PERMANENTAL RANK VERSUS DETERMINANTAL RANK OF RANDOM MATRICES OVER FINITE FIELDS

FATEMEH GHASEMI, GAL GROSS, AND SWASTIK KOPPARTY

ABSTRACT. This paper is motivated by basic complexity and probability questions about permanents of random matrices over small finite fields, and in particular, about properties separating the permanent and the determinant.

Fix $q = p^m$ some power of an odd prime, and let $k \leq n$ both be growing. For a uniformly random $n \times k$ matrix A over $\mathbb{F}_q$, we study the probability that all $k \times k$ submatrices of A have zero permanent; namely that A does not have full *permanental rank*.

When $k = n$, this is simply the probability that a random square matrix over $\mathbb{F}_q$ has zero permanent, which we do not understand. We believe that the probability in this case is $\frac{1}{q} + o(1)$, which would be in contrast to the case of the determinant, where the answer is $\frac{1}{q} + \Omega_q(1)$.

Our main result is that when k is $O(\sqrt{n})$, the probability that a random $n \times k$ matrix does not have full permanental rank is essentially the same as the probability that the matrix has a 0 column, namely $(1 + o(1))\frac{k}{q^n}$. In contrast, for determinantal (standard) rank the analogous probability is $\Theta(\frac{q^k}{q^n})$.

At the core of our result are some basic linear algebraic properties of the permanent that distinguish it from the determinant.

1. INTRODUCTION

Like the determinant, the permanent of an $n \times n$ matrix $A = [a_{ij}] \in M_n(\mathbb{F})$ is defined as a sum over permutations:

$$\operatorname{per} A := \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

Despite their apparent similarity, the permanent function and the determinant function conjecturally lie at vastly different locations in the computational landscape. It is thus of great interest to prove unconditional separations between their properties.

Unlike the determinant, which can be computed efficiently (e.g., via Gaussian elimination), the permanent is notoriously difficult to compute.[1] This has been vaguely understood for a long time, starting with Pólya's question [14] about the impossibility of reducing the computation of a permanent to that of a determinant. More recently, Valiant [18] defined the counting complexity class #P, proved that this complexity class is at least as hard as NP, and that the permanent of a matrix with entries in $\{0, 1\}$ is #P-complete. He also showed

[1]Unless we are in characteristic 2, in which case the permanent equals the determinant.

that over fields of odd characteristic p, the permanent is complete for the modular counting class $\mathsf{MOD}_p\mathsf{P}$.

In a separate line of work, Valiant [17] defined algebraic analogues of the P and NP complexity classes—called VP and VNP—and proved that computing the permanent is VNP-complete. In contrast, the determinant lies in VP (and is complete for the further subclass VBP). This makes the permanent one of the main candidates for an explicit polynomial which may separate VP from VNP and so it has been the subject of intense research in algebraic complexity. In particular, this has been studied under the title of "permanent vs. determinant", seeking to understand the intrinsic differences between the permanent polynomial and the determinant polynomial.

The permanent has also played an important role in average case complexity. Over fields of size $\text{poly}(n)$, building on the work of Beaver-Feigenbaum [2], Lipton [13] showed the random self reducibility of permanents under the uniform distribution. This implies that if there is no algorithm that efficiently computes the permanent of arbitrary $n \times n$ matrices (this is widely believed, since the permanent over $\mathbb{F}_p$ for p prime is a $\mathsf{MOD}_p\mathsf{P}$-hard problem), then there is no algorithm that can efficiently compute the permanent even if it is allowed to err on some $\frac{1}{\text{poly}(n)}$ fraction of $n \times n$ matrices. Using error-correcting properties of low-degree polynomials over large fields, the size of this fraction was improved by Gemmel-Lipton-Rubinfeld-Sudan-Wigderson [10] (to $\Omega(1)$), Gemmel-Sudan [11] (to $1/2 - o(1)$), and Cai-Pavan-Sivakumar [6] (to $1 - o(1))^2$. While these results do not hold over smaller fields, Feigenbaum-Fortnow [9] showed that for fields $\mathbb{F}_q$ with q a fixed prime power, there is a nonuniform distribution under which this kind of average case hardness holds. Today it is not known whether for a fixed prime p, computing the permanent for 99% of all $n \times n$ matrices over $\mathbb{F}_p$ is $\mathsf{MOD}_p\mathsf{P}$-hard. This gives further motivation for studying the distribution of the permanent of a uniformly random matrix over $\mathbb{F}_p$ which, surprisingly, is not well understood.

If the entries of an $n \times n$ matrix A are chosen uniformly at random from a finite field $\mathbb{F}_q$ of characteristic p and order $q = p^m$, it is well known that the probability of $\det A$ being 0 is

$$1 - \prod_{k=0}^{n-1} \left(1 - \frac{q^k}{q^n} \right) = \frac{1}{q} + \Omega(q^{-2}).$$

A simple argument shows that the exact same expression is also an upper bound for the probability that $\text{per } A$ is 0, and other than a $(1 + o(1))$ factor improvement [4], this is the best known upper bound. We believe, however, that something much stronger is true; that the permanent behaves like a random polynomial.

Conjecture. Let p be an odd prime and $q = p^m$ some fixed power. Let A be a uniformly random $n \times n$ matrix with entries in $\mathbb{F}_q$. Then:

$$\Pr[\text{per } A = 0] = \frac{1}{q} + o(1).$$

Since the permanent function enjoys a cofactor expansion formula similar to that of the determinant, one simple way for the permanent to vanish is if each of its cofactors vanishes.

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[2]Average case hardness results under stronger assumptions were given by Feige-Lund [8]. The list version of the question was proved hard by Cai-Hemachandra [5].

This observation naturally leads to the consideration of the *permanental rank*. First introduced by Yu [19] in connection with the Alon-Jaeger-Tarsi conjecture [1], the *permanental rank* perrank(A) of an $n \times k$ matrix A is the largest r such that A has an $r \times r$ submatrix with nonzero permanent. Note that if 'nonzero permanent' is replaced with 'nonzero determinant', the result is an equivalent definition of the usual notion of the rank of a matrix, rank A. For $k = n$ we have per $A = 0 \iff$ perrank$(A) < k$, i.e. if A is of non-full permanental rank.

We have the simple observation, based on cofactor expansion, that if A is a uniformly random $n \times n$ matrix, and B is a uniformly random $n \times (n-1)$ matrix, then:

$$\Pr[\text{per } A = 0] = \frac{1}{q} + \left(1 - \frac{1}{q}\right) \cdot \Pr[\text{perrank}(B) < n - 1].$$

This relates the conjecture to the problem of understanding the probability that the rectangular matrix B has non-full permanental rank: the above conjecture is equivalent to the statement that $\Pr[\text{perrank}(B) < n - 1] = o(1)$.

1.1. **Main Result.** In this paper we prove bounds on the probability that a random $n \times k$ matrix with $k \ll n$ has non-full permanental rank, i.e., the probability that all $k \times k$ submatrices have zero permanents.

The trivial way such an $n \times k$ matrix A may fail to have full permanental rank is if it has a 0-column. Viewing the columns as random vectors in $\mathbb{F}_q^n$, the probability of a zero column is easily seen to be $kq^{-n} + O((kq^{-n})^2)$. Therefore, the probability Z of perrank$(A) < k$ is at least as large:

$$\Pr[Z] \geq \frac{k}{q^n} + o(q^{-n}).$$

Our main result, Theorem 2.1, proves a matching upper bound for $k = O(\sqrt{n})$:

$$\Pr[Z] = \frac{k}{q^n} + o(q^{-n}).$$

Thus for $k = O(\sqrt{n})$ the only significant obstruction to full permanental rank is the trivial one of a zero column.

For determinants, we note the probability that a random $n \times k$ matrix has non-full determinantal (i.e. standard) rank is about $\Theta_q(\frac{q^k}{q^n})$ — it is within a constant factor of the probability that the last column lies in the span of the first $k - 1$ columns. This establishes a separation in the behaviours of the permanental rank and the (determinantal) rank. We hope that our ideas can lead to a proof of our conjecture above.

1.2. **Techniques.** Our probabilistic argument relies on novel structural results. We introduce the notion of a *permanull subspace*, which is a subspace $S \leq \mathbb{F}_q^n$ such that any $n \times n$ matrix A whose columns lie in S has vanishing permanent, per $A = 0$. Theorem 4.1 shows that the only codimension-1 permanull subspaces are the trivial ones $\{\mathbf{e}_i\}^\perp$. Generalizing this notion, we call a list of n-subspaces $S_1, \ldots, S_n$ *jointly-permanull* if every $n \times n$ matrix A such that the i-th column of A is an element of S_i has per $A = 0$.[3] Our main structural

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[3]Permuting the columns of A does not affect the vanishing of the permanent, so one could consider *n-multisets* of subspaces instead. We opted for lists for ease of readability.

result, Theorem 4.8, generalizes Theorem 4.1 by showing that if $n \geq 3$ and each S_i has codimension at most 1 then $S_1 = \cdots = S_n = \{\mathbf{e}_i\}^\perp$.

Both these facts do not hold for the determinant. This is the key source of how our results distinguish between the permanent and the determinant. Understanding when spaces $S_1, \ldots, S_n$ of codimension 2 or larger are jointly-permanull is very interesting, and seems highly relevant for relaxing our requirement that $k \leq O(\sqrt{n})$. In Sections 5 and 6 we make some small steps in this direction.

1.3. **Other related work.** For characteristics $p \geq 3$, Dolinar et al. [7] have shown that for a sufficiently large order q, the probability $\Pr[\text{per } A = 0]$ is strictly less than $\Pr[\det A = 0]$ by recursively computing upper bounds for the number of matrices with vanishing permanents; Budrevich and Guterman [4] removed the condition that the order of the field be sufficiently large. Several papers improved the upper bounds on the number of matrices with vanishing permanents, the most recent being Budrevich's paper [3] (see also the references therein). These results present significant progress in understanding the vanishing behaviour of the permanent, but are not yet sufficient to prove that for fixed q and growing n, the difference between the probability that the permanent of a random matrix is zero and the probability that the determinant of a random matrix is zero is at least a positive constant.

Very recently, Scheinerman [15] developed new methods for computing astonishingly large permanents over the field $\mathbb{F}_3$ on present day computers. He used his algorithms to compute permanents of random 30×30 matrices over $\mathbb{F}_3$, and found that the probability of zero permanent does seem to be very close to $1/3$.

Over the real numbers, Tao and Vu [16] showed that the permanent of a random ± 1 matrix is almost surely nonzero (and they further determined its magnitude). More recently, Kwan and Sauermann [12] extended these results to symmetric random ± 1 matrices. There seem to be some quantitative forms of permanental rank that appear in these works; it would be interesting to see if there are tools from there that can help resolve our main conjecture.

2. Results

In this section we state our main theorem, showing that for $k = O(\sqrt{n})$, a random $n \times k$ matrix has non-full permanental rank with probability about k/q^n.

Theorem 2.1. *Let $k, n \in \mathbb{N}$ with $k \leq 0.1\sqrt{n}$, .*

Let X be a uniformly random $n \times k$ matrix with entries in $\mathbb{F}_q$. Let Z be the event that all $k \times k$ submatrices of X have permanent equal to 0. Then:

$$\Pr[Z] = \frac{k}{q^n} + O(q^{-1.1n}).$$

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Remark. For much smaller k, the error term gets smaller. For example, for $k = o(\sqrt{n})$, we have

$$\Pr[Z] \leq \frac{k}{q^n} + O(q^{-(2-o(1))n}).$$

The precise relationship is given in the proof of Claim 3.1 below.

The main ingredient in our proof is a structural result about the permanent. In order to state the result we need the following definitions.

Definition 2.2. The list of subspaces $S_1, \ldots, S_n \leq \mathbb{F}^n$ is said to be *jointly-permanull* if every $n \times n$ matrix A whose i-th column is a vector in S_i (for $1 \leq i \leq n$) has per $A = 0$.

Our main structural result classifies codimension-1 jointly-permanull lists.

Theorem 4.8 (Classification of codimension-1 jointly-permanull lists)**.** *Let $n \geq 3$ and $S_1, \ldots, S_n \leq \mathbb{F}_q^n$ be subspaces, each of dimension $\geq n - 1$. Then $S_1, \ldots, S_n$ is a jointly-permanull list if and only if $S_1 = S_2 = \cdots = S_n = \{\mathbf{e}_i\}^\perp$ for some $1 \leq i \leq n$.*

Note that the "if" direction of the theorem is obvious, the interesting part is the "only if" direction. A particularly interesting aspect of this theorem is that it gives a separation between the permanent and the determinant: n subspaces in $\mathbb{F}^n$ of codimension at most 1 are jointly "determinull" if an only if they are the same codimension 1 subspace.

The proof of Theorem 4.8 proceeds by induction; the base case $n = 3$ is Lemma 4.5, and relies on some direct computations. For $n = 2$ the statement is false! The inductive step, remarkably, only uses the cofactor expansion formula of the permanent, and would have also gone through for the determinant — it is the base case that makes all the difference! The inductive step relies on a simple graph-theoretic observation which is Lemma 4.7. In the next section we show how Theorem 2.1 follows from 4.8. We defer the proof of Theorem 4.8 to Section 4.

Understanding jointly-permanull subspaces of larger codimension looks like a very interesting problem, and seems relevant to extending the range of k in our main theorem. An interesting special case of jointly-permanull subspaces is when all the subspaces in the list are the same subspace S.

Definition 2.3. The subspace $S \leq \mathbb{F}^n$ is said to be *permanull* if every $n \times n$ matrix A whose columns are elements of S has per $A = 0$.

This case plays an important role in the theoretical development leading to the proof of our main structural result. In Sections 5 and 6, we characterize permanullness in larger codimension. In Section 5, we show that over fields of sufficiently high characteristic, the only permanull subspaces are the trivial ones, i.e. the ones having a common zero coordinates (Theorem 5.8). In Section 6 we give a full characterization of permanull subspaces (for any characteristic of the ambient field) in terms of an auxiliary polynomial which we call the *permanental polynomial* (Theorem 6.7). This allows us to check if a given codimension-k subspace is permanull with $O_k(n^k)$ field operations, and a general subspace with $2^{O(n)}$ field operations.

3. Proof of the Main Theorem

In the current section we show how to use Theorem 4.8 to prove Theorem 2.1.

Proof of Theorem 2.1. Let $\mathsf{SC0}$ be the event that some column of X is the zero vector $\mathbf{0}$. We will estimate the probability of Z by showing that it almost coincides with the event $\mathsf{SC0}$ (up to probability $O(q^{-1.1n})$).

Note that $\mathsf{SC0} \implies Z$. To get some sort of reverse implication, we will define a "well-spreadedness" event WS and show that:

- WS occurs with probability $1 - o(q^{-n})$.
- $\mathsf{WS} \wedge Z \implies \mathsf{SC0}$.

To deduce $\mathsf{SC0}$ in the second step, we will use the classification of codimension-1 jointly-permanull lists, Theorem 4.8. The event WS will help us meet the hypothesis of Theorem 4.8. We now work towards defining WS.

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{F}_q^k$ be the rows of X. We first observe that if Z happens, any partition of the rows of X into k parts will give us a construction of a jointly-permanull list. Indeed, if $N_1, \ldots, N_k$ is a partition of $[n] = \{1, 2, \ldots, n\}$, then the linear spaces $S_1, \ldots, S_k \leq \mathbb{F}_q^k$ defined by
$$S_i = \mathrm{span}\{\mathbf{x}_j : j \in N_i\}$$
form a jointly-permanull list; this follows immediately from the hypothesis that any $k \times k$-submatrix of X has vanishing permanent, by the facts that the permanent is a multilinear function of the rows. To apply Theorem 4.8 to this, we will need the subspaces S_i to have dimension at least $k - 1$. This motivates the following definition of the well-spreadedness event WS: it is the event that there exists a partition $N_1, \ldots, N_k$ of $[n]$ such that for each i we have $\dim(S_i) \geq k - 1$. It is immediate from Theorem 4.8 that $\mathsf{WS} \wedge Z \implies \mathsf{SC0}$.

We make the following claim, which we prove immediately after showing how to use it to complete current proof.

Claim 3.1.
$$\Pr[\mathsf{WS}] \geq 1 - O(q^{-1.1n}).$$

Assuming the claim, we complete the proof of the theorem. By the claim, we have:
$$\Pr[Z] \leq \Pr[\mathsf{WS} \wedge Z] + \Pr[\neg \mathsf{WS}] \leq \Pr[\mathsf{SC0}] + O(q^{-1.1n}).$$
On the other hand, since $\mathsf{SC0} \implies Z$, we have:
$$\Pr[\mathsf{SC0}] \leq \Pr[Z].$$

Finally, since $\Pr[\mathsf{SC0}] = \frac{k}{q^n} + O(k^2 \cdot q^{-2n})$, we get the desired result. $\qquad\square$

Proof of Claim 3.1. We give an algorithm for finding the partition $N_1, \ldots, N_k$ witnessing the well-spreadedness event WS, and then show that when k is small enough compared to n, the algorithm succeeds with very high probability.

(1) Initialize $N_1 = N_2 = \cdots = N_k = \emptyset$.
(2) Initially all $i \in [k]$ are marked ACTIVE.
(3) For each $i \in [k]$, let S_i denote $\mathrm{span}\{\mathbf{x}_j : j \in N_i\}$.
(4) For $\ell = 1, 2, \ldots, n$, do the following:

- Pick one ACTIVE $i \in [k]$ for which $\mathbf{x}_\ell$ does not lie in S_i, and:
  - Add ℓ into the set N_i.
  - If $\dim(S_i) \geq k - 1$, then mark i INACTIVE.
- If no such ACTIVE i exists, mark ℓ as INEFFECTIVE, and we add ℓ into one of the N_i.

We now prove that with very high probability by the end of the algorithm each S_i has dimension $\geq k - 1$. This will prove that with very high probability WS occurred.

Observe that in any iteration of the ℓ loop, either $\sum_{i \in [k]} \dim(S_i)$ increases by 1, or ℓ is INEFFECTIVE. Thus at the end of the algorithm, if some $i \in [k]$ has $\dim(S_i) \leq k - 2$ (so that i is ACTIVE), then at most $(k-1)^2 + k - 2 = k(k-1) - 1$ of the ℓ's were "effective", and we must have strictly more than $n - k(k-1)$ INEFFECTIVE ℓ's.

Fix some $\ell^* \in [n]$. Suppose we are just about to start the ℓ^*-th iteration of the loop $\ell \leftarrow \ell^*$. So far only $\mathbf{x}_1, \ldots, \mathbf{x}_{\ell^*-1}$ have been revealed. What would it take for ℓ^* to be marked INEFFECTIVE?

This will happen only if $\mathbf{x}_{\ell^*}$ lies inside S_i for every ACTIVE i, namely if

$$\mathbf{x}_{\ell^*} \in \bigcap_{i \text{ ACTIVE}} S_i.$$

Observe that for any ACTIVE i, we have $\dim(S_i) \leq k - 2$. Thus, if there is at least one ACTIVE i, the probability that ℓ^* is marked INEFFECTIVE is at most:

$$\frac{q^{k-2}}{q^k} = q^{-2}.$$

If there are no ACTIVE i, then we already have $\dim(S_i) \geq k - 1$ for all i.

Suppose we have $L > n - k(k-1)$ INEFFECTIVE ℓ's. Using the union bound, the probability that some i remains ACTIVE at the end of the algorithm is at most

$$
\begin{aligned}
\binom{n}{L} \cdot \left(q^{-2}\right)^L = \binom{n}{n-L} \cdot q^{-2L} \\
\leq \binom{n}{n-L} \cdot q^{-2(n-k(k-1))} \\
\leq \binom{n}{k(k-1)} q^{2k(k-1)} q^{-2n} \\
\leq \binom{n}{k^2} q^{2k^2} q^{-2n}
\end{aligned}
$$

Thus for $k = 0.1\sqrt{n}$, the probability that some i remains ACTIVE is at most $O(q^{-1.1n})$.

If none of the i is ACTIVE at the end of the algorithm, then the partition $N_1, \ldots, N_k$ produced by the algorithm is a witness for the event WS. Thus

$$\Pr[\mathsf{WS}] \geq 1 - \Pr[\text{Some } i \text{ remains ACTIVE}] \geq 1 - O(q^{-1.1n}).$$

$\square$

4. Structural Results

In this section we prove the structural results we have relied on in the proof of Theorem 2.1. Our main structural result is

Theorem 4.8 (Classification of codimension-1 jointly-permanull lists)**.** *Let $n \geq 3$ and $S_1, \ldots, S_n \leq \mathbb{F}_q^n$ be subspaces, each of dimension $\geq n - 1$. Then $S_1, \ldots, S_n$ is a jointly-permanull list if and only if $S_1 = S_2 = \cdots = S_n = \{\mathbf{e}_i\}^\perp$ for some $1 \leq i \leq n$.*

Note that the "if" direction of the theorem is obvious, and below we shall prove the "only if" direction. The proof proceeds by induction; the base case $n = 3$ is Lemma 4.5. The inductive step relies on a simple graph-theoretic observation which is Lemma 4.7. In the remainder of this section we prove the necessary lemmas and show how Theorem 4.8 follows from them.

4.1. Codimension-1 permanull subspaces.

In this subsection, we focus on the case where all the subspaces S_i are the same, namely subspaces $S \subseteq \mathbb{F}^n$ which are *permanull*.

Clearly $S = \mathbb{F}^n$ cannot be permanull (e.g., consider the identity matrix), which leads naturally to the question of whether there are any permanull codimension-1 spaces. Since a matrix with a $\mathbf{0}$-row has a vanishing permanent, we see that any subspace of the form $\{\mathbf{e}_i\}^\perp$ is permanull. We now show that these are the only codimension-1 permanull spaces.

Theorem 4.1 (Classification of codimension-1 permanull subspaces)**.** *Let $S \leq \mathbb{F}^n$ be a codimension-1 subspace. Then S is permanull if and only if $S = \{\mathbf{e}_i\}^\perp$ for some $1 \leq i \leq n$.*

Proof. We prove the "only if" direction by induction on n. To avoid confusion, we use superscripts to denote the dimension of a vector, e.g., $\mathbf{e}_1^2 = (1,0)^T$ and $\mathbf{e}_1^3 = (1,0,0)^T$.

The claim is trivial for $n = 1$. For $n = 2$, we show that if $\{\mathbf{v}\}^\perp$ is permanull, then $\mathbf{v}$ is a scalar multiple of $\mathbf{e}_1^2$ or $\mathbf{e}_2^2$. Indeed, the only nonzero vectors that are not a scalar multiple of either $\mathbf{e}_1^2$ or $\mathbf{e}_2^2$ are of the form $(\lambda, \mu)^T$ with $\lambda, \mu \neq 0$. Note that

$$\mathrm{per} \begin{pmatrix} \mu & \mu \\ -\lambda & -\lambda \end{pmatrix} = -2\lambda\mu \neq 0$$

since $\lambda, \mu \neq 0$ and the characteristic $p \neq 2$. Since this matrix has columns in $\{\mathbf{v}\}^\perp$, we conclude that if $S = \{\mathbf{v}\}^\perp$ is codimension-1 permanull space, then $\mathbf{v} = \lambda \mathbf{e}_i^2$.

Suppose now that $n \geq 2$ and we have proved the claim for all values $1, 2, \ldots, n$; we shall prove it for $n + 1$. Let $\mathbf{v} = (v_1, \ldots, v_{n+1})^T$ be some nonzero vector in $\mathbb{F}^{n+1}$ not of the form $\lambda \mathbf{e}_i^{n+1}$, we prove that $\{\mathbf{v}\}^\perp$ is not permanull by finding a matrix with columns in $\{\mathbf{v}\}^\perp$ and non-zero permanent.

Case 1: $\mathbf{v}$ has at least one zero coordinate. Assume without loss of generality $v_{n+1} = 0$ and denote $\widehat{\mathbf{v}} = (v_1, \ldots, v_n)^T$. Note that $\widehat{\mathbf{v}}$ is nonzero and not of the form $\lambda \mathbf{e}_i^n$. By the induction hypothesis, there exist $\widehat{\mathbf{r}}_1, \ldots, \widehat{\mathbf{r}}_n \in \{\widehat{\mathbf{v}}\}^\perp$ such that

$$\mathrm{per} \begin{pmatrix} \widehat{\mathbf{r}}_1 & \cdots & \widehat{\mathbf{r}}_n \end{pmatrix} \neq 0.$$

Let $\mathbf{r}_i = (\widehat{\mathbf{r}}_i, 0)^T$. Then $\mathbf{e}_{n+1}^{n+1}, \mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n \in \{\mathbf{v}\}^\perp$ and

$$\mathrm{per} \begin{pmatrix} \mathbf{r}_1 & \cdots & \mathbf{r}_n & \mathbf{e}_{n+1} \end{pmatrix} = \mathrm{per} \begin{pmatrix} \widehat{\mathbf{r}}_1 & \cdots & \widehat{\mathbf{r}}_n \end{pmatrix} \neq 0.$$

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This proves that $\{\mathbf{v}\}^{\perp}$ is not permanull.

Case 2: $\mathbf{v} \in \mathrm{span}\{(1, 1, \ldots, 1)^T\}$. We prove directly that

$$
A_n =
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 1 \\
0 & 0 & 0 & \cdots & -1 & -1
\end{pmatrix}
$$

(with columns in $\{\mathbf{v}\}^{\perp}$) has a nonvanishing permanent. The proof proceeds by induction:

$$
\mathrm{per}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = -2 \neq 0
$$

and for $n \geq 3$ we simply expand A_n by the first row to obtain $\mathrm{per}\, A_n = \mathrm{per}\, A_{n-1}$.

Case 3: None of the coordinates of $\mathbf{v}$ is 0 and $\mathbf{v} \notin \mathrm{span}\{(1, 1, \ldots, 1)^T\}$. In this case, there are two coordinates (i, j) such that $(v_i, v_j) = (\lambda, \mu)$ for some distinct nonzero scalars $\lambda \neq \mu$, and we assume without loss of generality that $(v_n, v_{n+1}) = (\lambda, \mu)$.

Case 3a $n + 1 = 3$: In this case $\mathbf{v} = (\alpha, \lambda, \mu)^T$ for some nonzero scalar α.

$$
\mathrm{per}\begin{pmatrix} \lambda & 0 & 0 \\ -\alpha & \mu & \mu \\ 0 & -\lambda & -\lambda \end{pmatrix} = -2\mu\lambda^2 \neq 0
$$

so that $\{\mathbf{v}\}^{\perp}$ is not permanull.

Case 3b $n + 1 \geq 4$: In this case we denote $\widehat{\mathbf{v}} = (v_1, v_2, \ldots, v_{n-1})^T$ (dropping the last two coordinates of $\mathbf{v}$) and note that it is a nonzero vector not of the form $\lambda \mathbf{e}_i^{n-1}$ (because it has at least two nonzero coordinates). By the induction hypothesis there exist $\widehat{\mathbf{r}}_1, \ldots, \widehat{\mathbf{r}}_{n-1} \in \{\widehat{\mathbf{v}}\}^{\perp}$ such that

$$
\mathrm{per}\begin{pmatrix} \widehat{\mathbf{r}}_1 & \widehat{\mathbf{r}}_2 & \cdots & \widehat{\mathbf{r}}_{n-1} \end{pmatrix} \neq 0.
$$

Let $\mathbf{r}_i = (\widehat{\mathbf{r}}_i, 0, 0)^T$ and $\mathbf{s} = (0, 0, \ldots, 0, \lambda, -\mu)^T$. Then $\mathbf{s}, \mathbf{r}_1, \ldots, \mathbf{r}_{n-1} \in (\mathrm{span}\{\mathbf{v}\})^{\perp}$ and

$$
\mathrm{per}\begin{pmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \cdots & \mathbf{r}_{n-1} & \mathbf{s} & \mathbf{s} \end{pmatrix} =
$$
$$
-2\mu\lambda \,\mathrm{per}\begin{pmatrix} \widehat{\mathbf{r}}_1 & \widehat{\mathbf{r}}_2 & \cdots & \widehat{\mathbf{r}}_{n-1} \end{pmatrix} \neq 0.
$$

$\square$

4.2. **The base case of the induction.** In this subsection we prove Lemma 4.5, the base case ($n = 3$) of the classification of jointly-permanull subspaces. We proceed via a sequence of lemmas.

Lemma 4.2 (Classification of $n \times 2$ matrices). *Let $n \geq 2$. An $n \times 2$ matrix $A \in \mathbb{F}^{n \times 2}$ has non-full permanental rank $\mathrm{perrank}(A) < 2$ if and only if exactly one of these three possibilities holds:*

(i) *the matrix has a zero column;*
(ii) *there is a single nonzero row, and that row is a scalar multiple of $(1, x)$ (for some nonzero $x \in \mathbb{F}$);*

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(iii) there are exactly two nonzero rows: one a scalar multiple of $(1, x)$ and the other a scalar multiple of $(1, -x)$ (for some nonzero $x \in \mathbb{F}$).

Proof. Let A be an $n \times 2$ matrix with perrank$(A) < 2$. Each row of A is a scalar multiple of one of the $|\mathbb{F}| + 2$ vectors

$$(0, 0) \qquad\qquad (0, 1) \qquad\qquad (1, x)$$

where $x \in \mathbb{F}$ is an arbitrary field element. Which of these vectors may appear together as rows in A? First, consider vectors of the form $(1, x)$ with $x \neq 0$. Since

$$\operatorname{per} \begin{pmatrix} 1 & x \\ 1 & y \end{pmatrix} = x + y = 0 \iff y = -x$$

we see that $(1, x)$ and $(1, -x)$ may appear in A together or individually, each appearing at most once. Since

$$\operatorname{per} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = 1$$

no vector of the form $(1, x)$ may appear with $(1, 0)$. $\qquad\square$

Next we need some basic facts about codimension-1 spaces in $\mathbb{F}_q^3$. We shall consider a function $\operatorname{Per}_{3\times 2} : \mathbb{F}_q^{3\times 2} \to \mathbb{F}_q^3$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \mapsto \left(\operatorname{per} \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \quad \operatorname{per} \begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix} \quad \operatorname{per} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right)^T$$

(Note that the order of the subpermanents corresponds to expansion by the first column of a 3×3 permanent.) Given some $U, V \leq \mathbb{F}_q^3$ subspaces of dimension ≥ 2, we are interested in the dimension of

$$P(U, V) := \operatorname{span} \left\{ \operatorname{Per}_{3\times 2} \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix} : \mathbf{u} \in U, \mathbf{v} \in V \right\}.$$

Lemma 4.2 implies that $\dim P(U, V) > 0$.

Lemma 4.3. *Let $U, V \leq \mathbb{F}_q^3$ be subspaces, each of dimension ≥ 2. Then $\dim P(U, V) = 1$ if and only if $U = V = \{\mathbf{e}_i\}^\perp$ for some $1 \leq i \leq 3$.*

Proof. The "if" direction is immediate: if $U = V = \{\mathbf{e}_i\}^\perp$, then it is easy to see that $P(U, V) = \operatorname{span}\{\mathbf{e}_i\}$. For the "only if" direction, we assume that $\dim P(U, V) = 1$ and start by noting that it is not possible for either U or V to be full dimensional. Indeed, assume without loss of generality $V = \mathbb{F}_q^3$. Choosing any two linearly independent vectors $\mathbf{a}, \mathbf{b} \in U$ we know that $\begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix}$ has rank 2, so there exists some coordinate $1 \leq i \leq 3$ such that the vectors $\widehat{\mathbf{a}}, \widehat{\mathbf{b}} \in \mathbb{F}_q^2$ obtained from $\mathbf{a}, \mathbf{b}$ by deleting the i-th coordinate are linearly independent. For the purpose of illustration, say that $i = 3$. Then,

$$\operatorname{Per}_{3\times 2} \begin{pmatrix} \mathbf{a} & \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} a_2 & a_1 & 0 \end{pmatrix}^T \qquad\qquad \operatorname{Per}_{3\times 2} \begin{pmatrix} \mathbf{b} & \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} b_2 & b_1 & 0 \end{pmatrix}^T$$

so that $\dim P(U, V) \geq 2$.

Therefore, we may assume that $\dim U = \dim V = 2$. Since $\dim P(U, V) = 1$, we have $P(U, V) = \operatorname{span}\{\mathbf{p}\}$ for some nonzero vector $\mathbf{p} = (p_1, p_2, p_3)^T \in \mathbb{F}_q^3$.

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Claim 4.4. *If* $\mathbf{e}_i$ *is an element of* U *or* V *(for some* $1 \leq i \leq 3$*) then* $U = V = \{\mathbf{e}_j\}^{\perp}$ *for some* $1 \leq j \leq 3$.

Proof of claim. Without loss of generality, suppose $\mathbf{e}_i \in U$, and for concreteness suppose $\mathbf{e}_3 \in U$ (the other cases are analogous).

For every $\mathbf{v} = (v_1, v_2, v_3)^T \in V$ there exists some scalar $\lambda \in \mathbb{F}$ such that we have

$$\mathrm{Per}_{3 \times 2} \begin{pmatrix} \mathbf{v} & \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} v_2 & v_1 & 0 \end{pmatrix}^T = \lambda \mathbf{p}.$$

In particular, $p_3 = 0$, and $V = \mathrm{span}\left\{ \mathbf{e}_3, \begin{pmatrix} p_2 & p_1 & 0 \end{pmatrix} \right\}$. But then $\mathbf{e}_3 \in V$ and the same reasoning shows that $U = \mathrm{span}\left\{ \mathbf{e}_3, \begin{pmatrix} p_2 & p_1 & 0 \end{pmatrix} \right\} = V$. Consider now

$$\mathrm{Per}_{3 \times 2} \begin{pmatrix} p_2 & p_2 \\ p_1 & p_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2p_1 p_2 \end{pmatrix}^T \in \mathrm{span}\{\mathbf{p}\} \iff p_1 p_2 = 0.$$

If $p_1 = 0$ we have $U = V = \{\mathbf{e}_2\}^{\perp}$, whereas if $p_2 = 0$ we have $U = V = \{\mathbf{e}_1\}^{\perp}$. $\quad\square$

Therefore, it remains to show that $\mathbf{e}_i$ is an element of U or V (for some $1 \leq i \leq 3$). Suppose towards contradiction that for any $1 \leq i \leq 3$, $\mathbf{e}_i \notin U, V$. We start by finding nonzero vectors $\mathbf{u} \in U$ and $\mathbf{v} \in V$ such that $\mathrm{Per}_{3 \times 2} \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix} = 0$.

To find such vectors, fix $\mathbf{u}, \mathbf{u}'$ a basis for U and some nonzero V-vector $\mathbf{0} \neq \mathbf{v} \in V$. The map $H : \mathbb{F}_q^3 \times \mathbb{F}_q^3 \to \mathbb{F}_q^3$ taking a pair of vectors to their $\mathrm{Per}_{3 \times 2}$ function $(\mathbf{u}, \mathbf{v}) \mapsto \mathrm{Per}_{3 \times 2} \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix}$ is bilinear. We have

$$H(\mathbf{u}, \mathbf{v}) = \lambda_{uv} \mathbf{p} \qquad\qquad H(\mathbf{u}', \mathbf{v}) = \lambda_{u'v} \mathbf{p}.$$

For some scalars $\lambda_{uv}, \lambda_{u'v}$. If one of $\lambda_{uv}, \lambda_{u'v}$ is zero, we are done. Otherwise, bilinearity gives

$$H(\lambda_{uv}^{-1} \lambda_{u'v} \mathbf{u} - \mathbf{u}', \mathbf{v}) = \mathbf{0},$$

and we know that $\lambda_{uv}^{-1} \lambda_{u'v} \mathbf{u} - \mathbf{u}' \neq \mathbf{0}$ since $\mathbf{u}, \mathbf{u}'$ are linearly independent. We conclude that it is possible to find some nonzero vectors $\mathbf{u} \in U$ and $\mathbf{v} \in V$ such that $\mathrm{Per}_{3 \times 2} \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix} = 0$.

Next, we apply the conclusion of Lemma 4.2 to the matrix $\begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix}$. Since $\mathbf{e}_i \notin U, V$ for any $1 \leq i \leq 3$, we must have

$$\begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a & ax \\ b & -bx \end{pmatrix}$$

for some nonzero scalars $a, b, x \neq 0$, up to a permutation of the rows.

In particular, up to some permutation of the coordinates, we may choose $\mathbf{u} = (0, 1, c)^T \in U$ and $\mathbf{v} = (0, 1, -c)^T \in V$ for some $c \neq 0$. Since U is a 2-dimensional space and $\mathbf{e}_i \notin U$, we know that there exists a vector of the form $(1, 0, d)^T \in U$ (with $d \neq 0$). Similarly, there exists a vector of the form $(1, 0, e)^T \in V$ (with $e \neq 0$). We now observe that

$$\mathrm{Per}_{3 \times 2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ a & c \end{pmatrix} = \begin{pmatrix} c & a & 1 \end{pmatrix}^T \qquad \mathrm{Per}_{3 \times 2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ b & -a \end{pmatrix} = \begin{pmatrix} b & -a & 1 \end{pmatrix}^T$$

are both elements of span$\{\mathbf{p}\}$, and so are nonzero scalar multiples of each other. Now,

$$\begin{pmatrix} c & a & 1 \end{pmatrix}^T = \lambda \begin{pmatrix} b & -a & 1 \end{pmatrix}^T$$

if and only if $\lambda = 1$, $a = 0$, and $b = c$. In particular, $U \ni \begin{pmatrix} 0 & 1 & a \end{pmatrix}^T = \mathbf{e}_2$, contradicting the assumption that $\mathbf{e}_i \notin U, V$ for any $1 \leq i \leq 3$. $\qquad\square$

The proof of the base case of Theorem 4.8 follows easily from Lemma 4.3.

Lemma 4.5. *Let $S_1, S_2, S_3 \leq \mathbb{F}_q^3$ be three subspaces, each of dimension at least 2. Then S_1, S_2, S_3 is a jointly-permanull list if and only if $S_1 = S_2 = S_3 = \{\mathbf{e}_i\}^\perp$ for some $1 \leq i \leq 3$.*

Proof. If the three subspaces are identical, it is clear they cannot be of full dimension, and so the claim follows from the classification of codimension-1 permanull subspaces, Theorem 4.1. Assume for contradiction the three spaces are not identical, and without loss of generality that S_3 is different from both S_1 and S_2 (while S_1 and S_2 may or may not be the same subspace).

Since S_1, S_2, S_3 are jointly-permanull, for any choice of $\mathbf{x} \in S_1$, $\mathbf{u} \in S_2$ and $\mathbf{v} \in S_3$, we have

$$\mathrm{per} \begin{pmatrix} \mathbf{x} & \mathbf{u} & \mathbf{v} \end{pmatrix} = 0.$$

In other words, $\mathbf{x} \in \left\{ \mathrm{Per}_{3 \times 2} \begin{pmatrix} \mathbf{u} & \mathbf{v} \end{pmatrix} \right\}^\perp$, or more generally $S_1 \perp P(S_2, S_3)$. This is a contradiction since $\dim S_1 \geq 2$ and by Lemma 4.3, $\dim P(S_2, S_3) \geq 2$. $\qquad\square$

4.3. **A graph-theoretic lemma.** The inductive step of the proof of the classification of jointly-permanull subspaces (Theorem 4.8) relies on identifying a suitable coordinate to delete, so the inductive hypothesis can be invoked for the resulting vectors. In order to guarantee the existence of such a coordinate we need a graph-theoretic lemma.

Definition 4.6. Let $G = (V, E)$ be a bipartite graph with partition $V = A \sqcup B$. We say G is a *B-star* if B has a unique vertex of positive degree—called the *centre* of the star. (See Figure 1.)

We now prove that if the original graph G has no isolated vertices, then it is possible to delete an A-vertex and a B-vertex such that the resulting induced graph is not a B-star.

Lemma 4.7 (Star-avoidance lemma). *Let $G = (V, E)$ be a bipartite graph with partition $V = A \sqcup B$ such that*

(P1): $|A|, |B| \geq 4$;
(P2): *every $a \in A$ has a positive degree;*
(P3): *every $b \in B$ has a positive degree.*

Then it is possible to delete two vertices, $a \in A$ and $b \in B$, such that:

(R1): *b is not the unique neighbour of a in G;*
(R2): *after deleting a and b, the resulting induced graph G' is neither empty nor a B-star.*
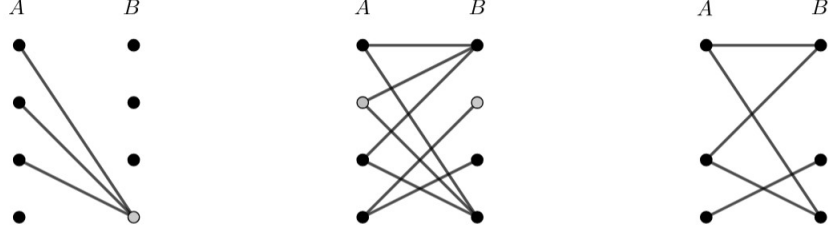
FIGURE 1. (Left) Example of a B-star, the lighter vertex is the centre. (Middle) A bipartite graph; the two lighter vertices are the ones marked for removal in the proof below. Note that removing the top B-vertex and the bottom A-vertex creates the star on the left. (Right) The induced subgraph resulting from removing the two lighter vertices of the graph in the middle.

Proof. Note that if G is a full bipartite graph, then we may take any vertex in B for b and any vertex in A for a; so we may assume that G is not full.

We now claim that we may take b to be a minimum degree vertex in B and a a minimum degree vertex outside the neighbourhood $\Gamma(b)$ of b, i.e. a minimum degree vertex among $\{a \in A : ab \notin E\}$. Clearly, the choice of b and a satisfies condition (R1); it remains to show that it also satisfies condition (R2).

First note that G' is not empty, for otherwise it means that every vertex in A other than a has b as its unique neighbour, and this contradicts the assumption that b is a minimal-degree vertex in B.

Next, assume for contradiction that G' is a B-star with centre r. Since $|B| \geq 4$, we know that in addition to b and r there are at least two more vertices $s, t \in B$. Likewise, since $|A| \geq 4$, in addition to a, there are at least three more vertices $x, y, z \in A$.

Since G' is a B-star with centre r, we have $\deg_{G'}(s) = \deg_{G'}(t) = 0$. Since by (P2) $\deg_G(s), \deg_G(t) > 0$, we see that $as, at \in E$, so that $\deg_G(a) \geq 2$. By the choice of a, every vertex $a' \in A$ not connected to b has $\deg_G(a') \geq 2$. In particular, if x, y, z are not connected to b, then $2 \leq \deg_G(x) = \deg_{G'}(x)$ and similarly for y, z. However, since G' is a star, we have $\deg_{G'}(x) = \deg_{G'}(y) = \deg_{G'}(z) \leq 1$. This proves that $xb, yb, zb \in E$.

We conclude that $\deg_G(b) \geq 3$. By the choice of b, every vertex $b' \in B$ has $\deg_G(b') \geq 3$. In particular, $\deg_G(s) \geq 3$. We now have the contradiction $0 = \deg_{G'}(s) \geq \deg_G(s) - 1 = 2$. $\square$

4.4. **Putting everything together.** Having completed the necessary groundwork, we are ready to prove our main structural result.

Theorem 4.8 (Classification of codimension-1 jointly-permanull lists). *Let $n \geq 3$ and $S_1, \ldots, S_n \leq \mathbb{F}_q^n$ be subspaces, each of dimension $\geq n - 1$. Then $S_1, \ldots, S_n$ is a jointly-permanull list if and only if $S_1 = S_2 = \cdots = S_n = \{\mathbf{e}_i\}^\perp$ for some $1 \leq i \leq n$.*

Proof. By induction on n. The base case $n = 3$ is Lemma 4.5. Suppose we have proved the theorem for some $n \geq 3$ and we wish to prove it for $n + 1$. Towards that end, let $S_1, \ldots, S_{n+1} \leq \mathbb{F}_q^{n+1}$ be a jointly-permanull list of subspaces, each of dimension $\geq n$ and assume for contradiction that they are not all of the form $\{\mathbf{e}_i\}^\perp$ for the same i. To derive a contradiction, we show how to construct an $(n + 1) \times (n + 1)$ matrix A whose j-th column

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is an element of S_j such that per $A \neq 0$, contradicting the assumption that $S_1, \ldots, S_{n+1}$ is a jointly-permanull list.

We may assume without loss of generality that each S_j is of codimension 1 (otherwise, $S_j = \mathbb{F}_q^{n+1}$ and we replace it with a subspace of codimension 1). In particular, we may assume that each $S_j = \{\mathbf{v}_j\}^\perp$ for some nonzero $\mathbf{v}_j \in \mathbb{F}_q^{n+1}$, and we may choose each $\mathbf{v}_j$ such that its first nonzero coordinate is 1. By hypothesis, there is no $1 \leq i \leq n+1$ such that all the $\mathbf{v}_j$ equal $\mathbf{e}_i$.

Case: Consider first the possibility that all the $\mathbf{v}_j$ have some common coordinate that is 0; say the last coordinate. Let $\widehat{\mathbf{v}}_j \in \mathbb{F}_q^n$ be the vector $\mathbf{v}_j$ with the last coordinate removed, and let $\widehat{S}_j = \{\widehat{\mathbf{v}}_j\}^\perp$. Then $\widehat{S}_1, \ldots, \widehat{S}_n$ is a jointly-permanull list.

To see this, suppose $\widehat{A}$ is a matrix with j-th column $\widehat{\mathbf{a}}_j \in \widehat{S}_j$, for $1 \leq j \leq n$. We build a matrix A whose j-th column is in S_j and such that per $\widehat{A} = $ per A which vanishes by the hypothesis that $S_1, \ldots, S_{n+1}$ is a jointly-permanull list. For $1 \leq j \leq n$, the j-th column of A is $(\widehat{\mathbf{a}}_j, 0)^T$, and the $(n+1)$-st column of A is $\mathbf{e}_{n+1}$.

This proves that $\widehat{S}_1, \ldots, \widehat{S}_n$ is a jointly-permanull list, so the induction hypothesis implies that there exists some $1 \leq i \leq n$ such that $\widehat{\mathbf{v}}_j = \mathbf{e}_i \in \mathbb{F}_q^n$ for all $1 \leq j \leq n$.

Repeating the same argument with $\widehat{S}_2, \ldots, \widehat{S}_{n+1}$ we find that $\widehat{\mathbf{v}}_{n+1} = \mathbf{e}_i$. Since the last coordinate of $\mathbf{v}_j$ is 0 (for $1 \leq j \leq n+1$), we see that $\mathbf{v}_j = \mathbf{e}_i \in \mathbb{F}_q^{n+1}$, contradicting the assumption that there is no $1 \leq i \leq n+1$ such that all the S_j equal $\{\mathbf{e}_i\}^\perp$.

Case: the $\mathbf{v}_j$'s do not have a common coordinate which is 0. In other words, for every coordinate $1 \leq i \leq n+1$ there is some $\mathbf{v}_j$ whose i-th coordinate is nonzero.

Consider the bipartite graph $G = (V, E)$ with partition $V = A \sqcup B$ where $A \coloneqq \{\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}\}$ is the set of vectors and $B \coloneqq \{1, 2, \ldots, n+1\}$ the set of coordinates; a vector is connected by an edge to its support, i.e. $\mathbf{v}_j k \in E$, if and only if the k-th coordinate of $\mathbf{v}_j$ is nonzero. Note that the three premises of the Star-avoidance Lemma (Lemma 4.7) are satisfied:

(P1): $|A|, |B| \geq 4$, since $n+1 \geq 4$;

(P2): every $a \in A$ has a positive degree, since the vectors $\mathbf{v}_j$ are all nonzero;

(P3): every $b \in B$ has a positive degree, since the vectors $\mathbf{v}_j$ do not have a common coordinate zero.

By the Star-avoidance Lemma it is possible to find a coordinate $b \in B$ and a vector $\mathbf{v} \in A$ such that

(R1): b is not the unique neighbour of a in G, i.e. $\mathbf{v} \neq \mathbf{e}_b$;

(R2): after deleting a and b, the resulting induced graph G' is neither empty nor a B-star. In other words, letting $\widehat{\mathbf{v}}_j \in \mathbb{F}_q^n$ denote the vector resulting from $\mathbf{v}_j$ by deleting the b-th coordinate, there is no $1 \leq i \leq n$ such that all of the $\widehat{\mathbf{v}}_j$ (not counting $\widehat{\mathbf{v}}$) are in the span of $\mathbf{e}_i^n$.

For ease of writing, we shall assume that $\mathbf{v} = \mathbf{v}_{n+1}$ and $b = n+1$.

Letting $\widehat{S}_j \coloneqq \{\widehat{\mathbf{v}}_j\}^\perp$ (for $1 \leq j \leq n$), the induction hypothesis together with (R2) imply that the $\widehat{S}_1, \ldots, \widehat{S}_n$ is *not* a jointly-permanull list. In particular, it is possible to find vectors $\widehat{\mathbf{a}}_j \in \widehat{S}_j$ (for $1 \leq j \leq n$) such that the matrix $\widehat{A}$ whose j-th column is $\widehat{\mathbf{a}}_j$ ($1 \leq j \leq n$) has per $\widehat{A} \neq 0$. We shall now construct a matrix A whose j-th column

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$\mathbf{a}_j$ in S_j $(1 \leq j \leq n+1)$ and such that $\operatorname{per} A = \operatorname{per} \widehat{A}$, contradicting the assumption that $S_1, \ldots, S_{n+1}$ is a jointly-permanull list.

For $1 \leq j \leq n$ we define $\mathbf{a}_j \coloneqq (\widehat{\mathbf{a}}_j, 0)$ (for $1 \leq j \leq n$) and note that $\mathbf{a}_j \in S_j$. For $\mathbf{a}_{n+1}$ we take any vector in S_{n+1} with 1 in the b-th coordinate; note that this is possible since by (R1) we know that $\mathbf{v} \neq \mathbf{e}_b$. Expanding by the b-th row we have $\operatorname{per} A = \operatorname{per} \widehat{A} \neq 0$.

$\square$

We would like to draw the reader's attention to the fact that the only property of the permanent that is used in the proof above is the cofactor expansion formula. As this formula is shared with the determinant, the inductive step would have also worked for the determinant, even though the result is false for the determinant — as we remarked in the introduction, it is the base case that makes all the difference.

5. PERMANULL SUBSPACES IN SUFFICIENTLY LARGE CHARACTERISTIC

Jointly-permanull subspaces of codimension 1 were a crucial component in our proof of Theorem 2.1. Understanding jointly-permanull subspaces of larger codimension looks like a very interesting problem, and seems relevant to extending the range of k in that theorem.

In this section and the next, we make some small steps in this direction. We begin with a very simple proof showing that for fields of characteristic 0 or $> n$, the only permanull subspaces of $\mathbb{F}^n$ are the trivial ones, i.e. subspaces of $\{\mathbf{e}_i\}^\perp$ for some i (Theorem 5.6).

With a more refined argument, we show how to do better for permanull subspaces of bounded codimension; we prove that this characterization continues to hold with the weaker hypothesis $\operatorname{char} \mathbb{F} > k + 1$ where k is the codimension of the permanull subspace, for any ambient dimension n (Theorem 5.8). This generalizes Theorem 4.1, the characterization for the $k = 1$ case, where the only hypothesis was that $\operatorname{char} \mathbb{F} > 2$.

Once $\operatorname{char} \mathbb{F} \leq k+1$, there can be nontrivial permanull spaces of codimension k. In Section 6 we go beyond the $\operatorname{char} \mathbb{F} > k+1$ restriction and give a characterization of permanull subspaces in terms of the vanishing of the *permanental polynomial* (Theorem 6.7). This also allows gives a computational criterion for determining whether a given subspace is permanull. We conclude the paper by showing how to use this polynomial to give an alternative, simplified proof of Theorem 5.8.

5.1. **A simple proof for** $\operatorname{char} \mathbb{F} > n$. We start with the following observation, which is an immediate consequence of the definition of permanull subspaces.

Lemma 5.1. *Any subspace of a permanull subspace is a permanull subspace.*

Proof. Immediate from the definition of permanull subspaces. $\square$

The necessity of some assumption on the characteristic is apparent in the proof of the following easy lemma.

Lemma 5.2. *Let $V \leq \mathbb{F}^n$ be a permanull subspace of dimension 1. If $\mathbb{F}$ is of characteristic 0 or $\operatorname{char} \mathbb{F} > n$, then $V \subseteq \{\mathbf{e}_i\}^\perp$ for some $1 \leq i \leq n$.*

Proof. Any vector $\mathbf{v}$ all of whose coordinates are nonzero induces a matrix

$$\begin{pmatrix} \mathbf{v} & \mathbf{v} & \cdots & \mathbf{v} \end{pmatrix}$$

with nonzero permanent. Indeed, if $\mathbf{v} = (v_1, \ldots, v_n)$, then by multilinearity

$$\mathrm{per} \begin{pmatrix} \mathbf{v} & \mathbf{v} & \cdots & \mathbf{v} \end{pmatrix} = v_1 \cdots v_n \cdot \mathrm{per} \begin{pmatrix} \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} \end{pmatrix} = v_1 \cdots v_n \cdot n!$$

(where $\mathbf{1}$ is the vector all of whose entries are 1). Since $v_1, \ldots, v_n$ are nonzero, this product is nonzero as long as $n! \neq 0$, i.e. as long as $\mathrm{char}\,\mathbb{F} = 0$ or $\mathrm{char}\,\mathbb{F} \geq n + 1$. $\square$

Corollary 5.3. *Let $V \leq \mathbb{F}^n$ be a permanull subspace. If $\mathbb{F}$ is of characteristic 0 or $\mathrm{char}\,\mathbb{F} > n$, then every $\mathbf{v} \in V$ has at least one zero coordinate.*

Proof. By Lemma 5.1, $\mathrm{span}\{\mathbf{v}\}$ is a permanull subspace so by Lemma 5.2 it must lie within $\{\mathbf{e}_i\}^{\perp}$ for some $1 \leq i \leq n$. $\square$

Using this corollary, we may prove directly that for fields of characteristic zero (or sufficiently large characteristic) the only permanull spaces are the trivial ones.

Theorem 5.4. *Let $V \leq \mathbb{F}^n$ be a permanull space. If $\mathbb{F}$ is of characteristic 0 then $V \subseteq \{\mathbf{e}_i\}^{\perp}$ for some $1 \leq i \leq n$.*

Proof. Assume for contradiction that $V \not\subseteq \{\mathbf{e}_i\}^{\perp}$ for any $1 \leq i \leq n$. Fix some basis $\mathbf{v}_1, \ldots, \mathbf{v}_\ell$ for V; then for any coordinate $1 \leq i \leq n$ at least one of these basis vectors has a nonzero i-th coordinate. Consider the n polynomials defined by the n coordinates of the vector

$$\mathbf{v}_1 + x\mathbf{v}_2 + x^2\mathbf{v}_3 + \cdots + x^{n-1}\mathbf{v}_n.$$

Let Z be the set of field elements for which at least one of the above polynomials vanishes. Since each polynomial equation in this system has at most $n - 1$ roots in $\mathbb{F}$, we have $|Z| \leq n(n-1)$. Therefore, if $|\mathbb{F}| > n(n-1)$ there is some x for which none of these polynomials vanishes, i.e.,

$$\mathbf{v}_1 + x\mathbf{v}_2 + \cdots + x^{n-1}\mathbf{v}_n$$

is a vector of the permanull subspace V none of whose entries is 0, contradicting Corollary 5.3. $\square$

Let us focus again on finite fields. The preceding proof shows that if $\mathrm{char}\,\mathbb{F} > n$ (a necessary hypothesis for Corollary 5.3) and $|\mathbb{F}| > n(n-1)$, then any permanull subspace of $\mathbb{F}^n$ is trivial. We can circumvent the restriction on the field size $|\mathbb{F}|$ via a straightforward application of the probabilistic method. We make use of the following simple observation.

Lemma 5.5. *Let $\mathbb{F}$ be a finite field and let $x_1, \ldots, x_\ell \in \mathbb{F}$ be some fixed field elements, at least one of which is nonzero. If $\alpha_1, \ldots, \alpha_\ell \in \mathbb{F}$ are selected uniformly at random, then*

$$x := \alpha_1 x_1 + \cdots + \alpha_\ell x_\ell$$

is a uniformly random field element; that is, for every $\alpha \in \mathbb{F}$, $\Pr[x = \alpha] = 1/|\mathbb{F}|$.

Proof. Let d be the number of nonzero elements among $x_1, \ldots, x_\ell$. Then

$$\left\{ (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{F}^\ell : \sum_{i=1}^{\ell} \alpha_i x_i = 0 \right\}$$

is an $(\ell - d)$-dimensional linear subspace of $\mathbb{F}^\ell$. Moreover, for any $\alpha \in \mathbb{F}$

$$\left\{ (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{F}^\ell : \sum_{i=1}^{\ell} \alpha_i x_i = \alpha \right\}$$

is a coset of that subspace. In particular, all of these sets have the same size. $\hfill\square$

Theorem 5.6. *Let $V \leq \mathbb{F}^n$ be a permanull space. If $\operatorname{char} \mathbb{F} > n$, then $V \subseteq \{\mathbf{e}_i\}^\perp$ for some $1 \leq i \leq n$.*

Proof. Assume for contradiction that $V \not\subseteq \{\mathbf{e}_i\}^\perp$ for any $1 \leq i \leq n$. Fix some basis $\mathbf{v}_1, \ldots, \mathbf{v}_\ell$ for V; then for any coordinate $1 \leq i \leq n$ at least one of these basis vectors has a nonzero i-th coordinate. If the coefficients of the linear combination

$$\mathbf{v} := \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$$

are chosen uniformly at random, the i-th coordinate of $\mathbf{v}$ is also uniformly random; in particular, it is 0 with probability $1/|\mathbb{F}|$. Using the union bound, the probability that at least one coordinate of $\mathbf{v}$ is 0 is at most $n/|\mathbb{F}|$. Since $|\mathbb{F}| \geq \operatorname{char} \mathbb{F} > n$, there exists a linear combination such that all of the coordinates of $\mathbf{v}$ are nonzero, contradicting Corollary 5.3. $\hfill\square$

5.2. Improved results for bounded codimension.

We have just seen that under some circumstances, the only permanull spaces are trivial. The proof of Lemma 5.2 shows that $\operatorname{char} \mathbb{F} > n$ is a necessary condition for permanull subspaces of dimension 1. On the other hand, Theorem 4.1 on permanull subspaces of codimension 1 required only that the characteristic is > 2, independent of n. This suggests that "sufficiently large" characteristic should depend on the codimension $n - \dim V$ rather than on the ambient dimension n alone. Indeed, these two extremes of codimension 1 and codimension $n - 1$ suggest that a suitable bound would be $\operatorname{char} \mathbb{F} > (n - \dim V) + 1$. Our goal for the remainder of the section is to prove such a bound.

The proof proceeds by induction on the ambient dimension n. The base case of $n = 1$ is trivial, whereas for $n = 2$, any nontrivial subspace is of codimension 1, and so trivial by Theorem 4.1. For the inductive proof, we shall make use of the following claim, where we show that Option (ii) below never holds!

Claim 5.7. *Suppose some $n \geq 1$ has the following property: for every permanull subspace $U \leq \mathbb{F}^n$, if $\operatorname{char} \mathbb{F} > \operatorname{codim} U + 1$, then U is trivial (i.e., a subspace of $\{\mathbf{e}_i\}^\perp$ for some $1 \leq i \leq n$). Then every permanull subspace $V \leq \mathbb{F}^{n+1}$ has the following property: if $\operatorname{char} \mathbb{F} > \operatorname{codim} V + 1$ then either:*

(i) there exists some index $1 \leq i \leq n+1$ such that $V \subseteq \{\mathbf{e}_i\}^\perp$; or
(ii) for every index $1 \leq i \leq n+1$ we have $\dim(V \cap \{\mathbf{e}_i\}^\perp) = \dim V - 1$. Moreover, there exists a different index $1 \leq j \leq n+1$ (with $j \neq i$) such that

$$V \cap \{\mathbf{e}_i\}^\perp = V \cap \{\mathbf{e}_j\}^\perp.$$

Proof. Suppose $V \subseteq \mathbb{F}^{n+1}$ is a permanull codimension-k subspace. Note that if $k = 1$, then V is trivial by Theorem 4.1. Similarly, if $k = n$, then the conclusion is satisfied by Lemma 5.2 above. We may therefore assume that $1 < k < n$. In particular, $d := \dim V = n + 1 - k > 1$.

Suppose Option (i) does not hold, i.e., for each index $1 \leq i \leq n + 1$ there exists some $\mathbf{v} \in V$ whose i-th coordinate is nonzero. For convenience, we denote $V_i := V \cap \{\mathbf{e}_i\}^\perp$.

First, note that since $V \not\subseteq \{\mathbf{e}_i\}^\perp$, we have $V + \{\mathbf{e}_i\}^\perp = \mathbb{F}^{n+1}$. Therefore, $\dim V_i = \dim V - 1$.

Next, let us fix some index $1 \leq i \leq n + 1$. Let $\mathbf{v}_1 \in V$ be a vector whose i-th coordinate is nonzero, and by replacing it with a scalar multiple if necessary, we may assume that the i-th coordinate is 1. Using linear combinations, we may complete $\mathbf{v}_1$ to a basis of V by choosing vectors $\mathbf{v}_2, \ldots, \mathbf{v}_d$ whose i-th coordinate is 0. Let $\widehat{\mathbf{v}}_2, \ldots, \widehat{\mathbf{v}}_d \in \mathbb{F}^n$ be the vectors obtained from $\mathbf{v}_2, \ldots, \mathbf{v}_d$ (respectively) by deleting the i-th coordinate. We claim that $\widehat{V} := \mathrm{span}\{\widehat{\mathbf{v}}_2, \ldots, \widehat{\mathbf{v}}_d\}$ is a permanull subspace of $\mathbb{F}^n$. This can be seen as follows: any $\widehat{\mathbf{u}} \in \mathrm{span}\{\widehat{\mathbf{v}}_2, \ldots, \widehat{\mathbf{v}}_d\} \subseteq \mathbb{F}^n$ is a linear combination of the basis elements

$$\widehat{\mathbf{u}} = \sum_{\ell=2}^{d} \alpha_\ell \widehat{\mathbf{v}}_\ell.$$

By introducing a new zero coordinate, we obtain the corresponding vector $\mathbf{u} \in \mathrm{span}\{\mathbf{v}_2, \ldots, \mathbf{v}_d\} \subseteq \mathbb{F}^{n+1}$:

$$\mathbf{u} = \sum_{\ell=2}^{d} \alpha_\ell \mathbf{v}_\ell.$$

If $\widehat{\mathbf{u}}_2, \ldots, \widehat{\mathbf{u}}_{n+1} \in \mathrm{span}\{\widehat{\mathbf{v}}_2, \ldots, \widehat{\mathbf{v}}_d\}$ then

$$\mathrm{per} \begin{pmatrix} \mathbf{v}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_{n+1} \end{pmatrix} = \mathrm{per} \begin{pmatrix} \widehat{\mathbf{u}}_2 & \cdots & \widehat{\mathbf{u}}_{n+1} \end{pmatrix}$$

as can immediately be seen by expanding the $(n + 1) \times (n + 1)$ matrix on the left by its i-th row. Since $V = \mathrm{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ is assumed to be a permanull subspace, so must be $\widehat{V} = \mathrm{span}\{\widehat{\mathbf{v}}_2, \ldots, \widehat{\mathbf{v}}_{k+1}\}$.

Since $\widehat{V}$ is a codimension-k permanull subspace of $\mathbb{F}^n$, the hypothesis of the claim applies: if $\mathrm{char}\,\mathbb{F} > k + 1$ there is some index $1 \leq j' \leq n$ such the j'-th coordinate of $\widehat{\mathbf{v}}_2, \ldots, \widehat{\mathbf{v}}_d$ is 0. This means there is some index $1 \leq j \leq n$ different from i such that the j-th index of $\mathbf{v}_2, \ldots, \mathbf{v}_d$ is zero[4]. Now, $\dim V_i = \dim V_j = d - 1$, and $\mathbf{v}_2, \ldots, \mathbf{v}_d$ is a list of linearly independent vectors belonging to each of these spaces. Therefore,

$$V \cap \{\mathbf{e}_i\}^\perp = V \cap \{\mathbf{e}_j\}^\perp$$

so that Option (ii) holds for V, as claimed. □

We are now ready to prove the relaxation on the characteristic.

Theorem 5.8. *Let $V \leq \mathbb{F}^n$ be a permanull subspace of codimension k. If $\mathrm{char}\,\mathbb{F} > k + 1$, then $V \subseteq \{\mathbf{e}_i\}^\perp$ for some $1 \leq i \leq n$.*

Proof. We proceed by induction on the ambient dimension n. The cases $n = 1, 2$ are straightforward. Suppose the theorem holds for all dimensions up to and including some fixed n, and we shall prove it for $n + 1$. Towards that end, let $V \leq \mathbb{F}^{n+1}$ be a permanull subspace of codimension k and suppose that $\mathrm{char}\,\mathbb{F} > k + 1$. Assume for contradiction that V is not trivial; in particular, invoking Theorem 4.1 and Lemma 5.2 we have $1 < k < n$.

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[4]More precisely, if $j' \leq i$ then $j = j' + 1$ and otherwise $j' = j$.

Claim 5.7 applies with Option (ii) guaranteeing that for every index $1 \le i \le n+1$ we have a different index $1 \le j \le n+1$ (with $j \ne i$) such that
$$V \cap \{\mathbf{e}_i\}^\perp = V \cap \{\mathbf{e}_j\}^\perp.$$
Let I denote the set of indices which vanish together with the first:
$$I := \left\{ 1 \le j \le n+1 \ : \ V \cap \{\mathbf{e}_1\}^\perp = V \cap \{\mathbf{e}_j\}^\perp \right\}.$$
Let $\iota := |I|$. Without loss of generality, we may assume that I consists of the first ι indices. For convenience, let us denote $V' := V \cap \{\mathbf{e}_1\}^\perp$. From Option (ii) we know that $\iota \ge 2$ and $\dim V' = \dim V - 1 = n - k$. Moreover, since all vectors in this $(n-k)$-dimensional subspace $V' \le \mathbb{F}^{n+1}$ have their first ι entries 0, we must have $\iota \le k+1$. We shall now derive a contradiction by showing that $|I| > \iota$.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_\ell$ be a basis for V' (so $\ell = n - k$). Let $\mathbf{u}$ complete this list to a basis for V. Since $\mathbf{u} \notin V'$, for each $i \in I$ the i-th coordinate of $\mathbf{u}$ is nonzero; for otherwise $\mathbf{u} \in V \cap \{\mathbf{e}_i\}^\perp = V'$.

For any $\mathbf{w} \in \mathbb{F}^{n+1}$, let $\widehat{\mathbf{w}} \in \mathbb{F}^{n+1-\iota}$ denote the projection of $\mathbf{w}$ on the last $n+1-\iota$ coordinates. We claim that $\mathrm{span}\{\widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_\ell\} \subseteq \mathbb{F}^{n+1-\iota}$ is a permanull subspace. To see this, consider the embedding sending any vector $\widehat{\mathbf{w}} \in \mathbb{F}^{n+1-\iota}$ to the vector $\mathbf{w} \in \mathbb{F}^{n+1}$ obtained from $\widehat{\mathbf{w}}$ by inserting ι zeroes as the first ι coordinates (and then the entries of $\widehat{\mathbf{w}}$ for the remaining coordinates). If $\widehat{\mathbf{u}}_1, \ldots, \widehat{\mathbf{u}}_{n+1-\iota} \in \mathrm{span}\{\widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_\ell\}$ is any collection of vectors, then the $(n+1) \times (n+1)$ matrix
$$\begin{pmatrix} \mathbf{u} & \cdots & \mathbf{u} & \mathbf{u}_1 & \cdots & \mathbf{u}_{n+1-\iota} \end{pmatrix}$$
has columns in V and therefore must have zero permanent. On the other hand, this is a block diagonal matrix; if $\mathbf{u} = (u_1, \ldots, u_{n+1})$ then the permanent is simply
$$\mathrm{per} \begin{pmatrix} \mathbf{u} & \cdots & \mathbf{u} & \mathbf{u}_1 & \cdots & \mathbf{u}_{n+1-i} \end{pmatrix} = u_1 \cdots u_i \cdot i! \cdot \mathrm{per} \begin{pmatrix} \widehat{\mathbf{u}}_1 & \cdots & \widehat{\mathbf{u}}_{n+1-\iota} \end{pmatrix}.$$
Since by assumption $\mathrm{char}\, \mathbb{F} > k+1 \ge \iota$, we see that $\iota! \ne 0$ so that
$$\mathrm{per} \begin{pmatrix} \widehat{\mathbf{u}}_1 & \cdots & \widehat{\mathbf{u}}_{n+1-\iota} \end{pmatrix} = 0,$$
proving that $\mathrm{span}\{\widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_\ell\} \subseteq \mathbb{F}^{n+1-\iota}$ is a permanull subspace.

Since $\iota \ge 1$, we may invoke the induction hypothesis to conclude that $\mathrm{span}\{\widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_\ell\} \subseteq \mathbb{F}^{n+1-\iota}$ must be trivial, so there exists some $1 \le j \le n+1-\iota$ such that $\mathrm{span}\{\widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_\ell\} \subseteq \{\mathbf{e}_j\}^\perp \subseteq \mathbb{F}^{n+1-\iota}$. It follows that
$$V' = \mathrm{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_\ell\} \subseteq \{\mathbf{e}_{j+\iota}\}^\perp \subseteq \mathbb{F}^{n+1}$$
and therefore that $j + \iota \in I$ (in addition to the first ι indices), contradicting the fact that $|I| = \iota$. $\qquad\square$

6. PERMANULL SUBSPACES IN ARBITRARY CHARACTERISTIC

The goal of this section is to provide a complete characterization of permanull subspaces, regardless of the characteristic of the field, in terms of an auxiliary polynomial which we call the *permanental polynomial* (Theorem 6.7). This allows us to check if a given codimension-k subspace is permanull with $O_k(n^k)$ field operations, and a general subspace with $2^{O(n)}$ field operations. In contrast, checking permanullness directly by fixing a basis $B = \{b_1, \ldots, b_{n-k}\}$ and checking the vanishing of all permanents of matrices with columns from B takes $O((n-k)^n)$ time.

In Section 5 we have found that the only permanull codimension-k subspaces of $\mathbb{F}^n$ are trivial, provided that $\operatorname{char} \mathbb{F} > k + 1$ (Theorem 5.8). Without the assumption on the characteristic, there are other possible permanull subspaces, as the following example shows.

Example 6.1 (Nontrivial permanull subspace). The codimension-2 subspace $S \le \mathbb{F}_3^4$ spanned by

$$\mathbf{v}_1 = \begin{pmatrix} 1 & 0 & 1 & 1 \end{pmatrix}^T \qquad\qquad \mathbf{v}_2 = \begin{pmatrix} 0 & 1 & 1 & -1 \end{pmatrix}^T$$

is a permanull subspace which is not contained in any permanull subspace of codimension-1 (i.e., S is non-trivial). Note that over $\mathbb{F}_{p^\ell}^4$ with $p > 3$, S is *not* a permanull subspace, as guaranteed by Theorem 5.8; one way of seeing this is to compute, for example,

$$\operatorname{per} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix} = 6 \ne 0.$$

In order to characterize permanull subspaces in this general setting, we introduce an auxiliary algebraic object: the *permanental polynomial.* To motivate the definition, let us start with codimension-2 permanull subspaces and reprove that for fields of characteristic $p > 3$, the only codimension-2 permanull subspaces are the trivial ones; namely, subspaces of $\{\mathbf{e}_i\}^\perp$.

Suppose S is a codimension-2 permanull subspace. Permuting coordinates if necessary (note that the permanent is invariant under permutation of columns), we may fix a basis $\mathbf{v}_1, \ldots, \mathbf{v}_{n-2}$ for S where

$$\mathbf{v}_i = (\mathbf{e}_i^{n-2}, a_i, b_i)$$

for some scalars a_i, b_i.

Since S is permanull, for any $\mathbf{u}, \mathbf{w} \in S$, we have

$$\operatorname{per} \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{n-2} & \mathbf{u} & \mathbf{w} \end{pmatrix} = 0.$$

Expressing $\mathbf{u}, \mathbf{w}$ as linear combinations of the basis vectors:

$$\mathbf{u} = x_1 \mathbf{v}_1 + \cdots + x_{n-2} \mathbf{v}_{n-2},$$
$$\mathbf{w} = y_1 \mathbf{v}_1 + \cdots + y_{n-2} \mathbf{v}_{n-2},$$

we may rewrite the matrix above as

$$\begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_{n-2} & \mathbf{u} & \mathbf{w} \end{pmatrix} = \begin{pmatrix} & & & & & x_1 & y_1 \\ & & & & & x_2 & y_2 \\ & & I_{n-2} & & & x_3 & y_3 \\ & & & & & \vdots & \vdots \\ & & & & & x_{n-2} & y_{n-2} \\ a_1 & a_2 & a_3 & \cdots & a_{n-2} & \mathbf{a}^T\mathbf{x} & \mathbf{a}^T\mathbf{y} \\ b_1 & b_2 & b_3 & \cdots & b_{n-2} & \mathbf{b}^T\mathbf{x} & \mathbf{b}^T\mathbf{y} \end{pmatrix}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{x}, \mathbf{y} \in \mathbb{F}^{n-2}$ are the vectors with corresponding coordinates a_i, b_i, x_i, y_i for $1 \le i \le n-2$. In order to understand the spaces S with basis $\mathbf{v}_1, \ldots, \mathbf{v}_{n-2}$, we seek conditions on $\mathbf{a}, \mathbf{b}$ which are consistent with matrices of this form having a vanishing permanent. The vectors $\mathbf{x}, \mathbf{y}$ represent the coefficients of possible linear combinations of basis vectors and are therefore allowed to vary freely. That is, we treat the permanent of the matrix above as a

polynomial in $\mathbf{x}, \mathbf{y}$ with coefficients $\mathbf{a}, \mathbf{b}$. As long as this polynomial is not identically 0, it is possible to find some values of $\mathbf{x}, \mathbf{y}$ which result in a nonzero permanent, implying that S is not permanull.

Using the definition of the permanent as a sum over permutations, we obtain

$$\operatorname{per} \begin{pmatrix} \mathbf{a}^T \mathbf{x} & \mathbf{a}^T \mathbf{y} \\ \mathbf{b}^T \mathbf{x} & \mathbf{b}^T \mathbf{y} \end{pmatrix} + \sum_{1 \leq i \leq n-2} x_i \operatorname{per} \begin{pmatrix} a_i & \mathbf{a}^T \mathbf{y} \\ b_i & \mathbf{b}^T \mathbf{y} \end{pmatrix} + y_i \operatorname{per} \begin{pmatrix} a_i & \mathbf{a}^T \mathbf{x} \\ b_i & \mathbf{b}^T \mathbf{x} \end{pmatrix} +$$
$$+ \sum_{\substack{1 \leq i,j \leq n-2 \\ i \neq j}} x_i y_j \operatorname{per} \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}$$

and after collecting like terms we obtain

$$(\star) \qquad\qquad 6 \sum_{1 \leq i \leq n-2} x_i y_i a_i b_i + 4 \sum_{\substack{1 \leq i,j \leq n-2 \\ i \neq j}} x_i y_j \operatorname{per} \begin{pmatrix} a_i & a_j \\ b_i & b_j \end{pmatrix}.$$

For this to be the 0 polynomial, we must have $\operatorname{Per}_{(n-2) \times 2} \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} = 0$ (recall that this notation means that every 2×2 subpermanent vanishes). If the characteristic is $p = 3$, the first term of $(\star)$ vanishes and this is the only condition, so the values $\mathbf{a}, \mathbf{b}$ are completely determined by the conditions in Lemma 4.2.

On the other hand, if the characteristic is $p > 3$, for the first term of $(\star)$ to vanish we must also have $0 \in \{a_i, b_i\}$ for each $1 \leq i \leq n - 2$. This means that conditions (ii) and (iii) of Lemma 4.2 cannot be met, so that condition (i) must be satisfied, i.e. $\mathbf{a} = 0$ or $\mathbf{b} = 0$. In the former case, $S \leq \{\mathbf{e}_3\}^\perp$ and in the latter $S \leq \{\mathbf{e}_4\}^\perp$. This reproves the conclusion of Theorem 5.8 for codimension-2 subspaces.

Example 6.2 (Example 6.1 revisited). Note that the basis vectors $\mathbf{v}_1, \mathbf{v}_2$ for $S \leq \mathbb{F}_3^4$,

$$\mathbf{v}_1 = \begin{pmatrix} 1 & 0 & 1 & 1 \end{pmatrix}^T \qquad\qquad \mathbf{v}_2 = \begin{pmatrix} 0 & 1 & 1 & -1 \end{pmatrix}^T$$

are in "standard form" with the first two coordinates $\mathbf{e}_1^2$ and $\mathbf{e}_2^2$ respectively. The last two coordinates satisfy the conditions of Lemma 4.2, as the third row of $\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}$ is of the form $(1, x)$ for $0 \neq x \in \mathbb{F}$ and the fourth row is of the form $(1, -x)$.

One advantage of this proof over the one in Section 5 is that it also shows us what "goes wrong" if the characteristic is 3. This will lead us to a full characterization of permanull subspaces in any characteristic via the vanishing of the *permanental polynomial*, a generalization of the polynomial $(\star)$.

For ease of computation, it will be convenient to introduce some notation; we continue to write vectors as column vectors. Fix some positive integers $d < k$. Because we shall need subscripts to distinguish between vectors, we write the components of a subscripted vector $\mathbf{a}_i \in \mathbb{F}^k$ as $\mathbf{a}_i = (a_{i,1}, a_{i,2}, \ldots, a_{i,k})^T$. As with the proof of the codimension-2 case, let $\mathbf{x}_1, \ldots, \mathbf{x}_d \in \mathbb{F}^k$ be some coefficient vectors and $\mathbf{a}_1, \ldots, \mathbf{a}_d \in \mathbb{F}^k$ variable vectors. The *permanental polynomial* corresponding to this basis is the permanent of the $(k+d) \times (k+d)$ matrix M below:

$$P := \operatorname{per} M := \operatorname{per} \begin{pmatrix} I_k & \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_d \\ \mathbf{a}_1^T & \mathbf{a}_1^T\mathbf{x}_1 & \mathbf{a}_1^T\mathbf{x}_2 & \cdots & \mathbf{a}_1^T\mathbf{x}_d \\ \mathbf{a}_2^T & \mathbf{a}_2^T\mathbf{x}_1 & \mathbf{a}_2^T\mathbf{x}_2 & \cdots & \mathbf{a}_2^T\mathbf{x}_d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_d^T & \mathbf{a}_d^T\mathbf{x}_1 & \mathbf{a}_d^T\mathbf{x}_2 & \cdots & \mathbf{a}_d^T\mathbf{x}_d \end{pmatrix}$$

viewed as a polynomial in the variable vectors $\mathbf{x}_1, \ldots, \mathbf{x}_d$ (i.e. in the dk variables $x_{1,1}, \ldots, x_{d,k}$); note that it is a multilinear homogeneous polynomial of degree d, with each monomial containing at most one variable $x_{i,j}$ for each $1 \leq i \leq d$, and is also "block-symmetric" with respect to the variables $\mathbf{x}_1, \ldots, \mathbf{x}_d$. We next introduce notation for the monomials of P.

We shall use Greek letters for ordered tuples. Given some tuple $\alpha \in [k]^d$ with components $\alpha = (\alpha_1, \ldots, \alpha_d)$, we denote by $\mathbf{x}^\alpha$ the monomial $x_{1,\alpha_1} x_{2,\alpha_2} \cdots x_{d,\alpha_d}$.

Lemma 6.3. *The coefficient of $\mathbf{x}^\alpha$ in P is the same as the coefficient of $\mathbf{x}^\alpha$ in*

$$P^\alpha := \operatorname{per} M^\alpha := \operatorname{per} \begin{pmatrix} I_k & \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_d \\ \mathbf{a}_1^T & a_{1,\alpha_1}x_{1,\alpha_1} & a_{1,\alpha_2}x_{2,\alpha_2} & \cdots & a_{1,\alpha_d}x_{d,\alpha_d} \\ \mathbf{a}_2^T & a_{2,\alpha_1}x_{1,\alpha_1} & a_{2,\alpha_2}x_{2,\alpha_2} & \cdots & a_{2,\alpha_d}x_{d,\alpha_d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_d^T & a_{d,\alpha_1}x_{1,\alpha_1} & a_{d,\alpha_2}x_{2,\alpha_2} & \cdots & a_{d,\alpha_d}x_{d,\alpha_d} \end{pmatrix}$$

Proof. This is clear by examining the generalized diagonals: fix some $\sigma \in S_{k+d}$; the only difference between M and M^α is the bottom right $d \times d$ submatrix, so suffice to examine those terms. Assume therefore that $M_{i,\sigma_i} = \mathbf{a}_i^T\mathbf{x}_{\sigma(i)} = \sum_{j=1}^d a_{i,j}x_{\sigma(i),j}$, after expanding the sums the only summand which contributes to $\mathbf{x}^\alpha$ is when $j = \alpha_{\sigma(i)}$, i.e. $a_{i,\alpha_{\sigma(i)}}x_{\sigma(i),\alpha_{\sigma(i)}} = M_{i,\sigma_i}^\alpha$. $\square$

To determine P, we need to compute the coefficient of $\mathbf{x}^\alpha$ for every $\alpha \in [k]^d$. This coefficient turns out to be a multiple of

$$A^\alpha := \operatorname{per} \begin{pmatrix} \mathbf{a}_1^\alpha & \mathbf{a}_2^\alpha & \cdots & \mathbf{a}_d^\alpha \end{pmatrix}.$$

The exact multiple depends on the choice of the tuple $\alpha \in [k]^d$ in the following manner: viewing α as a function $\alpha : [d] \to [k]$ assigning $i \in d$ to $\alpha_i \in [k]$, we obtain a partition of $[d]$ with blocks $\alpha^{-1}(j)$ (for $j \in \operatorname{img}(\alpha)$). Let n_α denote the number of ways of choosing at most one element from each block; if we denote the size of each block by $n_j := |\alpha^{-1}(j)|$, then

$$n_\alpha = \prod_{j \in \operatorname{img}(\alpha)} (n_j + 1).$$

For proving this claim, we introduce some helpful concepts. By a *partial configuration from n_α* we mean a selection of at most one representative from each block $\alpha^{-1}(j)$ (for each $j \in \operatorname{img}(\alpha)$). (Thus there are precisely n_α many partial configurations.) The *weight* of the partial configuration is the total number of representatives chosen. We say that a permutation $\tau \in S_{k+d}$ *corresponds* to a partial configuration exactly when: the element $i \in \alpha^{-1}(j)$ was selected as part of the partial configuration if and only if $\tau(\alpha_i) = k+i$. Note that every permutation corresponds to some partial configuration (possibly the empty one).

Theorem 6.4. *The coefficient of $\mathbf{x}^\alpha$ in P is $n_\alpha A^\alpha$.*

Proof. Fix some $\alpha \in [k]^d$. By Lemma 6.3, suffice it to determine the coefficient of $\mathbf{x}^\alpha$ in P^α. By the definition of the permanent, $P^\alpha = \sum_{\sigma \in S_{k+d}} \prod_{i=1}^{k+d} M^\alpha_{i,\sigma(i)}$. For the variable x_{i,α_i} to appear in the term $\prod_{i=1}^{k+d} M^\alpha_{i,\sigma(i)}$ exactly one of the following two possibilities must be satisfied:

- either $k + i = \sigma(\alpha_i)$; or
- there exists some $k + 1 \leq j \leq k + d$ such that $k + i = \sigma(j)$.

Call a permutation *valid* if it satisfies exactly one of these conditions for each $1 \leq i \leq d$, and let S^α_{k+d} be the set of valid permutations.

Fix some partial configuration of n_α. We shall prove that the sum over all valid permutations $\sigma \in S^\alpha_{k+d}$ corresponding to this partial configuration is precisely

$$\mathbf{x}^\alpha A^\alpha = \mathbf{x}^\alpha \operatorname{per} \begin{pmatrix} \mathbf{a}_1^\alpha & \mathbf{a}_2^\alpha & \cdots & \mathbf{a}_d^\alpha \end{pmatrix}.$$

Since every permutation corresponds to some partial configuration and there are n_α partial configurations, this immediately implies that the coefficient of $\mathbf{x}^\alpha$ in P is precisely $n_\alpha A^\alpha$, as we wanted to show.

To prove the claim, suppose the partial configuration is of weight ℓ and assume without loss of generality that the partial configuration is such that $1, \ldots, \ell$ are the chosen representatives (the permanent is invariant under permutation of columns) so any corresponding permutation must have $\sigma(\alpha_1) = k + 1$, $\sigma(\alpha_2) = k + 2$, ..., $\sigma(\alpha_\ell) = k + \ell$. This gives us $x_{1,\alpha_1} \cdots x_{\ell,\alpha_\ell}$, and we must determine the remainder of σ.

We claim that for $1 \leq r \leq k$ such that $r \notin \{\alpha_1, \ldots, \alpha_\ell\}$, we must have $\sigma(r) = r$. To prove this, suffice it to show that $1 \leq \sigma(r) \leq k$, for then $M^\alpha_{r,\sigma(r)}$ is an entry of I_k and must be 1 (i.e., $\sigma(r) = r$) lest the whole term be zero. However, if $\sigma(r) \geq k + 1$, say $k + t = \sigma(r)$, then (since σ is a valid permutation and $1 \leq r \leq k$) we must have $r = \alpha_t$ and this is impossible since $r \notin \{\alpha_1, \ldots, \alpha_\ell\}$ and σ corresponds to the partial configuration.

We have now accounted for the first k rows of M^α and also for some k columns: the columns $k + 1, \ldots, k + \ell$ and the columns $\{1, \ldots, k\} \setminus \{\alpha_1, \ldots, \alpha_\ell\}$. The remainder of σ can be chosen freely (under the constraint that the result is a permutation); thus $\prod_{j=k+1}^{k+d} M^\alpha_{j,\sigma(j)}$ is a generalized diagonal of the matrix

$$B = \begin{pmatrix} a_{1,\alpha_1} & \cdots & a_{1,\alpha_\ell} & a_{1,\alpha_{\ell+1}} x_{\ell+1,\alpha_{\ell+1}} & \cdots & a_{1,\alpha_d} x_{d,\alpha_d} \\ a_{2,\alpha_1} & \cdots & a_{2,\alpha_\ell} & a_{2,\alpha_{\ell+1}} x_{\ell+1,\alpha_{\ell+1}} & \cdots & a_{2,\alpha_d} x_{d,\alpha_d} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{d,\alpha_1} & \cdots & a_{d,\alpha_\ell} & a_{d,\alpha_{\ell+1}} x_{\ell+1,\alpha_{\ell+1}} & \cdots & a_{d,\alpha_d} x_{d,\alpha_d} \end{pmatrix}$$

and summing over the possibilities for σ we get $x_{\ell+1,\alpha_{\ell+1}} \cdots x_{d,\alpha_d} A^\alpha$. $\qquad\square$

Corollary 6.5. *For any injective $\alpha : [d] \to [k]$, the coefficient of $\mathbf{x}^\alpha$ in*

$$P = \begin{pmatrix} I_k & \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_d \\ \mathbf{a}_1^T & \mathbf{a}_1^T \mathbf{x}_1 & \mathbf{a}_1^T \mathbf{x}_2 & \cdots & \mathbf{a}_1^T \mathbf{x}_d \\ \mathbf{a}_2^T & \mathbf{a}_2^T \mathbf{x}_1 & \mathbf{a}_2^T \mathbf{x}_2 & \cdots & \mathbf{a}_2^T \mathbf{x}_d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_d^T & \mathbf{a}_d^T \mathbf{x}_1 & \mathbf{a}_d^T \mathbf{x}_2 & \cdots & \mathbf{a}_d^T \mathbf{x}_d \end{pmatrix}$$

is $2^d A^\alpha$.

Example 6.6. To illustrate Theorem 6.4, let us compute the permanental polynomial

$$P_3 := \operatorname{per} \begin{pmatrix} & & & & & x_1 & y_1 & z_1 \\ & & & & & x_2 & y_2 & z_2 \\ & & I_{n-3} & & & x_3 & y_3 & z_3 \\ & & & & & \vdots & \vdots & \vdots \\ & & & & & x_{n-3} & y_{n-3} & z_{n-3} \\ a_1 & a_2 & a_3 & \cdots & a_{n-3} & \mathbf{a}^T\mathbf{x} & \mathbf{a}^T\mathbf{y} & \mathbf{a}^T\mathbf{z} \\ b_1 & b_2 & b_3 & \cdots & b_{n-3} & \mathbf{b}^T\mathbf{x} & \mathbf{b}^T\mathbf{y} & \mathbf{b}^T\mathbf{z} \\ c_1 & c_2 & c_3 & \cdots & c_{n-3} & \mathbf{c}^T\mathbf{x} & \mathbf{c}^T\mathbf{y} & \mathbf{c}^T\mathbf{z} \end{pmatrix}$$

The monomials are of the form $x_r y_s z_t$ and there are three types of terms according to the number of distinct indices r, s, t:

$$P_3 = 4 \sum_{1 \le r \le n-3} x_r y_r z_r \operatorname{per} \begin{pmatrix} a_r & a_r & a_r \\ b_r & b_r & b_r \\ c_r & c_r & c_r \end{pmatrix}$$

$$+ 6 \sum_{\substack{1 \le r,s \le n-3 \\ r \ne s}} (x_r y_r z_s + x_r y_s z_r + x_s y_r z_r) \operatorname{per} \begin{pmatrix} a_r & a_r & a_s \\ b_r & b_r & b_s \\ c_r & c_r & c_s \end{pmatrix}$$

$$+ 8 \sum_{\substack{1 \le r,s,t \le n-3 \\ r,s,t \text{ distinct}}} x_r y_s z_t \operatorname{per} \begin{pmatrix} a_r & a_s & a_t \\ b_r & b_s & b_t \\ c_r & c_s & c_t \end{pmatrix}.$$

Suppose now that we are working over a field of characteristic $p > 3$ (and therefore of characteristic $p > 3+1$), so that the coefficient of each monomial above is 0 if and only if the permanent of the corresponding 3×3 matrix is 0. One can show, via case analysis, that one of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ must be 0. This would reprove the conclusion of Theorem 6.1 for codimension-3 subspaces; i.e. that if $S \le \mathbb{F}_q^n$ is a permanull space of codimension 3, then it is trivial. We shall not go through the case analysis, but instead derive this from a more general theorem below.

Theorem 6.7. *Let $S \le \mathbb{F}^n$ be a subspace of codimension d. Permuting the coordinates of $\mathbb{F}^n$ if necessary, fix a* standard basis *for S of the form*

$$(\mathbf{e}_1, \mathbf{a}_1), \dots, (\mathbf{e}_{n-d}, \mathbf{a}_{n-d})$$

(where $\mathbf{e}_i \in \mathbb{F}^{n-d}$ and $\mathbf{a}_i \in \mathbb{F}^d$). Let P be the corresponding $n \times n$ permanental polynomial

$$P := \operatorname{per} \begin{pmatrix} & & & x_{1,1} & \cdots & x_{d,1} \\ & I_{n-d} & & \vdots & \ddots & \vdots \\ & & & x_{1,n-d} & \cdots & x_{d,n-d} \\ a_{1,1} & \cdots & a_{1,n-d} & \mathbf{a_1}^T\mathbf{x_1} & \cdots & \mathbf{a_1}^T\mathbf{x_d} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{d,1} & \cdots & a_{d,n-d} & \mathbf{a_d}^T\mathbf{x_1} & \cdots & \mathbf{a_d}^T\mathbf{x_d} \end{pmatrix}.$$

Then S is permanull if and only if P is the zero polynomial.

Proof. Let us start with the forward direction and suppose P is identically zero. Given any n vectors $\mathbf{u}_1, \ldots, \mathbf{u_n} \in S$ let us express them as a linear combination of the basis elements $\mathbf{v}_1, \ldots, \mathbf{v}_{n-d}$

$$\mathbf{u}_i = \sum_{j=1}^{n-d} \lambda_{i,j} \mathbf{v}_j.$$

We wish to prove $\mathrm{per} \begin{pmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{pmatrix} = 0$. By the multilinearity of the permanent, suffice it to show that

$$\mathrm{per} \begin{pmatrix} \mathbf{v}_{j_1} & \cdots & \mathbf{v}_{j_n} \end{pmatrix} = 0$$

for every choice of $j_1, \ldots, j_n \in \{1, \ldots, n-d\}$ (where distinct j_k may correspond to the same index).

Note that if one of the indices $1, \ldots, n-d$ is not selected, then the resulting matrix has a row of 0 and so vanishing permanent. Suppose therefore that each of $1, \ldots, n-d$ is selected. Therefore, after permuting the columns, we obtain a permanent of the form

$$P \coloneqq \mathrm{per} \begin{pmatrix} & & & x_{1,1} & \cdots & x_{d,1} \\ & I_{n-d} & & \vdots & \ddots & \vdots \\ & & & x_{1,n-d} & \cdots & x_{d,n-d} \\ a_{1,1} & \cdots & a_{1,n-d} & \mathbf{a}_1^T \mathbf{x}_1 & \cdots & \mathbf{a}_1^T \mathbf{x}_d \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{d,1} & \cdots & a_{d,n-d} & \mathbf{a}_d^T \mathbf{x}_1 & \cdots & \mathbf{a}_d^T \mathbf{x}_d \end{pmatrix}$$

with $\mathbf{x}_1, \ldots, \mathbf{x}_d \in \{\mathbf{e}_1, \ldots, \mathbf{e}_{n-d}\}$. Since P is the zero polynomial, we are done.

For the backward direction, suppose S is permanull subspace. Then the polynomial P vanishes for each choice of the $d(n-d)$ variables $x_{1,1}, \ldots, x_{d,n-d}$. But P is multilinear and homogeneous of degree d and so must be the zero polynomial. $\qquad \square$

Corollary 6.8. *Let $S \leq \mathbb{F}^n$ be a subspace of codimension d. It is possible to determine whether S is permanull using at most $O_d(n^d)$ operations.*

Proof. Permuting the coordinates of $\mathbb{F}^n$ if necessary, fix a *standard basis* for S of the form

$$(\mathbf{e}_1, \mathbf{a}_1), \ldots, (\mathbf{e}_{n-d}, \mathbf{a}_{n-d})$$

(where $\mathbf{e}_i \in \mathbb{F}^{n-d}$ and $\mathbf{a}_i \in \mathbb{F}^d$), and let P be the corresponding $n \times n$ permanental polynomial. By Theorem 6.7, S is permanull if and only if P is the zero polynomial. By Theorem 6.4 we can verify this by computing n^d permanents of size $d \times d$. $\qquad \square$

Corollary 6.9. *Let $S \leq \mathbb{F}^n$ be a subspace of codimension d. Permuting the coordinates of $\mathbb{F}^n$ if necessary, fix a* standard basis *for S of the form*

$$\mathbf{v}_1 = (\mathbf{e}_1, \mathbf{a}_1), \ldots, \mathbf{v}_{n-d} = (\mathbf{e}_{n-d}, \mathbf{a}_{n-d})$$

(where $\mathbf{e}_i \in \mathbb{F}^{n-d}$ and $\mathbf{a}_i \in \mathbb{F}^d$). Let $\widehat{S} \coloneqq \mathrm{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_{n-d}\} \leq \mathbb{F}^d$. If $\widehat{S}$ is permanull then S is permanull.

Proof. Suppose $\widehat{S}$ is permanull, so that every $d \times d$ permanent with columns from $\widehat{S}$ vanishes. By Theorem 6.4 every monomial in the permanental polynomial corresponding to the basis $\mathbf{v}_1, \ldots, \mathbf{v}_{n-d}$ vanishes, so that by Theorem 6.7 S is permanull. $\qquad \square$

Remark. Example 6.1 shows that the sufficient condition in Corollary 6.9 is not necessary, as $\widehat{S}$ in that example is all of $\mathbb{F}^d$, which cannot be permanull.

Theorem 6.7 "explains" the characteristic bound $\operatorname{char} \mathbb{F} > k + 1$ of Theorem 5.8. We conclude the paper with another proof of this theorem.

Second proof of Theorem 5.8. The cases $n = 1, 2$ are straightforward. Suppose the theorem holds for all dimensions up to (but not including) some fixed n, and we shall prove it for n. Towards that end, let $S \leq \mathbb{F}^n$ be a permanull subspace of codimension d and suppose that $\operatorname{char} \mathbb{F} > d + 1$.

Permuting the coordinates of $\mathbb{F}^n$ if necessary, fix a standard basis for S of the form

$$\mathbf{v}_1 = (\mathbf{e}_1, \mathbf{a}_1), \ldots, \mathbf{v}_{n-d} = (\mathbf{e}_{n-d}, \mathbf{a}_{n-d})$$

(where $\mathbf{e}_i \in \mathbb{F}^{n-d}$ and $\mathbf{a}_i \in \mathbb{F}^d$). Let $\widehat{S} := \operatorname{span}\{\mathbf{a}_1, \ldots, \mathbf{a}_{n-d}\} \leq \mathbb{F}^d$. Since we may assume $d < n$ (the case $d = n$ being trivial) suffice it to show that $\widehat{S}$ is a permanull subspace of $\mathbb{F}^d$, since then inductive hypothesis implies that $\mathbf{a}_1, \ldots, \mathbf{a}_{n-d}$ share a common zero coordinate, which immediately implies that $\mathbf{v}_1, \ldots, \mathbf{v}_{n-d}$ share a common zero coordinate.

In order to show that $\widehat{S}$ is permanull, let $\mathbf{b}_1, \ldots, \mathbf{b}_d$ be arbitrary vectors of $\widehat{S}$ and write each as a linear combination of the spanning vectors

$$\mathbf{b}_i = \sum_{j=1}^{n-d} \lambda_{i,j} \mathbf{a}_j.$$

As with the proof of Theorem 5.8, in order to show $\operatorname{per} \begin{pmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_d \end{pmatrix} = 0$, by the multilinearity of the permanent, suffice it to show that each permanent of the following form vanishes

$$\operatorname{per} \begin{pmatrix} \mathbf{a}_{j_1} & \cdots & \mathbf{a}_{j_d} \end{pmatrix} = 0$$

for every choice of $j_1, \ldots, j_d \in \{1, \ldots, n-d\}$ (where distinct j_k may correspond to the same index). Since $\operatorname{char} \mathbb{F} > d + 1$ Theorem 6.4 implies that the coefficient of each monomial in the permanental polynomial corresponding to the basis $\mathbf{v}_1, \ldots, \mathbf{v}_{n-d}$ is nonzero. Since S is assumed to be permanull, Theorem 6.7 implies that this permanental polynomial is the zero polynomial, so it must be that each monomial is zero. Again, by Theorem 6.4 this means that

$$\operatorname{per} \begin{pmatrix} \mathbf{a}_{j_1} & \cdots & \mathbf{a}_{j_d} \end{pmatrix} = 0$$

for every choice of $j_1, \ldots, j_d \in \{1, \ldots, n-d\}$. $\qquad\square$

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Fatemeh Ghasemi, Department of Mathematics, University of Toronto

Email address: fatemeh.ghasemi@mail.utoronto.ca

Gal Gross, Department of Mathematics, University of Toronto

Email address: g.gross@mail.utoronto.ca

Swastik Kopparty, Department of Mathematics and Department of Computer Science, University of Toronto

Email address: swastik.kopparty@utoronto.ca