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The Log-Rank Conjecture: New Equivalent Formulations

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Abstract

The log-rank conjecture is a longstanding open problem with multiple equivalent formulations in complexity theory and mathematics. In its linear-algebraic form, it asserts that the rank and partitioning number of a Boolean matrix are quasi-polynomially related.

We propose a relaxed but still equivalent version of the conjecture based on a new matrix parameter, signed rectangle rank: the minimum number of all-1 rectangles needed to express the Boolean matrix as a ± 1-sum. Signed rectangle rank lies between rank and partition number, and our main result shows that it is in fact equivalent to rank up to a logarithmic factor. Additionally, we extend the main result to tensors. This reframes the log-rank conjecture as: can every signed decomposition of a Boolean matrix be made positive with only quasi-polynomial blowup?

As an application, we prove an equivalence between the log-rank conjecture and a conjecture of Lovett and Singer–Sudan on cross-intersecting set systems.

1 Introduction

The log-rank conjecture is one of the most well-known problems in complexity theory that despite extensive work it remains unsolved. It asserts that for a Boolean matrix its communication complexity and the logarithm of its matrix rank over the reals are polynomially related. An equivalent linear-algebraic formulation of the conjecture is that for Boolean matrices, the matrix rank and partitioning number (sometimes called the binary rank) are quasi-polynomially related.

More formally, for a Boolean matrix M let $\mathrm{r}(M)$ denote its rank over the reals, and let $\mathrm{p}(M)$ denote its partitioning number — the minimum number of all-1 submatrices that partition the 1-entries of M. Equivalently, the partitioning number is the minimum number p such that $M = \sum_{i=1}^{p} R_i$, where each R_i is a *primitive* matrix – an all-1 submatrix, possibly after adding all-zero rows and columns. If we relax the decomposition to allow general rank-1 matrices instead of primitive ones, we recover the standard matrix rank. That is, $\mathrm{r}(M)$ is the minimum number r such that $M = \sum_{i=1}^{r} M_i$, with each M_i of rank-1. In this sense, matrix rank can be seen as a relaxation of the partitioning number, and trivially $\mathrm{r}(M) \le \mathrm{p}(M)$. The log-rank conjecture asks how well this relaxation estimates the partitioning number:

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Conjecture 1.1 (Log-rank conjecture [LS88]). *For any Boolean matrix M,*

$$\log \mathrm{p}(M) \leq (\log \mathrm{r}(M))^{O(1)}.$$

The log-rank conjecture was first posed by Lovász and Saks [LS88] in the context of communication complexity: is the deterministic communication complexity of a Boolean matrix polynomially related to the logarithm of its rank over the reals? A closely related question had appeared even earlier in graph theory. There, the log-rank conjecture is equivalent to asking whether the logarithm of a graph's chromatic number is polynomially related to the rank of its adjacency matrix [vN76, Faj88, LS88].

Regarding the state of the art, the best known upper bound is due to Sudakov and Tomon [ST24], improving on previous work of Lovett [Lov16], and shows that

$$\log \mathrm{p}(M) \leq O\left(\sqrt{\mathrm{r}(M)}\right).$$

The largest known separation is due to Göös, Pitassi, and Watson [GPW15], who constructed a matrix M satisfying $\log \mathrm{p}(M) \geq \Omega(\log^2 \mathrm{r}(M))$. For a more detailed overview of the log-rank conjecture and its equivalent formulations, we refer the reader to the survey by Lee and Shraibman [LS23], as well as the textbooks by Jukna [Juk12] and Rao and Yehudayoff [RY20].

In this paper, we consider a "gradual relaxation" from partitioning number to rank via an intermediate complexity measure: the *signed rectangle rank*. This notion allows decomposition of the matrix into *signed* primitive matrices. Formally, let the *signed rectangle rank* of M, denoted by $\mathrm{srr}(A)$, be the minimum number t such that $M = \sum_{i=1}^{t} \varepsilon_i R_i$, where $\varepsilon_i \in \{1, -1\}$ and each R_i is a primitive matrix. Trivially, we have

$$\mathrm{r}(M) \leq \mathrm{srr}(M) \leq \mathrm{p}(M).$$

A promising approach to the log-rank conjecture is to study how "close" is signed rectangle rank to rank and partitioning number. To resolve the conjecture, it would suffice to either prove both of the following statements or disprove one of them:

- $\mathrm{srr}(M)$ is quasi-polynomially related to $\mathrm{r}(M)$, and
- $\mathrm{srr}(M)$ is quasi-polynomially relate to $\mathrm{p}(M)$.

The main result of the paper is that rank and signed rectangle rank are tightly related.

Theorem 1.2 (Main Theorem). *Every Boolean matrix M of rank r can be written as a ± 1-linear-combination of at most $O(r \log r)$ primitive matrices, that is,*

$$\mathrm{srr}(M) \leq O(r \log r).$$

From this we immediately get an equivalent formulation of the log-rank conjecture:

Conjecture 1.3. *For every Boolean matrix M, $\log \mathrm{p}(M) \leq (\log \mathrm{srr}(M))^{O(1)}$.*

Corollary 1.4. *The log-rank conjecture (Conjecture 1.1) is equivalent to Conjecture 1.3.*

Informally, this means that proving the log-rank conjecture reduces to converting a signed decomposition into a positive one, with at most a quasi-polynomial increase in the number of primitive matrices. On the other hand, to disprove the conjecture, it suffices to find a Boolean matrix for which the partitioning number significantly exceeds the signed rectangle rank.

In addition to the matrix case we extend Theorem 1.2 to Boolean tensors, showing analogous relation between the tensor rank and signed rectangle rank of a tensor.

Finally, a natural question left open here is whether the bound in Theorem 1.2 can be improved, or if it is already tight:

Question 1.5. *Is it true that for every Boolean matrix M, $\mathrm{srr}(M) \le O(\mathrm{r}(M))$?*

Equivalent conjecture on cross-intersecting set systems. Let $\mathcal{S}, \mathcal{T} \subseteq 2^{[d]}$ be two set families. We say the pair $(\mathcal{S}, \mathcal{T})$ is *L-cross-intersecting* for some $L \subseteq \{0, \ldots, d\}$ if for all $S \in \mathcal{S}$ and $T \in \mathcal{T}$, we have $|S \cap T| \in L$; that is, the size of every pairwise intersection belongs to L. Cross-intersecting set families have been widely studied in combinatorics, with much of the work focusing on their extremal properties [FR87, Sga99, KS05, HMST24].

The following conjecture about $\{a, b\}$-cross-intersecting set systems was independently proposed by Lovett [Lov21] and Singer–Sudan [SS22], who both observed that it is implied by the log-rank conjecture.

Conjecture 1.6. *Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ and $\mathcal{T} = \{T_1, \ldots, T_n\}$ be an $\{a, b\}$-cross-intersecting pair of families from $2^{[d]}$, where $a, b \in \{0, \ldots, d\}$. Then there exist subfamilies $\mathcal{A} \subseteq \mathcal{S}$ and $\mathcal{B} \subseteq \mathcal{T}$ such that $(\mathcal{A}, \mathcal{B})$ is either $\{a\}$- or $\{b\}$-cross-intersecting, and*

$$|\mathcal{A}|, |\mathcal{B}| \ge 2^{-\operatorname{polylog}(d)} \cdot |\mathcal{S}||\mathcal{T}|.$$

As an application of Theorem 1.2, we show that the log-rank conjecture is equivalent to this conjecture[1].

Theorem 1.7. *The log-rank conjecture (Conjecture 1.1) is equivalent to Conjecture 1.6.*

Organization. We give the proof of Theorem 1.2 in Section 2, then prove its extension to tensors in Section 3. Finally, in Section 4, we prove the equivalence of log-rank conjecture to Conjecture 1.6.

2 Proof of the Main Theorem

Let $M = (m_{i,j})$ be a $m \times n$ Boolean matrix of rank r. For a subset S of the columns, define its *column-sum* to be the column vector c such that for $i \in [m]$, $c_i = \sum_{j \in S} m_{i,j}$. That is, c is the entrywise sum of the all the columns of S along all the rows. Denote the column-sum of S as $\mathrm{sum}(S)$. Call a subset S of columns *independent* if all the subsets of S have distinct column-sums.

Claim 2.1. *Let S be an independent set of columns of M. Then $|S| \le O(r \log r)$, where r is the rank of M.*

[1] This equivalence was claimed in [SS22] as a parenthetical remark. However, this was a typo. The remark was meant to claim the implication from the log-rank conjecture [SS25].

Proof. Let A_S be the matrix whose columns are the column-sums of S, so A_S has $2^{|S|}$ columns. The entries of A_S are in $\{0, \ldots, |S|\}$ as they are sums of at most $|S|$ entries of M, which take value $\{0, 1\}$. The rank of A_S is at most r because its columns are in the span of the columns of M, which itself has rank r. Thus, there is a set of rows R with $|R| = r$ such that, for any column c of A_S, the values $(c_i)_{i\in R}$ are sufficient to determine every entry of c. Therefore, there are at most $(|S| + 1)^r$ unique columns of A_S, and since every column of A_S is unique, we have that the subsets of S have at most $(|S| + 1)^r$ column-sums. Therefore, $2^{|S|} \leq (|S| + 1)^r$, which implies $|S| \leq O(r \log r)$. ☐

Claim 2.2. *Let S be a maximal independent set of columns of M. Then every column of M can be expressed as a ± 1-linear combination of columns in S.*

Proof. This trivially holds for the columns in S. Fix a column $c \notin S$. Since S is a maximal independent set, $S \cup \{c\}$ is not independent. This means that there are two subsets A and B of $S \cup \{c\}$ with the same column-sum. We can assume that these subsets are disjoint; if they are not, then removing the common columns still results into equal column-sums. Note A and B cannot both exclude c, as this would contradict S being independent. Assume $c \in B$, and let $B' = B \setminus \{c\}$. Combining this with $\mathrm{sum}(A) = \mathrm{sum}(B)$, we get $\mathrm{sum}(A) = \mathrm{sum}(B') + c$. Noting that $A \cap B' = \emptyset$ and $A, B' \subseteq S$ concludes that $c = \mathrm{sum}(A) - \mathrm{sum}(B')$ is the desired linear combination. ☐

Combining these two claims concludes the proof of Theorem 1.2 as follows. Let S be the largest independent set of columns of M. By Claim 2.2 every column y can be expressed as a ± 1-linear combination of columns in S. Thus, M can be written as

$$M(x, y) = \sum_{c\in S} \alpha_c(y)c(x),$$

where each coefficient $\alpha_c(y) \in \{-1, 0, 1\}$. Decompose $\alpha_c(y) = \alpha_c^+(y) - \alpha_c^-(y)$, where $\alpha_c^+(y), \alpha_c^-(y) \in \{0, 1\}$. For each column $c \in S$, define,

$$R_c^+(x, y) = \alpha_c^+(y)c(x) \qquad \text{and} \qquad R_c^-(x, y) = \alpha_c^-(y)c(x)$$

Each of R_c^+ and R_c^- is either an all-zeroes matrix or a primitive matrix, since they are outer products of $\{0, 1\}$-valued vectors. Hence, $M = \sum_{c\in S}(R_c^+ - R_c^-)$, which expresses M as ± 1-sum of at most $2|S|$ primitive matrices. Finally, applying the bound on $|S|$ from Claim 2.1, we conclude that $\mathrm{srr}(M) \leq O(r \log r)$.

3 Generalizing to tensors

Theorem 1.2 can be generalized to hold for the tensor rank of constant-order Boolean tensors. Let T be an order-ℓ Boolean tensor: a multilinear map $[n_1] \times \ldots \times [n_\ell] \to \{0, 1\}$ (for some natural numbers $n_1, \ldots, n_\ell$). In other words, T can be expressed as an ℓ-dimensional array whose entries are all in $\{0, 1\}$.

We say T has tensor rank 1 if there are maps $v_i : [n_i] \to \mathbb{R}$ for $i \in [\ell]$ such that

$$T(x_1, \ldots, x_\ell) = v_1(x_1) \cdot \ldots \cdot v_\ell(x_\ell).$$

The *tensor rank* of a tensor is the minimum number r such that the tensor can be expressed as the sum of r rank-1 tensors.

A *primitive tensor* is the natural generalization of a primitive matrix: it is a rank-1 tensor where v_i's map to $\{0,1\}$. That is, a primitive tensor can be expressed as a multidimensional array which is all-1 on some product set $Q_1 \times \ldots \times Q_\ell$ for $Q_i \subseteq [n_i]$ and is all-0 elsewhere. We prove the following theorem, which generalizes Theorem 1.2 to tensors.

Theorem 3.1. *Let $T : [n_1] \times \ldots \times [n_\ell] \to \{0,1\}$ be an order-ℓ Boolean tensor with $\ell \geq 2$. If the tensor rank of T is r, then T can be written as a ± 1-linear-combination of at most $(cr \log r)^{(\ell-1)}$ primitive tensors, where c is an absolute constant.*

A *slice* of an order-ℓ tensor is the order-$(\ell-1)$ tensor obtained by taking a coordinate and setting to a fixed value. For $\lambda \in [\ell]$, a λ-*slice* is a slice where we specify that λ is the coordinate to be fixed. Similar to the proof of Theorem 1.2, we define a slice-sum to be the entrywise sum of slices, and an independent set of slices is one where all of its subsets have distinct slice-sums.

Claim 3.2. *Let S be an independent set of λ-slices of T. Then $|S| \leq O(r \log r)$, where r is the tensor rank of T.*

Proof. Let $m = \prod_{i \neq \lambda} n_i$. Consider the flattening of T to a matrix $M_T \in \{0,1\}^{m \times n_\lambda}$. Then M_T has rank at most r and, following the bijection between λ-slices of T and columns of M_T, there is an independent set of columns S' in M_T with $|S'| = |S|$. Apply Claim 2.1 to obtain $|S'| \leq O(r \log r)$. □

Claim 3.3. *Let S be a maximal independent set of λ-slices of T. Then every λ-slice of T can be expressed as a ± 1-linear combination of λ-slices in S.*

We omit the proof of Claim 3.3 as it is analogous to Claim 2.2.

Proof of Theorem 3.1. The proof is by induction. The base case is $\ell = 2$, which is proven by Theorem 1.2. For $\ell > 2$, arbitrarily choose a coordinate λ and use Claim 3.3 to express T as a ± 1-linear combination of a maximal independent set of λ-slices S. Each λ-slice of S has tensor rank at most r, and so by the inductive hypothesis, each λ-slice in S can be expressed as a ± 1-linear combination of $(cr \log r)^{(\ell-2)}$ order-$(\ell-1)$ primitive tensors.

In a similar fashion to the proof of Theorem 1.2, we therefore can write T as a ± 1-sum of $|S| \cdot (cr \log r)^{(\ell-2)}$ order-ℓ primitive tensors. Conclude by substituting $|S| = O(r \log r)$ using Claim 3.2. □

Remark 3.4. We note that the proof of Theorem 3.1 only uses the tensor rank as an upper bound for the *flattening rank* of a tensor, which is the maximal rank of any flattening of it as an $n_\lambda \times \left(\prod_{i \neq \lambda} n_i\right)$ matrix over $\lambda \in [\ell]$. If this flattening rank is r, then the conclusion of Theorem 3.1 still holds.

Moreover, in this formulation, the quantitative bound of $O((r \log r)^{\ell-1})$ primitive tensors is near optimal. Indeed, consider all tensors $T : [r]^\ell \to \{0,1\}$. Any such tensor has flattening rank $\leq r$. On the other hand, a simple counting argument shows that most such tensors need $\Omega(r^{\ell-1}/\ell)$ primitive tensors in their decomposition.

If we return however to the original formulation of Theorem 3.1 using tensor rank, it is unclear if the bound is close to tight. In fact, we suspect that it can be significantly improved.

Question 3.5. *Can the bound in Theorem 3.1 be improved to $o(r^{\ell-1})$?*

4 Equivalence to the cross-intersecting set systems conjecture

In order to prove Theorem 1.7, we will use the equivalent version of the log-rank conjecture by Nisan and Wigderson [NW95]. A submatrix of a matrix is called a *monochromatic rectangle* if all of its entries have the same value.

Conjecture 4.1 ([NW95]). *Any Boolean matrix M has a monochromatic rectangle of density $2^{-\operatorname{polylog}(\mathrm{r}(M))}$.*

Sgall [Sga99] considered Conjecture 1.6 in the case of $\{a, a+1\}$-cross-intersecting set systems and noted that it would follow from the log-rank conjecture. Sgall noted that any $\{a, b\}$-cross-intersecting set pair of families $(\mathcal{S}, \mathcal{T})$ from $2^{[d]}$ can be represented as an $m \times n$ matrix $M_{\mathcal{S},\mathcal{T}}$ over $\{a, b\}$ with rank at most d, where $M_{\mathcal{S},\mathcal{T}}[i,j] := |S_i \cap T_j|$. One can write $M_{\mathcal{S},\mathcal{T}}$ as a sum of d primitive matrices R_k indexed by elements of $[d]$, where $R_k[i,j] = 1$ iff $k \in S_i \cap T_j$. Each R_k has rank 1, so the total rank of $M_{\mathcal{S},\mathcal{T}}$ is at most d. Conversely, any $\{a, b\}$-valued matrix that is a sum of d primitive matrices corresponds to an $\{a, b\}$-cross-intersecting set system over a universe of size d.

Theorem 1.7. *The log-rank conjecture (Conjecture 1.1) is equivalent to Conjecture 1.6.*

Proof. As mentioned above, the fact that Conjecture 1.6 is implied by the log-rank conjecture was proved by Sgall [Sga99]. We include the proof here for completeness. For an $\{a, b\}$-cross-intersecting family pair $(\mathcal{S}, \mathcal{T})$ with $a \le b$, we can write $M_{\mathcal{S},\mathcal{T}} = (b-a)B + aJ$, where B is some Boolean matrix and J is the all-ones matrix. Then, $\mathrm{r}(B) - 1 \le \mathrm{r}(M_{\mathcal{S},\mathcal{T}}) \le \mathrm{r}(B) + 1$. By Conjecture 4.1, B has a monochromatic rectangle of density $2^{-\operatorname{polylog}(\mathrm{r}(B))}$, which yields a monochromatic rectangle in $M_{\mathcal{S},\mathcal{T}}$ of density $2^{-\operatorname{polylog}(\mathrm{r}(M))} \ge 2^{-\operatorname{polylog}(d)}$ from the discussion above.

For the reverse direction, assume Conjecture 1.6 holds. Let A be a Boolean $m \times n$ matrix with rank r and signed rectangle rank u. By Theorem 1.2 and Conjecture 4.1, it suffices to find a monochromatic rectangle in A of density at least $2^{-\operatorname{polylog}(u)}$. We reduce this to the problem of finding a large $\{a\}$- or $\{b\}$-cross-intersecting subfamily in an $\{a, b\}$-cross-intersecting set system. Let a, b be integers chosen later. Define a matrix $A' \in \{a, b\}^{m \times n}$ as follows:

$$A'[i,j] := \begin{cases} a & \text{if } A[i,j] = 1, \\ b & \text{if } A[i,j] = 0. \end{cases}$$

Our goal is to show that A' corresponds to an $\{a, b\}$-cross-intersecting set system over a universe of size $d = \Theta(u)$.

Let $A = \sum_{i=1}^{u} \varepsilon_i R_i$ be the signed rectangle rank decomposition for A. We now construct a set of $2u$ primitive matrices $\{R'_i\}$ such that $A' = \sum_{i=1}^{2u} R'_i$, by replacing each R_i with a pair of primitive matrices R'_{2i-1}, R'_{2i} depending on the sign of ε_i.

- If $\varepsilon_i = 1$, let $R'_{2i-1} = J$ (the all-ones matrix), and $R'_{2i} = R_i$. Then, $R'_{2i-1} + R'_{2i} = J + R_i$
- If $\varepsilon_i = -1$, let $A_i \subseteq [m]$, $B_i \subseteq [n]$ be such that $R_i = 1$ on $A_i \times B_i$ and zero elsewhere. Define R'_{2i-1} to be 1 on $([m] \setminus A_i) \times [n]$ and 0 elsewhere, and R'_{2i} to be 1 on $A_i \times ([n] \setminus B_i)$. Then $R'_{2i-1} + R'_{2i} = J - R_i$.

In either case, the pair (R'_{2i-1}, R'_{2i}) replaces $\varepsilon_i R_i$ by $J + \varepsilon_i R_i$. Thus, each entry of A' satisfies $A'[i,j] = A[i,j] + u$, and therefore A' corresponds to a $\{u, u+1\}$-cross-intersecting set family over $[d]$ for $d = 2u$. Applying Conjecture 1.6, we obtain a large monochromatic rectangle in A', and hence in A, of density at least $2^{-\operatorname{polylog}(u)}$. This proves the log-rank conjecture for A. □

Remark 4.2. The above proof shows that the log-rank conjecture is equivalent to a special case of Conjecture 1.6 for $\{k, k+1\}$-cross-intersecting set systems.

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