\mathrm{Res}(\oplus)$ refutation of $\varphi \circ g$ and denote it by Π.

Claim 6.2. *Assume that Π contains a linear clause C_0 such that*

- *$\mu(C_0) \leq r$, where $r < t$;*
- *there exists $\rho_0 \in \mathcal{A}$ such that $C_0 \sim \rho_0$ and $\Delta_{\mathcal{A},H}(\epsilon, \rho_0) \leq h - w$.*

Let $S_{t-r}(C_0)$ denote the set of all linear clauses C from Π such that

- *there is a path from C_0 to C of weighted length $t-r$ in the graph of Π (computing length, we compute weakening rules with weight zero and resolution rules with weight one);*
- *there exists $\rho \in \mathcal{A}$ such that $C \sim \rho$ and $\Delta_{\mathcal{A},H}(\rho_0, \rho) \leq t-r$.*

Assume that for every $C \in S_{t-r}(C_0)$, $\mu(C) \geq r + w\ell$. Then, the size of the refutation Π is at least 2^w.

Proof. Notice that $|\rho_0| = |\operatorname{Cl}(L(C_0))| \overset{\text{(Lemma 2.6)}}{\leq} |\widetilde{\operatorname{Cl}}(L(C_0))| = \mu(C_0) \leq r$.

Let Σ be the set of all assignments π such that π satisfies $\neg C_0$ and $\hat{\pi}|_{\operatorname{Cl}(L(C_0))} = \rho_0$. Since $C_0 \sim \rho_0$, $\Sigma \neq \emptyset$.

Let a linear clause C denote the result of the $(\Pi, C_0, \Sigma, t-r)$-random walk defined by a random variable σ distributed uniformly on Σ. Notice that $t - r \leq (t - |\rho_0|) + |\rho_0| - |\widetilde{\operatorname{Cl}}(L(C_0))|$. Let $\rho = \hat{\sigma}|_{\operatorname{Cl}(L(C))}$. By Theorem 5.1, with probability at least $2^{-(\ell-1)w}$, $\rho \in \mathcal{A}$, $C \sim \rho$ and $\Delta_{\mathcal{A},H}(\rho_0, \rho) \leq (t-r)$. By Lemma 2.9, C is not a clause of $\phi \circ g$, hence, the length of the path between C_0 and C is exactly $t-r$, hence $C \in S_{t-r}(C_0)$. Thus, $\mu(C) \geq r + w\ell$.

Consider some linear clause D such that $\mu(D) \geq r + w\ell$. By Lemma 2.8, $\Pr_{\sigma\in\Sigma}[\sigma \text{ satisfies } \neg D] \leq 2^{|\widetilde{\operatorname{Cl}}(L(C_0))| - |\widetilde{\operatorname{Cl}}(L(D))|} = 2^{\mu(C_0) - \mu(D)} \leq 2^{-w\ell}$.

Hence, the refutation Π contains at least $\frac{2^{-(\ell-1)w}}{2^{-\ell w}} = 2^w$ clauses D with $\mu(D) \geq r + w\ell$. □

Assume that the size of Π is less than 2^w. Our goal is to show that under this assumption, the depth of Π is at least $\frac{ws^2}{4\ell}$.

Let D_0 denote the empty clause from Π. $D_0 \sim \rho_0$, where ρ_0 equals the empty assignment ϵ and, thus, $\rho_0 \in \mathcal{A}$. Since $|\Pi| < 2^w$, by Claim 6.2, there is a clause D_1 in Π such that there is a path from D_0 to D_1 of length t and $\mu(D_1) \leq w\ell$ and there is $\rho_1 \in \mathcal{A}$ such that $D_1 \sim \rho_1$ and $\Delta_{\mathcal{A},H}(\epsilon, \rho_1) \leq t$.

Let $k := \lfloor \frac{t}{2w\ell} \rfloor$, then $w\ell k \leq t/2$. We repeat the same reasoning $k-1$ more times for all i from 1 to $k-1$, maintaining invariant $\mu(D_i)| \leq w\ell i$. Since $|\Pi| < 2^w$, by Claim 6.2 there is a linear clause D_{i+1} in Π such that there is a path from D_i to D_{i+1} of length $t - w\ell i$ and there is $\rho_i \in \mathcal{A}$ such that $D_i \sim \rho_i$ and $\Delta_{\mathcal{A},H}(\rho_i, \rho_{i+1}) \leq t$ and $\mu(D_{i+1})| \leq w\ell(i+1)$. Note that for all $i \in [k-1]$, by triangle inequalities, $\Delta(\epsilon, \rho_i) \leq kt \leq \frac{t^2}{2w\ell} = \frac{ws^2}{2\ell} \leq h - w$; the last inequality holds since by the conditions of the theorem $s \geq 2$ and $h \geq w(s^2-1) \geq w(\frac{s^2}{2\ell}+1) = \frac{ws^2}{2\ell} + w$. So the distance conditions in applications of Claim 6.2 are satisfied.

So under the assumption $|\Pi| < 2^w$ we get that the depth of Π is at least the length of the path from D_0 to D_1, from D_1 to D_2, etc, from D_{k-1} to D_k which is at least $kt/2 \geq \frac{ws^2}{4\ell}$. □

We now examine two specific cases of Theorem 6.1.

Theorem 6.3. *For every natural $K \geq 2$ and $n \geq 2$ such that $K \leq \sqrt{\frac{n}{10\log^3 n}}$ there is a CNF formula $\varphi_{n,K}$ that contains $5Kn\lfloor \log n \rfloor$ variables, the formula $\varphi_{n,K}$ is in $O(K\log^2 n)$-CNF and of size $n^{O(K\log n)}$. The formula $\varphi_{n,K}$ has resolution refutation of size $n^{O(K\log n)}$ and of width $O(K\log^2 n)$ but every its* Res(⊕) *refutation has either size at least $2^{\lfloor n/40\log n \rfloor}$ or depth at least $\Omega(K^2 n\log n)$.*

Proof. Let Ψ_n be a formula from Corollary 2.11.

Let us define $\varphi_{n,K} := \Psi_n \circ \oplus_{K\lfloor \log n \rfloor} \circ Maj_5$. Then $\varphi_{n,K}$ that contains $5Kn\lfloor \log n \rfloor$ variables, of the formula $\varphi_{n,K}$ is in $O(K \log^2 n)$-CNF and of size $n^{O(K \log n)}$. By Lemma 2.2, the formula $\varphi_{n,K}$ has resolution refutation of size $n^{O(K \log n)}$ and of width $O(K \log^2 n)$.

Let $w = \lfloor n/40 \log n \rfloor$ and $h = \lfloor n^2/400 \log^2 n \rfloor$. The formula Ψ_n does not have resolution refutations of width at most w and of depth at most h. Let $s = K\lfloor \log n \rfloor$. It is easy to see that $s \geq 2$.

If $w < 1$, then the conclusion of the theorem is trivial. Assume that $w \geq 1$. Notice that $h + w \geq h + 1 \geq \frac{n^2}{400 \log^2 n} = \frac{n}{10 \log^3 n} \frac{n}{40 \log n} \cdot \log^2 n \geq K^2 \log^2 nw \geq ws^2$, by Theorem 6.1, any Res($\oplus$) refutation of $\varphi_{n,K}$ has either size at least $2^w = 2^{\lfloor n/40 \log n \rfloor}$ or has depth at least $\frac{w^2 s}{20} = \Omega(K^2 n \log n)$. □

Corollary 6.4. *For every $\delta > 0$, Depth-$\frac{n^{4/3}}{\log^{4/3+\delta} n}$ Res($\oplus$) does not p-simulate resolution.*

Proof. Consider $K = \left\lfloor \sqrt{\frac{n}{10 \log^3 n}} \right\rfloor$ and the formula $\varphi_{n,K}$ from Theorem 6.3. $\varphi_{n,K}$ contains $m = \Theta\left(\frac{n^{3/2}}{\log^{1/2} n}\right)$ variables and has resolution refutation of size at most $n^{O(\sqrt{n/\log n})}$ and the size of $\varphi_{n,K}$ is also $n^{O(\sqrt{n/\log n})}$. There is a constant c such that every Res($\oplus$) refutation of $\varphi_{n,K}$ of depth at most $cn^2/\log^2 n$ has size at least $2^{\Omega(n/\log n)}$.

Notice that for n large enough, $\frac{m^{4/3}}{\log^{4/3+\delta} m} = \Theta\left(\frac{n^2}{\log^{2+\delta} n}\right) < cn^2/\log^2 n$. Thus, for n large enough every Res($\oplus$) refutation of $\varphi_{n,K}$ of depth at most $\frac{m^{4/3}}{\log^{4/3+\delta} m}$ has size at least $2^{\Omega(n/\log n)}$. And $2^{\Omega(n/\log n)}$ can not be bounded by a polynomial in $n^{O\left(\sqrt{n/\log n}\right)}$. □

Theorem 6.5. *Let ψ_n be a family of unsatisfiable $O(1)$-CNF formulas such that ψ_n has n variables and the resolution width of ψ_n is $w(n)$. For every natural $K \geq 2$ consider a formula $\Psi_{n,K} := \psi_n \circ \oplus_K \circ Maj_5$; it has $5nK$ variables, $\Psi_{n,K}$ is an $O(K)$-CNF formula of size at most $\mathrm{poly}(n)2^K$ and any Res($\oplus$) refutations of Ψ_n has either depth at least $\Omega(w(n)K^2)$ or size at least $2^{\Omega(w(n))}$.*

Proof. Let $w(n)$ be the resolution width of ψ_n, take $s = K$. Let $h = w(n)s^2$. There are no resolution refutations of ψ_n of width at most $w(n) - 1$ and depth h. Maj_5 is a 2-stifling gadget. Then by Theorem 6.1, any Res($\oplus$) refutation of Ψ_n has either size at least $2^{w(n)-1} = 2^{\Omega(n)}$ or depth at least $\frac{ws^2}{4\ell} = \Omega(nK^2)$. □

7 Open questions

Two main avenues for improving our results are:

1. Construct a polynomial-sized CNF formula that admits a polynomial-sized resolution refutation, yet any Res($\oplus$) refutation of it must have either superlinear depth or exponential size. One approach to achieving this is by strengthening the supercritical tradeoff between width and depth in resolution. Specifically, it suffices to construct an $O(1)$-CNF formula with n variables that has a resolution refutation of constant width, but for which any resolution refutation must have either superlinear depth or width $\Omega(n)$.

2. Establish a truly supercritical tradeoff between size and depth for Res($\oplus$), in which the depth is superlinear with respect to the size of the formula.

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