



A \mathbb{Z}_2 -Topological Framework for Sign-rank Lower Bounds

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Abstract

We develop a topological framework for proving lower bounds on sign-rank via \mathbb{Z}_2 -equivariant topology, and use it to resolve the sign-rank of the Gap Hamming Distance problem up to lower-order terms.

For every (partial) sign matrix A , we associate a free \mathbb{Z}_2 -simplicial complex $S(A)$ and show that sign-rank of A is characterized by the linear analog of \mathbb{Z}_2 -index of $S(A)$. As a consequence, the classical \mathbb{Z}_2 -index of $S(A)$ lower bounds the sign-rank of A , which reduces sign-rank lower bounds to topological obstructions. This reduction allows us to use various tools from \mathbb{Z}_2 -equivariant topology, particularly in regimes where classical lower-bound techniques break down.

As the main application, we consider the Gap Hamming Distance function GHD_k^n (defined for $k < n/2$), which distinguishes pairs of strings in $\{0, 1\}^n$ with Hamming distance at most k from pairs with distance at least $n - k$. We prove an essentially tight lower bound and show that for any k ,

$$\text{sign-rank}(\text{GHD}_k^n) = (1 - o_k(1)) 2k.$$

where the $o_k(1)$ term is $O\left(\sqrt{\frac{\log k}{k}}\right)$. This improves on the previous lower bound of Hatami, Hosseini, and Meng (STOC 2023) who proved that sign-rank of GHD_k^n is at least $\Omega(k/\log(n/k))$.

A key technical ingredient is a new analysis of the \mathbb{Z}_2 -coindex (which lower bounds \mathbb{Z}_2 -index) of the Vietoris–Rips complex of the hypercube in the sparse regime which yields an essentially tight lower bound. Previously, no results were known in the sparse regime.

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1 Introduction

The sign-rank of a partial matrix A , denoted $\text{rank}^\pm(A)$, is the minimum rank of any real matrix B such that $\text{sign}(B_{ij}) = A_{ij}$ whenever $A_{ij} \in \{-1, +1\}$. Sign-rank is a fundamental complexity measure in theoretical computer science and mathematics, yet despite decades of study, it remains notoriously difficult to analyze. There are three known general lower bound tools for sign-rank, which can be summarized as *VC dimension* [PS86], variants of Forster’s method based on *spectral norm/average margin* [For02, FKL⁺01, FS06, RS10, She11, LMSS07, HHP⁺22], and *rectangle based* methods [APP⁺05, HHP⁺22]. However, these lower bounds are effective only when the given matrix contains a large *pseudorandom* piece (i.e., has large VC dimension, or spectral norm is small, or it does not contain large monochromatic rectangles) and there are N by N matrices of sign-rank $\text{poly}(N)$ where these methods cannot produce any $\Omega(1)$ lower bound [HHP⁺22]. The reader is referred to [HHP⁺22] for an in-depth discussion of limitations of these methods. This lack of flexible tools to lower bound sign-rank without requiring pseudorandomness has stalled progress on many fundamental problems about sign-rank.

A benchmark problem to study sign-rank is the Gap Hamming Distance problem, which is a partial function $\text{GHD}_k^n: \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{-1, 1, *\}$ defined for a parameter $k < n/2$ by:

$$\text{GHD}_k^n(x, y) = \begin{cases} 1 & \text{if } d_H(x, y) \leq k, \\ -1 & \text{if } d_H(x, y) \geq n - k, \\ * & \text{otherwise.} \end{cases}$$

Here $d_H(x, y)$ is the Hamming distance between x and y . Gap Hamming Distance is a fundamental object and has been intensively studied in different parameter regimes, used as a lower bound primitive in randomized communication complexity and streaming algorithms [IW03, BCR⁺10, CR11, She12, JW13], separating the power of communication models [PSS14, CLV19, Gav25, BHH⁺25], separating partial and total learnability of half-spaces with margin [AHHM22, CMW25, BHH⁺25, CCMY24], and lower bounds for dimensionality reduction [HHM23]. Further more, Gap Hamming Distance is the XOR lift of the *approximate majority* function with a broad range of applications in circuit complexity and query complexity, and it is also the discrete analog of the concept class of *half-spaces with margin* which is a canonical object in learning theory.

Next we briefly discuss the sign-rank of GHD_k^n in different parameter regimes. In the case of odd n and $k = (n - 1)/2$, GHD_k^n is the XOR lift of the exact Majority function, and has VC dimension n which implies $\text{rank}^\pm(\text{GHD}_k^n) = n$. More generally, the function GHD_k^n has been studied in two parameter regimes:

- (1) $k \approx n/2 - \sqrt{n}$: in this regime one can get $\Omega(n)$ sign-rank lower bounds using VC dimension or Forster’s method as GHD_k^n still exhibits strong mixing behavior.
- (2) $k = \varepsilon n$ for constant $\varepsilon \in (0, 0.49)$: in this regime, it is known that the VC dimension and average margin are bounded and hence unable to prove super-constant lower bounds on its sign-rank. However, using isoperimetric inequalities of hypercube [FF81, FR87], it is easy to see that GHD_k^n is still mildly pseudorandom in the sense that it does not contain large monochromatic rectangles, which implies $\Omega(n)$ sign-rank lower bounds [HHH⁺23]. However, once k is $O(\sqrt{n})$, this flicker of pseudorandom behavior entirely dies down and none of these methods can produce any super-constant lower bounds.

Recently, Hatami, Hosseini, and Meng [HHM23] introduced a new lower bound on the sign-rank

of GHD_k^n by using a topological argument based on the *Borsuk–Ulam theorem*¹ and showed that

$$\text{rank}^\pm(\text{GHD}_k^n) = \Omega\left(\frac{k}{\log\left(\frac{n}{k}\right)}\right) \quad (1)$$

which is nontrivial for $k = \Omega(\log n)$. The idea of [HHM23] was to first consider the analog of GHD_k^n defined on $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ which is amenable to Borsuk–Ulam theorem, give a tight lower bound on its sign-rank², and then apply a randomized reduction to transfer the lower bound to GHD_k^n .

One limitation of the Borsuk–Ulam approach of [HHM23] is that because of the inevitable randomized reduction, it cannot be improved to handle the regime $k = o(\log n)$. In this paper, we take a different approach and resolve the sign-rank of GHD_k^n for any choice of $k < n/2$, up to lower order terms.

$$\text{rank}^\pm(\text{GHD}_k^n) \geq (1 - o_k(1))2k. \quad (2)$$

where the $o_k(1)$ term is $O\left(\sqrt{\frac{\log k}{k}}\right)$. It is easy to see that $\text{rank}^\pm(\text{GHD}_k^n) \leq 2k + 1$ (See Fact 16). To the best of our knowledge, this is the first finite matrix family whose sign-rank is resolved up to lower order terms.

As the main technical ingredient, we give a new analysis of the \mathbb{Z}_2 -coindex of *Vietoris–Rips (VR) complex* of the n -hypercube. These are central objects in metric topology which we briefly discuss now. For a metric space X and scale parameter $k \geq 0$, the Vietoris–Rips complex $\text{VR}(X, k)$ is the simplicial complex of all finite subsets of X of *diameter* at most k . Thus this complex provides a combinatorial-topological encoding of pairwise metric proximity. It was first introduced by Vietoris [Vie27] to define a homology theory for metric spaces, and later used by Rips and Gromov [Gro87] in the context of geometric group theory to study hyperbolic groups. More recently, topological properties of VR complexes have been studied in the computational topology community in the context of persistent homology [Car09, EH10, Ghr08]. Despite their widespread use and popularity, the topology of VR complexes is poorly understood: only partial results are known for VR complexes of various manifolds [ABV25] and of hypercubes, for example. For hypercubes [AA22, AV24, Fen25, Shu23] exact results are only known in the regime $k \leq 3$, along with asymptotic results for $k \geq n/2$. Here we prove essentially tight lower bounds for the \mathbb{Z}_2 -coindex of VR complexes of n -hypercubes for all $k < n/2$.

Another limitation of the Borsuk–Ulam argument of [HHM23] is that it is highly tailored to the specific structure of GHD_k^n and it is not clear whether one can use topology to study the sign-rank of arbitrary sign matrices. This naturally raises the following meta-question:

Is there a systematic topological framework for sign-rank lower bounds of arbitrary sign matrices?

In this paper we build a framework using \mathbb{Z}_2 -topology to lower bound the sign-rank of any sign matrix. First, we introduce a *correspondence* between partial sign matrices and *free \mathbb{Z}_2 -simplicial complexes*, and reinterpret sign-rank as the linear analogue of \mathbb{Z}_2 -index. This turns lower bounds on sign-rank into questions about equivariant topology. \mathbb{Z}_2 -index is a global topological invariant amenable to a broad range of equivariant methods far beyond the Borsuk–Ulam theorem, which will be discussed later in the paper.

¹The Borsuk–Ulam theorem [Bor33] states that for every continuous map on the d -dimensional unit sphere $f: \mathbb{S}^d \rightarrow \mathbb{R}^d$ there exists a point $x \in \mathbb{S}^d$ such that $f(x) = f(-x)$.

²The class of half-spaces with margin ε is defined as $\mathbb{G}_\varepsilon^n: \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \rightarrow \{-1, 1, *\}$ by $\mathbb{G}_\varepsilon^n(x, y) = \text{sign}(\langle x, y \rangle)$ if $|\langle x, y \rangle| \geq \varepsilon$, and $*$ otherwise. It was shown in [HHM23] that $\text{rank}^\pm(\mathbb{G}_\varepsilon^n) = n$

After applying our framework to sign-rank of GHD_k^n , we also reinterpret VC dimension topologically and obtain a chain of inequalities between VC dimension, \mathbb{Z}_2 -coindex, \mathbb{Z}_2 -index, and sign-rank, along with separations between all these parameters.

(1) *VC vs. \mathbb{Z}_2 -coindex*: for partial matrices, our result for Gap Hamming already yields a 1 -vs- $\Omega(\log N)$ separation between VC dimension and \mathbb{Z}_2 -coindex. We get improved separations by translating state of the art constructions of small triangulations of the real projective space due to Adiprasito et al. [AAK22]. For *total matrices*, we leave the possibility that VC dimension and coindex are *equivalent in the constant regime*, which if true, entirely reframes PAC learnability of total concept classes as a topological property and has important implications on VC dimension of disambiguations of Gap Hamming Distance, which we discuss later.

(2) *\mathbb{Z}_2 -index vs. sign-rank*: we also establish separations between \mathbb{Z}_2 -index and sign-rank by developing a general method to upper bound \mathbb{Z}_2 -index. For total matrices, we leave the possibility that \mathbb{Z}_2 -index and sign-rank are equivalent in the constant regime, which if true will imply that *constant sign-rank* is a topological phenomenon and would allow one to reformulate many fundamental open problems about sign-rank in terms of \mathbb{Z}_2 -index of \mathbb{Z}_2 -complexes.

To state our results formally, we briefly introduce the main framework and the notion of *sign complex* of a sign matrix. For more details see Section 3. The reader is referred to Section 2.2 for all the necessary background on \mathbb{Z}_2 -equivariant topology.

1.1 The main framework

Given a matrix $A \in \{-1, 1, *\}^{M \times N}$ with columns indexed by $[N]$, we construct an abstract simplicial complex called the *sign complex* denoted by $S(A)$. This is a simplicial complex built on two copies of the column set, namely

$$\{1^-, 1^+, \dots, N^-, N^+\}.$$

For each row $r \in \{-1, 1, *\}^N$, define the simplex

$$\sigma_r^+ := \{i^+ : r_i = +1\} \cup \{i^- : r_i = -1\},$$

and its antipodal copy

$$\sigma_r^- := \{i^- : r_i = +1\} \cup \{i^+ : r_i = -1\}.$$

The simplices of $S(A)$ are all subsets of σ_r^+ or σ_r^- for some row r of A . There is a natural free \mathbb{Z}_2 -action on $S(A)$ that exchanges i^- and i^+ . This turns $S(A)$ to a *free \mathbb{Z}_2 -simplicial complex*. For more detail, see Section 3. We then consider a geometric realization $|S(A)| \subset \mathbb{R}^N$ by exploiting its antipodal symmetry given by

$$j^+ \mapsto e_j, \quad j^- \mapsto -e_j$$

where e_j is the j th standard basis element in \mathbb{R}^N , and extending affinely on each simplex. We show the sign-rank of A is exactly the smallest dimension d such that $|S(A)|$ can be mapped linearly into \mathbb{R}^{d+1} while avoiding the origin. More formally, for a free \mathbb{Z}_2 -complex K , let $\text{ind}_{\mathbb{Z}_2}^{\text{lin}}(K)$ denote the smallest integer d such that there exists a linear map $g: \mathbb{R}^N \rightarrow \mathbb{R}^{d+1}$ such that

$$0 \notin g(|K|).$$

Such a linear map g induces a simplex-wise linear \mathbb{Z}_2 -equivariant map $|K| \rightarrow \mathbb{R}^{d+1}$ and every simplex-wise linear \mathbb{Z}_2 -equivariant map $|K| \rightarrow \mathbb{R}^{d+1}$ comes about in this way. We prove the following.

Lemma 1 (Main lemma 1). *For every partial sign matrix A ,*

$$\text{ind}_{\mathbb{Z}_2}^{\text{lin}}(S(A)) = \text{rank}^\pm(A) - 1.$$

See Section 3.2 for details and proof. In particular, sign-rank can be viewed as a linear analogue of a global topological invariant called the \mathbb{Z}_2 -index. The \mathbb{Z}_2 -index of a free \mathbb{Z}_2 -complex K , denoted by $\text{ind}_{\mathbb{Z}_2}(K)$, is the smallest integer d for which there exists a \mathbb{Z}_2 -equivariant continuous map (i.e., continuous maps that respect the antipodal symmetry $f(-x) = -f(x)$)

$$f: |K| \rightarrow \mathbb{S}^d.$$

Since the map g induces a \mathbb{Z}_2 -equivariant map $f: |K| \rightarrow \mathbb{S}^d$, defined by $f(x) := \frac{g(x)}{\|g(x)\|}$, we have that for any free \mathbb{Z}_2 -complex K

$$\text{ind}_{\mathbb{Z}_2}(K) \leq \text{ind}_{\mathbb{Z}_2}^{\text{lin}}(K)$$

which combined with Lemma 1 implies the following inequality.

Corollary 2 (Main inequality). *For every partial sign matrix A ,*

$$\text{ind}_{\mathbb{Z}_2}(S(A)) \leq \text{rank}^{\pm}(A) - 1.$$

Corollary 2 allows one to use the full machinery of \mathbb{Z}_2 -equivariant topology to prove sign-rank lower bounds. Index lower bound techniques include Borsuk–Ulam type arguments [Bor33, Lov78, Mat03], nerve lemmas and connectivity [Mat03], the Fadell–Husseini cohomological index / Stiefel–Whitney classes [FH88, BZ17, Mat03], equivariant obstruction theory [BZ17], and discrete Morse theory [For98]. In particular, our framework allows one to apply topological methods in settings where classical sign-rank techniques are ineffective.

Moreover, the correspondence between free \mathbb{Z}_2 -simplicial complexes and partial sign-matrices is in fact two-way: *every free \mathbb{Z}_2 -simplicial complex K is isomorphic to $S(A_K)$ for some partial sign matrix A_K . If K has $2N$ vertices and M facets, then A_K has N columns and M rows. Moreover, A_K is a total sign matrix iff all facets of K have size N .* Hence, in order to upper bound the index of K , it is enough to upper bound the sign-rank of A_K . This viewpoint can be beneficial in settings where topology of K is difficult to study but A_K is amenable to sign-rank *upper bound* methods including sign alternating number [AFR85], polynomial threshold degree [KS01], support rank [GHIS25], communication protocols [PS86], protocols with Hamming oracles [HHP⁺22, GHIS25], and analytic parameters such as factorization norm [HHP⁺22].

As our main application, we combine several tools from \mathbb{Z}_2 -topology to obtain an essentially tight lower bound on the sign-rank of GHD_k^n for all $k < n/2$ which is discussed next.

1.2 Main application: sign-rank of Gap Hamming Distance

It is easy to see that, by projecting to the first $2k + 1$ coordinates:

$$\text{rank}^{\pm}(\text{GHD}_k^n) \leq 2k + 1.$$

Indeed, identify rows and columns with $\{\pm 1\}^n$ and map each $x, y \in \{\pm 1\}^n$ into $x', y' \in \mathbb{R}^{2k+1}$ by only keeping the first $2k + 1$ coordinates and observe that $\text{sign}(\langle x', y' \rangle) = 1$ if $d_H(x, y) \leq k$ and -1 if $d_H(x, y) \geq n - k$. See Fact 16 for details. In this paper we show that this projection map is optimal up to lower order terms.

Theorem 3 (Main application). *For any $k < n/2$,*

$$\text{rank}^{\pm}(\text{GHD}_k^n) \geq (1 - o_k(1)) \cdot 2k.$$

where the $o_k(1)$ term is $O\left(\sqrt{\frac{\log k}{k}}\right)$.

See Section 4 for the proof. As mentioned before, the best previous bound, due to Hatami, Hosseini, and Meng [HHM23] was that

$$\text{rank}^\pm(\text{GHD}_k^n) = \Omega\left(\frac{k}{\log\binom{n}{k}}\right). \quad (3)$$

Theorem 3 is proved by lower bounding the \mathbb{Z}_2 -coindex which can be viewed as a dual of \mathbb{Z}_2 -index. The \mathbb{Z}_2 -coindex of a \mathbb{Z}_2 -complex K is the largest d for which there is a \mathbb{Z}_2 -equivariant continuous map

$$\mathbb{S}^d \rightarrow |K|.$$

Moreover, the definitions imply that

$$\text{coind}_{\mathbb{Z}_2}(K) \leq \text{ind}_{\mathbb{Z}_2}(K). \quad (4)$$

An instructive analogy is to compare the coindex and index with *clique number* and *chromatic number* of graphs. While clique number (i.e., homomorphism from largest clique to the graph) is like coindex and detects large forced local structure, but the chromatic number (i.e., smallest homomorphism to a clique) is like index and captures global obstructions and can be significantly higher than clique number.

Nevertheless, in the case of $S(\text{GHD}_k^n)$ we prove that coindex and index are essentially equal and large, which implies the bound on sign-rank.

Theorem 4. *For any $k < n/2$,*

$$\text{coind}_{\mathbb{Z}_2}(S(\text{GHD}_k^n)) \geq (1 - o_k(1)) \cdot 2k.$$

In particular, the same lower bound holds for the sign-rank of GHD_k^n . In Section 4.1 we give a weaker lower bound of k on the \mathbb{Z}_2 -coindex of $S(\text{GHD}_k^n)$ by exploiting the topological connectivity of the k -skeleton of the cubical *CW-complex* structure on the boundary of the hypercube. While this approach does not give the optimal bound and is shy of a factor of 2, it shows the topological connectivity mechanism in a more straightforward way.

We prove Theorem 4 by exploiting more than cubical faces of the hypercube. The right object to study is called the Vietoris-Rips complex of the hypercube, denoted by $\text{VR}(Q_n, k)$. The Vietoris-Rips complex of the hypercube $\text{VR}(Q_n, k)$ is the simplicial complex on vertex set $\{0, 1\}^n$ whose faces are subsets of vertices of pairwise Hamming distance at most k . As discussed previously, very little is known about the topology of $\text{VR}(Q_n, k)$ beyond $k \leq 3$ or the dense regime $k > n/2$, and essentially no sharp topological information is available in the sparse regime $k \ll n/2$.

As our main technical lemma, we prove a sharp lower bound on coindex of $\text{VR}(Q_n, k)$ and then give a \mathbb{Z}_2 -equivariant map from $\text{VR}(Q_n, k)$ to $S(\text{GHD}_k^n)$ which implies Theorem 4.

Lemma 5 (Main lemma 2). *For any $k < \frac{n}{2}$,*

$$\text{coind}_{\mathbb{Z}_2}(\text{VR}(Q_n, k)) \geq (1 - o_k(1)) 2k.$$

Proof idea of Lemma 5. A standard way to lower bound $\text{coind}_{\mathbb{Z}_2}(\text{VR}(Q_n, k))$ is to prove a lower bound on the homotopy connectivity of $\text{VR}(Q_n, k)$. However, this does not work in our setting, since k may be arbitrarily small compared to n , making the complex extremely sparse and poorly connected. Instead, the key step is to identify a *suitable symmetric subcomplex* K of $\text{VR}(Q_n, k)$ and prove that its connectivity is nearly $2k$ which implies a lower bound on $\text{coind}_{\mathbb{Z}_2}(K)$ and then by monotonicity of $\text{coind}_{\mathbb{Z}_2}(\cdot)$ we get the lower bound on $\text{coind}_{\mathbb{Z}_2}(\text{VR}(Q_n, k))$. To lower

bound connectivity of K , the idea is to cover it by pieces supported on lower-dimensional faces of the hypercube, each isomorphic to $\text{VR}(Q_t, k)$ for $t \approx 2k - \sqrt{k}$. A result of Bendersky and Grbic [BG23] implies that each such piece has high connectivity. We then glue these pieces together using two applications of the nerve lemma, which allows us to transfer this local connectivity to the connectivity of K , and finally transfer this to a lower bound on the \mathbb{Z}_2 -coindex of $\text{VR}(Q_n, k)$.

1.3 VC dimension and coindex

The VC dimension of a matrix $A \in \{-1, 1, *\}^{M \times N}$ is the largest number d for which there exists a set of columns $S \subseteq [N]$ with $|S| = d$ such that every sign pattern $P \in \{-1, 1\}^S$ appears as the restriction of some row of A to S . It is well known that by Radon's theorem, $\text{VC}(A)$ provides a lower bound on sign-rank. However, this bound is typically weak as VC dimension only captures local combinatorial structure: there exist matrices with constant VC dimension whose sign-rank is polynomially large [AMY16].

Here, VC dimension of our sign complex framework has a natural topological interpretation. Recall that the vertex set of the sign complex $S(A)$ is

$$\{1^+, 1^-, \dots, N^+, N^-\},$$

and that $S(A)$ is a subcomplex of the boundary of the crosspolytope $\partial\Diamond_N$. Let $\omega_\Diamond(K)$ of a free \mathbb{Z}_2 -complex be the largest d for which the complex K has $\partial\Diamond_d$ as an induced free \mathbb{Z}_2 -subcomplex. Then it is easy to see that

$$\frac{\omega_\Diamond(S(A))}{2} \leq \text{VC}(A) \leq \omega_\Diamond(S(A))$$

See Theorem 26. Since $\partial\Diamond_d$ is homeomorphic to the sphere \mathbb{S}^{d-1} with the antipodal action, its presence inside $S(A)$ immediately implies a lower bound on the \mathbb{Z}_2 -coindex, namely, the largest k for which there is a continuous \mathbb{Z}_2 -equivariant map from \mathbb{S}^k to $|S(A)|$.

Proposition 6. *For every partial sign matrix A ,*

$$\text{VC}(A) - 1 \leq \text{coind}_{\mathbb{Z}_2}(S(A)).$$

Overall, we obtain the following chain of inequalities:

$$\text{VC}(A) - 1 \leq \text{coind}_{\mathbb{Z}_2}(S(A)) \leq \text{ind}_{\mathbb{Z}_2}(S(A)) \leq \text{rank}^\pm(A) - 1. \quad (5)$$

Hence the classical VC lower bound on sign-rank appears in our framework as a very special case of a topological obstruction: it corresponds (up to a factor of 2) to the existence of a large antipodal crosspolytope embedded inside the sign complex. However, the topological viewpoint is considerably more flexible. While VC dimension only detects the presence of a crosspolytope boundary inside $S(A)$, the \mathbb{Z}_2 -index can capture much more complicated global topology. This additional power is crucial for obtaining strong lower bounds for sign-rank in settings where VC dimension remains constant.

A similar inequality as in Proposition 6 in has also been independently introduced by [CMW25] in the context of learning theory where they associate a simplicial complex similar to the sign complex, to a total concept class and relate its coindex to VC dimension, list replicability, and many other important learnability parameters. Also in [BHH⁺25], coindex has been proven to lower bound *simplicial covering dimension* which is another fundamental parameter characterizing list replicability. Both of these works have led to powerful applications in learning theory.

Nevertheless, even though coindex is useful for lower bounding fundamental learning parameters, but \mathbb{Z}_2 -index captures more *global obstructions* and can give stronger sign-rank lower bounds. In Section 6.3 we use classical constructions in topology that give sign complexes with coindex equal to 1, but index arbitrarily large. This is done via another topological invariant called *Stiefel-Whitney height* that sits between coindex and index.

1.4 Separation results

A basic remaining question we need to understand is to what extent the complexity measures in the chain of inequalities in Equation (5) can be separated.

Separating \mathbb{Z}_2 -index and sign-rank. We split to the case of total matrices and the more general case of partial matrices.

1. **Total matrices.** We show that a random $N \times N$ total sign matrix A has $\text{ind}_{\mathbb{Z}_2}(S(A)) = O(\log N)$ with high probability, while its sign-rank is $\Omega(N)$ [AFR85]. Similarly we show that for $N = 2^n$, the N by N Hadamard matrix H_N has $\text{ind}_{\mathbb{Z}_2}(S(H_N))$ at most $2 \log N$. However, it is already known, from seminal work of Forster [For02] that its sign-rank is at least \sqrt{N} .

More generally, we introduce a combinatorial parameter, for an arbitrary free \mathbb{Z}_2 -complex K , as the *height* of its associated partial sign matrix A_K , and show that $\text{ind}_{\mathbb{Z}_2}(K)$ is at most twice its height. This provides a systematic way to obtain logarithmic upper bounds on \mathbb{Z}_2 -index even when the underlying linear dimension is large. For more details, see Section 6.1.

2. **Partial matrices.** We construct random $N \times N$ partial matrices built on top of the incidence matrix of the finite projective plane that has \mathbb{Z}_2 -index equal to 1, but has sign-rank $\Omega(\sqrt{N})$. For more details, see Section 6.2.

We leave the following remaining important question as open.

Question 7. Does there exist a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that for every total sign matrix A ,

$$\text{rank}^{\pm}(A) \leq f(\text{ind}_{\mathbb{Z}_2}(S(A)))?$$

If the answer to Question 7 is positive, then theories of \mathbb{Z}_2 -index and sign-rank for total matrices will be equivalent in the $O(1)$ regime. This will have major consequences, as one can reformulate a large number of fundamental open problems related to sign-rank entirely in the language of topology. Some of these problems will be discussed in conclusion in Section 7.

Separating VC dimension and coindex. Next, we address the question of separating VC and coindex. We obtain the following separation using a result of Adiprasito et al. [AAK22] on triangulations of the projective space with few vertices. For details see Section 6.4.

Proposition 8 (Separating VC and coindex). *There exists a partial matrix $A \in \{-1, 1, *\}^{M \times N}$ such that $\text{VC}(M) = 1$ but*

$$\text{coind}_{\mathbb{Z}_2}(S(A)) = \Omega\left(\frac{\log^2 N}{\log \log N}\right).$$

However, in the case of total matrices we leave the possibility that coindex and VC dimension are qualitatively equivalent which if true, will entirely reframe learnability of total concept classes (constant VC) as a topological property of having bounded coindex.

Question 9. Does there exist a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every total sign matrix A ,

$$\text{coind}_{\mathbb{Z}_2}(S(A)) \leq f(\text{VC}(A))? \quad (6)$$

We will give another interesting implication of Question 9 in the concluding remarks in Section 7 on the VC dimension of disambiguations of Gap Hamming Distance to total functions.

1.5 Paper organization

In Section 2 we review the necessary background on sign-rank and \mathbb{Z}_2 -equivariant topology. In Section 3 we introduce the main framework. In Section 4 we show the lower bound for the Gap Hamming Distance matrix. In Section 5 we discuss VC dimension and other basic properties of the sign complex. In Section 6 we show quantitative gaps between VC dimension, coindex, index and sign-rank. Finally, in Section 7 we conclude with some open problems.

2 Background

In this section we introduce the necessary background on sign-rank, Gap Hamming Distance, and \mathbb{Z}_2 -topology.

2.1 Sign-rank

The *sign-rank* of a total sign matrix $A_{M \times N} \in \{-1, 1\}^{M \times N}$, is the minimum rank of a real matrix $B_{M \times N}$ such that

$$\text{sign}(B_{x,y}) = A_{x,y} \quad \text{for all } (x,y) \in [M] \times [N].$$

This definition extends naturally to partial sign matrices: if $A_{x,y} = *$, then the corresponding entry $B_{x,y}$ may be any real number. In other words, the sign-rank of $A \in \{-1, 1, *\}^{M \times N}$, denoted by $\text{rank}^\pm(A)$, is the smallest integer d for which there exist vectors

$$u_1, \dots, u_m \in \mathbb{R}^d \setminus \{0\} \quad \text{and} \quad v_1, \dots, v_n \in \mathbb{R}^d \setminus \{0\}$$

such that for every specified entry $A_{ij} \in \{\pm 1\}$ we require

$$\text{sign}(\langle u_i, v_j \rangle) = A_{ij}.$$

Equivalently, each row vector u_i defines the homogeneous hyperplane

$$H_i := \{x \in \mathbb{R}^d : \langle u_i, x \rangle = 0\},$$

and it strictly separates the column-point sets

$$C_i^+ := \{v_j : A_{ij} = +1\}, \quad C_i^- := \{v_j : A_{ij} = -1\},$$

in the sense that $\langle u_i, v \rangle > 0$ for all $v \in C_i^+$ and $\langle u_i, v \rangle < 0$ for all $v \in C_i^-$. This viewpoint is used in learning theory, where sign-rank is known as *dimension complexity*, capturing the smallest dimension d in which a concept class can be realized as a family of half-spaces and points in \mathbb{R}^d .

2.2 Background on \mathbb{Z}_2 -Topology

Here we discuss a minimal background on \mathbb{Z}_2 -topology that we need to use, and refer the reader to [Mat03, Koz08, DL13] for more details.

Abstract simplicial complex. An *abstract simplicial complex* K on a finite vertex set V is a family $K \subseteq 2^V$ such that:

1. $\{v\} \in K$ for every $v \in V$,
2. if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$.

Elements $\sigma \in K$ are called *simplices*. The dimension of σ is $\dim(\sigma) = |\sigma| - 1$, and $\dim(K)$ is the maximum dimension of a simplex in K . A *facet* of K is a maximal simplex under inclusion.

As a basic example, the *full d -dimensional simplex* is the complex

$$\Delta^d := 2^{[d+1]},$$

i.e., it consists of all subsets of a $(d + 1)$ -element vertex set.

The *k -skeleton* of K is the subcomplex

$$K^{(k)} := \{\sigma \in K : \dim(\sigma) \leq k\}.$$

Let K and L be simplicial complexes on vertex sets $V(K)$ and $V(L)$. A map on vertices $\varphi: V(K) \rightarrow V(L)$ induces a *simplicial map* $\varphi: K \rightarrow L$ if for every simplex $\sigma \in K$ the image $\varphi(\sigma) := \{\varphi(v) : v \in \sigma\}$ is a simplex of L (i.e. $\varphi(\sigma) \in L$). The simplicial map φ is called an *isomorphism* if

$$\sigma \in K \iff \varphi(\sigma) \in L.$$

In which case, we say that K and L are *isomorphic* and write $K \cong L$.

A \mathbb{Z}_2 -*action* on K is a simplicial map $\tau: K \rightarrow K$ such that $\tau^2 = \text{id}$ (equivalently, $\tau(\tau(v)) = v$ for all vertices v). Furthermore, the map τ is *free* if $\tau(\sigma) \neq \sigma$ for any simplex $\sigma \in K$. A free \mathbb{Z}_2 -action is also called an *involution*.

Let (K, τ_K) and (L, τ_L) be simplicial complexes equipped with free \mathbb{Z}_2 -actions. A simplicial map $f: K \rightarrow L$ is called *\mathbb{Z}_2 -equivariant* if it commutes with the involutions, that is,

$$f(\tau_K(v)) = \tau_L(f(v)) \quad \text{for every vertex } v \in V(K).$$

Equivalently,

$$f \circ \tau_K = \tau_L \circ f$$

as maps of simplicial complexes.

Let K be a simplicial complex with an involution $\tau: K \rightarrow K$. It follows that no simplex of K contains both v and $\tau(v)$ for any vertex v , hence K can be viewed as a subcomplex of the boundary of the crosspolytope which is discussed next.

Given simplicial complexes K and L on disjoint vertex sets, their *join* $K * L$ is defined as

$$K * L := \{\sigma \cup \tau : \sigma \in K, \tau \in L\}.$$

If K and L do not necessarily have disjoint vertex sets, their *deleted join* is

$$K *_{\Delta} L := \{(\sigma \times \{1\}) \cup (\tau \times \{2\}) : \sigma \in K, \tau \in L, \text{ and } \sigma \cap \tau = \emptyset\}.$$

The boundary of the crosspolytope. The n -dimensional crosspolytope \diamond_d is the convex hull of $\{\pm e_1, \dots, \pm e_d\} \subset \mathbb{R}^d$. Its boundary complex, denoted by $\partial \diamond_d$ is viewed here as an abstract simplicial complex: the vertices are

$$\{1^+, 1^-, \dots, d^+, d^-\},$$

and a subset forms a simplex if it does not contain both i^+ and i^- for any i . The involution exchanging i^+ and i^- defines a free \mathbb{Z}_2 -action. Hence $\partial \diamond_d$ is a free \mathbb{Z}_2 -simplicial complex. In fact, any free \mathbb{Z}_2 simplicial complex on $2n$ vertices is a subcomplex of $\partial \diamond_d$. Equivalently, $\partial \diamond_d$ can be expressed as the deleted join of two copies of the simplex:

$$\partial \diamond_d \cong \Delta^{d-1} *_\Delta \Delta^{d-1}.$$

Finally, any free \mathbb{Z}_2 -complex can be viewed as a subcomplex of the crosspolytope.

Lemma 10. *Let K be a simplicial complex on $2N$ vertices with a \mathbb{Z}_2 -action τ that is free. Then K is (isomorphic to) a subcomplex of $\partial \diamond_N$.*

Proof. Since the action is free, the vertices of K split into disjoint orbits of size 2; choose one representative v_i from each orbit so that the vertex set of K decomposes as

$$V(K) = \{v_1, \tau(v_1), \dots, v_N, \tau(v_N)\}.$$

Denote the vertices of $\partial \diamond_N$ with $\{1^+, 1^-, \dots, N^+, N^-\}$. Define a map $\phi : V(K) \rightarrow \{1^+, 1^-, \dots, N^+, N^-\}$ by

$$\phi(v_i) = i^+, \quad \phi(\tau(v_i)) = i^-$$

and note that ϕ extends to an injective and \mathbb{Z}_2 equivariant map $\phi : K \hookrightarrow \partial \diamond_N$.

It remains to check that ϕ extends to a simplicial map. Let $\sigma \in K$ be any simplex. Strong freeness implies $\phi(\sigma)$ contains at most one of $\{i^+, i^-\}$ for each i , hence $\phi(\sigma)$ is a simplex of $\partial \diamond_N$. \square

Geometric realization in the crosspolytope. Let K be a free \mathbb{Z}_2 simplicial complex on $2n$ vertices. To talk about various topological properties of K such as connectivity, homotopy, and \mathbb{Z}_2 -index, we pass to a geometric realization of K , denoted by $|K|$. By Lemma 10, we may identify K with a subcomplex of $\partial \diamond_N$ with vertex set

$$\{1^+, 1^-, \dots, N^+, N^-\}.$$

We realize K geometrically inside \mathbb{R}^N . Let $\phi : V(K) \rightarrow \mathbb{R}^N$ be the embedding defined by

$$\phi(i^+) = e_i, \quad \phi(i^-) = -e_i$$

where $\{e_1, \dots, e_N\}$ is the standard basis of \mathbb{R}^N . For each simplex $\sigma \in K$, define its geometric realization as

$$|\sigma| = \left\{ \sum_{v \in \sigma} \lambda_v \phi(v) : \lambda_v \geq 0, \sum_{v \in \sigma} \lambda_v = 1 \right\}.$$

The geometric realization of K is

$$|K| = \bigcup_{\sigma \in K} |\sigma| \subseteq |\partial \diamond_N| \subset \mathbb{R}^N.$$

The involution τ exchanging i^+ and i^- corresponds to the map

$$\tau(x) = -x,$$

and hence induces a free continuous \mathbb{Z}_2 -action on $|K|$.

\mathbb{Z}_2 -index and coindex. Let K be a free \mathbb{Z}_2 -complex. The \mathbb{Z}_2 -index of K is defined as

$$\text{ind}_{\mathbb{Z}_2}(K) := \min\{d \geq 0 : \exists \mathbb{Z}_2\text{-equivariant map } |K| \rightarrow \mathbb{S}^d\}.$$

Here \mathbb{S}^d is equipped with the antipodal action $x \mapsto -x$. In particular, if K admits no equivariant map to \mathbb{S}^{d-1} , then $\text{ind}_{\mathbb{Z}_2}(K) \geq d$. There is also the dual notion of \mathbb{Z}_2 -coindex:

$$\text{coind}_{\mathbb{Z}_2}(K) := \max\{d \geq 0 : \exists \text{ a } \mathbb{Z}_2\text{-equivariant map } \mathbb{S}^d \rightarrow |K|\}.$$

It is immediate from the definitions that

$$\text{coind}_{\mathbb{Z}_2}(K) \leq \text{ind}_{\mathbb{Z}_2}(K).$$

Connectivity. A standard method to provide a lower bound for \mathbb{Z}_2 -index of a topological space X is to show that X is highly-connected. A topological space X is r -connected if for every $k \leq r$, every continuous map

$$f: \mathbb{S}^k \longrightarrow X$$

admits a continuous extension

$$\tilde{f}: B^{k+1} \longrightarrow X$$

on the unit ball B^{k+1} . The following classical fact relates connectivity with \mathbb{Z}_2 -coindex.

Proposition 11 ([Mat03], Proposition 5.3.2). *Let X be a r -connected \mathbb{Z}_2 -space. Then $\text{coind}_{\mathbb{Z}_2}(X) \geq (r + 1)$.*

Moreover, if it is possible to equivariantly map such X to some other space Y then the bound carries over as well. This is particularly useful in a scenario where Y is a \mathbb{Z}_2 -space of low or unknown (global) connectivity yet one can identify a highly connected symmetric subspace of Y .

Corollary 12. *Let Y be a \mathbb{Z}_2 -space. Suppose that there exists a r -connected \mathbb{Z}_2 -space X admitting \mathbb{Z}_2 -equivariant map $f: X \rightarrow Y$. Then $\text{ind}_{\mathbb{Z}_2}(Y) \geq (r + 1)$.*

Covering and nerve lemmas. An important approach to establish connectivity of a space is to cover it via highly connected pieces and then apply a nerve lemma.

Let $X = |K|$ be the geometric realization of a simplicial complex K . A family of subcomplexes $\mathcal{C} = \{C_i\}_{i \in I}$ is called a *cover* of X if

$$X = \bigcup_{i \in I} C_i.$$

Furthermore, \mathcal{C} is called a *good cover* if for every nonempty finite intersection

$$C_{i_1} \cap \cdots \cap C_{i_s}$$

the intersection is contractible. The *nerve* of \mathcal{C} , denoted by $\text{Nrv}(\mathcal{C})$, is the simplicial complex with vertex set I where a finite subset $\{i_1, \dots, i_s\} \subset I$ spans a simplex if and only if

$$C_{i_1} \cap \cdots \cap C_{i_s} \neq \emptyset.$$

Lemma 13. *Let $X = |K|$ be the geometric realization of a simplicial complex K and let $\mathcal{C} = \{C_i\}_{i \in I}$ be a finite good cover of X by subcomplexes. Then the geometric realization of $\text{Nrv}(\mathcal{C})$ is homotopy equivalent to X : $|\text{Nrv}(\mathcal{C})| \simeq X$. In particular $|\text{Nrv}(\mathcal{C})|$ and X have the same connectivity.*

Recall that an N -dimensional *CW-complex* is defined inductively starting from a discrete space $X^{(0)}$, the 0-skeleton or vertex set, and for $n \leq N$ constructing the n -skeleton $X^{(n)}$ from $X^{(n-1)}$ by gluing a disjoint union of n -dimensional balls into $X^{(n-1)}$ via continuous maps from their boundaries to $X^{(n-1)}$. The following more general nerve lemma is useful for us [Bjö03].

Lemma 14. *Let $\mathcal{C} = \{C_i\}_{i \in I}$ be a finite cover of a CW-complex X by subcomplexes C_i . Suppose there is k such that*

1. *The nerve $\text{Nrv}(\mathcal{C})$ is k -connected.*
2. *Every nonempty finite intersection $C_{i_1} \cap \dots \cap C_{i_s}$ is $(k - s + 1)$ -connected.*

Then X is k -connected.

Order Complex and Barycentric subdivision. Let (P, \leq) be a finite poset. The *order complex* of P , denoted $\Delta(P)$, is the simplicial complex whose simplices are the finite chains in P :

$$\Delta(P) = \{\{x_0, \dots, x_k\} \subseteq P : x_0 < \dots < x_k\}.$$

Vertices correspond to elements of P , and simplices correspond to strictly increasing chains.

For a simplicial complex K , let $\mathcal{F}(K)$ be its face poset (nonempty simplices ordered by inclusion). The *barycentric subdivision* of K is

$$\text{sd}(K) := \Delta(\mathcal{F}(K)).$$

Thus vertices of $\text{sd}(K)$ are simplices of K , and a simplex of $\text{sd}(K)$ is a chain

$$\sigma_0 \subsetneq \sigma_1 \subsetneq \dots \subsetneq \sigma_k.$$

There is a canonical homeomorphism

$$|\text{sd}(K)| \cong |K|,$$

so barycentric subdivision preserves homotopy type. If K carries a simplicial \mathbb{Z}_2 -action, then the induced action on $\mathcal{F}(K)$ makes $\text{sd}(K)$ a simplicial \mathbb{Z}_2 -complex. Freeness is preserved.

3 The sign complex framework

In this section we introduce the main framework to assign a free \mathbb{Z}_2 simplicial complex $S(A)$ to a given partial sign matrix A and vice versa. Then we show that sign-rank is equal to a linear analog of \mathbb{Z}_2 -index, and in particular $\text{ind}_{\mathbb{Z}_2}(S(A))$ is a lower bound on $\text{rank}^{\pm}(A)$.

3.1 Sign complex

Let $A \in \{-1, 1, *\}^{M \times N}$ be a partial sign matrix. For each row $r \in \{-1, 1, *\}^n$, define the simplex

$$\sigma_r^+ := \{i^+ : r_i = +1\} \cup \{i^- : r_i = -1\},$$

and its antipodal copy

$$\sigma_r^- := \{i^- : r_i = +1\} \cup \{i^+ : r_i = -1\}.$$

The *sign complex* $S(A)$ is the simplicial complex on vertex set

$$\{1^+, 1^-, \dots, N^+, N^-\}$$

whose simplices are all subsets of σ_r^+ or σ_r^- for some row r of A . The involution exchanging i^+ and i^- defines a free \mathbb{Z}_2 -action on $S(A)$. This action turns $S(A)$ into a *finite free \mathbb{Z}_2 -simplicial complex*. This is because by construction, no face contains both j^+ and j^- .

Lemma 15 (Universality of sign complexes). *Every finite free \mathbb{Z}_2 simplicial complex K is isomorphic to $S(A_K)$ for some partial sign matrix A_K .*

Proof. Let τ be the involution on K , hence vertices come in disjoint pairs $\{v, \tau(v)\}$. Index these pairs by $[N]$, labeling them j^+, j^- . Let \mathcal{F} be the set facets of K . Note that if $F \in \mathcal{F}$, then $\tau(F) \in \mathcal{F}$. Hence facets come in τ -pairs. Pick an arbitrary facet from each orbit of this action and let F_1, \dots, F_m be such representatives.

Define a matrix $A_K \in \{-1, 1, *\}^{M \times N}$ by

$$(A_K)_{i,j} = \begin{cases} +1 & \text{if } j^+ \in F_i, \\ -1 & \text{if } j^- \in F_i, \\ * & \text{otherwise.} \end{cases}$$

Since K is free, each facet contains at most one of j^+, j^- , hence A_K is well-defined. By construction the facets of $S(A_K)$ are precisely the F_i and their τ -images. Thus $K \cong S(A_K)$. \square

3.2 Sign-rank is a linear analog of \mathbb{Z}_2 -index

Next we prove the main lemma that reformulates sign-rank as the linear analog of \mathbb{Z}_2 -index. For a free \mathbb{Z}_2 -complex K realized as $|K| \subset \mathbb{R}^N$, define

$$\text{ind}_{\mathbb{Z}_2}^{\text{lin}}(K) := \min \left\{ d \geq 0 : \exists \text{ linear } g: \mathbb{R}^N \rightarrow \mathbb{R}^{d+1} \text{ such that } 0 \notin g(|K|) \right\}.$$

Lemma 1 (Main lemma 1). *For every partial sign matrix A ,*

$$\text{ind}_{\mathbb{Z}_2}^{\text{lin}}(S(A)) = \text{rank}^{\pm}(A) - 1.$$

Since the map g induces a \mathbb{Z}_2 -equivariant map $f: |K| \rightarrow \mathbb{S}^d$, defined by $f(x) := \frac{g(x)}{\|g(x)\|}$, we have that for any free \mathbb{Z}_2 -complex K

$$\text{ind}_{\mathbb{Z}_2}(K) \leq \text{ind}_{\mathbb{Z}_2}^{\text{lin}}(K)$$

An immediate corollary of Lemma 1 is as follows.

Corollary 2 (Main inequality). *For every partial sign matrix A ,*

$$\text{ind}_{\mathbb{Z}_2}(S(A)) \leq \text{rank}^{\pm}(A) - 1.$$

Proof of Lemma 1. We prove both inequalities.

Part 1: $\text{ind}_{\mathbb{Z}_2}^{\text{lin}}(S(A)) \leq \text{rank}^{\pm}(A) - 1$. Assume $\text{rank}^{\pm}(A) \leq d$. By definition, there exist nonzero vectors

$$u_1, \dots, u_N \in \mathbb{R}^d \setminus \{0\}, \quad v_1, \dots, v_M \in \mathbb{R}^d \setminus \{0\}$$

such that for every specified entry $M_{rj} \in \{\pm 1\}$ we have

$$\text{sign}(\langle u_r, v_j \rangle) = A_{rj}.$$

Define a linear map $g: \mathbb{R}^N \rightarrow \mathbb{R}^d$ by setting

$$g(e_j) = v_j \quad (j \in [N]).$$

Since g is linear, we also have $g(-e_j) = -v_j$. We claim that $0 \notin g(|S(A)|)$. Note that $|S(A)|$ is the crosspolytope realization of $S(A)$.

Let $x \in |S(A)|$. Then x lies in some simplex of $S(A)$. Hence, there exists a row $r \in [M]$ and disjoint sets $J_r^+, J_r^- \subseteq [N]$ such that

$$M_{rj} = 1 \text{ for } j \in J_r^+, \quad M_{rj} = -1 \text{ for } j \in J_r^-,$$

and

$$x = \sum_{j \in J_r^+} \alpha_j e_j - \sum_{j \in J_r^-} \beta_j e_j$$

for some coefficients $\alpha_j, \beta_j > 0$ satisfying $\sum_{j \in J_r^+} \alpha_j + \sum_{j \in J_r^-} \beta_j = 1$. Therefore

$$g(x) = \sum_{j \in J_r^+} \alpha_j v_j - \sum_{j \in J_r^-} \beta_j v_j$$

and

$$\langle u_r, g(x) \rangle = \sum_{j \in J_r^+} \alpha_j \langle u_r, v_j \rangle - \sum_{j \in J_r^-} \beta_j \langle u_r, v_j \rangle.$$

For $j \in J_r^+$, we have $\langle u_r, v_j \rangle > 0$, and for $j \in J_r^-$, we have $\langle u_r, v_j \rangle < 0$. Hence every term on the right-hand side is strictly positive, therefore, $\langle u_r, g(x) \rangle > 0$ so $g(x) \neq 0$. By definition $\text{ind}_{\mathbb{Z}_2}^{\text{lin}}(S(A)) \leq d - 1$.

Part 2: $\text{ind}_{\mathbb{Z}_2}^{\text{lin}}(S(A)) \geq \text{rank}^\pm(A) - 1$. Assume $\text{ind}_{\mathbb{Z}_2}^{\text{lin}}(S(A)) \leq d - 1$. Then there exists a linear map $g : \mathbb{R}^N \rightarrow \mathbb{R}^d$ such that $0 \notin g(|S(A)|)$. Set

$$v_j := g(e_j) \in \mathbb{R}^d \quad (j \in [N]).$$

Fix a row $r \in [M]$, and define

$$J_r^+ := \{j \in [N] : M_{rj} = 1\}, \quad J_r^- := \{j \in [N] : M_{rj} = -1\}.$$

We claim that

$$\text{conv}\{v_j : j \in J_r^+\} \cap \text{conv}\{v_j : j \in J_r^-\} = \emptyset. \quad (7)$$

If not, there exist coefficients $\alpha_j \geq 0$ for $j \in J_r^+$ and $\beta_j \geq 0$ for $j \in J_r^-$ such that $\sum_{j \in J_r^+} \alpha_j = 1$, $\sum_{j \in J_r^-} \beta_j = 1$, and

$$\sum_{j \in J_r^+} \alpha_j v_j = \sum_{j \in J_r^-} \beta_j v_j.$$

Consider the point

$$x = \frac{1}{2} \sum_{j \in J_r^+} \alpha_j e_j + \frac{1}{2} \sum_{j \in J_r^-} \beta_j (-e_j).$$

Since the coefficients are nonnegative and sum to 1, the point x lies in the simplex of $|S(A)|$ corresponding to row r . But then

$$g(x) = \frac{1}{2} \sum_{j \in J_r^+} \alpha_j v_j - \frac{1}{2} \sum_{j \in J_r^-} \beta_j v_j = 0,$$

contradicting the assumption that g avoids 0 on $|S(A)|$, proving Equation (7).

By the hyperplane separation theorem, there exists a vector $u_r \in \mathbb{R}^d$ such that

$$\langle u_r, v_j \rangle > 0 \quad \text{for all } j \in J_r^+, \quad \langle u_r, v_j \rangle < 0 \quad \text{for all } j \in J_r^-.$$

Hence if $M_{rj} \in \{\pm 1\}$, then $\text{sign}(\langle u_r, v_j \rangle) = M_{rj}$. Doing this for each row r gives a sign-rank realization of M in \mathbb{R}^d . □

4 Main Application: Sign-rank of Gap Hamming Distance

Let $k < n/2$ and define $\text{GHD}_k^n : \{0, 1\}^n \times \{0, 1\}^n \rightarrow \{-1, 1, *\}$ by

$$\text{GHD}_k^n(x, y) = \begin{cases} 1 & \text{if } d_H(x, y) \leq k, \\ -1 & \text{if } d_H(x, y) \geq n - k, \\ * & \text{otherwise.} \end{cases}$$

The following fact is essentially trivial.

Fact 16. For any k and n , $\text{rank}^\pm(\text{GHD}_k^n) \leq 2k + 1$.

Proof. Define

$$u(x) := ((-1)^{x_1}, \dots, (-1)^{x_{2k+1}}) \in \{-1, 1\}^{2k+1}, \quad v(y) := ((-1)^{y_1}, \dots, (-1)^{y_{2k+1}}) \in \{-1, 1\}^{2k+1}.$$

Suppose $d_H(x, y) \leq k$. Then the total number of disagreements between $u(x)$ and $v(y)$ is at most k , hence the total number of agreements is at least $k + 1$. Therefore, $\text{sign}(\langle u(x), v(y) \rangle) = 1$. The case of $d_H(x, y) \geq n - k$ is similar. □

The main result of this section is to prove an almost sharp lower bound on $\text{rank}^\pm(\text{GHD}_k^n)$ for any choice of $k < n/2$.

Theorem 3 (Main application). *For any $k < n/2$,*

$$\text{rank}^\pm(\text{GHD}_k^n) \geq (1 - o_k(1)) \cdot 2k.$$

where the $o_k(1)$ term is $O\left(\sqrt{\frac{\log k}{k}}\right)$.

4.1 Weaker bound via CW-complex of the hypercube

We prove the weaker bound as a warm up.

Theorem 17 (Weak bound). *For any $k < n/2$, $\text{coind}_{\mathbb{Z}_2}(S(\text{GHD}_k^n)) \geq k$.*

The main idea is to give a \mathbb{Z}_2 equivariant map from the k -skeleton of the hypercube, denoted by $H_n^{(k)}$, which is known to be $(k - 1)$ -connected, to our complex $S(\text{GHD}_k^n)$. This approach is not able to improve the bound beyond k because the k -skeleton is not k -connected.

Before proving Theorem 17, we need to discuss connectivity of the skeleton of the hypercube.

The k -skeleton of the hypercube. Let H_n denote the boundary of the CW-complex realization of the hypercube graph Q_n . Denote by $H_n^{(r)}$ the r -skeleton of H_n . For the background on CW-complexes the reader is referred to [Mat03]. Informally, facets of $H_n^{(r)}$ are all the r -dimensional faces of the n -dimensional hypercube. Formally, let $I \subseteq [n]$ with $|I| = r$, and let $a \in \{0, 1\}^{[n] \setminus I}$. The corresponding r -dimensional face of the cube $[0, 1]^n$ is

$$F_{I,a} = \{x \in [0, 1]^n : x_i = a_i \text{ for all } i \notin I\}.$$

which is homeomorphic to $[0, 1]^r$. Also its vertex set is $F_{I,a} \cap \{0, 1\}^n$. We need the following lemma (see, e.g., [Bjö95, Sec. 11]).

Lemma 18. $H_n^{(r)}$ is $(r-1)$ -connected.

Proof of Theorem 17. For $x \in \{0, 1\}^n$ define the Hamming ball

$$B(x, k) := \{y \in \{0, 1\}^n : d_H(x, y) \leq k\}.$$

Let \bar{x} denote the bitwise complement of x . The complex $S(\text{GHD}_k^n)$ is defined on the vertex set $V^- \sqcup V^+$, where V^-, V^+ are two disjoint copies of $\{0, 1\}^n$. By definition of GHD_k^n , its maximal simplices are precisely the joins

$$B(x, k)^- * B(\bar{x}, k)^+, \quad x \in \{0, 1\}^n.$$

We are going to construct a \mathbb{Z}_2 -equivariant map h from $|H_n^{(k)}|$ to $|S(\text{GHD}_k^n)|$, where $|H_n^{(k)}|$ is equipped with a flip map $x \mapsto \bar{x}$. This will finish the proof, because by Lemma 18, $H_n^{(k)}$ is $(k-1)$ -connected, hence

$$\text{coind}_{\mathbb{Z}_2}(S(\text{GHD}_k^n)) \geq \text{coind}_{\mathbb{Z}_2}(H_n^{(k)}) \geq k.$$

where the last inequality is via Proposition 11.

The map h . Define a \mathbb{Z}_2 -map

$$h : H_n^{(k)} \longrightarrow |S(\text{GHD}_k^n)|$$

on vertices by

$$h(y) := \frac{1}{2} y^- + \frac{1}{2} (\bar{y})^+ \in |V^- * V^+|, \quad y \in \{0, 1\}^n,$$

and extend linearly over each face of $H_n^{(k)}$ using any symmetric triangulation without additional vertices. Moreover, for every vertex y we have

$$h(\bar{y}) = \frac{1}{2} (\bar{y})^- + \frac{1}{2} y^+ = \tau\left(\frac{1}{2} y^- + \frac{1}{2} (\bar{y})^+\right) = \tau(h(y)),$$

so h is \mathbb{Z}_2 -equivariant.

It remains to check that h lands in $|S(\text{GHD}_k^n)|$. Let F be any face of $H_n^{(k)}$ of dimension at most k . Choose a vertex $x_0 \in F$. Since F is a face of dimension at most k in the cube, there exists a set $J \subseteq [n]$ with $|J| \leq k$ such that only coordinates in J vary on F . Hence for every vertex $y \in F$,

$$d_H(x_0, y) \leq k,$$

so $y \in B(x_0, k)$ and similarly $\bar{y} \in B(\bar{x}_0, k)$. Therefore, for every vertex $y \in F$,

$$h(y) \in |B(x_0, k)^- * B(\bar{x}_0, k)^+|,$$

which is a simplex of $S(\text{GHD}_k^n)$. Because h is affine on F , we obtain

$$h(F) \subseteq |B(x_0, k)^- * B(\bar{x}_0, k)^+| \subseteq |S(\text{GHD}_k^n)|.$$

Thus h is a well-defined \mathbb{Z}_2 -map $H_n^{(k)} \rightarrow |S(\text{GHD}_k^n)|$. □

4.2 Strong bound via Vietoris–Rips complex of the hypercube

In this section, we obtain an essentially sharp bound on the $\text{coind}_{\mathbb{Z}_2}(S(\text{GHD}_k^n))$.

Theorem 4. *For any $k < n/2$,*

$$\text{coind}_{\mathbb{Z}_2}(S(\text{GHD}_k^n)) \geq (1 - o_k(1)) \cdot 2k.$$

To prove this result, we work with the Vietoris–Rips complex of the hypercube graph. We give a brief description as follows. The reader is referred to [EH10, Ghr14, CM21] for more details.

Definition 19 (Vietoris–Rips complex of the hypercube.). For $r \in \mathbb{N}$, the Vietoris–Rips complex $\text{VR}(Q_n, k)$ is the simplicial complex on vertex set $\{0, 1\}^n$ whose simplices are all subsets of Hamming diameter at most k . Equivalently, for a subset $\sigma \subseteq \{0, 1\}^n$,

$$\sigma \in \text{VR}(Q_n, k) \iff \forall x, y \in \sigma, d_H(x, y) \leq k.$$

It is easy to see that the map h , given in Theorem 17, gives a \mathbb{Z}_2 -equivariant map from $\text{VR}(Q_n, k)$ to $S(\text{GHD}_k^n)$. Hence we immediately get the following lemma.

Lemma 20. *For any $k < \frac{n}{2}$,*

$$\text{coind}_{\mathbb{Z}_2}(S(\text{GHD}_k^n)) \geq \text{coind}_{\mathbb{Z}_2}(\text{VR}(Q_n, k)).$$

Proof. Define a \mathbb{Z}_2 -map

$$h: |\text{VR}(Q_n, k)| \longrightarrow |S(\text{GHD}_k^n)|$$

on vertices by

$$h(y) := \frac{1}{2}y^- + \frac{1}{2}(\bar{y})^+ \in |V^- * V^+|, \quad y \in \{0, 1\}^n,$$

extend it linearly on each face, and the same argument as in Theorem 17 implies that such extension still lies in $|S(\text{GHD}_k^n)|$. \square

Therefore the goal of the rest of this section is to prove an essentially sharp lower bound on $\text{coind}_{\mathbb{Z}_2}(\text{VR}(Q_n, k))$, which could be of independent interest.

Lemma 5 (Main lemma 2). *For any $k < \frac{n}{2}$,*

$$\text{coind}_{\mathbb{Z}_2}(\text{VR}(Q_n, k)) \geq (1 - o_k(1)) 2k.$$

Note that the bound in Lemma 5 is essentially sharp

$$\text{coind}_{\mathbb{Z}_2}(\text{VR}(Q_n, k)) \leq \text{ind}_{\mathbb{Z}_2}(\text{VR}(Q_n, k)) \leq \text{rank}^\pm(\text{GHD}_k^n) - 1 \leq 2k$$

The key advantage of the $\text{VR}(Q_n, k)$ is that it reflects the Hamming metric on $\{0, 1\}^n$ rather than only the cubical face structure as captured by $H_n^{(k)}$. As k increases, in particular when k is larger than $\frac{n}{2}$ by more than a standard deviation, k -neighborhood of each vertex starts to cover all but small portion of the hypercube, which implies strong connectivity properties of $\text{VR}(Q_n, k)$. In particular, the following recent result of Bendersky and Grbic [BG23] quantifies this property.

Theorem 21 ([BG23], Theorem 2.3). *Let $\alpha_{n,k} = \frac{2^{n-1}}{\sum_{i=k+1}^n \binom{n}{i}}$. Then $\text{VR}(Q_n, k)$ is $(\alpha_{n,k} - 2)$ -connected.*

However, a limitation of the above is that the value of $\alpha_{n,k}$ becomes large only when the upper binomial tail $\sum_{i>k} \binom{n}{i}$ is small. By Chernoff bound, this happens precisely when

$$k \geq \frac{n}{2} + \Theta(\sqrt{n \log n}).$$

However, in Lemma 5 we are working in the opposite regime where $k < \frac{n}{2}$ can be arbitrarily small and Theorem 21 does not directly give nontrivial connectivity bounds for $\text{VR}(Q_n, k)$ itself. In fact, if say, $k = \frac{n}{2} - \Omega(n)$, balls of radius k contain only a $2^{-\Omega(n)}$ fraction of the vertices. Hence the link of each vertex x is only a tiny fraction of the cube, and the complex behaves like a union of many locally dense but globally poorly interacting clusters. Hence, one should not expect high connectivity. However, remember that we only need a lower bound on \mathbb{Z}_2 -index of $\text{VR}(Q_n, k)$. To do so, instead of lower bounding the connectivity of $\text{VR}(Q_n, k)$, we lower bound the connectivity of a carefully chosen \mathbb{Z}_2 -equivariant subcomplex of $\text{VR}(Q_n, k)$ which is sufficient to imply Lemma 5, since unlike connectivity, \mathbb{Z}_2 -coindex is monotone.

Lemma 22. *Let $k < \frac{n}{2}$. Then there exists a $(1 - o_k(1))2k$ -connected \mathbb{Z}_2 -complex that admits a \mathbb{Z}_2 -equivariant embedding into $\text{VR}(Q_n, k)$.*

4.3 Proof of Lemma 22

The main idea is to still exploit Theorem 21 by applying it not in dimension n , but inside carefully chosen lower-dimensional faces of the cube. Then gluing together these highly connected pieces via the nerve lemma produces a highly connected \mathbb{Z}_2 -equivariant subcomplex of $\text{VR}(Q_n, k)$.

We need the following as a corollary of Theorem 21.

Claim 23. Let $k \geq 5$ and $t \geq 2$ be chosen such that

$$k > \frac{t}{2} + 2\sqrt{t \log t}.$$

Then for any $t' \leq t$, $\text{VR}(Q_{t'}, k)$ is $(t-1)$ -connected.

Proof. Fix $t' \leq t$.

Case 1: $t' \leq k$. Then every subset of $\{0, 1\}^{t'}$ has Hamming diameter at most k , hence $\text{VR}(Q_{t'}, k)$ is the full simplex on $2^{t'}$ vertices. In particular, $\text{VR}(Q_{t'}, k)$ is contractible.

Case 2: $t' > k$. Since $t' \leq t$ and $k > \frac{t}{2} + 2\sqrt{t \log t}$, we also have

$$2k - t' \geq 2k - t > 4\sqrt{t \log t}.$$

Applying Chernoff bound gives

$$\sum_{i=k+1}^{t'} \binom{t'}{i} \leq 2^{t'} \exp\left(-\frac{(2k-t')^2}{2t'}\right).$$

Using $t' \leq t$ and $2k - t' \geq 2k - t$ we obtain

$$\frac{(2k-t')^2}{2t'} \geq \frac{(2k-t)^2}{2t} > \frac{(4\sqrt{t \log t})^2}{2t} = 8 \log t,$$

and hence

$$\sum_{i=k+1}^{t'} \binom{t'}{i} \leq 2^{t'} e^{-8 \log t} = 2^t t^{-8}.$$

Therefore

$$\alpha_{t',k} = \frac{2^{t'-1}}{\sum_{i=k+1}^{t'} \binom{t'}{i}} \geq \frac{2^{t'-1}}{2^{t'} t^{-8}} = \frac{1}{2} t^8.$$

In particular, for $t \geq 2$ we have $\alpha_{t',k} \geq t + 1$ (e.g. since $\frac{1}{2}t^8 - (t + 1) \geq 0$ for $t \geq 2$), and thus by Theorem 21 the complex $\text{VR}(Q_{t'}, k)$ is $(\alpha_{t',k} - 2)$ -connected, hence at least $(t - 1)$ -connected. \square

Returning to the proof of Lemma 22, let t be maximal such that

$$k > \frac{t}{2} + 2\sqrt{t \log t}. \quad (8)$$

Maximality of t implies that

$$k \leq \frac{t+1}{2} + 2\sqrt{(t+1) \log(t+1)}.$$

Rearranging gives

$$t \geq 2k - 1 - 4\sqrt{2k \log(2k)} \geq (1 - o_k(1))2k.$$

The subcomplex $\text{VR}^t(Q_n, k)$. Let $\text{VR}^t(Q_n, k)$ denote the subcomplex of $\text{VR}(Q_n, k)$ consisting of those simplices whose vertices

1. have diameter at most k , and
2. that subset lies in some t -dimensional face of the hypercube H_n .

In other words, each simplex of $\text{VR}^t(Q_n, k)$ is of the form $\sigma = \tau \cap F_{I,a}$, where τ is a simplex of $\text{VR}(Q_n, k)$ and

$$F_{I,a} = \{x \in \{0, 1\}^n : x_j = a_j \text{ for all } j \notin I\},$$

for some index set $I \subseteq [n]$ with $|I| = t$ and some fixed $a \in \{0, 1\}^{[n] \setminus I}$. Thus $\text{VR}^t(Q_n, k)$ is obtained by restricting attention to simplices supported on t -faces of the hypercube. Furthermore, observe that $\text{VR}^t(Q_n, k)$ is a \mathbb{Z}_2 -free subcomplex of $\text{VR}(Q_n, k)$. Indeed, the antipodal action $x \mapsto \bar{x}$ on Q_n preserves Hamming distance and sends t -dimensional faces to t -dimensional faces. Hence it preserves the defining conditions of $\text{VR}^t(Q_n, k)$, so $\text{VR}^t(Q_n, k)$ is \mathbb{Z}_2 -invariant.

In the rest of the proof we show that $\text{VR}^t(Q_n, k)$ is $(t - 1)$ -connected.

Connectivity of $\text{VR}^t(Q_n, k)$. The connectivity is established in the following steps.

1. Cover $\text{VR}^t(Q_n, k)$ with a suitable family \mathcal{C} and show every $\sigma \in \mathcal{C}$ is $(t - 1)$ -connected.
2. Show that the nerve of the cover \mathcal{C} itself is $(t - 1)$ -connected.
3. Apply the connectivity version of nerve lemma, to conclude that $\text{VR}^t(Q_n, k)$ is $(t - 1)$ -connected.

Step 1. For each t -dimensional face $F = F_{I,a}$ of the cube, define

$$U(F) := \text{VR}(Q_n, k)|_F.$$

Let

$$\mathcal{C} = \{U(F) : F \text{ is a } t\text{-face of } Q_n\}.$$

By definition of $\text{VR}^t(Q_n, k)$, \mathcal{C} forms a cover:

$$\text{VR}^t(Q_n, k) = \bigcup_F U(F).$$

Each F is canonically isomorphic to $\{0, 1\}^t$ by restricting to the free coordinates of F , and under this identification

$$U(F) \cong \text{VR}(Q_t, k).$$

Now consider a finite intersection

$$U(F_0) \cap \cdots \cap U(F_s).$$

If it is nonempty, then

$$F_0 \cap \cdots \cap F_s = F'$$

is a (possibly lower-dimensional) face of Q_n of dimension $t' \leq t$, and

$$U(F_0) \cap \cdots \cap U(F_s) = \text{VR}(Q_n, k)|_{F'} \cong \text{VR}(Q_{t'}, k).$$

The last complex is highly connected by Claim 23

Step 2. Here we show that the nerve $\mathcal{N} = \text{Nrv}(\mathcal{C})$ is $(t-1)$ -connected. A finite collection $\{F_0, \dots, F_s\}$ spans a simplex in \mathcal{N} iff

$$U(F_0) \cap \cdots \cap U(F_s) \neq \emptyset,$$

which by Step 1 holds iff

$$F_0 \cap \cdots \cap F_s \neq \emptyset.$$

which is true iff

$$\bar{F}_0 \cap \cdots \cap \bar{F}_s \neq \emptyset.$$

where $\bar{F} \subseteq [0, 1]^n$ is the face of the CW-complex of the geometric realization of hypercube H_n with vertex set $F = \bar{F} \cap \{0, 1\}^n$. Define a cover of the t -skeleton $(H_n)^{(t)}$

$$\mathcal{C}' = \{ \bar{F} : \bar{F} \text{ is a } t\text{-face of } H_n \}.$$

Since we know that $\text{Nrv}(\mathcal{C}')$ is isomorphic to $\text{Nrv}(\mathcal{C})$, it is enough to show that $\text{Nrv}(\mathcal{C}')$ is $(t-1)$ -connected. To show this, note that

$$(H_n)^{(t)} = \bigcup_{\bar{F} \in \mathcal{C}'} \bar{F}.$$

Each \bar{F} is contractible, and any nonempty finite intersection of t -faces is again a (lower-dimensional) face, hence contractible. Thus \mathcal{C}' is a *good cover* of $(H_n)^{(t)}$, hence by the good cover nerve Lemma 13

$$|\text{Nrv}(\mathcal{C}')| \simeq (H_n)^{(t)}.$$

Finally by Lemma 18, $(H_n)^{(t)}$, is $(t-1)$ -connected.

Step 3. We now apply the connectivity version of the Nerve Lemma, Lemma 14, to the cover \mathcal{C} of $\text{VR}^t(Q_n, k)$. We have established that

- From Step 1: every nonempty intersection of members of \mathcal{C} is $(t-1)$ -connected.
- From Step 2: the nerve $\text{Nrv}(\mathcal{C})$ is $(t-1)$ -connected.

Therefore Lemma 14 implies that

$$\text{VR}^t(Q_n, k) = \bigcup_{\sigma \in \mathcal{C}} \sigma$$

is $(t-1)$ -connected.

5 Basic properties of sign complex

5.1 Transposing the matrix

Let $A \in \{-1, 1, *\}^{M \times N}$ and write

$$S(A) = \bigcup_{i=1}^M (K_i^+ \cup K_i^-),$$

where K_i^+ and K_i^- are the two the subcomplexes induced by row i . Let \mathcal{N}_A be the nerve of this cover. Then we have the following.

Claim 24. $\mathcal{N}_A \cong S(A^t)$.

Proof. Vertices of the nerve correspond to the simplices K_i^+ and K_i^- , so we identify them with i^+ and i^- respectively. A face of \mathcal{N}_A has the form $R^+ \sqcup T^-$ with $R, T \subseteq [M]$. By definition of the nerve, this is a face iff

$$\bigcap_{i \in R} K_i^+ \cap \bigcap_{j \in T} K_j^- \neq \emptyset. \quad (9)$$

A point lies in this intersection iff there exists a column $c \in [N]$ such that:

$$A_{ic} = +1 \text{ for all } i \in R, \quad A_{jc} = -1 \text{ for all } j \in T,$$

or the same with all signs flipped (which corresponds to choosing c^- instead of c^+).

Vertices of $S(A)$ are c^+ and c^- for $c \in [N]$. Fix $c \in [N]$. Then, by unpacking the definitions of K_i^\pm :

$$c^+ \in K_i^+ \iff A_{ic} = +1, \quad c^+ \in K_j^- \iff A_{jc} = -1,$$

and similarly

$$c^- \in K_i^+ \iff A_{ic} = -1, \quad c^- \in K_j^- \iff A_{jc} = +1.$$

Therefore the intersection in (9) is nonempty if and only if there exists some column $c \in [N]$ such that either

$$A_{ic} = +1 \forall i \in R \quad \text{and} \quad A_{jc} = -1 \forall j \in T, \quad (10)$$

or the same condition with all signs flipped (corresponding to choosing c^- instead of c^+).

Now consider $S(A^t)$. Each column c of A becomes a row of A^t , producing a simplex whose positive and negative parts are

$$\{i : M_{ic} = +1\}, \quad \{i : A_{ic} = -1\}.$$

Hence $R^+ \sqcup T^-$ is a face of $S(A^t)$ iff there exists $c \in [N]$ such that

$$A_{ic} = +1 \text{ for all } i \in R, \quad A_{jc} = -1 \text{ for all } j \in T.$$

This is exactly the same condition as above. Therefore

$$R^+ \sqcup T^- \in \mathcal{N}_A \iff R^+ \sqcup T^- \in S(A^t).$$

□

5.2 VC dimension vs. \mathbb{Z}_2 -coindex

In this section, we show that VC dimension of A gives a lower bound on $\text{coind}_{\mathbb{Z}_2}(A)$. Let $A \in \{-1, 1, *\}^{M \times N}$, such that for each pattern $P \in \{-1, 1\}^S$, there is a row whose restriction to S is P .

Definition 25. For a free \mathbb{Z}_2 -complex K , let $\omega_\diamond(K)$ be the largest k such that there is a copy of $\partial\Diamond_k$ as a free \mathbb{Z}_2 -subcomplex of K .

It is easy to see that $\omega_\diamond(S(A))$, is equal to the notion of *dual sign-rank* which was introduced by Alon, Moran, and Yehudayoff [AMY16], as the largest number of columns that are ‘antipodally shattered’.

Theorem 26. *We have*

$$\frac{\omega_\diamond(S(A))}{2} \leq \text{VC}(A) \leq \omega_\diamond(S(A)) \leq \text{coind}_{\mathbb{Z}_2}(S(A)) + 1.$$

Proof. First we show $\text{VC}(A) \leq \omega_\diamond(S(A))$. Let $S \subseteq [N]$ be shattered. Consider the induced subcomplex of $S(A)$ on

$$V_S := \{j^+, j^- : j \in S\}.$$

For each $P \in \{-1, 1\}^S$, set

$$F_P := \{j^{P(j)} : j \in S\}.$$

Since S is shattered, for every P there exists a row i with $M_{i|S} = P$, and hence $F_P \subseteq \sigma_i^+ \subseteq S(A)$. Hence every choice of one vertex from each pair $\{j^+, j^-\}$ is a face of the induced subcomplex on V_S which is simplicially isomorphic to $\partial\Diamond_{|S|}$ and is invariant under $j^+ \leftrightarrow j^-$.

Now we show $\omega_\diamond(S(A)) \leq 2 \text{VC}(A)$. Suppose $S(A)$ contains a \mathbb{Z}_2 -invariant subcomplex $\partial\Diamond_d$. Then there exists $S \subseteq [N]$, $|S| = d$, such that for every $P \in \{-1, 1\}^S$ the face

$$F_P := \{j^{P(j)} : j \in S\}$$

lies in $S(A)$. By construction of $S(A)$, for each P there is a row whose restriction to S equals either P or $-P$. Hence at least one element from each antipodal pair $\{P, -P\}$ is realized, so the number of distinct patterns on S is at least 2^{d-1} . If $\text{VC}(A) < d/2$, then by Sauer–Shelah lemma the number of patterns realizable on S is at most

$$\sum_{i=0}^{\lfloor d/2 \rfloor - 1} \binom{d}{i} < 2^{d-1},$$

which is a contradiction.

Finally, to see $\omega_\diamond(S(A)) \leq \text{coind}_{\mathbb{Z}_2}(S(A)) + 1$, let $\omega_\diamond(S(A)) = d$ and note that $|\partial\Diamond_d|$ is \mathbb{Z}_2 -homeomorphic to \mathbb{S}^{d-1} with the antipodal action, hence

$$\text{coind}_{\mathbb{Z}_2}(\partial\Diamond_d) = d - 1$$

and by monotonicity of coindex,

$$d - 1 = \text{coind}_{\mathbb{Z}_2}(\partial\Diamond_d) \leq \text{coind}_{\mathbb{Z}_2}(S(A)).$$

□

6 Separations

The main goal of this section is to provide separation among parameters in the inequality

$$\text{VC}(A) \leq \text{coind}_{\mathbb{Z}_2}(S(A)) \leq \text{ind}_{\mathbb{Z}_2}(S(A)) \leq \text{rank}^\pm(A).$$

6.1 Index vs. sign-rank for total matrices

We work with two cases of total and partial matrices separately. For total matrices, we give a separation of $O(\log N)$ -vs- $\Omega(N)$ by considering random matrices where a linear lower bound on sign-rank is known from the work of Alon et. al, [AFR85]. For the well-known Hadamard matrix, we get a similar separation of $O(\log N)$ -vs- $\Omega(\sqrt{N})$ based on the work of Forster [For02].

For partial matrices we obtain a stronger separation of $\text{ind}_{\mathbb{Z}_2}(S(A)) = 1$ and $\text{rank}^\pm(A) = \Omega(\sqrt{N})$ by considering a random labeling of the incidence matrix of the finite projection plane.

We develop a general technique to upper bound \mathbb{Z}_2 -index, and apply it to two well-known families of matrices of high sign-rank: random matrices, and the Hadamard matrix.

Let $A \in \{\pm 1, *\}^{M \times N}$ be a partial sign matrix with column set $[N]$. For each row $i \in [M]$ define

$$R_i^+ := \{j \in [N] : A_{ij} = +1\}, \quad R_i^- := \{j \in [N] : A_{ij} = -1\}.$$

Let

$$\mathcal{R}(A) := \{R_i^+, R_i^- : i \in [M]\}$$

be the family of shore-parts of facets induced by A . Define the *intersection family* of $\mathcal{F}(A)$ by

$$\mathcal{I}(A) := \left\{ \bigcap_{t=1}^k F_t : k \geq 1, F_t \in \mathcal{R}(A) \right\} \setminus \{\emptyset\}.$$

Let

$$h(A) := \max \left\{ k : \exists S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k \text{ with each } S_i \in \mathcal{I}(A) \right\}$$

be the height (maximum strict chain length) of $\mathcal{I}(A)$ under inclusion. The main result of this section is the following.

Theorem 27 (Chain height bounds \mathbb{Z}_2 -index). *For any partial sign matrix A ,*

$$\text{ind}_{\mathbb{Z}_2}(S(A)) \leq 2h(A) - 1.$$

Later we apply the above to random matrices and the Hadamard matrix and show these matrices have logarithmic height. Before proving Theorem 27, we need one lemma about \mathbb{Z}_2 -equivariant maps between order complexes.

Lemma 28 (A poset map induces a simplicial \mathbb{Z}_2 -map on order complexes). *Let P, Q be finite posets with \mathbb{Z}_2 -actions τ and ν respectively. Let $\phi : P \rightarrow Q$ be order-preserving and \mathbb{Z}_2 -equivariant. Then ϕ induces a simplicial \mathbb{Z}_2 -equivariant map*

$$\phi : \Delta(P) \rightarrow \Delta(Q).$$

Proof. A simplex of $\Delta(P)$ is a chain

$$x_0 < x_1 < \cdots < x_t$$

in P . Since ϕ is order-preserving,

$$\phi(x_0) \leq \phi(x_1) \leq \cdots \leq \phi(x_t).$$

After deleting repetitions, this becomes a chain in Q , hence a simplex of $\Delta(Q)$. Therefore ϕ induces a simplicial map $\Delta(\phi) : \Delta(P) \rightarrow \Delta(Q)$.

Since ϕ is \mathbb{Z}_2 -equivariant, for every $x \in P$ we have

$$\phi(\tau(x)) = \nu(\phi(x)),$$

and therefore the induced simplicial map commutes with the \mathbb{Z}_2 -actions. So $\Delta(\phi)$ is a simplicial \mathbb{Z}_2 -equivariant map. \square

Proof of theorem 27. Let P be the poset of nonempty faces of $S(A)$, ordered by inclusion. Then the order complex $\Delta(P)$ coincides with $\text{sd}(S(A))$, and since the barycentric subdivision preserves \mathbb{Z}_2 -index, it is enough to show that

$$\text{ind}_{\mathbb{Z}_2}(\Delta(P)) \leq 2h(A) - 1.$$

Let \mathcal{G} be the family of simplices:

$$\mathcal{G} = \{R_i^+ * R_i^- \mid i \in [N]\} \cup \{R_i^- * R_i^+ \mid i \in [M]\}.$$

Define map $\phi: P \rightarrow P$ by

$$\phi(X) := \bigcap_{\substack{G \in \mathcal{G} \\ X \subseteq G}} G,$$

where X is an arbitrary face of $S(A)$. We have the following observations:

- We know that \mathcal{G} generates all simplices of $S(A)$, so ϕ is well-defined.
- Family \mathcal{G} is symmetric under exchanging i^- with i^+ , so ϕ preserves symmetry as well.
- Map ϕ is order-preserving. Indeed, if $X \subseteq X'$ then $\{G \in \mathcal{G} : X' \subseteq G\} \subseteq \{G \in \mathcal{G} : X \subseteq G\}$, hence $\phi(X) \subseteq \phi(X')$.

These observations, combined with Lemma 28 imply that ϕ induces a \mathbb{Z}_2 -equivariant map between corresponding order complexes.

$$\phi: \Delta(P) \rightarrow \Delta(\phi(P))$$

hence

$$\text{ind}_{\mathbb{Z}_2}(\Delta(P)) \leq \text{ind}_{\mathbb{Z}_2}(\Delta(\phi(P))).$$

So it remains to bound $\text{ind}_{\mathbb{Z}_2}(\Delta(\phi(P)))$. Since $\text{ind}_{\mathbb{Z}_2}(K) \leq \dim(K)$ for every free \mathbb{Z}_2 -complex K , it suffices to bound the dimension of $\Delta(\phi(P))$. A simplex of $\Delta(\phi(P))$ is a strict chain

$$X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_t \quad (X_i \in \phi(P)).$$

Each X_i has the form

$$X_i = A_i^+ \cup B_i^-,$$

with $A_i, B_i \in \mathcal{I}(A)$. Because the positive and negative vertices are disjoint,

$$X_i \subseteq X_{i+1} \iff A_i \subseteq A_{i+1} \text{ and } B_i \subseteq B_{i+1}.$$

Moreover, since the chain is strict, at each step at least one of these two inclusions is strict. Hence along the whole chain, the A_i 's can strictly increase at most $h(A) - 1$ times, and the B_i 's can strictly increase at most $h(A) - 1$ times. Hence

$$t - 1 \leq (h(A) - 1) + (h(A) - 1) = 2h(A) - 2,$$

so

$$t \leq 2h(A).$$

Therefore every simplex of $\Delta(\phi(P))$ has at most $2h(A)$ vertices, and so

$$\dim(\Delta(\phi(P))) \leq 2h(A) - 1.$$

□

Application I: Random matrices. Next we apply Theorem 27 to random matrices, and show with high probability, random matrices have logarithmic height.

Theorem 29. *Let $A \in \{\pm 1\}^{N \times N}$ be a random sign matrix whose entries are independent and uniform. Then there exists an absolute constant $C > 0$ such that with probability $1 - o(1)$:*

$$h(A) \leq C \log_2 N$$

which in particular implies

$$\text{ind}_{\mathbb{Z}_2}(S(A)) \leq 2C \log_2 N.$$

Proof. For $T \subseteq [N]$ and a sign pattern $s \in \{\pm 1\}^T$ define the *cell*

$$C(T, s) := \{j \in [N] : A_{ij} = s(i) \text{ for all } i \in T\}.$$

Every element of $\mathcal{I}(A)$ equals $C(T, s)$ for some (T, s) . Moreover, if

$$C(T', s') \subsetneq C(T, s),$$

then necessarily $T \subsetneq T'$. Hence along any strict chain in $\mathcal{I}(A)$, the parameter $|T|$ strictly decreases at each step.

By Lemma 30 (applied with $\varepsilon = \frac{1}{2}$), there exists an absolute constant $C_1 > 0$ such that with probability $1 - o(1)$:

$$\forall T \subseteq [N] \text{ with } |T| \geq \frac{3}{2} \log_2 N, \forall s \in \{\pm 1\}^T, \quad |C(T, s)| \leq C_1 \log N. \quad (11)$$

Fix an outcome matrix A where Equation (11) holds. Now consider a strict chain in $\mathcal{I}(A)$,

$$S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_M, \quad \text{where } S_j = C(T_j, s_j).$$

As observed above,

$$|T_1| > |T_2| > \cdots > |T_M|.$$

We split the chain indices $j \in [M]$ into two ranges according to $|T_j|$.

Range I: $|T_j| \geq \frac{3}{2} \log N$. By Equation (11), every S_j in this range has size at most $C_1 \log N$. Since the chain of S_j 's is strict, their sizes must increase by at least 1 each step; hence the number of indices j in Range I is at most $C_1 \log N$.

Range II: $|T_j| < \frac{3}{2} \log N$. In this range, we use the fact that the parameter $|T_j|$ decreases by at least 1 each step. Hence the number of indices j in Range II is at most $\frac{3}{2} \log N$. Overall we get that the length of the chain, m is at most

$$\left(\frac{3}{2} + C_1\right) \log N.$$

By choosing $C = \frac{3}{2} + C_1$ and applying Theorem 27 we get that $\text{ind}_{\mathbb{Z}_2}(S(A)) \leq 2h(A) - 1 \leq 2C \log N$. \square

Lemma 30. *Fix $\varepsilon > 0$. There exists a constant $C_1 = C_1(\varepsilon) > 0$ such that for a random matrix $A \in \{\pm 1\}^{N \times N}$ with independent uniform entries, with probability $1 - o(1)$, we have that simultaneously for all $T \subseteq [N]$ satisfying $|T| \geq (1 + \varepsilon) \log N$ and all $s \in \{\pm 1\}^T$:*

$$|C(T, s)| \leq C_1 \log N.$$

Proof. Fix $T \subseteq [N]$ with $|T| = t$ and $s \in \{\pm 1\}^T$. Then

$$|C(T, s)| \sim \text{Bin}(N, 2^{-t}).$$

Let $L := C_1 \log_2 N$ where C_1 is chosen later. By a union bound over L -subsets of columns,

$$\Pr [|C(T, s)| \geq L] \leq \binom{N}{L} (2^{-t})^L \leq \left(\frac{eN}{L}\right)^L 2^{-tL}.$$

Now union bound over all (T, s) with $|T| = t$: there are $\binom{N}{t} 2^t$ such pairs, hence

$$\Pr [\exists T, s : |T| = t, |C(T, s)| \geq L] \leq \binom{N}{t} 2^t \left(\frac{eN}{L}\right)^L 2^{-tL}.$$

Using $\binom{N}{t} \leq (en/t)^t$ and $t \geq (1 + \varepsilon) \log_2 n$, we get

$$\Pr [\exists T, s : |T| = t, |C(T, s)| \geq L] \leq \left(\frac{2eN}{t}\right)^t \left(\frac{eN}{L}\right)^L 2^{-tL}.$$

Summing this bound over all integers $t \geq (1 + \varepsilon) \log_2 n$, and choosing $C_1 = C_1(\varepsilon)$ sufficiently large, the factor 2^{-tL} dominates the other factors, giving total probability $o(1)$. \square

Application II: The Hadamard matrix. Let $N = 2^n$ and let $H = H_{N \times N}$ be the Hadamard matrix indexed by \mathbb{F}_2^n , defined by

$$H_{x,y} := (-1)^{\langle x,y \rangle}, \quad x, y \in \mathbb{F}_2^n,$$

where $\langle \cdot, \cdot \rangle$ is the standard dot product over \mathbb{F}_2 . It is known since Forster [For02] that

$$\text{rank}^\pm(H) \geq \sqrt{N} = 2^{n/2}.$$

Let $K = S(H)$ be the sign complex associated with H .

Lemma 31. *Let $H_{N \times N}$ be the Hadamard matrix as above. Then $h(H) = n + 1 = \log N + 1$.*

Proof. For each $x \in \mathbb{F}_2^n$, the row H_x defines the partition

$$R_x = \{y \in \mathbb{F}_2^n : \langle x, y \rangle = 0\}, \quad R_x^c = \{y \in \mathbb{F}_2^n : \langle x, y \rangle = 1\}.$$

Thus every member of $\mathcal{F}(H)$ is an affine hyperplane in \mathbb{F}_2^n .

Lower bound $h(H) \geq n + 1$. Let e_1, \dots, e_n be the standard basis of \mathbb{F}_2^n . The chain

$$\{0\} \subsetneq \langle e_1 \rangle \subsetneq \langle e_1, e_2 \rangle \subsetneq \dots \subsetneq \langle e_1, \dots, e_n \rangle$$

has length $n + 1$.

Upper bound $h(H) \leq n + 1$. Any set in $\mathcal{I}(H)$ is an affine subspace of \mathbb{F}_2^n . Along any strict inclusion chain of nonempty affine subspaces, the dimension drops by at least 1 at each step, so the chain length is at most n . Thus $h(H) \leq n$. \square

Combining Lemma 31 and Theorem 27 gives the following theorem.

Theorem 32. *Let H be the $N \times N$ Hadamard matrix with $N = 2^n$. Then*

$$n \leq \text{ind}_{\mathbb{Z}_2}(S(H)) \leq 2n + 1.$$

For the lower bound, note that, it is easy to see that the VC dimension of $H_{2^n \times 2^n}$ is n . Hence n is a lower bound on $\text{ind}_{\mathbb{Z}_2}(K)$.

6.2 Index vs. sign-rank for partial matrices

In this section, we show how to construct a partial matrix $M \subset \{-1, 1, *\}^{N \times N}$ such that

$$\text{ind}_{\mathbb{Z}_2}(S(A)) \leq 1 \quad \text{but} \quad \text{rank}^\pm(A) \geq \Omega(\sqrt{N}).$$

The construction is randomized and is based on the incidence matrix of the finite projective plane. The lower bound on sign-rank is based on a counting argument using Warren's theorem from real algebraic geometry. This is similar to the lower bound arguments in [AFR85, BDL98].

Theorem 33 (Warren[War68]). *Let f_1, \dots, f_t be real polynomials of degree at most k in m variables each, where $t \geq m$. Then the number of distinct sign patterns*

$$(\text{sign}(f_1(x)), \dots, \text{sign}(f_t(x))) \in \{-1, 0, 1\}^t$$

is at most $(\frac{ckt}{m})^m$ for some absolute constant c .

Construction. Let q be a prime power and let $N = q^2 + q + 1$. Let $\text{PG}(2, q)$ denote the finite projective plane of order q . Recall that: (1) there are N points and N lines, (2) each line contains exactly $q + 1$ points, and each point lies on exactly $q + 1$ lines, (3) every two distinct lines intersect in exactly one point.

Let $A \in \{0, 1\}^{N \times N}$ be the point–line incidence matrix, where rows correspond to lines and columns to points:

$$A_{\ell, p} = \begin{cases} 1 & \text{if } p \in \ell, \\ 0 & \text{otherwise.} \end{cases}$$

We define a random partial sign matrix $A \in \{-1, 1, *\}^{N \times N}$ as follows:

- if $A_{\ell, p} = 0$, set $M_{\ell, p} = *$,

- if $A_{\ell,p} = 1$, assign $M_{\ell,p} \in \{\pm 1\}$ independently and uniformly at random.

Proposition 34. *Let A be the random matrix constructed above. Then with probability $1 - o(1)$,*

$$\text{ind}_{\mathbb{Z}_2}(S(A)) \leq 1 \quad \text{but} \quad \text{rank}^{\pm}(A) \geq \Omega(\sqrt{n}).$$

Lemma 35. *Let K be a free \mathbb{Z}_2 -complex where every pair of facets intersects in at most one vertex. Then $\text{ind}_{\mathbb{Z}_2}(K) \leq 1$.*

Proof. Let G be the vertex-facet incidence graph of K . The \mathbb{Z}_2 -action on K induces a \mathbb{Z}_2 -action on G , and this action on G is again free: Indeed, no face of K is fixed by the freeness of the action, so no vertex of G is fixed. No edge is fixed either since the action respects the bipartition of G .

Every face σ of K of dimension at least one lies in a unique facet $F(\sigma)$ since facets intersect in at most one vertex. Define the simplicial map $f: \text{sd}(K) \rightarrow G$ from the barycentric subdivision of K to G by $f(\sigma) = \sigma$ if σ is a vertex and $f(\sigma) = F(\sigma)$ otherwise. To see that f is a simplicial map let $\sigma_0 \subset \dots \subset \sigma_k$ be a chain of faces in K . If σ_0 is not a vertex, then $f(\sigma_i) = F(\sigma_i) = F(\sigma_0)$, so the face $\{\sigma_0, \dots, \sigma_k\}$ of $\text{sd}(K)$ is mapped to the vertex $F(\sigma_0)$ in G . If σ_0 is a vertex and $k \geq 1$ (the case $k = 0$ being trivial) then $F(\sigma_1) = \dots = F(\sigma_k)$ and the image of the face $\{\sigma_0, \dots, \sigma_k\}$ is the edge $\{\sigma_0, F(\sigma_1)\}$ of G .

Clearly, the map f is \mathbb{Z}_2 -equivariant and G has dimension one. Thus $\text{ind}_{\mathbb{Z}_2}(K) \leq \text{ind}_{\mathbb{Z}_2}(G) \leq \dim(G) \leq 1$. \square

Proof of Prop. 34. The Index upper bound follows immediately from Lemma 35. For the sign-rank lower bound, fix $d \geq 1$. We bound the probability that

$$\Pr[\text{rank}^{\pm}(A) \leq d].$$

Suppose $\text{rank}^{\pm}(A) \leq d$. Then there exist vectors $u_{\ell}, v_p \in \mathbb{R}^d$ such that for every $\{\pm 1\}$ entry (ℓ, p) ,

$$M_{\ell,p} = \text{sign}(\langle u_{\ell}, v_p \rangle).$$

Hence $M_{\ell,p}$ is given by a degree-2 polynomial in the $2nd$ variables $\{u_{\ell}\}, \{v_p\}$. Let $t = N(q+1) = \Theta(N^{3/2})$ be the number of $\{\pm 1\}$ entries in A .

By Warren's theorem, with $k = 2$ and $m = 2Nd$, the number of realizable sign patterns on the $\{\pm 1\}$ entries is at most

$$\left(\frac{ckt}{m}\right)^m = \left(\frac{2ct}{2Nd}\right)^{2Nd} = \left(\frac{c(q+1)}{d}\right)^{2Nd}.$$

provided $2Nd \leq t$. On the other hand, the $\{\pm 1\}$ entries of A are independent and uniformly random. Hence

$$\Pr[\text{rank}^{\pm}(A) \leq d] \leq \frac{\left(\frac{c(q+1)}{d}\right)^{2Nd}}{2^t} \leq \exp\left(O\left(Nd \log\left(\frac{q}{d}\right)\right) - t\right).$$

Since $t = \Theta(N^{3/2})$, this probability is $o(1)$ if

$$Nd \log\left(\frac{q}{d}\right) \ll t,$$

which holds for $d \ll q$. Therefore, with high probability,

$$\text{rank}^{\pm}(A) \geq \Omega(\sqrt{N}).$$

\square

6.3 Index vs. coindex

For the sake of completeness, we include the following separation between coindex and index. The odd-dimensional real projective spaces $\mathbb{R}P^{2N-1}$ is the space of 1-dimensional real subspaces of \mathbb{C}^N , which carries a free $\mathbb{Z}/2$ -action induced by multiplication with i . Stolz [Sto89] determined the $\mathbb{Z}/2$ -index of $\mathbb{R}P^{2N-1}$ as $N + 1$, $N + 2$, or $N + 3$, depending on the remainder of N modulo 8. The coindex of a \mathbb{Z}_2 -space X space is bounded from above by the Stiefel–Whitney height, that is, the largest integer N such that the N th power of the Stiefel–Whitney class of the double cover $X \rightarrow X/\mathbb{Z}_2$ is non-zero in mod-2 cohomology. The Stiefel–Whitney height of $\mathbb{R}P^{2N-1}$ with a free \mathbb{Z}_2 -action is one; see Singh [Sin11].

On the other hand, Matsushita [Mat17] constructs free \mathbb{Z}_2 -spaces with coindex one and arbitrarily large Stiefel–Whitney height. The Stiefel–Whitney height in turn is a lower bound for the \mathbb{Z}_2 -index. To summarize, denoting Stiefel–Whitney height of a \mathbb{Z}_2 -space X by $\text{swh}(X)$ all inequalities in

$$\text{coind}_{\mathbb{Z}_2}(X) \leq \text{swh}(X) \leq \text{ind}_{\mathbb{Z}_2}(X)$$

have arbitrarily large separation.

6.4 VC dimension vs. coindex

Here we prove the following stronger separation between VC dimension and coindex.

Proposition 8 (Separating VC and coindex). *There exists a partial matrix $A \in \{-1, 1, *\}^{M \times N}$ such that $\text{VC}(M) = 1$ but*

$$\text{coind}_{\mathbb{Z}_2}(S(A)) = \Omega\left(\frac{\log^2 N}{\log \log N}\right).$$

Here we construct a simplicial complex K on $2N$ vertices, such that K does not contain any $\partial\Delta_2$, yet the $\text{coind}_{\mathbb{Z}_2}(K) = \Omega(\log^2 N)$. Then we can consider the partial matrix $A = A_K$ associated with K , and conclude that $\text{VC}(A) = 1$ but $\text{coind}_{\mathbb{Z}_2}(S(A)) = \Omega(\log^2 N)$. The complex K comes from state-of-the-art triangulations of the projective space with small number of vertices, due to Adiprasito et. al., [AAK22].

Theorem 36 (Efficient triangulations of $\mathbb{R}P^d$). *For every integer $d \geq 1$, there exists a triangulation of $\mathbb{R}P^d$ with N vertices where*

$$N \leq \exp\left(\left(\frac{1}{2} + o(1)\right)\sqrt{d \log d}\right).$$

The above implies the following separation between VC and coindex.

Proof. Let T be a triangulation of $\mathbb{R}P^d$ with N vertices given by Theorem 36. Let

$$\pi: \mathbb{S}^d \rightarrow \mathbb{R}P^d$$

be the antipodal double cover. Lifting the triangulation T along π gives a simplicial complex K together with a free involution τ such that

$$|K| \cong \mathbb{S}^d \quad \text{and} \quad |K|/\tau \cong |T| \cong \mathbb{R}P^d.$$

Equivalently, K is a free \mathbb{Z}_2 -triangulation of \mathbb{S}^d whose quotient is T . Since every vertex of T has exactly two lifts, the complex K has exactly $2N$ vertices, arranged in N antipodal pairs.

Now let $A := A_K$ be the partial sign matrix associated to K by Lemma 15. Then

$$S(A) \cong K.$$

Coindex. Since $|K|$ is \mathbb{Z}_2 -homeomorphic to \mathbb{S}^d with the antipodal action, we have

$$\text{coind}_{\mathbb{Z}_2}(S(A)) = \text{coind}_{\mathbb{Z}_2}(K) = \text{coind}_{\mathbb{Z}_2}(\mathbb{S}^d) = d.$$

VC dimension. We claim that K does not contain $\partial\Diamond_2$ as a free \mathbb{Z}_2 -subcomplex, in other words,

$$\omega_{\Diamond}(K) \leq 1.$$

Therefore, by the inequality proved earlier,

$$\text{VC}(A) \leq \omega_{\Diamond}(S(A)) = \omega_{\Diamond}(K) \leq 1.$$

Suppose $\partial\Diamond_2 \subseteq K$. Let

$$v_1 \sim v_2 \sim v_3 \sim v_4 \sim v_1$$

be the 4-cycle with $\tau(v_i) = v_{i+2}$. Set $x_i := \pi(v_i) \in T$ and let

$$\gamma := (x_1, x_2, x_3, x_4, x_1)$$

be the induced loop in T .

Since $\pi : |K| \rightarrow |T|$ is a 2-fold cover, the number of lifts of γ is at most 2 and each lift is determined by its starting point. However, the cycle above gives four distinct lifts:

$$(v_1, v_2, v_3, v_4, v_1), \quad (v_2, v_3, v_4, v_1, v_2), \quad (v_3, v_4, v_1, v_2, v_3), \quad (v_4, v_1, v_2, v_3, v_4),$$

which are pairwise distinct modulo the involution τ . Thus γ admits ≥ 4 lifts, a contradiction. Hence $\partial\Diamond_2 \not\subseteq K$, so

$$\omega_{\Diamond}(K) \leq 1.$$

Finally, by Theorem 36,

$$N \leq \exp\left(\left(\frac{1}{2} + o(1)\right)\sqrt{d \log d}\right).$$

Taking logarithms and rearranging gives

$$d = \Omega\left(\frac{\log^2 N}{\log \log N}\right)$$

proving the theorem. □

7 Concluding remarks and open questions

Consider the chain of inequalities

$$\text{coind}_{\mathbb{Z}_2}(S(A)) \leq \text{ind}_{\mathbb{Z}_2}(S(A)) \leq \text{ind}_{\mathbb{Z}_2}^{\text{lin}}(S(A)) = \text{rank}^{\pm}(A) - 1.$$

A downside is that all of these parameters are very hard to compute for a given input matrix A . Indeed, it is known that checking whether sign-rank is 3 or not is complete for the existential theory of the reals [Mnë06, BFG⁺09, BK15] which is a class including PSPACE. Also index and coindex are known to be hard [Mat03].

However, it is possible to insert more useful topological invariants between $\text{coind}_{\mathbb{Z}_2}$ and $\text{ind}_{\mathbb{Z}_2}$ that are efficiently computable. For example, the Stiefel–Whitney height of a \mathbb{Z}_2 -complex K , denoted by $\text{swh}(K)$, also known as cohomological index satisfies

$$\text{coind}_{\mathbb{Z}_2}(K) \leq \text{swh}(K) \leq \text{ind}_{\mathbb{Z}_2}(K)$$

and we discussed in Section 6.3 that there are K with $\text{coind}_{\mathbb{Z}_2}(K) = 1$ and arbitrarily large $\text{swh}(K)$. It is known that $\text{swh}(K)$ is efficiently computable in the number of faces of K (see p. 106 of [Mat03]) and therefore, it can be a useful computable lower bound on sign-rank of arbitrary matrices that is provably stronger than analytic lower bounds, as in the case of Gap Hamming Distance.

Upper bounds on \mathbb{Z}_2 -index. We showed that in the case of N by N random matrices and the Hadamard matrix, \mathbb{Z}_2 -index of the associated complex is at most $O(\log N)$. We leave the following as a question, which if true, indicates that the sign complex is not able to prove a strong lower bound on sign-rank of square matrices, and one needs to construct significantly richer complexes than the sign complex to prove strong bounds. We phrase the question more generally for strongly regular free \mathbb{Z}_2 -CW complexes, which are a broader class than free \mathbb{Z}_2 -simplicial complexes. In a strongly regular CW complex the intersection of any two faces is a (possibly empty) face.

Question 37. Is it true that any finite strongly regular free \mathbb{Z}_2 -CW-complex K with N vertices and M inclusion maximal cells, satisfies $\text{ind}_{\mathbb{Z}_2}(K) \leq O(\log N \log M)$?

If true, it implies that for any N by N sign matrix A , $\text{ind}_{\mathbb{Z}_2}(S(A)) \leq O(\log^2 N)$ since $S(A)$ has $2N$ vertices and at most $2N$ facets.

Question 37 may be viewed as a topological generalization of the classical Figiel–Lindenstrauss–Milman inequality for centrally symmetric polytopes [FLM76].

Theorem 38 (Figiel–Lindenstrauss–Milman theorem [FLM76]). *There exists an absolute constant $c > 0$ such that for every centrally symmetric polytope $P \subset \mathbb{R}^d$ with N vertices and M facets,*

$$d \leq c \cdot \log N \cdot \log M$$

To see the connection, let $K = \partial P$ be the boundary complex of a centrally symmetric polytope $P \subset \mathbb{R}^d$, equipped with the antipodal \mathbb{Z}_2 -action. Then K is a strongly regular \mathbb{Z}_2 -CW complex with N vertices and M facets, and

$$\text{coind}_{\mathbb{Z}_2}(K) = \text{ind}_{\mathbb{Z}_2}(K) = d - 1.$$

Indeed radial projection from ∂P to \mathbb{S}^{d-1} gives a \mathbb{Z}_2 -equivariant map $|K| = \partial P \rightarrow \mathbb{S}^{d-1}$.

Hence a positive answer to Question 37 (even with $\text{coind}_{\mathbb{Z}_2}$ instead of $\text{ind}_{\mathbb{Z}_2}$) would recover Theorem 38. However, it is significantly stronger, because boundary complexes of centrally symmetric polytopes form a highly constrained subclass of strongly regular free \mathbb{Z}_2 -CW complexes with strong local structure.

Equivalence of index and sign-rank for total matrices. Regardless of the answer to Question 37, one may still hope for an equivalence between sign-rank and index. We have already showed that for partial matrices this is impossible, since in Proposition 34 we constructed a partial matrix with $\text{ind}_{\mathbb{Z}_2}(S(A)) = 1$ while $\text{rank}^{\pm}(A) \geq \Omega(\sqrt{N})$. However, we only have an exponential separation in the case of total matrices and it is still possible that in this case, there is a qualitative equivalence between index and sign-rank.

Question 7. Does there exist a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that for every total sign matrix A ,

$$\text{rank}^\pm(A) \leq f(\text{ind}_{\mathbb{Z}_2}(S(A)))?$$

If the question above has a positive answer, a large number of fundamental problems in sign-rank literature can be entirely formulated in terms of \mathbb{Z}_2 -index of free \mathbb{Z}_2 -simplicial complexes. Some of these problems include bounding closure properties of sign-rank and equivalence to semi-algebraic complexity [Lov19, HHP⁺22], the possibility of upper bounding sign-rank based on margin [LMSS07, HHM23], and upper bounding sign-rank based on factorization norm [HHP⁺22].

Equivalence of coindex and VC dimension for total matrices. In the case of partial matrices, we showed how to obtain a strong separation between VC dimension and coindex, and we stated the following interesting question for total matrices which would imply learnability is a topological phenomenon for total concept classes.

Question 9. Does there exist a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that for every total sign matrix A ,

$$\text{coind}_{\mathbb{Z}_2}(S(A)) \leq f(\text{VC}(A))? \tag{6}$$

Here we discuss another implication a positive answer to the above question. If the above question has a positive answer, combined with our lower bound on coindex of GHD_k^n , it immediately implies the following, which is a strong form of a major open problem on learnability of disambiguations of the concept class of half-spaces with margin conjectured by Alon et. al., [AHHM22]

Conjecture 39. *There exists a function $g(k) \rightarrow \infty$ such that the following is true: for every completion of GHD_k^n to a total sign matrix A by changing all the $*$ outputs of GHD_k^n arbitrarily to $-1, 1$,*

$$\text{VC}(A) \geq g(k).$$

Now we explain how Question 9 implies Conjecture 39 with the function $g(k) = f^{-1}(k)$. Here we can assume f is strictly increasing hence invertible. Indeed, suppose the answer to Question 9 is positive and let A be an arbitrary total completion of GHD_k^n . Then $S(\text{GHD}_k^n)$ is a \mathbb{Z}_2 -subcomplex of $S(A)$. By combining Equation (6) with monotonicity of $\text{coind}_{\mathbb{Z}_2}(\cdot)$, we have:

$$f(\text{VC}(A)) \geq \text{coind}_{\mathbb{Z}_2}(S(A)) \geq \text{coind}_{\mathbb{Z}_2}(S(\text{GHD}_k^n)) = k.$$

The last inequality is using our weak bound on $\text{coind}_{\mathbb{Z}_2}(S(\text{GHD}_k^n))$.

In fact, in a surprising result, recently Chornomaz, Moran, and Waknine [CMW25] have defined a notion of *spherical dimension*, as the \mathbb{Z}_2 -coindex of a topological \mathbb{Z}_2 -space associated with a total concept class on the sphere, and proved that bounding spherical dimension in terms of VC dimension is *equivalent* to the corresponding disambiguation-type statement for half-spaces with margin. It may be possible to similarly here show that Conjecture 39 implies Question 9.

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