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# A Note on Conditional Complexity Hardness of Matrix Rigidity and Tensor Rank

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**Abstract**

Recently, together with Kulikov, Mihajlin, and Smirnova [CKMS26], we gave conditional constructions of functions with large monotone circuit complexity, matrices with high rigidity, and 3-dimensional tensors of strongly superlinear rank. In this note, I strengthen the rigidity construction under the same assumption and, as a direct consequence, immediately obtain a slightly improved trade-off theorem for tensor rank.

[CKMS26] established the following connection between algorithmic lower bounds and matrix rigidity.

**Theorem 1.** *[CKMS26] If, for every $\varepsilon > 0$,* MAX-3-SAT *cannot be solved in co-nondeterministic time $O(2^{(1-\varepsilon)n})$, then, for all $\delta > 0$, there is a generator $g\colon \{0,1\}^{\log^{O(1)} k} \to \mathbb{F}^{k\times k}$ computable in time polynomial in $k$ such that, for infinitely many $k$, there exist a seed $s$ for which $g(s)$ has $k^{\frac{1}{2}-\delta}$-rigidity $k^{2-\delta}$.*

In addition, they showed the following result, which either yields matrices of very high rigidity or 3-dimensional tensors of superlinear rank.

**Theorem 2.** *[CKMS26] If, for any $\varepsilon > 0$,* MAX-3-SAT *cannot be solved in co-nondeterministic time $O(2^{(1-\varepsilon)n})$, then, for all $\delta > 0$ and some $\Delta > 0$, there are two generators $g_1\colon \{0,1\}^{\log^{O(1)} k} \to \mathbb{F}^{k\times k}$ and $g_2\colon \{0,1\}^{\log^{O(1)} k} \to \mathbb{F}^{k\times k\times k}$ computable in time polynomial in $k$ such that, for infinitely many $k$, at least one of the following is satisfied:*

- *$g_1(s)$ has $k^{1-\delta}$-rigidity $k^{2-\delta}$, for some $s$;*
- *$\mathrm{rank}(g_2(s))$ is at least $k^{1+\Delta}$, for some $s$.*

In this note, I improve this result by strengthening Theorem 1 to the following parameters, by utilizing a matrix multiplication scheme instead of the simpler version employed in [CKMS26].

**Theorem 3.** *If, for every $\varepsilon > 0$,* MAX-3-SAT *cannot be solved in co-nondeterministic time $O(2^{(1-\varepsilon)n})$, then, for all $\delta > 0$, there is a generator $g\colon \{0,1\}^{\log^{O(1)} k} \to \mathbb{F}^{k\times k}$ computable in time polynomial in $k$ such that, for infinitely many $k$, there exist a seed $s$ for which $g(s)$ has $k^{\frac{1}{\omega-1}-\delta}$-rigidity $k^{2-\delta}$.*

Since the best known value for $\omega$ is 2.371339 [ADV+25], this construction yields a conditional construction of a 0.729-rigid $n^{2-\delta}$ matrix. Moreover, this theorem immediately implies the following trade-off.

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- If $\omega \geqslant 2+\varepsilon$ for some $\varepsilon > 0$, then the tensor of matrix multiplication has rank at least $k^{1+\varepsilon/2}$, where $k$ denotes the size.
- Otherwise, for any $\delta > 0$, the generator described above outputs a matrix with $k^{1-\delta}$-rigidity $k^{2-\delta}$.

Hence, a slightly stronger version of Theorem 2 follows immediately from Theorem 3.

Moreover, this observation allows to improve the conditional result on canonical circuits.

**Corollary 1.** *If, for every $\varepsilon > 0$, MAX-3-SAT cannot be solved in co-nondeterministic time $O(2^{(1-\varepsilon)n})$, then, for any $\delta > 0$, one can construct an explicit family of $2^{\log^{O(1)} n}$ functions such that, for infinitely many $n$, at least one of them is bilinear and requires canonical circuits of size $2^{\Omega(n^{2/3-\delta})}$.*

# 1 Preliminaries

For the completeness of the presentation, I include the following reduction from MAX-3-SAT to the problem of finding a 4-clique in a 3-uniform hypergraph.

A subset of $l$ nodes in a $k$-hypergraph is called an $l$-*clique*, if any $k$ of them form an edge in the graph.

**Theorem 4** ([Wil07, LVW18, CKMS26])**.** *There exists an algorithm that, given a 3-CNF formula $F$ with $n$ variables and an integer $t$, outputs a 4-partite 3-uniform hypergraph $G$ with parts of size $k = n^{O(1)}2^{n/4}$ such that $G$ has a 4-clique if and only if it is possible to satisfy exactly $t$ clauses of $F$.*

*Proof.* To construct the graph $G$, partition the variables of $F$ into four groups $A_0, A_1, A_2, A_3$ of size $n/4$. Then, label each clause of $F$ by some $i \in \{0,1,2,3\}$ such that the clause does not contain variables from $A_i$. Then, for any $i \in \{0,1,2,3\}$, assigning variables from all parts except from $A_i$, determines how many clauses labeled $i$ are satisfied. Define tensors $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \in \mathbb{Z}_{\geqslant 0}^{2^{\frac{n}{4}} \times 2^{\frac{n}{4}} \times 2^{\frac{n}{4}}}$ as follows: for $U_1, U_2, U_3 \in \{0,1\}^{\frac{n}{4}}$, let $\mathcal{T}_i[U_1, U_2, U_3]$ be equal to the number of clauses with label $i$ when the variables from groups $A_{(i+1) \bmod 4}$, $A_{(i+2) \bmod 4}$, and $A_{(i+3) \bmod 4}$ are assigned as in $U_1$, $U_2$, and $U_3$. Then, it is possible to satisfy $t$ clauses in $F$ if and only if there exist $U_0, U_1, U_2, U_2 \in \{0,1\}^{\frac{n}{4}}$ such that

$$\mathcal{T}_0[U_1, U_2, U_3] + \mathcal{T}_1[U_2, U_3, U_0] + \mathcal{T}_2[U_3, U_0, U_1] + \mathcal{T}_3[U_0, U_1, U_2] = t.$$

As $F$ contains at most $O(n^3)$ clauses, $t = O(n^3)$. This makes it possible to enumerate in polynomial time all values of the four terms above. For an integer $q$ and $i \in \{0,1,2,3\}$, define a tensor $\mathcal{A}_{i,q} \in \{0,1\}^{2^{\frac{n}{4}} \times 2^{\frac{n}{4}} \times 2^{\frac{n}{4}}}$ as

$$\mathcal{A}_{i,q}[U_1, U_2, U_3] = [T_i[U_1, U_2, U_3] = q].$$

Then, one can satisfy $t$ clauses in $F$ if and only if

$$\sum_{\substack{U_0, U_1, U_2, U_3 \in \{0,1\}^{\frac{n}{4}} \\ t_0+t_1+t_2+t_3=t}} \mathcal{A}_{0,t_0}[U_1, U_2, U_3] \cdot \mathcal{A}_{1,t_1}[U_2, U_3, U_0] \cdot \mathcal{A}_{2,t_2}[U_3, U_0, U_1] \cdot \mathcal{A}_{3,t_3}[U_0, U_1, U_2] > 0.$$

For fixed $t_0, t_1, t_2, t_3$, such that $t_0 + t_1 + t_2 + t_3 = t$, the tensors $\mathcal{A}_{0,t_0}, \mathcal{A}_{1,t_1}, \mathcal{A}_{2,t_2}, \mathcal{A}_{3,t_3}$ may be viewed as a description of edges of a 4-partite 3-uniform hypergraph $G_{t_0,t_1,t_2,t_3}$. There is a 4-clique in $G_{t_0,t_1,t_2,t_3}$ if and only if one can satisfy $t_0$ clauses with label 0, $t_1$ clauses with label 1, and so on. Let $G$ be a superimposition of all such graphs. Then, $G$ contains a 4-clique if and only if one can satisfy exactly $t$ clauses of $F$. $\square$

The next two lemmas shows how rectangular matrix multiplication can be reduced to square matrix multiplication.

**Lemma 1** ([CKMS26]). *For $a, b \geqslant n$, the product of two matrices $A \in \mathbb{F}^{a \times n}$ and $B \in \mathbb{F}^{n \times b}$ can be computed in time $O(abn^{\omega-2})$.*

*Proof.* Partition $A$ and $B$ into $n \times n$-matrices $A_1, \ldots, A_{a/n}$ and $B_1, \ldots, B_{b/n}$. Then, to compute $A \cdot B$, it suffices to compute $A_i \cdot B_j$, for all $i \in [a/n]$ and $j \in [b/n]$. The resulting running time is

$$O\left(\frac{a}{n} \cdot \frac{b}{n} \cdot n^{\omega}\right) = O(abn^{\omega-2}).$$

$\square$

**Lemma 2.** *For $k \geqslant r$, the product of two matrices $A \in \mathbb{F}^{r \times k}$ and $B \in \mathbb{F}^{k \times r}$ can be computed in time $O(kr^{\omega-1})$.*

*Proof.* Partition $A$ and $B$ into $r \times r$ matrices $A_1, \ldots, A_{k/r}$ and $B_1, \ldots, B_{k/r}$, respectively. Then, to compute $A \cdot B$, it suffices to compute each product $A_i \cdot B_i$ and sum the resulting matrices. The resulting running time is

$$O\left(\frac{k}{r} \cdot r^{\omega}\right) = O\left(kr^{\omega-1}\right).$$

$\square$

### 1.1 Rigidity and Tensor Rank [CKMS26]

For a field $\mathbb{F}$, by $\mathbb{F}^{a \times b}$ we denote the set of all matrices of size $a \times b$ over $\mathbb{F}$. For a matrix $M \in \mathbb{F}^{a \times b}$, by $|M|$ we denote the number of nonzero entries of $M$.

For a matrix $M \in \mathbb{F}^{a \times b}$, we say that it has $r$-rigidity $s$ if it is necessary to change at least $s$ entries of $M$ to reduce its rank to $r$. That is, for each decomposition $M = R + S$ such that $\operatorname{rank}(R) \leqslant r$, it holds that $|S| \geqslant s$.

## 2 Improvement of Rigidity

**Theorem 3.** *If, for every $\varepsilon > 0$, MAX-3-SAT cannot be solved in co-nondeterministic time $O(2^{(1-\varepsilon)n})$, then, for all $\delta > 0$, there is a generator $g \colon \{0,1\}^{\log^{O(1)} k} \to \mathbb{F}^{k \times k}$ computable in time polynomial in $k$ such that, for infinitely many $k$, there exist a seed $s$ for which $g(s)$ has $k^{\frac{1}{\omega-1}-\delta}$-rigidity $k^{2-\delta}$.*

*Proof.* Take a 3-CNF formula over $n$ variables and an integer $t$, and transform it, using Theorem 4, into a 4-partite 3-uniform hypergraph with parts $H_0, H_1, H_2, H_3$ of size $k = n^{O(1)} 2^{\frac{n}{4}}$. Recall that $G$ contains a 4-clique if and only if one can satisfy $t$ clauses of $F$.

Let $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3 \in \{0,1\}^{k \times k \times k}$ be three-dimensional tensors encoding the edges of $G$. The tensor $\mathcal{A}_i$ is responsible for storing edges spanning vertices from all parts except $H_i$. For example, G has an edge $(u_1, u_2, u_3)$, where $u_1 \in H_1, u_2 \in H_2$, and $u_3 \in H_3$, if and only if $\mathcal{A}_0[u_1, u_2, u_3] = 1$. Let

$$R = \sum_{j_0,j_1,j_2,j_3 \in [k]} \mathcal{A}_0[j_1, j_2, j_3] \cdot \mathcal{A}_1[j_2, j_3, j_0] \cdot \mathcal{A}_2[j_3, j_0, j_1] \cdot \mathcal{A}_3[j_0, j_1, j_2].$$

Then G has a 4-clique if and only if $R > 0$. For $j_0, j_1 \in [k]$, define

$$R_{j_0,j_1} = \sum_{j_2,j_3 \in [k]} \mathcal{A}_0[j_1, j_2, j_3] \cdot \mathcal{A}_1[j_2, j_3, j_0] \cdot \mathcal{A}_2[j_3, j_0, j_1] \cdot \mathcal{A}_3[j_0, j_1, j_2].$$

Thus,

$$R = \sum_{j_0,j_1 \in [k]} R_{j_0,j_1}.$$

Fix $j_0, j_1 \in [k]$. Define vectors $v, u \in \{0,1\}^k$ by $v[i] = \mathcal{A}_3[j_0, j_1, i]$ and $u[\ell] = \mathcal{A}_2[\ell, j_0, j_1]$. Define matrices $M, L \in \{0,1\}^{k \times k}$ by $M[i, \ell] = \mathcal{A}_0[j_1, i, \ell]$ and $L[i, \ell] = \mathcal{A}_1[i, \ell, j_0]$. Then

$$R_{j_0,j_1} = \sum_{j_2,j_3 \in [k]} M[j_2, j_3] \cdot L[j_2, j_3] \cdot u[j_3] \cdot v[j_2].$$

Assume that $M$ and $L$ have $r$-rigidity $s$, that is,

$$M = R_M + S,$$
$$L = R_L + T,$$

where $R_M, R_L, S, T \in \mathbb{F}^{k \times k}$, $\text{rank}(R_M), \text{rank}(R_L) \leqslant r$, and $|S|, |T| \leqslant s$.

Since $\text{rank}(R_M), \text{rank}(R_L) \leqslant r$, there exist matrices $U, V, P, Q \in \mathbb{F}^{k \times r}$ such that

$$M = UV^T + S,$$
$$L = PQ^T + T.$$

We guess such representations for $M$ and $L$. In addition, we guess only the nonzero entries of $S$ and $T$, since they are sparse.

For fixed $j_0$, the time complexity to guess and verify the decomposition of $L$ is $O(k^2 r^{\omega-1} + s + k^2)$, which includes the time required to multiply the matrices using Lemma 1, sum them, add $T$, and verify equality with $L$. An analogous bound holds when fixing $j_1$. Thus, the overall time complexity over all $j_0, j_1$ is

$$O(k^3 r^{\omega-1} + ks + k^3) = O(k^3 r^{\omega-1} + ks).$$

We decompose

$$R_{j_0,j_1} = \sum_{j_2,j_3 \in [k]} (UV^T + S)[j_2, j_3] \cdot (PQ^T + T)[j_2, j_3] \cdot u[j_3] \cdot v[j_2]$$
$$= \sum_{j_2,j_3 \in [k]} (UV^T)[j_2, j_3] \cdot (PQ^T)[j_2, j_3] \cdot u[j_3] \cdot v[j_2] + R'_{j_0,j_1},$$

where $R'_{j_0,j_1}$ collects all terms containing at least one of $S$ or $T$. Since $S$ and $T$ are sparse, $R'_{j_0,j_1}$ can be computed in time $O(s)$ for fixed $j_0, j_1$.

It remains to evaluate

$$R_{j_0,j_1} - R'_{j_0,j_1} = \sum_{j_2,j_3 \in [k]} (UV^T)[j_2,j_3] \cdot (PQ^T)[j_2,j_3] \cdot u[j_3] \cdot v[j_2].$$

Define diagonal matrices

$$D_v \coloneqq \mathrm{diag}(v[1],\dots,v[k]), \qquad D_u \coloneqq \mathrm{diag}(u[1],\dots,u[k]).$$

Then

$$R_{j_0,j_1} - R'_{j_0,j_1} = \sum_{a,b \in [r]} (U^T D_v P)[a,b] \cdot (V^T D_u Q)[a,b].$$

To compute $U^T D_v P$, one multiplies matrices of dimensions $r \times k$, $k \times k$, and $k \times r$, where $k \geqslant r$. The products $D_v P$ and $D_u Q$ can be computed in time $O(kr)$ since $D_v$ and $D_u$ are diagonal. The remaining multiplication of an $r \times k$ matrix by a $k \times r$ matrix can be performed in time $O(kr^{\omega-1})$ using Lemma 2.

Thus, the total running time over all $j_0, j_1$ is

$$O(k^3 r^{\omega-1} + ks).$$

If $r = k^{\frac{1}{\omega-1}-\delta}$ and $s = k^{2-\delta}$ for some $\delta > 0$, the running time becomes $O(k^{4-\delta})$.

We construct a generator $g$ that takes as input the encoding of a 3-CNF formula, an integer $t$, indices $j_0, j_1$, and a bit in $\{0,1\}$. If the bit equals 0, the generator outputs $M$; otherwise, it outputs $L$. If the input is invalid, the generator outputs an empty family. Therefore, if MAX-3-SAT cannot be solved in co-nondeterministic time $O(2^{(1-\varepsilon)n})$ for any $\varepsilon > 0$, then for infinitely many $n$ the generator outputs at least one matrix with $k^{\frac{1}{\omega-1}-\delta}$-rigidity $k^{2-\delta}$, for any $\delta > 0$, where $k = n^{O(1)} 2^{\frac{n}{4}}$. The generator uses $n^{O(1)} = \log^{O(1)}(k)$ input bits, runs in time polynomial in $k$, and outputs a $k \times k$ matrix. □

## References


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