Electronic Colloquium on Computational Complexity, Report No. 213 (2025)

Improved Small Set Expansion in High Dimensional Expanders

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December 10, 2025

Abstract

Small set expansion in high dimensional expanders is of great importance, e.g., towards proving cosystolic expansion, local testability of codes and constructions of good quantum codes.

In this work we improve upon the state of the art results of small set expansion in high dimensional expanders. Our improvement is either on the expansion quality or on the size of sets for which expansion is guaranteed.

One line of previous works [KM22, DD24] has obtained weak expansion for small sets, which is sufficient for deducing cosystolic expansion of one dimension below. We improve upon their result by showing strong expansion for small sets.

Another line of works [KKL14, EK16, KM21] has shown strong expansion for small sets. However, they obtain it only for very small sets. We get an exponential improvement on the size of sets for which expansion is guaranteed by these prior works.

Interestingly, our result is obtained by bridging between these two lines of works. The works of [KM22, DD24] use global averaging operators in order to obtain expansion for larger sets. However, their method could be utilized only on sets that are cocycle-like. We show how to combine these global averaging operators with ideas from the so-called "fat machinery" of [KKL14, EK16, KM21] in order to apply them for general sets.

1 Introduction

High dimensional expansion. High dimensional expansion is a generalization of expansion in graphs to high dimensional objects called simplicial complexes. A d-dimensional simplicial complex is a hypergraph with hyperedges of size $d+1$ which is downwards closed, i.e., if σ is a hyperedge and $\tau \subset \sigma$ then τ is also a hyperedge. A hyperedge of size $k+1$ is called a k-face of the complex.

There are two main notions of high dimensional expansion, which generalize expansion in graphs into higher dimensions. The notion of local spectral expansion is considered with the spectral expansion of the underlying graphs of each local piece of the complex. These local pieces are called links, where the link of a face $\sigma \in X$ is the subcomplex $X_\sigma = \{\tau \setminus \sigma \mid \sigma \subset \tau \in X\}$ obtained by taking all faces $\tau \supset \sigma$ and removing σ from them.

The high dimensional analog of edge expansion in graphs is called coboundary or cosystolic expansion, and it can be described as follows. Consider a k-cochain $f : X(k) \to \mathbb{F}_2$, which is a $\{0,1\}$-assignment to the k-faces of the complex. The coboundary of f is a $\{0,1\}$-assignment to the $(k+1)$-faces defined by $\delta f(\sigma) = \sum_{v \in \sigma} f(\sigma \setminus \{v\})$. The coboundary of f can be thought of as a set of unsatisfied equations, where each $(k+1)$-face corresponds to an equation that the sum

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ISSN 1433-8092

of its sub-k-faces equals 0. A k-cochain that satisfies all the equations is called a k-cocycle, and a k-cochain that is obtained by the coboundary of a $(k-1)$-cochain is called a k-coboundary.

Coboundary and cosystolic expansion[1] deal with the proportion between the fraction of unsatisfied equations and the distance of f from a satisfying assignment. More formally, a complex is said to be a β-cosystolic expander if for any cochain f

$$\|\delta f\| \geq \beta \min_{g\,:\,\delta(g)=0} \|f-g\|,$$

where $\|f\|$ is the normalized hamming weight of f.

When X is a graph and f is a function on the vertices, i.e., $f : X(0) \to \mathbb{F}_2$, this notion coincides with edge expansion. It can be seen by looking at $\mathrm{supp}(f) = S$ as a set of vertices and $\mathrm{supp}(\delta f) = E(S, \overline{S})$ as the set of edges that leave S.

On the question of small set expansion. The celebrated results of [KKL14, EK16] have shown that cosystolic expansion of a complex can be deduced from small set expansion of one dimension above. To be more precise, a cochain is said to be minimal if its weight cannot be reduced by adding a coboundary to it. It is said to be locally minimal if it is minimal at every link. [KKL14, EK16] have shown that expansion of small locally minimal cochains implies cosystolic expansion of one dimension below. Interestingly, even weak expansion is sufficient in order to imply cosystolic expansion, i.e., $\|\delta f\| > 0$ for small locally minimal cochains implies cosystolic expansion in one dimension below.

Expansion of small locally minimal cochains has been very useful in many other applications, such as proving Gromov's topological overlapping property [KKL14, EK16], proving local testability of codes and constructions of good quantum codes [PK22, DEL+22].

In this work we ask the natural question: **What is the strongest expansion that can be obtained for small sets?**

Namely, what is the optimal constant $c > 0$ for which $\|\delta f\| \geq c\|f\|$ holds for small locally minimal cochains? This question has not been addressed in this form by previous works, as $\|\delta f\| > 0$ is sufficient for applications regarding one dimension below in the complex. We put forward the question of what quality of expansion can be obtained when there are no more dimensions above in the complex. Similar to graphs, where $\|\delta f\| > 0$ implies connectivity, or some weak expansion, and it is desired to have strong expansion for small sets. We ask the same question for small sets in high dimensional expanders.

As opposed to graphs, where small sets of vertices easily expand, when moving to higher dimensions things get more complicated. For instance, fix an arbitrary vertex $u \in X(0)$ and take the set of edges on this vertex, i.e., define the 1-cochain $f : X(1) \to \mathbb{F}_2$ by $f(e) = 1$ for every $e \ni u$, and $f(e) = 0$ for the all the other edges. It is easy to see that f is a small set of edges. However, $\|\delta f\| = 0$, since every triangle contains either zero or two edges that contain u. This is why the extra condition of local minimality is required. Notice that in this example, f is not locally minimal, since its weight on X_u can be reduced by adding the constant 1 function to it, which is a coboundary.

Our result: Strong expansion for small locally minimal cochains. Previous works on cosystolic expansion can be divided into two main lines. One line of works [KKL14, EK16, KM21] use a so-called "fat machinery" and show strong expansion for locally minimal cochains

[1]The difference between coboundary and cosystolic expansion is whether there exist cocycles that are not coboundaries, i.e., satisfying assignments which are not obtained by the coboundary of an assignment of one dimension below. For the sake of introduction we can think of them as similar notions.

of very small weight. More formally, they show that there exists a constant $c > 0$ such that any small locally minimal cochain f satisfies $\|\delta f\| \geq c\|f\|$. However, this fat machinery is affected significantly by the weight of the cochain, hence they could show expansion only for cochains of weight $\|f\| \lesssim (\beta^k/k!)^{2^k}$, where β is the coboundary expansion of the links of the complex.

Another line of works [KM22, DD24] use local-to-global averaging operators, which improve upon the weight of cochains that expand to $\|f\| \lesssim \beta^k/(k+1)!$ but at the cost of the expansion quality. More precisely, they show that any locally minimal *cocycle* f satisfies $\|f\| \gtrsim \beta^k/(k+1)!$, which in turn implies that any locally minimal cochain of weight $\|f\| \lesssim \beta^k/(k+1)!$ is not a cocycle, i.e., it satisfies $\|\delta f\| > 0$.

In this work we obtain "the best of both worlds" and show strong expansion for locally minimal cochains of weight $\lesssim \beta^k/(k+2)!$. We achieve an exponential improvement in the weight of expanding cochains upon the first line of works, and an improvement in the quality of expansion upon the second line of works.

Theorem 1.1 (Main theorem, informal)**.** *Let X be a local spectral expander whose links are β-coboundary expanders. For every group Γ and every locally minimal cochain $f : X(k) \to \Gamma$, if $\|f\| \lesssim \beta^k/(k+2)!$ then $\|\delta f\| \geq c\|f\|$, where $c = \beta^k/(k!(k+1)^4)$.*

Our technique. In order to describe our technique, let us discuss briefly the strategies of previous works. The first line of works [KKL14, EK16, KM21] can be thought of as deducing the expansion of small locally minimal cochains from their local expansion in the links. Their analysis divided faces of smaller dimensions into "fat faces" and "non-fat faces". Their main argument (with a few variations between the works) builds upon the observation that whenever a cochain contains many fat faces, their local expansion is actually a global expansion. The main caveat of this line of works is that in order to benefit from these "fat faces", the cochain must be very small.

The second line of works [KM22, DD24] can be thought of as deducing expansion from some global structure of the cochain. They utilize (with a few variations between the works) certain local-to-global averaging operators, which are applicable only to locally minimal cocycles. This way they obtain a larger lower bound on the weight of a locally minimal cocycle, which in turn implies weak expansion for small locally minimal cochains, i.e., $\|\delta f\| > 0$ for any small locally minimal f. The main caveat of this line of works is the dependency on the global structure of the cochain being a cocycle.

We found a bridge between these techniques that allows us to use the local-to-global averaging operators for any cochain which is not necessarily a cocycle.

Key Idea: We argue that any general cochain either has a "global" structure in a sense that it looks much like a cocycle or it has a "local" structure in a sense that it is concentrated on links of some smaller dimension. This way we can apply the local-to-global averaging operators even on general cochains and not only on cocycles.

Decoupling between spectral and coboundary expansion. Interestingly, our technique decouples between the coboundary and spectral expansion of the links. In the "global" case we deduce expansion merely by spectral arguments, and in the "local" case we deduce expansion merely by coboundary expansion of the links. This is in contrast to all previous works who have used both coboundary and spectral expansion of the links simultaneously in their proofs.

1.1 Organization

In section 2 we provide required preliminaries. In section 3 we prove our strong small set expansion theorem. Propositions 3.3 and 3.4 demonstrate our case analysis of general cochains: If the cochain is "local", i.e., it is concentrated on links of some smaller dimension, we get expansion from proposition 3.3, and if the cochain is "global", i.e., it looks like a cocycle, we get expansion from proposition 3.4. We prove our main theorem with these two propositions and then we turn to prove each proposition.

As this work bridges between two lines of works, some lemmas with slight variations appear in previous works and some others appear with different notations. We comment through the work regarding the similarities and differences with previous works.

2 Preliminaries

2.1 Simplicial complexes

A pure d-dimensional simplicial complex X is a downwards closed $(d+1)$-hypergraph, namely, X is a collection of sets (hyperedges) of size $d+1$ together with all of their subsets. A hyperedge of size $k+1$ is called a *k-face*, and the set of all k-faces of X is denoted by $X(k)$.

We also consider ordered faces, where an *ordered k-face* is a tuple $(v_0, \dots, v_k)$, and the set of all ordered k-faces of X is denoted by $\overrightarrow{X}(k) = \{(v_0, \dots, v_k) \mid \{v_0, \dots, v_k\} \in X(k)\}$. Let us fix an arbitrary ordering of the vertices of the complex, so that for any k-face $\{v_0, \dots, v_k\} \in X$ there is a unique corresponding ordered k-face $(v_0, \dots, v_k) \in \overrightarrow{X}(k)$. For convenience sake, when it is clear from the context, we might refer to σ both as a face or as its unique corresponding ordered face. When we want to emphasize that we refer to an ordered face, we will denote it by $\vec{\sigma}$.

Let $\vec{\sigma} = (v_0, \dots, v_\ell)$ and $\vec{\tau} = (v_{\ell+1}, \dots, v_k)$ be two disjoint ordered faces. We denote by $\overrightarrow{\sigma\tau} = (v_0, \dots, v_k)$ the ordered k-face that is obtained by the concatenation of $\vec{\sigma}$ and $\vec{\tau}$. For any $0 \le i \le \ell$, we denote by $\vec{\sigma_i} = (v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_\ell)$ the ordered $(\ell-1)$-face that is obtained by removing v_i from $\vec{\sigma}$.

2.2 Coboundary and cosystolic expansion

Cochains, cocycles and coboundaries. Let X be a d-dimensional simplicial complex and Γ an abelian group[2]. A *k-cochain* over Γ is an antisymmetric function $f : \overrightarrow{X}(k) \to \Gamma$, where f is said to be antisymmetric if for any permutation $\pi \in \mathrm{Sym}(k+1)$,

$$f((v_{\pi(0)}, v_{\pi(1)}, \dots, v_{\pi(k)})) = \mathrm{sgn}(\pi) f((v_0, v_1, \dots, v_k)).$$

The space of all k-cochains over Γ is denoted by $C^k(X, \Gamma)$.

For any $-1 \le k \le d-1$, any k-cochain $f \in C^k(X, \Gamma)$ induces a $(k+1)$-cochain by the coboundary operator $\delta : C^k(X, \Gamma) \to C^{k+1}(X, \Gamma)$, which is defined by

$$\delta f(\vec{\sigma}) = \sum_{i=0}^{k+1} (-1)^i f(\vec{\sigma_i}),$$

where the sum is performed over the group Γ. A direct calculation shows that if f is antisymmetric then δf is also antisymmetric, so indeed $\delta f \in C^{k+1}(X, \Gamma)$.

[2]For simplicity we assume in this paper that the group is abelian. We note that all of the results also hold for non-abelian groups in the dimensions they are defined.

We can view the complex as inducing a system of equations, where the equations are determined by the coboundary operator, i.e., each $(k+1)$-face $\vec{\sigma} \in \overrightarrow{X}(k+1)$ defines the equation $\delta f(\vec{\sigma}) = 0$. The kernel of δ, i.e., the assignments that satisfy all the equations are called the *k-cocycles* and denoted by

$$Z^k(X,\Gamma) = \{f \in C^k(X,\Gamma) \mid \delta(f) = \mathbf{0}\}.$$

One can check that $\delta\delta f = \mathbf{0}$ always holds, i.e., every assignment that is obtained as the coboundary of one dimension below satisfies all the equations. These assignments are called the *k-coboundaries* and denoted by

$$B^k(X,\Gamma) = \{\delta f \mid f \in C^{k-1}(X,\Gamma)\}.$$

It follows that $B^k(X,\Gamma) \subseteq Z^k(X,\Gamma) \subseteq C^k(X,\Gamma)$.

Probabilities and weights. For a d-dimensional simplicial complex X, let $P_d : X(d) \to [0,1]$ be a probability distribution over the d-faces of the complex. This probability distribution over the d-faces induces a probability distribution $P_k : X(k) \to [0,1]$ for every dimension $k < d$ by selecting a d-face σ according to P_d and then selecting a k-face $\tau \subset \sigma$ uniformly at random.

The weight of any k-cochain $f \in C^k(X,\Gamma)$ is defined by

$$\|f\| = \Pr_{\sigma \sim P_k}[f(\sigma) \neq 0].$$

The distance between two k-cochains $f, g \in C^k(X,\Gamma)$ is defined by $\mathrm{dist}(f,g) = \|f - g\|$.

Since the weight of a cochain is dependent only on its non-zero elements, it is often convenient to consider the set $\mathrm{supp}(f)$, i.e., the set of non-zero elements in f, and define equivalently

$$\|f\| = \|\mathrm{supp}(f)\| = \Pr_{\sigma \sim P_k}[\sigma \in \mathrm{supp}(f)].$$

Cosystolic expansion. A d-dimensional simplicial complex X is called a *(β,ε)-cosystolic expander* over a group Γ if for every $k < d$ the following two conditions hold.

1. For any $f \in C^k(X,\Gamma) \setminus Z^k(X,\Gamma)$ it holds that
$$\frac{\|\delta(f)\|}{\mathrm{dist}(f, Z^k(X,\Gamma))} \geq \beta,$$
where $\mathrm{dist}(f, Z^k(X,\Gamma)) = \min\{\mathrm{dist}(f,g) \mid g \in Z^k(X,\Gamma)\}$.
2. For any $f \in Z^k(X,\Gamma) \setminus B^k(X,\Gamma)$ it holds that $\|f\| \geq \varepsilon$.

The second condition ensures that there are no small cocycles which are not coboundaries, i.e., any satisfying assignment that is not obtained by the coboundary of one dimension below must be large.

Coboundary expansion. Coboundary expansion is similar to cosystolic expansion with the difference that the only satisfying assignments are the coboundaries. Formally, a d-dimensional simplicial complex is called a *β-coboundary expander* over a group Γ if for every $k < d$ and $f \in C^k(X,\Gamma) \setminus B^k(X,\Gamma)$ it holds that

$$\frac{\|\delta(f)\|}{\mathrm{dist}(f, B^k(X,\Gamma))} \geq \beta,$$

where $\mathrm{dist}(f, B^k(X,\Gamma)) = \min\{\mathrm{dist}(f,g) \mid g \in B^k(X,\Gamma)\}$.

2.3 Local properties of complexes

Links and localizations. The *link* of a k-face $\sigma \in X(k)$ is a $(d-k-1)$-dimensional complex defined by $X_\sigma = \{\tau \setminus \sigma \mid \sigma \subseteq \tau \in X\}$. The probability distribution over faces of X_σ is induced from the probability distribution of X conditioned only on faces that contain σ.

For any k-face $\sigma \in X(k)$ and a $(k+\ell+1)$-cochain $f \in C^{k+\ell+1}(X,\Gamma)$, the *localization* of f to the link of σ, denoted by $f_\sigma \in C^\ell(X_\sigma,\Gamma)$, is an ℓ-cochain in X_σ defined as follows. For any ordered ℓ-face $\vec{\tau} \in \overrightarrow{X_\sigma}(\ell)$, $f_\sigma(\vec{\tau}) = f(\overrightarrow{\sigma\tau})$, where $\overrightarrow{\sigma\tau}$ is the concatenation of $\vec{\sigma}$ (i.e., the unique ordered k-face that corresponds to σ) and $\vec{\tau}$.

Minimal and locally minimal cochains. A k-cochain $f \in C^k(X,\Gamma)$ is said to be *minimal* if its weight cannot be reduced by adding a coboundary to it. Namely, for every coboundary $g \in B^k(X,\Gamma)$ it holds that $\|f\| \le \|f-g\|$. Recall that the distance of f from the coboundaries is defined by $\operatorname{dist}(f, B^k(X,\Gamma)) = \min\{\|f-g\| \mid g \in B^k(X,\Gamma)\}$. Since $\mathbf{0} \in B^k(X,\Gamma)$, it follows that $\|f\| \ge \operatorname{dist}(f, B^k(X,\Gamma))$ for every $f \in C^k(X,\Gamma)$. Hence, f is said to be minimal if and only if $\|f\| = \operatorname{dist}(f, B^k(X,\Gamma))$.

A k-cochain $f \in C^k(X,\Gamma)$ is said to be *locally minimal* if for every non-empty face $\sigma \in X$, the localization of f to the link of σ is minimal in the link, i.e., f_σ is minimal in X_σ for every non-empty $\sigma \in X$.

Local spectral expansion. A graph is called a λ-one-sided spectral expander if the second largest eigenvalue of its normalized adjacency matrix is upper bounded by λ

A d-dimensional simplicial complex X is said to be a λ-one-sided local spectral expander if for every $k \le d-2$ and every face $\sigma \in X(k)$, the underlying graph of X_σ[3] is a λ-one-sided spectral expander.

The following is a very useful lemma which follows by the well-known Cheeger inequality (see e.g. [KKL14] for a proof):

Lemma 2.1. *Let $G=(V,E)$ be a λ-one-sided spectral expander graph. For any set of vertices $S \subseteq V$ it holds that*

1. $\left\|E(S,\overline{S})\right\| \ge 2(1-\lambda)\|S\|\|\overline{S}\|$,
2. $\|E(S,S)\| \le \|S\|^2 + \lambda\|S\|$,

where $E(S,T)$ is the set of edges with one endpoint in S and one endpoint in T.

3 Small set expansion

Our goal in this section is to prove the following theorem.

Theorem 3.1 (Main theorem)**.** *Let X be a d-dimensional λ-one-sided local spectral expander such that for every $\sigma \in X(\ell)$, $0 \le \ell \le d-2$, X_σ is a β-coboundary expander. For any $f \in C^k(X,\Gamma)$, if f is locally minimal and $\|f\| \le \left(\frac{1}{k+2} - \lambda\right)\frac{\beta^k}{(k+1)!} - e\lambda$ then*

$$\|\delta f\| \ge \frac{\beta^k}{k!(k+1)^4}\|f\|.$$

[3] The graph whose vertices are $X_\sigma(0)$ and edges are $X_\sigma(1)$.

The main idea is to deal separately with two types of small cochains. If the cochain has a "global" structure in a sense that it looks like a cocycle, we would gain expansion by spectral arguments. Otherwise, if the cochain has a "local" structure in a sense that it is concentrated on links of some smaller dimension, we would gain expansion by the coboundary expansion of these links.

In order to distinguish between the two types of cochains, we provide another way of looking at neighborhoods of faces, in addition to the links.

Let X be a d-dimensional simplicial complex, $f \in C^k(X,\Gamma)$, and $\sigma \in X(\ell)$, $\ell \leq k \leq d$. Recall that the link of σ is a $(d-\ell-1)$-simplicial complex defined by $X_\sigma = \{\tau \setminus \sigma \mid \sigma \subseteq \tau \in X\}$, and the localization of f to X_σ is defined by $f_\sigma(\vec{\tau}) = f(\overrightarrow{\sigma\tau})$.

We define the set $Y_\sigma^k = \{\tau \setminus \{v\} \mid v \in \sigma \subset \tau \in X(k+1)\}$. In words, Y_σ^k is the set of k-faces that complete σ to a $(k+1)$-face which does not contain σ. We also define probability distribution on Y_σ^k according to the probability distribution of X, as follows. The probability of $\tau \setminus \{v\} \in Y_\sigma^k$ is the probability to choose $\tau \setminus \sigma$ in X_σ and then to choose v uniformly at random from σ, i.e., $\Pr_{Y_\sigma^k}[\tau \setminus \{v\}] = \Pr_{X_\sigma}[\tau \setminus \sigma]/(\ell+1)$. We denote by $f^\sigma = f|_{Y_\sigma^k}$ the restriction of f to the k-faces in Y_σ^k.

The idea behind the definition of f^σ is the following observation. Consider a $(k+1)$-face τ that contains σ. Note that any k-face that is contained in τ either contains σ or does not contain σ. Hence, the k-faces that are contained in τ can be decomposed to $(k-\ell-1)$-faces in X_σ and k-faces in Y_σ^k. We note that this observation and the definition of f^σ have appeared in [KM22] and later also in [DD24] with a different notation.

Informally speaking, we expect that in a cochain f that looks like a cocycle, i.e., a "global" cochain, for most faces σ it would be the case that $\|f_\sigma\| \lesssim \|f^\sigma\|$. The following definition allows us to distinguish between cochains which are "global" and cochains which are concentrated on links of some lower dimension, i.e., "local" cochains.

Definition 3.2 (Heavy faces[4]). Let X be a d-dimensional simplicial complex, $f \in C^k(X,\Gamma)$, and $\sigma \in X(\ell)$, $\ell \leq k$. We say that f is *heavy* on σ if $\|f_\sigma\| > (\ell+2)\beta^{-1}\|f^\sigma\|$, where β is the coboundary expansion of X_σ. Denote by $\mathrm{Heavy}_\ell(f) = \{\sigma \in X(\ell) \mid f \text{ is heavy on } \sigma\}$ the set of heavy ℓ-faces.

The two aforementioned cases are dealt in the following two propositions.

Proposition 3.3 (Expansion in the "local" case). *Let X be a d-dimensional simplicial complex such that for every $\sigma \in X(\ell)$, $0 \leq \ell \leq d-2$, X_σ is a β-coboundary expander. For any locally minimal $f \in C^k(X,\Gamma)$ and every $\ell < k$ it holds that*

$$\|\delta f\| \geq \frac{\beta}{\ell+2} \mathop{\mathbb{E}}_{\sigma \in \mathrm{Heavy}_\ell(f)}[\|f_\sigma\|] \cdot \|\mathrm{Heavy}_\ell(f)\|.$$

Proposition 3.4 (Expansion in the "global" case). *Let X be a d-dimensional λ-one-sided local spectral expander such that for every $\sigma \in X(\ell)$, $0 \leq \ell \leq d-2$, X_σ is a β-coboundary expander. For any $f \in C^k(X,\Gamma)$, if $\|f\| \leq \left(\frac{1}{k+2} - \lambda\right)\frac{\beta^k}{(k+1)!} - e\lambda$ then*

$$\|\delta f\| \geq \|f\| - \sum_{\ell=0}^{k-1} \frac{(k+1)(k+2)!}{\beta^{k-\ell-1}(\ell+2)!} \mathop{\mathbb{E}}_{\sigma \in \mathrm{Heavy}_\ell(f)}[\|f_\sigma\|] \cdot \|\mathrm{Heavy}_\ell(f)\|.$$

[4]We use the new term "heavy faces" in order to distinguish it from the notion of "fat faces" of previous works. We say that a face σ is "heavy" if $\|f_\sigma\|$ is large *relative* to $\|f^\sigma\|$, whereas a face σ is said to be "fat" according to previous works if $\|f_\sigma\|$ is larger than some *absolute* constant.

We note that β-coboundary expansion in the links is not required for the proof of this proposition, but only so heavy faces would be defined. In other words, one could use any arbitrary β and proposition 3.4 would still hold.

We first show how Theorem 3.1 is implied by these two propositions and then we will prove them.

Proof of Theorem 3.1. If there exists $0 \leq \ell \leq k-1$ such that

$$\mathop{\mathbb{E}}_{\sigma \in \mathrm{Heavy}_\ell(f)}[\|f_\sigma\|] \cdot \|\mathrm{Heavy}_\ell(f)\| \geq \frac{\beta^{k-\ell-1}(\ell+2)!}{k(k+1)(k+2)! + \beta^k}\|f\|,$$

then by proposition 3.3

$$\|\delta f\| \geq \frac{\beta}{\ell+2} \cdot \frac{\beta^{k-\ell-1}(\ell+2)!}{k(k+1)(k+2)! + \beta^k}\|f\| \geq \frac{\beta^k}{k!(k+1)^4}\|f\|.$$

Otherwise, it holds that

$$\sum_{\ell=0}^{k-1} \frac{(k+1)(k+2)!}{\beta^{k-\ell-1}(\ell+2)!} \mathop{\mathbb{E}}_{\sigma \in \mathrm{Heavy}_\ell(f)}[\|f_\sigma\|] \cdot \|\mathrm{Heavy}_\ell(f)\| \leq \frac{k(k+1)(k+2)!}{k(k+1)(k+2)! + \beta^k}\|f\|.$$

Thus, by proposition 3.4

$$\begin{aligned}
\|\delta f\| &\geq \|f\| - \sum_{\ell=0}^{k-1} \frac{(k+1)(k+2)!}{\beta^{k-\ell-1}(\ell+2)!} \mathop{\mathbb{E}}_{\sigma \in \mathrm{Heavy}_\ell(f)}[\|f_\sigma\|] \cdot \|\mathrm{Heavy}_\ell(f)\| \\
&\geq \|f\| - \frac{k(k+1)(k+2)!}{k(k+1)(k+2)! + \beta^k}\|f\| \\
&\geq \frac{\beta^k}{k!(k+1)^4}\|f\|,
\end{aligned}$$

□

Proof of Proposition 3.3

In this case, when the cochain is concentrated on links of some lower dimension $\ell < k$, we can show that the local expansion from heavy links of dimension ℓ is actually a global expansion of the cochain.

Consider a heavy ℓ-face $\sigma \in \mathrm{Heavy}_\ell(f)$ and a $(k-\ell)$-face $\tau \in X_\sigma(k-\ell)$ such that $\delta(f_\sigma)(\tau) \neq 0$. Denote $\sigma = (v_0, v_1, \ldots, v_\ell)$ and $\tau = (v_{\ell+1}, v_{\ell+2}, \ldots, v_{k+1})$. By definition

$$0 \neq \delta(f_\sigma)(\tau) = \sum_{i=\ell+1}^{k+1} (-1)^i f_\sigma(\tau_i) = \sum_{i=\ell+1}^{k+1} (-1)^i f((\sigma\tau)_i).$$

Now, if $f((\sigma\tau)_i) = 0$ for all $0 \leq i \leq \ell$ then

$$\delta f(\sigma\tau) = \sum_{i=0}^{k+1} (-1)^i f((\sigma\tau)_i) = \sum_{i=\ell+1}^{k+1} (-1)^i f((\sigma\tau)_i) \neq 0.$$

In such a case, the local coboundary is actually a global coboundary, i.e.,

$$\delta(f_\sigma)(\tau) \neq 0 \ \wedge \ \forall_{0\leq i\leq \ell}\ f((\sigma\tau)_i) = 0 \quad \Rightarrow \quad \delta f(\sigma\tau) \neq 0. \tag{3.1}$$

Let $\sigma = (v_0, v_1, \ldots, v_\ell)$ be a heave ℓ-face. We will show that many $(k-\ell)$-faces $\tau \in X_\sigma(k-\ell)$ satisfy (3.1). Recall the probability distribution of Y_σ^k: Choose a $(k-\ell)$-face $\tau \in X_\sigma(k-\ell)$ according to the probability distribution of X_σ and $0 \leq i \leq \ell$ uniformly at random, and take $(\sigma\tau)_i$. It holds that

$$\begin{aligned}
\|f^\sigma\| &= \Pr_{\substack{\tau\in X_\sigma(k-\ell)\\ 0\leq i\leq \ell}}[f((\sigma\tau)_i) \neq 0] \\
&= \frac{1}{\ell+1}\sum_{i=0}^{\ell} \Pr_{\tau\in X_\sigma(k-\ell)}[f((\sigma\tau)_i) \neq 0] \\
&\geq \frac{1}{\ell+1} \Pr_{\tau\in X_\sigma(k-\ell)}[\exists_{0\leq i\leq \ell} \text{ s.t. } f((\sigma\tau)_i) \neq 0].
\end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2) yields

$$\begin{aligned}
\Pr_{\tau\in X_\sigma(k-\ell)}[\delta f(\sigma\tau) \neq 0] &\geq \Pr_{\tau\in X_\sigma(k-\ell)}[\delta(f_\sigma)(\tau) \neq 0 \ \wedge \ \forall_{0\leq i\leq \ell}\ f((\sigma\tau)_i) = 0] \\
&\geq \Pr_{\tau\in X_\sigma(k-\ell)}[\delta(f_\sigma)(\tau) \neq 0] - \Pr_{\tau\in X_\sigma(k-\ell)}[\exists_{0\leq i\leq \ell} \text{ s.t. } f((\sigma\tau)_i) \neq 0] \\
&\geq \|\delta(f_\sigma)\| - (\ell+1)\|f^\sigma\| \\
&\geq \beta\|f_\sigma\| - \frac{(\ell+1)\beta}{\ell+2}\|f_\sigma\| \\
&= \frac{\beta}{\ell+2}\|f_\sigma\|,
\end{aligned}$$

where the first inequality follows by (3.1), the second inequality follows by the law of total probability, the third inequality follows by (3.2), and the last inequality follows since X_σ is a β-coboundary expander and $\sigma \in \text{Heavy}_\ell(f)$.

It follows that

$$\begin{aligned}
\|\delta f\| &= \Pr_{\sigma\tau\in X(k+1)}[\delta f(\sigma\tau) \neq 0] \\
&\geq \mathop{\mathbb{E}}_{\sigma\in\text{Heavy}_\ell(f)}\left[\Pr_{\tau\in X_\sigma(k-\ell)}[\delta f(\sigma\tau) \neq 0]\right] \cdot \|\text{Heavy}_\ell(f)\| \\
&\geq \frac{\beta}{\ell+2} \mathop{\mathbb{E}}_{\sigma\in\text{Heavy}_\ell(f)}[\|f_\sigma\|] \cdot \|\text{Heavy}_\ell(f)\|,
\end{aligned}$$

which completes the proof.

Proof of Proposition 3.4

In this case, when there are only a few heavy links in every dimension, we can show by spectral arguments that there are many $(k+1)$-faces that contain exactly one k-face σ for which $f(\sigma) \neq 0$.

The proof of Proposition 3.4 will follow from the following two lemmas.

Lemma 3.5. *Let X be a d-dimensional λ-one-sided local spectral expander. For any group Γ and $f \in C^k(X, \Gamma)$, $1 \le k < d$,*

$$\|\delta f\| \ge (k+2)\left(\|f\| - (k+1)\lambda\|f\| - (k+1)\mathop{\mathbb{E}}_{\sigma\in X(k-1)}[\|f_\sigma\|^2]\right).$$

Lemma 3.6. *Let X be a d-dimensional λ-one-sided local spectral expander. For any group Γ, $f \in C^k(X, \Gamma)$, $1 \le k < d$ and $0 \le \ell < k$,*

$$\mathop{\mathbb{E}}_{\sigma\in X(\ell)}[\|f_\sigma\|\|f^\sigma\|] \le \mathop{\mathbb{E}}_{\sigma\in X(\ell-1)}[\|f_\sigma\|^2] + \lambda\|f\|.$$

We note that both lemmas are inspired from previous works. A slight variation of lemma 3.5 has appeared in [KM22], and lemma 3.6, with a different notation, has appeared in [DD24]. For completeness, we provide proofs for both lemmas after we show that Proposition 3.4 follows from them.

Proof of Proposition 3.4. For any $0 \le \ell < k$,

$$\begin{aligned}
\mathop{\mathbb{E}}_{\sigma\in X(\ell)}[\|f_\sigma\|^2] &= \mathop{\mathbb{E}}_{\sigma\in \overline{\mathrm{Heavy}_\ell(f)}}[\|f_\sigma\|^2]\cdot\left\|\overline{\mathrm{Heavy}_\ell(f)}\right\| + \mathop{\mathbb{E}}_{\sigma\in \mathrm{Heavy}_\ell(f)}[\|f_\sigma\|^2]\cdot\|\mathrm{Heavy}_\ell(f)\| \\
&\le \frac{\ell+2}{\beta}\mathop{\mathbb{E}}_{\sigma\in \overline{\mathrm{Heavy}_\ell(f)}}[\|f_\sigma\|\|f^\sigma\|]\cdot\left\|\overline{\mathrm{Heavy}_\ell(f)}\right\| + \mathop{\mathbb{E}}_{\sigma\in \mathrm{Heavy}_\ell(f)}[\|f_\sigma\|^2]\cdot\|\mathrm{Heavy}_\ell(f)\| \\
&\le \frac{\ell+2}{\beta}\mathop{\mathbb{E}}_{\sigma\in X(\ell)}[\|f_\sigma\|\|f^\sigma\|] + \mathop{\mathbb{E}}_{\sigma\in \mathrm{Heavy}_\ell(f)}[\|f_\sigma\|^2]\cdot\|\mathrm{Heavy}_\ell(f)\| \\
&\le \frac{\ell+2}{\beta}\left(\mathop{\mathbb{E}}_{\sigma\in X(\ell-1)}[\|f_\sigma\|^2] + \lambda\|f\|\right) + \mathop{\mathbb{E}}_{\sigma\in \mathrm{Heavy}_\ell(f)}[\|f_\sigma\|^2]\cdot\|\mathrm{Heavy}_\ell(f)\|,
\end{aligned} \tag{3.3}$$

where the first inequality holds by the definition of non-heavy faces, the second inequality holds by the law of total expectation, and the last inequality holds by lemma 3.6.

Applying (3.3) over and over for $\ell = k-1, k-2, \dots, 0$ yields

$$\begin{aligned}
\mathop{\mathbb{E}}_{\sigma\in X(k-1)}[\|f_\sigma\|^2] &\le \frac{(k+1)!}{\beta^k}\left(\|f\|^2 + \sum_{\ell=0}^{k-1}\frac{\beta^\ell}{(\ell+1)!}\lambda\|f\|\right) + \\
&\qquad \sum_{\ell=0}^{k-1}\frac{(k+1)!}{\beta^{k-\ell-1}(\ell+2)!}\mathop{\mathbb{E}}_{\sigma\in \mathrm{Heavy}_\ell(f)}[\|f_\sigma\|^2]\cdot\|\mathrm{Heavy}_\ell(f)\| \\
&\le \frac{(k+1)!}{\beta^k}(\|f\| + e\lambda)\|f\| + \\
&\qquad \sum_{\ell=0}^{k-1}\frac{(k+1)!}{\beta^{k-\ell-1}(\ell+2)!}\mathop{\mathbb{E}}_{\sigma\in \mathrm{Heavy}_\ell(f)}[\|f_\sigma\|^2]\cdot\|\mathrm{Heavy}_\ell(f)\| \\
&\le \left(\frac{1}{k+2} - \lambda\right)\|f\| + \\
&\qquad \sum_{\ell=0}^{k-1}\frac{(k+1)!}{\beta^{k-\ell-1}(\ell+2)!}\mathop{\mathbb{E}}_{\sigma\in \mathrm{Heavy}_\ell(f)}[\|f_\sigma\|^2]\cdot\|\mathrm{Heavy}_\ell(f)\|,
\end{aligned} \tag{3.4}$$

where the second inequality holds since $\beta^\ell \le 1$ for every $\ell \ge 0$, and the last inequality holds since $\|f\| \le \left(\frac{1}{k+2} - \lambda\right)\frac{\beta^k}{(k+1)!} - e\lambda$. Substituting (3.4) in lemma 3.5 completes the proof. $\square$

Proofs of lemmas 3.5 and 3.6

Proof of Lemma 3.5. Let $f \in C^k(X,\Gamma)$. For ease of notation, we identify f with its support, i.e., we write that $\sigma \in f$ when $f(\sigma) \neq 0$.

For any $0 \leq i \leq k+2$ denote by $\delta_i(f)$ the set of $(k+1)$-faces that contain exactly i k-faces in f. Formally,

$$\delta_i(f) = \big\{\tau \in X(k+1) \;\big|\; |\{\sigma \subset \tau \mid \sigma \in f\}| = i\big\}.$$

Consider two k-faces $\sigma_1, \sigma_2 \subset \tau$. If $\sigma_1 \in f$ and $\sigma_2 \notin f$ then τ is seen in the link of $\sigma_1 \cap \sigma_2$ as an edge between a vertex in f and vertex not in f, i.e., $\tau \setminus (\sigma_1 \cap \sigma_2) \in \delta_1(f_{\sigma_1 \cap \sigma_2})$. Similarly, if both $\sigma_1 \in f$ and $\sigma_2 \in f$ then τ is seen in the link of $\sigma_1 \cap \sigma_2$ as an edge between two vertices in f, i.e., $\tau \setminus (\sigma_1 \cap \sigma_2) \in \delta_2(f_{\sigma_1 \cap \sigma_2})$.

Now, consider a $(k+1)$-face $\tau \in \delta_i(f)$. By definition, τ contains i k-faces in f and $(k+2-i)$ k-faces not in f. By the explanation above, each pair of k-faces $\sigma_1, \sigma_2 \subset \tau$ such that $\sigma_1 \in f$ and $\sigma_2 \notin f$ yields a $(k-1)$-face $\rho = \sigma_1 \cap \sigma_2$ such that $\tau \setminus \rho \in \delta_1(f_\rho)$. Therefore, the probability to choose a $(k-1)$-face $\rho \subset \tau$ such that $\tau \setminus \rho \in \delta_1(f_\rho)$ equals $i(k+2-i)/\binom{k+2}{2}$. It follows that

$$\begin{aligned}
\mathop{\mathbb{E}}_{\rho \in X(k-1)} \|\delta_1(f_\rho)\| &= \sum_{\rho \in X(k-1)} \Pr_{\tau \in X(k+1)}[\tau \setminus \rho \in \delta_1(f_\rho) \mid \tau \supset \rho] \Pr[\rho] \\
&= \sum_{\tau \in X(k+1)} \Pr_{\rho \in X(k-1)}[\tau \setminus \rho \in \delta_1(f_\rho) \mid \rho \subset \tau] \Pr[\tau] \\
&= \sum_{i=0}^{k+2} \sum_{\tau \in \delta_i(f)} \Pr_{\rho \in X(k-1)}[\tau \setminus \rho \in \delta_1(f_\rho) \mid \rho \subset \tau] \Pr[\tau] \\
&= \sum_{i=0}^{k+2} \sum_{\tau \in \delta_i(f)} \frac{i(k+2-i)}{\binom{k+2}{2}} \Pr[\tau] \\
&= \sum_{i=0}^{k+2} \frac{i(k+2-i)}{\binom{k+2}{2}} \sum_{\tau \in \delta_i(f)} \Pr[\tau] \\
&= \sum_{i=1}^{k+1} \frac{i(k+2-i)}{\binom{k+2}{2}} \|\delta_i(f)\|,
\end{aligned} \tag{3.5}$$

where the second equality follows by the law of total probability, the third equality follows by decomposition of the $(k+1)$-faces into $\delta_i(f)$ for $0 \leq i \leq k+2$, and the fourth equality follows by the explanation above.

Similarly, each pair of k-faces $\sigma_1, \sigma_2 \subset \tau$ such that $\sigma_1 \in f$ and $\sigma_2 \in f$ yields a $(k-1)$-face $\rho = \sigma_1 \cap \sigma_2$ such that $\tau \setminus \rho \in \delta_2(f_\rho)$. Thus, the probability to choose a $(k-1)$-face $\rho \subset \tau$ such that $\tau \setminus \rho \in \delta_2(f_\rho)$ equals $\binom{i}{2}/\binom{k+2}{2}$. By an exact same argument as in (3.5), it follows that

$$\mathop{\mathbb{E}}_{\rho \in X(k-1)} \|\delta_2(f_\rho)\| = \sum_{i=2}^{k+2} \frac{\binom{i}{2}}{\binom{k+2}{2}} \|\delta_i(f)\|. \tag{3.6}$$

Multiplying (3.6) by $2k$ yields

$$2k \mathop{\mathbb{E}}_{\rho \in X(k-1)} \|\delta_2(f_\rho)\| = \sum_{i=2}^{k+2} \frac{i(i-1)k}{\binom{k+2}{2}} \|\delta_i(f)\| \geq \sum_{i=2}^{k+2} \frac{i(k+2-i)}{\binom{k+2}{2}} \|\delta_i(f)\|. \tag{3.7}$$

Subtracting (3.7) from (3.5) yields

$$\mathop{\mathbb{E}}_{\rho\in X(k-1)}\|\delta_1(f_\rho)\| - 2k \mathop{\mathbb{E}}_{\rho\in X(k-1)}\|\delta_2(f_\rho)\| \le \frac{2}{k+2}\|\delta_1(f)\|.$$

Multiplying both sides by $(k+2)/2$ yields

$$\|\delta_1(f)\| \ge (k+2)\left(\frac{1}{2}\mathop{\mathbb{E}}_{\rho\in X(k-1)}\|\delta_1(f_\rho)\| - k\mathop{\mathbb{E}}_{\rho\in X(k-1)}\|\delta_2(f_\rho)\|\right). \tag{3.8}$$

Now, since the support of f is a set of k-faces, it is seen as a set of vertices in any link of a $(k-1)$-face $\rho \in X(k-1)$. Thus, by lemma 2.1 it holds that

$$\|\delta_1(f_\rho)\| \ge 2\big((1-\lambda)\|f_\rho\| - \|f_\rho\|^2\big), \tag{3.9}$$

$$\|\delta_2(f_\rho)\| \le \|f_\rho\|^2 + \lambda\|f_\rho\|. \tag{3.10}$$

Substituting (3.9) and (3.10) in (3.8) completes the proof. □

Proof of Lemma 3.6. Let $f \in C^k(X,\Gamma)$. For ease of notation, as in the proof of previous lemma, we identify f with its support, i.e., we write that $\sigma \in f$ when $f(\sigma) \neq 0$.

By the definition of f^σ,

$$\mathop{\mathbb{E}}_{\sigma\in X(\ell)}[\|f_\sigma\|\|f^\sigma\|] = \mathop{\mathbb{E}}_{\sigma\in X(\ell)}\mathop{\mathbb{E}}_{u\in\sigma}[\|f_\sigma\|\|(f_{\sigma\setminus u})^u\|] = \mathop{\mathbb{E}}_{\sigma\in X(\ell-1)}\mathop{\mathbb{E}}_{u\in X_\sigma(0)}[\|f_{\sigma u}\|\|(f_\sigma)^u\|]. \tag{3.11}$$

Let $\sigma \in X(\ell-1)$. Define the following two random walks on the $(k-\ell)$-faces of X_σ:

1. Move from a face $\tau \in X_\sigma(k-\ell)$ to a vertex $u \in \tau$ and then to a face $\tau' \in X_\sigma(k-\ell)$ such that $u \mathbin{\dot\cup} \tau' \in X_\sigma(k-\ell+1)$.
2. Move from a face $\tau \in X_\sigma(k-\ell)$ to a vertex $u \in X_\sigma(0)$ such that $u \mathbin{\dot\cup} \tau \in X_\sigma(k-\ell+1)$ and then to a face $\tau' \in X_\sigma(k-\ell)$ such that $\tau' \ni u$.

Random walk #1 can be described by the following operators: $A_1 : \mathbb{R}^{X_\sigma(k-\ell)} \to \mathbb{R}^{X_\sigma(0)}$ which moves from a $(k-\ell)$-face in X_σ to a vertex that is contained in it, and $B_1 : \mathbb{R}^{X_\sigma(0)} \to \mathbb{R}^{X_\sigma(k-\ell)}$ which then moves from a vertex to a $(k-\ell)$-face that completes it to a $(k-\ell+1)$-face. Similarly, random walk #2 can be described by the following operators: $A_2 : \mathbb{R}^{X_\sigma(k-\ell)} \to \mathbb{R}^{X_\sigma(0)}$ which moves from a $(k-\ell)$-face in X_σ to a vertex that completes it to a $(k-\ell+1)$-face, and $B_2 : \mathbb{R}^{X_\sigma(0)} \to \mathbb{R}^{X_\sigma(k-\ell)}$ which then moves from a vertex to a $(k-\ell)$-face that contains it.

Therefore, random walk #1 is given by applying the operator $B_1A_1 : \mathbb{R}^{X_\sigma(k-\ell)} \to \mathbb{R}^{X_\sigma(k-\ell)}$ and random walk #2 is given by applying the operator $B_2A_2 : \mathbb{R}^{X_\sigma(k-\ell)} \to \mathbb{R}^{X_\sigma(k-\ell)}$. Note that if we consider the random walks which are defined by the same operators but with opposite order, i.e., $A_1B_1 : \mathbb{R}^{X_\sigma(0)} \to \mathbb{R}^{X_\sigma(0)}$ and $A_2B_2 : \mathbb{R}^{X_\sigma(0)} \to \mathbb{R}^{X_\sigma(0)}$ we get (in both cases!) the well-known non-lazy random walk on the underlying graph of X_σ. For instance, A_1B_1 moves from a vertex $u \in X_\sigma(0)$ to a $(k-\ell)$-face $\tau \in X_\sigma(k-\ell)$ such that $u \mathbin{\dot\cup} \tau \in X_\sigma(k-\ell+1)$ and then to a vertex $v \in \tau$, namely, it moves from u to a random neighbor $v \neq u$.

Since A_1, B_1, A_2 and B_2 are linear operators, the eigenvalues of B_1A_1 and B_2A_2 equal to the eigenvalues of A_1B_1 and A_2B_2 (up to the multiplicity of the 0 eigenvalue). Therefore, $\lambda_2(B_1A_1) = \lambda_2(A_1B_1) = \lambda_2(A_2B_2) = \lambda_2(B_2A_2)$, where the equality in the middle is because both A_1B_1 and A_2B_2 describe the same random walk.

Now, consider the undirected graph $G(X_\sigma) = (V(X_\sigma), E(X_\sigma))$, where $V(X_\sigma)$ correspond to the $(k-\ell)$-faces of X_σ and $E(X_\sigma)$ correspond to a step in either of the random walks #1 and

#2, i.e., its random walk operator is given by $\frac{1}{2}(B_1A_1 + B_2A_2)$. Since X is a λ-one-sided local spectral expander and by the explanation above, it follows that $\lambda_2\big(\frac{1}{2}(B_1A_1 + B_2A_2)\big) = \lambda$.

It follows that

$$\begin{aligned}
\mathop{\mathbb{E}}_{u\in X_\sigma(0)}[\|f_{\sigma u}\|\|(f_\sigma)^u\|] &= \mathop{\mathbb{E}}_{u\in X_\sigma(0)}\left[\Pr_{\substack{\tau\in X_\sigma(k-\ell)\\ u\in\tau}}[\tau\in f_\sigma]\cdot \Pr_{\substack{\tau'\in X_\sigma(k-\ell)\\ u\cup\tau'\in X_\sigma(k-\ell+1)}}[\tau'\in f_\sigma]\right] \\
&= \Pr_{\{\tau,\tau'\}\in E(X_\sigma)}[\tau\in f_\sigma \wedge \tau'\in f_\sigma] \qquad (3.12) \\
&\leq \|f_\sigma\|^2 + \lambda\|f_\sigma\|,
\end{aligned}$$

where the second equality follows by the probability distribution of $E(X_\sigma)$, and the inequality follows by lemma 2.1.

Finally, substituting (3.12) in (3.11) completes the proof. □

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ECCC ISSN 1433-8092

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