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Hyperdeterminants are hard in four dimensions

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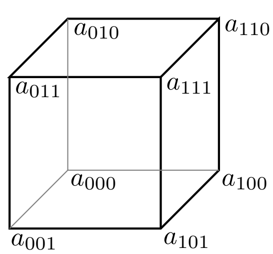
Abstract. Hyperdeterminants are high dimensional analogues of determinants, associated with tensors of formats generalizing square matrices. First conceived for $2 \times 2 \times 2$ tensors by Cayley, they were developed in generality by Gelfand, Kapranov and Zelevinsky. Yet, hyperdeterminants in three or more dimensions are long conjectured to be VNP-Hard to compute, akin to permanents and unlike determinants. Even deciding if the hyperdeterminant of a given tensor is zero is conjectured to be NP-Hard. We prove this decision problem is NP-Hard under randomised reductions, in four or more dimensions. In quantum information, hyperdeterminants measure quantum entanglement, under the name "tangle". Our reduction implies that it is hard to tell if four or more qudits are tangled, unless quantum computers can efficiently solve NP-complete problems.

1 Introduction

1.1 Hyperdeterminants

The determinant of a square matrix is a homogeneous polynomial in the matrix coordinates with integer coefficients that vanishes if and only if the matrix is singular. Cayley discovered an analogue of the determinant for three dimensional $2 \times 2 \times 2$ format tensors, called the hyperdeterminant, depicted below [3].



$$\begin{aligned}\text{Hyperdeterminant} = {} & a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{011}^2 a_{100}^2 \\ & -2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} + a_{000}a_{011}a_{100}a_{111} \\ & + a_{001}a_{010}a_{101}a_{110} + a_{001}a_{011}a_{110}a_{100} + a_{010}a_{011}a_{101}a_{100}) \\ & +4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111}).\end{aligned}$$

Cayley's hyperdeterminant is evidently a homogeneous polynomial in the tensor coordinates with integer coefficients. Remarkably, it vanishes precisely when the tensor is singular. Gelfand, Kapranov and Zelevinsky generalised hyperdeterminants to arbitrary dimensions and built a vast theory around them [10,9]. The property of the hyperdeterminant vanishing precisely when the tensor is singular persists in all dimensions. Tensor singularity is a geometric notion defined through projective duality. The hyperdeterminant is defined as the polynomial, uniquely defined up to sign, whose vanishing defines the hypersurface of singular tensors. We defer to [9] for formal definitions of singularity and the hyperdeterminant.

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1.2 Degeneracy

Instead, our exposition relies on the algebraic notion of degeneracy (see § 2), which is equivalent to singularity. Consider as r-dimensional tensors, elements A in the tensor product of (dual) vector spaces of dimensions $k_1+1, k_2+1, \ldots, k_r+1$ over complex numbers. We also think of such a tensor as a multilinear form, and identify it with a $(k_1+1) \times (k_2+1) \times \ldots \times (k_r+1)$ format r-dimensional matrix of coordinates. We will call k_j+1 as the length in the j-th dimension. Square matrices correspond to the $r=2$, $k_1=k_2$ with the same length in both dimensions. Consider the following algebraic definition of degeneracy of square matrices: A matrix A is degenerate if there is a pair of non zero vectors $(w^{\text{left}}, w^{\text{right}})$ such that $w^{\text{left}}A$ and Aw^{right} are both zero vectors. The definition may seem atypical, but is clarified by thinking of $(w^{\text{left}}, w^{\text{right}})$ as a pair of left and right kernel vectors. It is this motif that easily generalises in the following definition for three dimensional tensors. A three dimensional tensor A (trilinear form) is degenerate, if there exists a triple $(w^{(1)}, w^{(2)}, w^{(3)})$ of non zero kernel vectors such that evaluating the trilinear form at all but one (so, two) of the vectors results in the all zero dual vector. That is,

$$A\left(*, w^{(2)}, w^{(3)}\right) = 0,\ A\left(w^{(1)}, *, w^{(3)}\right) = 0,\ A\left(w^{(1)}, w^{(2)}, *\right) = 0,$$

are zero (dual) vectors in the first, second and third dimension respectively. Likewise, an r-dimensional tensor is degenerate if there is an r-tuple of non zero vectors such that evaluating the r-linear form at all but one of the vectors gives the zero (dual) vector.

1.3 Tensors generalising square matrices

Not all matrices have associated determinants, only square matrices do. Likewise, not all tensor formats have associated hyperdeterminants. Gelfand, Kapranov and Zelevinsky delineated this dichotomy as follows. A tensor has a well defined hyperdeterminant if and only if it is of a format satisfying the convexity constraint

$$\forall j \in \{1, 2, \ldots, r\}, \qquad k_j \leq \sum_{\ell \neq j} k_\ell. \tag{1.1}$$

Formats with the further assurance that there is at least one j satisfying the equation with equality are called as boundary formats. Formats that satisfy equation 1.1 that are not boundary are called as interior. We call all other formats exterior, though this is not common terminology. In two dimensions, the convexity constraint simplifies to $k_1 \leq k_2$ and $k_2 \leq k_2$, meaning only square matrices have determinants. Boundary formats generalise square matrices to higher dimensions in the strictest sense. To quote Gelfand, Kapranov and Zelevinsky [10], *"It is instructive to think of matrices of boundary format as proper high dimensional analogs of ordinary square matrices"*. To summarise, interior and boundary formats come with a hyperdeterminant, whose vanishing characterises degenracy. Exterior formats do not have a well defined hyperdeterminant. Informally, exterior formats have one dimension whose length (projectively) exceeds the sum of lengths of the other dimensions. This affords enough freedom that the variety of degenerate tensors has co-dimension more than one, preventing a single polynomial from carving it. Yet, the notion of degeneracy remains perfectly sound, even for exterior formats. Informally, when we refer to the hyperdeterminant of a format, we mean a polynomial in the coordinates ring. But the hyperdeterminant of a tensor refers to a field element, resulting from evaulating the aforementioned polynomial at the tensor coordinates.

1.4 Conjectures on the hardness of hyperdeterminants

Linear algebraic problems that are computationally easy in two dimensions, often transition in three or more dimensions into multilinear analogues that are hard. Hillar and Lim catalogued many such problems, and proved new hardness results on some more [12]. They conjectured that computing the hyperdeterminant is hard, as one transitions from two dimensions (where computing determinants is easy) to three. The conjectured hardness depends on the model of computation: #P-hard in the counting model, VNP-hard in the arithmetic circuit model and NP-hard to zero test [12][Conjecture 1.9, Conjecture 13.1]. In the discussion following [12][Conjecture 13.1], they cite reasons for their conviction behind the conjecture. Among them is the apparent complexity of hyperdeterminants even in small examples: the $2 \times 2 \times 2 \times 2$ hyperdeterminant has nearly 2.9 million monomials [13]. Another reason they propose is that testing the vanishing of general multivariate resultants (for systems with as many polynomials as variables) is known to be NP-hard.

The obstruction to previous hardness proof attempts: In fact, they show that polynomials for the multivariate resultant decision problem can be taken to be bilinear forms [12][Theorem 3.7]. The general multivariate system in [12][Theorem 3.7], translated appropriately to our notation, is exactly the problem of testing if a tensor is degenerate! But the instances they generate while encoding an NP-complete problem (namely, 3-Color) are three dimensional exterior format, with no associated hyperdeterminant to speak of. In essence, known encodings of NP-complete problems as tensor degeneracy inflate the tensor in one dimension (the dimension that indexes constraints) to land in exterior formats, beyond the relevance of hyperdeterminants.

Some tangential results: There is some evidence for hardness of hyperdeterminants over finite fields from cryptography. Tensors that are keys in tensor isomorphism based cryptography (such as the MEDS post-quantum signature scheme [4]) are weak if they are degenerate, and there are cryptanalytic algorithms to test degeneracy in three dimensional cubical (which are, interior) formats [23]. These algorithms take time exponential in the length of the cube. Likewise, Grobner basis and other algebraic methods that attempt to solve the degeneracy equation take exponential time in the length of one of the dimensions [8,24,25,26]. This is merely evidence of the difficulty of known techniques, and not insightful towards a proof of hardness. There is also some evidence for VNP-hardness of hyperdeterminants [14], using a Cramer's rule that translates an arithmetic circuit for the hyperdeterminant to one that solves the homogeneous multilinear system defining degeneracy. But again, solving degeneracy was not known to be hard in any format for which hyperdeterminants exist. The combinatorial hyperdeterminant is a different attempt at a generalisation of determinants to higher dimensions, by extrapolating the Leibniz formula for determinants. Barvinok proved NP-hardness of zero testing [1] and Gurvits proved #P-hardness and VNP-hardness of the combinatorial hyperdeterminant [11]. Curiously, both these hardness results were in four dimensions, leaving the three dimensional case open. Yet, it is dubious if the hardness proof of the combinatorial hyperdeterminant informs the status of hardness of the hyperdeterminant, for the combinatorial version does not have the rich algebraic and geometric structure intrinsic to the hyperdeterminant.

1.5 Our results

We prove that the decision version of problem is hard.

Theorem 1. *Fix a dimension $r \geq 4$. Deciding if a rational r-dimensional tensor (of a format for which hyperdeterminants exist, that is, interior or boundary) has zero hyperdeterminant is NP-hard under randomised reduction.*

By rational, we mean that the coordinates of the tensor are rational numbers. The result holds over arbitrary number fields too, since the statement over the rational numbers is stronger. The theorem only considers constant dimensions, which allow the input tensor to be presented by writing down all its coordinates. If we allow the dimension to grow, the input size grows exponentially in the dimension and is not considered by the theorem.

The reduction to four dimensions: We initially follow the reduction of Hillar and Lim [12][Theorem 3.7]: starting from 3-COLOR, expressing it as a homogeneous polynomial system, which is then encoded as a degeneracy problem. This initial phase maps a graph G to a three dimensional tensor A^{3G} such that the graph has a proper 3-colouring if and only if A^{3G} is degenerate. As hinted before, this tensor A^{3G} is of exterior format, without a hyperdeterminant. In fact, it is wildly exterior, with the length in the longest dimension being quadratic in the number of vertices, while the length in the other two dimensions is linear. Our main insight is to construct a tensor A^{4G} in four dimensions of a boundary format enveloping this exterior format in three dimensions, as follows. Append a fourth dimension of the smallest possible length to make the format boundary. Call this the enveloping four dimensional boundary format. Plant A^{3G} as one of the slices of A^{4G} (in the fourth dimension) and draw the rest of the coordinates of A^{4G} independently and uniformly from a large finite set of integers.

In boundary formats, the Cayley trick implies that the hyperdeterminant equals the resultant of the multilinear systems defined by the slices (in the longest dimension) [9][Theorem 3.1][1]. This helps relate the hyperdeterminant of A^{4G} to the degeneracy of one of its slices, namely A^{3G}. In particular, while the three dimensional format of A^{3G} does not have a hyperdeterminant, it does have a Chow form, which Gelfand, Kapranov and Zelevinsky prove to be the hyperdeterminant of the enveloping four dimensional boundary format [9][Theorem 3.10]. As a consequence, A^{3G} is degenerate if and only if every enveloping boundary format tensor that has A^{3G} as a slice (in the fourth dimension) has vanishing hyperdeterminant [9][Corollary 3.11]. Therefore, if A^{3G} is degenerate, then the hyperdeterminant of A^{4G} vanishes. Say A^{3G} is not degenerate. Consider the hyperdeterminant of the enveloping four dimensional boundary format, partially evaluated at the A^{3G} slice. This partial evaluation is a non zero polynomial (with coordinates outside the slice as indeterminates), for otherwise we show it contradicts [9][Corollary 3.11]. The fact that permuting parallel slices preserves the hyperdeterminant up to sign is crucial in reaching this conclusion. The hyperdeterminant of A^{4G} is the evaluation of this non zero polynomial at the randomly chosen coordinates outside the planted slice. With a degree bound on this non zero polynomial, the Schwartz-Zippel lemma ensures A^{4G} does not vanish with high probability. Therefore, one can test if G has a proper 3-colouring by testing if A^{4G} has zero hyperdeterminant.

Reduction to higher dimensions: We present a deterministic reduction that lifts hardness to higher dimensions in lemma 3. We reduce the hyperdeterminant zero testing problem (restricted to boundary formats) from r-dimensions to the hyperdeterminant zero testing

[1] The longest dimension will turn out to be the first, not the fourth dimension of A^{4G}

problem in a desired higher dimension s. Again, we think of s as a constant, since the reduction will write down an s-dimensional tensor in its entirety. Let A_r be the r-dimensional tensor whose hyperdeterminant is in question. We construct a $(s-r+2)$-dimensional boundary format tensor B_{s-r+2} designed to be of non vanishing hyperdeterminant, whose longest dimension is of the same length as one of A_r's dimensions. To this end, we rely on explicit constructions of Gelfand, Kapranov and Zelvinsky (identity tensors) or Weyman and Zelevinsky (diagonal tensors, Vandermonde-Weyman-Zelevinsky tensors)(see also [22]). The reduction maps A_r to the tensor convolution $A_r \star B_{s-r+2}$. Tensor convolution is a generalisation of inner products, matrix-vector products and matrix multiplication. Multiplying by a vector (inner product) reduces dimension, multiplying by matrices preserves dimension, while convolving with tensors (of three or more dimensions) increases dimensions. The result of our convolution $A_r \star B_{s-r+2}$ has dimension s. Dionisi and Ottaviani proved a high dimensional analogue of the Binet-Cauchy theorem for hyperdeterminants, which for boundary formats generalises the multiplicativity of determinants [6]. Multiplicativity implies that the hyperdeterminant of $A_r \star B_{s-r+2}$ vanishes if and only if that of A_r or B_{s-r+2} vanish. Thus, the hyperdeterminant of A_r is zero if and only if that of $A_r \star B_{s-r+2}$ is.

Hyperdeterminants modulo large primes: Our hardness results seem to extend to the hardness of testing if the hyperdeterminant of a given tensor is zero modulo a given large prime. A natural path to such results is to rely on the theory of hyperdeterminants over (algebraic closures of) finite fields. The foundational theory however, was developed by Gelfand, Kapranov and Zelevinsky [10][9, Chap. 14] over the complex numbers. We need the theory to hold in positive characteristic. Further, the very definition of hyperdeterminants uses geometric tools (such as tangency and projective duality) that need great care while translating to positive characteristic [16,17]. Kaji proved that the hyperdeterminant theory does translate to positive characteristic [15], with theorems such as those ensuring hyperdeterminants exists for interior/boundary formats, that their vanishing characterises degeneracy, etc. continuing to hold true. Taking for granted that other theorems regarding hyperdeterminants we invoke also hold, we have hardness results for computing hyperdeterminants modulo primes. The pursuit of hardness modulo primes will not be our focus of this work, as we merely remark in passing the modifications needed to this end. The reason for the exponential blow up in the prime modulus, is that certain polynomials derived from the hyperdeterminant have exponential degree, which forces the Schwartz-Zippel lemma to demand exponentially large fields. Nevertheless, such a prime can be written down in size polynomial in the size of the tensor. Further, in cryptographic contexts [23,4], where hardness over finite fields is useful, the prime moduli are exponential for other reasons.

1.6 Open problems and directions

The obvious question is if *hyperdeterminants are hard in three dimensions.* Even in four or more dimensions, our reductions always land eventually in boundary formats, perhaps due to a richer known theory in these formats. Thus, our hardness results actually hold if the hyperdeterminant zero testing problem is restricted to boundary formats. A natural open question is *if there is an infinite family of interior formats, restricted to which zero testing hyperdeterminants is hard.* A technical question is if our reduction mapping G to A^{4G} is parsimonious (counting the number of projective solutions to the degeneracy equation of A^{4G}). If so, hyperdeterminants are #P-hard in four dimensions.

1.7 Applications

Hyperdeterminants arise in quantum information as a measure of entanglement, when the amplitudes of quantum states are considered as normalised tensors in a projective space. Curiously, (the absolute value of) Cayley's $2 \times 2 \times 2$ hyperdeterminant was rediscovered by physicists Coffman, Kundu and Wooters as a tripartite entanglement measure of three qubits, generalising concurrence (the usual determinant) of bipartite systems [5]. Further significance of hyperdeterminants to quantum information was identified by Miyake and Wadati [21], through projective duality between separability and singularity. In particular, the hyperdeterminant is invariant under stochastic local operations and classical communication (SLOCC). It has good distinguishing power as an invariant (called in general, tangle), for instance separating the Greenberger-Horne-Zeilinger state from the W state [7]. Further, its non vanishing certifies that the state is in a generic orbit, ensuring entanglement distillation algorithms starting from that state succeed [2][Section 5], with applications to the quantum marginals and N-representability problems [18]. Our hardness result proves that it is hard to decide if a given four (or more) qudit system is tangled (that is, has non zero tangle), even for small systems where the (exponentially larger) classical description of the state is given as input. The phrase "four qudits" allows the length "d" across the four qudits to be different. An open problem is if there is quantum version of deciding tangle that is QMA-hard. Remark 4 suggests that our reduction motif may help prove hardness in increasing dimensions, when the classical description is exponential, but there is a succinct quantum state. For instance, in deciding the tangle of one qudit and d qubits, as d grows. Hyperdeterminants are also a sought after invariant in cryptanalysis, as mentioned before. Therefore their hardness is reassuring for tensor based cryptosystems (as a first step towards average case hardness, which cryptography demands).

1.8 Organisation

In section § 2, we recall the notions of degeneracy, hyperdeterminants and state the problem. In subsections § 3.1 and § 3.2, we recount the initial phase of the reduction mapping 3-Color to the degeneracy problem, due to Hillar and Lim [12], which builds on [20,19]. Part of the reason is for the exposition to be self contained. The other reason is that we do not use the first phase as a blackbox, but take notation from it for certain bounds. In subsection § 3.3, we present the reduction to four dimensional hyperdeterminants. The mapping of the reduction is in the statement of theorem 2. In subsection § 3.4, we lift the hardness from four to higher dimensions. These combine to give the main theorem 3. Many of the mathematical ingredients we need are in the book of Gelfand, Kapranov and Zelevinsky [9], all citations to which will implicitly mean to chapter 14.

2 Degeneracy of tensors and hyperdeterminants

We define degeneracy and hyperdeterminants over an arbitrary field $\mathbb{F}$ that is embedded in the field of complex numbers $\mathbb{C}$, but later restrict to the rational numbers $\mathbb{Q}$ or a number field in computational contexts. Part of the reason is to distinguish the role of $\mathbb{F}$ from its algebraic closure (when translating the results to finite fields, etc), which are identical for complex numbers. The choice of field will not be explicit in the notation, but understood with context. For positive numbers $k_1, k_2, \ldots, k_r$, an r-dimensional tensor over $\mathbb{F}$ of format

$(k_1+1)\times(k_2+2)\ldots\times(k_r+1)$ is an element

$$A \in \left(\mathbb{F}^{k_1+1}\right)^* \otimes \left(\mathbb{F}^{k_2+1}\right)^* \otimes \ldots \otimes \left(\mathbb{F}^{k_r+1}\right)^*$$

in the tensor product of dual vector spaces. We will use j exclusively to index dimensions $\{1,2,\ldots,r\}$. Fix a coordinate system $x^{(j)} = (x_0^{(j)}, x_1^{(j)}, \ldots, x_{k_j}^{(j)})$ for the j^{th}-vector space $\mathbb{F}^{k_j+1}$, or equivalently an ordered basis for the dual $\left(\mathbb{F}^{k_j+1}\right)^*$. Then, identify A with the r-dimensional matrix

$$A = \left(a_{i_1,i_2,\ldots,i_r}, 0 \le i_1 \le k_1, 0 \le i_2 \le k_2, \ldots, 0 \le i_r \le k_r\right).$$

When a tensor A is the input to algorithm or a computation problems, we mean that it is presented by writing down the r-dimensional matrix in its entirety. There is a trichotomy of tensor formats depending on the convexity constraint

$$\forall j \in \{1,2,\ldots,r\}, \qquad k_j \le \sum_{\ell \ne j} k_\ell. \tag{2.1}$$

Formats that satisfy equation 2.1, with the further assurance that there is at least one j satisfying the equation with equality are called as boundary formats. Formats that satisfy equation 2.1 that are not boundary are called as interior. We call all other formats exterior. We follow (but not demand) the common convention to place the longest dimension (that is, j with the largest k_j) first. Most of the cited results also follow this convention, with the notable exception of [10][Section 3C], where the longest dimension is placed last. Therefore care must be taken in translating cited results to our context. Associated with A is the multilinear form over the algebraic closure $\bar{\mathbb{F}} = \mathbb{C}$ (which we also denote by A)

$$\begin{aligned} A : \mathbb{C}^{k_1+1} \times \mathbb{C}^{k_2+1} \times \ldots \times \mathbb{C}^{k_r+1} &\longrightarrow \mathbb{C} \\ \left(w^{(1)}, w^{(2)}, \ldots, w^{(r)}\right) &\longmapsto \sum_{\substack{0 \le i_1 \le k_1 \\ \cdots \\ 0 \le i_r \le k_r}} a_{i_1,i_2,\ldots,i_r} w_{i_1}^{(1)} w_{i_2}^{(2)} \ldots w_{i_r}^{(r)}. \end{aligned}$$

Evaluating the multilinear form in all but one of the vectors yields a dual vector. A tensor is degenerate if there is a tuple of vectors, each non zero, such that all these evaluations $\left\{A\left(w^{(1)}, w^{(2)}, \ldots, w^{(j-1)}, *, w^{(j+1)}, \ldots, w^{(r)}\right)\right\}_j$ yield the zero dual vector, as stated below.

Definition 1. *(Degeneracy) Call the tensor A degenerate if and only if there is an r-tuple of non zero vectors $\left(w^{(1)}, w^{(2)}, \ldots, w^{(r)}\right) \in \mathbb{C}^{k_1+1} \times \mathbb{C}^{k_2+1} \times \ldots \times \mathbb{C}^{k_r+1}$ such that in every dimension $j \in \{1,2,\ldots,r\}$,*

$$\sum_{0 \le i_j \le k_j} \left(\sum_{\substack{0 \le i_1 \le k_1 \\ \cdots \\ 0 \le i_r \le k_r}} a_{i_0,i_1,\ldots,i_r} w_{i_0}^{(0)} w_{i_1}^{(1)} \ldots w_{i_{j-1}}^{(j-1)} w_{i_{j+1}}^{(j+1)} \ldots w_{i_r}^{(r)} \right) x_{i_j}^{(j)} = 0 \left(\in \left(\mathbb{C}^{k_j+1}\right)^* \right). \tag{2.2}$$

The inner summation is over all dimensions except j. That is,

$$\sum_{\substack{0 \le i_0 \le k_0 \\ \cdots \\ 0 \le i_r \le k_r}} a_{i_0,i_1,\ldots,i_r} w_{i_0}^{(0)} w_{i_1}^{(1)} \ldots w_{i_{j-1}}^{(j-1)} w_{i_{j+1}}^{(j+1)} \ldots w_{i_r}^{(r)} = 0, \ \forall 0 \le j \le r, \ 0 \le i_j \le k_j, \tag{2.3}$$

where again the summation is over all dimensions except j.

This notion of degeneracy is identical to that in [9], except that there it is (i) stated in terms of an r-tuple of projective vectors instead of non zero vectors, (ii) and stated for $\mathbb{F} = \bar{\mathbb{F}} = \mathbb{C}$.

Hyperdeterminants. Consider a format $(k_1+1)\times(k_2+1)\ldots\times(k_r+1)$ that is either interior or boundary. Introduce a set of commuting indeterminates

$$\mathfrak{a}_{i_1,i_2,\ldots,i_r}, 0\le i_1\le k_1, 0\le i_2\le k_2,\ldots,0\le i_r\le k_r,$$

corresponding to the tensor coordinates $a_{i_1,i_2,\ldots,i_r}$ of the format. The hyperdeterminant of format $(k_1+1)\times(k_2+2)\ldots\times(k_r+1)$ is a homogeneous integer polynomial

$$Det_{k_1+1,k_2+1,\ldots,k_r+1}\in\mathbb{Z}[\mathfrak{a}_{i_1,i_2,\ldots,i_r}, 0\le i_1\le k_1, 0\le i_2\le k_2,\ldots,0\le i_r\le k_r]$$

which can be viewed as an element in the coordinate ring of the tensor. The polynomial is only defined up to a sign, but the sign does not matter in the problems we address. Besides, there are natural ways to resolve the sign [27]. The choice of format on the hyperdeterminant notation is implicit in [9,10], but we write it explicitly as a subscript, for we will jump across dimensions and formats in the reduction. Denote the evaluation of the hyperdeterminant at the tensor A as

$$Det_{k_1+1,k_2+1,\ldots,k_r+1}(A)\in\mathbb{F},$$

obtained by substituting the coordinates $a_{i_1,i_2,\ldots,i_r}$ of the tensors for the indeterminates $\mathfrak{a}_{i_1,i_2,\ldots,i_r}$. If the format of A is clear from context, we may drop the subscript and denote $Det_{k_1+1,k_2+1,\ldots,k_r+1}(A)$ by $Det(A)$. A property we will repeatedly appeal to is that the tensor A is degenerate if and only if

$$Det_{k_1+1,k_2+1,\ldots,k_r+1}(A)=0.$$

Definition 2. *Fix a dimension $r\ge 3$. Define r-*Hyperdeterminant *to be the problem of deciding if $Det(A)=0$ for a given r-dimensional tensor A (of a format that has a hyperdeterminant) with entries in the rational numbers.*

The hardness results we prove for r-Hyperdeterminant remain true if the entries of the input tensor are from a number field, instead of rational numbers.

3 Reductions

We will reduce from the NP-Complete problem 3-Color, consisting of all undirected graphs $G=(V,E)$ that have a proper three colouring. A three colouring is a map $c:V\to S$ to some set S of size three. It is proper, if and only if for all edges $(u,v)\in E$, $c(u)\ne c(v)$.

3.1 Encoding graph colouring as polynomial systems

Definition 3. *For a graph $G=(V,E)$ let*

$$C_G:=\begin{cases} x_vy_v-z^2 & \forall v\in V,\\ y_vz-x_v^2 & \forall v\in V,\\ x_vz-y_v^2 & \forall v\in V,\\ \sum_{(u,v)\in E}x_u^2+x_ux_v+x_v^2 & \forall v\in V.\end{cases}$$

denote the homogeneous system of polynomial equations, with each vertex $v\in V$ indexing a pair (x_v,y_v) of variables and an extra variable z for homogenisation.

Lemma 1. *[Hillar-Lim [12]] For every field with $\mathbb{F}$ of characteristic zero with a primitive cube root of unity, the graph $G = (V, E)$ is in 3-COLOR if and only if the homogeneous system has a non-trivial solution over $\mathbb{F}$. By non-trivial, we mean not all variables are assigned zero.*

Proof. Fix a primitive cube root of unity $\omega \in \mathbb{F}$. Then, the other primitive cube root of unity is ω^2. Without loss of generality take the set of three colours to be the set $\{1, \omega, \omega^2\} \subset \mathbb{F}$ of cube roots of unity.

Let $G = (V, E)$ be in 3-COLOR with a proper three colouring $c : V \to \{1, \omega, \omega^2\}$. For every vertex $v \in V$, set $x_v \leftarrow c(v)$ and $y_v = x_v^{-1}$. Set $z \leftarrow 1$, ensuring the first three sets of equations in C_G are satisfied. Since every edge has distinctly coloured vertices, for all $(u, v) \in E$,

$$x_u \neq x_v \Rightarrow x_u^2 + x_u x_v + x_v^2 = \frac{x_u^3 - x_v^3}{x_u - x_v} = 0$$

implying

$$\sum_{(u,v)\in E} x_u^2 + x_u x_v + x_v^2 = 0, \ \forall v \in V,$$

ensuring the last set of equations are satisfied.

Conversely, consider a non-trivial solution $((x_v, y_v)_{v\in V}, z) \in \mathbb{F}^{2|V|+1}$ to C_G. If $z = 0$, then for all $v \in V$, $x_v^2 = y_v^2 = 0 \Rightarrow x_v^2 = y_v^2 = 0$, contradicting non-triviality. Therefore, $z \neq 0$, and due to the homogeneity, we may assume $z = 1$. Then, for all $v \in V$, $x_v y_v = 1 \Rightarrow y_v = x_v^{-1}$ and $y_v = x_v^2$. Therefore, for all $v \in V$, $x_v^3 = 1$. Assign the vertex $v \in V$ the colour x_v, which is valid since x_v is a cube root of unity. All that remains is to prove that it is a proper colouring. For an edge $(u, v) \in E$, if the incident vertices are coloured differently, then $x_u \neq x_v \Rightarrow x_u^2 + x_u x_v + x_v^2 = 0$, as before. But if for an an edge $(u, v) \in E$ the incident vertices are coloured the same, $x_u = x_v \Rightarrow x_u^2 + x_u x_v + x_v^2 = 3x_v^2$. Therefore, for all vertices $v \in V$,

$$\sum_{(u,v)\in E} x_u^2 + x_u x_v + x_v^2 = 3s_v x_v^2 = 0$$

where s_v counts the number of vertices adjacent to v that are coloured the same as v. But since x_V^2 is a root of unity and $3 \neq 0$ in a field of characteristic zero, s_v has to be zero, ensuring the colouring is proper. □

Remark 1. Lemma 1 works in characteristic p, if $3s_v \neq 0$. This is true, if $p \neq 3$ and $p \geq |V|$. We can do better, and reduce from 3-COLOR restricted to graphs of degree at most 4. Despite the degree constraint, this problem remains NP-complete and the condition $p \neq 3$ and $p \geq |V|$ can be relaxed to $p \geq 5$. A finite field $\mathbb{F}_q$ of size q has a primitive cube root of unity if and only if $q = 1 \bmod 3$. For $q \neq 1 \bmod 3$, we can extend the field by adjoining a root of $x^2 + x + 1$.

3.2 Encoding the graph colouring polynomial system as degeneracy

Hillar and Lim embed the polynomial system C_G associated with a graph $G = (V, E)$ (of $n = |V|$ vertices) into a three dimensional tensor A^{3G} as follows [2]. The second and third dimensions both index the variables, while the first dimension indexes constraints.

[2] Superscripts such as $3G$ on the tensors A^{3G} are merely labels indicating dependence on the graph G and dimension 3, and do not denote any operation on the tensor.

Bilinear to homogeneous quadratic forms: Denote the solution vectors (of the degeneracy system of the tensor A^{3G} we will soon write down) in the second and third dimensions as $w^{(2)}$ and $w^{(3)}$ respectively. Evaluating the tensor in the second and third dimensions yields a bilinear form in $w^{(2)}$ and $w^{(3)}$. For these bilinear forms to capture the quadratic constraints in C_G, it suffices (i) to identify the coordinates of $w^{(2)}$ with the variables $((x_v)_{v\in V}, (y_v)_{v\in V}, z)$ and (ii) to constrain the solutions $w^{(2)}$ and $w^{(3)}$ to be projectively equivalent. To this end, the first set of constraints are that the 2×2 minors of the matrix $(2n+1) \times 2$ obtained by placing $w^{(2)}$ and $w^{(3)}$ next to each other vanishes[3]. Explicitly, these constraints are

$$w^{(2)}_\ell w^{(3)}_m - w^{(2)}_m w^{(3)}_\ell = 0, \quad \forall \ell, m \in \{0, 1, \ldots, 2n\} \text{ such that } \ell \neq m.$$

The constraint $w^{(2)}_\ell w^{(3)}_m - w^{(2)}_m w^{(3)}_\ell = 0$ for $\ell = m$ is satisfied trivially and does not need to be enforced. Say two non zero vectors $w^{(2)}$ and $w^{(3)}$ are projectively equivalent, that is, there is a non zero constant $c \in \mathbb{C}$ such that $w^{(2)} = cw^{(3)}$. Then clearly, these constraints are satisfied. The converse is also true. Say, two non zero vectors $w^{(2)}$ and $w^{(3)}$ satisfy these constraints. Hence, there exists an index m such that $w^{(3)}_m \neq 0$, which implies

$$w^{(2)}_\ell w^{(3)}_m = w^{(2)}_m w^{(3)}_\ell \Rightarrow w^{(2)}_\ell = \frac{w^{(2)}_m}{w^{(3)}_m} w^{(3)}_\ell, \ \ \forall \ell \in \{0, 1, \ldots, 2n\} \Rightarrow w^{(2)} = \frac{w^{(2)}_m}{w^{(3)}_m} x^{(3)},$$

proving the converse. To encode these $n(2n+1)$ constraints as tensor coordinates, fix an enumeration $\iota : \{(\ell, m) \mid \ell \neq m\} \longrightarrow \{0, 1, \ldots, n(2n+1) - 1\}$ of the constraints, and set the matrix $A_{\iota(\ell,m)} \in \{-1, 0, 1\}^{(2n+1)\times(2n+1)}$ such that

$$(w^{(2)})^t A_{\iota(\ell,m)} w^{(3)} = w^{(2)}_\ell w^{(3)}_m - w^{(2)}_m w^{(3)}_\ell$$

as the $\iota(\ell, m)$-th slice of the tensor A^{3G} in the first dimension. That is,

$$A^{3G}_{\iota(\ell,m),i_2,i_3} := A_{\iota(\ell,m)}(i_2, i_3), 0 \leq \iota(\ell, m) \leq n(2n+1) - 1, 0 \leq i_2 \leq 2n, 0 \leq i_3 \leq 2n.$$

Encoding the C_G system: Likewise, fix an enumeration of the $4n$ constraints in C_G, to index the first dimension running from $n(2n+1)$ to $n(2n+4)-1$. Let $A_{i_1} \in \{-1, 0, 1\}^{(2n+1)\times(2n+1)}$ be the matrix such that $(w^{(2)})^t A_{i_1} w^{(3)}$ is the i_1-th constraint and set i_1)-th slice of the tensor A in the first dimension. That is,

$$A^{3G}_{i_1,i_2,i_3} := A_{i_1}(i_2, i_3), n(2n+1) \leq i_1 \leq n(2n+5) - 1, 0 \leq i_2 \leq 2n, 0 \leq i_3 \leq 2n.$$

Thus, A^{3G} is of $n(2n+5) \times (2n+1) \times (2n+1)$ format with coordinates in $\{-1, 0, 1\}$.

Lemma 2. *[Hillar-Lim [12]] G is in 3-Color if and only if A^{3G} is degenerate.*

Proof. By lemma 1 and the constraints translating bilinear constraints to quadratic forms, G is in 3-Color if and only if there exists non zero vectors $w^{(2)}$ and $w^{(3)}$ such that $A^{3G}(*, w^{(2)}, w^{(3)}) = 0$. It remains to show that there is a non zero $w^{(1)} \in \mathbb{C}^{n(2n+5)}$ compatible with the choice of $w^{(2)}, w^{(3)}$, that is, satisfying $A^{3G}(w^{(1)}, w^{(2)}, *) = 0$ and $A^{3G}(w^{(1)}, *, w^{(3)}) = 0$. But for fixed $w^{(2)}, w^{(3)}$, these constitute homogeneous linear constraints, $2(2n+1)$ in number. Since $n(2n+5) > 2(2n+1)$, there exists such a non zero vector $w^{(1)}$. □

[3] These quadratically many constraints to enforce the projective equivalence of the solution vectors across two dimensions are the culprits in inflating the length in one dimension, pushing to external formats. A succinct enforcement of equivalence may help prove hardness in three dimensions

3.3 Lifting to four dimensional boundary formats

Theorem 2. *Let $G = (V, E)$ be a graph with $n = |V|$ vertices. Construct a $n(2n+5) \times (2n+1) \times (2n+1) \times n(2n+1)$ boundary format tensor A^{4G} as follows. Set the zeroth slice of A^{4G} in the fourth dimension to be A^{3G}, that is*

$$i_4 = 0 \;\Rightarrow\; a^{4G}_{i_1,i_2,i_3,i_4} := a^{3G}_{i_1,i_2,i_3}.$$

Draw the remaining coordinates, that is $a^{4G}_{i_1,i_2,i_3,i_4}$ with $i_4 > 0$, independently and uniformly from $\left\{1, 2, \ldots, \frac{2((n(2n+5))!)}{(2n+1)!(2n+1)!(n(2n+1))!}\right\} \subset \mathbb{Z}$.

- *If G is in 3-Color, then*

$$Det_{n(2n+5),2n+1,2n+1,n(2n+1)}(A^{4G}) = 0.$$

- *If G is not in 3-Color, then*

$$\Pr\left[Det_{n(2n+5),2n+1,2n+1,n(2n+1)}(A^{4G}) \neq 0\right] \geq \frac{1}{2}.$$

Therefore, 3-Color $\leq_p$ 4-Hyperdeterminant under randomised reductions.

Proof. Say G is in 3-Color. Then, by lemma 2, A^{3G} is degenerate. This does NOT imply $Det_{n(2n+5),2n+1,2n+1}(A^G) = 0$, since $n(2n+5)\times(2n+1)\times(2n+1)$ is an exterior format and the hyperdeterminant $Det_{n(2n+5),2n+1,2n+1}$ does not exist. But the enveloping four dimensional format $n(2n+5) \times (2n+1) \times (2n+1) \times n(2n+1)$ is a boundary format, which has an associated hyperdeterminant. By [9][Corollary 3.11],

$$Det_{n(2n+5),2n+1,2n+1,n(2n+9)}(A^{4G}) = 0,$$

since A^{4G} contains the exterior format A^{3G} as a slice in the fourth dimension.

Say G is not in 3-Color. Recall the four dimensional boundary format hyperdeterminant

$$Det_{n(2n+5),2n+1,2n+1,n(2n+9)} \in \mathbb{Z}[\mathfrak{a}_{i_1,i_2,i_3,i_4}, i_1, i_2, i_3, i_4],$$

considered as a polynomial in the coordinate indeterminates. Partially evaluate it on the zeroth slice of A^{4G} in the fourth dimension, by substituting the coordinates of the tensor

$$\mathfrak{a}_{i_1,i_2,i_3,i_4} \leftarrow A^{4G}_{i_1,i_2,i_3,i_4}, \forall i_4 = 0,$$

and (by mild abuse of notation) denote the resulting polynomial as

$$Det(A^{3G}, *) \in \mathbb{Z}[\mathfrak{a}_{i_1,i_2,i_3,i_4}, i_1, i_2, i_3, i_4 > 0].$$

As the partial evaluation of an integer coefficient polynomial at integers, $Det(A^{3G}, *)$ is a polynomial with integer coefficients. We claim that $Det(A^{3G}, *)$ is a non zero polynomial. Assume otherwise, that $Det(A^{3G}, *)$ is identically zero. Then, for every $n(2n+5) \times (2n+1) \times (2n+1) \times n(2n+1)$ boundary format tensor B that contains A^{3G} as the zeroth slice in the fourth dimension, $Det_{n(2n+5),2n+1,2n+1,n(2n+1)}(B)$ is obtained by substituting in $Det(A^{3G}, *)$ the coordinates of B that are not on the zeroth slice in the fourth dimension. If $Det(A^{3G}, *)$ were identically zero, then

$$Det_{n(2n+5),2n+1,2n+1,n(2n+1)}(B) = 0.$$

Further more is true, $Det_{n(2n+5),2n+1,2n+1,n(2n+1)}(B) = 0$ for all $n(2n+5) \times (2n+1) \times (2n+1) \times n(2n+1)$ boundary format tensor B that contains A^{3G} as a slice in the fourth dimension, irrespective of if the slice is in the zeroth position. This is true since permuting parallel slices preserves the hyperdeterminant, up to sign [9][Corollary 1.5a]. But then, by [9][Corollary 3.11], A^{3G} is degenerate. By lemma 2, A^{3G} being degenerate contradicts that G is not in 3-COLOR. Therefore, our assumption is wrong, and $Det(A^{3G}, *)$ is a non zero polynomial, as claimed. We wish to bound the probability of vanishing of the hyperdeterminant $Det_{n(2n+5),2n+1,2n+1,n(2n+1)}(A^{4G})$, which is obtained by substituting in the non zero polynomial $Det_{A^{3G}}$, the coordinates of A^{4G} that are not on the zeroth slice in the fourth dimension. Therefore, we may view $Det_{n(2n+5),2n+1,2n+1,n(2n+1)}(A^{4G})$ as the evaluation of the of the non zero polynomial $Det(A^{3G}, *)$ at points drawn independently and uniformly from the set $\left\{1, 2, \ldots, \frac{2((n(2n+5))!)}{(2n+1)!(2n+1)!(n(2n+1))!}\right\}$ of integers. To bound the probability of non vanishing, we invoke the Schwartz-Zippel lemma over the integers. To apply the Schwartz-Zippel lemma, we only need a bound on the degree of $Det(A^{3G}, *)$, since the bound is apathetic to the size of the coefficients. Since substitutions do not increase degree,

$$\deg\left(Det(A^{3G}, *)\right) \leq \deg\left(Det_{n(2n+5),2n+1,2n+1,n(2n+1)}\right).$$

Applying the degree bound for boundary formats [9][Corollary 2.6],

$$\deg\left(Det(A^{3G}, *)\right) \leq \frac{(n(2n+5))!}{(2n+1)!(2n+1)!(n(2n+1))!}.$$

Since the sample domain we draw each coordinate from is at least twice $\deg\left(Det(A^{3G}, *)\right)$, the Schwartz-Zippel lemma implies

$$\Pr\left[Det_{n(2n+5),2n+1,2n+1,n(2n+9)}(A^{4G}) \neq 0\right] \geq \frac{1}{2}.$$

All that remains is to show that the reduction is indeed polynomial time. Since the number of coordinates (roughly $16n^6$) of A^{4G} and the number

$$\log_2\left(\frac{2((n(2n+5))!)}{(2n+1)!(2n+1)!(n(2n+1))!}\right) = O(n \log n)$$

of bits to write down each coordinate are both polynomial in n, we can write down A^{4G} in its entirety in polynomial time. Therefore, the mapping in the statement of theorem 2 is a polynomial time reduction. □

Remark 2. For theorem 2 to be valid over a finite field $\mathbb{F}_q$, the finite field must have at least

$$q > \frac{2((n(2n+5))!)}{(2n+1)!(2n+1)!(n(2n+1))!}$$

elements. Otherwise, the Schwartz-Zippel bound does not suffice. If all the properties of the hyperdeterminants we invoked in the proof of theorem 2 hold in positive characteristic, then in four dimensions, testing if the hyperdeterminant of a given tensor is zero modulo a given prime that is exponentially large (in the smallest dimension length) is NP-hard under randomised reductions.

3.4 Lifting to higher dimensions

We present a deterministic reduction to lift hardness to higher dimensions, that leverages the multiplicativity of boundary format determinants. The reduction needs two new ingredients, (i) the notion of tensor convolution (which generalises matrix multiplication) and (ii) structured boundary format tensors that by design have non vanishing hyperdeterimant.

Definition 4. *(Tensor convolution.) Let A be a $(k_1+1)\times(k_2+1)\times\ldots\times(k_r+1)$ boundary format tensor with $k_1=k_2+k_3+\ldots+k_r$. Let B be $(\ell_1+1)\times(\ell_2+1)\times\ldots\times(\ell_s+1)$ boundary format tensor with $\ell_1=\ell_2+\ell_3+\ldots+\ell_s$. The convolution $A\star_j B$ with respect to a dimension j such that $k_j=\ell_1$ is the $(k_1+1)\times(k_2+1)\times\ldots\times(k_{j-1}+1)\times(k_{j+1}+1)\ldots\times(k_r+1)\times(\ell_2+1)\times(\ell_3+1)\times\ldots\times(\ell_s+1)$ format $r+s-2$-dimensional tensor with entries*

$$\sum_{i_j=0}^{k_j}\sum_{i'_1=0}^{\ell_1} a_{i_1,i_2,\ldots,i_j,\ldots,i_r} b_{i'_1,i'_2,\ldots,i'_s},$$
$$0\le i_1\le k_1, 0\le i_2\le k_2,\ldots,0\le i_{j_1-1}\le k_{j_1-1}, 0\le i_{j_1+1}\le k_{j_1+1},\ldots,0\le i_r\le k_r,$$
$$0\le i'_2\le \ell_2, 0\le i'_3\le\ell_3,\ldots,0\le i'_s\le\ell_s.$$

The resulting convolution is again of boundary format with the first dimension being the longest, $k_1=k_2+k_3+\ldots+k_{j-1}+k_{j+1}+\ldots+k_r+\ell_2+\ell_3+\ldots+\ell_s$. The definition of tensor convolution is identical to that of [9], except we have specialised it to boundary formats to ease the cumbursome notation.

Definition 5. *(Identity tensor) For a boundary format $(k_1+1)\times(k_2+1)\times\ldots\times(k_r+1)$ with $k_1=k_2+k_3+\ldots+k_r$, define $D^{k_1+1,k_2+1,\ldots,k_r+1}$ to be the tensor with entries*

$$d^{k_1+1,k_2+1,\ldots,k_r+1}_{i_1,i_2,\ldots,i_r} := \begin{cases}1, & i_1=i_2+i_3+\ldots+i_r\\ 0, & i_1\ne i_2+i_3+\ldots+i_r.\end{cases}$$

Lemma 3. *Fix dimensions r,s with $r<s$. Restricted to boundary formats, r-Hyperdeterminant $\le_p$ s-Hyperdeterminant.*

Proof. Let A^r be an r-dimensional $(k_1+1)\times(k_2+1)\ldots\times(k_r+1)$ boundary format tensor, whose degeneracy is in question. Without loss of generality, let $k_1=k_2+k_3+\ldots+k_r$. We may assume that there is a $j\ne 1$ such that $s-r+2\le k_j$, for otherwise, A^r is of a format of constant size and can be ignored in the reduction. Convolve A^r and the $(s-r+2)$-dimensional diagonal tensor of

$$(k_j+1)\times(k_j-s+r+1)\times\underbrace{2\times2\times\ldots\times2}_{s-r},$$

boundary format with respect to the j-th dimension, resulting in the s dimensional tensor

$$A^s := A^r\star_j D^{k_j+1,k_j-s+r+1,2,2,\ldots,2}.$$

By the multiplicativity of boundary format hyperdeterminants in Dionisi and Ottaviani [6],

$$Det(A^s)=Det(A^r)^e Det\left(D^{k_j+1,k_j-s+r+1,2,2,\ldots,2}\right)^{e'},$$

where the exponents e,e' are certain easy to compute multinomial coefficients in the lengths of the format. We only need the fact that they are both positive integers. For brevity of notation, we have suppressed the subscripts on the hyperdeterminants signifying the format, but the format should be clear from the tensors in the argument. The identity tensor has hyperdeteminant plus or minus one [9][Lemma 3.4], implying

$$Det(A^s) = \pm Det(A^r)^e.$$

In particular, $Det(A^s)$ is zero if and only if $Det(A^r)$ is zero, proving that $A \longmapsto A^s$ is a deterministic polynomial time reduction. □

In fact, the reduction is stronger. Given $Det(A^s)$, one can efficiently compute $Det(A^r)$ up to an e-th root of unity, which may help lift VNP or #P hardness to high dimensions.

Remark 3. The role of the identity tensor in the reduction in lemma 3 can be recast with other explicit constructions of boundary format tensors that are guaranteed to have non vanishing hyperdeterminant. Two options, both due to Weyman and Zelevinsky [27], are diagonal tensors with non zero entries on the diagonal and Vandermonde-Weyman-Zelevinsky tensors. For Vandermonde-Weyman-Zelevinsky tensors, the field has to have at least as many elements as $k_j + 1$ to make sure the vectors that define the tensor can be made out of distinct field elements. They have the advantage of a proof of non vanishing that is elementary, irrespective of the characteristic of the field [22].

Theorem 3. *For a fixed dimension $r \geq 4$, r-*Hyperdeterminant *is NP-hard under randomised reductions.*

Proof. Compose the reduction in theorem 2 with that of lemma 3. □

The idea underlying the reduction in lemma 3 in lifting hardness to higher dimensions s applies to a broad spectrum of formats. For instance, we can prove hardness when the dimension s grows with the problem size. Such a reduction may not be polynomial time, since the tensor A^s mapped to, is too big to write down. However, such reductions may be interesting for quantum information, since such large exponentially large tensors can be encoded efficiently as quantum states.

Remark 4. Consider the $n(2n+5) \times (2n+1) \times (2n+1) \times n(2n+1)$ boundary format tensor A^{4G} that a graph with n vertices gets mapped to in theorem 2. Successively convolve it with three diagonal matrices to get

$$A^{n(2n+9)G} := \left(\left(\left(A^{4G} \star D^{(n(2n+5),2,2,\ldots,2)} \right) \star D^{2n+1,2,2,\ldots,2} \right) \star D^{2n+1,2,2,\ldots,2} \right).$$

Here, we have suppressed the subscripts on the convolution operator $\star$, but it is understood that the convolution happens at a dimension of appropriate length. The resulting tensor $A^{n(2n+5)G}$ is of boundary format

$$n(2n+5) \times \underbrace{2 \times 2 \times \ldots \times 2}_{n(2n+5)-1}.$$

Successively invoking the multiplicativity of boundary format hyperdeterminants [6],

$$Det(A^{n(2n+5)G}) = \pm Det(A^{4G})^e,$$

for some positive integer exponent e. Therefore,

$$Det(A^{n(2n+5)G}) = 0 \Leftrightarrow Det(A^{4G}) = 0.$$

embedding 3-Color as an instance of $n(2n+5)$-Hyperdeterminant, restricted to $n(2n+5)\times 2\times 2\times \ldots \times 2$ boundary formats. The tensor $A^{n(2n+5)G}$ can be encoded in the amplitudes of a pure state of a one qudit, d qubit system, with $d = n(2n+5)-1$ (after projectivising and normalising). The hyperdeterminant becomes a measure of multipartite entanglement of such systems, called as tangle. Assuming this embedding is easy to compute on quantum computers, deciding if a one qudit, d qubit system is tangled is NP-hard under randomised reductions. Such quantum information applications will be expanded on in a longer version.

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